Motion of incompressible fluids

As we have seen, if a fluid moves in such a way that the velocity is everywhere sufficiently lower than the local speed of sound (say $M \le 0.3$), the variations of density are small enough to allow the fluid to be considered as *incompressible*, i.e. as characterized by the simple equation of state $\rho = \text{constant}$.

This assumption has significant consequences, from both the physical and the mathematical points of view. From the physical point of view, the use of the above equation of state implies that the velocity of sound is everywhere infinite, i.e. that, theoretically, the Mach number is always zero, irrespective of the value of the velocity. Note that this means that a perturbation in a point is instantaneously "*felt*" in all the flow field. Furthermore, the temperature variations in the flow will be due only to dissipation, and very limited. Now, as the coefficient of viscosity is not a strong function of temperature (varying approximately as $\sqrt{T}$), we may assume with sufficient accuracy that μ be a constant, which, as will be seen, has important consequences on the resulting set of equations.

From the mathematical point of view, the equations of motion are considerably simplified (even if not necessarily as regards their solution, specially if numerical methods are used). In particular, the constancy of density (in space and time) reduces *the mass balance equation* to a very simple form, i.e. that the velocity be *solenoidal*:

$$\mathrm{div}\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This is a *linear* equation, and it allows many further terms appearing in the flow equations to be simplified.

In particular, the components of the viscous stress tensor become:

$$\tau_{ik} = \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

If now we substitute these expressions in the viscous term of, say, the x-component of the momentum equation, i.e. $\mathrm{div}\vec{\tau}_x$, we have (considering also the constancy of μ):

$$\begin{aligned}
\mathrm{div}\vec{\tau}_x &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \\
&= \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \\
&= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] = \\
&= \mu\left[\nabla^2 u + \frac{\partial}{\partial x}\left(\mathrm{div}\vec{V}\right)\right] = \mu\nabla^2 u
\end{aligned}$$

The same procedure may be applied to the remaining components, so that we finally obtain the vector relation:

$$\mathrm{div}\boldsymbol{\tau} = \mu\nabla^2\vec{V}$$

and thus, due to the constancy of μ, the viscous term becomes a linear one.

With all these simplifications, we may now write the following complete set of the equations of motion of an incompressible fluid, introducing also the kinematic viscosity $\nu = \mu/\rho$:

Mass balance:

$$\mathrm{div}\vec{V} = 0$$

Momentum balance:

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \mathrm{grad}\vec{V} = \vec{f} - \mathrm{grad}\left(\frac{p}{\rho}\right) + \nu\nabla^2\vec{V}$$

Energy balance:

$$\rho\frac{De}{Dt} = \rho\left(\frac{\partial e}{\partial t} + \vec{V} \cdot \mathrm{grad}e\right) = \Phi - \mathrm{div}\vec{q} = \rho T\frac{DS}{Dt}$$

Constitutive equations:

$$\rho = \text{constant};\ e = e(T);\ \vec{q} = -k\mathrm{grad}T$$

$$\tau_{ik} = \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

Dissipation function:

$$\Phi = \sum_i \vec{\tau}_i \cdot \mathrm{grad}u_i = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] +$$

$$+ \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right]$$

The energy balance has been written in terms of internal energy balance, which, as can be seen, in this case is strictly connected with the entropy balance.

The first point to note is that the five unknowns in the equations have now become the three components of the velocity vector, $\vec{V} \equiv (u_i)$, the temperature T and the pressure p, (instead of the density).

However, it should be pointed out that the pressure is a rather peculiar unknown. First, in this case it can no longer be considered as a thermodynamic state function, but only as the opposite of the *mean value* of the normal stresses acting on three orthogonal planes passing through the considered point (mechanical definition of pressure). Moreover, a closer analysis shows that, actually, only *the gradient of pressure* appears in the equations, so that a space-constant variation of pressure does not influence the flow. Finally, no explicit time derivative of the pressure is present in the equations, and this may cause difficulties when a numerical method is used for their solution.

Perhaps the most interesting feature of the set of equations of motion of an incompressible fluid is that, due to the assumed constancy of μ (and thus also of ν), the temperature appears only in the energy equation, which is then decoupled from the mass and momentum equations. The latter may then be solved independently for the velocity and the pressure, and, subsequently, (and only if necessary) the obtained functions may be substituted into the energy equation to derive the temperature field.

The consequence is that, if, for instance, the interest is on the derivation of the forces acting on a body due to its motion in a fluid, the energy equation is not taken into account in the analysis, because all what is needed is the knowledge of the pressure and of the velocity (from which the viscous stresses on the body surface may be derived).

Manipulation of momentum equation

For an incompressible fluid we may rewrite the momentum equation in a different form, in which the vorticity vector $\vec{\omega} = \mathrm{curl}\vec{V}$ appears. To this end we use the following vector identities:

$$\vec{V}\cdot \mathrm{grad}\vec{V} = \mathrm{curl}\vec{V}\wedge\vec{V} + \mathrm{grad}\frac{V^2}{2} = \vec{\omega}\wedge\vec{V} + \mathrm{grad}\frac{V^2}{2}$$

$$\nabla^2\vec{V} = \mathrm{grad}\left(\mathrm{div}\vec{V}\right) - \mathrm{curl}\left(\mathrm{curl}\vec{V}\right) \overset{inc.}{=} -\mathrm{curl}\vec{\omega}$$

In the latter we have used the solenoidality of the velocity field for an incompressible fluid.

Furthermore, let us assume that the mass forces are conservative, i.e. that they may be derived from a potential so that

$$\vec{f} = -\mathrm{grad}\Omega$$

By introducing all these relations, we finally obtain the following form of the momentum balance equation:

$$\boxed{\frac{\partial\vec{V}}{\partial t} + \vec{\omega}\wedge\vec{V} = -\mathrm{grad}\left(\frac{p}{\rho} + \frac{V^2}{2} + \Omega\right) - \nu\,\mathrm{curl}\vec{\omega}}$$

This form of the equation of motion of an incompressible fluid is particularly important. Indeed, it allows the conditions of validity of widely-used relations to be clearly defined, and shows that the kinematics of the flow, and in particular the existence or not of a non-zero value of the vorticity vector, may have a profound influence on the forces acting on a fluid particle, and thus on its motion.

First of all, it may be seen that if the motion is *irrotational*, i.e. with zero vorticity, or even if $\mathrm{curl}\vec{\omega} = 0$, then the last term in the equation, connected with viscosity, vanishes.

The consequence is that

> *The equation of the irrotational motion of a viscous incompressible fluid coincides with the equation of motion of an incompressible non-viscous fluid.*

It should be pointed out that *this does not mean that in an irrotational motion the viscous stresses are zero, but that the resultant force per unit volume acting on an elementary volume of fluid is zero*. Therefore, a viscous fluid in irrotational motion is still *dissipative*, because, in general, the dissipation function is not zero, so that the internal energy increases. However, as for the velocity field and the paths followed by a fluid particle, the fluid behaves as if it were non-viscous.

