Image Denoising - Ex 1

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1. Theoretical Questions:

1.1

$p(x) = \frac{1}{z_x} exp(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))$
$p(y|x) = \frac{1}{z_{y|x}} exp(-\frac{1}{2\sigma^2}(x-y)^T(x-y))$

where z_x and $z_{y|x}$ are normalization factors. Therfore:

$p(x,y) = \frac{1}{z_x z_{y|x}} exp(-\frac{1}{2\sigma^2}(x-y)^T(x-y) - \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))$

As shown in class, in the case that x, y are gaussian, we have that:

$E(x|y) = \underset{x}{argmax} p(x,y) = \underset{x}{argmin}(\frac{1}{2\sigma^2}(x-y)^T(x-y) + \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))$

Taking the derivative w.r.t x and equating to 0 yields the desired solution:

$\frac{\partial}{\partial x}(\frac{1}{2\sigma^2}(x-y)^T(x-y) + \frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)) = \Sigma^{-1}(x-\mu) + \frac{1}{\sigma^2}(x-y) = 0$
$\Rightarrow (\Sigma^{-1} + \frac{1}{\sigma^2}I)x* = (\Sigma^{-1}\mu + \frac{1}{\sigma^2}y) \Rightarrow x* = (\Sigma^{-1} + \frac{1}{\sigma^2}I)^{-1}(\Sigma^{-1}\mu + \frac{1}{\sigma^2}y)$

1.2

Noting that $U^T = U^{-1}$ in the case of a unitary (orthogonal) matrix, we have that:
$B\Sigma B^T = P^T \Sigma P = P^T(P\Lambda P^T)P = \Lambda$
Which is indeed a diagonal matrix (the eigenvalues of the covariance matrix).

1.3

$t = P^T y = P^T(x+\eta_1) = P^T((P^T)^{-1}s + \eta_1) = s + P^T\eta_1 = s + \eta_2$

Since P is an orthogonal matrix, it merely performs a rotation of its inputs, and

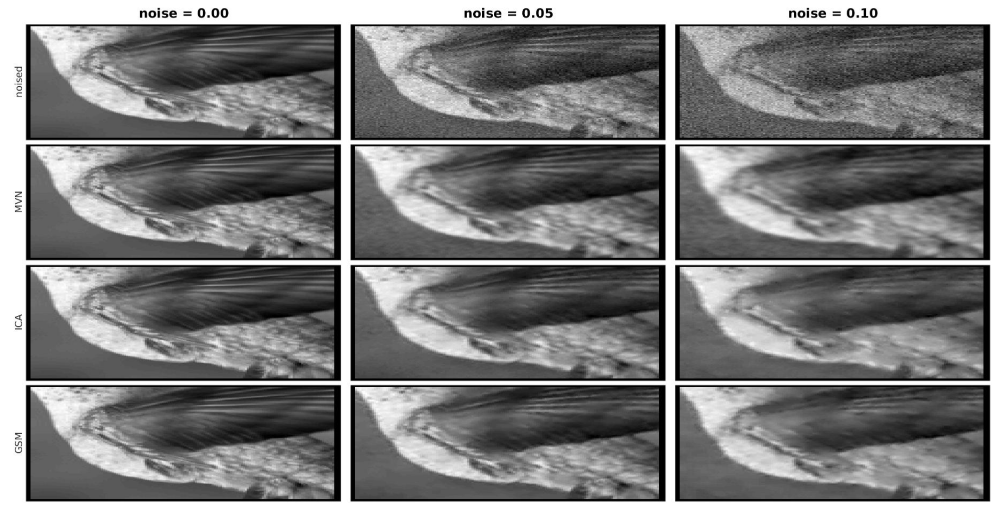
therefore, a zero-symmetric random vector (η) remains a zero symmetric random
vector ($P^T\eta_1 = \eta_2$).
Now, since $x = Ps$, $y = Pt$ and using the linearity of the expectation, we have that:

