Problem set for 27-Feb-2025

- 选自去年的习题课讲义, 见 `参考资料/高代习题课_强基.pdf`.

Exercise 1.1 Let $P_n := \{f(x \in \mathbb{F}[x]) \mid \deg f(x) < n\}$. Pick $\{a_i\}_{i=1}^n \in \mathbb{F}$ such that $a_i \neq a_j$ for any $i \neq j$. Show that

$$f_j(x) := \prod_{i \neq j}(x - a_i) \quad (1 \leq j \leq n)$$

form a basis of P_n.

Exercise 4.1 Assume $f(x) = x^3 + px + q \in \mathbb{Z}[x]$ is irreducible and $\alpha \in \mathbb{C}$ is a root of f.

1. Prove that $\mathbb{Q}[\alpha] := \{g(\alpha) \mid g(x) \in \mathbb{Q}[x]\}$ is a linear space over $\mathbb{Q}$ and $1, \alpha, \alpha^2$ form a basis.
2. Prove that $\varphi : \beta \mapsto f'(\alpha)\beta$ gives a linear map on $\mathbb{Q}[\alpha]$ and find its matrix under $1, \alpha, \alpha^2$.