Furthermore, if the motion is irrotational, i.e. if $\vec{\omega} = 0$, then the velocity vector may be written as

$$\vec{V} = \mathrm{grad}\varphi(x, y, z, t)$$

where φ is the so-called *velocity potential*; in this case the flow is said to be a *potential flow*. The momentum equation reduces then to:

$$\mathrm{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + \frac{V^2}{2} + \Omega\right) = 0$$

or, equivalently,

$$\frac{\partial\varphi}{\partial t} + \frac{p}{\rho} + \frac{V^2}{2} + \Omega = f(t)$$

If, moreover, the motion is *steady* ($\partial/\partial t = 0$), then we obtain the well-known *Bernoulli theorem* relation:

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + \Omega = \text{constant}}$$

Therefore, *it is not necessary that a fluid be non-viscous for the Bernoulli theorem to apply, but only that the motion be irrotational and steady*.

(Actually, it is easy to see that even if $\vec{\omega} \neq 0$ but $\text{curl}\vec{\omega} = 0$, then in a steady flow the Bernoulli trinomial is constant both along lines parallel to $\vec{\omega}$ and along lines parallel to $\vec{V}$)

The irrotationality of the motion of an incompressible fluid has now an immediate and fundamental consequence on the form of the mass balance equation. Indeed, by introducing the velocity potential, we obtain:

$$\boxed{\text{div}\vec{V} = \text{div}(\text{grad}\varphi) = \nabla^2\varphi = 0}$$

i.e., the continuity equation reduces to the linear Laplace equation for the single <u>*scalar*</u> function φ (in other terms we may also say that the potential is a *harmonic* function).

Therefore, we have arrived to a perfect decoupling of the equations. Indeed, in principle, the velocity field may now be obtained by solving the Laplace equation for the potential, and the pressure may then be immediately derived from the Bernoulli relation.

Obviously, this suggests that it is extremely important to understand if, and in which conditions, the motion of an incompressible viscous fluid may be irrotational.

Let us consider a body in translational motion; in a reference frame fixed with the body, the boundary condition for a viscous fluid requires that the velocity of a particle be zero on the body surface. Thus, if the flow were all irrotational, the differential problem defining the velocity field would be

$$\begin{cases} \nabla^2\varphi = 0 & \text{in the flow field} \\ \text{grad}\varphi = 0 & \text{on the body surface} \end{cases}$$

Now, it may be seen that the only solution to this problem is $\vec{V} = \text{grad}\varphi = 0$ everywhere, i.e. a condition of rest.

This demonstrates that

A <u>completely</u> irrotational flow of a viscous incompressible fluid in presence of solid surfaces is impossible.

However, it should be pointed out that this does not mean that vorticity must be present everywhere in the flow. In other words, conditions in which the vorticity is non-zero only in limited parts of the flow are allowed.

In any case, the above mentioned decoupling of the mass and momentum equations seems to be impossible.

Nevertheless, it is interesting to see that, *if the fluid were non-viscous*, the mathematical problem would change considerably. Indeed, in that case the boundary condition would be that only the normal component of the relative velocity be zero at the body surface, which implies:

$$\begin{cases} \nabla^2\varphi = 0 & \text{in the flow field} \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on the body surface} \end{cases}$$

It may be shown that the solution of this problem exists, is unique, and corresponds to a non-zero velocity field, which then describes the irrotational flow of a *non-viscous* fluid around a solid body, i.e. with a velocity that is tangent to its surface.

It should be noted that the condition of irrotationality of the flow is in perfect agreement with the assumption of the fluid being non-viscous. Indeed, we have seen that vorticity is equal to twice the angular velocity of an elementary volume of fluid; however, in an ideal non-viscous fluid only normal surface stresses would exist, so that, in a motion starting from rest, no physical mechanism would act that could produce a rotation of a fluid particle.

As we have seen that non-viscous fluids do not exist, it is appropriate to analyse in which conditions the assumption of non-viscosity might be considered as an acceptable one. To this end it is useful to return to the original form of the momentum balance equation, and to analyse what is the order of magnitude of the ratio between the inertial term and the viscous term. We will assume that L is a reference length of the problem and U a reference value of the velocity (e.g. a typical dimension and the translational velocity of a body in a fluid), and evaluate the order of magnitude of the above-mentioned ratio (we indicate with square brackets the order of magnitude):

$$\frac{\left[\vec{V}\cdot \mathrm{grad}\vec{V}\right]}{\left[\nu\nabla^2\vec{V}\right]} = \frac{UUL^{-1}}{\nu UL^{-2}} = \frac{UL}{\nu} = \frac{\rho UL}{\mu} = Re$$

where the parameter Re is the *Reynolds number*, and characterizes the particular considered problem.

Now, if we consider, say, a body with a typical dimension $L = 1.5$ m, moving in air at a velocity $U = 10$ m/s, we obtain $Re \cong 10^6$. This implies that in many (or most) engineering applications the viscous term in the equation of motion is smaller than the inertial term by several orders of magnitude. More properly, we should say that the forces *per unit volume* due to viscosity are much smaller than the inertia forces *per unit volume* acting on a fluid particle.

Therefore, we may conclude that neglecting viscosity seems to be a quite reasonable assumption.

Nevertheless, if we now solve the mathematical problem describing the potential flow of a non-viscous fluid around a generic three-dimensional body, and then evaluate the pressure field by using the Bernoulli theorem, we find that *the resultant of the pressure forces acting on the body surface is zero*. This result, which is known as *D'Alembert's paradox*, is in clear contrast with common experience.

Actually, for a two-dimensional flow (say in the x-y plane) the conclusion is slightly different. Indeed, if x is the direction of motion of the body and $-U$ its translational velocity, we obtain for the components in the x and the y directions of the force on the body:

$$F_x = 0; \quad F_y = -\rho U \Gamma$$

where Γ is the *circulation* around a curve C enclosing the body (having line element $d\vec{l}$), and is defined as

$$\Gamma = \oint_C \vec{V}\cdot d\vec{l}$$

(Note that Γ is positive for a counter-clockwise rotation)

This result shows that, in principle, a force in the cross-flow direction is possible, and this is an important result, e.g. for the evaluation of the *lift* force on a wing section.

However, it is easy to see that the production of a non-zero circulation around a body moving from rest in an incompressible non-viscous fluid is impossible.

