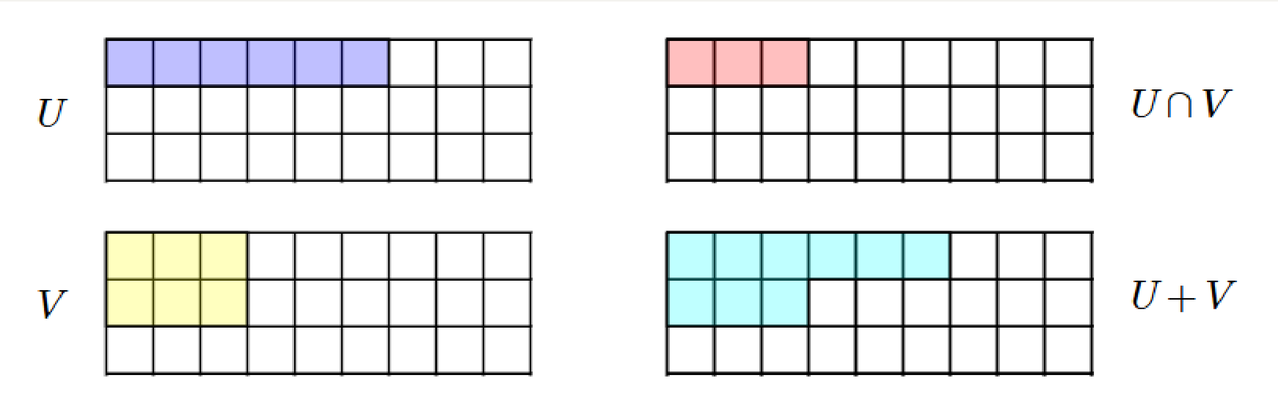
A kindergarten puzzle on Zassenhaus lemma

> *Throughout, it is INVALID to take basis.*

Example Let U and V be linear subspaces. Determine (guess) what the following diagram represents.

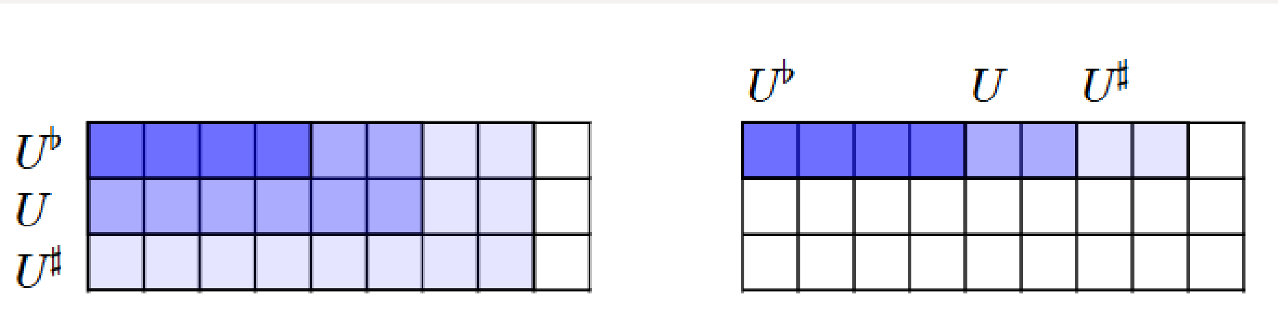


Exercise Prove the following identities of subspaces

1. $(U \cap V) \cap W = U \cap (V \cap W)$;
2. $U + V = V + U$.

