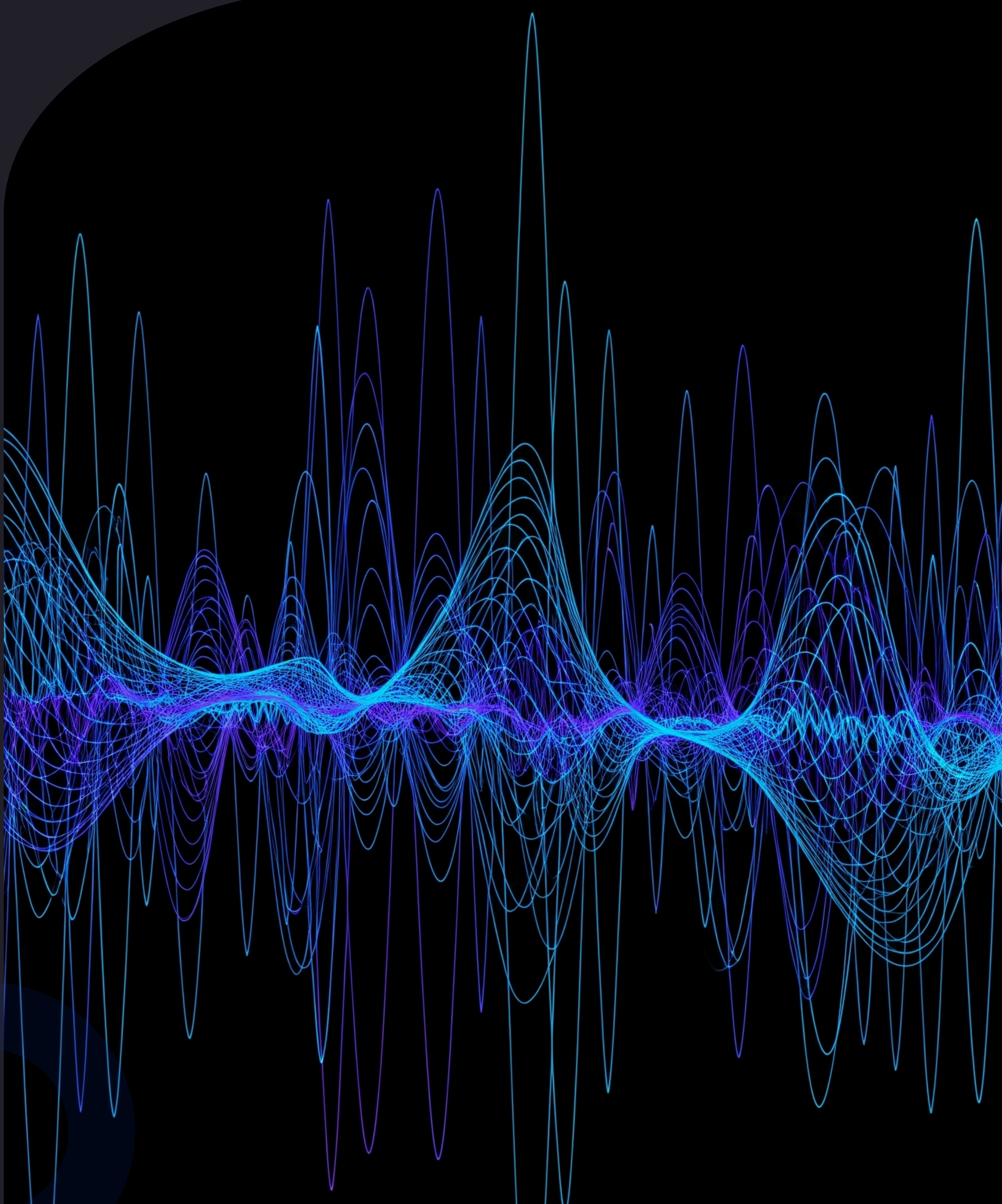
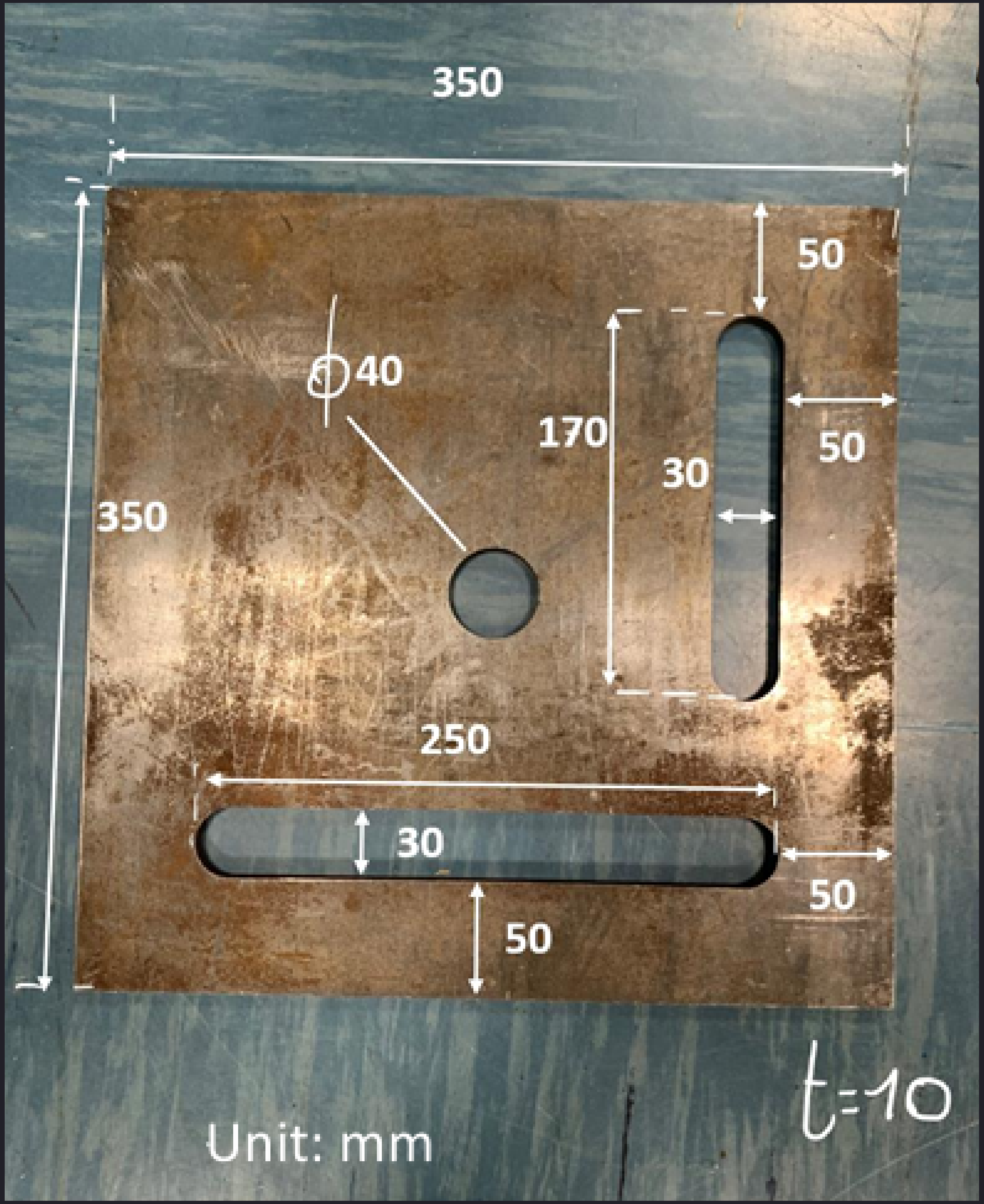
DATA ANALYSIS
ASSIGNEMENT A.A.2024/2025

Modal analysis

Group 8

Avi Gabiriele
Bregni Enrico
Dinulescu Dorin-Mihail
Kebede Fitsum Moges
Pisacane Francesco
Taboni Michele
Tundo Carlo





Description of the problem

Perform a modal analysis on a plate with defined geometry

PROJECT TIMELINE

Data Analysis Assignement
a.a.2024/2025

FEM → AUTOMAC → EXPERIMENTAL TESTS → PRE PROCESSING → MODAL ANALYSIS → PRESENTATION

1-4 May 2025
FE Model
Computation of numerical mode shapes

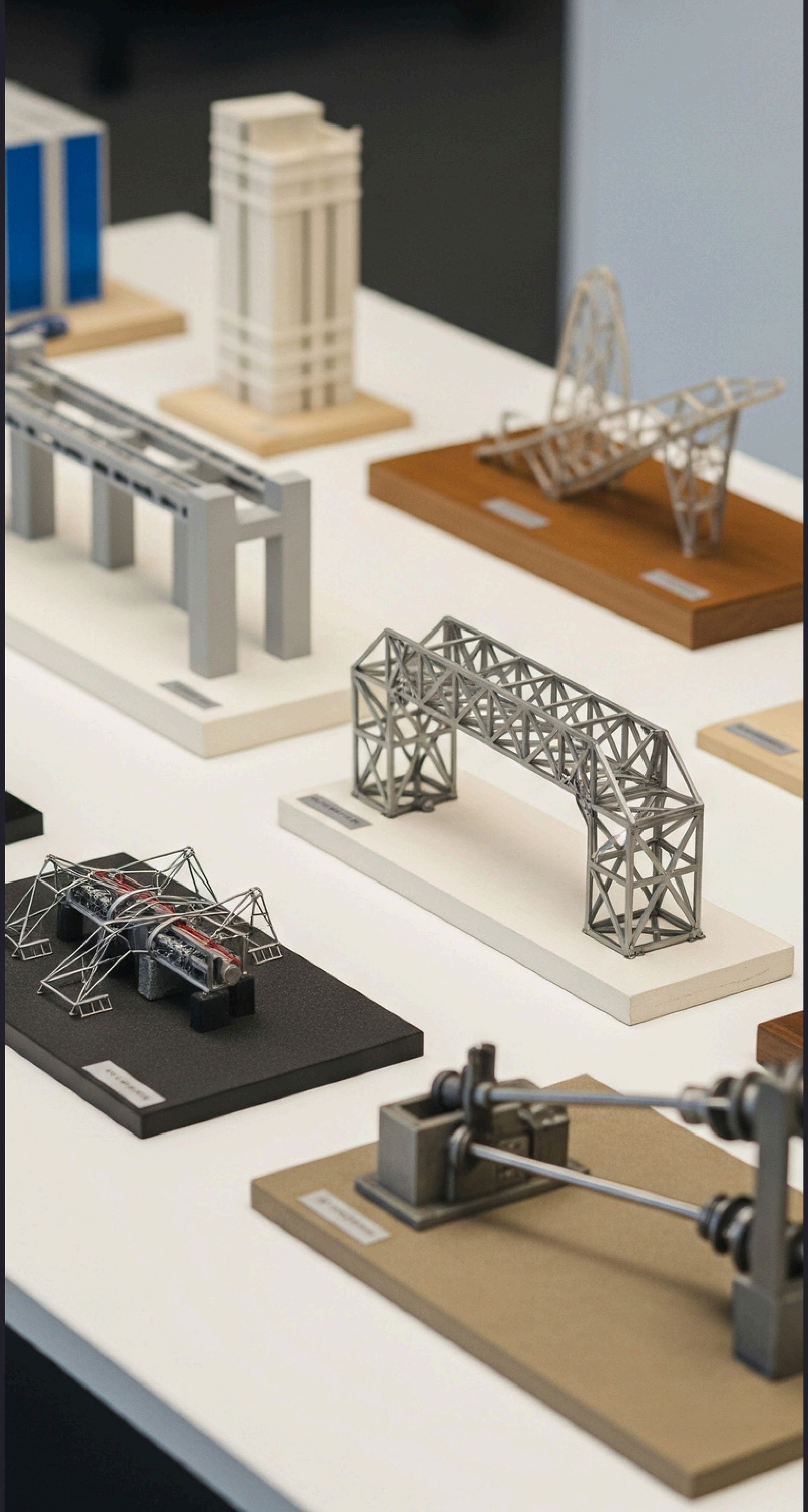
6-15 May 2025
AutoMAC computation
Definition of accelerometers position for experimental tests

19 May 2025
Laboratory B13
Hammer impulse on the plate structure

20 May 2025
Data preprocessing
FRF computation

22-30 May 2025
Modal analysis
SDOF and MDOF

26 June 2025
Building B22
Project presentation



STEPS

01 FE Model

02 AUTOMAC Computation

03 Experimental Tests

04 Data preprocessing

05 Modal Extraction (SDOF, MDOF)

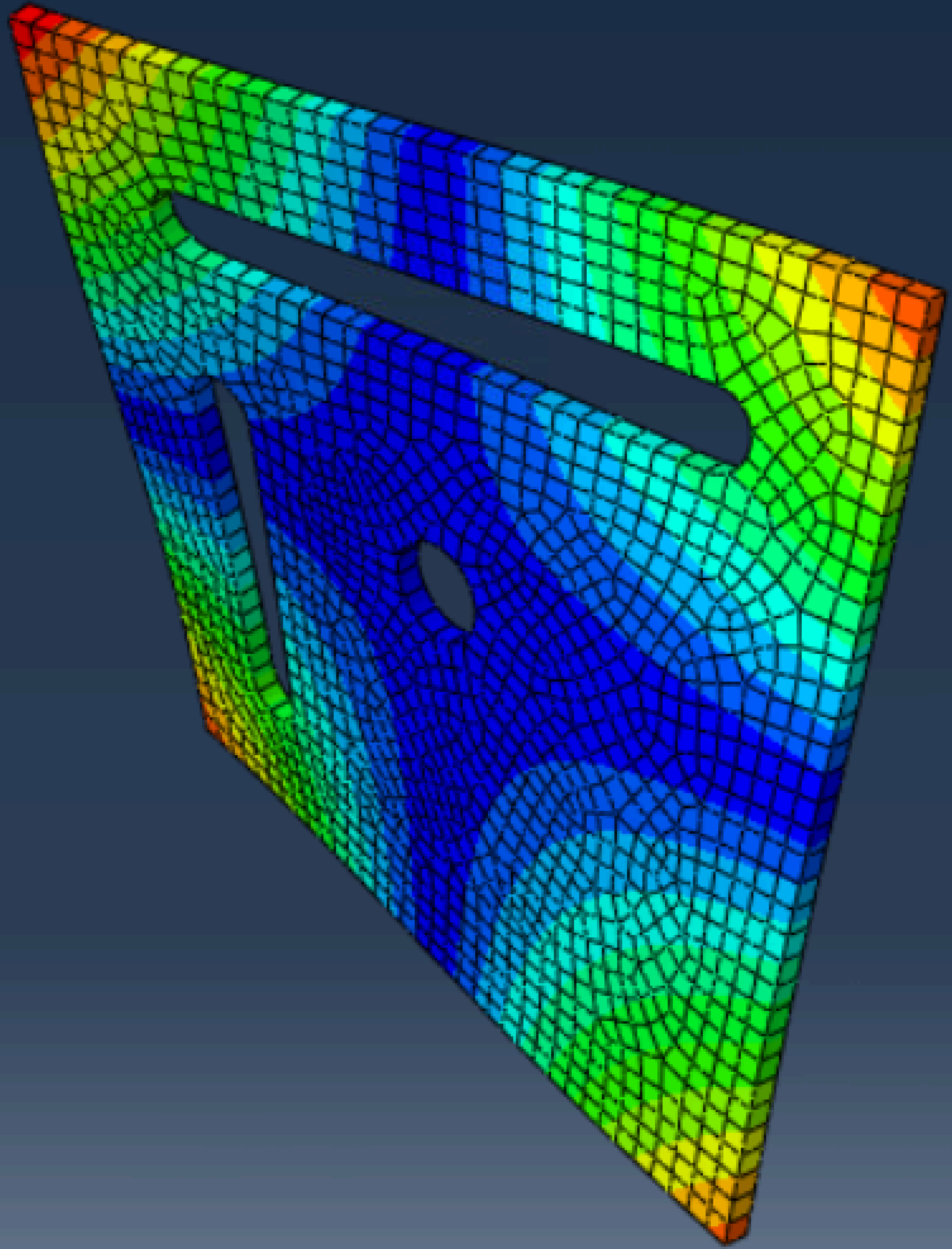
06 Comparison between numerical and experimental datas