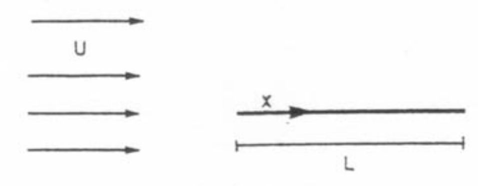
On the other hand, the fact that the force component on the body in the direction of motion, i.e. the *drag*, be zero in a non-viscous fluid is in perfect agreement with the absence, in such an ideal fluid, of mechanisms of dissipation. In other terms, it may be seen that the kinetic energy of the whole flow field does not change with the translation of the body, so that no work is being done by the body on the fluid, and, by definition, the only component that can do work is the drag (or better, the opposite of the drag).

In spite of these deductions, it will be seen that the results that may be obtained through the assumption of non-viscosity of the fluid may be useful, in certain conditions, for the evaluation of the pressure distribution and of the forces (particularly of lift) acting on a class of bodies.

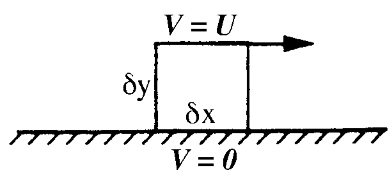
More properly, however, one should say that, *for the evaluation of the forces acting on a certain class of bodies with particular types of motion in a real viscous fluid, a procedure may be devised which, in certain stages, implies the solution of the same equations that would have to be solved if the fluid had been considered as non-viscous*; however, in general, conditions will be imposed to the problem that could not have been derived with the non-viscosity assumption. In the following the meaning of this statement will (hopefully) become clearer.

Origin of vorticity in an incompressible flow

The importance of the presence of vorticity in the flow has already been pointed out, as well as the impossibility of a completely irrotational flow of a viscous incompressible fluid in presence of solid boundaries. It will now be shown that, as could be deduced from what already explained, vorticity is introduced in the flow by the no-slip boundary condition. To this end, let us first consider the simplest possible flow, i.e. the flow produced by a flat plate of length L and negligible thickness moving impulsively from rest with velocity U in a direction parallel to its plane.
In a frame of reference fixed with the plate the situation at the start of the motion is thus as in the figure:



Obviously, due to the negligible thickness, the only perturbation to the flow will be given by the no-slip condition at the plate surface. This implies that, considering a small element of fluid δx-δy (in two dimensions), the particles on the plate will have zero velocity, while those at distance δy will have velocity equal to U; in practice, the situation over the upper surface will be as follows:



We may now apply the *Stokes theorem* to the small δx-δy element. In doing so, we must remember that the circulation must be evaluated by following the circuit corresponding to the boundary of the element in the counter-clockwise direction:

$$\oint_{\partial S} \vec{V} \cdot d\vec{l} = \iint_S \mathrm{curl}\vec{V} \cdot \vec{n} dS = \iint_S \vec{\omega} \cdot \vec{n} dS$$

If we now call ω the *average* vorticity crossing the surface of the element δx-δy (remember that in a 2-D flow the only component of vorticity is in the z-direction normal to the flow plane), we have

$$\omega \delta x \delta y = -U \delta x$$

A layer of vorticity of thickness δy will thus be generated over upper surface of the plate. The *global* quantity of vorticity contained in this layer over a generic point of coordinate x is given by

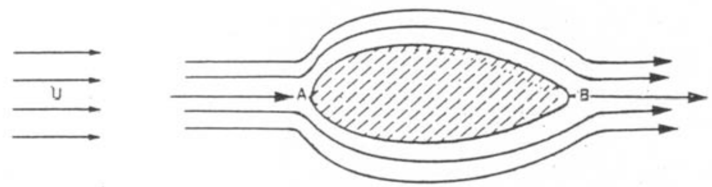
$$\omega \delta y = -U$$

(Note that this formula is equivalent to $\omega = -\partial u / \partial y$, because the variation of velocity in the y-direction is U)

Therefore, due to the no-slip boundary condition, two layers of vorticity will be generated over the upper and lower surface of the plate, whose total values will be, respectively, $-UL$ and UL (remind that positive vorticity is counter-clockwise). Note that the global amount of vorticity is still zero, as it was before the start of the motion.

The subsequent behaviour of the layers of vorticity after their generation will be analysed in the following, by deriving and discussing the vorticity dynamics equation.

The above described mechanism for the generation of vorticity applies also in the case of the impulsive motion of a body having a non-zero volume, but with a difference as regards the velocity field produced at the start of the motion. Consider, e.g., a 2-D symmetrical body in motion in a direction parallel to its symmetry axis (the more general case of non-symmetric bodies or motion will be considered later on):



The difference with respect to the previous case is that now, due to its thickness, the body will also displace the fluid particles, which, obviously, cannot penetrate its solid surface. In other words, the condition that the relative velocity at the body surface have zero normal component will act to produce a flow field satisfying the following irrotational flow problem (as there is no vorticity in the field at the start of the motion):

$$\begin{cases} \nabla^2 \varphi = 0 & \text{in the flow field} \\ \dfrac{\partial \varphi}{\partial n} = 0 & \text{on the body surface} \end{cases}$$

As can be seen, the resulting flow is tangent to the body, and is characterized by two stagnation points, A and B, one at the front and the other at the rear of the body, separating the streamlines passing above and below the body.

If we now call *V(s)* the velocity tangent to the body surface (resulting from the solution of the previous problem) at the generic point defined by the curvilinear coordinate *s*, the no-slip boundary condition will also act at the start of the motion to bring this component to zero, similarly to what described for the flat plate, and two layers of vorticity will be generated on the upper and lower surfaces of the body. For instance, if δn is the thickness of the layer in the normal direction, on a point of the upper surface we will have

$$\omega \delta n = -V(s)$$

Therefore, the total amount of vorticity generated at the body surface by the no-slip boundary condition may be evaluated in this case by the following relations (where $V_u(s)$ and $V_l(s)$ are, respectively, the potential flow velocity components tangent to the upper and lower surfaces):

$$-\int_A^B V_u(s)ds \text{ on the upper surface}$$

$$\int_A^B V_l(s)ds \text{ on the lower surface}$$

Again, the integral of the vorticity in the whole flow is zero, as the total amounts of positive and negative vorticity are equal. Actually, this is not accidental, because it may be shown that a conservation theorem applies for the global vorticity in an incompressible flow. The demonstration of this theorem is relatively simple in the case of a three-dimensional flow, but more involved in a two-dimensional flow. Indeed, in this case the *vorticity inside any closed solid bodies* present in the flow *must be included*, by defining it *as twice the angular velocity of the solid bodies* (see J.C. Wu, 1981).

In other terms, we have:

$$\int_R \omega dR = \text{constant}$$

where the integration domain *R* includes both the fluid and the solid bodies.

Obviously, in the above example the body moves only with translational velocity, so that the vorticity inside it is zero. Consequently, considering that before the start of the motion the total vorticity in the fluid was zero, it will remain zero for all subsequent times.

