Fundamental Aerodynamics
Boundary Layers (2)

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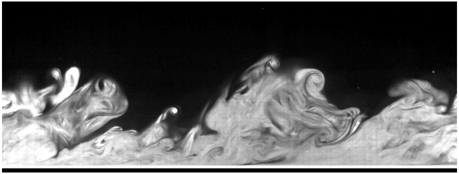

Statistical description

Turbulent boundary layer equations

Turbulent velocity profile

TBL with pressure gradients

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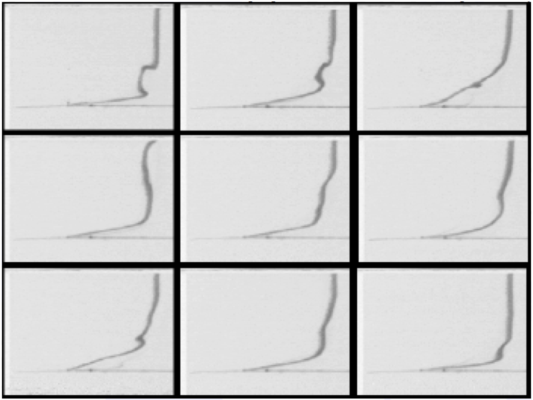
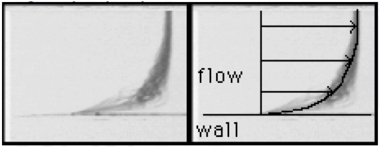
Statistical description of a turbulent flow





- Choose an averaging operator (ensemble, time, space)
- Decompose instantaneous quantities in a mean and a fluctuating part (Reynolds decomposition):

$$\boxed{\varphi = \overline{\varphi} + \varphi'} \tag{1}$$

- Properties of the average operator (1):

$$\overline{\varphi'} = 0$$

$$\overline{\varphi\psi} = \overline{\varphi}\,\overline{\psi} + \overline{\varphi'\psi'}$$

$$\overline{\frac{\partial\varphi}{\partial x}} = \frac{\partial\overline{\varphi}}{\partial x}$$

Reynolds equations:

- start from Navier-Stokes
  (simplifying hypothesis : incompressible flow)
- apply Reynolds decomposition to the velocity and pressure fields
- by applying the average to the NS equations,
  the Reynolds-Averaged NS equations are obtained (RANS)

Remark: for compressible flow, the density also needs to be decomposed; for the other variable, we use a mass-averaged mean, called Favre average.

- Reynolds-Averaged Navier-Stokes equations (RANS):

$$\begin{cases} \dfrac{\partial \overline{u_i}}{\partial x_i} = 0 \\ \dfrac{\partial \overline{u_i}}{\partial t} + \dfrac{\partial(\overline{u_i}\,\overline{u_j})}{\partial x_j} = -\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial x_i} + \dfrac{1}{\rho}\dfrac{\partial}{\partial x_j}\left(\mu \dfrac{\partial u_i}{\partial x_j} - \rho \overline{u_i' u_j'}\right) \end{cases}$$

$$\begin{aligned} \tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} &= \text{ viscous stress tensor} \\ \tau_{ij}^R = -\rho \overline{u_i' u_j'} &= \text{ turbulent stress tensor} \\ &= \text{ Reynolds tensor} \end{aligned}$$

- System of equations formally identical to NS, but not closed, because of the unknown Reynolds stresses.

Turbulent BL: statistical approach

- Turbulent BL equations: formally identical to laminar BL equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = u_e\dfrac{du_e}{dx} + \dfrac{1}{\rho}\dfrac{\partial \tau}{\partial y} \end{cases}$$

- no-slip condition: $u = v = 0$ for $y = 0$
- matching with external layer: $u(x, \infty) = u_e(x)$

- ... The total shear stress is given by:

$$\tau = \tau^{\text{lam}} + \tau^{\text{turb}} = \mu\frac{\partial u}{\partial y} - \rho\overline{u'v'}$$

- von Kármán equation remains the same:

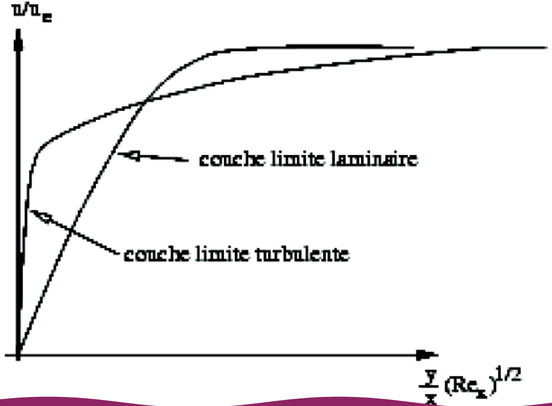
$$\frac{C_f}{2} = \frac{d\theta}{dx} + (H+2)\frac{\theta}{u_e}\frac{du_e}{dx}$$

