Evaluation of Feynman integrals with arbitrary complex masses via series expansions.

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Abstract

We present an algorithm to evaluate multiloop Feynman integrals with an arbitrary number of internal massive lines, with the masses being in general complex valued. The implementation solves by series expansions the system of differential equations satisfied by the Feynman integrals. At variance with respect to other existing codes, the analytical continuation of the solution is performed in the complex plane associated to each kinematical invariant. We present the results of the evaluation of the Master Integrals relevant for the NNLO QCD-EW corrections to the neutral-current Drell-Yan processes.

1. Introduction

The paper is organised as follows. In Section 2 we provide a pedagogical introduction to the solution of a system of first order linear differential equations by series expansions. In Section 3 we discuss a general procedure to exploit the information available through the differential equation, to identify the an-

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alytical structure of the solution. We eventually present how we implement in practice the analytic continuation with a one-loop example, comparing the real and complex mass cases. In Section 4 we outline the implementation in the Mathematica package SeaFire of the algorithm that solves a generic system of differential equations, for arbitrary complex-valued kinematical variables and weights. We illustrate the different computational strategies available. In Section 5 we discuss the solution of the Master Integrals needed to evaluate the two-loop QCD-EW virtual corrections to the neutral- and charged-current Drell-Yan processes. The latter constitute an original result of this paper.

2. Solving differential equations by series expansion

The solution of a system of linear differential equations by series expansions is well known in the mathematical literature. The analytic continuation of the solution of a differential equation, from its initial region of convergence to an external arbitrary point is the topic that we want to discuss, applied to the specific case of Feynman integrals. In this Section we present some basic definitions to set the stage of our discussion.

2.1. An introductory example

The method can be illustrated with a simple example. Let us consider the differential equation

$$\begin{cases} f'(x) + \frac{1}{2x(1+x)} f(x) = \frac{1-2x}{8x(1+x)} \\ f(0) = \frac{1}{4} \end{cases} \tag{1}$$

with x a real variable. We introduce $f(x) = x^r \sum_{k=0}^{\infty} c_k x^k$, as an expansion around $x = 0$, with c_k arbitrary coefficients. We replace $f(x)$ with the power series in the associated homogeneous differential equation, we collect all the terms with the same power of x and obtain an infinite set of algebraic equations in the unknowns c_k. One of the equations, associated to the lowest power of x and called indicial equation, assigns the value of the exponent r, which in this example turns out to be $r = -1/2$, while the other relations determine all the

c_k but one; we choose arbitrarily the latter, and set e.g. $c_0 = 1$. We eventually obtain

$$f_{hom}(x) = \frac{1}{\sqrt{x}} \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + ... \right) \tag{2}$$

A particular solution is obtained by applying the variation of the constant method, where the inverse of the homogeneous solution, multiplied by the inhomogeneous term, is expanded about $x = 0$ and easily integrated

$$f_{part}(x) = f_{hom}(x) \int_0^x dz\, \frac{1-2z}{8z(1+z)}\, f_{hom}^{-1}(z) = \frac{1}{4} - \frac{1}{6}x + \frac{1}{15x^2} - \frac{4}{105}x^3 + ... \tag{3}$$

The general solution is given by $f(x) = f_{part}(x) + C f_{hom}(x)$. In order to satisfy the boundary condition $f(0) = 1/4$ we set $C = 0$.

The solution eq. 3 is valid in the proximity of $x = 0$, with a convergence radius of 1, determined by the presence of a singularity at $x = -1$. The latter can be already read from the homogeneous equation, with the $1+x$ factor in the denominator. The regularity of the solution at $x = 0$ is enforced by the boundary condition, which effectively discards the homogeneous solution, with its singular $1/\sqrt{x}$ behaviour. Also the latter could be read from the homogeneous equation, from the $2x$ factor in the denominator.

2.2. Singularities and branch cuts

We consider now the same first-order linear differential equation presented in eq. 1, now as a function of a complex variable z. The series representation obtained in eq.3, replacing $x \to z$, converges in the complex plane z inside a disc Γ_0 centered around the boundary condition point $z_0 = 0$. We can analytically continue the solution into a new disc Γ_1, centered at a point z_1 internal to Γ_0, provided that Γ_1 does not include any singular point, in our example of $z = -1$, as illustrated in Figure 1 by the blue and red circles. It is possible to demonstrate that this procedure is unique.

The presence of singularities in the homogeneous or also in the inhomegeneous coefficients of the differential equation requires a careful definition of the analytic continuation. In the frequent physical case of simple poles in the homogeneous coefficient, we obtain a logarithmic behaviour of the solution. The

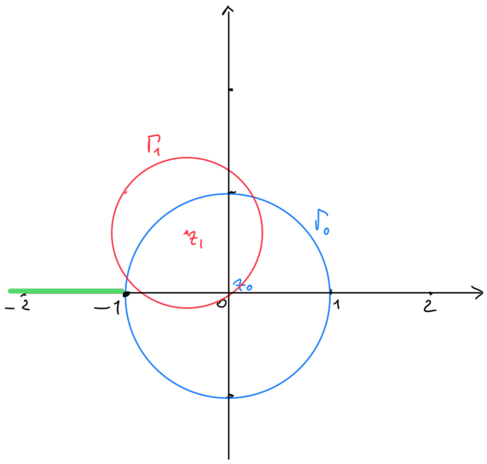

Figure 1: Example of analytic continuation

latter can be made single-valued by adding cuts in the complex plane, so that we specify the Riemann sheet where we evaluate the function. This feature is clearly not present in the power series representation, but must be introduced to allow for a physical interpretation of the results.

