Fundamental Aerodynamics
Boundary Layers (1)

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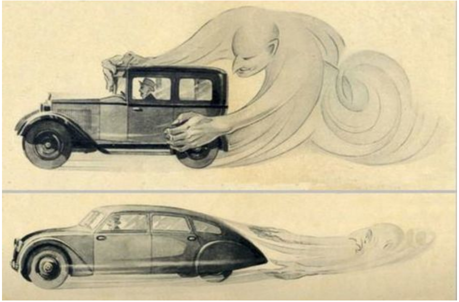

Introduction

Boundary layer equations

- Prandtl's BL equations
- Definitions of skin friction and BL thicknesses

Solutions for Laminar boudary layers

- Inviscid/viscous coupling
- Blasius solutions
- von Kármán integral equation

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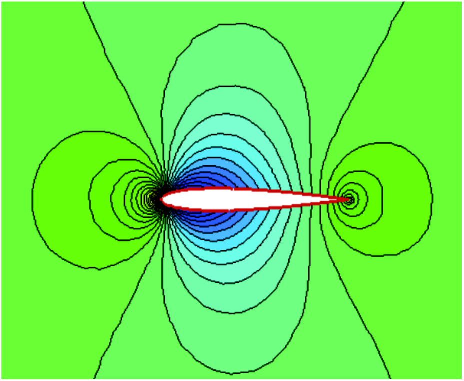
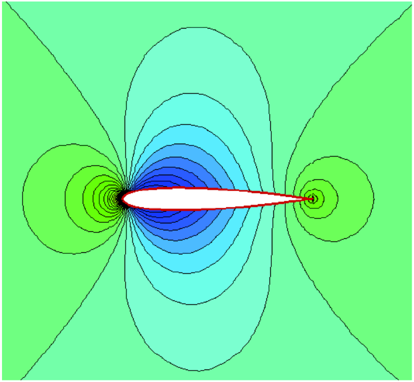
Pressure field around an airfoil is sensibly the same, both in inviscid and in viscous flow. **Why?**

Pressure contours for NACA0012, M=0.5, α=0°

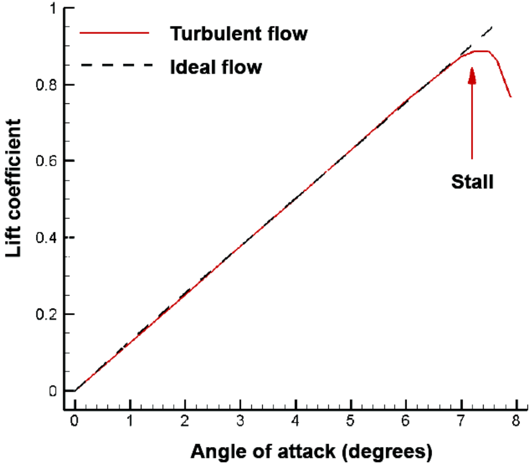


Ideal flow



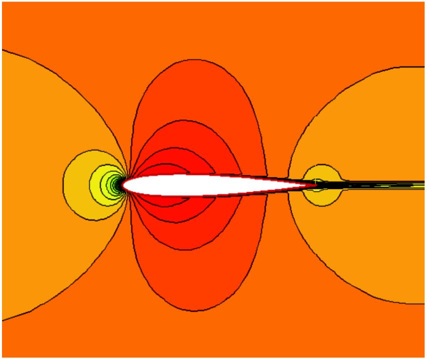
Viscous flow

Moreover, inviscid flow theories provide an excellent estimate of the lift coefficient, at least for low values of the angle of attack

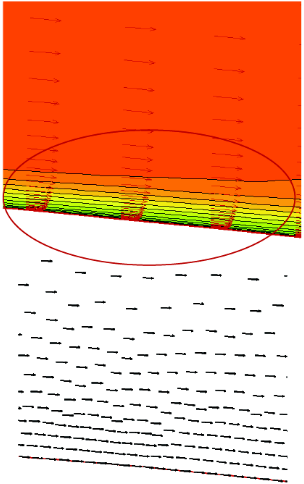


Explanation: presence of a **boundary layer**

Velocity contours



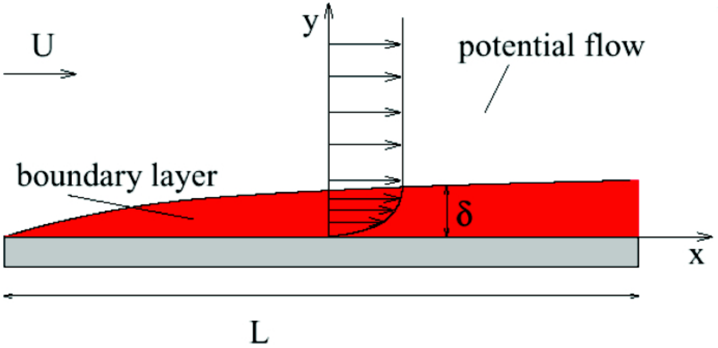
Viscous flow



Ideal flow

Thin layer close to the wall where viscous effects cannot be neglected

- The higher *Re*, the thinner the boundary layer
- Close to the wall, viscous effects are always important
- Flow has to satisfy the no-slip condition