- no explicit expression for C_f

Turbulent BL equations

- using von Kármán equation (EVK) requires :
  - an approximate velocity profile (as in the laminar case)
  - an approximated law for C_f

- choice of the velocity profile
  - very different from laminar case

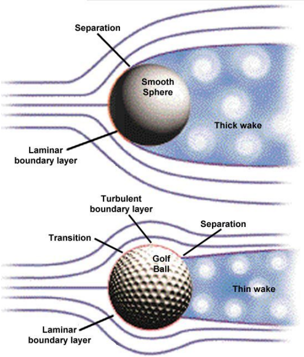


Turbulent BL

A turbulent boundary layer is dominated by vortical structures (turbulent eddies)

- increased mixing, momentum transfer from high-speed regions to lowspeed regions much more efficient than in the laminar case
  $\hookrightarrow$ turbulent BL is thicker than the laminar one

- average velocity profiles increase much more quickly with the distance to the wall:
  - turbulent friction much higher than laminar friction
    $\hookrightarrow$ need for flow laminarity control
  - higher "endurance" of turbulent BL to adverse pressure gradients
    $\hookrightarrow$ delay of flow separation
- riblets, vortex generators, morfing surfaces, … may be used to control both laminarity and separation



- Typical average turbulent velocity profiles in a BL are approximated through a power law
  - fitting of velocity profiles from experimental data:

$$\frac{u}{u_e} = \left(\frac{y}{\delta}\right)^{1/n}$$

$\hookrightarrow$ typical value for n (moderate pressure gradients): $n = 7$

- Friction law :
  - generally given as: $C_f = C_f(Re_\delta)$ or $C_f = C_f(Re_\theta)$

Example Prandtl law based on head losses in cylindrical pipes for fully turbulent flow regime

$$C_f = \frac{0.0464}{Re_\delta^{1/4}}$$

Typical application of EVK for turbulent BL

- consider a power-law velocity profile
  - compute displacement and momentum thicknesses in terms of boundary layer thickness
  - compute shape factor $H(n)$
- consider an empirical friction law of the form $C_f = C_f(Re_\theta)$
  - apply EVK = leads to an ordinary differential eq. for $\theta(x)$

$$\frac{d\theta}{dx} + \left[(H(n) + 2)\,\frac{1}{u_e}\frac{du_e}{dx}\right]\theta = C_f(Re_\theta)$$

  - solve for $\theta(x)$ and compute $C_f(x)$

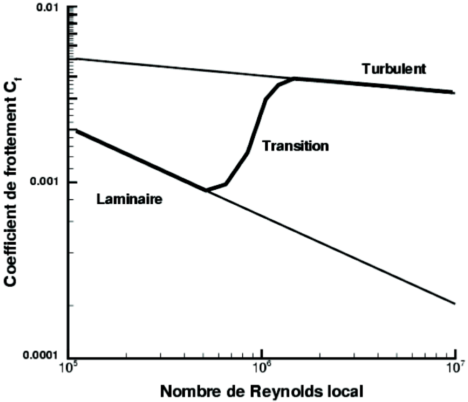
Example: using a power law with (n=7) and Prandtl's friction law we get:

$$(C_f)_{\text{turbulent}} = \frac{0.0592}{Re_x^{1/5}}$$

Comparison of laminar and turbulent skin friction:

$$(C_f)_{\text{laminar}} = \frac{0.664}{Re_x^{1/2}}$$

$$(C_f)_{\text{turbulent}} = \frac{0.0592}{Re_x^{1/5}}$$



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Turbulent velocity profile

TBL with pressure gradients

Analytical study of velocity profiles in a turbulent BL

- momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)$$

- in the wall neighborhood:
  laminar friction dominates all other terms
  (average velocity and velocity fluctuations $\to 0$ for $y \to 0$

$$\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) \approx 0 \quad \text{for} \quad y \to 0 \qquad \Leftrightarrow \qquad \boxed{u = Cy = \frac{\tau_w}{\mu}y}$$