A simple algorithm to assign the cuts can be devised by noticing that we can group all possible singular points with respect to z in two categories, those lying on the z real axis and those with a non-vanishing imaginary part. *i*) We collect all the singularities on the real axis, we identify the rightmost one and we cut from that point to $-\infty$. *ii*) For a singular point located at c in the complex plane, we cut parallel to the imaginary axis from c upwards to $(\mathrm{Re}(c), +i\infty)$ if $\mathrm{Im}(c) > 0$ and downwards from c to $(\mathrm{Re}(c), -i\infty)$ if $\mathrm{Im}(c) < 0$. An illustration is given in Figure 2 by the green lines, assuming that $z = -1, -3/2, -1+i, -1-i, 1/2+i/4, 1/2-i/4$ are singular points and that there are no other singularities on the real axis for $\mathrm{Re}(z) > -1$. This procedure leaves a simply connected Riemann surface, because all the cuts are applied "outwards" and never intersect each other. Given a solution for the unknown function $f(z)$ olomorphic inside a disc Γ_0, the uniqueness of the analytic continuation and the simply connected topology of the domain make the solution single-valued in the

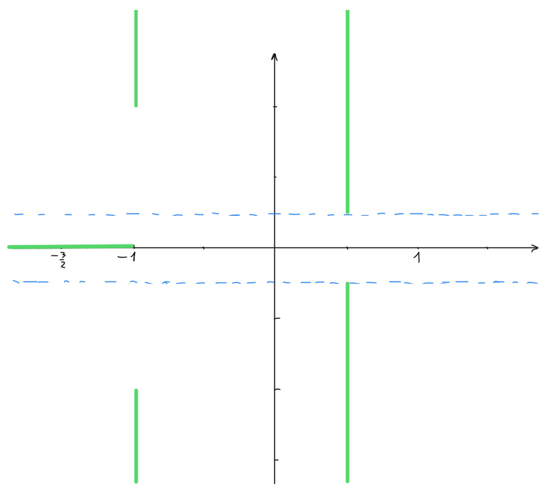

Figure 2: Example of cuts assignment.

whole domain. In Figure 2 we illustrate the above example drawing in green the cuts. In dashed-blue we highlight the presence of a strip of the z complex plane, which is by construction free of singularities and will turn out to be central in the implementation of the algorithm: it is defined by looking for the complex-valued singular point with the smallest (in absolute value) imaginary part; the latter defines the width of the strip, which spans the whole plane, parallel to the real axis.

A choice for the cuts on the real z axis different[1] than the one described in i) can lead to a different determination of the solution. This ambiguity can be discussed and solved, in the light of the Feynman prescription for the particle propagators, which enforces the causality requirement of the theory. If we consider, for the sake of simplicity, the propagator $\Pi(s) = i/(s - m^2 + i\varepsilon)$ of a massive scalar field, we see that the kinematic invariant s is real, but the Feynman prescription shifts it indeed to $s + i\varepsilon$ in the upper half of the complex s plane. We adopt the same prescription for the kinematic variables of the

[1]We can e.g. choose to put some of the cuts from the singular point to $+\infty$, yielding a different inequivalent Riemann surface.

differential equation, which implies that the complete evolution of the solution from the BCs to the point of interest is performed not on the real axis, but in the complex plane of the kinematic variable, and, more precisely, in one specific half of that complex plane. Having a prescription which pushes the solution of the differential equation in one specific half of the z complex plane, we implicitly choose the determination of the solution compatible with the BCs. The remark that we study the solution in the whole complex plane sets in a natural way the framework to handle Feynman integrals with internal complex masses.

2.3. Evaluating the solution for arbitrary values of its arguments

The discussion of the analytical continuation of a function of several variables can be a tremendous challenge and we find it convenient to discuss one variable at a time, while keeping all the others constant with a given numerical value. With this choice, all the singular points with respect to the kinematic variable under discussion are uniquely identifed and are constant.

The solution is conveniently expressed in terms of adimensional variables, which can be obtained by rescaling the kinematic variables with a dimensional constant, typically a mass. We discuss first the case by a real-valued mass. We define $x = (s + i\varepsilon)/m^2$, and we remark that dividing by a positive real number clearly does not change the sign of the infinitesimal imaginary part of the Feynman prescription. In this case, we find two sets of potential singularities, depending whether the points are real- or complex-valued. The former are typically associated with physical (pseudo-)thresholds, while the latter often appear in the linearisation of square root factors in the differential equations.

