课程编号: MATH1406H

考试时间: 2024 年 6 月 12 日, 15:40 至 17:40

高等代数 (荣誉) II 期末测试

试卷内容 + 参考答案

问题 1 (30 points)**.** Fill-in-the-Blank Questions (No need to write the process)

1. For any vector space V, what is the relation between V^{**} and V.
2. Is $(x-1)^2(x-2)^2\cdots(x-2024)^2+1$ irreducible over $\mathbb{Q}$?
3. What is the *Fundamental Theorem of Algebra*?
4. Let V be an n-dimensional Euclidean space and $v_1 \neq v_2 \in V$ with $\|v_1\| = \|v_2\|$. Find a v such that the linear transformation
$$\varphi : V \to V, \quad u \mapsto u - 2(u,v)v$$
maps v_1 to v_2.
5. Write down the definition of tensor product of $V \otimes U$ of vector spaces.
6. Give the construction of the tensor product $V \otimes U$.
7. Find $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$, that is, the dimension of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$.
8. Find $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$.
9. For matrices $A \in \mathbb{C}^{2\times 2}$ and $B \in \mathbb{C}^{3\times 3}$ with $\det(A) = 2$ and $\det(B) = 4$, find $\det(A \otimes B)$.
10. Let $A := \operatorname{diag}(0,1,1,2,2) \in \mathbb{C}^{5\times 5}$. Define
$$\varphi : \mathbb{C}^{5\times 5} \to \mathbb{C}^{5\times 5}, \quad X \mapsto AX - XA^T.$$
Find $\dim_{\mathbb{C}}(\ker \varphi)$.

解答 1. 解答如下.

1. 写出不依赖坐标的典范态射

$$\theta_V : V \to V^{**}, \quad v \mapsto \{f \mapsto v(f)\}_{f \in V^*}.$$

特别地, $\dim V < \infty$ 时 θ_V 是线性同构.

若 V 是无限维: 在承认公理 "$\forall v \in V \exists f \in V^*(f(v) \neq 0)$" 时 θ_V 是单射, 在承认选择公理时 θ_V 是严格的单射.

2. 不可约. 下采用反证法证明. 若该多项式在 $\mathbb{Q}[x]$ 意义下可约, 则该多项式在 $\mathbb{Z}[x]$ 意义下可约 (Gauss 引理). 因此存在因式 $g(x) \in \mathbb{Z}[x]$ 满足 $1 \leq \deg g \leq 2024$. 由于 $\{g(i)\}_{i=1}^{2024} \subset \{\pm 1\}$, 且 g 没有零点, 因此 $g(x)-1$ 或者 $g(x)+1$ 有 2024 个相异零点. 遂有

$$g(x) = \lambda \cdot (x-1)(x-2)\cdots(x-2024) \pm 1 \quad (\lambda \in \mathbb{Q}).$$

容易检验, 此时 g 与原多项式互素, 故矛盾.

3. 代数闭域 (或 $\mathbb{C}$) 上的非常值多项式存在该域上的根.

4. $\|v_1 - v_2\|^{-1} \cdot (v_1 - v_2)$. 几何地, v 是单位反射向量, 其方向是 v_2 指向 v_1.

5. $V \otimes U$ 即双线性映射 $\iota : V \times U \to V \otimes U$, 定义作如下普适映射问题的解: 对任意线性空间 W, 所有双线性映射 $F : V \times U \to W$ 双射对应于所有线性映射 $f : V \otimes U \to W$, 对应方式是 $F = f \circ \iota$.

6. $V \otimes U$ 是集合的 Cartesian 积 $V \times U$ 在约束 R 下的自由化线性空间. R 取遍如下三类式子:

$$(u_1+u_2, v_1) - (u_1, v_1) - (u_2, v_1), \quad (u_1, v_1+v_2) - (u_1, v_1) - (u_1, v_2),$$
$$(u_1, \lambda v_1) - (\lambda u_1, v_1) \qquad (\forall u_i \in U,\ v_i \in V,\ \lambda \in \mathbb{F}).$$

注: 集合 X 的自由化 k-线性空间 (FX) 的一种定义是 $i(X)$ 的线性张成, 此处

$$i : X \hookrightarrow (\mathrm{Hom}_{\mathrm{Sets}}(X, k))^*, \quad x \mapsto [f \mapsto f(x)]. \tag{1}$$

此时 $V \otimes U := F(V \times U)/N$, 子空间 $N \subset F(V \times U)$ 由以上三类线性组合式张成.

7. 2. 考虑 $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \mathbb{R}i)$. 依照张量积与直和的分配律得 $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R})^{\oplus 2} \cong \mathbb{C}^{\oplus 2}$.

8. 4. 将 $\mathbb{C}$ 视作二维 $\mathbb{R}$-线性空间, 再依照 $\dim(U, V) = (\dim U) \cdot (\dim V)$ 即可.

9. 128. 依照 $\det(A_{m \times m} \otimes B_{n \times n}) = (\det A)^n \cdot (\det B)^m$ 即可.