- characteristic velocity scale = friction velocity

$$\boxed{u_\tau = \sqrt{\frac{\tau_w}{\rho}}}$$

Analytical study of velocity profiles in a turbulent BL

- relevant non-dimensional variables in the near-wall region:

$$u^+ = \frac{u}{u_\tau}\ , \quad y^+ = \frac{u_\tau y}{\nu}$$

- viscous sub-layer solution valid in the range $y^+ \in [5, 10]$

$$\boxed{u^+ = y^+}$$

- outside of the viscous sub-layer, turbulent friction becomes relevant compared to laminar friction, but inertia terms are still small enough to be neglected (small average speeds)
    - in this region, shear stress approximately constant and equal to τ_w
    - average velocity remains of the order of friction velocity
    - characteristic length scale is y^+
    - solution of the form

$$\boxed{u^+ = f(y^+)}$$

Analytical study of velocity profiles in a turbulent BL

- going further from the wall, turbulent friction and inertia terms dominate laminar friction, but the speeds are still lower than the outer one
  - velocity defect of the order of friction velocity
  - characteristic length scale of the order of BL thickness
  - look for a solution of the form

$$\boxed{\frac{u_e - u}{u_\tau} = g\left(\frac{y}{\delta}\right) = g(\eta)}$$

- outside BL the inertia term dominates all other terms and viscous effects may be neglected

Analytical study of velocity profiles in a turbulent BL

- consider an inner and an outer BL region
  - characteristic variables in the inner region: $u^+, \ y^+$
  - characteristic variables in the outer region: $(u_e - u)/u\tau, \ \eta$
- inner solution for $y^+ \to 0$ has to match the outer solution for $\eta \to 0$

$$u^+ = f(y^+) = \frac{u_e}{u_\tau} - g(\eta)$$

- ratio of the inner and outer variables :
- matching conditions + derivation + variable separation:

$$f(Re_\tau \eta) = \frac{u_e}{u_\tau} - g(\eta) \ \Rightarrow \ y^+ f'(y^+) = -\eta g'(\eta)$$

which gives

$$y^+ f'(y^+) = \text{cste} \quad \text{and} \quad -\eta g'(\eta) = \text{cste}$$

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The preceding relations may be integrated analytically:

- velocity profile for $y^+ \to \infty$:

law of the wall
(universal logarithmic law)

$$\boxed{u^+ = \frac{1}{\kappa}\ln(y^+) + B}$$

- $\kappa=$ von Kármán constant ($\approx$ 0.40–0.41)
- B=intercept ($\approx$ 5.0–5.2)
- validity domain $y^+ \in [50, 500]$

- defect velocity profile in the outer layer for $\eta \to 0$:

defect law

$$\boxed{\frac{u_e - u}{u_\tau} = -\frac{1}{\kappa}\ln(\eta) + A}$$

- $A \approx 2.35$
- validity domain depends on Re_τ (and so on Re): typically from $y \approx 0.1\delta$

The velocity profile is completed by:

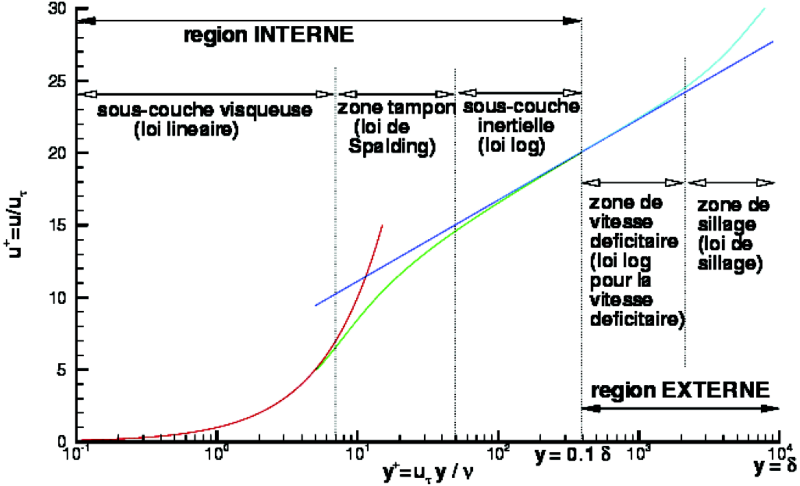
- a buffer region in between the viscous and the logarithmic sublayers: *e.g.* implicit relation between u^+ and y^+ (Spalding's law, 1961)
- a wake region at the outer border of BL: the wake law accounts for the outer pressure gradient *e.g.* Coles' law (1952)

$$\boxed{u^+ = \frac{1}{\kappa}\ln(y^+) + B + \frac{\Pi}{\kappa}W(\eta)}$$

with Π the wake parameter and W the wake law.
Written in outer variables:

