来自 Gilbert's text book (5th edition).

§ 1.1

27 How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? A typical corner is $(0, 0, 1, 0)$. A typical edge goes to $(0, 1, 0, 0)$.

30 The linear combinations of $\boldsymbol{v} = (a, b)$ and $\boldsymbol{w} = (c, d)$ fill the plane unless _____. Find four vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, $\boldsymbol{z}$ with four components each so that their combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} + f\boldsymbol{z}$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

31 Write down three equations for c, d, e so that $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$. Can you somehow find c, d, e for this $\boldsymbol{b}$?

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

§ 1.2

30 Can three vectors in the xy plane have $\boldsymbol{u} \cdot \boldsymbol{v} < 0$ and $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ and $\boldsymbol{u} \cdot \boldsymbol{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would certainly be impossible ...).

⋆ **Prove the following statement in Exercise 1.2.30.**

- four of those vectors in the plane would certainly be impossible...

31 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\boldsymbol{v} = (x, y, z)$ and the vector $\boldsymbol{w} = (z, x, y)$. Challenge question: Explain why $\boldsymbol{v} \cdot \boldsymbol{w} / \|\boldsymbol{v}\| \|\boldsymbol{w}\|$ is always $-\frac{1}{2}$.

⋆ **Finish the challenge question** (possibly not a challenge...)

§ 1.3

3 Solve these three equations for y_1, y_2, y_3 in terms of c_1, c_2, c_3:

$$S\boldsymbol{y} = \boldsymbol{c} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Write the solution $\boldsymbol{y}$ as a matrix $A = S^{-1}$ times the vector $\boldsymbol{c}$. Are the columns of S independent or dependent?

⋆ You can omit the details and **just write down the answers**.

5 The rows of that matrix W produce three vectors (*I write them as columns*):

$$\boldsymbol{r}_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad \boldsymbol{r}_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \qquad \boldsymbol{r}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

Linear algebra says that these vectors must also lie in a plane. There must be many combinations with $y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + y_3\boldsymbol{r}_3 = \boldsymbol{0}$. Find two sets of y's.

$\star$ You can omit the details and **just write down the answers**.

6 Which numbers c give dependent columns so a combination of columns equals zero ?

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \quad \begin{array}{l} \text{maybe} \\ \text{always} \\ \text{independent for } c \neq 0\,? \end{array}$$

$\star$ You can omit the details and **just write down the answers**.

来自第一节习题课.

以下仅涉及 2×2 矩阵.

Exercise 1. 给定矩阵 $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

1. 按照通常的方法计算 $A \cdot B$.
2. 在第 1 问的计算中, 我们使用了几次乘法, 几次加法?
3. 给出以下 7 个乘法式:

$$\begin{aligned} S_1 &= (a_{11} + a_{21})(b_{11} + b_{12}), \\ S_2 &= (a_{12} + a_{22})(b_{21} + b_{22}), \\ S_3 &= (a_{11} - a_{22})(b_{11} + b_{22}), \\ S_4 &= a_{11}(b_{12} - b_{22}), \\ S_5 &= (a_{21} + a_{22})b_{11}, \\ S_6 &= (a_{11} + a_{12})b_{22}, \\ S_7 &= a_{22}(b_{21} - b_{11}), \end{aligned}$$

请验证

$$A \cdot B = \begin{pmatrix} S_2 + S_3 - S_6 - S_7 & S_4 + S_6 \\ S_5 + S_7 & S_1 - S_3 - S_4 - S_5 \end{pmatrix}.$$

4. 考虑第 3 问就 $A \cdot B$ 的算法, 我们使用了几次乘法, 几次加法?

Exercise 2 给定矩阵 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

1. 记 $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, 计算 $A \cdot B$ 与 $B \cdot A$.
2. 给定 $S = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$, 找到矩阵 T 使得 $S \cdot T = T \cdot S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
3. 求出 λ_1 与 λ_2, 使得
$$S \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$
4. 给定数列首项 $a_0 = a_1 = 1$, 通项 $a_{n+2} = a_{n+1} + a_n$. 证明
$$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}.$$
5. 求出 $\{a_n\}$ 通项.
6. 如果你在中学学过线性递推数列的特征根法, 请与上法比较.

Exercise 3 回忆习题课上的类比: $a + ib = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. 记 S 是所有形如右式的矩阵构成的集合.

1. 请问 S 中矩阵的加法, 乘法, 转置, 行列式, 迹分别对应复数中的什么运算?
2. 构造矩阵 A, 使得 $A^{2023} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq A$.
3. 依照原点处的 Taylor 展开 $e^z = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \cdots$, 计算 $e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}}$. 尝试使用极限语言以说明这一矩阵级数和是收敛的.
4. 查询四元数的定义 (使用记号 $i^2 = j^2 = k^2 = ijk = -1$, 此处 i 可直接视作复数). 能否将四元数表示成 2-阶复矩阵?