10. 9. 即, 求 25-阶对角映射 $A \otimes I - I \otimes A$ 的零空间维数. 观察得 $1^2 + 2^2 + 2^2 = 9$.

问题 2 (20 points)**.** Suppose $\sigma \in \mathrm{Hom}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $\tau \in \mathrm{Hom}(V/U, W)$ such that $\sigma = \tau\pi$ if and only if $U \subseteq \ker \sigma$.

解答 2. 本题考察余核泛性质之刻画. 式 "$\sigma = \tau\pi$" 意即 $\sigma(v) = \tau(v+U)$ 恒成立. 依照线性性, 上式亦等价于 $\sigma(u) = \sigma(0)$ $(\forall u \in U)$, 即 $U \subseteq \ker(\sigma)$.

问题 3. Proof or disproof: if an orthogonal transformation $\mathscr{A}$ on an n-dimensional Euclidean space V has two different eigenvalues, then the eigenvectors of $\mathscr{A}$ corresponding to different eigenvalues are orthogonal.

解答 3. 记 $(-,-)$ 为内积, $\mathscr{A}v_i = \lambda_i v_i$ 是特征向量 $(i = 1, 2)$, 满足 $\lambda_1 \neq \lambda_2$. 由于 $\|\mathscr{A}v_i\| = \|v_i\|$, 从而 $\lambda_i \in \{\pm 1\}$. 此时

$$(v_1, v_2) = (\mathscr{A}v_1, \mathscr{A}v_2) = \lambda_1\lambda_2(v_1, v_2) = -(v_1, v_2).$$

遂有 $(v_1, v_2) = 0$.

问题 4. Set $V := \mathbb{R}[x]$ and $V_0 := \{f \in \mathbb{R}[x] \mid f(0) = f(1)\}$.

1. Prove that $V \times V \to \mathbb{R}, \quad (f, g) \mapsto \int_0^1 f(x)g(x)\,\mathrm{d}\,x$ is an inner product.
2. Set $\mathscr{D} : V_0 \to V, \quad f(x) \mapsto f'(x)$. Find $\dim\ker(\mathscr{D})$ and $\dim\operatorname{coker}(\mathscr{D})$.
3. Define the inner product restricted on the subspace

$$(\cdot,\cdot)_0 : V_0 \times V_0 \to \mathbb{R}, \quad (f, g) \mapsto \int_0^1 f(x)g(x)\,\mathrm{d}\,x.$$

   Is there any linear map $\mathscr{D}^* : V \to V_0$ such that for any $h \in V_0$ and $g \in V$,

$$(\mathscr{D}^*g, h)_0 = (g, \mathscr{D}h)?$$

解答 4. 解答如下.

1. 容易验证双线性性, 等式 $(f, g) = (g, f)$, 以及 $(f, f) \geq 0$. 正定性之证明: 若 $(f, f) = 0$, 则 f 是定义区间 $[0, 1]$ 上恒零的多项式, 从而是 $\mathbb{R}[x]$ 中的零元.
2. 记 $D : V \to V$ 为 V 上导数, $i : V_0 \to V$ 为子集的包含映射, 故 $Di = \mathscr{D}$.
   - 往证 $\dim\ker(\mathscr{D}) = 1$. 由于 i 是单射, 从而 $\dim\ker(Di) \leq \dim\ker(D) = 1$. 由于常函数属于 $\ker(\mathscr{D})$, 故上式取等.
   - 往证 $\dim\operatorname{coker}(\mathscr{D}) = 1$. 由于 D 是满射, 从而 $\dim\operatorname{coker}(Di) \leq \dim\operatorname{coker}(i) = 1$. 由于常函数不属于 $\operatorname{im}(\mathscr{D})$, 故上式取等.
3. $\mathscr{D}^*$ 不存在. 下采用反证法证明: 若 $\mathscr{D}^*$ 存在, 限定多项式 h 满足 $h(0) = h(1) = 0$, 则

$$(\mathscr{D}^*, h) = (g, \mathscr{D}h) = (-g', h).$$

   若能证明 $\mathscr{D}^*g + g'$ 恒零, 则 $g' \in V_0$ 与假设矛盾. 故只需证明

$$V_1 : \{h \in \mathbb{R}[x] \mid h(0) = h(1) = 0\} = \{x(1-x)f(x) \mid f \in \mathbb{R}[x]\}$$

在 V 中的正交补是 0. 对任意 $p(x) \in V_1^{\perp}$, 考虑

$$0 = (x(1-x)p(x), p(x)) = \int_0^1 x(1-x)p(x)^2 \,\mathrm{d}\, x$$

从而 $p(x) = 0$. 因此, V_1 在 V 中补空间是 0.

问题 5. The vector spaces in this problem are all finite dimensional.

1. Given linear maps $\varphi_i \in \mathrm{Hom}(U_i, V_i)$ $(i = 1, 2)$, show that the following map is well-defined.

$$\varphi_1 \otimes \varphi_2 : U_1 \otimes U_2 \to V_1 \otimes V_2, \quad \sum_{\text{finite}} u_1^{(i)} \otimes u_2^{(i)} \mapsto \sum_{\text{finite}} \varphi_1(u_1^{(i)}) \otimes \varphi_2(u_2^{(i)})$$

2. Let $p : U \to V$ be a surjective linear map. Show that for any linear space W,

$$p \otimes \mathrm{id}_W : U \otimes W \to V \otimes W, \quad \sum_{\text{finite}} u^{(i)} \otimes w^{(i)} \mapsto \sum_{\text{finite}} p(u^{(i)}) \otimes w^{(i)}$$

   is also surjective.