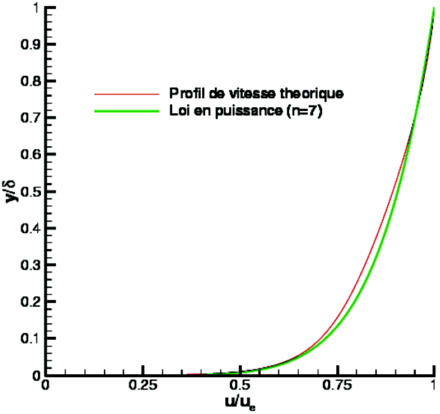
$$\frac{u_e - u}{u_\tau} = -\frac{1}{\kappa}\ln(\eta) - \frac{\Pi}{\kappa}W(\eta) + 2\frac{\Pi}{\kappa}$$

Summary of the turbulent velocity profile



Comparison with empirical velocity profiles

- power law profile + EVK (green)
- analytical law: Coles' law (red)



Use of the velocity profile to estimate of skin friction:

- use of the wake law to obtain a relation between Re_θ and $\lambda = \sqrt{\frac{2}{C_f}} = \frac{u_e}{u_\tau}$, which can be approximated simply by:

$$C_f \approx 0.012 Re_\theta^{-1/6}$$

- application of EVK for zero pressure gradient BL:

$$C_f(Re_\theta) = 2\frac{d\theta}{dx} \quad \Leftrightarrow \quad \frac{d\,Re_\theta}{d\,Re_x} = 0.006 Re_\theta^{-1/6}$$

$$\rightarrow \quad \boxed{C_f \approx \frac{0.0244}{Re_x^{1/7}}}$$

Inner functions (1)

There are numerous empirical correlations for u^+ or its gradient du^+/dy^+.
Piecewise solutions have also been proposed (*e.g.* Thomson, 1965).
Five versions of the inner functions are listed below:

- Spalding's law

$$y^+ = U^+ + \exp(-\kappa B)\left[\exp(\kappa U^+) - 1 - \kappa U^+ - (\kappa U^+)^2/2 - (\kappa U^+)^3/6\right]$$

- Van Driest's law

$$U^+ = \int_0^{y^+} \frac{2dy^+}{1 + \sqrt{1 + 4\kappa^2 y^{+2}[1 - \exp(y^+/A_p)]^2}} \quad \text{with} \quad A_p = 21.36$$

- Original Musker's law

$$U^+ = \int_0^{y^+} \frac{\kappa + C y^{+2}}{\kappa + C y^{+2} + C\kappa y^{+3}} \quad \text{with} \quad C = 0.001093$$

which yields the following closed form after integration:

$$U^+ = 5.424 \tan^{-1}\left[\frac{2y^+ - 8.15}{16.7}\right] + \log_{10}\left[\frac{(y^+ + 10.6)^{9.6}}{(y^{+2} - 8.15y^+ + 86)^2}\right] - 3.52$$

Inner functions (2)

- Musker's law adapted by Chauhan et al. (2007,2009)

$$U^+ = \frac{1}{\kappa}\ln\left(\frac{y^+ - a}{-a}\right) + \frac{R^2}{a(4\alpha - a)}\left[(4\alpha + a)\ln\left(-\frac{a}{R}\frac{\sqrt{(y^+ - \alpha)^2 + B^2}}{y^+ - a}\right) + \frac{\alpha}{B}(4\alpha + 5a)\left(\arctan\left(\frac{y^+ - \alpha}{B}\right) + \arctan\left(\frac{\alpha}{B}\right)\right)\right] \quad (2)$$

where $\alpha = (-1/\kappa - a)/2$, $B = \sqrt{-2a\alpha - \alpha^2}$ and $R = \sqrt{\alpha^2 + B^2}$.
The parameter a can be adjusted to a given combination of κ and B.
For instance, a=-10.5531 corresponds to the classical values (κ=0.41, B=5).
Chauhan et al. chose a=-10.3061 for the couple (κ=0.384, B=4.17), obtained for KTH and IIT experiments at high Reynolds numbers.