Following this distinction, we introduce cuts in the complex plane, as anticipated in Section 2.2. When moving from one point to another of the kinematic variable, we can always choose a path inside the strip free of singular points, where we can compute, step by step, a convergent Taylor expansion of the solution.

The nice example with the colored circles.

We consider now the rescaling of the kinematic invariants by a complex mass, which induces a small deformation of the setup described above in the real-mass case. We define $z = s/\mu^2$ by dividing the kinematic invariant s by a complex mass squared μ^2, with $\mu^2 = m^2 - i\Gamma m$ where m and Γ are the pole values of the propagator. The variable z will obviously be complex valued, and we observe the consistency between the sign of the imaginary part of the Feynman prescription and the sign of the imaginary part of this adimensional complex variable. All the singular points are shifted. Those which are complex valued simply change their position, while the ones on the real axis acquire an imaginary part. The single cut on the real axis is now replaced by one horizontal cut for each singular point, each with its imaginary part. Provided that we choose a path that avoids the intersection with all these horizontal cuts, we can still draw a path, inside the singularity-free strip, connecting any two points associated to two values of the kinematic variable.

Illustration of the cut position with complex masses.

We conclude that we can always draw, with either real or complex masses and thanks to the existence of a singularity-free strip, a trajectory in the complex plane that connects the BC to the point of interest. This construction considerably simplifies the realisation of an algorithm which implements the analytic continuation in the case of a generic multi-scale Feynman integral.

In summary, the case of real masses becomes a limiting case of the more general problem of Feynman integrals with arbitrary complex-valued masses.

2.4. Taylor vs logarithmic expansions

All the steps to obtain the homogeneous and eventually the general solution of the differential equation do not depend on the choice of the expansion point z_0 of the unknown function $f(z) = (z - z_0)^r \sum_{k=0}^{\infty} c_k (z - z_0)^k$. When we find, from the solution of the indicial equation, $r \geq 0$, we recover the standard Taylor expansion around z_0 and we can impose the BCs directly in z_0. When instead $r < 0$ with r half-integer or integer, we have a singular behaviour in z_0, which is clearly not available to impose the BCs. We have to choose another point, z_1, to assign the BCs and this last choice uniquely fixes the solution. We stress that the coefficients c_k can be determined in either case.

When $r < 0$ the general solution contains a term singular in z_0, a square root or a logarithm, expressed in exact form. Having extracted the problematic component (the name of logarithmic expansion stems from this specific feature), the rest of the series expansion has a good convergence, faster than the one which we could obtain by using a Taylor expansion centered in z_1.

2.5. Example

Initial differential equation with different rescalings.

2.6. Reading the singular points from the differential equations

When we study a scalar Feynman integral, we write the linear first-order differential equation with respect to one of the invariants which the integral depends upon. In the inhomogeneous term, we find additional integrals with simpler topologies, which we assume to be already known. An equivalent way to formulate the problem is to write the complete system of differential equations satisfied by the integral of interest and all the relevant subtopologies, collectively represented by $\vec{I}(\mathbf{s})$, where $\mathbf{s}$ represents all the kinematic variables. We have

$$\frac{\partial}{\partial s_\alpha} \vec{I}(\mathbf{s}) = \mathbf{A}_\alpha(\varepsilon, \mathbf{s})\, \vec{I}(\mathbf{s})\,, \tag{4}$$

where s_α is a generic kinematic variable and the coefficient matrix $\mathbf{A}_\alpha$ contains rational functions of all the invariants. The system considerably simplifies, if it is possible, by an appropriate change of basis $\vec{I} \to \mathbf{B}\vec{I}$, to bring the coefficient matrix into the form $\mathbf{A} \to \varepsilon\tilde{\mathbf{A}}$, called the canonical form. In this case we can write the system, in differential form, as

$$d\vec{I} = \varepsilon\, d\tilde{\mathbf{A}}\, \vec{I}, \qquad \tilde{\mathbf{A}} = \sum_l \tilde{\mathbf{A}}_l \log l \tag{5}$$

with $\tilde{\mathbf{A}}_l$ a matrix of rational numbers and l the letters, i.e. the combination of kinematic variables which parameterise the singular structure of the scattering amplitude and in particular the various internal thresholds of the Feynman integral. The complete information about the singular structure of the problem under study, when it can be written in canonical form, is thus given by the set of the letters. The latter can be read from the matrix $\tilde{\mathbf{A}}$, in general after the application of a partial fractioning simplification procedure.

If the system is not in canonical form, the elements of the matrix $\mathbf{A}$ can anyhow be inspected and all the points which are potentially singular in the solution can be identified. This is sufficient to perform in a robust way the analytic continuation of the solution.

3. Description of the package

4. Two-loop MIs for the neutral-current Drell-Yan

4.1. Checks

Comparison against GiNaC and DiffExp.

4.2. Comparison of real and complex mass evaluations

results

4.3. Boundary conditions

list of the exact BCs values

5. Conclusions