An interesting application of the constancy of the global vorticity is a circular cylinder of radius *r* starting to rotate impulsively counter-clockwise with angular velocity Ω in a fluid at rest. In this case it is easy to see that over each point of the surface of the cylinder an amount of clockwise vorticity corresponding to $-V(r) = -\Omega r$ will be generated, so that the total vorticity over the whole circumference will be:

$$-\Omega r \cdot 2\pi r = -2\Omega \pi r^2$$

which is exactly the opposite of the counter-clockwise vorticity generated inside the body by its rotation, $2\Omega \cdot \pi r^2$.

Again, the subsequent evolution of the generated vorticity will be described by the vorticity dynamics equation.

Vorticity dynamics

The equation describing the dynamics of vorticity may be obtained by taking the curl of both sides of the momentum equation written in the form:

$$\frac{\partial \vec{V}}{\partial t} + \vec{\omega} \wedge \vec{V} = -\text{grad}\left(\frac{p}{\rho} + \frac{V^2}{2} + \Omega\right) + \nu \nabla^2 \vec{V}$$

and by using the following vector identities (together with the solenoidality of the velocity field):

$$\text{curl}\left(\vec{\omega} \wedge \vec{V}\right) = \vec{\omega}\text{div}\vec{V} - \vec{V}\text{div}\vec{\omega} + \vec{V} \cdot \text{grad}\vec{\omega} - \vec{\omega} \cdot \text{grad}\vec{V}$$

$$\text{div}\vec{\omega} = \text{div}\left(\text{curl}\vec{V}\right) = 0; \quad \text{curl}\left(\text{grad}[\bullet]\right) = 0$$

The vorticity equation is then:

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} + \vec{V} \cdot \text{grad}\vec{\omega} = \vec{\omega} \cdot \text{grad}\vec{V} + \nu \nabla^2 \vec{\omega}}$$

The two terms on the l.h.s. represent the material derivative of the vorticity. Therefore, the terms on the r.h.s. are the sources of the possible variation of the vorticity of a fluid particle during its motion, and we will discuss their physical meaning separately.

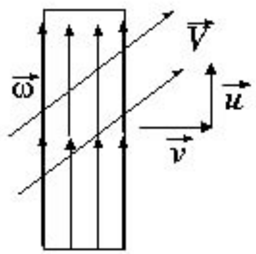
To understand the specific action of the term $\vec{\omega} \cdot \text{grad}\vec{V}$ we must first assume that it be the only one acting, i.e. that the viscous term be zero. In that condition, it may be demonstrated that *vortex-lines are material lines* (*second Helmholtz theorem*); in other words, a set of particles which composes a vortex-line (i.e. a line tangent to the vorticity vector) at one instant will continue to form the same (generally displaced and deformed) vortex-line at later instants.

Furthermore, if we take a *vortex-tube*, i.e. a tube defined by the vortex-lines passing through a small circuit inside the flow, and define the *strength* of this vortex tube as the circulation around the boundary of any of its cross-sections

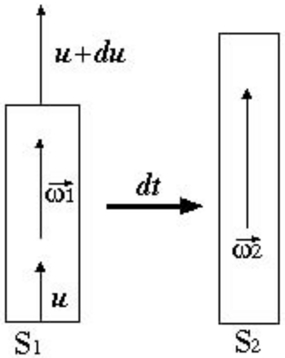
$$\Gamma = \oint_{\partial S} \vec{V} \cdot d\vec{l} = \iint_S \vec{\omega} \cdot \vec{n} dS$$

then we have that *the strength of the vortex tube is constant along the tube* (*first Helmholtz theorem*), and that *the strength of a vortex tube remains constant as the tube moves with the fluid* (*third Helmholtz theorem*).

Let us now consider in the flow a material volume coinciding with a portion of a vortex-tube having a small cross-section (sometimes called *vortex filament*), whose strength is ωS, if ω is the average vorticity value over the tube cross-section S. In general, the velocity vector will neither be parallel nor orthogonal to the vector $\vec{\omega}$. Without loss of generality, we may imagine the portion of vortex tube to be rectilinear, and decompose the velocity vector in two components, one parallel to $\vec{\omega}$, say $\vec{u}$, and the other orthogonal to $\vec{\omega}$, say $\vec{v}$. The situation is thus as in the following figure:

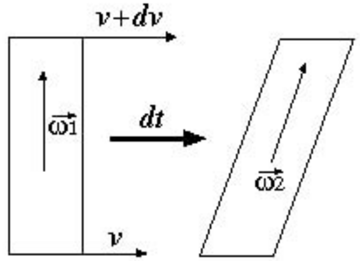


We will first consider the component of $\vec{\omega}\cdot\text{grad}\vec{V}$ in the direction of $\vec{\omega}$, i.e. $\vec{\omega}\cdot\text{grad}u$ (where u is the magnitude of $\vec{u}$). If this component is not zero, then grad$u$ has a non-zero component in the direction of $\vec{\omega}$, i.e. the situation is as in the figure below. Due to the existence of a difference between the components of velocity normal to the two extreme cross-sections of the vortex tube, in a small time interval dt these two cross-sections will be subjected to a different displacement. For instance if u increases in the $\vec{\omega}$ direction, the vortex tube (which coincides with the considered material volume) will be *stretched*, because its volume must remain constant (as the fluid is incompressible and thus $\text{div}\vec{V}=0$). Therefore, the cross-section will be reduced consequently. But the strength of the vortex tube must remain constant, so that $\omega_1 S_1 = \omega_2 S_2$, and $\omega_2 > \omega_1$.



This mechanism of variation of the magnitude of the vorticity vector due to the stretching of vortex tubes by the gradient of the velocity component parallel to $\vec{\omega}$ is extremely important in explaining the production, in certain cases, of regions of concentrated high-intensity vorticity, as happens, for instance, in tornados.

If we now consider the component of $\vec{\omega}\cdot\text{grad}\vec{V}$ in the lateral direction, $\vec{\omega}\cdot\text{grad}v$, and again assume v to increase in the $\vec{\omega}$ direction, we see that in a time interval dt the two extreme cross-sections of the material volume will move laterally in a different way, producing a *tilting* of the vortex tube, and thus a change in the *direction* of the $\vec{\omega}$ vector.



The above-mentioned mechanisms of variation of the vorticity field (which are completely independent of viscosity), act together, in general, to give significant contributions to the dynamics of vortex tubes in a three-dimensional flow. For instance, they are essential for explaining the exchange of kinetic energy between structures having different *scales* (i.e. characteristic sizes) in *turbulent* flows.

However, it is also clear that the term $\vec{\omega}\cdot\text{grad}\vec{V}$ is zero when the flow is two-dimensional. Indeed, if, for instance, the flow takes place in the *x-y* plane, the gradients of all the velocity components lie in the same plane, while the vorticity vector is normal to it.