Inner functions (3)

- Padé law (Monkewitz et al., 2007; Chauhan et al., 2007)

$$\frac{dU^+}{dy^+} = b_0 \frac{1+b_1 y^+ + b_2 {y^+}^2}{1+b_1 y^+ + b_2 {y^+}^2 + \kappa b_0 b_2 {y^+}^3} + (1-b_0)\frac{1+c_1 y^+ + c_2 {y^+}^2}{1+c_1 y^+ + c_2 {y^+}^2 + c_3 {y^+}^3 + c_4 {y^+}^4 + c_5 {y^+}^5}$$

with

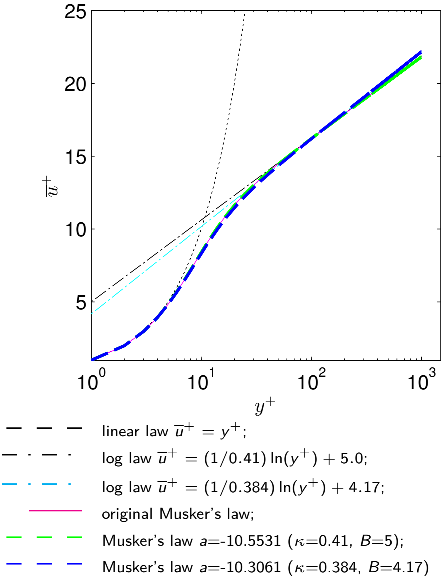
$$b_0 = \frac{0.01}{\kappa} \ , \quad b_1 = 1.1\,10^{-2} \ , \quad b_2 = 1.1\,10^{-4}$$

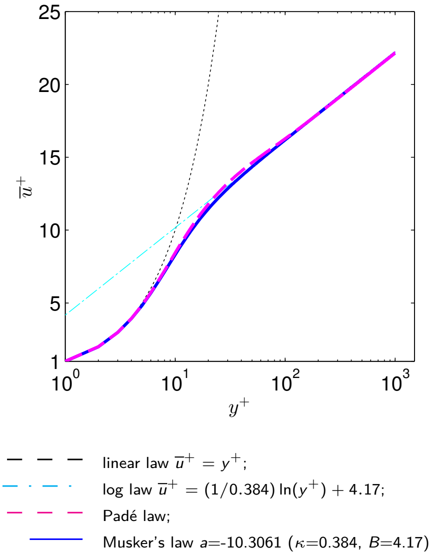
$$c_1 = -0.01 \ , \quad c_2 = 0.006 \ , \quad c_3 = 9.977\,10^{-4} \ , \quad c_4 = 2.2\,10^{-5} \ , \quad c_5 = 10^{-6}$$

which yields after integration:

$$\begin{aligned}
U^+ = {} & 0.68285472 \ln({y^+}^2 + 4.7673096 y^+ + 9545.9963) \\
& + 1.2408249 \arctan(0.010238083 y^+ + 0.024404056) \\
& + 1.2384572 \ln(y^+ + 95.232690) \\
& - 0.50435126 \ln({y^+}^2 - 7.8796955 y^+ + 78.389178) \\
& + 4.7413546 \arctan(0.12612158 y^+ - 0.49689982) \\
& - 2.77687771 \ln({y^+}^2 + 16.209175 y^+ + 933.16587) \\
& + 0.37625729 \arctan(0.033952353 y^+ + 0.27516982) \\
& + 6.5624567 \ln(y^+ + 13.670520) - 5.8178576
\end{aligned}$$

Inner functions (4)





Wake functions (1)

Different log-wake laws have been proposed. Some classical laws are:

- Coles' law (1956)

$$w_C(\eta) = 1 - \cos(\pi\eta) = 2\sin^2\left(\frac{\pi}{2}\eta\right) \quad \text{with} \quad \eta = y/\delta$$

$$\text{with} \qquad U^+_{outer} = \frac{\Pi_C}{\kappa} w_C(\eta)\ ; \quad \left.\frac{dU}{dy}\right|_{y=\delta} = \frac{u_\tau}{\kappa\delta} \neq 0\ ; \quad w_C(1) = 2$$

- Granville's law (1976)

$$w_G(\eta) = 1 - \cos(\pi\eta) + \frac{1}{\Pi_G}\eta^2(1-\eta)$$

This is the first introduction of a polynomial corrective term to satisfy $\left.\frac{dU}{dy}\right|_{y=\delta} = 0.$

- Lewkowicz' quartic law (1982)

$$w_L(\eta) = w_{\text{quartic}}(\eta) = 2\eta^2(3-2\eta) - \frac{1}{\Pi_L}\eta^2(1-\eta)(1-2\eta)$$

$$\text{with} \qquad U^+_{outer} = \frac{\Pi_L}{\kappa} w_L(\eta)\ ; \quad \left.\frac{dU}{dy}\right|_{y=\delta} = 0\ ; \quad w_L(1) = 2$$

