判断题

- 不选用的原因: 第一次测验的所有题目均考察过程. 没有判断填空之类的问题.

(1) Let v_1, v_2, and v_3 be vectors in a certain linear space V. Suppose that the pairs (v_1, v_2), (v_2, v_3), and (v_1, v_3) are all linearly independent. Claim: (v_1, v_2, v_3) is also linearly independent.

(2) Let U, V, and W be linear subspaces of a certain linear space. Claim: $(U + V) \cap W$ is a subspace of $(U \cap W) + (V \cap W)$.

(3) Let U be a proper linear subspace of V. Claim: $\operatorname{span}(V \setminus U)$ may be a proper subspace of V.

(4) Let A and B be matrices of the same size. Claim:

$$r(A^2) + r(B^2) \geq r(AB).$$

(5) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix such that

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r\begin{pmatrix} A & B \end{pmatrix} = r\begin{pmatrix} B \\ D \end{pmatrix} = r\begin{pmatrix} C & D \end{pmatrix}.$$

Claim: $r(M) = r\begin{pmatrix} A & B \end{pmatrix}$.

(6) Let A be a real square matrix. Suppose that $x^T A x = 0$ for any vector x. Claim: $A = O$.

(7) Let $\mathbb{F}$ be a number field containing π. Claim: $\mathbb{F} = \mathbb{R}$.

(8) Claim: $\dim N(AB) \leq \dim(N(A)) + \dim(N(B))$.

证明题 1

- 不选用此题的原因: 第一周的作业解答出现过类似的证明. 属于 Strang 自称不会做的题目.

Let $(v_1, v_2, \ldots, v_m)$ be distinct vectors in $\mathbb{R}^d$ such that

$$\cos\theta_{i,j} = \frac{v_i^T \cdot v_j}{\|v_i\| \cdot \|v_j\|} \equiv C$$

is a constant. Prove that $m_{\max} = d + 1$.

Since the construction of $m = d + 1$ is well-known, it suffices to prove that $m \leq d + 1$.

Hint: You can assume that $\|v_i\| \equiv 1$ for convenience. Set

$$Q := \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix} \in \mathbb{R}^{d\times m}.$$

Now consider the rank of $Q^T \cdot Q$.

证明题 2

- 不选用此题的原因: 这更适合作为学完 Jordan 标准型后的巩固题.

Problem: (The decreasing sequence of rank) Let $A \in \mathbb{F}^{n\times n}$.

(0) $r(A^p) < r(A^q)$ whenever $N(A^p) \subsetneq N(A^q)$. $r(A^p) = r(A^q)$ whenever $N(A^p) = N(A^q)$.

(1) Prove that if $r(A) = r(A^2)$, then $r(A) = r(A^3)$.

(2) Prove that if $r(A^k) = r(A^{k+1})$, then $r(A^k) = r(A^{k+t})$ for all $t \in \mathbb{N}$.

(3) Give an example such that $r(A) > r(A^2) > \cdots > r(A^{2024}) = r(A^{2025})$.

(4) Prove that $\{r(A^k) - r(A^{k+1})\}_{k\in\mathbb{N}}$ is also decreasing, with a limit of 0.

证明题 3

- 不选用此题的原因: 为了让试卷少一题. (其实这题挺适合考试的).

Problem: (Birkhoff algorithm for bi-stochastic matrices)

Let $\mathbb{F}$ be a number field. All matrices are in $\mathbb{F}^{3\times 3}$. Let j be the column vector $(1, 1, 1) \in \mathbb{F}^3$.

(1) A matrix A is a permutation matrix if $a_{i,j}$ is either 0 or 1, and $Aj = j = A^T j$. How many permutation matrices are there in $\mathbb{F}^{3\times 3}$?

(2) Is $V = \{X \mid Xj = X^T j = j\}$ a linear space? If not, find X_0 such that $\{X - X_0 \mid X \in V\}$ is a linear space.

(3) Find the dimension of the linear space in (2).

(4) Prove that $X \in V$ is always a linear combination of permutation matrices. Is such a combination unique?

(5) Prove that the sum of the coefficients in the linear combination from (4) equals 1.

奖励题 1

- 不选用此题的原因: 取消奖励题. 并且这题或许挺难的 (?), 有个数学系硕士不会做.

(供所有题的同学消遣用) 给定五个相同大小的勾三股四弦五三角形. 不重叠地拼出一个对称图形.

奖励题 2

- 不选用此题的原因: 取消奖励题. 并且这题或许挺难的 (?), 有个数学系博士不会做.

Let A and B be real matrices such that $AB = BA$. Determine whether it is true that

$$r(A^2) + r(B^2) \geq 2r(AB).$$

Counterexamples do exist!

奖励题 3

- 不选用此题的原因: 取消奖励题. 这题适合作为学完三角矩阵的特征值后的巩固题.

Let $\mathbb{F}$ be a number field. $\mathrm{M}_n(\mathbb{F})$ is the linear space of all $n \times n$ matrices over $\mathbb{F}$.

(1) The trace is a linear map defined as

$$\mathbf{tr} : \mathrm{M}_n(\mathbb{F}) \to \mathbb{F}, \quad X \mapsto \sum_{i=1}^{n} x_{i,i}.$$

Show that $V = \{Y \in \mathrm{M}_n(\mathbb{F}) \mid \mathbf{tr}(Y) = 0\}$ is a linear subspace of $\mathrm{M}_n(\mathbb{F})$.

(2) Find $\dim V$ and provide a basis for V.

(3) Prove that for any $X \in \mathrm{M}_n(\mathbb{F})$, there exist finitely many

$$A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m \in \mathrm{M}_n(\mathbb{F})$$

such that $X = \sum_{i=1}^{m}(A_i B_i - B_i A_i)$.

(4) (Bonus) Prove that $\mathbf{tr}(A) = 0$ if and only if there exist $B, C \in \mathrm{M}_n(\mathbb{F})$ such that $A = BC - CB$. You can prove this with the following steps:

(4.1) Let Λ be the diagonal part of A. Prove that there exists a permutation matrix P and a diagonal matrix D such that

- $\Lambda = P^{-1}DP - D$,
- D has distinct entries on its diagonal.

(4.2) Consider $A = L - U$, where U is upper triangular and $L - D$ is strictly lower triangular. Show that L, D, and U have the same eigenvalues (counting multiplicities). Hence, there exist invertible R and S such that

$$R^{-1}LR = S^{-1}US = D.$$

(4.3) $A = L - U = (RS^{-1}) \cdot (SR^{-1}L) - (SR^{-1}L) \cdot (RS^{-1})$.

(4.4) This shows that for any sufficiently large field $\mathbb{F}$,

$$\{X \in \mathrm{M}_n(\mathbb{F}) \mid \mathbf{tr}(X) = 0\} = \{AB - BA \mid A, B \in \mathrm{M}_n(\mathbb{F})\}.$$

Can you prove the existence of counterexamples over finite fields?