Therefore, for a 2-D flow the vorticity equation reduces to the simpler form:

$$\frac{\partial\vec{\omega}}{\partial t}+\vec{V}\cdot\text{grad}\vec{\omega}=\nu\nabla^2\vec{\omega}$$

It is immediate to recognize that this equation coincides with the equation describing the *diffusion* of a quantity, the only difference being that the partial derivative with respect to time is replaced by a material derivative. Therefore, the term $\nu\nabla^2\vec{\omega}$ represents the variation of the vorticity of a fluid particle (following its motion) due to diffusion by viscosity. In other words, vorticity may diffuse to or from neighbouring particles if their vorticity is different form that of the considered particle. As can be seen, the diffusivity of vorticity is equal to the kinematic viscosity of the fluid, which, by observing the momentum equation, may also be recognized to be the diffusivity of momentum in an incompressible fluid.

Therefore, we may now say that, after its production at the surface of a solid body due to the no-slip boundary condition, vorticity will enter in the flow (i.e. will be passed to neighbouring fluid particles) due to diffusion by viscosity. But once this happens, the vorticity is also carried downstream by the velocity field, i.e. by the moving particles to which it has diffused.

Perhaps this is better seen in an Eulerian frame of reference fixed with the body, i.e. by analysing the time variation of vorticity in a fixed point of space. To this end we may rewrite the vorticity equation in the following form:

$$\frac{\partial\vec{\omega}}{\partial t} = -\vec{V}\cdot\mathrm{grad}\vec{\omega} + \nu\nabla^2\vec{\omega}$$

which shows that the vorticity in a point of space may vary in time due to the action of viscous diffusion (the last term on the r.h.s.) and of *convection* by the velocity field (the first term on the r.h.s.).

In particular, a steady condition (i.e. $\partial\vec{\omega}/\partial t = 0$) may be reached only if both these terms are zero, or if they are equal and opposite.

From the above it is reasonable to infer that, after a certain time from the start of the motion of a body, the vorticity may be confined in a region around it. Indeed, while diffusion tends to spread vorticity inside the flow field, convection tends to carry it downstream, and thus to confine it in a limited region around the body. It is then interesting to analyse what is the order of magnitude of the distance the vorticity may penetrate inside the flow through diffusion before being carried downstream by convection.

To this end, it is useful to refer again to the simple case of a flat plate of length L which started impulsively from rest with a velocity U parallel to its plane.

We have seen that, immediately after the start of the motion, two small layers of vorticity of opposite sign are generated on the upper and lower surfaces of the plate. Now, a fundamental result of the diffusion equation is that, in principle, at subsequent times vorticity would be present everywhere, even if, obviously, with decreasing magnitude with increasing distance from the plate.

However, from a more practical point of view, we may be interested in knowing the value of the *length of penetration* of the vorticity in the flow field in a time interval t^*. In other words, considering, e.g., a line normal to the surface and extending to infinity, we seek an estimate of the distance δ from the plate within which a certain percentage (say 99%) of the total vorticity present along that line is confined after the time interval t^*.

The answer comes again from the solution of the diffusion equation, from which we obtain that the order of magnitude of δ is given by

$$\delta \cong A\sqrt{\nu t^*}$$

where the coefficient A depends on the chosen percentage (e.g., $A \cong 5$ for a percentage of 99%).

Now, as a characteristic time interval t^* we choose the reference time of our problem, viz. $t^* = L/U$, which represents the time required for a particle moving with velocity U to displace along the whole length of the plate. We thus obtain:

$$\delta \cong A\sqrt{\nu t^*} = A\sqrt{\nu \frac{L}{U}}$$

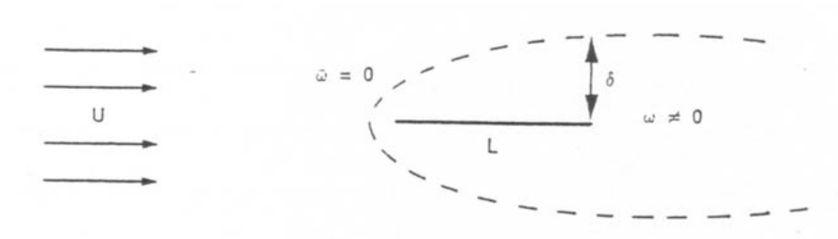
Actually, we are not interested in the value of δ, but rather on the ratio between δ and the length of the plate, which is:

$$\frac{\delta}{L} \cong A\sqrt{\frac{\nu}{UL}} = A\sqrt{\frac{1}{Re}}$$

The fundamental result is that we may consider vorticity to be confined in a layer adjacent to the plate whose thickness (in terms of the plate length) is inversely proportional to the square root of the Reynolds number, so that it will be small for high values of this parameter; outside this layer, the amount of vorticity present is so small that it may be neglected completely, and we may assume that $\vec{\omega} = 0$.

In practice, after a transient from the start of the motion, a steady condition will be reached in which there will be no more variations of the vorticity around the plate, due to a compensation between convection and diffusion, and the situation will be as in the following figure:

Fluid Dynamics - Lecture notes G. Buresti

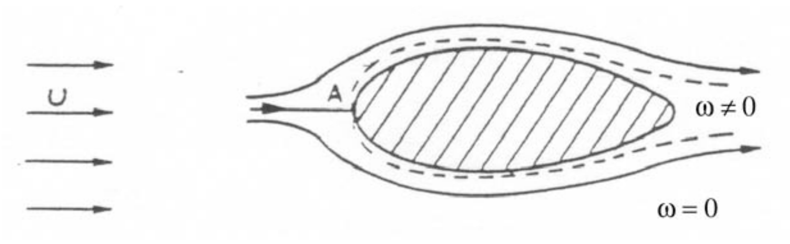


The effect of the Reynolds number on the dynamics of vorticity that was found from the previous analysis may become clearer by evaluating the ratio between the orders of magnitude of the convective and of the diffusive terms in the vorticity equation. Indeed we find:

$$\frac{\left[\vec{V}\cdot \mathrm{grad}\vec{\omega}\right]}{\left[\nu\nabla^2\vec{\omega}\right]} = \frac{U(UL^{-1})L^{-1}}{\nu(UL^{-1})L^{-2}} = \frac{UL}{\nu} = \frac{\rho UL}{\mu} = Re$$

Therefore, we have found a further interpretation of the Reynolds number, as a parameter giving the ratio between the orders of magnitude of the convective and of the diffusive terms in the vorticity equation. More precisely, it is easy to see that the square root of the Reynolds number is proportional to the ratio between the velocity of vorticity convection (which is of the order of U) and the velocity of vorticity diffusion (which may be estimated from $\partial\delta/\partial t$).