FEM

Finite element model - pre experiment

FEM

Finite Elements Model of a plate to estimate natural frequencies



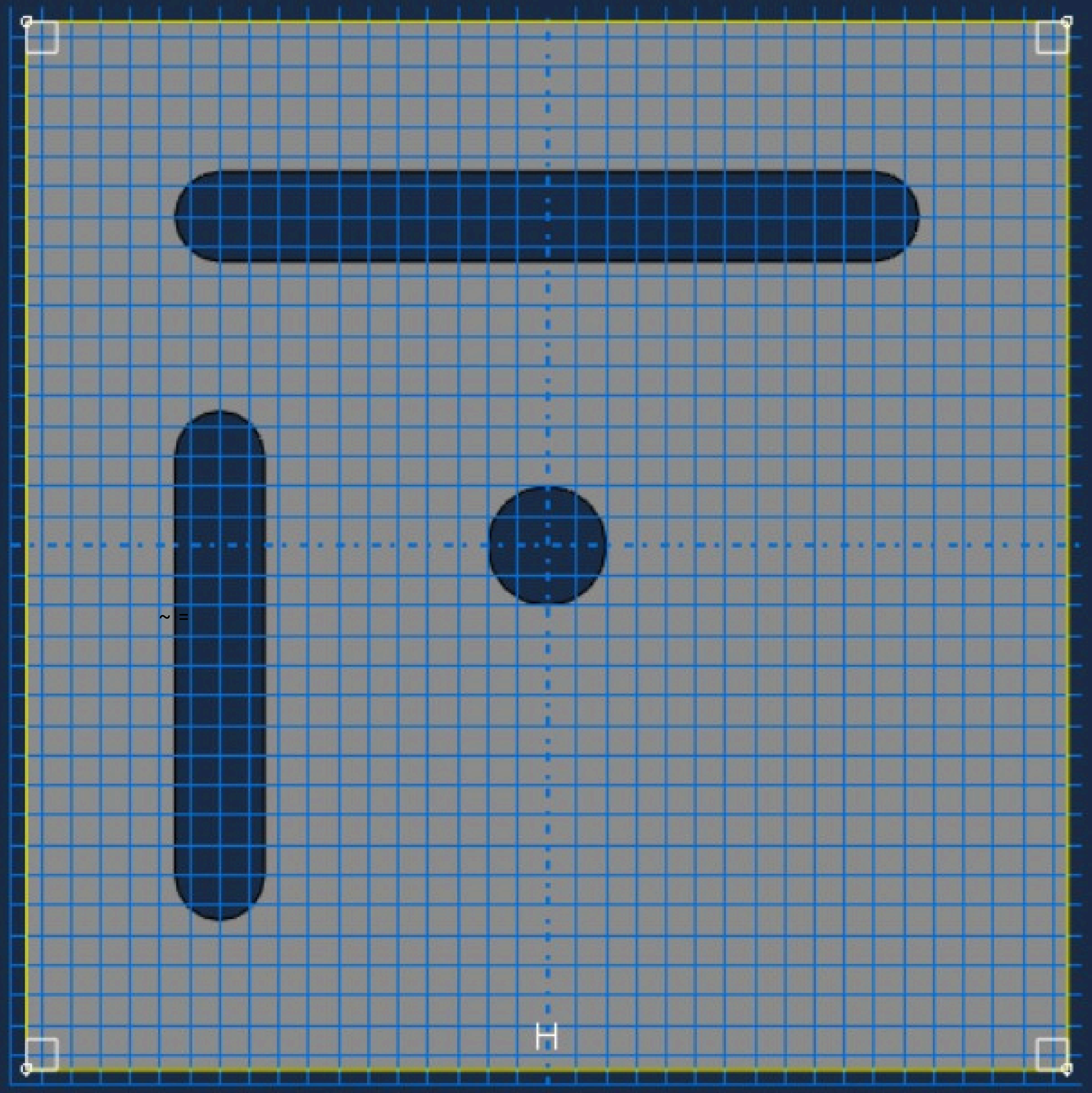
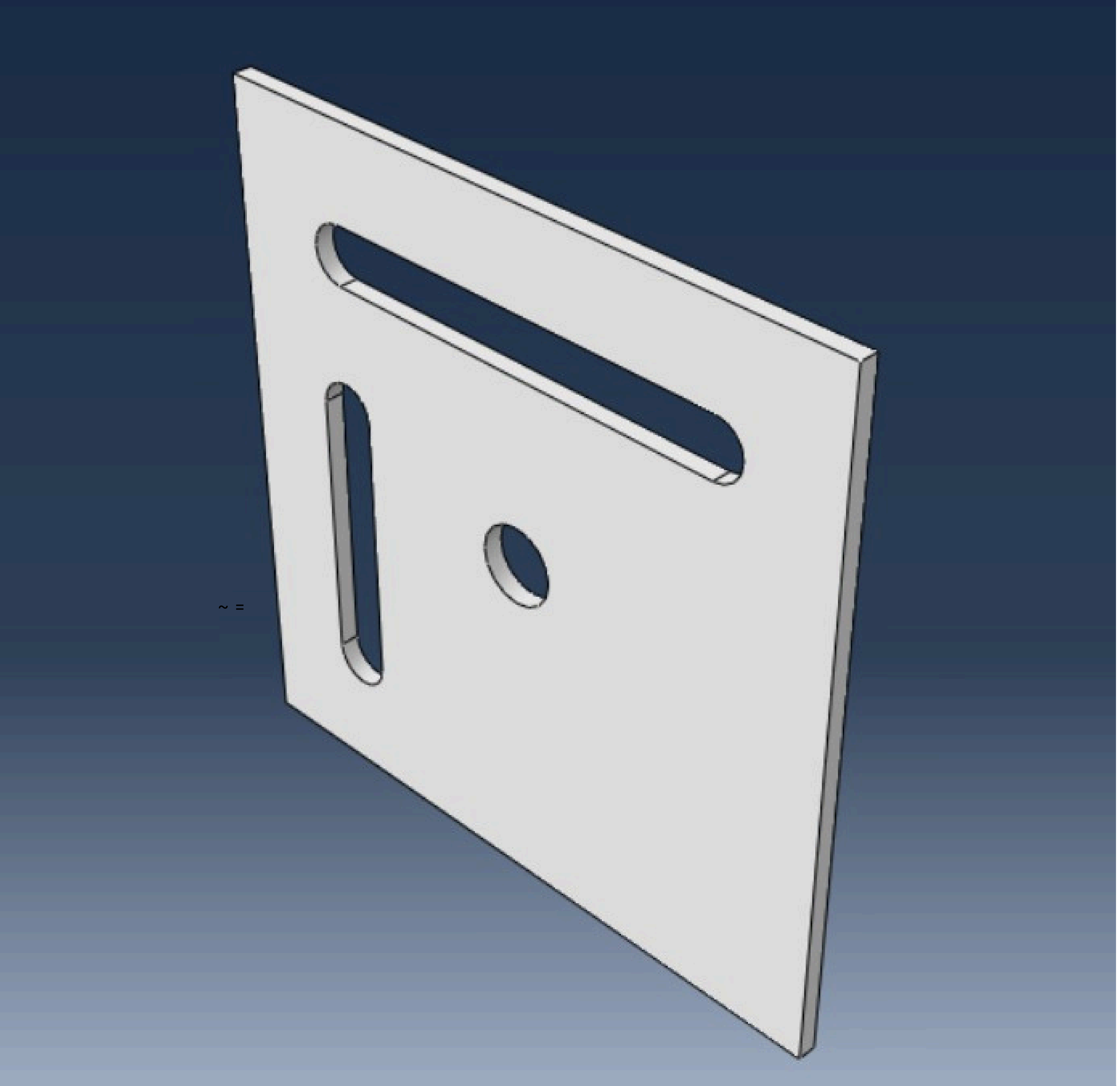


PLATE MODEL

Definition of perforated plate geometry



PERFORATED PLATE 3D MODEL

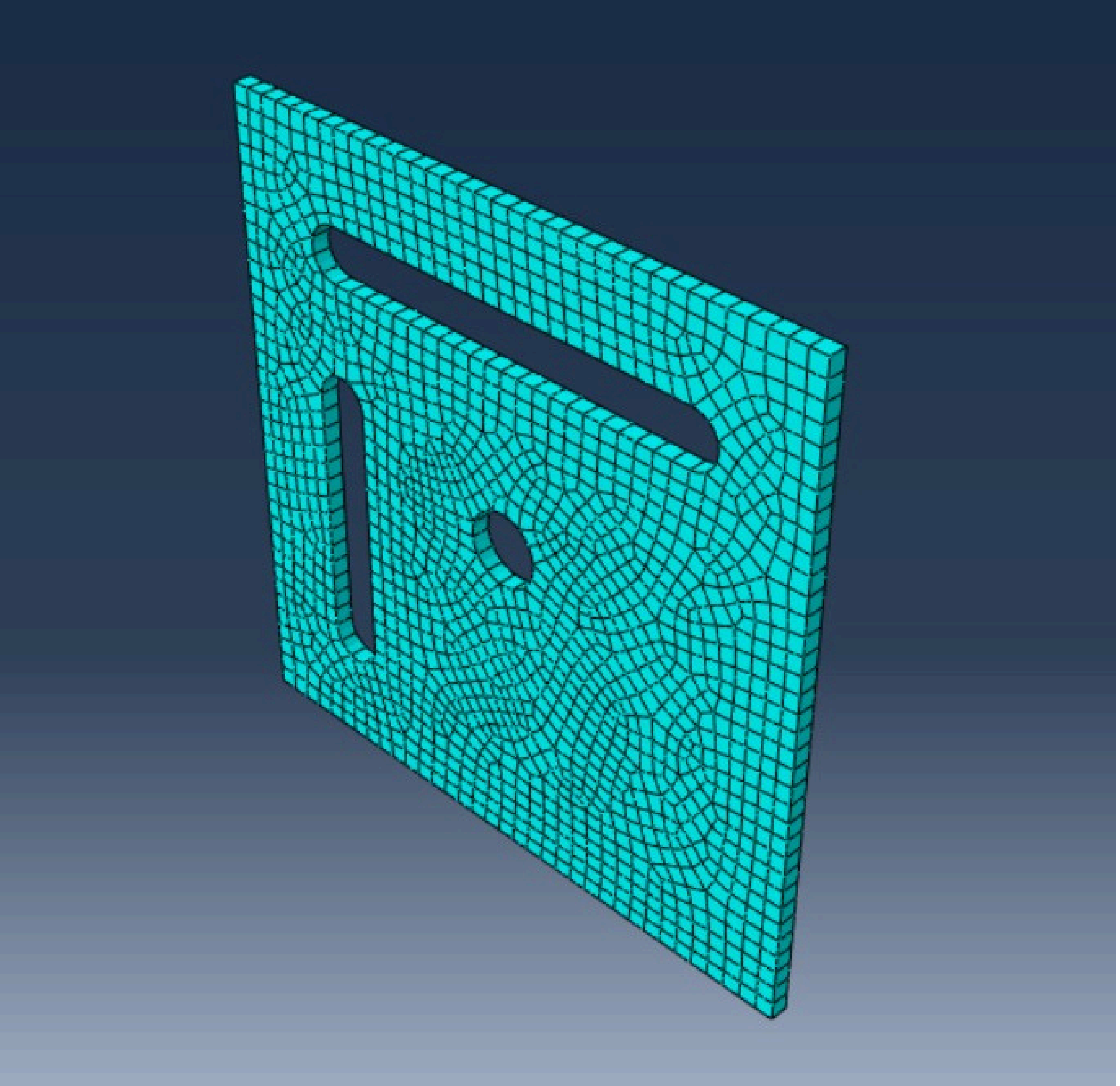
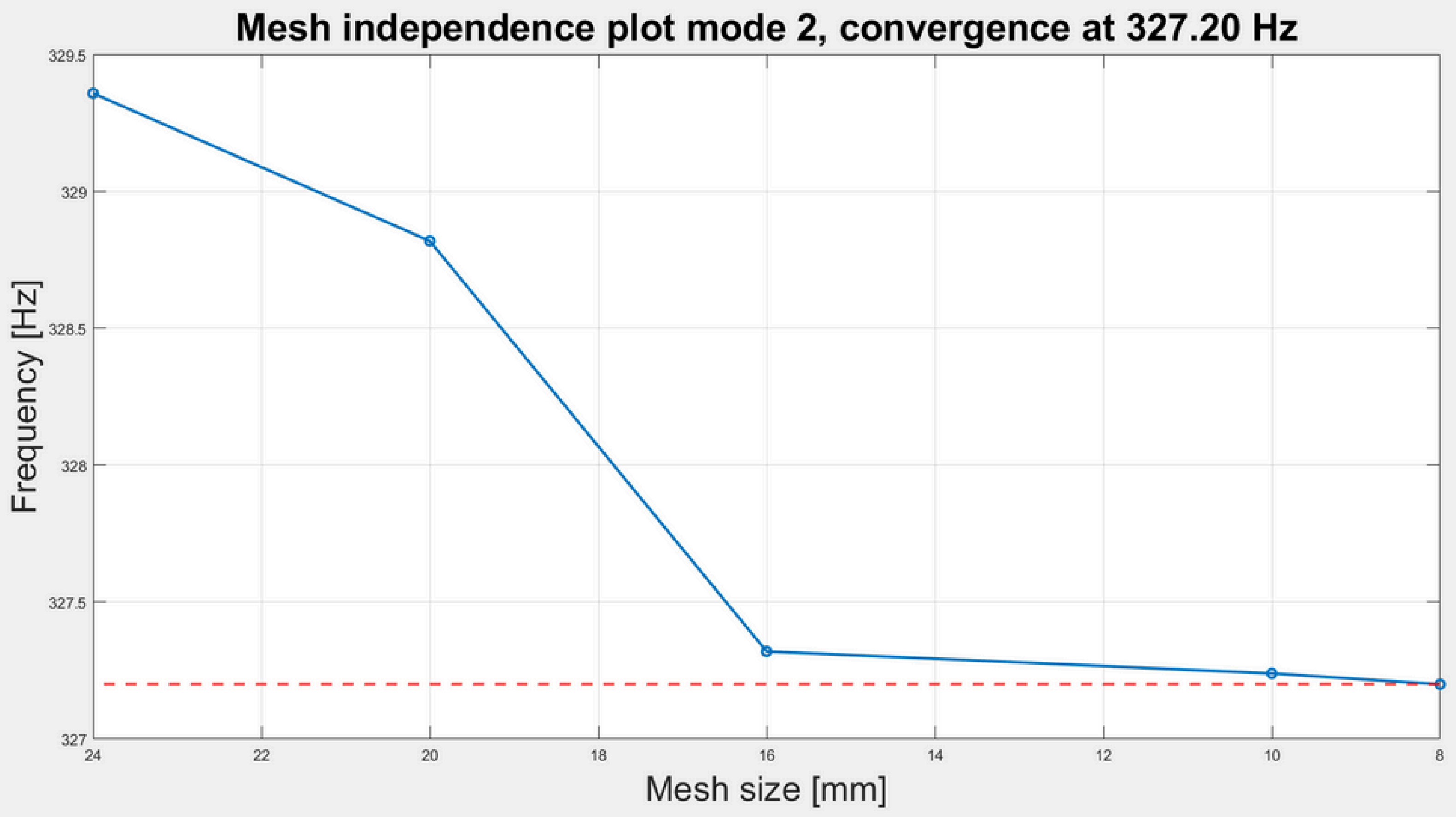
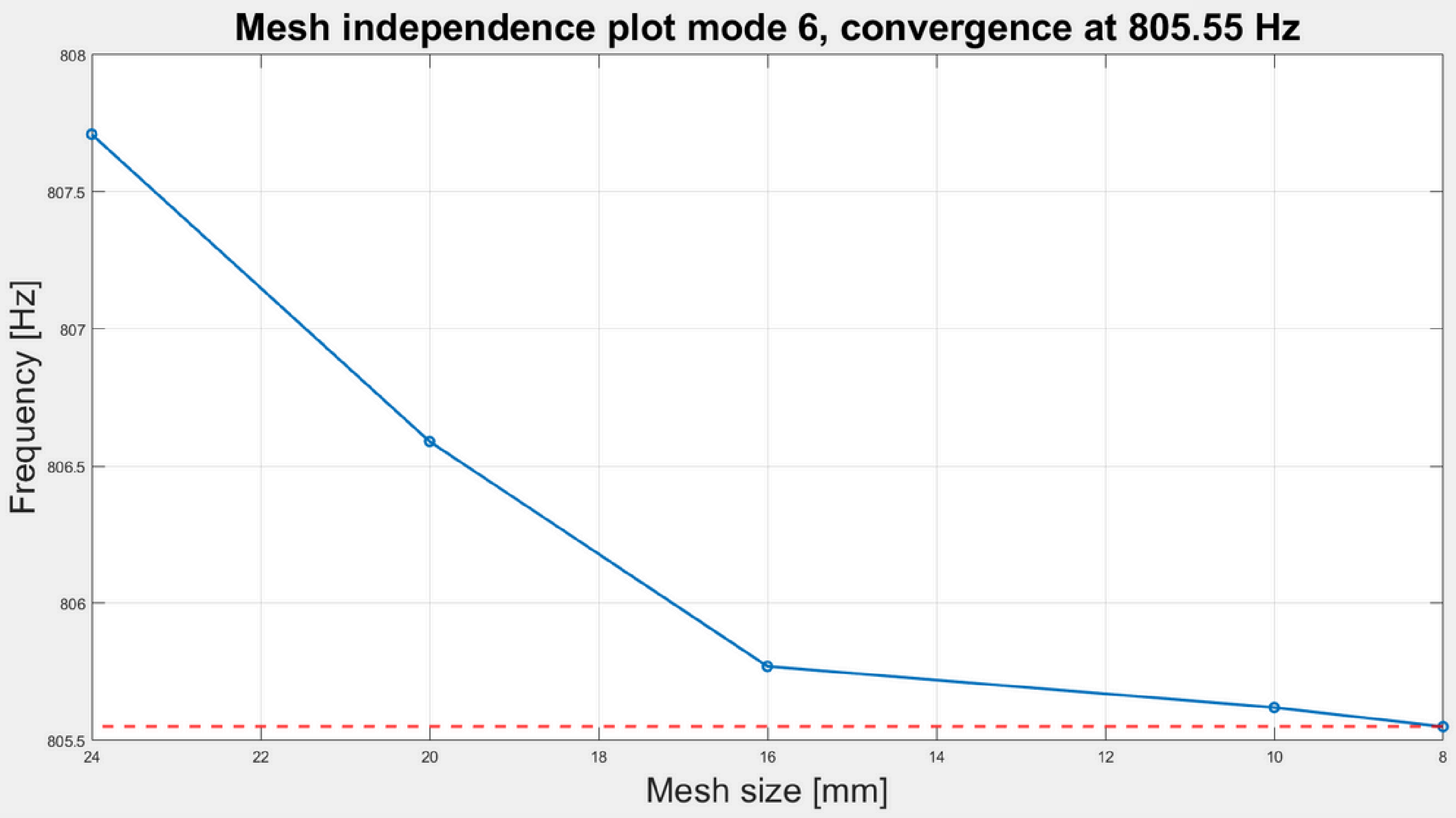


PLATE MODEL

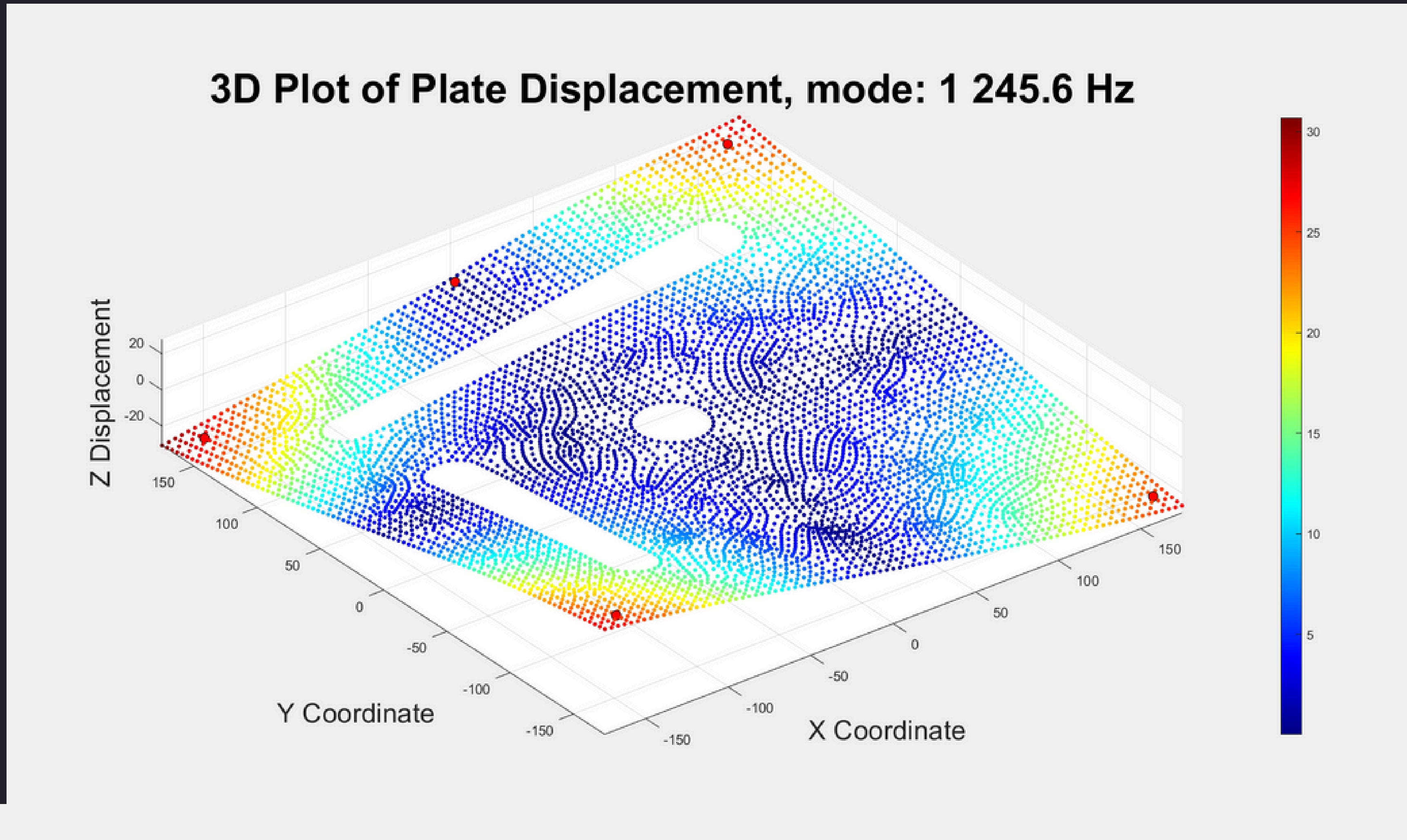
Is it the right mesh size?



