姓名: ______________________________

学号: ______________________________

Mock Final Exam (高等代数 (荣誉) II)

考试时间: 15 : 40 – 17 : 40, DD/June/2024

得分表	(1)	(2)	(3)	(4)	(5)	Problem 总得分
Problem 1						
Problem 2						
Problem 3						

- Throughout, k is an arbitrary field, and V be a vector space (not necessarily finite-dimensional).
- Feel free to answer the following questions in either 中文 or English; it is FORBIDDEN to mix both languages within a single sentence.

Problem 1 (35pt)**.** The following questions are about the ring of algebraic integers.

1. (8pt) Say $r \in \mathbb{C}$ is integral over $\mathbb{Z}$, whenever there exists some polynomial $f \in \mathbb{Z}[x]$ with leading coefficient 1 such that $f(r) = 0$. Let $\mathbb{Q}(r)$ be the smallest number field containing r. Prove that $\mathbb{Q}(r)$ is a finite dimensional $\mathbb{Q}$-vector space with basis $\{1, r, r^2, \ldots, r^{d-1}\}$. Here $d = \dim_{\mathbb{Q}} \mathbb{Q}(r) - 1$.

2. (8pt) Let $r \in \mathbb{C}$ be integral over $\mathbb{Z}$. Write down the matrix form of the $\mathbb{Q}$-linear endomorphism

$$m_r : \mathbb{Q}(r) \to \mathbb{Q}(r), \quad x \mapsto rx \tag{1}$$

with basis $\{1, r, \ldots, r^{d-1}\}$. Also write down the characteristic polynomial of m_r.

3. (4pt) Give an example: m_r has no Jordan canonical form over $\mathbb{Q}(r)$.

4. (5pt) If $r, s \in \mathbb{C}$ are integral over $\mathbb{Z}$, then so are $r + s$ and $r \cdot s$. Hint:

   - use the matrix forms of m_r and m_s;
   - show that $m_r \cdot m_s = m_{rs}$, $m_r + m_s = m_{r+s}$, which identifies $\mathbb{C}$ with $\{m_z\}_{z \in \mathbb{C}}$.

5. (10pt) Set $g(x) = x^{2024} + x + 1$. For any $f \in \mathbb{Z}[x]$ with leading coefficient 1, one has the factorisation

$$f(x) = (x - z_1)(x - z_2) \cdots (x - z_n) \quad \text{(over } \mathbb{C}\text{)}. \tag{2}$$

Deduce that

$$(x - g(z_1))(x - g(z_2)) \cdots (x - g(z_n)) \in \mathbb{Z}[x]. \tag{3}$$

Hint: consider the upper-triangulated form of the companion matrix of f.

Problem 2 (30pt)**.** The following problems are about real inner product space V (not necessary of finite dimension!). Suppose that $\varphi \in \mathrm{Hom}_{\mathbb{R}}(V, V)$ is invertible, and φ^* exists.

1. (5pt) Prove that φ^* is injective, and $(\mathrm{im}(\varphi^*))^\perp = 0$. Hence φ^* is surjective $\leftrightarrow$ φ^* is invertible.
2. (5pt) Prove that, if φ^* is surjective, then $(\varphi^{-1})^*$ exists and $(\varphi^{-1})^* = (\varphi^*)^{-1}$.
3. (5pt) Prove that, if $(\varphi^{-1})^*$ exists, then φ^* is invertible and $(\varphi^{-1})^* = (\varphi^*)^{-1}$.
4. (10pt) Let V be the space of real sequences with finitely many non-zero entries, which is isomorphic to $\mathbb{R}[x]$. The inner product is defined as the following pointwise product
$$V \times V \to \mathbb{R}, \quad (a_i)_{i\geq 1}, (b_i)_{i\geq 1} \mapsto \sum_{i\geq 1} a_i b_i. \tag{4}$$
Now set $L : V \to V, \quad (a_i)_{i\geq 1} \mapsto (a_{i+1})_{i\geq 1}$. Find the adjoint of $(\mathrm{id} + L)$.
5. (5pt) Give an example that φ^{-1} has no adjoint.

Problem 3 (35pt)**.** We shall find the tensor rank of multiplication of complex numbers with following steps.

1. (5pt) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-linear vector space. Find $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$.
2. (8pt) Prove that the following linear map is well-defined
$$\Phi : V \otimes U^* \to \mathrm{Hom}_k(U, V), \quad v \otimes f \mapsto [u \mapsto v \cdot f(u)]. \tag{5}$$
3. (8pt) Prove that Φ is an isomorphism if either U or V is finite dimensional. Then use tensor-hom adjunction to show the isomorphism $\theta : V^* \otimes_k U^* \cong (V \otimes_k U)^*$. Describe the image of simple tensor $f \otimes g$ under θ.
4. (10pt) We know that the multiplication operator $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is non-generated symmetric $\mathbb{R}$-bilinear. Explain how you identify such multiplication operator as a tensor in $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^* \otimes_{\mathbb{R}} \mathbb{C}$, which is isomorphic to $\mathbb{C}^{\otimes_{\mathbb{R}} 3}$. Hint:
   - the multiplication identifies $(1 \otimes 1 \otimes 1 + 1 \otimes i \otimes i + i \otimes 1 \otimes i + i \otimes i \otimes (-1)) \in \mathbb{C}^{\otimes_{\mathbb{R}}^3}$.
5. (4pt) Find the rank r of the above tensor. Show that, in general, any algorithm costs at least 3 steps of $\mathbb{R}$-multiplications when computing $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.