The above description of the dynamics of vorticity after its production obviously applies also in the more general condition of a body having a certain non-zero volume. Indeed, also in this case we may expect that, for sufficiently high Reynolds numbers, after a transient the vorticity be confined within a small layer around the body surface, as shown in the following figure:



Actually, we will see that, for this to be true, certain conditions on the shape and on the motion of the body must be satisfied. For the moment, we will assume that all these conditions (whose nature and origin will become clear in the following) be satisfied.

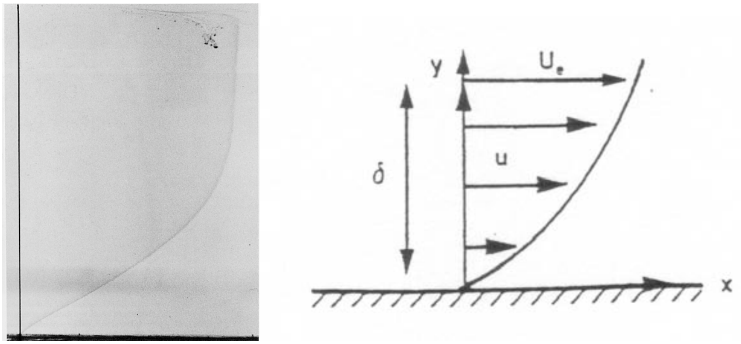
The consequence is that in the outer irrotational region the viscous term in the equation of motion disappears (i.e., the fluid behaves *as if* it were non-viscous), the velocity field is potential, and the Bernoulli theorem applies.

However, at this point of the treatment, this does not yet mean that we can evaluate the pressure forces on the body surface, because the latter is separated from the potential flow by the layer where vorticity is present, and where, in principle, the complete equations of motion must be used. In particular, in this layer the viscous term in the momentum equation does not disappear, and, conversely, may be expected to give a significant contribution to the dynamics of the fluid particles.

Nevertheless, if we may assume that the rotational layer is sufficiently small, some simplifications to the equations of motion will become acceptable, whose consequences may then be analysed in detail. This is what is done in the *boundary layer theory* developed by L. Prandtl in 1904.

Boundary layers

Prandtl suggested that, for flows *at high Reynolds numbers*, the effects of viscosity are important only in a *thin layer* close to solid boundaries (which he called *Grenzenschicht*, or *boundary layer* in English), while in the rest of the flow the fluid behaves as if it were non viscous. In this layer the velocity is brought to zero, from a value corresponding to potential flow, in order to satisfy the no-slip condition at the wall, and it is assumed that the viscous term in the equation of motion is of the same order as the inertia term.



Visualization of a boundary layer and coordinate system

Besides being incompressible, we assume the flow to be two-dimensional, and define an x - y reference system in which x is a local coordinate parallel to the wall (assumed to have negligible curvature), and y is normal to the wall. The assumption that the Reynolds number be high allows certain approximations to be introduced in the equations of motion. In particular, the thickness of the boundary layer, δ, is assumed to be small compared to any significant linear dimension L in the x - direction ($\delta/L << 1$), and, accordingly, the derivatives with respect to y are assumed to be much larger than those with respect to x.

The equations of motion, neglecting mass forces, are

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x - momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

y - momentum

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

However, from the assumptions, we have

$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} << \frac{\partial^2 v}{\partial y^2}$$

so that, in the momentum equations, the first terms of the Laplacians may be neglected with respect to the second ones.

We assume now x to be of the order of L, y to be of the order of δ, and the x - component of velocity, u, to be of the order of a reference velocity U (e.g. the velocity external to the boundary layer). The order of magnitude of v is then obtained from the continuity equation as

$$v \propto U\frac{\delta}{L}$$

If we analyse the order of magnitude of the terms on the l.h.s. of the x - momentum equation (considering that t is of the order of L/U), we now easily find that all of them are (at most) of the order of U^2/L.

Now, by definition, the boundary layer is the region where the order of magnitude of the inertia terms is the same as that of the viscous term, so that we have:

$$U^2/L \propto \nu U/\delta^2$$

From this relation we obtain

$$\delta^2/L^2 \propto \nu/(UL) = 1/Re$$

and we thus find confirmation both that the boundary layer thickness is small for high Reynolds numbers, and that its order of magnitude coincides with that of the region where vorticity has diffused from the wall. This is not accidental, as we have seen that viscous effects are “felt” only in the region containing vorticity, while in the irrotational zone the fluid behaves *as if* it were non-viscous.

From the above relation we may also derive the order of magnitude of the kinematic viscosity as a function of the other quantities

$$\nu \propto U\delta^2/L$$

In conclusion, the x - momentum equation becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

and allows us to say that the pressure variation in the x - direction is, *at most*, of the order of

$$\partial p/\partial x \propto \rho U^2/L$$

We may now analyse the order of magnitude of the terms appearing in the y - momentum equation. It is easy to see that the maximum order of magnitude of both the inertia and the viscous terms is

$$\left(U^2/L\right)(\delta/L)$$

We thus find the important result that *the maximum order of magnitude of the variation of pressure in the direction normal to the wall is negligible* for high values of the Reynolds number:

$$\frac{\partial p}{\partial y} \propto \rho \frac{U^2}{L}\frac{\delta}{L} = \frac{\delta}{L}\left[\frac{\partial p}{\partial x}\right]_{\max} \cong 0$$

This *result* of the boundary layer theory is absolutely fundamental. Indeed, it *allows the pressures on the body surface to be estimated from the values of the pressures at the border of the boundary layer, i.e. in the potential flow*:

$$p(x,0) \cong p(x,\delta)$$

In conclusion, the equations of a 2-D incompressible boundary layer flow are:

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x - momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

y - momentum

$$\frac{\partial p}{\partial y} \cong 0$$

If now suppose that the solution of the outer potential flow problem be available, in the second equation the variation of p with x, i.e. the term $\partial p/\partial x$, is no longer an unknown function, as it may be obtained from the equation of the outer flow at the edge of the boundary layer:

$$\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

where U_e is the x - component of the velocity in the potential flow just outside the boundary layer (where the term $v\,\partial U_e/\partial y$ may be neglected because both the y - component, v, and $\partial U_e/\partial y$ are small).

It should be recalled that if, as often happens, a steady case is considered, in the above equations all the derivatives with respect to time are not present.

In principle, the boundary layer equations may now be solved for the velocity components u and v, provided appropriate boundary conditions are imposed:

at $y = 0$: $u = v = 0$

at $y = \delta$: $u = U_e$

Often, further conditions are used to describe the velocity profile at the edge, as, for instance,

at $y = \delta$: $\partial u/\partial y = 0$

It should be noticed that the last relation is consistent with the interpretation of the boundary layer as the region where the vorticity is confined.