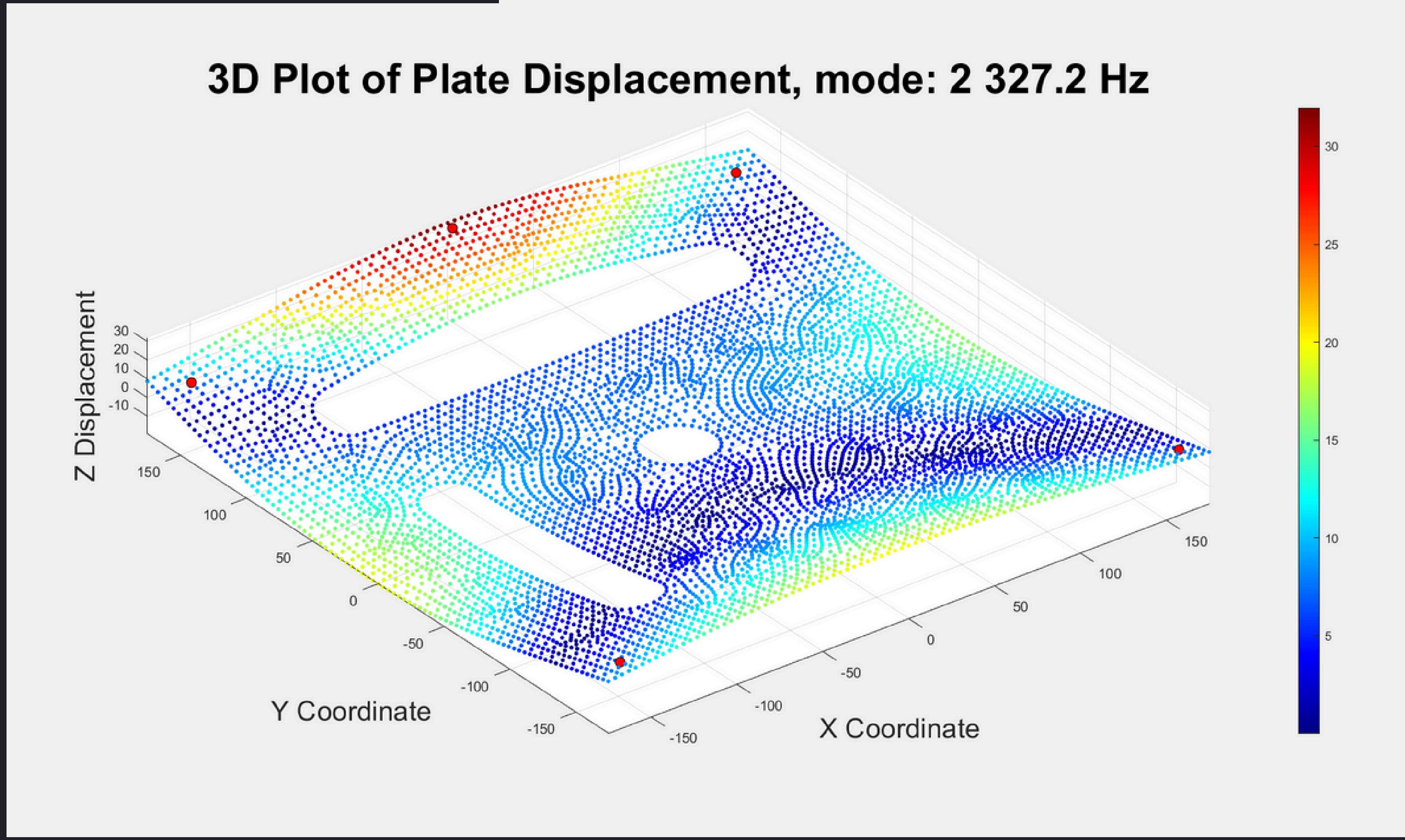


MESH INDEPENDENCE ANALYSIS

RESULTS

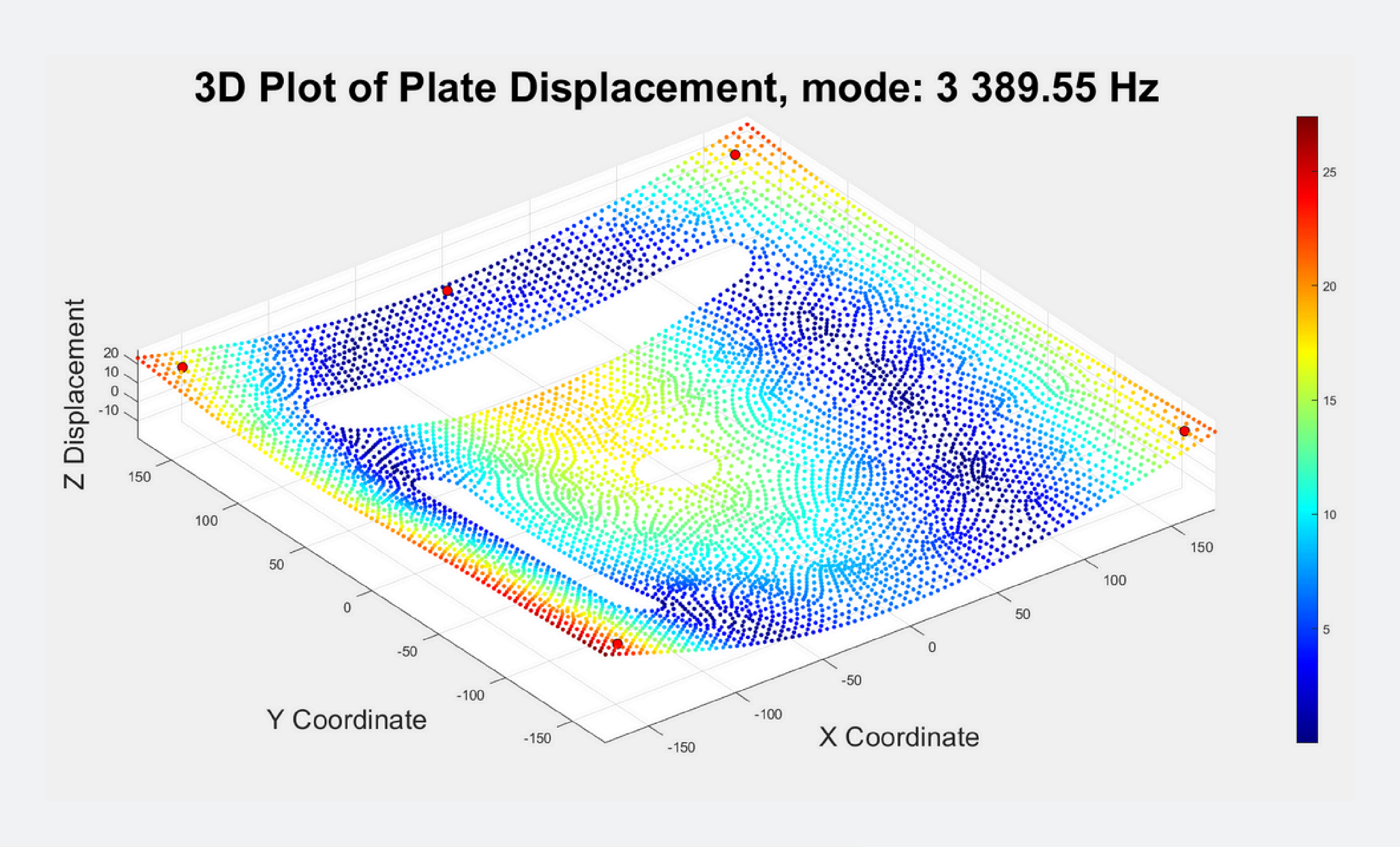


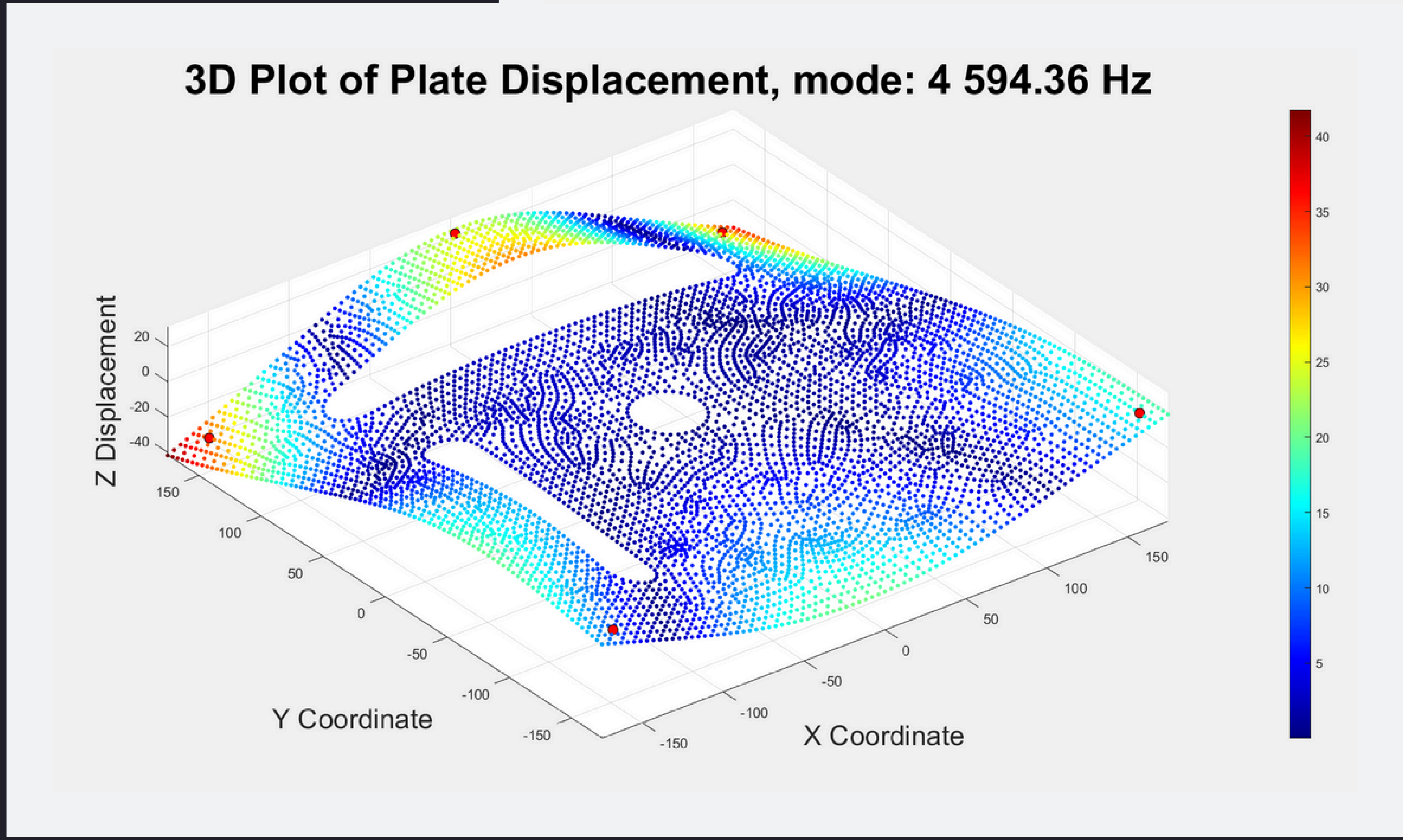
3D Plot of Plate Displacement, mode: 1 245.6 Hz
Z Displacement
Y Coordinate
X Coordinate



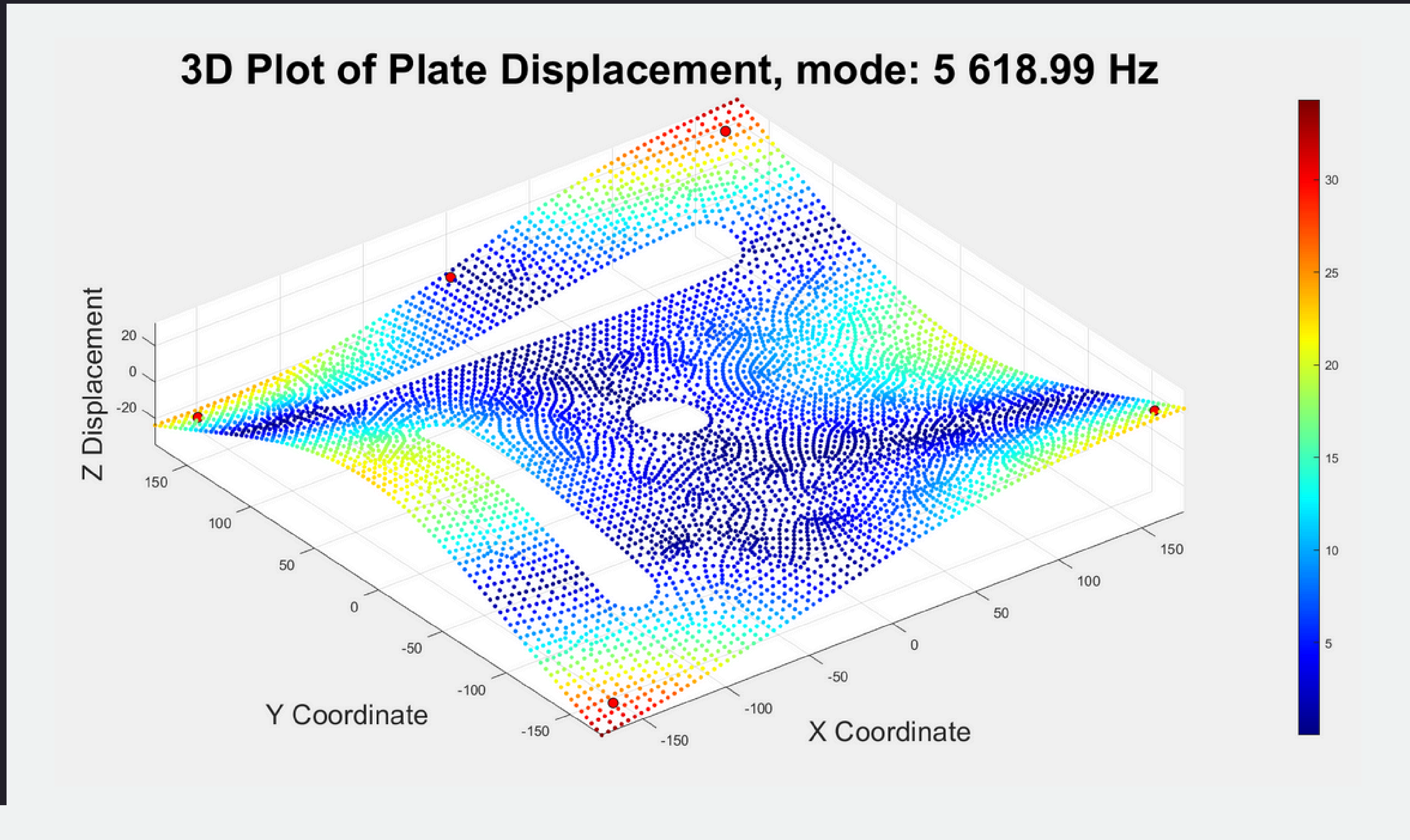
3D Plot of Plate Displacement, mode: 2 327.2 Hz
Z Displacement
Y Coordinate
X Coordinate