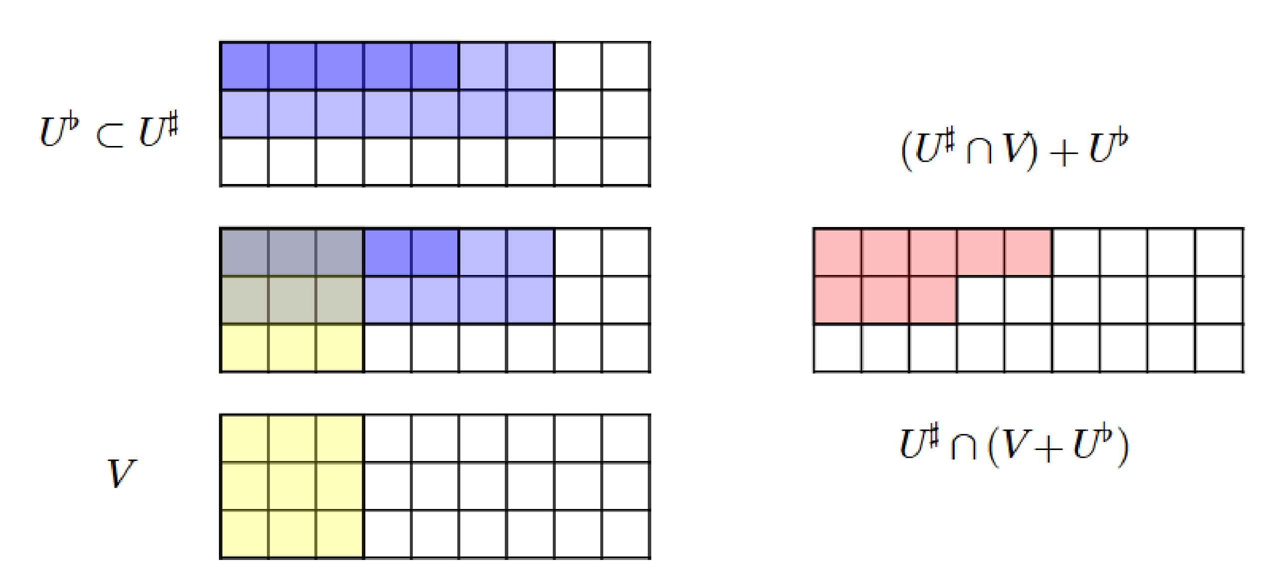
Exercise How would you denote the 0-subspace and the entire vector space in the diagram?

Definition Let $U^\flat \subset U \subset U^\sharp$ be inclusions of subspaces:



From now onwards, the symbol $U^\sharp$ (respectively $U^\flat$) denotes a subspace that contains (respectively is contained by) U in the rest of the adventure.

Theorem (**Modular identity**, recall the homework months ago)
$(U^\sharp \cap V) + U^\flat = U^\sharp \cap (V + U^\flat)$.

