Linear Algebar (Honorary) Homework

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ss1.1

0.1

We might as well construct a 4-dimension orthogonal coordinate system to contain this hyper-cube from $(0,0,0,0)$ to $(1,1,1,1)$. It's transparent that a corner is all this points and the number is $2^4 = 16$.

As for edges, It's obvious that an edge means a vector whose length is 1.We know from one point can lead to 4 vectors in different directions along the edge. Hence, the number of all vectors(with direction) is $16 \times 4 = 64$. Get rid of the direction, there are $64 \div 2 = 32$ edges.

A face means a set of points which has the same coordinate in one dimension. The number of faces is $2 \times 4 = 8$.

0.2

The linear combination of vector $\mathbf{v}$ and $\mathbf{w}$ can fill the plane unless $ad - bc = 0$.

You just need to choose $(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0)$ as $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$.

0.3

$$\begin{bmatrix} c & d & e \end{bmatrix} \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} = \mathbf{b}$$

That's to say,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

r(A) = r(A,$\mathbf{b}$)=n=3, this matrix has a specific solution.It is: .3 $c = \dfrac{3}{4}, d = \dfrac{1}{2}, e = \dfrac{1}{4}$

ss1.2

0.4

(1) Possible. We can take $\mathbf{a} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \mathbf{b} = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \mathbf{c} = (1, 0)$ for example.

(2) 4 is the maximum that is feasible.

(3) This problem could be regarded as "how to divide a round angle into 4 parts, ensuring they're all obtuse." But it's evident that if three of them are obtuse, the last has to be an acute angle.There aren't any solutions that divides a round angle into 4 obtuse angle.

0.5

证明. Since $x + y + z = 0$, s.t. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 0$.
s.t.

$$\frac{xz + zy + yx}{x^2 + y^2 + z^2} \equiv -\frac{1}{2}$$

It's transparent that above is the dot product of $\mathbf{v} = (x, y, z), \mathbf{w} = (z, x, y)$. □

ss1.3

0.6

Solution:$y_1 = c_1, y_2 = c_2 - c_1, y_3 = c_3 - c_2 - c_1$.
Written by matrix:

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The columns of S are independent.

0.7

$y_1\mathbf{r}_1 + y_2\mathbf{r}_2 + y_3\mathbf{r}_3 = \mathbf{0}$. Written in matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{y}$ is infinite. Two possible answer are: (0,0,0) and (1,-2,1).

0.8

(1) $c = 3$ makes its columns depentent.

(2) $c = -1$ makes its columns depentent.

(3) $c = 0$ makes its columns depentent. But if $c \neq 0$, its column can be permanently linear independent.

the rest

0.9

(1) calculate the multiplication by normal means:

$$A \times B = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{11} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

(2) Obviously, we did multiplication for 8 times, we did addition for 4 times.
(3) Verify: Apparent.
(4) According to this new method, we did multiplication for 7 times, we did addition (and subtraction) for 18 times.

0.10

(1) Do multiplication:

$$A \times B = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

(2) $|\det A| = \left| \frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} \right| = \sqrt{5}$,
According to the lemma above, if $S \times T = T \times S = I_2$,

$$T = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{\sqrt{5}+1}{2} \\ -1 & \frac{\sqrt{5}-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{\sqrt{5}+1}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{\sqrt{5}-1}{2\sqrt{5}} \end{bmatrix}$$

(3)

$$\lambda_1 = \frac{1-\sqrt{5}}{2} \quad \lambda_2 = \frac{1+\sqrt{5}}{2}$$

(4) Written in linear equations form:

$$\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_{n+1} = a_{n+1} \end{cases}$$

So it is evident that

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

(5)

$$\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

Eigen-decompose this matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

Thus,

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}^n \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

Calculate,

$$a_n = \frac{1}{\sqrt{5}} \left(\lambda_2^n - \lambda_1^n \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

0.11

(1) The addition, multiplication, transpose, determinant, trace in S respectively means addition, multiplication, get-conjugate, get-absolutevalue, get-realpart-squared in Complex Set.

(2) Corresponded with the insight mentioned above, we can refer to the matrix A as a complex number z. The following equality

$$A^{2023} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq A$$

can be referred to as

$$z^{2023} = 1 \qquad (z \neq 1) \tag{1}$$

There are 2023 solutions to this equation.One possible solution is

$$z = e^{\frac{2\pi}{2023} i} = \cos \frac{2\pi}{2023} + i \sin \frac{2\pi}{2023}$$

Correspondingly, the matrix A could be:

$$A = \begin{bmatrix} \cos \frac{2\pi}{2023} & \sin \frac{2\pi}{2023} \\ -\sin \frac{2\pi}{2023} & \cos \frac{2\pi}{2023} \end{bmatrix}$$

(3) I think it's a good technique to exert an eigen-decomposition on matrix A. This could help us understand the construction of the accordance between matrix and complex number to a further extent.

After calculation, the matrix is decomposed like the following:

$$A = P \Lambda P^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} a+bi & \\ & a-bi \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

You can easily notice that P and P^{-1} is the bridge connecting matrix with complex number. Evidently,we get:

$$A^n = P\Lambda^n P^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} a+bi & \\ & a-bi \end{pmatrix}^n \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

Thus,

$$\begin{aligned} e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \\ &= P\left(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots\right)P^{-1} \\ &= P\begin{pmatrix} e^{a+bi} & \\ & e^{a-bi} \end{pmatrix}P^{-1} \\ &= P\begin{pmatrix} e^a(\cos b + i\sin b) & \\ & e^a(\cos b - i\sin b) \end{pmatrix}P^{-1} \\ &= \begin{pmatrix} e^a\cos b & e^a\sin b \\ -e^a\sin b & e^a\cos b \end{pmatrix}. \end{aligned}$$

(4) We once have an assumption. If we can somehow refer to Quaternions as Complex numbers, this problem may be a lot easier. But how to get rid of j and k? First of all, The formula $ijk = -1$ can deprive us of k. Then we can see j in Quaternions as i in Complex numbers, and send real i into the matrix, like the following procedure:

$$q \in \mathbb{H}, \quad q = a + bi + cj + dk = (a+bi) + (c-di)j = A + Bj$$

Accordingly Quaternions can be confined into a 2*2 matrix. you just need to write a Quaternion(a, b, c, d) like this:

$$Q = \begin{bmatrix} a+bi & c-di \\ -c+di & a+bi \end{bmatrix}$$