Wake functions (2)

Different log-wake laws have been proposed. Some classical laws are:

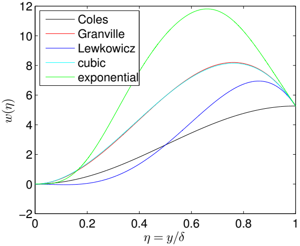
- Lewkowicz'cubic law (1982)
  There is a cubic version for the corrective term:

$$w_{\text{cubic}}(\eta) = 2\eta^2(3-2\eta) + \frac{1}{\Pi_{\text{cubic}}}\eta^2(1-\eta)$$

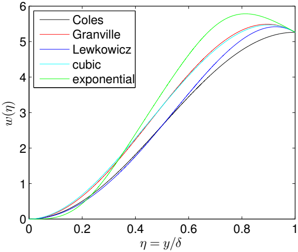
- exponential law Chauhan et al. (2007)

$$w_{\exp}(\eta) = \frac{1-\exp\left[-(1/4)(5a_2+6a_3+7a_4)\eta^4 + a_2\eta^5 + a_3\eta^6 + a_4\eta^7\right]}{1-\exp\left[-(a_2+2a_3+3a_4)/4\right]} \times \left[2 - \frac{1}{\Pi_{\exp}}\ln(\eta)\right]$$

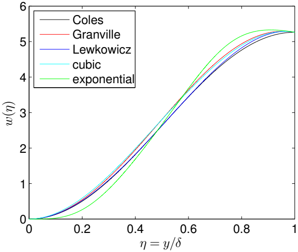
with $a_2 = 132.8410$, $a_3 = -166.2041$, $a_3 = 71.9114$, $U^+_{outer} = \frac{\Pi_{\exp}}{\kappa} w_{\exp}(\eta)$.



Π=0.1 (FPG TBL)



Π=0.6 (ZPG TBL)

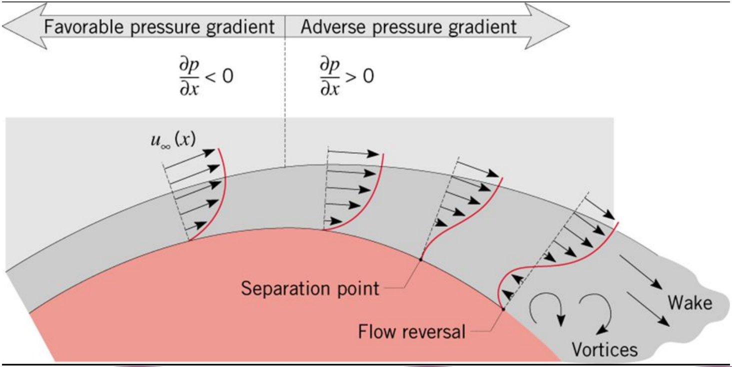


Π=1.9 (APG TBL)

- some variability is obtained for favorable gradient (Π=0.1) but the wake region is very small
- for adverse cases (e.g. Π=1.9), the wake region is important but all the approximate laws yield similar results

Adverse and favorable pressure gradients

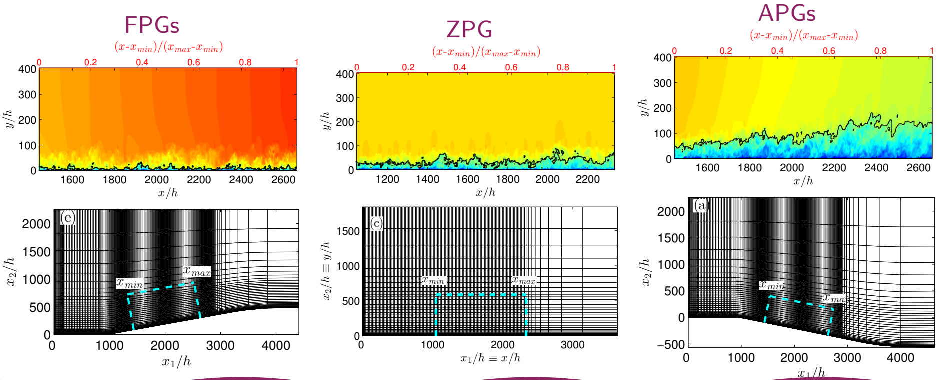
- FPG: Favorable pressure gradient $\dfrac{dp_e}{dx} < 0$
- ZPG: Zero pressure gradient $\dfrac{dp_e}{dx} = 0$
- APG: Adverse pressure gradient $\dfrac{dp_e}{dx} > 0$