RESULTS

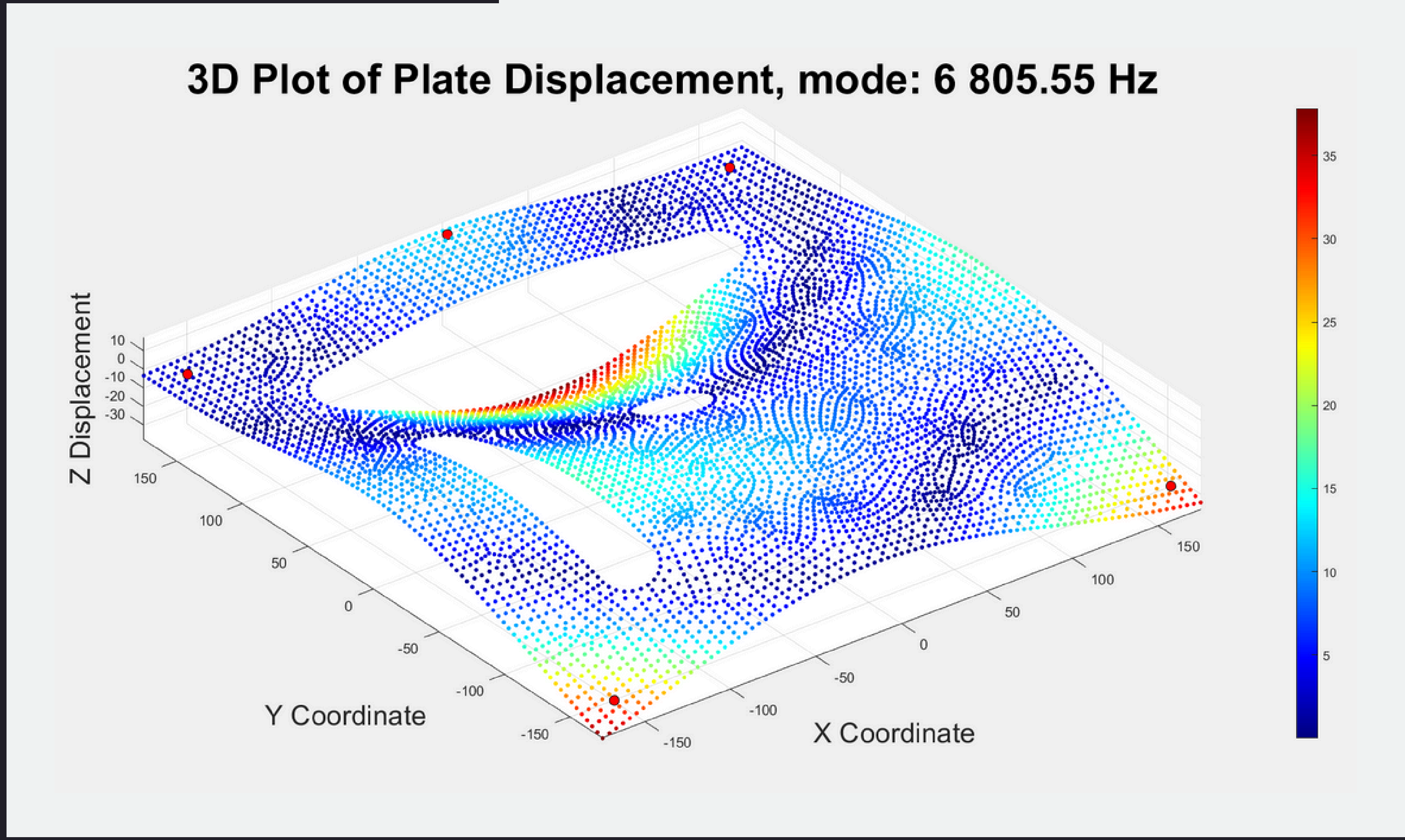

3D Plot of Plate Displacement, mode: 3 389.55 Hz
Z Displacement
Y Coordinate
X Coordinate


3D Plot of Plate Displacement, mode: 4 594.36 Hz
Z Displacement
Y Coordinate
X Coordinate

RESULTS



3D Plot of Plate Displacement, mode: 5 618.99 Hz
Z Displacement
Y Coordinate
X Coordinate

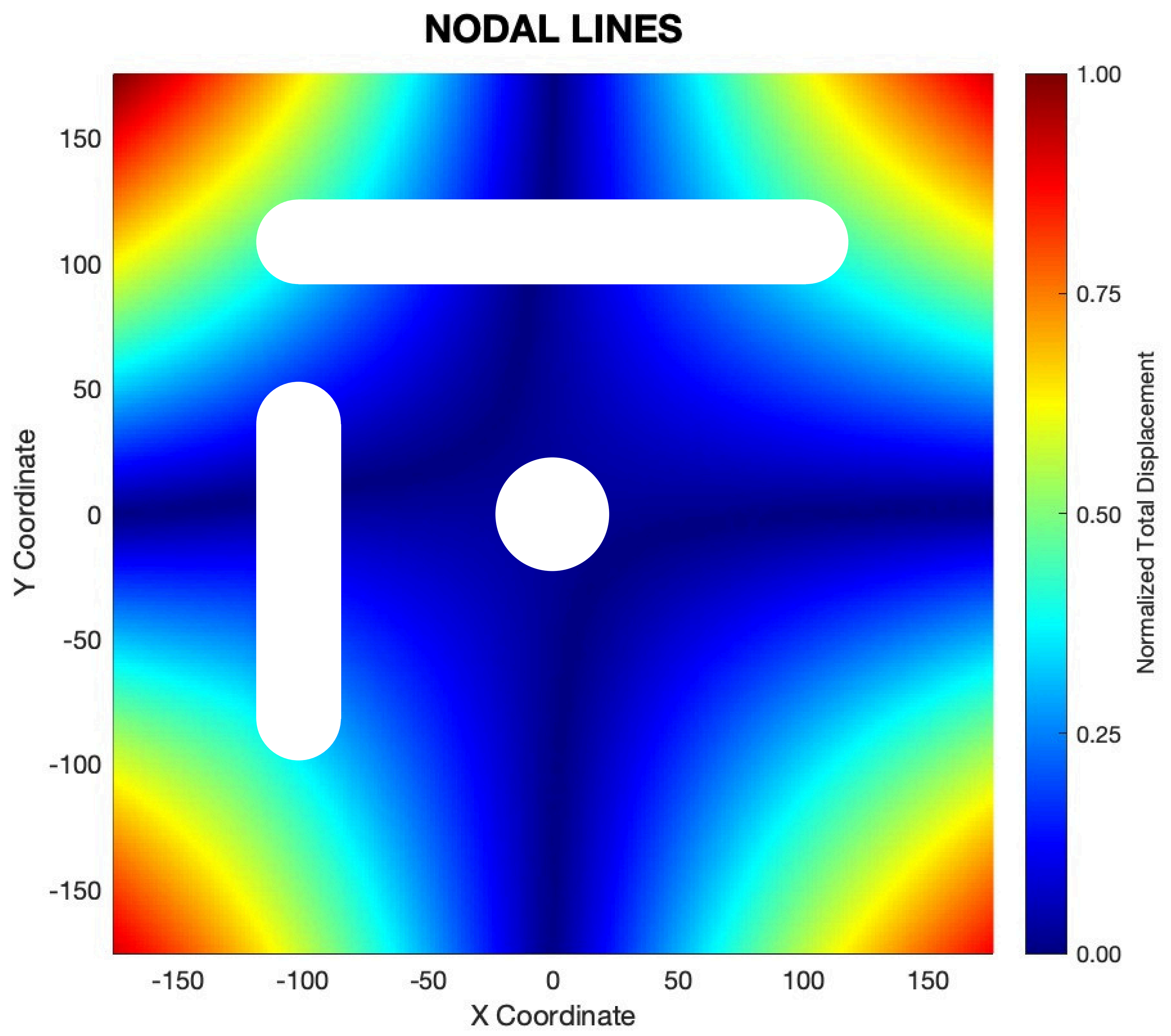


3D Plot of Plate Displacement, mode: 6 805.55 Hz
Z Displacement
Y Coordinate
X Coordinate

Data Analysis Assignement a.a. 2024/2025

AUTO-MAC

Auto Modal Assurance Criterion - pre experiment

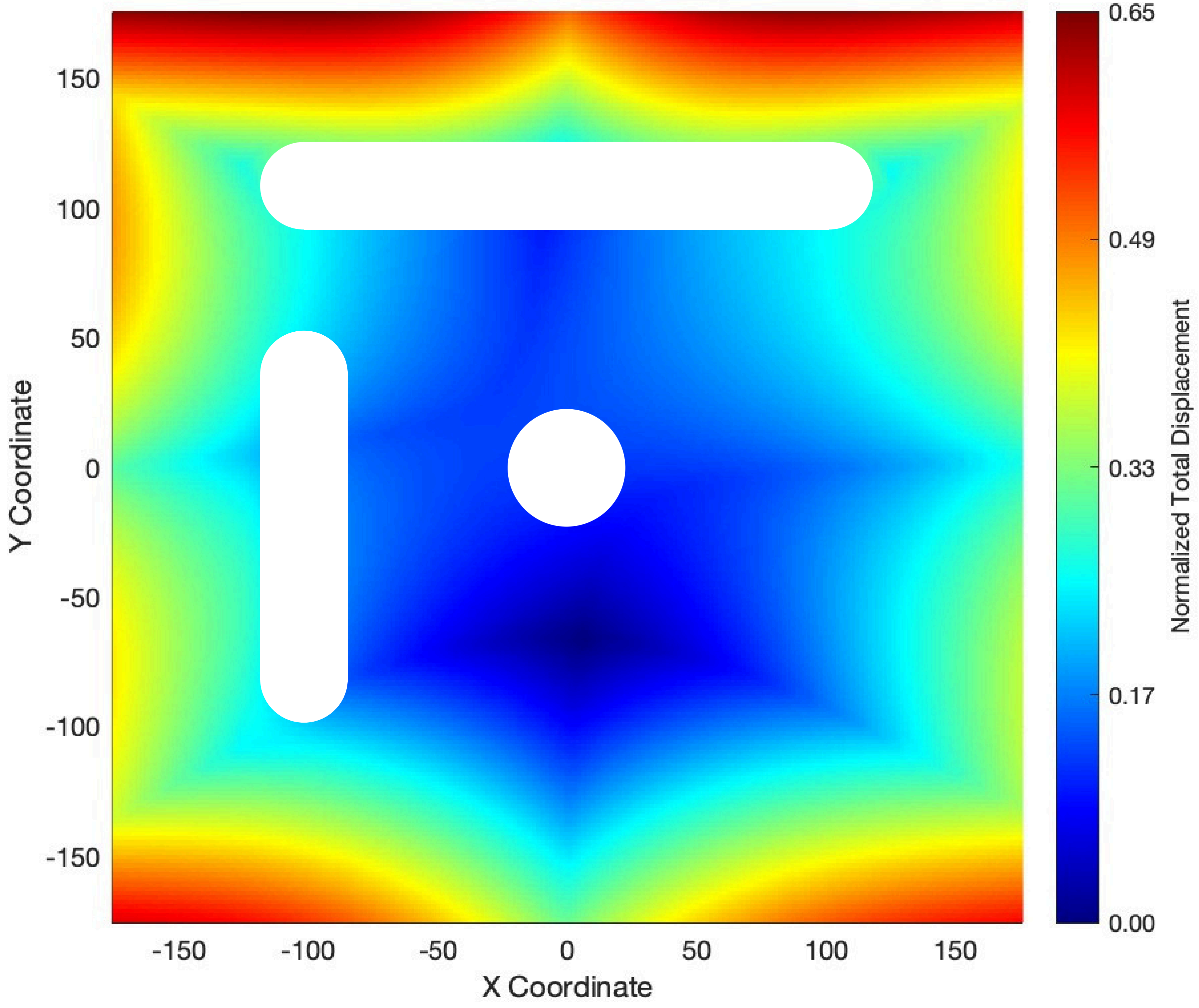


MODE 1

NODAL SURFACES DISTRIBUTION

SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES

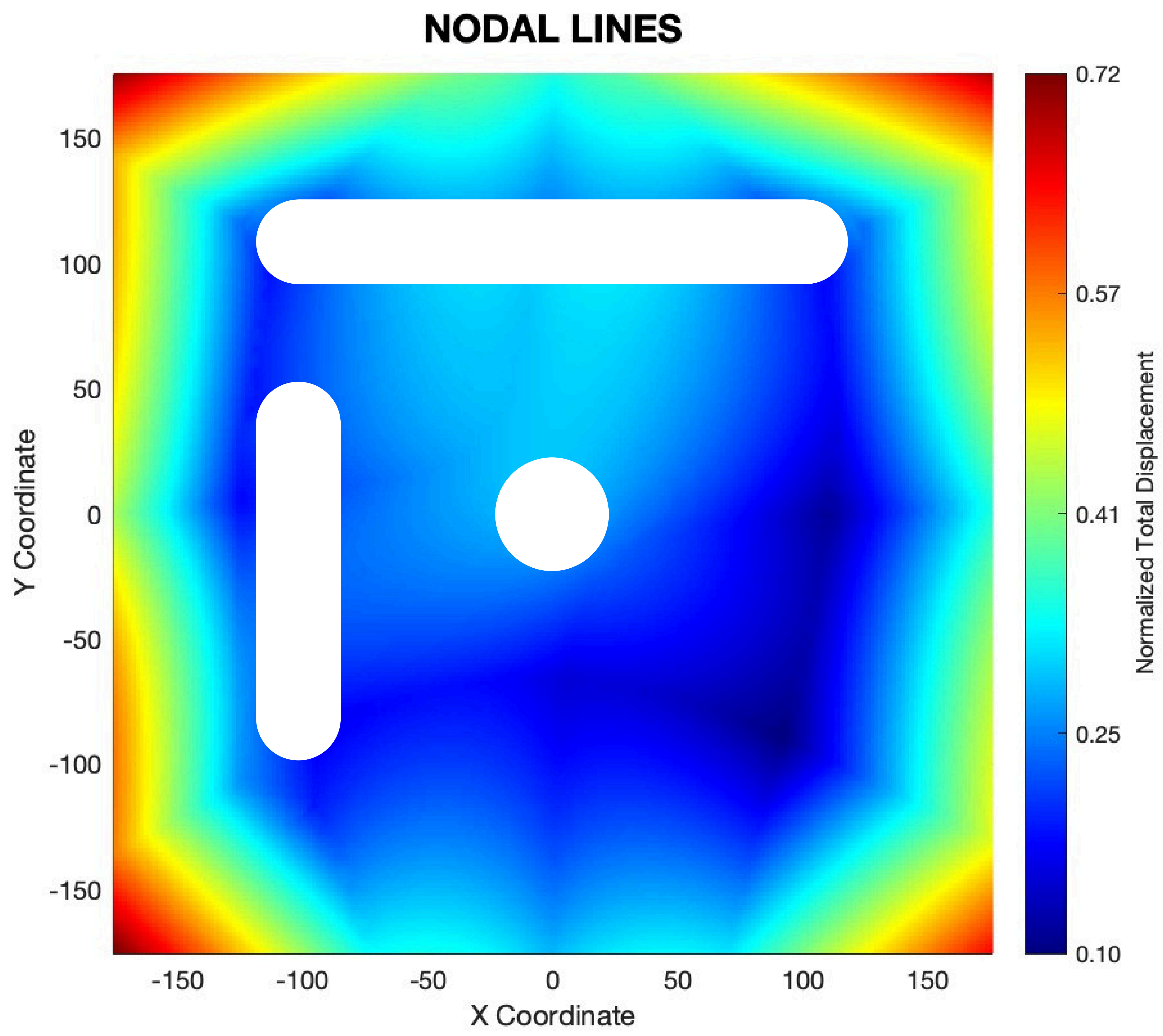
NODAL LINES



MODE 1+2

NODAL SURFACES DISTRIBUTION

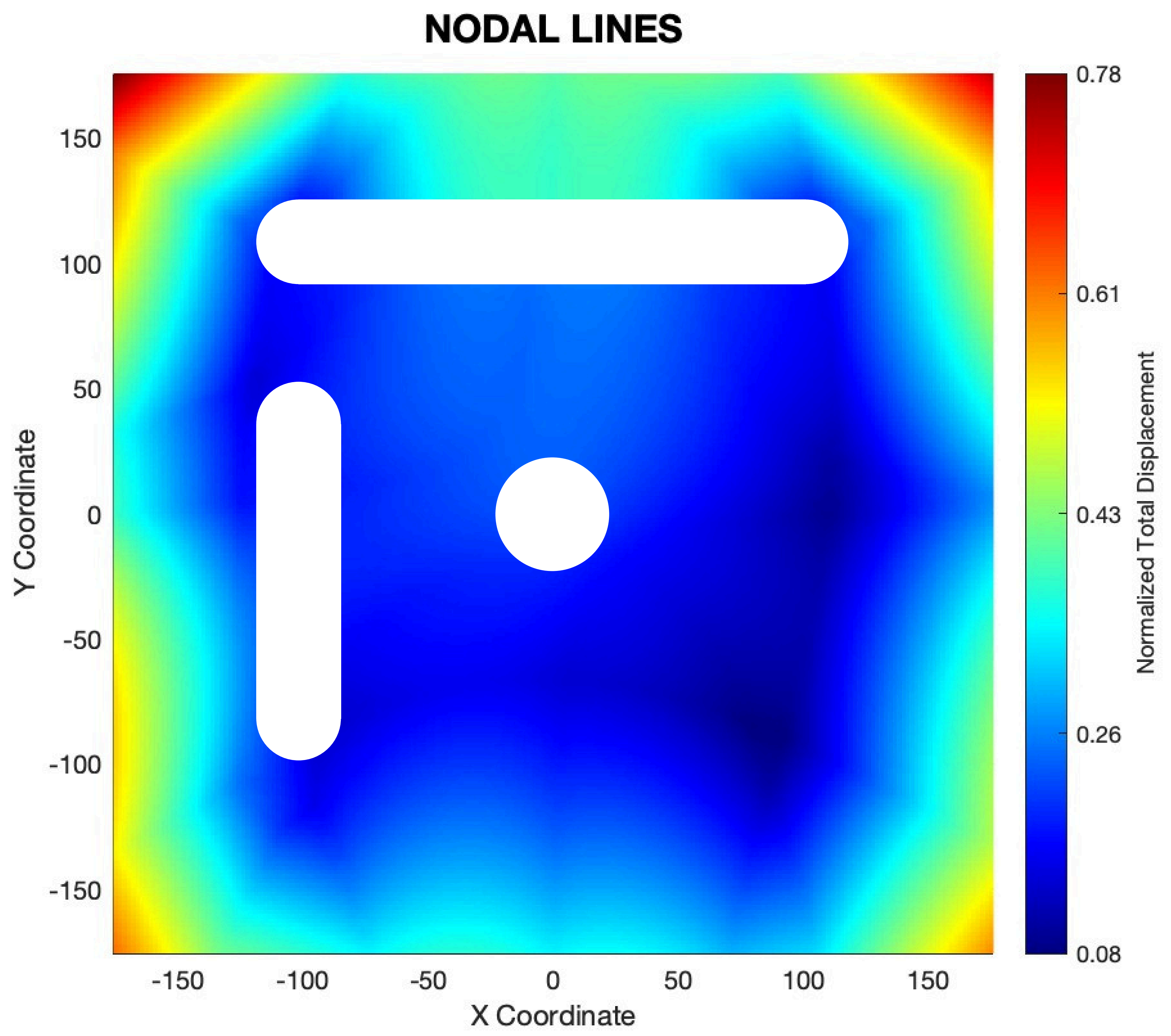
SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES