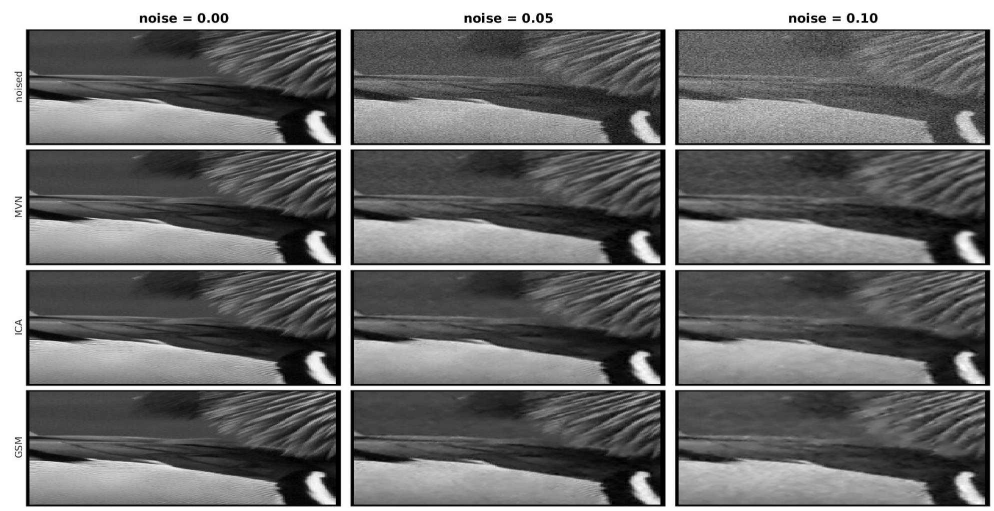
$\hat{x} = E(x|y) = P \cdot E(s|y) = P \cdot E(s|t)$
Where the last transition is a result of deterministic relation between t and y.
The above derivation shows that t is a noisy version of s, where the coordinates of
t are independently noised. Hence we can perform a coordinate wise denoising of
t_i and obtain the optimal estimation of s_i.

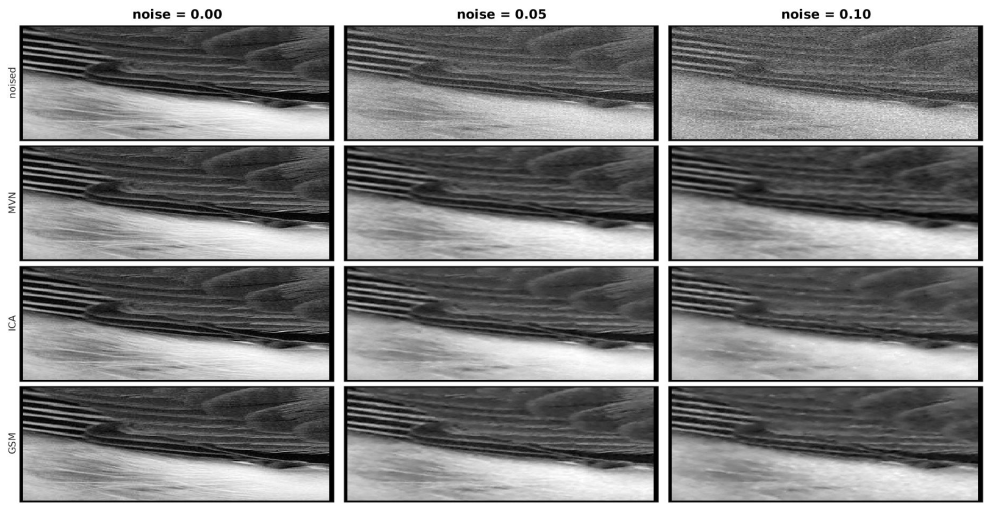
2. Models Comparison:

