Problem Set for 20-Feb-2025

Problem 1 Find two linear maps

$$\alpha, \beta : \mathbb{F}[x] \to \mathbb{F}[x],$$

such that

$$\alpha(\beta(f)) - \beta(\alpha(f)) = f$$

for any $f \in \mathbb{F}[x]$.

Is it possible to find such $\alpha, \beta : V \to V$ when V is of finite dimension?

> *答: α 是求导, $\beta(f) = x \cdot f(x)$. 那么*
>
> $$\alpha(\beta(f)) - \beta(\alpha(f)) = D(xf) - x(Df) = f.$$
>
> *有限维无解, 因为不存在 $AB - BA = I$ 的矩阵 (两侧取 tr).*

Problem 2 Here is a **clarification of irreducibility** over general polynomial rings. Let $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$. A polynomial $f \in \mathbb{A}[x]$ is **reducible** if and only if there exists some factorisation $f = g \cdot h$ such that $g^{-1} \notin \mathbb{A}[x]$ and $h^{-1} \notin \mathbb{A}[x]$. For instance:

- $2 \cdot x$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{Z}[x]$;
- $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{C}[x]$.

Now consider $f \in \mathbb{Z}[x]$. **Prove** the following:

1. If f is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$;

   > *答: 对任意 $d \in \mathbb{N}+$, 多项式 $d \cdot f(x)$ 在 $\mathbb{Z}[x]$ 中无法分解作两个非常值多项式的乘积. 继而使用反证法: 假定 $\mathbb{Q}[x]$ 中存在分解 $f = p \cdot q$, 满足 $\deg p \cdot \deg q \geq 1$. 取 d 使得 $\sqrt{d}p, \sqrt{d}q \in \mathbb{Z}[x]$, 则与 $d \cdot f$ 在 $\mathbb{Z}[x]$ 中的分解方式矛盾.*

2. If f is irreducible in $\mathbb{R}[x]$, then it is irreducible in $\mathbb{Q}[x]$.

   > *答: 考虑逆否命题即可.*

> ⚠️ *规范的表述是"多项式 $f(x)$ 在 $\mathbb{A}[x]$ 中可约", 而非"多项式 $f(x)$ 可约"; 类似地, 规范地表述是" f 是 $\mathbb{F}$-线性映射", 而非" f 是线性映射"。若无歧义, 可适当地选用后者以精简表述。*

In fact, one has

$$\text{(domain)} \quad \underbrace{\mathbb{Z}[x] \to (\mathbb{Z}[1/2])[x] \to \cdots \to \mathbb{Q}[x]}_{\text{more irreducible polynomials}} \quad \text{(fractional field)},$$

and

$$\text{(field)} \quad \underbrace{\mathbb{Q}[x] \to (\mathbb{Q}[\sqrt{2}])[x] \to \cdots \to \mathbb{C}[x]}_{\text{less irreducible polynomials}} \quad \text{(algebraic closure)}.$$

(Optional) Find **Gauß's lemma** in any of the textbooks and understand both the statement and the proof. The lemma states that:

> *For any $f(x) \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ if and only if f is both irreducible over $\mathbb{Q}[x]$ and f is primitive (i.e., the greatest common divisor of its coefficients is 1).*

Let f be **monic**, i.e., non-zero with leading coefficient 1. From Gauß's lemma, we learn that for any monic $f \in \mathbb{Z}[x]$, f is irreducible in $\mathbb{Z}[x]$ **if and only if** it is irreducible in $\mathbb{Q}[x]$.

Problem 3 Here are some criteria for the irreducibility of polynomials in $\mathbb{C}[x]$:

1. Let $f \in \mathbb{Z}[x]$ be a **monic** polynomial of degree n. Denote the zeros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if there is exactly one z_i such that $|z_i| \geq 1$ and $f(0) \neq 0$, then f is irreducible in $\mathbb{Q}[x]$.

   > *答: 依照Gauß 引理, 首一整系数多项式在$\mathbb{Z}[x]$ 与$\mathbb{Q}[x]$ 中的可约性等价. 若$\mathbb{Z}[x]$ 中存在分解 $f = pq$ $(\deg p \cdot \deg q \geq 1)$, 则不妨设 p 的所有根满足 $|z| < 1$. 根据 $p(0) \neq 0$, 以及根乘积的 Vieta 定理, 矛盾.*

2. Let $f \in \mathbb{Z}[x]$ be a polynomial such that $f(0)$ is prime. Denote the zeros of f in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if $|z_i| > 1$ for all i, then f is irreducible.

   > *答: 若有分解 $f = pq$ $(\deg p \cdot \deg q \geq 1)$, 则 $|p(0)| \cdot |q(0)|$ 是素数. 不妨设 $|p(0)| = 1$, 与根乘积的 Vieta 定理矛盾.*

3. Let $f(x) = \sum_{k=0}^n a_k \cdot x^k \in \mathbb{Z}[x]$ be a polynomial with $f(0)$ prime. Suppose that $|a_0| > \sum_{k=1}^n |a_k|$. Show that f is irreducible.