Indeed, using the boundary layer approximations ($v << u$, $\partial/\partial x << \partial/\partial y$), the expression for the vorticity component in the z - direction (the only one that is not zero in a 2-D flow) becomes, with very good accuracy,

$$\omega_z = \omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \cong -\frac{\partial u}{\partial y}$$

Therefore, the total amount of vorticity present inside a boundary layer at a given coordinate x is given by the value of the velocity at its edge:

$$\int_0^\delta \omega dy = \int_0^\delta -\frac{\partial u}{\partial y} dy = -U_e$$

Once the outer pressure distribution is given, the boundary layer equations may be solved, for instance through an appropriate numerical method. It should be pointed out, however, that the solution of the boundary layer equations is at least one order of magnitude more expensive (in terms of computer time) than the solution of a potential flow problem.

The output of the solution is, in principle, the whole velocity profile in the boundary layer at each coordinate x, from which the most interesting quantities may be derived. These are the value of the thickness of the boundary layer $\delta(x)$, and the value of the tangential stress at the surface, $\tau_w(x)$, with

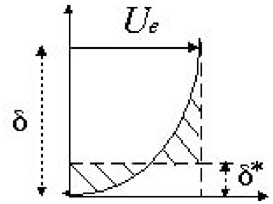
$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

However, the definition of the thickness δ is not immediate, as it may be seen that the extent of the boundary layer is, in theory, infinite. A conventional definition is then used, and δ is assumed to be the distance from the surface at which the velocity has reached the value $u = 0.99U_e$. More appropriately, one might define δ as the distance within which 99% of the vorticity (present above the considered point on the surface) is contained.

Nevertheless, other significant (and more useful) thicknesses may be defined in a boundary layer, the most important of which is the *displacement thickness* δ^*:

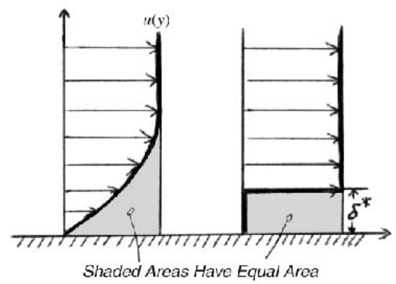
$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_e}\right) dy$$

The meaning of the displacement thickness may be immediately derived from the following figure and relation:



$$\rho U_e(\delta - \delta^*) = \int_0^\delta \rho u dy$$

from which (considering that ρ = constant) it is seen that the mass flow through the boundary layer is equal to the mass flow that would pass through a distance (δ - δ^*) if the velocity were constant and equal to the velocity at the outer border U_e. Therefore, δ^* is the distance by which the surface would have to be moved in an ideal potential flow to have the same mass flux that occurs in a real boundary layer between $y = 0$ and $y = \delta$. The importance of δ^* for the evaluation of the forces will become clear in the following.



Incidentally, the displacement thickness gives also the coordinate $y_{cg\omega}$ of the *centre of gravity of the vorticity* present inside the boundary layer over a certain point of the surface. Indeed, we have:

$$y_{cg\omega} = \frac{\int_0^\delta \omega y dy}{\int_0^\delta \omega dy} = \frac{-\int_0^\delta y \frac{\partial u}{\partial y} dy}{-\int_0^\delta \frac{\partial u}{\partial y} dy} = \frac{1}{U_e}\left[(uy)_0^\delta - \int_0^\delta u dy \right] =$$

$$= \frac{1}{U_e}\left(U_e \delta - \int_0^\delta u dy \right) = \frac{1}{U_e}\int_0^\delta (U_e - u) dy = \int_0^\delta \left(1 - \frac{u}{U_e}\right) dy = \delta^*$$

Another important boundary layer reference length is the so-called *momentum thickness*, defined as

$$\theta = \int_0^\delta \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right) dy$$

It may be shown that, if we consider a flat plate of length L translating parallel to itself with velocity U, the total friction force (per unit spanwise length) acting on one of its sides is given by

$$F_x = \int_0^L \tau_w(x) dx = \rho U^2 \theta(L)$$

An immediate relation exists then between θ and the global *friction coefficient* of the plate:

$$C_F = \frac{F_x}{\frac{1}{2}\rho U^2 L} = \frac{2\theta(L)}{L}$$

For a flat plate in a steady flow parallel to its plane the equations of the boundary layer may be solved by neglecting the effect of the displacement thickness on the outer stream, i.e. by assuming that the velocity at the border of the boundary layer remains equal to the free-stream velocity, U, and that, consequently, $\partial p / \partial x = 0$ all over the plate. The solution is first due to Blasius (1908).

If we place the origin of the coordinates at the leading edge of the plate, it may be shown that the non-dimensional velocity profiles are similar for all values of x, and the following expressions are found numerically for the variation of the most significant quantities:

$$\delta(x) = \frac{5.2x}{\sqrt{Re_x}}; \quad \delta^*(x) = \frac{1.72x}{\sqrt{Re_x}}; \quad \theta(x) = \frac{0.664x}{\sqrt{Re_x}}$$

where we have introduced the local Reynolds number

$$Re_x = \frac{Ux}{\nu}$$

To express the behaviour of the wall stress along the plate, a *local friction coefficient* is introduced, and for the flat plate it is found to be given by:

$$c_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}$$

Conversely, the global friction coefficient is ($Re_L = UL/\nu$):

$$C_F = \frac{\int_0^L \tau_w(x) dx}{\frac{1}{2}\rho U^2 L} = \frac{1.328}{\sqrt{Re_L}} = \frac{2\theta(L)}{L}$$

As can be seen, the local friction coefficient decreases in the downstream direction, being proportional to $1/\sqrt{x}$, and this is understandable, because the outer velocity remains equal to U all over the plate, while the thickness of the boundary layer increases as $\sqrt{x}$.

Even if they refer only to the flat plate case, the Blasius results are also a reference for obtaining approximate estimates of the friction forces acting on bodies that are elongated in the free-stream direction.

Coming back to the solution of the general problem of the flow around a body with a thin boundary layer all over its surface, we may now say that, if the outer potential flow is known, then the solution of the boundary layer equation provides an estimate of the friction drag force and of the variation of the thickness of the boundary layer, i.e. of the boundary of the potential flow region. However, the solution of the potential flow requires the knowledge of that boundary, so that, actually, the solution of the complete flow problem depends on its own solution.

To overcome this difficulty, we may devise an *iterative procedure*, in which we take advantage of the fact that, for high values of *Re*, the boundary layer thickness is small. Therefore, we may assume, as first tentative potential flow solution, the one that corresponds to *neglecting the thickness* of the boundary layer on the surface of the body. In practice this corresponds to solving, at the first step of the procedure, the mathematical problem

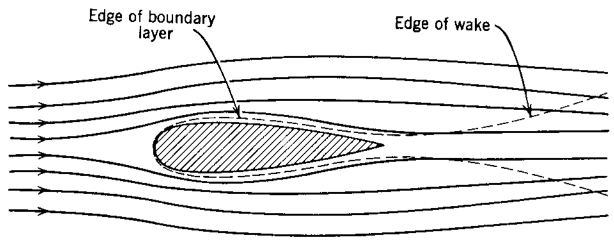
$$\begin{cases} \nabla^2 \varphi = 0 & \text{in the flow field} \\ \dfrac{\partial \varphi}{\partial n} = 0 & \text{on the body surface} \end{cases}$$

It is immediate to recognize that this is the same (very well-known) problem that would have to be solved if the fluid were taken as non-viscous, and we may then use all the techniques that have been devised for its solution.

