Exercise 1 in Algorithms

The Interdisciplinary Center, Herzelia

Due: 5/11/14

Problem 1 (20 pts):

Consider the following adjacency matrix M, which represents a directed graph G (M(i,j)=1 if and only if there is an edge from i to j). Empty entries in the figure correspond to M(i,j)=0.

	a	b	c	d	e	f	g
a		1		1	1		
b			1	1			
c				1			
d						1	
e				1			
f							1
g							

A. Determine the topological sort of G achieved by implementing the topological sort algorithm using a stack. Analyze the time complexity of this implementation (the analysis should be general, as a function of |V| and |E| and not specific to the given graph).

B. The edge (f,g) is removed from the graph. List all the different topological sorts the resulting graph has.

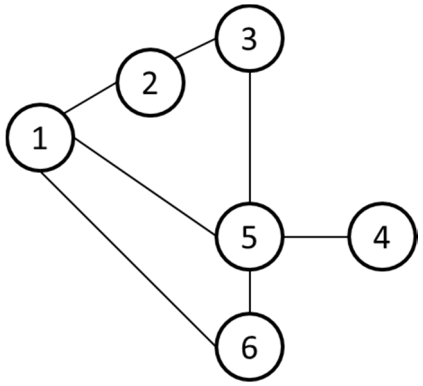
Problem 2 (20 pts):

Let A, B be two n * n binary matrices. Consider the entries of A,B as Boolean values.
The binary matrix multiplication $C = A \cdot B$ is defined by $C[i, j] = OR_{1 \le k \le n}(A[i,k] \text{ AND } B[k,j])$.
For a single binary matrix, A, and an integer k>0, the matrix A^k is defined as follows:
For k=1, $A^1=A$.
For k>1, $A^k=A^{k-1} \cdot A$.

Let A be the adjacency matrix of a simple directed graph G.
Prove by induction on k: $A^k[i,j]=1$ if and only if the graph G includes a path of length k from i to j.

Problem 3 (20 pts):
The undirected graph H is given in the figure.

a. Draw the representation of H by an adjacency matrix and by adjacency lists. What is the size of each of the data structures if it takes x bits to store a number or a letter, and a single bit to store a binary value? (use the actual number of vertices and edges in H to give your answer).



b. Convert H into a direct graph: each edge (i, j) is directed from i to j if and only if i<j. Let H' be the resulting graph.

   i. Does H' contain a directed cycle?

   ii. Repeat part ***a*** of the problem for H'.

Problem 4 (20 pts):

a. A $source$ in a directed acyclic graph (DAG) is a vertex with in-degree 0. Prove that every DAG contains at least one $source$.
b. Definition: A Hamiltonian path is a path that visits each vertex exactly once.
   Prove the following claim:
   Claim: A DAG has a unique topological sort if and only if it has a Hamiltonian path.

Problem 5 (20 pts):

1. Draw a directed graph G over seven vertices that has exactly three strongly connected components. The underlying graph of G should be connected. List the strongly connected components of the graph.

2. What is the maximal possible number of edges in a simple directed acyclic graph over n vertices? Explain your answer.

3. What is the minimal and maximal number of edges in a simple connected undirected bipartite with n_1 vertices in the left side and n_2 vertices in the right side? Explain your answer.