Qualitative aspects

- Ideal fluid: $\mathbf{T} = -p\mathbf{I} \quad \rightarrow$ Euler equations

- Actually, air = Newtonian viscous fluid

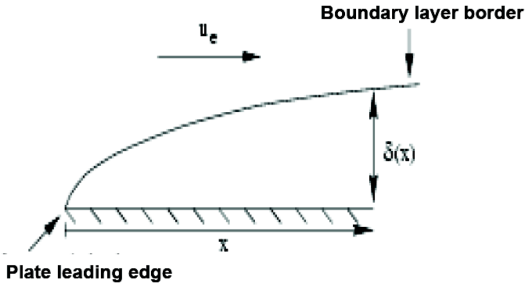
$$\begin{cases} \mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \\ \boldsymbol{\tau} = 2\mu\mathbf{D} - \dfrac{2}{3}\mathrm{Tr}(\mathbf{D})\mathbf{I} \end{cases}$$

  - μ= dynamic viscosity [$\sim 1.8 \times 10^{-5}$ kg/(m s)] at normal conditions for T and p
  - viscous stresses are not negligible in regions characterized by high velocity gradients (shear layers)

Evolution of boundary layer thickness along a wal

- convection: $\dfrac{\partial u}{\partial t} + u_e \dfrac{\partial u}{\partial x} = 0 \;\Rightarrow\; u_e \propto x/t$
- diffusion: $\dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial y^2} \;\Rightarrow\; 1/t \propto \nu/\delta^2$

$$\Rightarrow \quad \frac{\delta(x)}{x} \propto \frac{1}{\sqrt{u_e x/\nu}}$$



Nombre de Reynolds local:

$$\boxed{Re_x = \frac{u_e x}{\nu}}$$

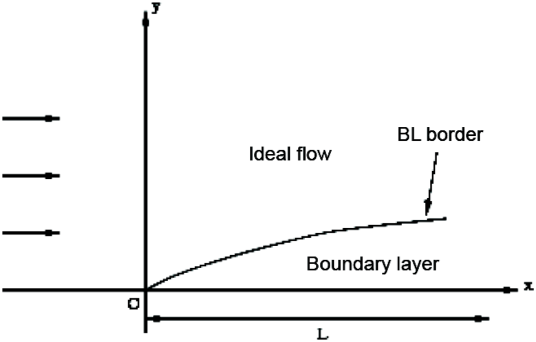
$$\boxed{\frac{\delta(x)}{x} \propto \frac{1}{\sqrt{Re_x}}}$$

- Orders of magnitude of BL thickness in Aeronautics :
  - cruise flight speed of an aircraft: *several hundreds* m/s
  - air kinematic viscosity of the order of 10^{-5} m^2/s

- Ratio u_e/ν of the order of 10^7 m^{-1}

- BL thickness :
  - some mm at a wing leading edge,
  - some cm at a wing trailing edge,
  - some dm at the rear of the fuselage

- General description of viscous fluid flow: Navier-Stokes equations
- **High-Reynolds** flow past an obstacle:
  Boundary layer thickness $\ll$ obstacle length (e.g., chord)
  $\hookrightarrow$ *simplified* flow description



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- Navier-Stokes equations for 2D incompressible steady flow:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{1}$$

- Non-dimensional equations: *Unique* length scale L and velocity scale

$$\begin{cases} x = L\bar{x}, \quad y = L\bar{y} \\ u = U_\infty \bar{u}, \quad v = U_\infty \bar{v} \\ p = \rho U_\infty^2 \bar{p} \end{cases}$$

- Non-dimensional equations:

$$\left\{\begin{array}{l}\dfrac{\partial \bar{u}}{\partial \bar{x}}+\dfrac{\partial \bar{v}}{\partial \bar{y}}=0\\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\dfrac{\partial \bar{u}}{\partial \bar{y}}=-\dfrac{\partial \bar{p}}{\partial \bar{x}}+\dfrac{1}{Re}\left(\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2}+\dfrac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)\\ \bar{u}\dfrac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\dfrac{\partial \bar{v}}{\partial \bar{y}}=-\dfrac{\partial \bar{p}}{\partial \bar{y}}+\dfrac{1}{Re}\left(\dfrac{\partial^2 \bar{v}}{\partial \bar{x}^2}+\dfrac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right)\end{array}\right. \quad \underset{Re\to\infty}{\longrightarrow} \quad \left\{\begin{array}{l}\dfrac{\partial \bar{u}}{\partial \bar{x}}+\dfrac{\partial \bar{v}}{\partial \bar{y}}=0\\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\dfrac{\partial \bar{u}}{\partial \bar{y}}=-\dfrac{\partial \bar{p}}{\partial \bar{x}}\\ \bar{u}\dfrac{\partial \bar{v}}{\partial \bar{x}}+\bar{v}\dfrac{\partial \bar{v}}{\partial \bar{y}}=-\dfrac{\partial \bar{p}}{\partial \bar{y}}\end{array}\right.$$

