Advanced Algebra (Honor) II

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Throughout $\mathbb{F}$ is a number field.

Exercise 1.1 *Let* $P_n := \{f(x) \in \mathbb{F}[x] \mid \deg f(x) < n\}$. *Pick* $a_1, \ldots, a_n \in \mathbb{F}$ *such that* $a_i \neq a_j$ *for any* $i \neq j$. *Show that*

$$f_j(x) := \prod_{i \neq j}(x - a_i)\ (1 \leq j \leq n)$$

form a basis of P_n.

Exercise 1.2 *Let* $V \subseteq \mathbb{F}^{n\times n}$ *be a subspace such that for any* $\mathbf{0} \neq A \in V$, A *is non-singular. Show that* $\dim V \leq n$.

Exercise 1.3 *Let* $f(x), g(x) \in \mathbb{F}[x]$ *such that* $f(x)g(x) \neq 0$. *Set*

$$\langle f(x)\rangle := \{h(x) \mid f(x)|h(x)\}.$$

Prove that

(1) $\langle f(x)\rangle$ *is a linear subspace of* $\mathbb{F}[x]$.

(2) $\langle f(x)\rangle \cap \langle g(x)\rangle = \langle [f(x), g(x)]\rangle$.

(3) $\langle f(x)\rangle + \langle g(x)\rangle = \langle (f(x), g(x))\rangle$.

Exercise 1.4 *Assume* $\mathbb{F}_1 \subseteq \mathbb{F}_2 \subseteq \mathbb{F}_3$ *such that* $\dim_{\mathbb{F}_1} \mathbb{F}_2 = m$ *and* $\dim_{\mathbb{F}_2} \mathbb{F}_3 = n$ *(as linear spaces). Show that* $\dim \mathbb{F}_3/\mathbb{F}_1 = mn$.

Exercise 1.5 *Let* $W, V_1, \ldots, V_m$ *be linear subspaces of* V *such that*

$$W \subseteq V_1 \cup \cdots \cup V_m.$$

Prove that $W \subseteq V_i$ *for some* i.

Exercise 1.6 *Assume that* $\{\alpha_1, \ldots, \alpha_n\}$ *and* $\{\beta_1, \ldots, \beta_n\}$ *are two basis of a linear space* V. *Prove that there exists* $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$ *such that*

$$\{\alpha_1, \ldots, \alpha_{j-1}, \beta_{i_j}, \alpha_{j+1}, \cdots, \alpha_n\}$$

is a basis of V *for any* j.

Exercise 1.7 (Quotient spaces) *Let V be a linear space over $\mathbb{F}$ and $U \subseteq V$ a subspace. For any $v \in V$, we fine the U-*coset *of v :*

$$v + U := \{v + u \mid u \in U\}.$$

Let $S := \{v + U \mid v \in V\}$ and define

$$(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U, k \cdot (v_1 + U) = kv_1 + U$$

for any $v_1, v_2 \in V$ and $k \in \mathbb{F}$. Prove"

(1) Either $v_1 + U = v_2 + U$ or $(v_1 + U) \cap (v_2 + U) = \emptyset$.

(2) $v_1 + U = v_2 + U$ iff $v_1 - v_2 \in U$.

(3) Both $+$ and $\cdot$ do not depend on the choice of v.

(4) $(S, +, \cdot)$ is a linear space. (S is the quotient space *V/U.)*

Exercise 1.8 (Dual spaces) *We call $f : V \to \mathbb{F}$ a* linear function *if*

- *$f(\alpha + \beta) = f(\alpha) + f(\beta)$ for any $\alpha, \beta \in V$, and*
- *$f(k\alpha) = kf(\alpha)$ for any $\alpha \in V$ and $k \in \mathbb{F}$.*

Denote by V^* the set of all linear functions on V. Prove the followings.

(1) A map f is a linear function iff

$$f(k\alpha + l\beta) = kf(\alpha) + lf(\beta)$$

for any $k, l \in \mathbb{F}, \alpha,\ \beta \in V$.

(2) If $f \in V^*$, then $f(\mathbf{0}) = 0$ and $f(-\alpha) = -f(\alpha)$.

