模拟卷

Problem 1

1. **(5pt)** State the Perron-Frobenius theorem, ensuring that your answer incorporates the following key concepts: matrices with all positive entries, eigenvalues with multiplicity, spectral radius, and eigenvectors.

2. **(5pt)** Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ be a real column vector satisfying $\sum_{i=1}^{n} v_i = 0$. Determine all eigenspaces of the matrix $(vv^T + \mathbf{1}\mathbf{1}^T) \in \mathbb{R}^{n\times n}$, where $\mathbf{1}$ denotes the vector of all ones.

3. **(10pt)** If, in addition, $\sum_{i=1}^{n} v_i^2 = n$, prove that there exist indices i and j such that $v_i v_j \leq -1$.

Problem 2

(10pt) Let A be a square matrix of size n with entries in $\{0, 1\}$, satisfying the following conditions:

- A has all diagonal entries equal to zero.
- $a_{ij} = 0$ whenever $a_{ji} = 1$.

Prove that $\dim N(A) \leq 1$, where $N(A)$ is the null space of A.

Problem 3

(20pt) Find invertible matrix P and Jordan canonical form J such that

$$\begin{bmatrix} -3 & -3 & -1 \\ 4 & 5 & 2 \\ -4 & -4 & -1 \end{bmatrix} = P^{-1}JP.$$

Problem 4

(20pt) Find the singular value decomposition (SVD) of the matrix

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix},$$

i.e., determine $U \in \mathrm{O}(2)$, $\Sigma = \begin{pmatrix} *_{\geq 0} & 0 & 0 \\ 0 & *_{\geq 0} & 0 \end{pmatrix}$, and $V \in \mathrm{O}(3)$ such that $U\Sigma V^T = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

Problem 5

(10pt) Let $\mathbb{F}$ be an arbitrary field, and let $\{A_i\}_{i=1}^{k} \subseteq \mathbb{F}^{m\times n}$ be a collection of matrices. Prove the following equality:

$$\dim\left(\sum_{i=1}^{k} N(A_i)\right) - \dim\left(\bigcap_{i=1}^{k} N(A_i)\right) = \dim\left(\sum_{i=1}^{k} R(A_i)\right) - \dim\left(\bigcap_{i=1}^{k} R(A_i)\right),$$

where $N(A_i)$ and $R(A_i)$ denote the null space and row space of A_i, respectively.

Problem 6 Let $A \in \mathbb{C}^{n\times n}$ be a complex matrix. Define its numerical range as:

$$W(A) := \{x^H A x \mid x^H x = 1\} \subseteq \mathbb{C}.$$

From calculus, it is known that if $\lambda, \mu \in W(A)$, then for any $t \in [0, 1]$, $t\lambda + (1-t)\mu \in W(A)$.

Now, assume further that $\mathrm{tr}(A) = 0$.

1. **(5pt)** Prove that there exists a vector x such that $x^H A x = 0$.

2. **(10pt)** Prove that there exists a unitary matrix U such that $U^H A U$ has all diagonal entries equal to zero.

3. **(5pt)** Prove that there exist complex matrices B and C such that $BC - CB = A$.