- Flow characteristic Reynolds number: $Re=\dfrac{U_\infty L}{\nu}$

$\rightarrow$ Euler equations / ideal flow *incompatible* with no-slip condition at solid walls

- Existence of a region with thickness tending to 0 when the Reynolds number becomes large, not governed by the Euler equations
  $\hookrightarrow$ solution of problem (1) is decomposed into the solution of two sub-problems:
  - External problem, governed by ideal flow equations (Euler equations)
    Characteristic length scale $= L$
  - Internal problem in the immediate neighborhood of solid walls governed by the boundary layer equations
    Characteristic length scale $=$ BL thickness $\delta \ll L$
- Both solution satisfy matching conditions, at the interface of both regions, so that the inner solution *matches* the external solution (FP)
  Characteristic velocity scale in the inner region $= V_0 \ll U_\infty$

- System non-dimensionalization:

$$\left\{\begin{array}{ll} x = L\bar{x}, & y = L\bar{y} \\ u = U_\infty \bar{u}, & v = U_\infty \bar{v} \\ p = \rho U_\infty^2 \bar{p} & \end{array}\right. \longrightarrow \left\{\begin{array}{ll} x = L\bar{x}, & y = \delta\tilde{y} \\ u = U_\infty \bar{u}, & v = V_0 \tilde{v} \\ p = \rho U_\infty^2 \bar{p} & \end{array}\right.$$

- Matching of the inner and outer solutions

$$\lim_{\tilde{y}\to\infty} \bar{u}(\bar{x},\tilde{y}) = \lim_{\bar{y}\to\infty} \bar{u}(\bar{x},\bar{y}) = \bar{u}_e(\bar{x})$$

Similar condition holds for the pressure p

Boundary layer equations

- Transformation of system (1):

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{L}{\delta}\dfrac{V_0}{U_\infty}\dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{L}{\delta}\dfrac{V_0}{U_\infty}\tilde{v}\dfrac{\partial \bar{u}}{\partial \tilde{y}} = -\dfrac{\partial \bar{p}}{\partial \bar{x}} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \left(\dfrac{L}{\delta}\right)^2 \dfrac{\partial^2 \bar{u}}{\partial \tilde{y}^2}\right) \\ \bar{u}\dfrac{\partial \tilde{v}}{\partial \bar{x}} + \dfrac{L}{\delta}\dfrac{V_0}{U_\infty}\tilde{v}\dfrac{\partial \tilde{v}}{\partial \tilde{y}} = -\dfrac{L}{\delta}\dfrac{U_\infty}{V_0}\dfrac{\partial \bar{p}}{\partial \tilde{y}} + \dfrac{1}{Re}\left(\dfrac{\partial^2 \tilde{v}}{\partial \bar{x}^2} + \left(\dfrac{L}{\delta}\right)^2 \dfrac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right) \end{cases}$$

- Minimal degeneracy principle: $\dfrac{\delta}{L} = \dfrac{V_0}{U_\infty}$ and $\dfrac{1}{Re}\left(\dfrac{L}{\delta}\right)^2 = 1$

  $\hookrightarrow$ BL characteristic lengths

$$\delta = \frac{L}{\sqrt{Re}}, \quad V_0 = \frac{U_\infty}{\sqrt{Re}}$$

- Transformation of system (1):

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \tilde{v}\dfrac{\partial \bar{u}}{\partial \tilde{y}} = -\dfrac{\partial \bar{p}}{\partial \bar{x}} + \dfrac{1}{Re}\dfrac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \dfrac{\partial^2 \bar{u}}{\partial \tilde{y}^2} \\ \dfrac{1}{Re}\left(\bar{u}\dfrac{\partial \tilde{v}}{\partial \bar{x}} + \tilde{v}\dfrac{\partial \tilde{v}}{\partial \tilde{y}}\right) = -\dfrac{\partial \bar{p}}{\partial \tilde{y}} + \dfrac{1}{Re^2}\dfrac{\partial^2 \tilde{v}}{\partial \bar{x}^2} + \dfrac{1}{Re}\dfrac{\partial^2 \tilde{v}}{\partial \tilde{y}^2} \end{cases}$$