(3) If $f \in V^*$, $\alpha_i \in V$ and $k_i \in \mathbb{F}$ $(1 \leq i \leq s)$, then

$$f(\sum_{i=1}^{s} k_i\alpha_i) = \sum_{i=1}^{s} k_i f(\alpha_i).$$

(4) Assume $f, g \in V^*$, and $\epsilon_1, \ldots, \epsilon_n$ is a basis of V, then $f = g$ iff

$$f(\epsilon_i) = g(\epsilon_i) \text{ for any } 1 \leq i \leq s.$$

(5) Assume that $\epsilon_1, \ldots, \epsilon_n$ is a basis of V, and $a_1, \ldots, a_n \in \mathbb{F}$. Then there exists a unique $f \in V^*$ such that

$$f(\epsilon_i) = a_i \text{ for any } 1 \leq i \leq n.$$

(6) For any $f, g \in V^*$ and $c \in \mathbb{F}$, define

$$(f + g)(\alpha) = f(\alpha) + g(\alpha), \text{ and } (cf)(\alpha) = c(f(\alpha)) \text{ for any } \alpha \in V.$$

Then V^* is a linear space and $\dim V = \dim V^*$.

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For a linear map $\phi : V \to U$, recall

$$\mathrm{Ker}(\phi) := \{v \in V \mid \phi(v) = 0\} \text{ and } \mathrm{Im}\ (\phi) := \{\phi(v) \mid v \in V\}.$$

Exercise 2.1 *Find the* $\mathrm{Ker}(\phi)$ *and* $\mathrm{Im}\ (\phi)$*.*

(1) $\phi : \mathbb{F}^2 \to \mathbb{F}^1,\ (x, y) \mapsto x.$

(2) $\phi : \mathbb{F}^2 \to \mathbb{F}^1,\ (x, y) \mapsto x - y.$

(3) $\phi : \mathbb{F}^2 \to \mathbb{F}^3,\ (x, y) \mapsto (x + y, x - y, 2x + 3y).$

Exercise 2.2 *Let V be a linear space with a basis $v_1, \ldots, v_n$. Define a linear map $\phi : V \to V$ by*

$$\phi(v_i) = v_{i+1}\ (1 \le i \le n-1) \text{ and } \phi(v_n) = 0.$$

(1) Find the matrix of ϕ under this basis.

(2) Show that $\phi^n = 0$ and $\phi^{n-1} \ne 0$.

Exercise 2.3 *Let V and U be linear spaces of dimensions n and m. Let $\phi : V \to U$ be a linear map of rank r. Prove the followings.*

(1) There is a basis $\{v_1, \cdots, v_n\}$ of V such that $\mathrm{Ker}(\phi) = \mathrm{Span}\{v_{r+1}, \ldots, v_n\}$ and $\phi(v_1), \ldots, \phi(v_r)$ form a basis of $\mathrm{Im}\ (\phi)$. In particular,

$$\dim \mathrm{Ker}(\phi) + \dim \mathrm{Im}\ (\phi) = n.$$

(2) If $W \subset V$ is a subspace, then $\dim V - \dim W \ge \dim \mathrm{Im}\ (\phi) - \dim \phi(W)$.

(3) There exist linear maps $\phi_i : V \to U$ $(1 \le i \le r)$ such that $r(\phi_i) = 1$ and $\phi = \phi_1 + \cdots + \phi_r$.

Exercise 2.4 *Let V and U be linear spaces of dimensions n and m. Let $\phi, \psi : V \to U$ be linear maps of rank r such that $\mathrm{Im}\ (\psi) \subset \mathrm{Im}\ (\phi)$, then $\psi = \phi \circ \xi$ for some linear map $\xi : V \to V$.*

Exercise 2.5 *Let $A, B \in \mathbb{F}^{m\times n}$. Define a linear map $\phi : \mathbb{F}^{n\times m} \to \mathbb{F}^{m\times n}$ by $\phi(X) := AXB$.*

(1) Prove that ϕ is not isomophic if $m \ne n$.

(2) Find $\dim \mathrm{Ker}(\phi)$ and give a basis of $\mathrm{Ker}(\phi)$.