MODE 1+2+3

NODAL SURFACES DISTRIBUTION

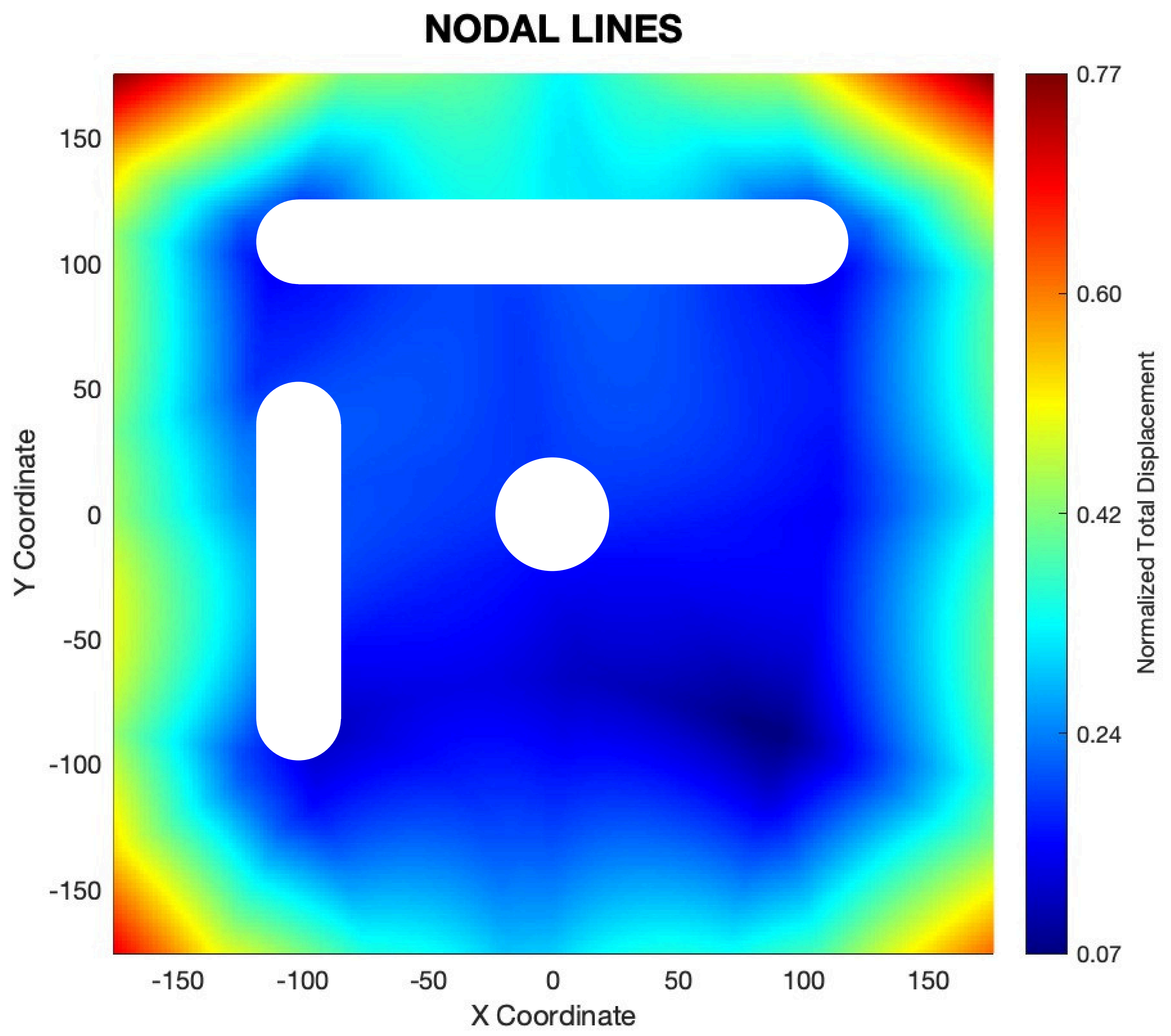
SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES



MODE 1+2+3+4

NODAL SURFACES DISTRIBUTION

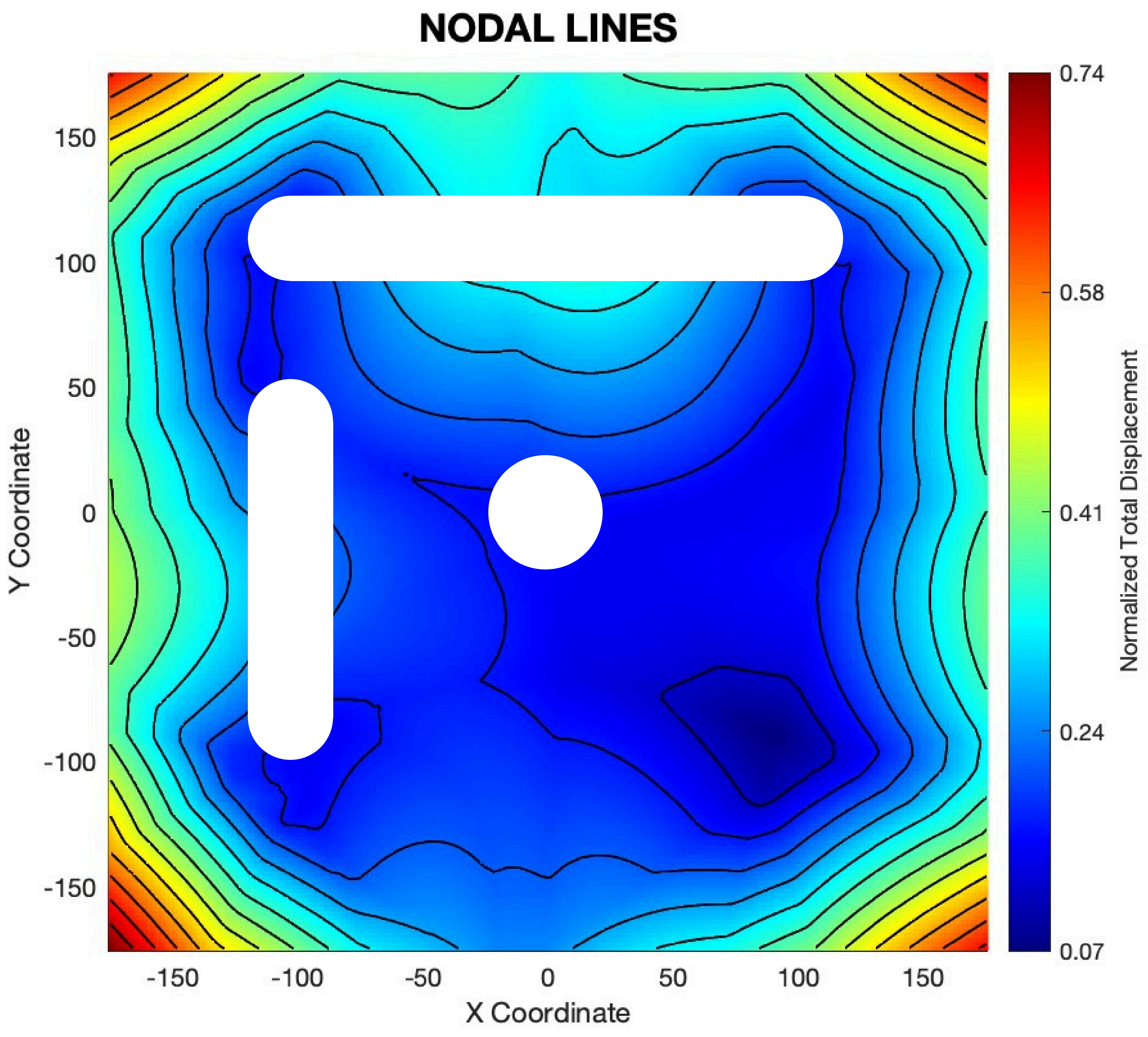
SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES



MODE 1+2+3+4+5

NODAL SURFACES DISTRIBUTION

SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES



ALL 6 MODES

NODAL SURFACES DISTRIBUTION

SUPERIMPOSITION OF THE NORMALIZED DISPLACEMENT OF THE DIFFERENT MODES

THEORETICAL DEFINITION

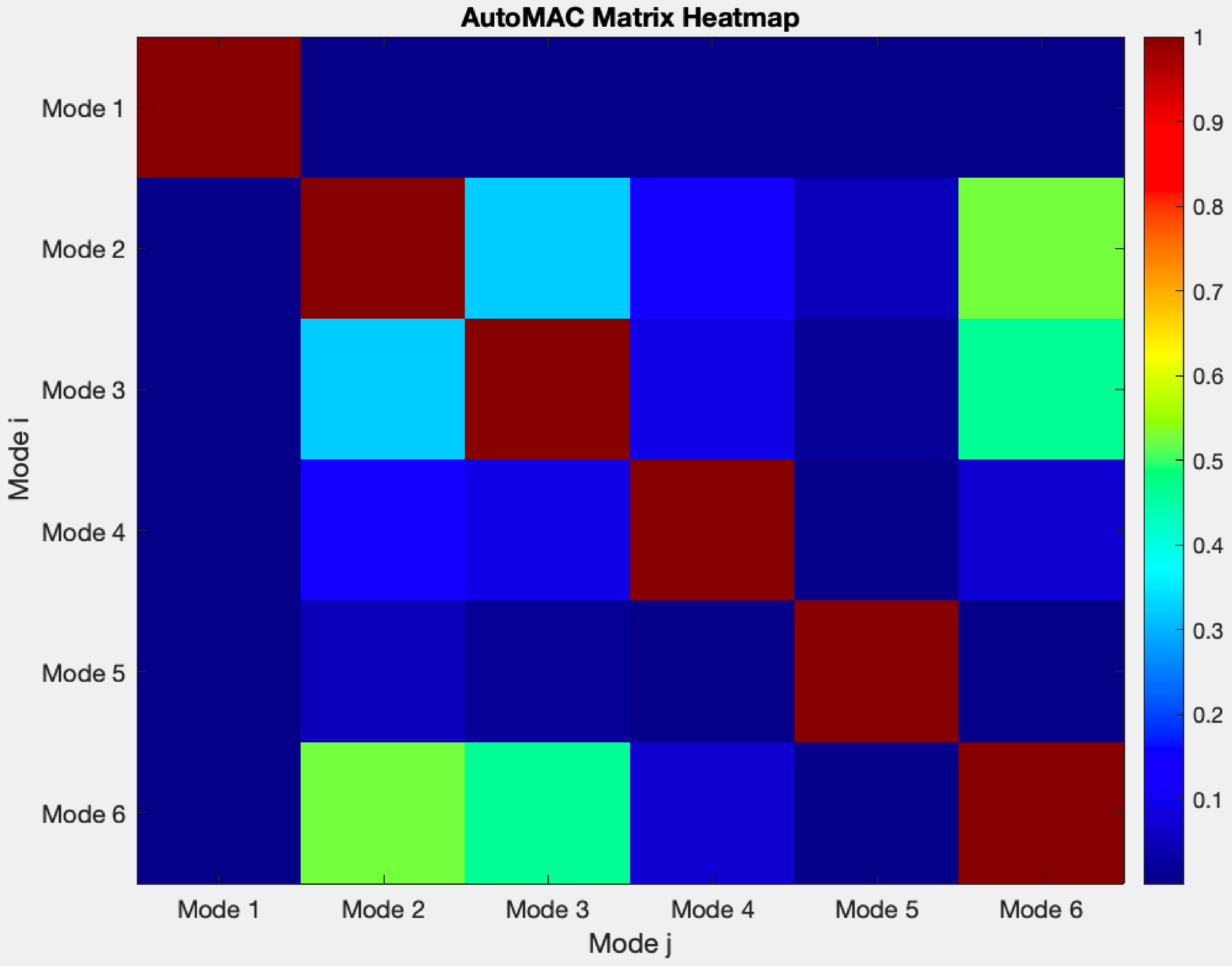
Used to find whether a certain measurement mesh is correct or produce spatial aliasing

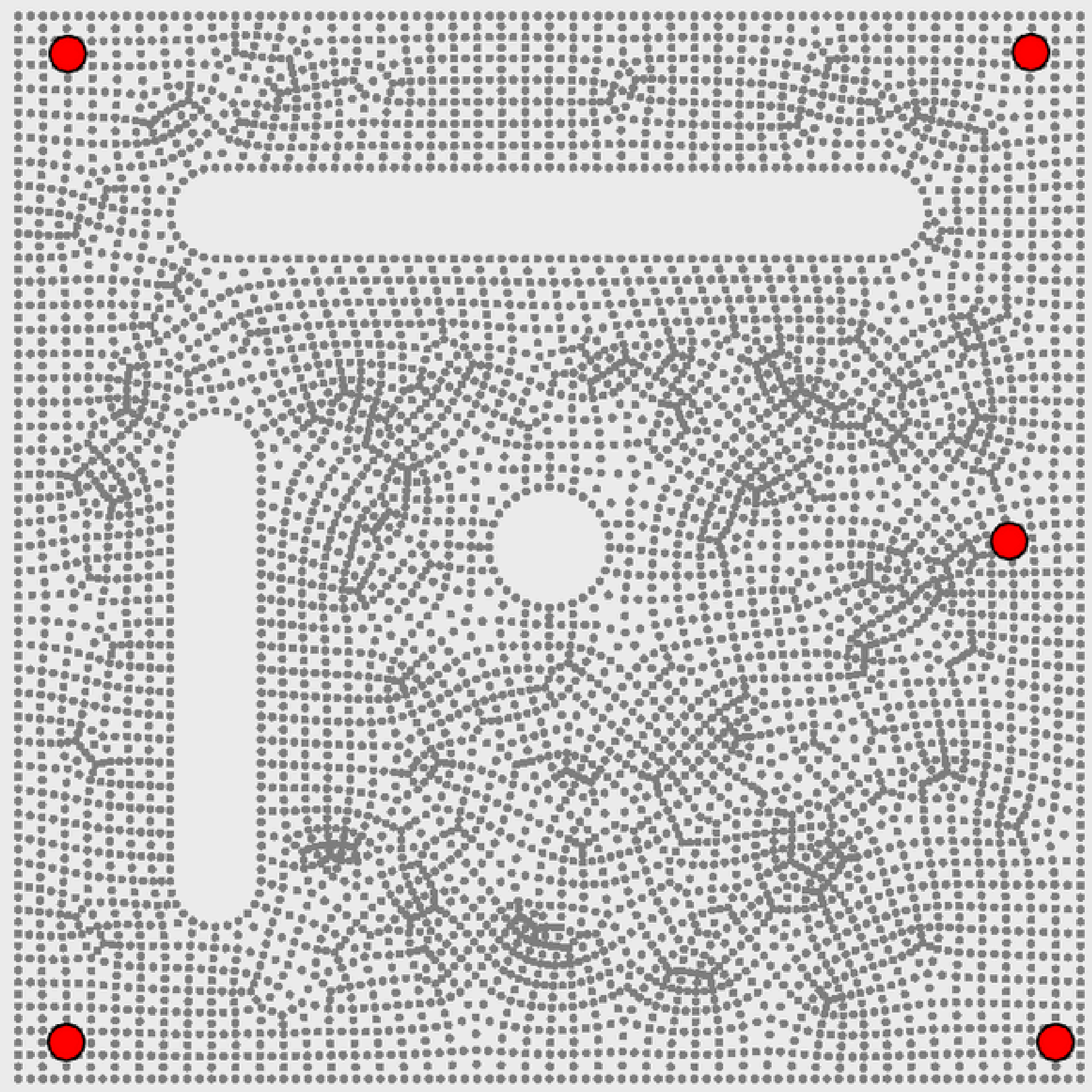
AutoMac

$$AutoMAC\left(A_i, A_j\right) = \frac{\left|\left\{\psi_{A,i}\right\}^T \left\{\psi_{A,j}\right\}\right|^2}{\left(\left\{\psi_{A,i}\right\}^T \left\{\psi_{A,i}\right\}\right) \cdot \left(\left\{\psi_{A,j}\right\}^T \left\{\psi_{A,j}\right\}\right)}$$

