Problem Set for 24-Feb-2025

Study Suggestions

- **Irreducibility (不可约)**: An element $x \in R$ is called irreducible in a ring if and only if it cannot be factored as the product of two non-invertible elements.
- **Primality (素)**: An irreducible element in a unique factorisation domain is termed prime element.

In particular, the polynomial ring $\mathbb{F}[x]$ over a field $\mathbb{F}$, and the polynomial ring $R[x]$ over a unique factorisation domain R (for example, a multivariate polynomial ring), are both unique factorisation domains, thus allowing the discussion of prime elements.

Problem 1 Let $\mathbb{F}$ be any field, and let $\{u_i\}_{i=1}^n$ and $\{v_j\}_{j=1}^m$ be bases of U and V, respectively. Define $\mathrm{Hom}_{\mathbb{F}}(U, V)$ as the set of $\mathbb{F}$-linear maps from U to V.

1. Endow $\mathrm{Hom}_{\mathbb{F}}(U, V)$ with appropriate additional structures to make it an $\mathbb{F}$-vector space. In simpler terms, how should we regard $\mathrm{Hom}_{\mathbb{F}}(U, V)$ as a vector space in practice?
2. Determine the dimension of $\mathrm{Hom}_{\mathbb{F}}(U, V)$ and provide a minimal description of one of its bases.

Problem 2 (Blank-Filling Questions) Throughout, $\mathbb{F}$ is an arbitrary field, and $\{u_i\}_{i=1}^l$, $\{b_j\}_{j=1}^m$, and $\{c_k\}_{k=1}^n$ are bases of U, V, and W, respectively. Carefully endow the following spaces with $\mathbb{F}$-linear structures, and write done their dimensions along with the corresponding *distinguished* bases.

1. $U \oplus V$ as subspaces.
2. $U \times V$ as the usual Cartesian product.
3. $\mathrm{Hom}_{\mathbb{F}}(U, V)$ as the set of linear maps.
4. $\mathrm{Hom}_{\mathrm{Sets}}(U, V)$ as the set of maps.
5. $\mathrm{Hom}_{\mathbb{F}}(U \times V, W)$, where $U \times V$ is defined a priori.
6. $\mathrm{Bil}_{\mathbb{F}}(U, V; W)$, the set of bilinear maps from U and V to W.
7. $\mathrm{Hom}_{\mathbb{F}}(U, \mathrm{Hom}_{\mathbb{F}}(V, W))$, also known as the Currying of bilinear maps.

> *Currying, named after Haskell Curry, is the technique of transforming a function that takes multiple arguments into a sequence of functions, each accepting a single argument.*

Problem 3 (Optional for the time being) This problem is inspired by *普通高中教科书数学（A版）必修第一册*. It is likely that you never read math textbooks in high school. We restate the notation of *maps between two sets* as used in this textbook:

$$Y^X := \{f \mid f : X \to Y \text{ is a set map}\} = \mathrm{Hom}_{\mathrm{Sets}}(X, Y).$$

Consider a finite dimensional vector space U with the following procedure:

$$U \xrightarrow{\text{take its basis}} S \xrightarrow{\text{count its basis}} n.$$

1. State the connection between the 0-vector space, the empty set, and the natural number 0.
2. State the connection between the direct sum operation $\oplus$ on vector subspaces, the disjoint union of sets, and the summation of natural numbers.
3. State the connection between complementary subspaces, the exclusion of sets, and the subtraction of natural numbers.
4. Define the *noumenal operation* of vector spaces, which refers to the Cartesian product of sets and the multiplication of natural numbers. This is the tensor product, and the basis of the tensor product space is clearly defined in this sense.
5. There is no connection between $\mathrm{Hom}_{\mathbb{F}}(U, V)$ and T^S. The noumenal operation of $\mathrm{Hom}_{\mathrm{Sets}}$ is a peculiar tensor product of several V's labelled by U.

Now we move on to isomorphisms. Throughout, the external direct sum (denoted by $\oplus$ in our class) is identified with the Cartesian product $\times$ in our textbook (*Done Right*).

1. From $(m \cdot n)^a = m^a \cdot n^b$, one obtains the set isomorphism $(S \times T)^X \simeq (S^X) \times (T^X)$. Construct an isomorphism between vector spaces (not necessarily of finite dimension):
$$\mathrm{Hom}_{\mathbb{F}}(U, V \times W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(U, V) \times \mathrm{Hom}_{\mathbb{F}}(U, W).$$
2. From $n^{a+b} = n^a \cdot n^b$, one obtains the set isomorphism $S^{X \sqcup Y} \simeq (S^X) \times (S^Y)$. Construct an isomorphism between vector spaces (not necessarily of finite dimension):
$$\mathrm{Hom}_{\mathbb{F}}(U \times V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(U, W) \times \mathrm{Hom}_{\mathbb{F}}(V, W).$$
3. From $(n^a)^b = n^{a \cdot b}$, one obtains the set isomorphism $(S^X)^Y \simeq S^{X \times Y}$, known as currying. Construct an isomorphism between vector spaces (not necessarily of finite dimension):
$$\mathrm{Hom}_{\mathbb{F}}(U \otimes_{\mathbb{F}} V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(U, \mathrm{Hom}_{\mathbb{F}}(V, W)).$$

> ***Hint***: *By the universal property,* $\mathrm{Hom}_{\mathbb{F}}(U \otimes_{\mathbb{F}} V, W)$ *identifies with* $\mathrm{Bil}(U, V; W)$. *Recall that a bi-function* $f : X \times Y \to S$ *identifies a set of usual functions* $f_y : X \to S$, *indexed by* $y \in Y$. *Hence, a bilinear function* $B : (U \& V) \to W$ *identifies a linear space of usual linear maps* $B_u : V \to W$, *indexed by* $u \in U$.

The adjunction of the product-object $\otimes$ and the Hom-object $\mathbf{Hom}_{\mathbb{F}}$ comes from Currying.