Exercise 2.6 *Let V, U, W be non-empty linear spaces of finite dimension. Suppose that $\phi : V \to U$ and $\psi : U \to W$ are linear maps such that $r(\phi) = r(\psi \circ \phi)$. Show that there exists a linear map $\xi : W \to U$ such that $\xi \circ \psi \circ \phi = \phi$.*

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Exercise 3.1 *Assume W is the solution of* $\left\{ \begin{array}{r} x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \end{array} \right.$. *Find the dimension and a basis of the quotient space $\mathbb{R}^3/W$.*

Exercise 3.2 *Let V_1 and V_2 be linear spaces. Let $W_1 \subset V_1$ and $W_2 \subset V_2$ be subspaces. Prove that $(V_1 \times V_2)/(W_1 \times W_2)$ is isomorphic to $(V_1/W_1) \times (V_2/W_2)$.*

Exercise 3.3 *Let $\phi : V \to V$ be a linear map such that $\phi^2 = \phi$. Then $\phi|_{\mathrm{Im}\ \phi} = id_{\mathrm{Im}\ \phi}$ and*

$$V = \mathrm{Im}\, \phi \oplus \mathrm{Ker}\, \phi.$$

Exercise 3.4 *Let $\phi, \psi : V \to V$ be linear maps such that $\phi\psi = \psi\phi$. Show that both $\ker \phi$ and $\mathrm{Im}\ \phi$ are ψ-invariant.*

Exercise 3.5 *Let V be a linear space of dimension n and $\varphi : V \to V$. Prove that the followings are equivalent.*

1. $V = \mathrm{Ker}\, \varphi + \mathrm{Im}\, \varphi$;
2. $V = \mathrm{Ker}\, \varphi \oplus \mathrm{Im}\, \varphi$;
3. $\mathrm{Ker}\, \varphi \cap \mathrm{Im}\, \varphi = 0$;
4. $\mathrm{Ker}\, \varphi = \mathrm{Ker}\, \varphi^2$ $\left(\Leftrightarrow \dim \mathrm{Ker}\, \varphi = \dim \mathrm{Ker}\, \varphi^2\right)$;
5. $\mathrm{Im}\, \varphi = \mathrm{Im}\, \varphi^2$ $\left(\Leftrightarrow \mathrm{r}(\varphi) = \mathrm{r}\left(\varphi^2\right)\right)$;
6. $\mathrm{Ker}\, \varphi$ *has a φ-invariant complement, i.e., there is a φ-invariant complement U such that $V = \mathrm{Ker}\, \varphi \oplus U$;*
7. $\mathrm{Im}\, \varphi$ *has a φ-invariant complement, i.e., there is a φ-invariant complement W such that $V = \mathrm{Im}\, \varphi \oplus W$.*

Exercise 3.6 *Let*

$$\cdots \longrightarrow V_{i-1} \xrightarrow{\varphi_{i-1}} V_i \xrightarrow{\varphi_i} V_{i+1} \longrightarrow \cdots$$

be a sequence of linear maps between linear spaces such that $\mathrm{Im}\ \varphi_{i-1} \subset \ker \varphi_i$. We say the sequence is **exact** *at V_i, if $\mathrm{Im}\ \varphi_{i-1} = \ker \varphi_i$. We say the sequence is exact if all V_i are exact. Prove the following.*

1. $0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2$ *is exact at V_1 if and only if φ_1 is injective;*
2. $V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} 0$ *is exact at V_2 if and only if φ_1 is surjective;*
3. *Assume $V_1 \to V_2$ is injective. Then $0 \to V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \to 0$ is a (short) exact sequence (i.e., exact at V_1, V_2, V_3) if and only if we have the isomorphism*

$$V_2/V_1 \xrightarrow[\cong]{\overline{\varphi_2}} V_3, \quad \overline{\varphi_2}(\bar{\alpha}) = \varphi_2(\alpha);$$

4. *If $0 \to V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \to 0$ is a short exact sequence, then*

$$V_2 = V_1' \oplus V_3',$$

and $V_1 \cong V_1'$ and $V_3' \cong V_3$ given by φ_1 and the restriction of φ_2.