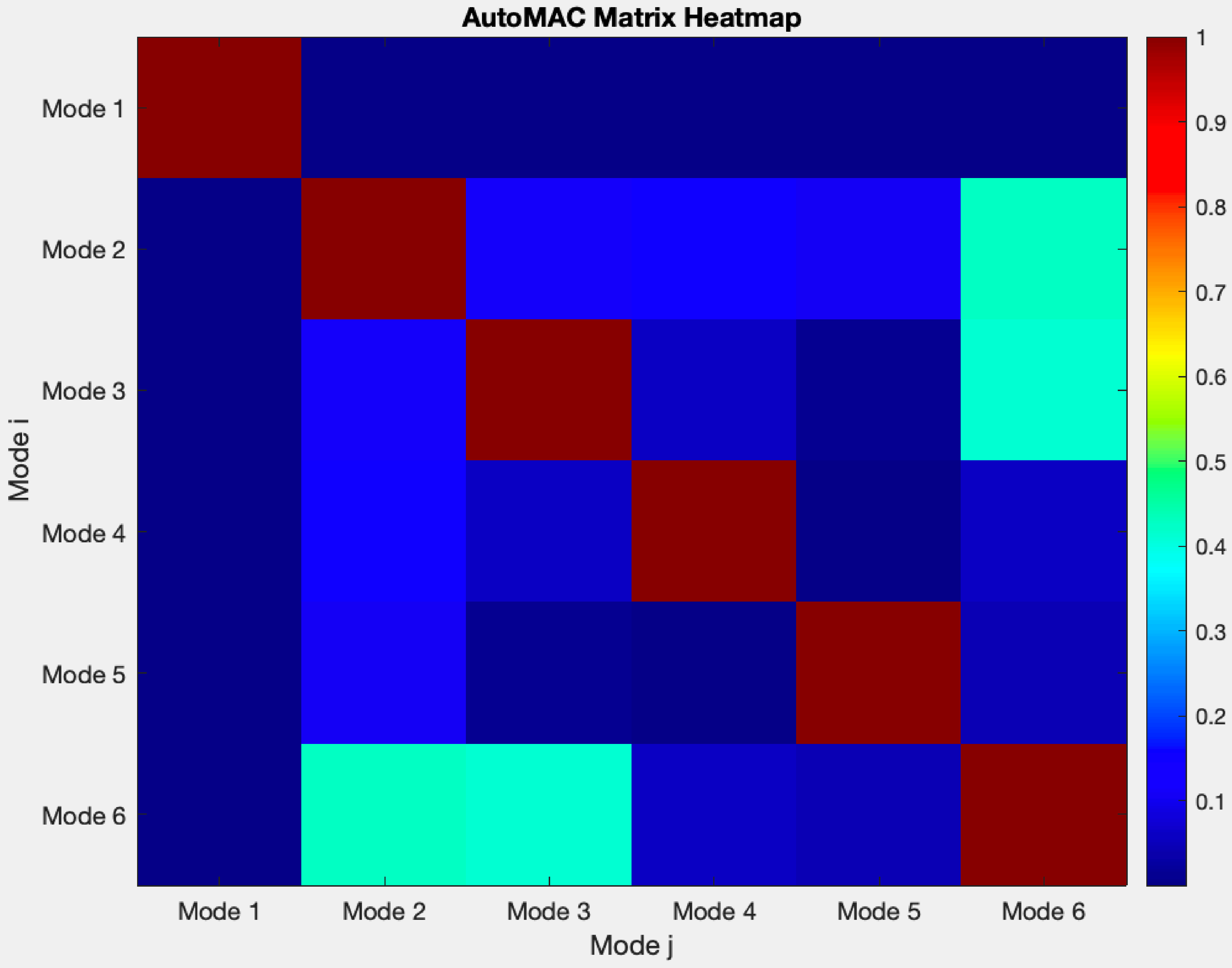
modes computed numerically

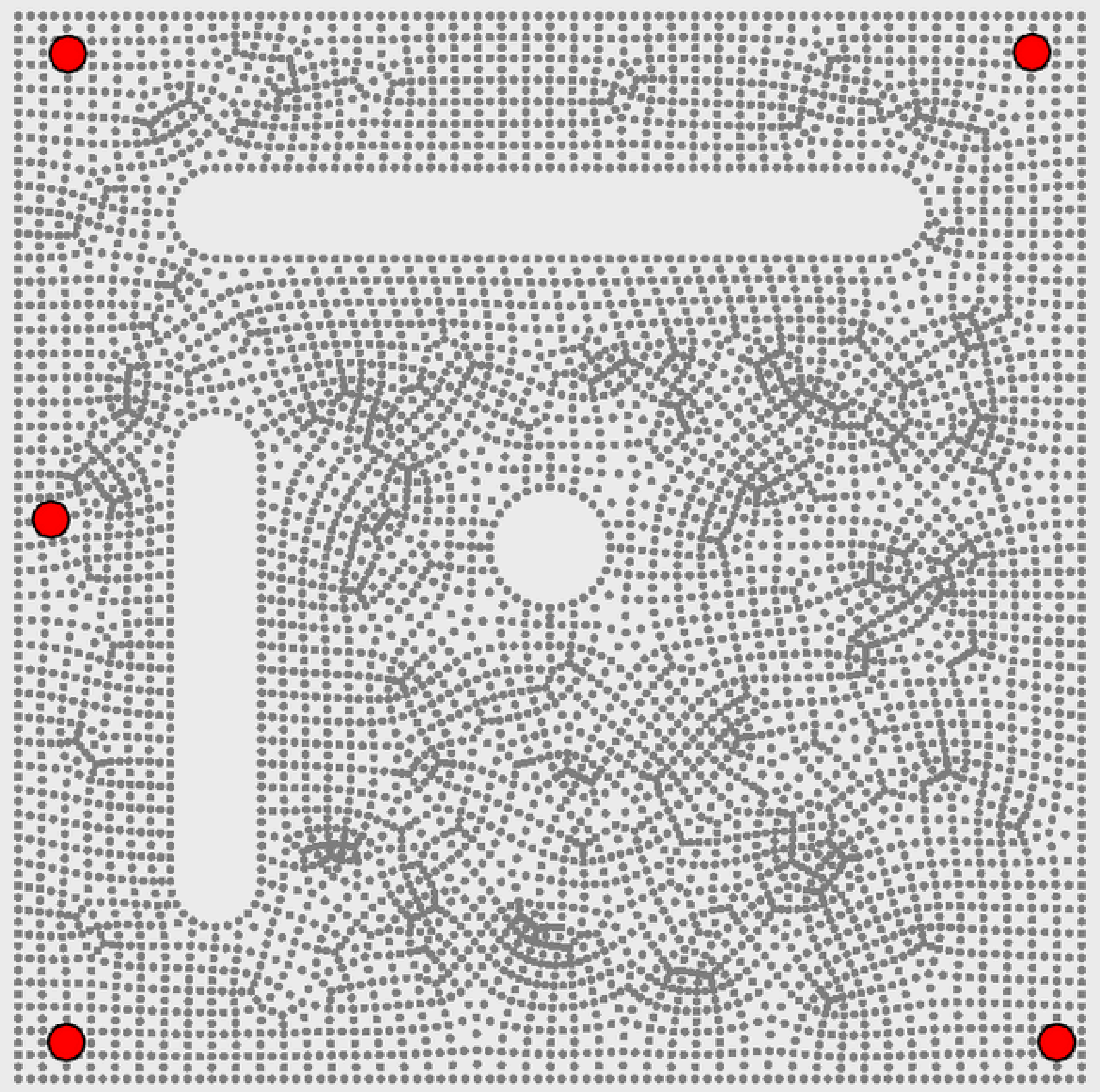
CONFIGURATION 1



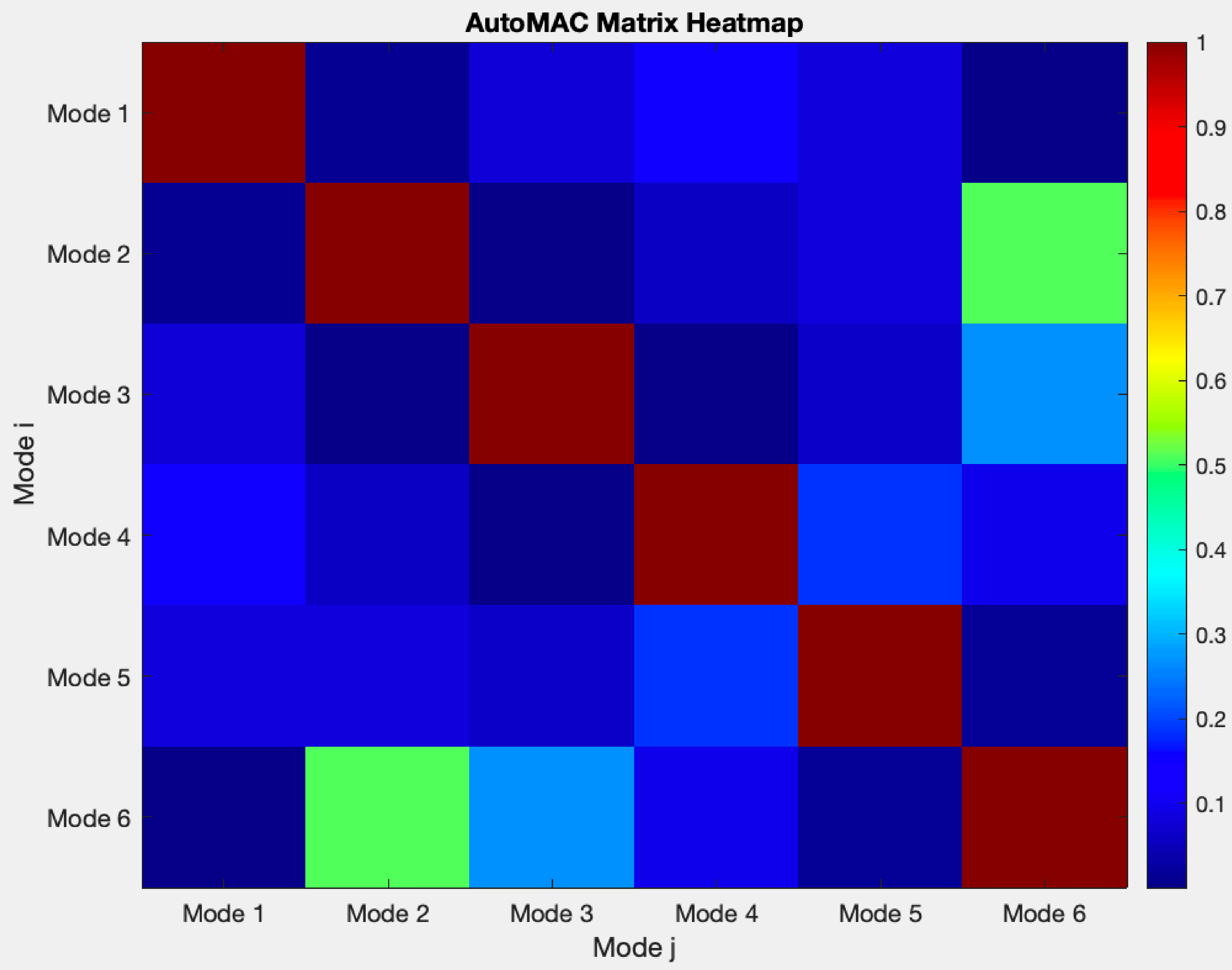


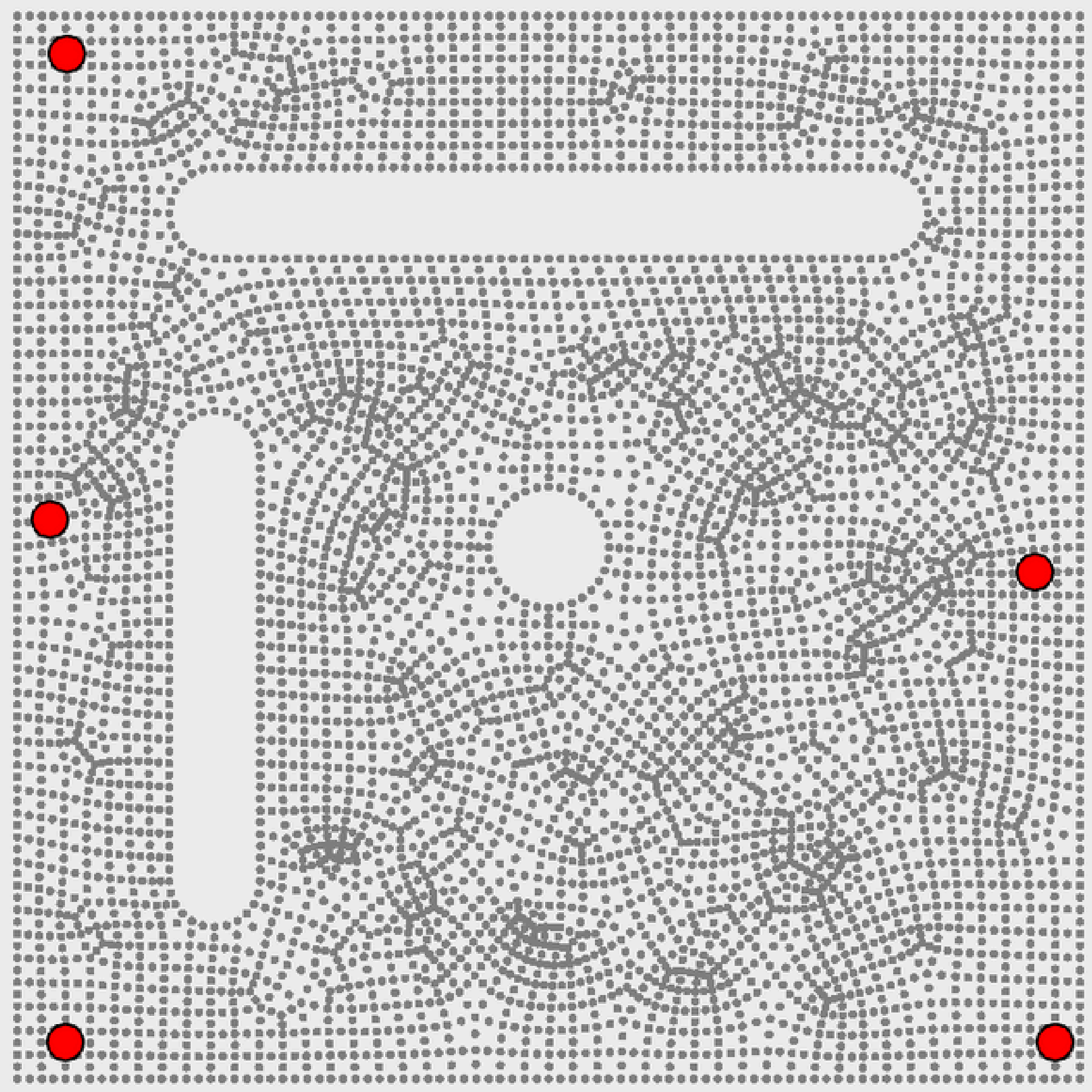
CONFIGURATION 2





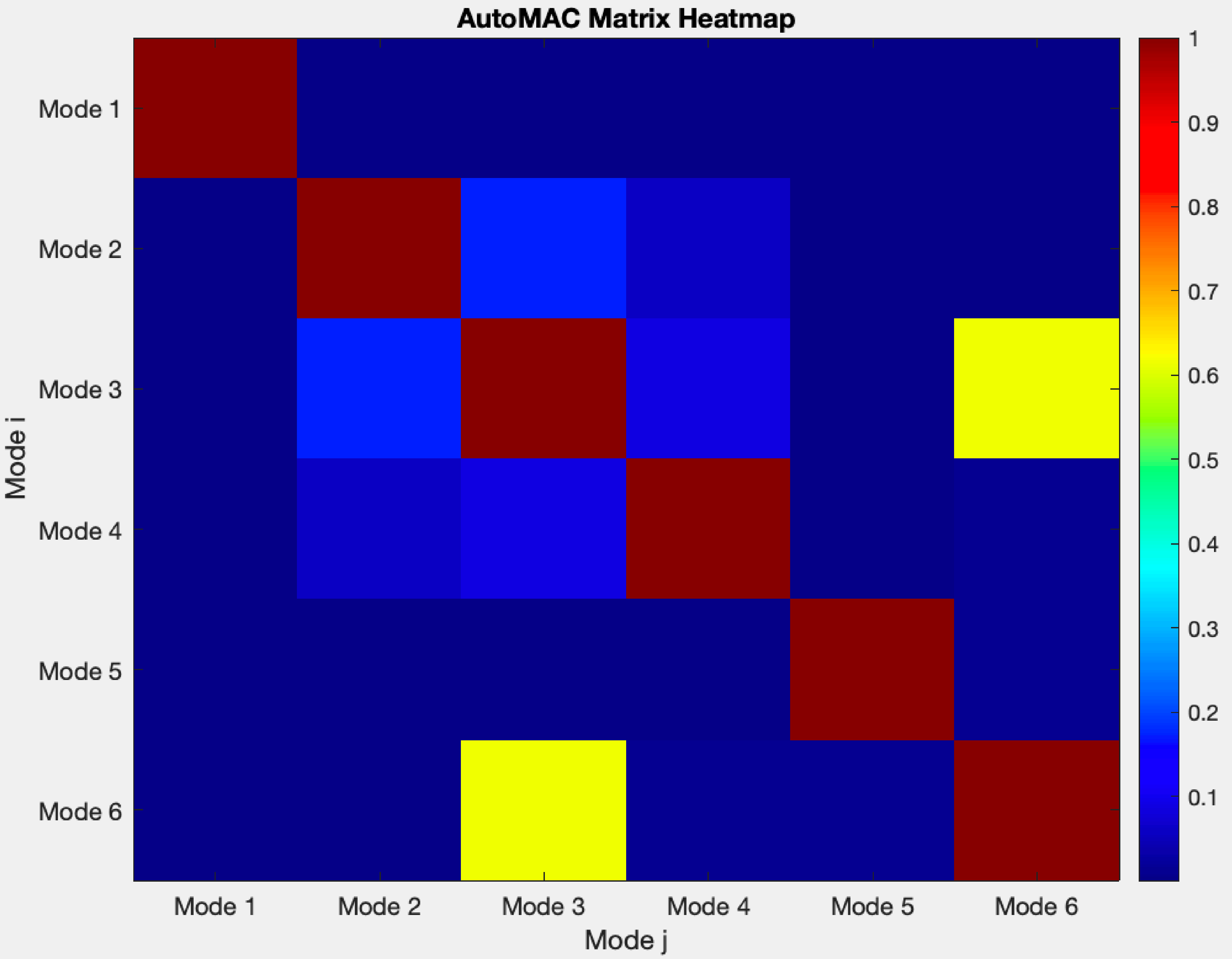
CONFIGURATION 3





CONFIGURATION 4

The chosen configuration



AUTOMAC

Configuration 4

Node 11299

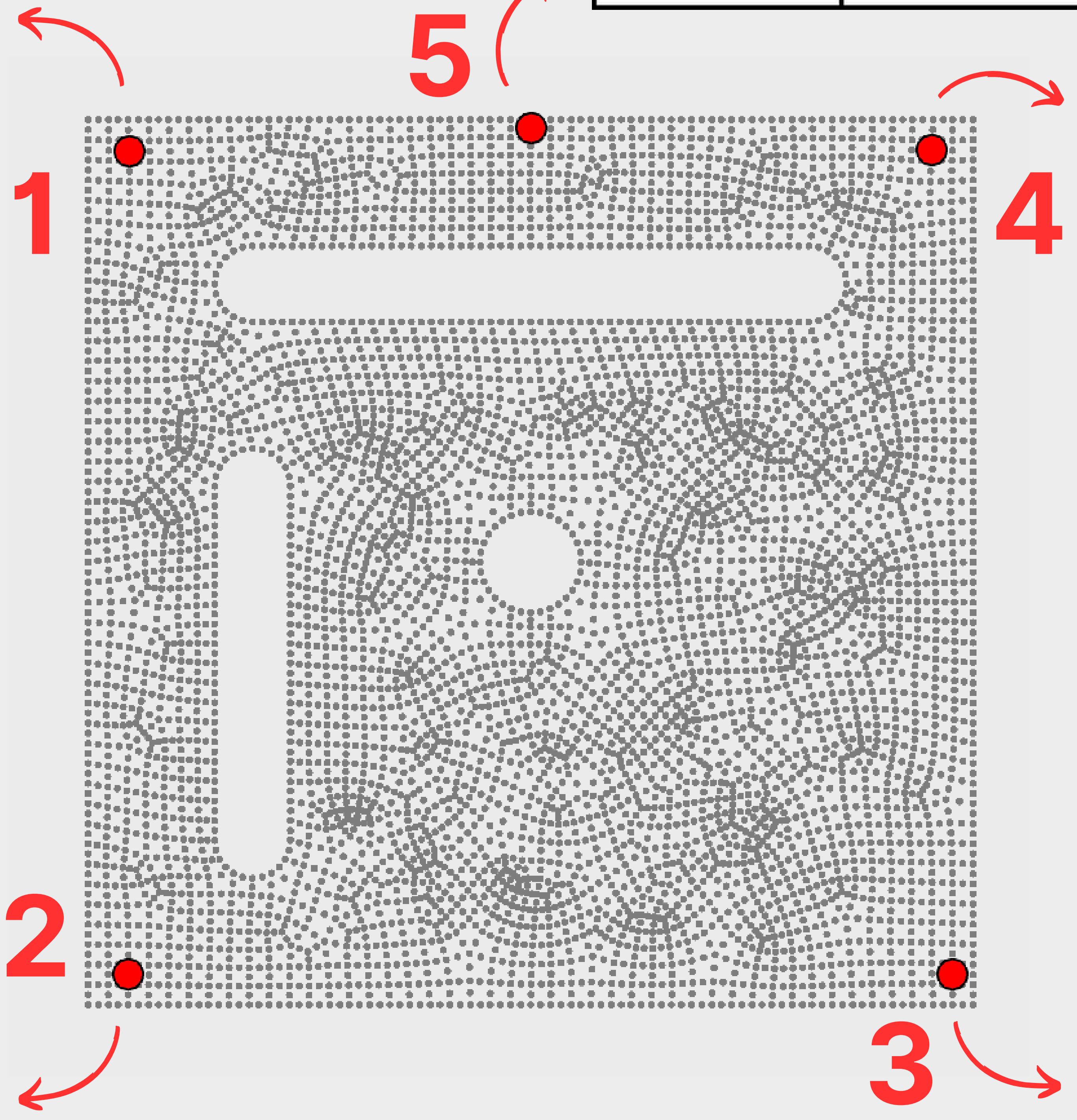
Mode	Successful acquisition index
1st mode	0%
2nd mode	96%
3rd mode	5%
4th mode	58%
5th mode	6%
6th mode	33%