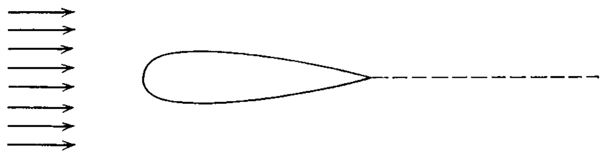
However, our point of view is somewhat different, because we do not assume the fluid to be non-viscous, but only that *we may use, as an appropriate first step of a calculation procedure to obtain the forces acting on a body moving in a viscous fluid, the solution of the problem obtained by squeezing the boundary layer to the surface of the body*. In other words, we may imagine that the boundary layer, even if of negligible thickness, is still present and contains the vorticity that is necessary to change the boundary condition on the wall from zero velocity to tangential velocity.

The usefulness or not of this solution depends then only on the fact that, *for the considered body and motion*, it be or not a good starting point for the iterative solution of the complete problem. In other words, this means that the potential flow tangent to the body surface must give a pressure distribution that is not completely different from the actual final pressure distribution that will be found when the boundary layer is taken into account.

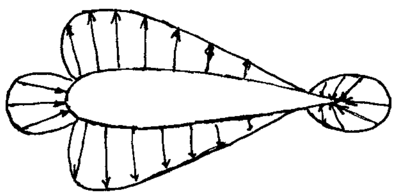
If, for instance, we take a symmetrical body elongated in the direction of motion, the real flow will be as follows



We now consider, as a first step, the problem of finding the potential flow tangent to the body:



and the solution of this problem will give a qualitative pressure distribution of this type



In this figure we have drawn, over the body surface, the values of the pressure coefficients, defined as

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2}$$

where p_∞ and U_∞ are the values of the undisturbed upstream pressure and velocity (i.e. at $x = -\infty$), and positive and negative values of C_p are indicated, as is usual, with arrows pointing towards or from the surface, respectively.

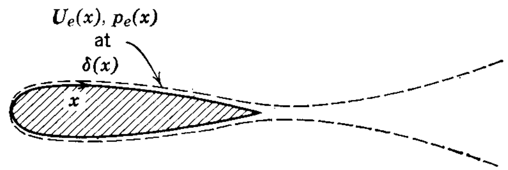
It should be noticed that, when the Bernoulli theorem applies (as in the present case), there is a direct relation between the value of the pressure coefficient and the velocity. Indeed, it is easy to see that, if p is the pressure and V the magnitude of the velocity at a certain point, the following relation holds:

$$C_p = 1 - \frac{V^2}{U_\infty^2}$$

which shows that the maximum value of C_p is 1 and is found at stagnation points (i.e. when $V = 0$). Conversely, C_p is zero when the local velocity is equal to U_∞ and negative when $V > U_\infty$.

As could be expected, the above pressure distribution corresponds to a zero resultant force, in agreement with D'Alembert's paradox.

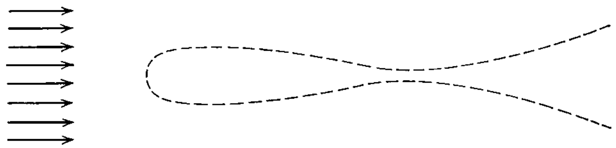
Now we use this first pressure distribution as an input for the evaluation of the boundary layer over the upper and lower surfaces of the body:



As a results, we obtain a first estimate of the evolution of the boundary layer thickness and of the tangential viscous stresses over the body. By integrating the latter, we may get a first estimate of the friction drag force:

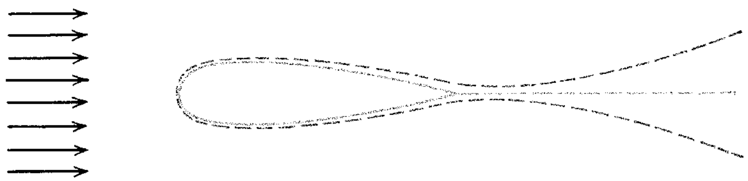
$$F_x = \int_S \vec{\tau}_w \cdot \vec{e}_x ds$$

Now the body may be changed by superposing over its surface a characteristic thickness of the boundary layer (the more appropriate one being the displacement thickness), and solve the problem of finding the potential flow that is tangential to this modified body.

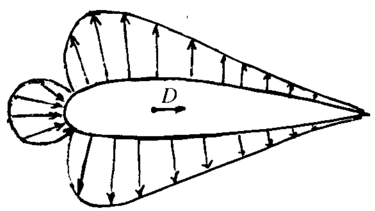


The use of the displacement thickness in this step assures that the potential flow solution for the modified body satisfies the same mass balance that is valid for the real flow around the original body.

Without going into the technical details of this step of the procedure, we may say that we will found a new pressure distribution, which we will assume to act on the original surface of the body, obviously only for the part that surrounds it, i.e. up to its rear end.



As may be inferred, now the velocity will no longer be zero at the points of the upper and lower surfaces of the modified body corresponding to the rear end of the original body, but will rather have a value near to $V \cong U_\infty$, which corresponds to $C_p \cong 0$. The new pressure distribution on the body will then be of the following type:



This pressure distribution is now qualitatively similar to the one that might be found experimentally for such a body, and as can be seen, gives rise to a non-zero drag component of the resultant pressure force, which may be derived from

$$F_x = \int_S -p\vec{n} \cdot \vec{e}_x ds$$

This drag component should be added to the previously calculated friction drag, to obtain a first estimate of the total drag acting on the body.

However, a better estimate may be obtained by continuing the iteration procedure. To this end, the newly calculated pressure distribution is used for another boundary layer calculation, providing a second estimate of the friction force and of the displacement thickness. The latter may then be *substituted* in place of the displacement thickness estimated at the first step, and a new modified body may be obtained, which will be used to obtain a third potential flow solution and a new pressure distribution.

The procedure may be stopped when two estimates of a chosen quantity at two successive steps differ by less than a given value. The quantity to be used for the check will depend on the objective. For instance, if the global force is of interest, then it may be appropriate to use this quantity, because, being an integral quantity, it will give a more rapid convergence of the procedure than using a local quantity at a given point of the surface, like for instance the value of the thickness of the boundary layer or of the friction stress.

We will analyse later on the changes to be applied to this procedure when the body and/or its orientation do not produce a symmetric flow field.