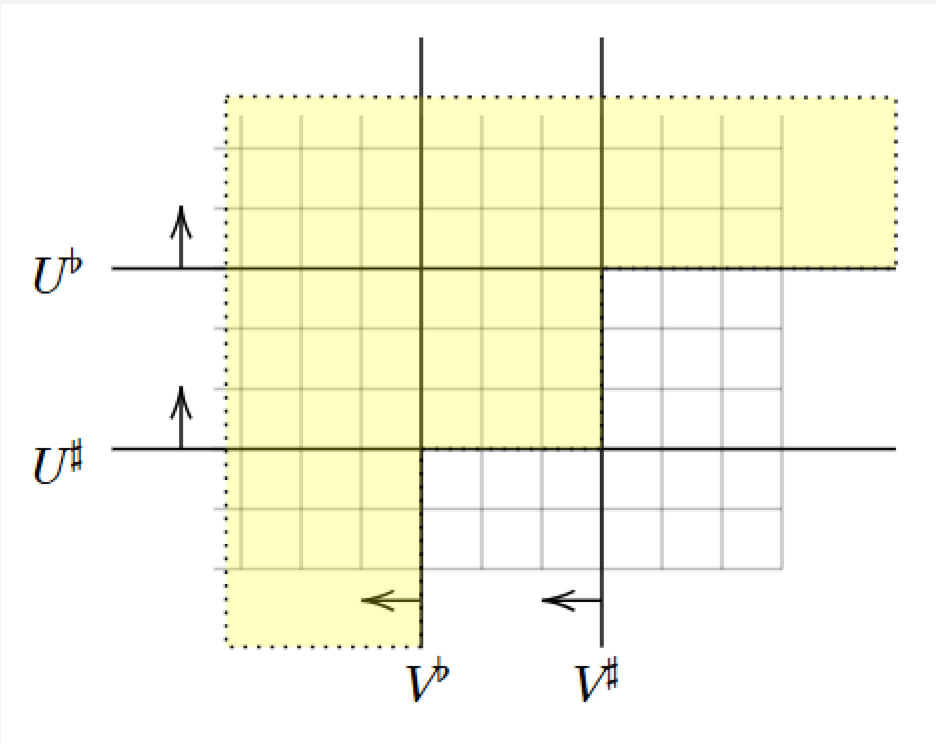


Exercise Prove (in diagrams) that the subspace

$$U^\flat + V^\sharp \cap U^\sharp + V^\flat$$

is well-defined. In other words, there is no need to add any round brackets (parentheses in Ameliclish).

Hint



Exercise Hunt for a prove **Isomorphism theorem A** in *done right*, which states that

- Let $f : U \to V$ be linear map. Then there exists an isomorphism

$$\frac{U}{\ker(f)} \to \mathrm{im}(f), \quad [u] \mapsto f(u),$$

where:

a. $U/\ker(f)$ is quotient set of U, the equivalence classes of U characterised by the relation $[u_1 \sim u_2] \iff [u_1 - u_2 \in \ker(f)]$, and

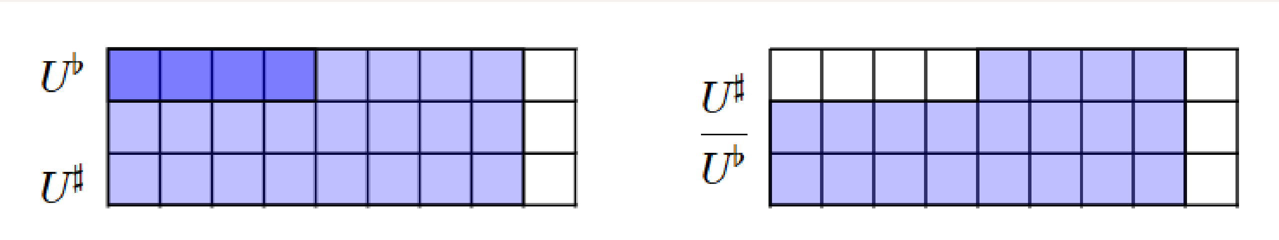
$$[u] := \{u' \in U \mid u \sim u'\}.$$

b. $\mathrm{im}(f)$ is the subset of V, where

$$[v \in \mathrm{im}(f)] \iff [\exists\, u \in U\ (v = f(u))].$$

- Notice that both the quotient set $U/\ker(f)$ and the subset $\mathrm{im}(f)$ are automatically linear spaces.

Definition (Quotient spaces in diagram) The rectangles are assumed to be unbounded in the left $\leftarrow$ and upward $\uparrow$ directions:



Exercise (**Isomorphism theorem B**) For any subspaces U and V, there exists an isomorphism:

$$\frac{U+V}{U} \simeq \frac{V}{U \cap V}.$$

Restate the theorem in diagrams.

> *If you are very familiar with quotient sets, then it is valid to use $=$ instead of $\simeq$. However, one must write $\simeq$ in **isomorphism theorem A**, since there is an external bijection induced by f.*