Node 12110

Mode	Successful acquisition index
1st mode	87%
2nd mode	25%
3rd mode	66%
4th mode	77%
5th mode	57%
6th mode	7%

Node 10814

Mode	Successful acquisition index
1st mode	80%
2nd mode	27%
3rd mode	67%
4th mode	65%
5th mode	77%
6th mode	9%



Node 11075

Mode	Successful acquisition index
1st mode	80%
2nd mode	24%
3rd mode	80%
4th mode	21%
5th mode	80%
6th mode	75%

Node 7884

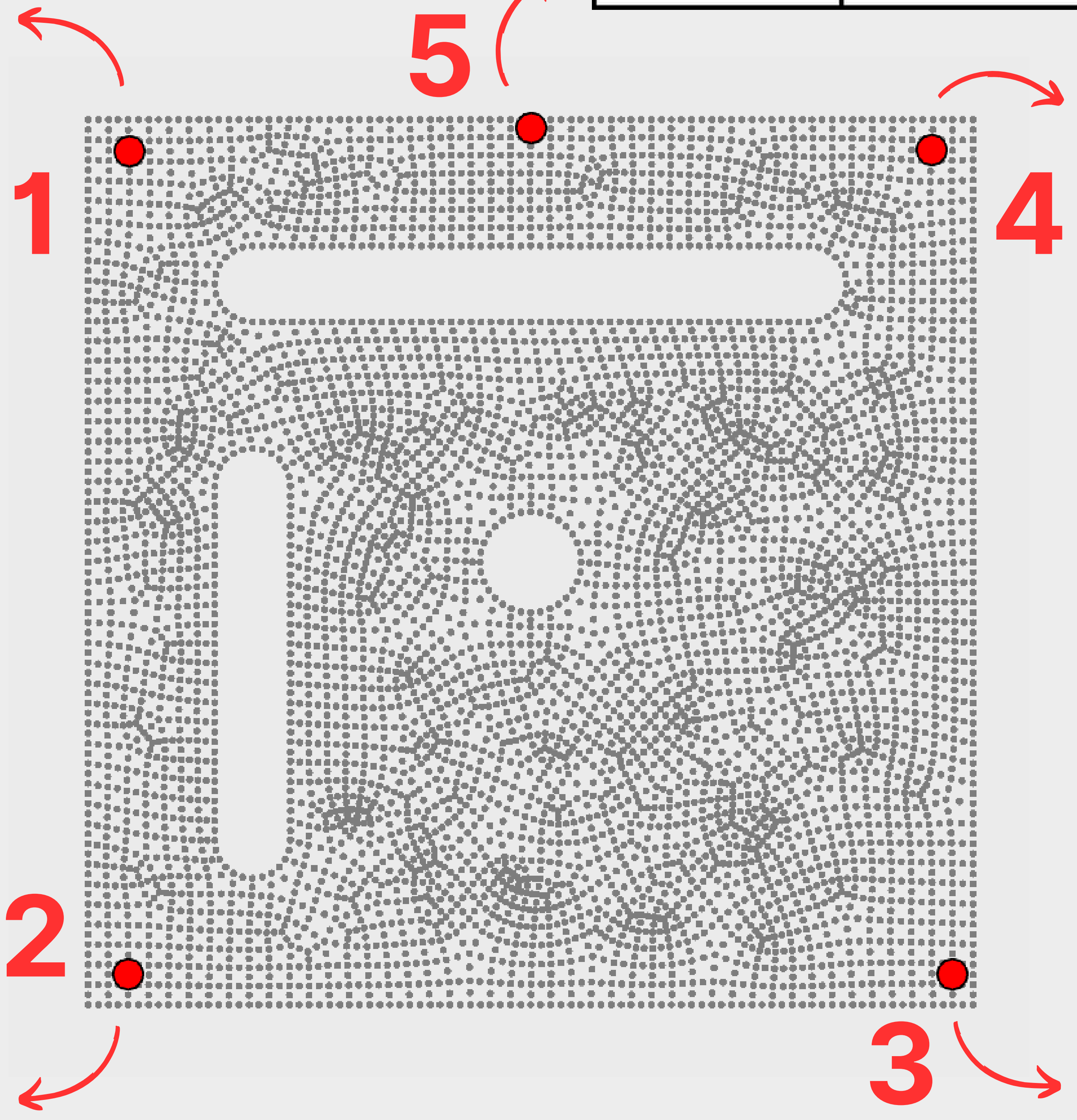
Mode	Successful acquisition index
1st mode	79%
2nd mode	21%
3rd mode	70%
4th mode	40%
5th mode	55%
6th mode	74%

Node 12110

Mode	Successful acquisition index
1st mode	87%
2nd mode	25%
3rd mode	66%
4th mode	77%
5th mode	57%
6th mode	7%

Node 11075

Mode	Successful acquisition index
1st mode	80%
2nd mode	24%
3rd mode	80%
4th mode	21%
5th mode	80%
6th mode	75%



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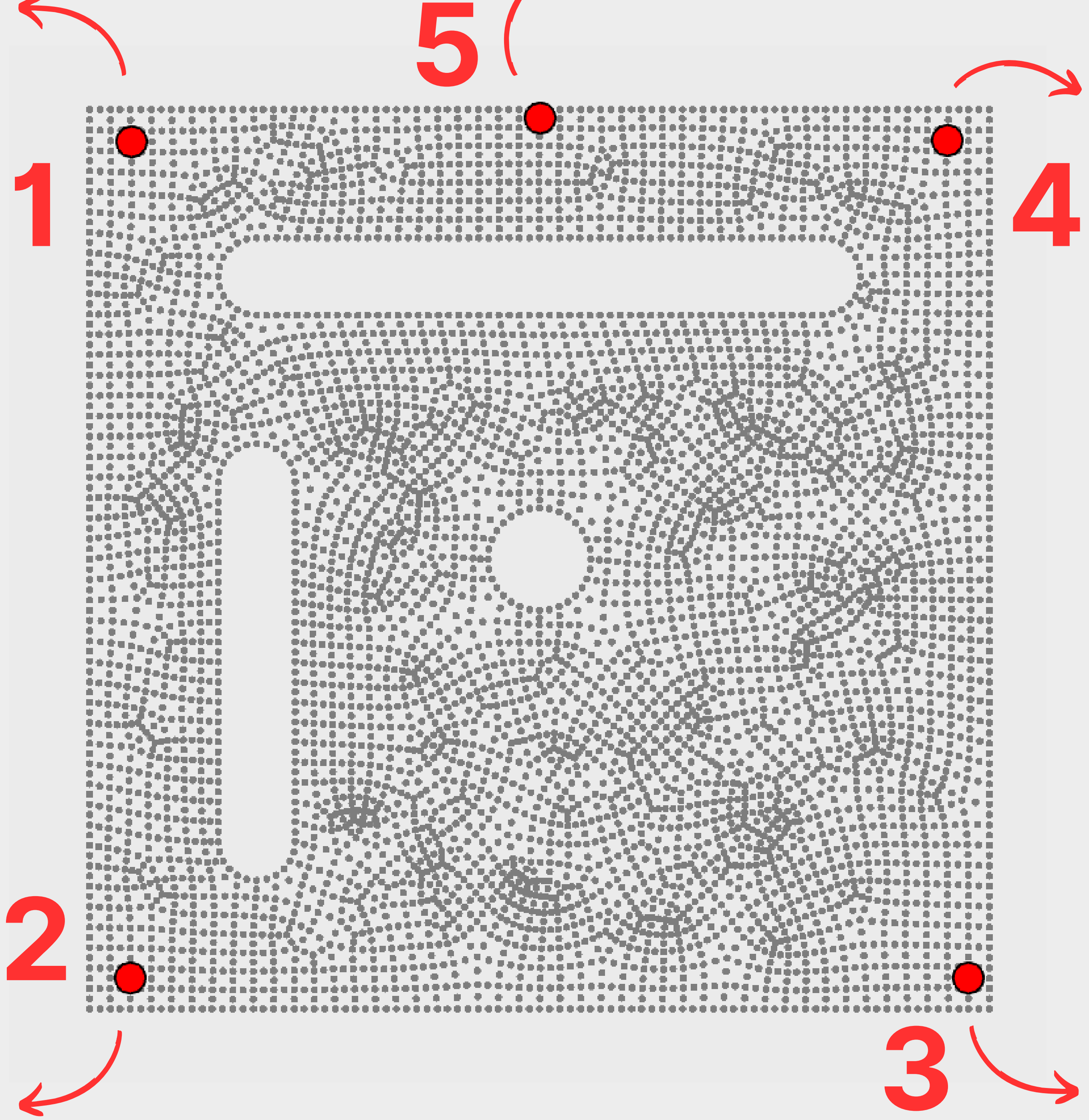
4th mode	77%
5th mode	57%
6th mode	7%

Node 11075

Mode	Successful acquisition index
1st mode	80%
2nd mode	24%
3rd mode	80%
4th mode	21%
5th mode	80%
6th mode	75%



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Node 7884

Mode	Successful acquisition index
1st mode	79%
2nd mode	21%
3rd mode	70%
4th mode	40%
5th mode	55%
6th mode	74%

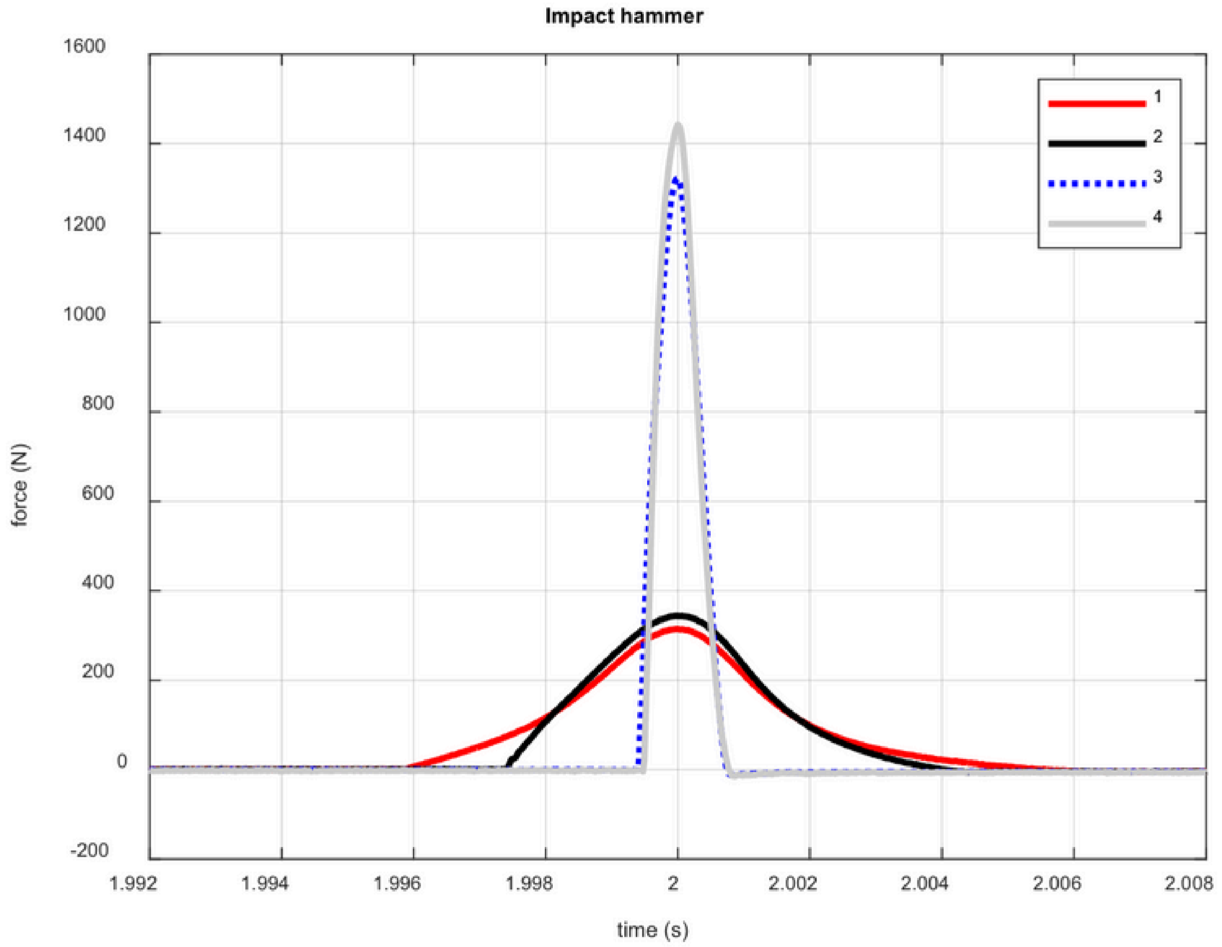
EXPERIMENTAL TESTS

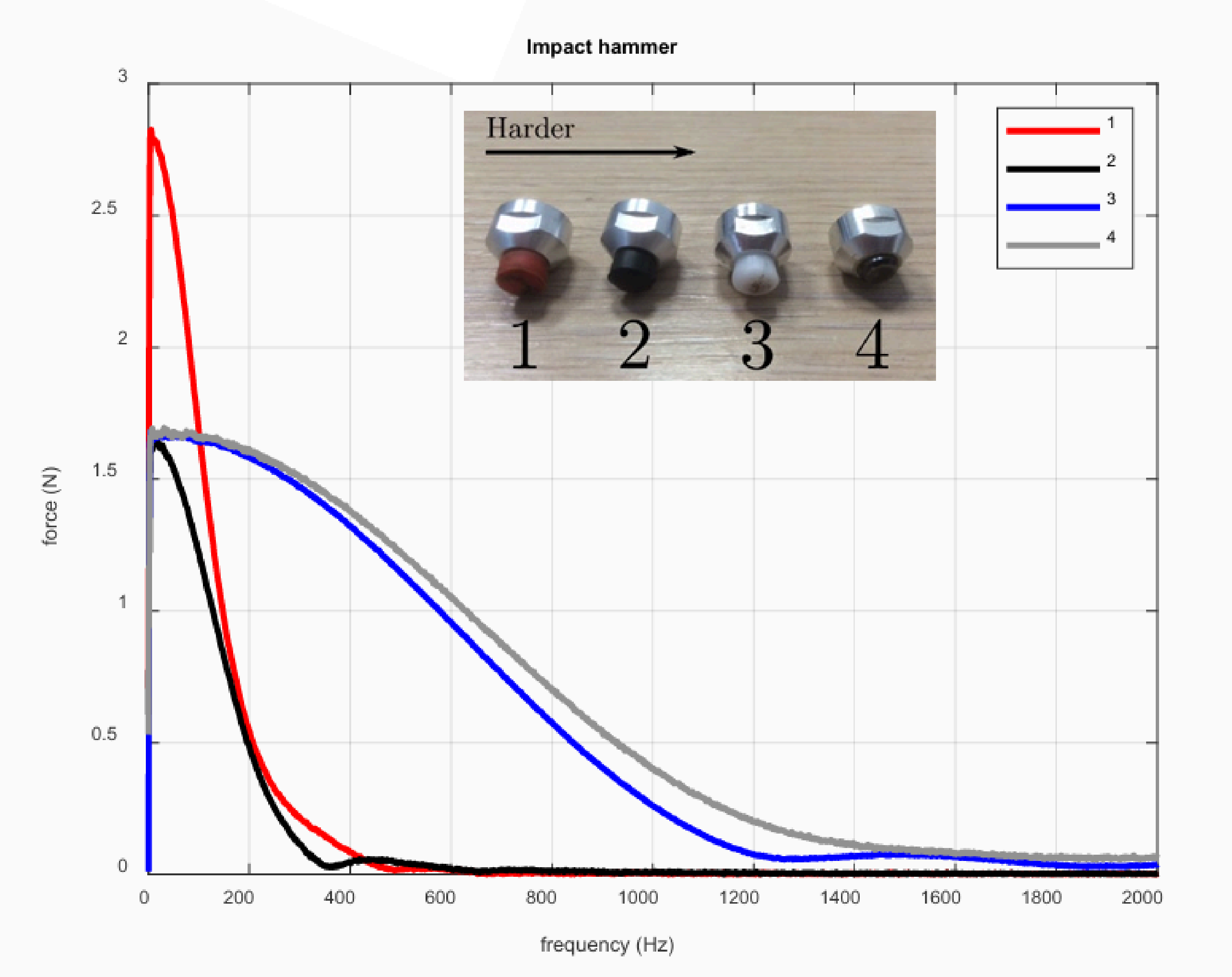
INPUT FORCE, EXCITER

Type:

Hammer (non attached exciter)

Sensitivity: 2.488 mV/N





Advantages:

- fastest approach
- no load effect
- cheap

Drawbacks

- input force control
- spectrum control

RESPONSE MEASUREMENT

Type:

