Problem Set for 17-Feb-2025

Suggestions for self-study in advance

1. Review the fundamental definitions of Abelian groups, fields, vector spaces (linear spaces), linear subspaces, linear maps, and related concepts.

2. It is highly recommended to summarise the linear maps encountered in *mathematical analysis*. For instance, the integral

$$\int_0^1 (-)\,\mathrm{d}\,x : R([0,1]) \to \mathbb{R}, \quad f \mapsto \int_0^1 f(x)\,\mathrm{d}\,x$$

is a well-defined $\mathbb{R}$-linear map. In your textbook on mathematical analysis, propositions such as $\int(f+g) = \int f + \int g$ demonstrate (a part of) the linearity property.

Problem 1. Let $\mathbb{F}$ be an arbitrary ground field, and let $\mathbb{F}[x]$ denote the polynomial ring (algebra) in one indeterminate. For the sake of convention, assume that $x^0 = 1$.

1. Demonstrate that $\mathbb{F}[x]$ forms a vector space over $\mathbb{F}$ with the basis $\{x^n\}_{n\geq 0}$.

2. Determine whether the set $\{x^n + 2 \cdot x^{n-1}\}_{n\geq 1}$ constitutes a basis for $\mathbb{F}[x]$, and provide your reasoning.

3. Investigate whether the series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ belongs to $\mathbb{F}[x]$, and provide your reasoning.

4. (Optional) Let $\mathbb{F}\langle x\rangle$ denote the linear space of *formal power series*, which takes the form

$$\mathbb{F}\langle x\rangle = \left\{ \sum_{k=0}^{\infty} a_k x^k \mid a_k \in \mathbb{F} \right\}.$$

One can identify $\mathbb{F}[x]$ as a proper linear subspace of $\mathbb{F}\langle x\rangle$. Let

$$\ell : \mathbb{F}[x] \to \mathbb{F}, \quad \sum_{k=0}^{n} a_k x^k \mapsto \sum_{k=0}^{n} a_k$$

be a linear map which sends a polynomial to the sum of its coefficients.
Is it possible to define a linear map $\mathcal{L} : \mathbb{F}\langle x\rangle \to \mathbb{F}$ such that $\mathcal{L}(f) = \ell(f)$ for any $f \in \mathbb{F}[x]$?

5. Let $U \subseteq V$ be an inclusion of a proper linear subspace (potentially infinite-dimensional), and let $\ell : U \to W$ be a linear map. Is it always possible to find a linear map $\mathscr{L} : V \to W$ such that $\mathscr{L} = \ell$ when restricted to U? A related question (though not exactly requires same system of axioms) is whether it is always possible to construct a direct sum $U \oplus W = V$ provided that $U \subseteq V$.
You may either offer a simple response, such as *it is quite difficult*, or provide a more detailed reflection of your thoughts. It is also recommended to finish *this subproblem* with the assistance of AI models.