- Thus, for $Re \to \infty$

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \tilde{v}\dfrac{\partial \bar{u}}{\partial \tilde{y}} = -\dfrac{\partial \bar{p}}{\partial \bar{x}} + \dfrac{\partial^2 \bar{u}}{\partial \tilde{y}^2} \\ 0 = -\dfrac{\partial \bar{p}}{\partial \tilde{y}} \end{cases}$$

Constant pressure across the boundary layer, i.e. $\bar{p} = \bar{p}_e(x)$

Pressure verifies Bernoulli's relation

$$\bar{p}_e + \frac{1}{2}\bar{u}_e^2 = \text{cste}$$

Boundary layer equations

- Boundary layer equations or Prandtl equations

$$\begin{cases} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \tilde{v}\dfrac{\partial \bar{u}}{\partial \tilde{y}} = \bar{u}_e \dfrac{d\bar{u}_e}{d\bar{x}} + \dfrac{\partial^2 \bar{u}}{\partial \tilde{y}^2} \end{cases}$$

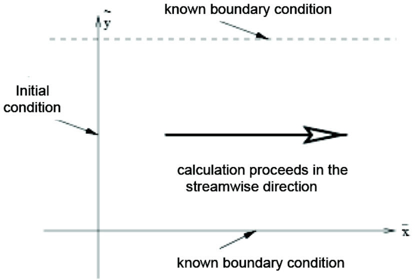
- no-slip condition: $\bar{u}(\bar{x},0)=0$, $\tilde{v}(\bar{x},0)=0$
- matching with external layer: $\bar{u}(\bar{x},\infty)=\bar{u}_e(\bar{x})$

- Dimensional boundary layer equations:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = u_e \dfrac{du_e}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2} \end{cases}$$

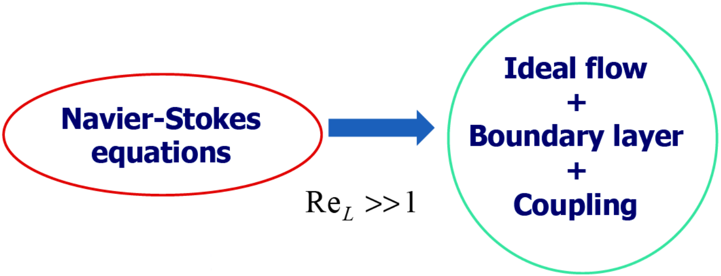
- no-slip condition: $u=v=0$ for $y=0$
- matching with external layer: $u(x,\infty)=u_e(x)$

- System of parabolic equations *i.e.* describing a marching problem in the streamwise flow direction x
  $\hookrightarrow$ the system of Prandtl equations admits an unique solution when the problem is posed in the following form :



- Solved by space marching from upstream to downstream
- Solution blowup if flow separates ($\bar{u}(\bar{x}, \tilde{y}) < 0$ becomes negative, so information propagates in the opposite sense)

Boundary layer equations



If:

- there is no flow separation
- no strong wall curvature

- Viscous stress tensor:

$$\boldsymbol{\tau} = \begin{pmatrix} 2\mu\dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & 2\mu\dfrac{\partial v}{\partial y} \end{pmatrix}$$

- Total stress tensor: $\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau}$
  for high Reynolds number flow

$$\boldsymbol{\tau} = \begin{pmatrix} -p & \mu\dfrac{\partial u}{\partial y} \\ \mu\dfrac{\partial u}{\partial y} & -p \end{pmatrix}$$

$\rightarrow$ Wall stress : $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$, where $\mathbf{n}=$ unit normal to the wall

- Total wall stress:
  - purely normal (pressure) for ideal flow
  - both normal and tangential (viscous friction) components are present for viscous flow

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{n} = \begin{pmatrix} \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0} = \tau_w \\ -p_e(x) \end{pmatrix}$$

with τ_w the viscous wall stress (skin friction)

- Skin friction normalized with wall pressure = friction coefficient

$$\boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2}}$$

Boundary layers and skin friction

- Total viscous friction force $=$ friction drag for instance, on the upper face of a flat plat of length L:

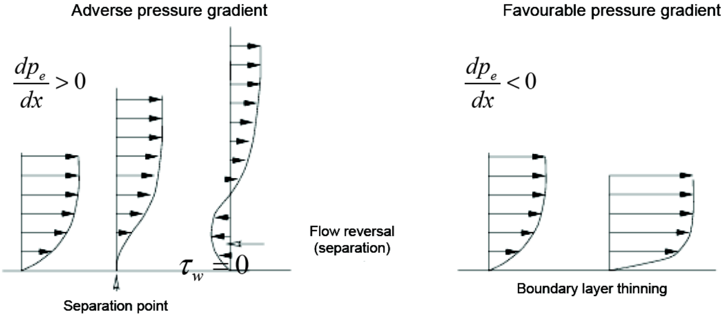
$$\boxed{D = \int_0^L \tau_w(x)\, dx}$$

- Friction drag coefficient:

$$C_D = \frac{D}{\frac{1}{2}\rho u_e^2 L} = \frac{1}{L}\int_0^L \tau_w(x)\, dx \propto \frac{1}{\sqrt{Re_L}}$$

