**Problem Set for 20-Feb-2025**

**Problem 1** Find two linear maps

$$\alpha, \beta : \mathbb{F}[x] \to \mathbb{F}[x],$$

such that

$$\alpha(\beta(f)) - \beta(\alpha(f)) = f$$

for any $f \in \mathbb{F}[x]$.

Is it possible to find such $\alpha, \beta : V \to V$ when $V$ is of finite dimension?

**Problem 2** Here is a **clarification of irreducibility** over general polynomial rings. Let $\mathbb{A} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots\}$. A polynomial $f \in \mathbb{A}[x]$ is **reducible** if and only if there exists some factorisation $f = g \cdot h$ such that $g^{-1} \notin \mathbb{A}[x]$ and $h^{-1} \notin \mathbb{A}[x]$. For instance:

- $2 \cdot x$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{Z}[x]$;
- $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$, yet reducible in $\mathbb{C}[x]$.

Now consider $f \in \mathbb{Z}[x]$. **Prove** the following:

1. If $f$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$;
2. If $f$ is irreducible in $\mathbb{R}[x]$, then it is irreducible in $\mathbb{Q}[x]$.

> ⚠ 规范的表述是“多项式 $f(x)$ 在 $\mathbb{A}[x]$ 中可约”，而非“多项式 $f(x)$ 可约”；类似地，规范地表述是“ $f$ 是 $\mathbb{F}$-线性映射”，而非“ $f$ 是线性映射”。若无歧义，可适当地选用后者以精简表述。

In fact, one has

$$(\text{domain}) \quad \underbrace{\mathbb{Z}[x] \to (\mathbb{Z}[1/2])[x] \to \cdots \to \mathbb{Q}[x]}_{\text{more irreducible polynomials}} \quad (\text{fractional field}),$$

and

$$(\text{field}) \quad \underbrace{\mathbb{Q}[x] \to (\mathbb{Q}[\sqrt{2}])[x] \to \cdots \to \mathbb{C}[x]}_{\text{less irreducible polynomials}} \quad (\text{algebraic closure}).$$

**(Optional)** Find **Gauß's lemma** in any of the textbooks and understand both the statement and the proof. The lemma states that:

> For any $f(x) \in \mathbb{Z}[x]$, $f$ is irreducible in $\mathbb{Z}[x]$ if and only if $f$ is both irreducible over $\mathbb{Q}[x]$ and $f$ is primitive (i.e., the greatest common divisor of its coefficients is 1).

Let $f$ be **monic**, i.e., non-zero with leading coefficient $1$. From Gauß's lemma, we learn that for any monic $f \in \mathbb{Z}[x]$, $f$ is irreducible in $\mathbb{Z}[x]$ **if and only if** it is irreducible in $\mathbb{Q}[x]$.

**Problem 3** Here are some criteria for the irreducibility of polynomials in $\mathbb{C}[x]$:

1. Let $f \in \mathbb{Z}[x]$ be a **monic** polynomial of degree $n$. Denote the zeros of $f$ in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if there is exactly one $z_i$ such that $|z_i| \geq 1$ and $f(0) \neq 0$, then $f$ is irreducible in $\mathbb{Q}[x]$.
2. Let $f \in \mathbb{Z}[x]$ be a polynomial such that $f(0)$ is prime. Denote the zeros of $f$ in $\mathbb{C}$ by $(z_i)_{i=1}^n$. Show that, if $|z_i| > 1$ for all $i$, then $f$ is irreducible.
3. Let $f(x) = \sum_{k=0}^n a_k \cdot x^k \in \mathbb{Z}[x]$ be a polynomial with $f(0)$ prime. Suppose that $|a_0| > \sum_{k=1}^n |a_k|$. Show that $f$ is irreducible.

**Problem 4** Find all $f(x) \in \mathbb{C}[x]$ such that

$$f(x) \equiv \begin{cases} 2x \mod (x-1)^2, \\ 3x \mod (x-2)^2. \end{cases}$$

**Exercises (optional)** The following problems are **optional** but some of the problems are very important.

1. Is there any irreducible $f(x) \in \mathbb{Z}[x]$ such that $f(f(x))$ is reducible?

2. Prove that $1+\prod_{k=1}^{2025}(x-k)^2$ is irreducible in $\mathbb{Z}[x]$;

3. Prove that $\prod_{k=1}^{n}(x-x_i)+1$ is either irreducible in $\mathbb{Z}[x]$, or a perfect square;

   - where $x_1 < x_2 < \cdots < x_n$ are integers.

4. ($f \in \mathbb{Z}[x]$) Prove that if $f(x)=1$ has $\geq 4$ solutions in $\mathbb{Z}$, then $f(x)=-1$ has no solutions in $\mathbb{Z}$.

5. Prove that the partial sum $(e^x)_{\deg \leq n}$ is always irreducible in $\mathbb{Q}[x]$.