3. Let $i : U \to V$ be an injective linear map. Show that for any linear space W,

$$p \otimes \mathrm{id}_W : U \otimes W \to V \otimes W, \quad \sum_{\text{finite}} u^{(i)} \otimes w^{(i)} \mapsto \sum_{\text{finite}} i(u^{(i)}) \otimes w^{(i)}$$

   is also injective.

解答 5. 解答如下.

1. 张量积 $\iota_V : V_1 \times V_2 \to V_1 \otimes V_2$ 是良定义的双线性型, 因此, 双线性型的拉回

$$U_1 \times U_2 \to V_1 \otimes V_2, \quad (u_1, u_2) \mapsto \iota_V(\varphi_1(u_1), \varphi_2(u_2))$$

   也是良定义的双线性型. 依照泛性质定义 $\varphi_1 \otimes \varphi_2$, 并检验简单张量的像

$$\varphi_1 \otimes \varphi_2 : u_1 \otimes u_2 \mapsto \varphi_1(u_1) \otimes \varphi_2(u_2).$$

   依照线性性, 如上良定义的 $\varphi_1 \otimes \varphi_2$ 满足题干条件.

2. 任意 $v^{(i)} \in V$ 有原像 $u^{(i)}$, 从而

$$\sum_{\text{finite}} v^{(i)} \otimes w^{(i)}$$

   有 $p \otimes \mathrm{id}_W$ 下的原像

$$\sum_{\text{finite}} u^{(i)} \otimes w^{(i)} \quad (p(u^{(i)}) = v^{(i)}).$$

3. 由于 $i \otimes \mathrm{id}_W$ 的良定义性, 不妨设定义域中的加和项数等于张量的秩, 亦即 $\{u^{(i)}\}$ 与 $\{w^{(i)}\}$ 是线性无关组. 由于单射保持线性无关组, 故 $\{i(u^{(i)})\}$ 也是线性无关的. 最后依照课堂引理

或是如下方法证明 0 的原像也是 0. 取一族与 $\{w^i\}$ 适配的示性对偶映射 $\{f_{(i)} \in W^*\}$, 满足 $f_{(i)}(w^{(j)}) = \delta_{i,j}$. 当 $\sum_{\text{finite}} u^{(i)} \otimes w^{(i)} = 0$ 时, 有

$$\mathrm{id}_V \otimes f_{(j)} : \sum_{\text{finite}} i(u^{(i)}) \otimes w^{(i)} = i(u^{(i)}) = 0.$$

这表明所有 $i(u^{(i)})$ 为零. 依照 i 是单射, 故所有 $u^{(i)}$ 为零. 这表明 $i \otimes \mathrm{id}_W$ 是单的.

问题 6. Let V be an n-dimensional Euclidean space and the set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ satisfy the following condition: if there exists non-negative real numbers $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \dots + \lambda_m\alpha_m = 0$, then it must be that $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$. Prove: there exists a vector $\alpha \in V$ such that $(\alpha, \alpha_i) > 0$ for all $1 \le i \le m$.

解答 6. 本题系凸集分离定理的弱化版本. 将向量族 $\{\alpha_i\}$ 的正项系数线性组合单位化成凸组合:

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \dots + \alpha_n\alpha_n \quad \mapsto \quad \frac{\lambda_1\alpha_1 + \lambda_2\alpha_2 + \dots + \alpha_n\alpha_n}{\lambda_1 + \lambda_2 + \dots + \lambda_n}.$$

定义 $\{\alpha_i\}$ 的凸包为有界闭集 (紧集)

$$S := \{\lambda_1\alpha_1 + \lambda_2\alpha_2 + \dots + \lambda_n\alpha_n \mid 0 \le \lambda_i \le 1, \lambda_1 + \lambda_2 + \dots + \lambda_n = 1\}.$$

显然 $0 \notin S$, 故 S 中存在模长极小元 x_m (紧集上的连续映射取达最值). 下证明 x_m 唯一: 若存在不同于 x_m 的 y_m 使得 $\|x_m\| = \|y_m\|$, 则依照 S 的构造知存在模长更小的元素 $\frac{1}{2}(x_m + y_m) \in S$, 矛盾. 最后断言不存在 $x \in S$ 使得 $(x, x_m) < (x_m, x_m)$; 若不然,

$$\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\|\lambda x + (1-\lambda)x_m\|^2 = 2\lambda(x,x) + 2(1-2\lambda)(x,x_m) + 2(\lambda-1)(x_m,x_m)$$

在 $\lambda = 0$ 处的导数小于 0, 与 x_m 模长最小矛盾. 由于 $\alpha_i \in S$, 故 $\alpha = x_m$ 即为所求.