   > *答: 三角不等式 $|a_0| > \sum_{k=1}^n |a_k|$ 说明所有根模长大于1, 往后同上一问.*

Problem 4 Find all $f(x) \in \mathbb{C}[x]$ such that

$$f(x) \equiv \begin{cases} 2x \mod (x-1)^2, \\ 3x \mod (x-2)^2. \end{cases}$$

> *答: 见张贤科 (1.17)*
>
> **17.** 求次数最低的多项式 $f(X)$,使得 $f(X)$被$(X-1)^2$ 除时余式为 $2X$,被$(X-2)^3$ 除时,余式为 $3X$.
>
> **解** 由已知,有
>
> $$f(X)=\varphi_1(X-1)^2+2X=\varphi_2(X-2)^3+3X=\varphi_2[(X-1)-1]^3+3X$$
> $$=\varphi_2[(X-1)^3-3(X-1)^2+3(X-1)-1]+3X.$$
>
> 于是应有 $\varphi_2(3X-4)+X$ 是$(X-1)^2$ 的倍式,$\deg\varphi_2\geqslant 1$,设 $\varphi_2=(aX+b)$,则可设 $(3X-4)(aX+b)+X=3a(X-1)^2$,比较两边同次系数得
>
> $$\begin{cases} -4a+3b+1=-6a \\ -4b=3a \end{cases} \Rightarrow \begin{cases} a=4 \\ b=-3 \end{cases}.$$
>
> 所以
>
> $$f(X)=(X-1)^3(4X-3)+3X=4X^4-27X^3+66X^2-65X+24.$$
>
> *此题题干和张贤科书中有出入, 正确答案是*
>
> $$-3x^3+14x^2-17x+8 \quad + P(x)(x-1)^2(x-2)^2.$$

Exercises (optional) The following problems are **optional** but some of the problems are very important.

1. Is there any irreducible $f(x) \in \mathbb{Z}[x]$ such that $f(f(x))$ is reducible?

> *答: 经耐心计算, 可以发现* $f(x) = x^2 + 10x + 17$, *且*
>
> $$f(f(x)) = \left(x^2+8x+14\right)\left(x^2+12x+34\right).$$

2. Prove that $1 + \prod_{k=1}^{2025}(x-k)^2$ is irreducible in $\mathbb{Z}[x]$;

> *答: 若可约, 则存在次数* ≤ 2025 *的子式* $f \in \mathbb{Z}[x]$. *对* $1 \leq k \leq 2025$, $f(k)$ *是* 1 *的约数. 由于原多项式无零点, 其因子* f *亦无零点. 此时, 不妨设假设* $f-1$ *以* $\{k\}_{k=1}^{2025}$ *为零点. 比较次数, 只能有* $f = \prod_{k=1}^{2025}(x-k)+1$, *矛盾.*

3. Prove that $\prod_{k=1}^{n}(x-x_i)+1$ is either irreducible in $\mathbb{Z}[x]$, or a perfect square;
   - where $x_1 < x_2 < \cdots < x_n$ are integers.

> *答: 若可约, 则* $f^2 = g^2h^2$. *不妨设* $\deg g^2 \leq n$. *由零点数量知* $(g^2-1) = (f-1)$. *从而* f *是完全平方式.*
>
> *由初等数学,* f *是平方式当且仅当* x_i *是长度为* 2 *或* 4 *的等差数列, 公差为* 1.

4. ($f \in \mathbb{Z}[x]$) Prove that if $f(x) = 1$ has ≥ 4 solutions in $\mathbb{Z}$, then $f(x) = -1$ has no solutions in $\mathbb{Z}$.

> *答: 也就是是说明 $(x-a)(x-b)(x-c)(x-d)g(x)=2$ 无解. 显然.*

5. Prove that the partial sum $(e^x)_{\deg\leq n}$ is always irreducible in $\mathbb{Q}[x]$.

> *答: 考虑多项式 $f_n(x)=x^n+nx^{n-1}+\cdots+n!$, 记素因子分解 $n=\prod p_i^{n_i}$.*
>
> *依照补充材料中的小技巧, 对任意素数 p_i, 相应的 $\bmod p$-凸包的底部折线段的斜率总是形如 $\frac{1-p_i^k}{p_i^k\cdot(p_i-1)}$, 其中 $k\geq n_i$. 用格点将凸包底线切分成最细单元, 由于 $\frac{(p^k-1)/(p-1)}{p^k}$ 是既约分数, 故这些最细单元的最短横向长度必然是 p^{n_i}. 上述最细单元的横向长度在多项式乘法下不会变得更长.*
>
> *作为推论, $f_n(x)$ 的每个不可约子式的次数一定是 p^{n_i} 的整数倍 (使用反证即可). 今遍历所有 $p_i^{n_i}$, 则 $f_n(x)$ 不可约子式的次数一定是各 $p_i^{n_i}$ 的倍数. 由是观之, f_n 不可约.*