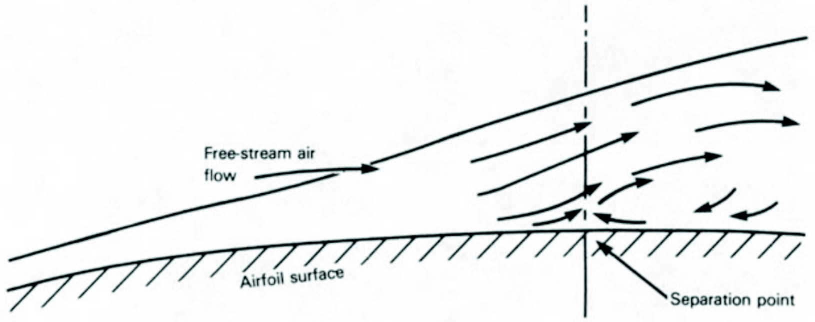
Friction drag $=$ substantial contribution to the total drag of modern aircraft
$\hookrightarrow$ friction drag reduction is a major challenge in Aeronautics

- Viscous friction slows down the flow in the vicinity of solid walls
- Positive pressure gradient $dp_e/dx \quad \Rightarrow$ further slow down the flow
  $\hookrightarrow$ fluid layer closest to the wall may be slowed down to zero velocity, and even reverse their direction: this phenomenon is called **separation**

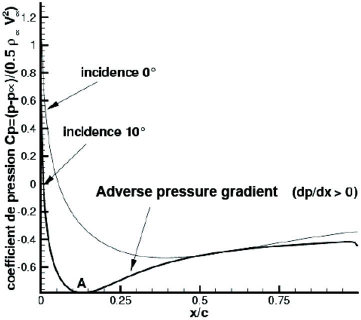


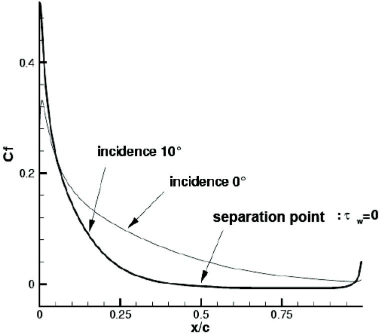
Boundary layer separation

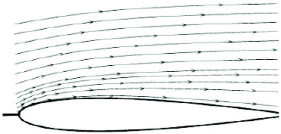
Sketch of streamline patterns around a separation point

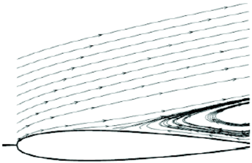


Flow separation past an airfoil





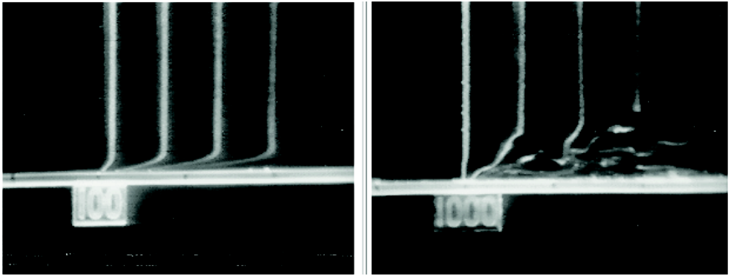




Laminar/turbulent transition

Experimental observations show that:

- Laminar BL are characterized by a regular flow motion, similar to the relative motion of thin fluid ”laminae”
- For Reynolds numbers in the range 300 000 to 3 000 000 BL flow is no longer regular (laminar) and tends to become chaotic → **transition**



Laminar/turbulent transition

- Origin of laminar/turbulent transition :
  - Solutions of the Navier-Stokes equations become unstable when the Reynolds number exceeds a "critical" value which strongly depend on the specific flow configuration: small perturbations (due, e.g. to wall roughness) become unstable and amplified, leading to laminar/turbulent transition
  - Exact transition mechanisms are ill known yet (very active research field)

- Laminar BL and turbulent BL have a very different behavior, namely in terms of viscous friction and flow separation

  $\hookrightarrow$ accurate localization of the transition point is of the utmost importance for predicting aerodynamic performance

