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CALCULATION PROBLEMS IN LINEAR ALGEBRA

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ABSTRACT. This document enumerates the types of calculation questions that may appear in the examination, serving as a reference for identifying potential knowledge gaps.

The document is (partly) written in Unicode-math.

1. Linear Systems

Example 1.1. Let

$$x_1^2 + x_2^2 + 5x_3^2 + 2tx_1x_2 - 2x_1x_3 + 4x_2x_3 \tag{1.1}$$

be a real quadratic form. Please give the value of t so that $f(x)$ is positive definite.

Example 1.2. Find the reduced row échelon form of the following matrix,

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15, \end{pmatrix} \tag{1.2}$$

and prove that the reduced row échelon form is always unique.

2. Linear Maps

3. Matrix Calculations

3.1. Determinants.

Example 3.1. Find the coefficient of x^3 in the following determinant

$$f(x) = \det \begin{pmatrix} 4x & 3x & 2 & 1 \\ 1 & x & 1 & -1 \\ 3 & 2 & 2x & 1 \\ 1 & 0 & 1 & x \end{pmatrix}. \tag{3.1}$$

Example 3.2. Summarise the techniques in HW7 (Canvas Page).

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4. Matrix Decompositions

4.1. Polar, $\mathbb{R}$-symmetric.

Example 4.1. Find a 3×3 real upper triangular matrix T with positive diagonal entries, such that

$$T^T \cdot T = \begin{pmatrix} 16 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & 14 \end{pmatrix}. \tag{4.1}$$

It is also worthwhile to prove the uniqueness of T.

5. Answers

Solution. 1.1 $(-\infty, -\frac{4}{5}) \cup (0, +\infty)$.

Solution. 4.1 The desired matrix is

$$\begin{pmatrix} 4 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}. \tag{5.1}$$

The uniqueness: if there is another L', then there exists some orthogonal matrix Q such that $L = Q \cdot L'$. Since $Q = L \cdot (L')^{-1}$ is upper triangular with all positive eigenvalues, $Q = I$.

Remark 5.1. What if $T^T \cdot T$ is not a square matrix? To what extent the decomposition is unique? Hint: a priori examination.

Solution. 3.1 -6. One should recognise it "efficiently".

Solution. 3.2 Some feasible techniques:

(1) Inductions.
(2) Eigenvalues.
(3) $\lambda^m \cdot \det(\lambda I_n - AB) = \lambda^n \cdot \det(\lambda I_m - BA)$, wherein the corollary
$$\det(I - AB) = \det(I - BA) \tag{5.2}$$
is more used.
(4) add-one-line trick: $\det \begin{pmatrix} 1 & * \\ 0 & A \end{pmatrix} = \det A$.
(5) Schur complement. when $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$?
(6) Laplace expansion, cofactor expansion, and neglect the linearly dependent sets when necessary:
   (a) for instance, $\det(A + u \cdot v^T) = \det A + \sum_{n\text{-terms}} + \sum_{2^n - n - 1\text{-zeros}}$;
   (b) another example: $\det(\cos(i \cdot \theta_j)) = \det(2^{1-i} \cdot \cos^i(\theta_j))$.

(7) Cauchy-Binet formula. An distinguished example:

$$\det(R^T R + S^T S) = \det(R \quad S)^T \cdot (R \quad S) \overset{\text{CB}}{=} \cdots \geq (\det R)^2, \tag{5.3}$$

equality holds whence $S = O$.

(8) By factors (e.g., Bi-Vandermonde determinant).

(9) ...

Solution. 1.2 Ans:

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}. \tag{5.4}$$

See HW2 for the proof of uniqueness.

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