Exercise (**Isomorphism theorem C**) There exists an isomorphism

$$\frac{U^\sharp/U^\flat}{U/U^\flat} \simeq \frac{U^\sharp}{U}.$$

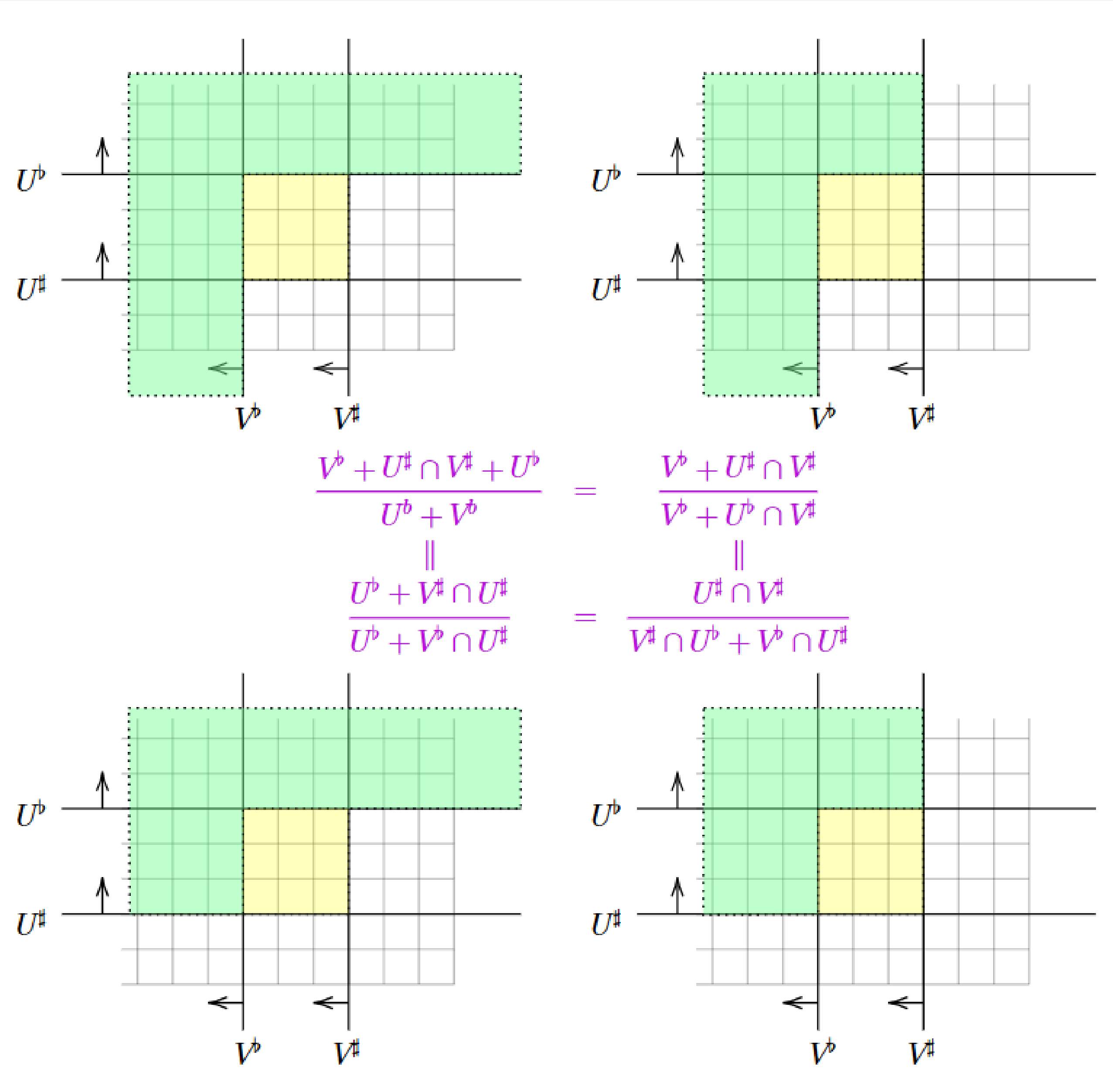
Restate the theorem in diagrams.

> *(Repeat) If you are familiar with quotient sets, then it is valid to use $=$ instead of $\simeq$.*

Exercise (**Isomorphism theorem D**) Let $U \subset V$. There is a bijective correspondence between:

1. the subspaces of V containing U, i.e. those W such that $U \subset W \subset V$,
2. the subspaces of V/U.