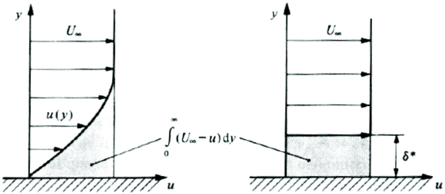
- Presently, predictions are mostly based on empirical criteria (experimental correlations):
  - implies a Reynolds number based on abscissa x and a Reynolds number based on boundary layer thickness

$$(Re_\theta)_{\text{trans}} \simeq 2.9(Re_x)^{0.4}_{\text{trans}} \qquad \text{Michel (1952)}$$

- **BL Thickness** (conventional definition): $\dfrac{u(x,\delta(x))}{u_e(x)} = 0.99$

- **Displacement thickness** δ^*: mass flow lost wrt ideal flow, due to the BL

$$\int_0^\delta \rho_e u_e dy - \int_0^\delta \rho u dy = \rho_e u_e \delta^* \quad \text{thus} \quad \boxed{\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy}$$



- **Momentum thickness** θ: momentum lost wrt to ideal flow (for a given mass flow rate), due to the BL

$$\left(\int_0^\delta \rho u dy\right) u_e - \int_0^\delta \rho u^2 dy = \rho_e u_e^2 \theta \quad \text{thus} \quad \boxed{\theta = \int_0^\delta \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dy}$$

- **Shape factor**:

$$\boxed{H = \frac{\delta^*}{\theta}}$$

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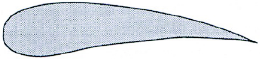
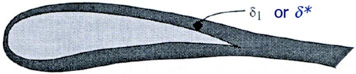
- Main objective of quantitative flow analyses :
  - estimate wall stress τ_w and found friction drag
  - predict boundary layer separation (if any)
  - estimate the laminar/turbulent transition region

- The outer velocity u_e is a datum from preliminary inviscid flow analysis

- Compute the pressure distribution for an ideal flow around an airfoil $\rightarrow \; p_e(x)$



- Solve the BL equations using $p_e(x)$ or $u_e(x)$ with
$$\rho u_e(x)\frac{du_e(x)}{dx} = -\frac{dp_e(x)}{dx}$$
$\hookrightarrow u(x,y)$ and $v(x,y) \rightarrow c_f$ (friction drag) and δ^*
- Construct a fictitious airfoil by augmenting the true airfoil by the computed displacement thickness + a wake



- Update the pressure distribution $\rightarrow$ find pressure drag

Remark: this kind of coupling is only possible for attached flow

Solution methods for the BL equations

- No known analytical solutions of the Prandtl equations for general configurations
  $\rightarrow$ Use of approximate solutions:
  numerical solution of local equations by means, e.g. of finite difference methods

- Existence of quasi-exact (self-similar) solutions for simple geometries, corresponding to simple outer pressure distributions
  $\rightarrow$ Can be used to validate approximation methods

- Calculation of approximate solutions using the integral equations + use of self-similar solutions where needed
  $\rightarrow$ this is the approach studied in the following

Self-similar solutions: Blasius solution

Flow over a semi-infinite flat plate with zero incidence in incompressible inviscid flow

- no geometrical length scale
- outer flow solution: $u_e = U_\infty$
- BL equations become:

$$\left\{\begin{array}{l} \dfrac{\partial \bar{u}}{\partial \bar{x}} + \dfrac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \\ \bar{u}\dfrac{\partial \bar{u}}{\partial \bar{x}} + \tilde{v}\dfrac{\partial \bar{u}}{\partial \tilde{y}} = \dfrac{\partial^2 \bar{u}}{\partial \tilde{y}^2} \end{array}\right. \quad \text{with} \quad \left\{\begin{array}{ll} \bar{u} = \tilde{v} = 0 & \text{for } \tilde{y} = 0 \\ \bar{u} = 1 & \tilde{y} \to \infty \end{array}\right.$$

- we look for a solution under the form

$$\bar{u} = h(\bar{x}, \tilde{y}) \quad \text{thus} \quad \frac{u}{U_\infty} = h\left(\frac{x}{L}, \frac{y}{L}\sqrt{\frac{U_\infty L}{\nu}}\right)$$

- L being arbitrary, we look for u/U_∞ under the form of a function $h(\eta)$ where η is a non-dimensional parameter built upon $\bar{x}$ and $\tilde{y}$ and independent on L:

$$\frac{u}{U_\infty} = h\left(\frac{\tilde{y}}{\sqrt{\bar{x}}}\right) = h\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) = h(\eta) = f'(\eta)$$

