课程编号: MATH1406H

考试时间: 2024 年 6 月 12 日, 15:40 至 17:40

高等代数 (荣誉) II 期末测试

试卷内容

问题 1 (30 points)**.** Fill-in-the-Blank Questions (No need to write the process)

1. For any vector space V, what is the relation between V^{**} and V.
2. Is $(x-1)^2(x-2)^2\cdots(x-2024)^2+1$ irreducible over $\mathbb{Q}$?
3. What is the *Fundamental Theorem of Algebra*?
4. Let V be an n-dimensional Euclidean space and $v_1 \neq v_2 \in V$ with $\|v_1\| = \|v_2\|$. Find a v such that the linear transformation
$$\varphi : V \to V, \quad u \mapsto u - 2(u,v)v$$
maps v_1 to v_2.
5. Write down the definition of tensor product of $V \otimes U$ of vector spaces.
6. Give the construction of the tensor product $V \otimes U$.
7. Find $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$, that is, the dimension of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ over $\mathbb{C}$.
8. Find $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$.
9. For matrices $A \in \mathbb{C}^{2\times 2}$ and $B \in \mathbb{C}^{3\times 3}$ with $\det(A) = 2$ and $\det(B) = 4$, find $\det(A \otimes B)$.
10. Let $A := \operatorname{diag}(0,1,1,2,2) \in \mathbb{C}^{5\times 5}$. Define
$$\varphi : \mathbb{C}^{5\times 5} \to \mathbb{C}^{5\times 5}, \quad X \mapsto AX - XA^T.$$
Find $\dim_{\mathbb{C}}(\ker \varphi)$.

问题 2 (20 points)**.** Suppose $\sigma \in \operatorname{Hom}(V,W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $\tau \in \operatorname{Hom}(V/U, W)$ such that $\sigma = \tau\pi$ if and only if $U \subseteq \ker \sigma$.

问题 3. Proof or disproof: if an orthogonal transformation $\mathscr{A}$ on an n-dimensional Euclidean space V has two different eigenvalues, then the eigenvectors of $\mathscr{A}$ corresponding to different eigenvalues are orthogonal.

问题 4. Set $V := \mathbb{R}[x]$ and $V_0 := \{f \in \mathbb{R}[x] \mid f(0) = f(1)\}$.

1. Prove that $V \times V \to \mathbb{R}, \quad (f, g) \mapsto \int_0^1 f(x)g(x)\,\mathrm{d}\,x$ is an inner product.

2. Set $\mathscr{D} : V_0 \to V, \quad f(x) \mapsto f'(x)$. Find $\dim\ker(\mathscr{D})$ and $\dim\operatorname{coker}(\mathscr{D})$.

3. Define the inner product restricted on the subspace

$$(\cdot,\cdot)_0 : V_0 \times V_0 \to \mathbb{R}, \quad (f, g) \mapsto \int_0^1 f(x)g(x)\,\mathrm{d}\,x.$$

   Is there any linear map $\mathscr{D}^* : V \to V_0$ such that for any $h \in V_0$ and $g \in V$,

$$(\mathscr{D}^* g, h)_0 = (g, \mathscr{D}h)?$$

问题 5. The vector spaces in this problem are all finite dimensional.

1. Given linear maps $\varphi_i \in \operatorname{Hom}(U_i, V_i)$ $(i = 1, 2)$, show that the following map is well-defined.

$$\varphi_1 \otimes \varphi_2 : U_1 \otimes U_2 \to V_1 \otimes V_2, \quad \sum_{\text{finite}} u_1^{(i)} \otimes u_2^{(i)} \mapsto \sum_{\text{finite}} \varphi_1(u_1^{(i)}) \otimes \varphi_2(u_2^{(i)})$$

2. Let $p : U \to V$ be a surjective linear map. Show that for any linear space W,

$$p \otimes \mathrm{id}_W : U \otimes W \to V \otimes W, \quad \sum_{\text{finite}} u^{(i)} \otimes w^{(i)} \mapsto \sum_{\text{finite}} p(u^{(i)}) \otimes w^{(i)}$$

   is also surjective.

3. Let $i : U \to V$ be an injective linear map. Show that for any linear space W,

$$p \otimes \mathrm{id}_W : U \otimes W \to V \otimes W, \quad \sum_{\text{finite}} u^{(i)} \otimes w^{(i)} \mapsto \sum_{\text{finite}} i(u^{(i)}) \otimes w^{(i)}$$

   is also injective.

问题 6. Let V be an n-dimensional Euclidean space and the set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ satisfy the following condition: if there exists non-negative real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_m\alpha_m = 0$, then it must be that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. Prove: there exists a vector $\alpha \in V$ such that $(\alpha, \alpha_i) > 0$ for all $1 \leq i \leq m$.