TBL with pressure gradients

LES of TBL at M=0.5

Case	α	Δx^+	Δy_w^+	Δy_e^+	Δz^+	Re_θ	Re^+	U_e (m/s)	u_τ (m/s)
APGs	-11.49°	26.9	0.7	18.8	8.4	3125	688	158	5.30
APGw	-6.05°	32.0	0.8	18.2	9.8	2462	692	165	6.22
ZPG	0°	37.6	1.0	15.6	12.6	1693	608	171	7.26
FPGw	5.66°	44.7	1.1	19.1	13.9	1771	745	199	8.66
FPGs	10.19°	45.1	1.2	18.0	14.2	1618	780	197	8.83



Pressure gradient parameters

Variety of non-dimensional parameters to characterize pressure gradients:

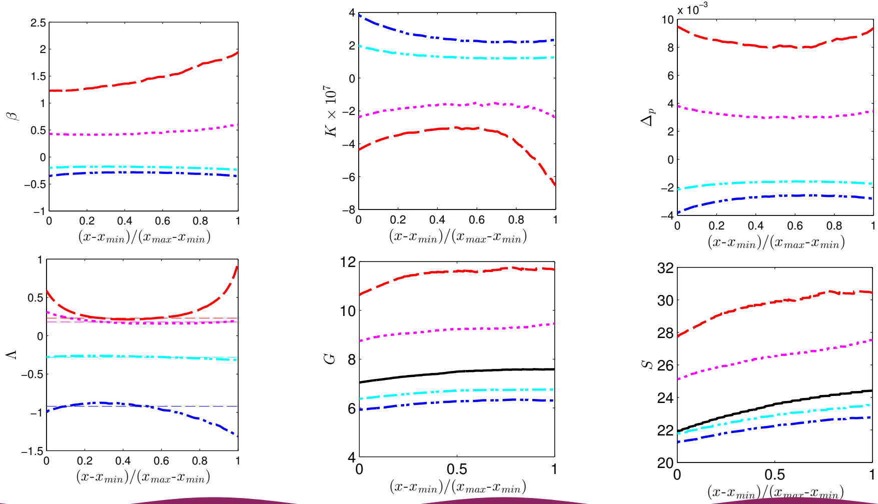
- the Clauser pressure-gradient parameter: $\boxed{\beta = \frac{\delta^*}{\tau_w}\frac{dp_e}{dx}}$
- the acceleration parameter: $\boxed{K = \frac{\nu}{u_e^2}\frac{du_e}{dx}}$
- the viscous-scaled pressure gradient $\Delta_p = \frac{\nu}{\rho u_\tau^3}\frac{dp_e}{dx}$
- the pressure-gradient of Castillo and George: $\Lambda = \frac{\delta}{\rho u_e^2 (d\delta/dx)}\frac{dp_e}{dx}$

The TBL is also characterized by:

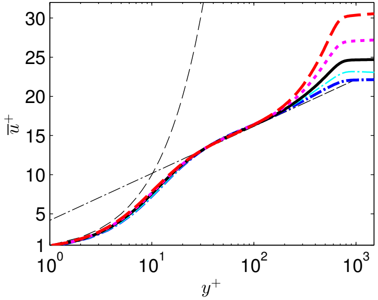
- the shape factor: $H = \frac{\delta^*}{\theta}$
- the ratio of outer and inner velocities: $S = \frac{u_e}{u_\tau} = \sqrt{\frac{2}{C_f}}$
- the defect shape factor: $G = S\left(1 - \frac{1}{H}\right)$

Pressure gradient parameters

Case	δ (mm)	H	β	$K\times10^7$	$\Delta_p\times10^3$	Λ	G	S	$C_f\times10^3$	Π
APGs	1.87	1.63	1.41	-3.0	8.0	0.21	11.5	29.8	2.24	1.9
APGw	1.61	1.53	0.44	-1.6	3.0	0.16	9.2	26.5	2.83	1.1
ZPG	1.21	1.46	0	0	0	0	7.4	23.5	3.59	0.6
FPGw	1.24	1.41	-0.18	1.2	-1.6	-0.27	6.7	22.9	3.80	0.2
FPGs	1.27	1.39	-0.28	2.3	-2.6	-0.91	6.2	22.2	4.04	0.1



