Reference: ExampleClass-240929.pdf.

If you find them difficult, you may skip all challenging problems.

(1) State the definition of a number field, and prove that number fields are $\mathbb{Q}$-linear spaces.

(2) Prove that the 3-dimensional $\mathbb{Q}$-linear space

$$V = \{a + b \cdot 2^{1/3} + c \cdot 2^{2/3} \mid a, b, c \in \mathbb{Q}\}$$

is a number field.

- **Challenge: generalise & prove.**

(3) Find a field K such that $\mathbb{C}$ is a **proper** subfield of K.

(4) Prove that $(1, e^x, e^{2x}, \ldots, e^{2024x})$ are linearly independent real-valued functions.

- Hint: take derivatives, and use the fact **Vandermonde matrix is invertible** as a shortcut.

(5) Challenge: find n such that $(\sin \frac{\pi}{2n}, \sin \frac{2\pi}{2n}, \ldots, \sin \frac{(n-1)\pi}{2n})$ are linearly dependent (over $\mathbb{Q}$).

- 此题源自著名习题集 *近世代数三百题* 的一处错误 (4.1.13). 今后学习近世代数时需要稍作留意.