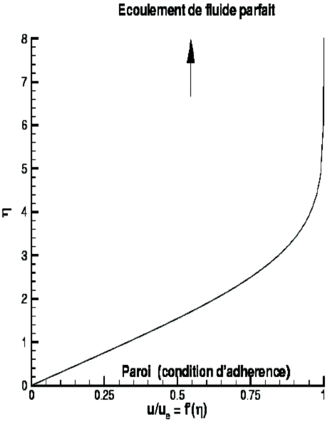
- $u =$ self-similar velocity profile
  $h =$ similarity variable
- we plug this expression into Prandtl equations

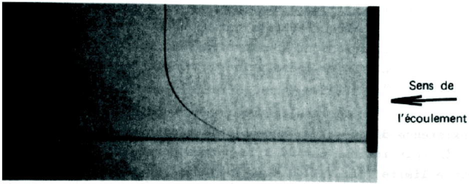
- After some manipulations, we come to the following ordinary differential equation:

$$\begin{cases} f'''(\eta) + \dfrac{1}{2} f(\eta) f''(\eta) = 0 \\ f(0) = 0, \quad f'(0) = 0 \\ f'(\infty) = 1 \end{cases}$$

- Requires numerical solution:
  - results for f and its derivatives can be tabulated
  - with f being known, velocity profiles can be derived

Excellent agreement between theory and experience





Visualisation d'une couche limite laminaire

ılisation est obtenue en photographiant à un instant donné la trace laissée par une particule de bleu de ıne tombant lentement dans un fluide en mouvement de droite à gauche : loin du fond (immobile), est sensiblement rectiligne, montrant que l'écoulement extérieur de fluide parfait est pratiquement e : par contre, cette trace s'incurve près de la paroi, décelant la présence d'une couche limite e.

Cliché réalisé au laboratoire de mécanique des fluides de l'ENSTA

The velocity profile being known

- calculation of the friction coefficient: $\tau_w = \mu \left.\dfrac{\partial u}{\partial y}\right|_w$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2} = \frac{0.664}{\sqrt{Re_x}} \quad \text{with} \quad Re_x = \frac{U_\infty x}{\nu}$$

- boundary layer thickness along the plate:

$$\frac{\delta_{0.99}}{x} \approx \frac{5}{\sqrt{Re_x}}$$

- displacement and momentum thicknesses along the plate:

$$\delta^* = 1.72\sqrt{\frac{\nu x}{U_\infty}}, \quad \theta = 0.664\sqrt{\frac{\nu x}{U_\infty}},$$

- Exact solutions of (laminar) BL equations are known for a few special outer velocity u_e distributions, which enable looking for self-similarity solutions

- For arbitrary velocity distributions, similarity solutions cannot be found in general $\rightarrow$ calculation of approximate solutions by means of von Kármán integral equation

- Steady incompressible BL equations

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = u_e\dfrac{du_e}{dx} + \dfrac{1}{\rho}\dfrac{\partial \tau}{\partial y} \end{array}\right.$$

with $\tau = \mu\frac{\partial u}{\partial y}$ the shear stress for laminar flow

- $+$ no-slip conditions and matching with the outer flow
- $+$ far-field conditions $\tau(y \to \infty) = 0$ and $v(y \to \infty) = 0$

- Transformation of the integral equation:

$$\int_{y=0}^{y\to\infty}\left\{\underbrace{\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-u_e\frac{du_e}{dx}-\frac{1}{\rho}\frac{\partial\tau}{\partial y}\right]}_{(ii)}+(u-u_e)\underbrace{\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]}_{(i)}\right\}dy=0$$

- Boundary conditions applied at integration limits

$$\frac{1}{\rho}\tau_w=\frac{du_e}{dx}\int_0^\infty(u_e-u)dy+\frac{\partial}{\partial x}\left[\int_0^\infty u(u_e-u)dy\right]$$

- Introduce displacement and momentum thickness

$$\frac{\tau_w}{\rho u_e^2}=\frac{1}{u_e}\frac{du_e}{dx}(\delta^*+2\theta)+\frac{d\theta}{dx}\quad\Leftrightarrow\quad\frac{C_f}{2}=\frac{d\theta}{dx}+(H+2)\frac{\theta}{u_e}\frac{du_e}{dx}$$

- An exact BL solution satisfies :

$$\boxed{\frac{C_f}{2} = \frac{d\theta}{dx} + (H+2)\frac{\theta}{u_e}\frac{du_e}{dx}} \tag{2}$$

- (2) may be used as a filter to obtain a good approximation of the friction coefficient C_f from a rough approximation of the true velocity profil u/u_e

Typical use of VK integral (2):

- Define an approximate BL velocity profile $\dfrac{u}{u_e} = f\left(\dfrac{y}{\delta}\right)$ verifying a given nomber of boundary conditions,

$$\text{for example, } \frac{u}{u_e}(y=0) = f(0) = 0 \text{ and } \frac{u}{u_e}(y=\delta) = f(1) = 1$$