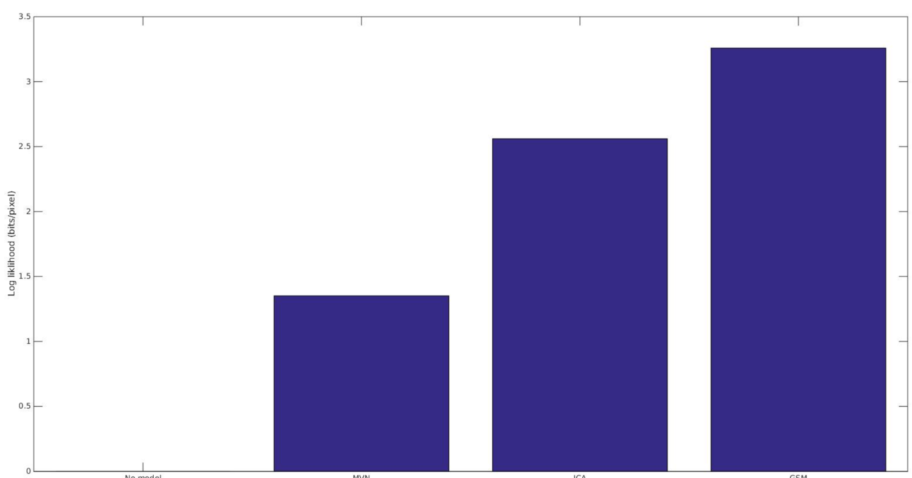
Models review:
The MVN model seems to retain the original image's edges well, however it comes at the expense of denoising quility.
Meaning that the MVN didn't succeed to denoise the image very well.
The ICA and GSM models seems to denoise the image much better than the MVN (especially when the noise variance is significant).
Although, the ICA and GSM models blurred the image much more than the MVN model.
From our human prespective - we think that the GSM model has slightly better denoising results than the ICA model.
In addision, the simpe MVN model has performed significantly worse than both the GSM and ICA models.

Here are 3 images whcih support our claim above:


noise = 0.00
noise = 0.05
noise = 0.10
noised
MVN
ICA
GSM


noise = 0.00
noise = 0.05
noise = 0.10
noised
MVN
ICA
GSM


noise = 0.00
noise = 0.05
noise = 0.10
noised
MVN
ICA
GSM


Log likelihood (bits/pixel)
No model
MVN
ICA
GSM

Loglikelihood measurement:
Looking on the logliklihood measurements we can see that the GSM model has the highest bin among the other models.
We also can see that the ICA model in his turn has an significantly higher likelihood than the MVN model

pSNR measurement:

pSNR table										
noise level		0.001			0.05			0.1		
img No.		1	2	3	1	2	3	1	2	3
Model	no model	59.963	60.065	60.070	26.002	26.041	26.011	19.980	19.984	20.031
	MVN	59.379	59.861	59.455	29.503	30.093	28.089	27.307	27.220	25.875
	ICA	59.935	60.121	59.973	30.188	30.310	28.222	27.798	22.195	25.895
	GSM	59.990	60.252	60.071	30.985	31.621	29.480	28.369	28.265	26.914

Looking on the pSNR table we can can conclude some interesting conclusions:
- When noising with 0.001 std noise we can see that no model has better pSNR score than the other models, it's
reasonable since when the noise is so minor the models seems to ruin more than being useful.
- When noising with larger std noises we can see that the no model has much worse pSNR score than the other models.
Which is a good sanity check that tells us that the models actually perform better denoising than doins nothing.
- Generally comparing the pSNR scores of the models we can see that the GSM model achieves the best score
in each of the images shown (when noising with noise of more that 0.001 std).

To sum up, we conclude that the GSM model lead in both our human prespective measure and in the logliklihood and
pSNR quantitative measures.
Therefore our conclusion is that The GSM model is the best model for the denoising task in the given framework
(gaussian noise with 0 mean and sigma * I covariance).