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Exercise 4.1 *Assume $f(x) = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible and $\alpha \in \mathbb{C}$ is a root of f.*

1. *Prove that $\mathbb{Q}[\alpha] := \{g(\alpha) \mid g(x) \in \mathbb{Q}[x]\}$ is a linear space over $\mathbb{Q}$ and $1, \alpha, \alpha^2$ form a basis.*
2. *Prove that $\varphi : \beta \mapsto f'(\alpha)\beta$ gives a linear map on $\mathbb{Q}[\alpha]$ and find its matrix under $1, \alpha, \alpha^2$.*

Exercise 4.2 *Assume that $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$. Assume that p is a prime integer such that $p \nmid a_0$, $p \mid a_i (1 \le i \le n)$, and $p^2 \nmid a_n$. Show that $f(x)$ is irreducible.*

Exercise 4.3 *We say $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ is* **primitive** *if $\gcd(a_0, \ldots, a_n) = 1$. Prove that if $f, g \in \mathbb{Z}[x]$ are primitive, then so is fg. As a corollary, if $f \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$, then it is reducible in $\mathbb{Z}[x]$.*

Exercise 4.4 *Assume n is odd. Pick $a_1, \ldots, a_n \in \mathbb{Z}$ such that $a_i \neq a_j$ for any $i \neq j$. Show that $f(x) = (x - a_1) \cdots (x - a_n) + 1$ are irreducible in $\mathbb{Q}[x]$.*

Exercise 4.5 *Let V be a finite-dimensional linear space over $\mathbb{F}$ and $S \subset V$ a non-empty subset. We call*

$$\operatorname{Ann}(S) := \{f \in V^* \mid f(x) = 0, \forall x \in S\}$$

the **annihilator** *of S. Show the following:*

1. $\operatorname{Ann}(S)$ *is a subspace of V^*.*
2. $\operatorname{Ann}(S) = \operatorname{Ann}(\operatorname{Span}(S))$.
3. $\dim \operatorname{Ann}(S) = \dim V - \dim(\operatorname{Span}(S))$.
4. *For any subspaces V_1, V_2 of V, we have*

   $$\operatorname{Ann}(V_1 \cap V_2) = \operatorname{Ann}(V_1) + \operatorname{Ann}(V_2) \text{ and } \operatorname{Ann}(V_1 + V_2) = \operatorname{Ann}(V_1) \cap \operatorname{Ann}(V_2).$$

5. *If $V = V_1 \bigoplus V_2$, then $V^* = \operatorname{Ann}(V_1) \bigoplus \operatorname{Ann}(V_2)$.*

Exercise 4.6 *Let V be a linear space over $\mathbb{F}$ and $0 \neq f \in V^*$. Prove that*

1. $\operatorname{Ker} f$ *is a maximal subspace of V (i.e., if $\operatorname{Ker} f \subset U$, then $U = \operatorname{Ker} f$ or $U = V$).*
2. *Fix arbitrary $\beta \notin \operatorname{Ker} f$, then for any $\alpha \in V$, there are unique $\eta \in \operatorname{Ker} f$ and $k \in \mathbb{F}$ such that*

   $$\alpha = \eta + k\beta.$$

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Exercise 5.1 *Prove the following vector spaces become inner product spaces under the given binary operations:*

(1) Let $V = \mathbb{R}[x]$*, which is the real vector space of all polynomials with real coefficients. For any polynomials*

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \text{ and } g(x) = b_0 + b_1 x + \cdots + b_m x^m$$

in V*, define*

$$(f(x), g(x)) := a_0 b_0 + a_1 b_1 + \cdots + a_k b_k,$$

where $k := \min\{n, m\}$*;*

(2) Let $V = M_n(\mathbb{R})$*, which is the real vector space consisting of all* $n \times n$ *real matrices. For any matrices* $A = (a_{ij})$ *and* $B = (b_{ij})$ *in* V*, define*

$$(A, B) := \operatorname{tr}(AB') = \sum_{i,j=1}^{n} a_{ij} b_{ij};$$

(3) Let $V = M_n(\mathbb{C})$*, which is the complex vector space consisting of all* $n \times n$ *complex matrices. For any matrices* $A = (a_{ij})$ *and* $B = (b_{ij})$ *in* V*, define*

$$(A, B) := \operatorname{tr}\left(A\overline{B}'\right) = \sum_{i,j=1}^{n} a_{ij} \overline{b_{ij}}.$$

Remark*: operations in (2) and (3) are called* ***Frobenius inner product****.*

Exercise 5.2 *Let* V *be an inner product space. Prove the following:*

(1) If $(\alpha, \beta) = 0$ *for all* $\beta \in V$*, then* $\alpha = 0$*; if* $(\alpha, \beta) = 0$ *for all* $\alpha \in V$ *, then* $\beta = 0$*.*