NB δ is one of the problem unknowns

- Compute the displacement and momentum thicknesses:

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy = \delta \int_0^1 (1 - f(\xi)) d\xi = k_1 \delta$$

$$\theta = \delta \int_0^1 f(\xi)(1 - f(\xi)) d\xi = k_2 \delta$$

Typical use of VK integral (2):

- For a laminar BL, skin friction is given by:

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_w = \frac{\mu u_e}{\delta}\, f'(\xi)\big|_w = \frac{\mu u_e}{\delta} f'(0) \;\Rightarrow\; \frac{C_f}{2} = \frac{\tau_w}{\rho u_e^2} = \frac{\nu}{u_e \delta} f'(0)$$

- Plug the preceding expressions in (2):

$$\boxed{\frac{\nu f'(0)}{u_e} = k_2 \delta \frac{d\delta}{dx} + (2k_2 + k_1)\frac{\delta^2}{u_e}\frac{du_e}{dx}}$$

- $u_e(x)$ is known $\rightarrow$ δ^2 calculated $\rightarrow$ C_f

$u/u_e = f(\xi)$ modelled as a polynomial of $\xi = y/\delta$ with a sufficiently high degree to satisfy all of the following conditions:

- no-slip condition at the wall : $u(y=0) = 0$ or $f(0) = 0$
- matching of the BL solution with outer flow: $u(y=\delta) = u_e$ or $f(1) = 1$
- vanishing of viscous stresses outside the BL: $\dfrac{\partial u}{\partial y}(y=\delta) = 0$ or $f'(1) = 0$
- momentum conservation inside BL: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = u_e\dfrac{du_e}{dx} + \nu\dfrac{\partial^2 u}{\partial y^2}$
  - $y=\delta$: $u = u_e$ and $\dfrac{\partial u}{\partial y} = 0 \rightarrow \dfrac{\partial^2 u}{\partial y^2}(y=\delta) = 0$ or $f''(1) = 0$
  - $y=0$: $u = 0$ and $v = 0 \rightarrow \dfrac{\partial^2 u}{\partial y^2}(y=0) = -\dfrac{u_e}{\nu}\dfrac{du_e}{dx}$ or $f''(0) = -\dfrac{\delta^2}{\nu}\dfrac{du_e}{dx} = -\Lambda$

$\hookrightarrow$ $f(\xi)$ can be modelled as a **4th-order polynomial**

- **Polhausen polynomial (1921):**

$$\boxed{\frac{u}{u_e} = \left[2\xi - 2\xi^3 + \xi^4\right] + \frac{\Lambda}{6}\xi(1-\xi)^3} \tag{3}$$

  - first term: velocity profile with no pressure gradient ($du_e/dx = 0 \;\rightarrow\; \Lambda = 0$)
  - second term: effect of a pressure gradient on the velocity profile

- This approach is not very accurate in general (generic u_e) and cannot predict correctly the separation point location for adverse pressure gradients ($\Lambda < 0$)

Walz-Thwaites method

- 1940: Holstein and Bohlen introduce a non-dimensional parameter for describing BL, by correlating the most significant non-dimensional quantities in VK integral eq. (2)

$$\lambda = \frac{\theta^2}{\nu}\frac{du_e}{dx} \quad : \quad H = H(\lambda) \quad \text{and} \quad \frac{\tau_w \theta}{\mu u_e} = S(\lambda)$$

$\hookrightarrow$ (2) rewritten as:

$$u_e \frac{d}{dx}\left(\frac{\lambda}{u_e'}\right) = 2\left[S(\lambda) - \lambda\left(2 + H(\lambda)\right)\right] = F(\lambda)$$

- 1941 : Walz shows that if $F(\lambda)$ is linear, the preceding expression can be integrated analytically, and leads to

$$\frac{\theta^2}{\nu} = \Phi(u_e)$$

- 1949 : after analyzing several analytic and exerimental results, Thwaites shows that data fit the following correlation:

$$F(\lambda) \approx 0.45 - 6\lambda$$

↪ then, momentum thickness of a BL is well approximated by:

$$\boxed{\theta^2(x) = \frac{0.45\nu}{u_e^6(x)} \int_0^x u_e^5(x')dx'} \tag{4}$$

Walz-Thwaites method

- If the outer velocity is known, then it is possible to write

$$\lambda = \frac{\theta^2}{\nu}\frac{du_e}{dx} \quad \text{with the preceding formula}$$

- With λ being known, we compute $H(\lambda)$ and $S(\lambda)$ from correlation tables given by Thwaites

- Wall friction is deduced from $S(\lambda)$ as well as the separation point ($S(\lambda) = 0$ for $\lambda = -0.082$)

- <u>Remark:</u> preceding information obtained without postulating anything about the BL velocity profile