Exercise State and prove Zassenhaus lemma



You are expected to explain how we obtain the diagram(s), quotient spaces, and all equalities (or isomorphisms, you decide).

> *Once you find Zessenhaus theorem trivial, proceed with the following exercises.*

Non-kindergarten part

> *The concept of linear maps is beyond the scope of students (precisely, babies) from kindergarten, as it concerns homomorphisms of diagrams.*

Definition (Linear maps) A linear map is an affine transformation of diagrams, which preserves 0. Let f be a linear map. We restate the definition of image and pre-image (as mentioned in mathematical analysis, I believe) in formal and symmetric terms:

1. (the image) $f_*U = \{f(u) \mid u \in U\}$, where U is the subspace of the domain (domain: where f starts/maps from/originates);
2. (the inverse image) $f^*V = \{u \mid f(u) \in V\}$, where V is the subspace of the codomain (codomain: where f ends/maps to/terminates).

Exercise The most elementary part of linear algebra is to study the operations $\{f_*, f^*, +, \cap\}$. It is an enjoyable time to observe how these simple operations generate identities.

Prove the following (when we write f_*X,it is assumed that the linear space X is a subspace of the domain, and similarly for f^*Y):

1. $U \subset f^*f_*U$, when does equality hold for all U?
2. $f_*f^*V \subset V$, when does equality hold for all V?
3. $f_*f^*f_* = f_*$,
4. $f^*f_*f^* = f^*$,
5. $f_*(U+V) = f_*U + f_*V$;
6. $f^*(U \cap V) = f^*U \cap F^*V$;
7. if $f_*(U \cap V) = f_*U \cap f_*V$ always holds, then f is surjective, does the converse hold?
8. if $f^*(U+V) = f^*U + f^*V$ always holds, then f is injective, does the converse hold?
9. $f_*((f^*U) \cap V) = U \cap (f_*V)$;
10. $f^*((f_*U) + V) = U + (f^*V)$.

> *The final two identities come from basic logical facts (guess what they are). There is an analogue known as the* ***projection formula*** *if you are familiar with sheaf theory.*

Theorem (Zassenhaus lemma with a linear map) The following are identities and isomorphisms

$$
\begin{tikzcd}
& \frac{V^\flat+f^*U^\sharp\cap V^\sharp+f^*U^\flat}{f^*U^b+V^b} \arrow[rr, equal] \arrow[dl, dashed, no head, "\sim"'] \arrow[dd, equal] & & \frac{V^\flat+f^*U^\sharp\cap V^\sharp}{V^\flat+f^*U^\flat\cap V^\sharp} \arrow[dl, dashed, no head, "\sim"'] \arrow[dd, equal] \\
\frac{f_*V^\flat+U^\sharp\cap f_*V^\sharp+U^\flat}{U^b+f_*V^b} \arrow[rr, equal] \arrow[dd, equal] & & \frac{f_*V^\flat+U^\sharp\cap f_*V^\sharp}{f_*V^\flat+U^\flat\cap f_*V^\sharp} \arrow[dd, equal] & \\
& \frac{f^*U^\flat+V^\sharp\cap f^*U^\sharp}{f^*U^\flat+V^\flat\cap f^*U^\sharp} \arrow[rr, equal] \arrow[dl, dashed, no head, "\sim"'] & & \frac{f^*U^\sharp\cap V^\sharp}{V^\sharp\cap f^*U^\flat+V^\sharp\cap f^*U^\sharp} \arrow[dl, dashed, no head, "\sim"'] \\
\frac{U^\flat+f_*V^\sharp\cap U^\sharp}{U^\flat+f_*V^\flat\cap U^\sharp} \arrow[rr, equal] & & \frac{U^\sharp\cap f_*V^\sharp}{f_*V^\sharp\cap U^\flat+f_*V^\sharp\cap U^\sharp} &
\end{tikzcd}
$$