Mean velocity profiles



$$U^+_{\text{composite}} = U^+_{\text{inner}} + \frac{\Pi}{\kappa}\mathcal{W}\left(\frac{y}{\delta}\right)$$

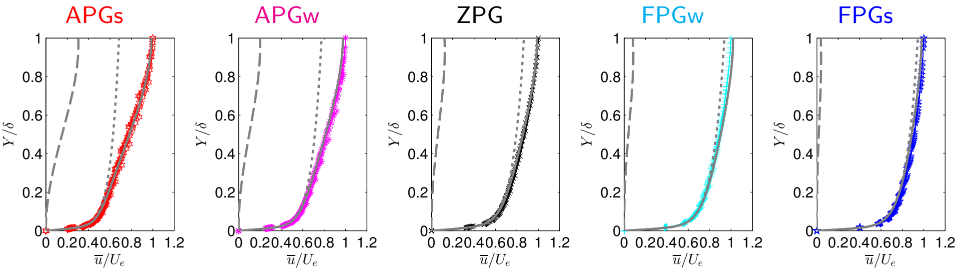
- Musker's law for inner function

$$U^+_{\text{inner}} = \frac{1}{\kappa}\ln\left(\frac{y^+ - a}{-a}\right) + \frac{R^2}{a(4\alpha - a)}\left[(4\alpha + a)\ln\left(-\frac{a}{R}\frac{\sqrt{(y^+ - \alpha)^2 + B^2}}{y^+ - a}\right) + \frac{\alpha}{B}(4\alpha + 5a)\left(\arctan\left(\frac{y^+ - \alpha}{B}\right) + \arctan\left(\frac{\alpha}{B}\right)\right)\right]$$

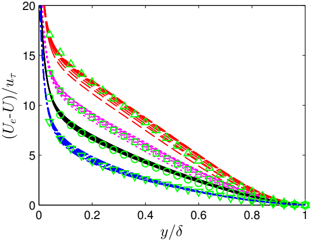
- Lewlowicz's wake law

$$\mathcal{W}\left(\frac{y}{\delta}\right) = 2\left(\frac{y}{\delta}\right)^2\left(3 - 2\frac{y}{\delta}\right) - \frac{1}{\Pi}\left(\frac{y}{\delta}\right)^2\left(1 - \frac{y}{\delta}\right)\left(1 - 2\frac{y}{\delta}\right)$$

Composite velocity profiles

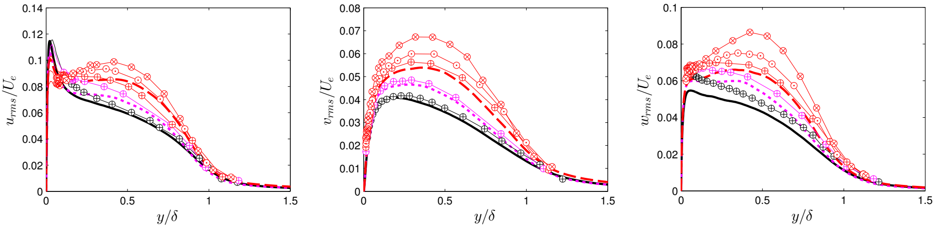


- wake parameter Π deduced from defect law

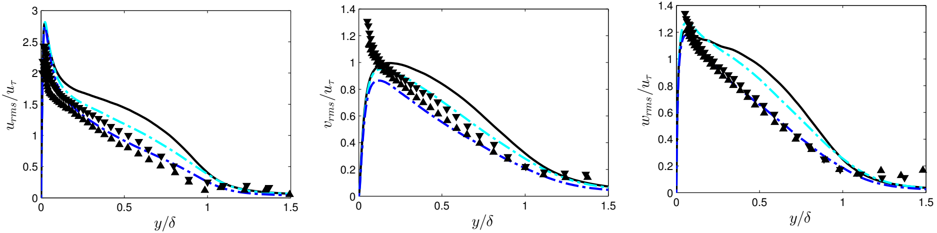


Turbulent intensities

Adverse gradient: APGs, APGw, ZPG



Favorable gradient: FPGs, FPGw, ZPG



Comparison with APG experiments Nagano et al. (1998) (β=0 ⊕, β=0.77 ⊕, β=2.19 ⊕, β=3.95 ⊙, β= 5.32 ⊗) and FPG experiments of Jones et al. (2001) (K=2.7×10^7, x=0.8 m ▼ & x=2.2 m ▲)