(2) Let $\{e_1, e_2, \cdots, e_n\}$ *be a basis for* V*. If* $(\alpha, e_i) = (\beta, e_i)$ *for all* i*, then* $\alpha = \beta$*.*

Exercise 5.3 (Gram matrix) *Let* $\{e_1, e_2, \cdots, e_n\}$ *be a basis for the inner product space* V*. Define* $g_{ij} = (e_i, e_j)$*. We call* $G = (g_{ij})_{n\times n}$ *the* ***Gram matrix*** *or* ***metric matrix*** *of the inner product space* V *with respect to the basis* $\{e_1, e_2, \cdots, e_n\}$*.*

(1) Prove that G *is invertible.*

(2) Suppose $\alpha, \beta \in V$ *have coordinate vectors* x, y *w.r.t this basis. Show that*

$$(\alpha, \beta) = \begin{cases} x^T G y & (\textit{if } V \textit{ is a Euclidean space}) \\ x^T G \overline{y} & (\textit{if } V \textit{ is a complex space}) \end{cases}.$$

(3) Suppose that $\{f_1, f_2, \cdots, f_n\}$ *is another basis with Gram matrix* H*. Suppose that the transition matrix from the basis* $\{e_1, e_2, \cdots, e_n\}$ *to the basis* $\{f_1, f_2, \cdots, f_n\}$ *is* C*. Prove that*

$$H = \begin{cases} C'GC & (\textit{if } V \textit{ is a Euclidean space}) \\ C'G\bar{C} & (\textit{if } V \textit{ is a complex space}) \end{cases}.$$

(4) Assume that H *a positive definite* $n \times n$ *real symmetric matrix (or positive definite Hermitian matrix). Prove that there exists a basis* $\{f_1, \ldots, f_n\}$ *for* V *such that its Gram matrix is* H*.*

Exercise 5.4 *Let $\boldsymbol{A}$ be an $n \times n$ positive semi-definite real symmetric matrix. Prove that for any n dimensional real column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the following inequality holds:*

$$(\boldsymbol{x}'\boldsymbol{A}\boldsymbol{y})^2 \leq (\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x})(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}).$$

Exercise 5.5 *Let φ be a linear transformation on a finite-dimensional inner product space V. We want to prove that if φ has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then the adjoint φ^* of φ has eigenvalues $\overline{\lambda_1}, \overline{\lambda_2}, \cdots, \overline{\lambda_n}$.*

Exercise 5.6 (Non-existence of adjoint) *Let $U = \mathbb{R}[x]$ with the inner product defined in Exercise 5.1(1). For any $f(x), g(x) \in U$, we can represent them in a unified form with certain coefficients possibly being zero:*

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \text{ and } g(x) = b_0 + b_1x + \cdots + b_nx^n.$$

(1) Define the linear transformation φ as

$$\varphi(f(x)) = a_1 + a_2x + \cdots + a_nx^{n-1}.$$

Find the adjoint φ^*.

(2) Define the linear transformation φ as

$$\varphi(f(x)) = a_0 + a_1(1+x) + a_2\left(1+x+x^2\right) + \cdots + a_n\left(\sum_{i=0}^{n} x^i\right).$$

Prove that the adjoint φ^* does not exist.

Exercise 5.7 (Reflection transformation) *Let v be a unit vector in an n-dimensional Euclidean space V. Define a linear transformation $\varphi : V \to V$ such that for any vector $u \in V$, we have*

$$\varphi(u) = u - 2(u, v)v.$$

Show that φ is an orthogonal transformation and $\det \varphi = -1$.

Remark*: The linear transformation above is called a **reflection transformation**.*

Exercise 5.8 *Considers an $n \times n$ matrix $M = I_n - 2\alpha\alpha'$, where α is an n-dimensional real column vector and $\alpha'\alpha = 1$. Such an M is called a **mirror matrix**.*

Let φ be a linear transformation on an n-dimensional Euclidean space V. Prove that φ is a reflection transformation if and only if the representation matrix of φ in any (or some) orthonormal basis of V is a mirror matrix.

Exercise 5.9 ($\star$) *In an n-dimensional Euclidean space, any orthogonal transformation can be expressed as a product of at most n reflection transformations.*