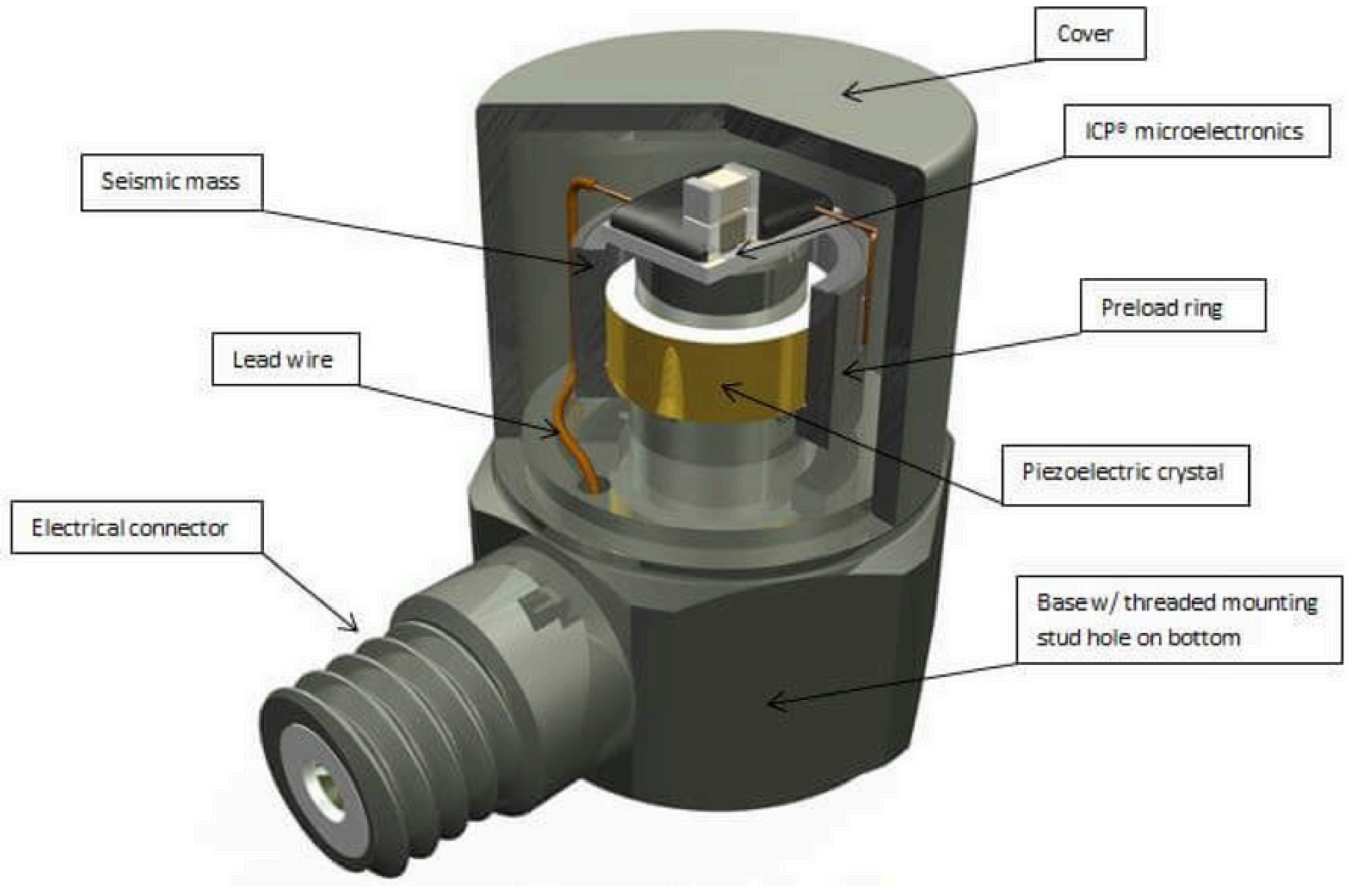
Piezoelectric accelerometer
Sensitivity: 10.2 mV/(m/s^2)
Frequency range: 0.5 - 3000 Hz

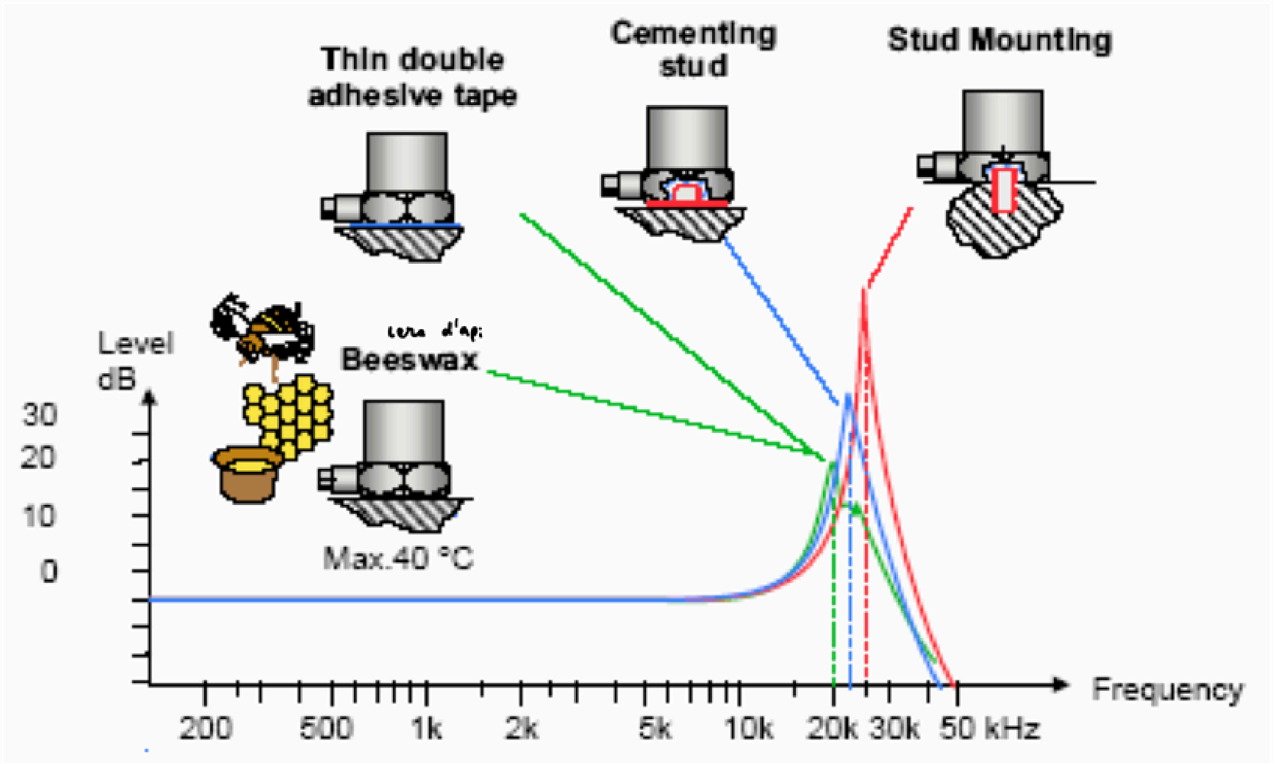
Mounting:

Glue

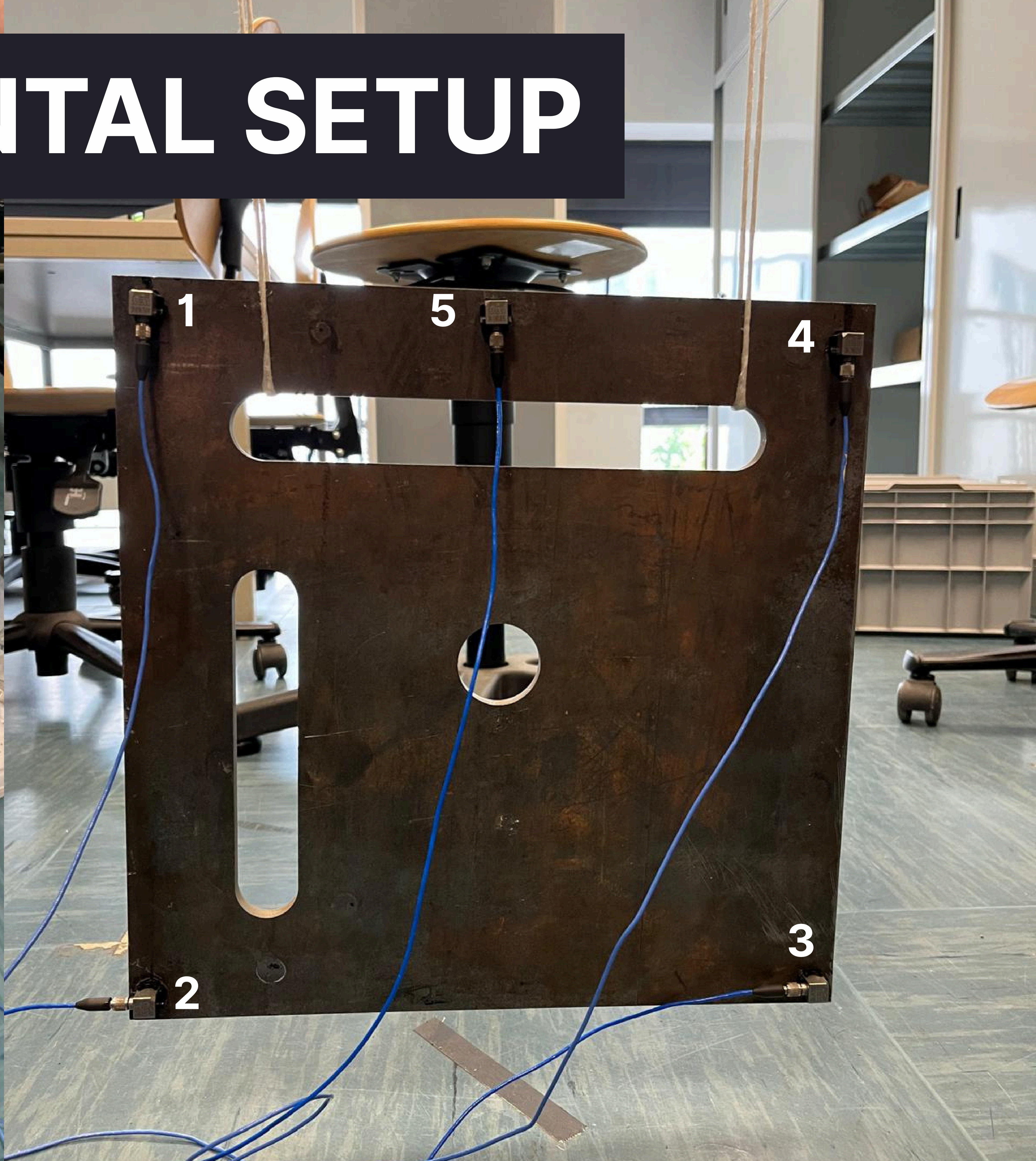
Advantages:

- not expensive
- absolute measurements





EXPERIMENTAL SETUP



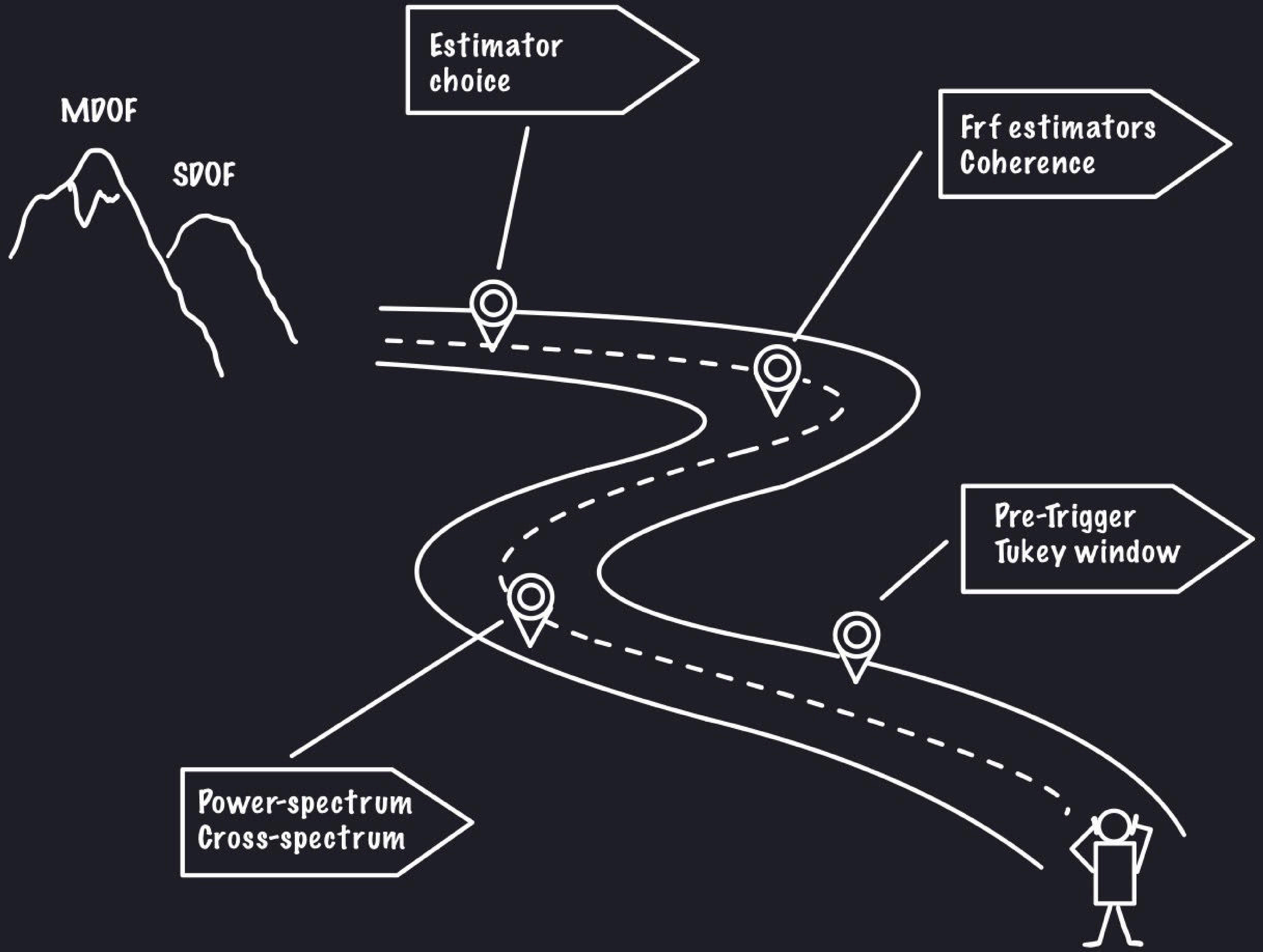
MAIN CHALLENGES DURING ACQUISITION

PROBLEM		SOLUTION
ALIASING	→	SAMPLING FREQUENCY: 2500 HZ
ECCITATION FREQUENCY RANGE	→	PROPER CHOICE OF THE HAMMER TIP: PLASTIC, NUMBER 3
HAMMER DOUBLE HITS CO-LOCATED HITS SATURATION OF ACCELEROMETERS	→	EXPERIENCE AND PATIENCE

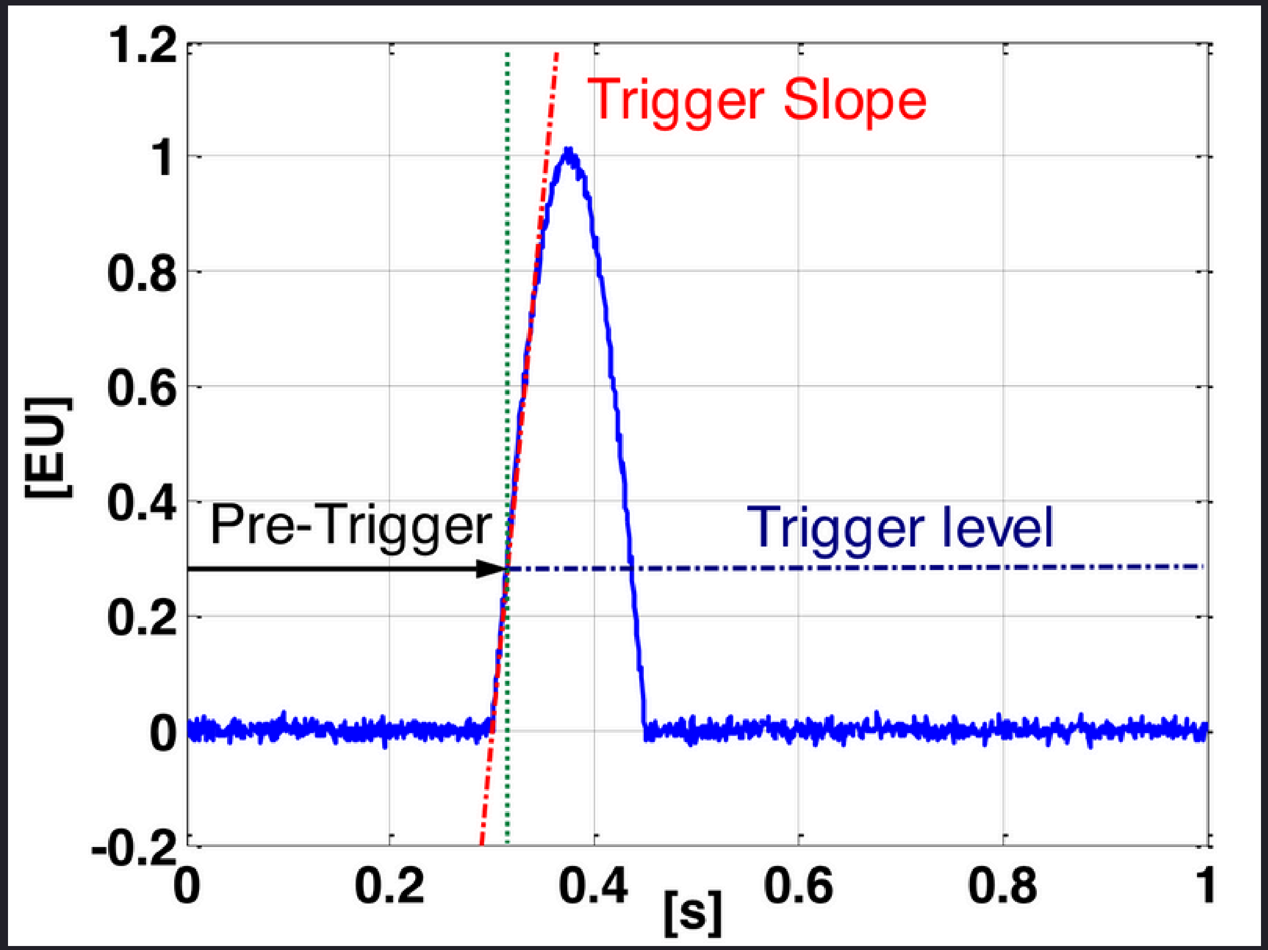
EXPERIMENTAL CAMPAIGN

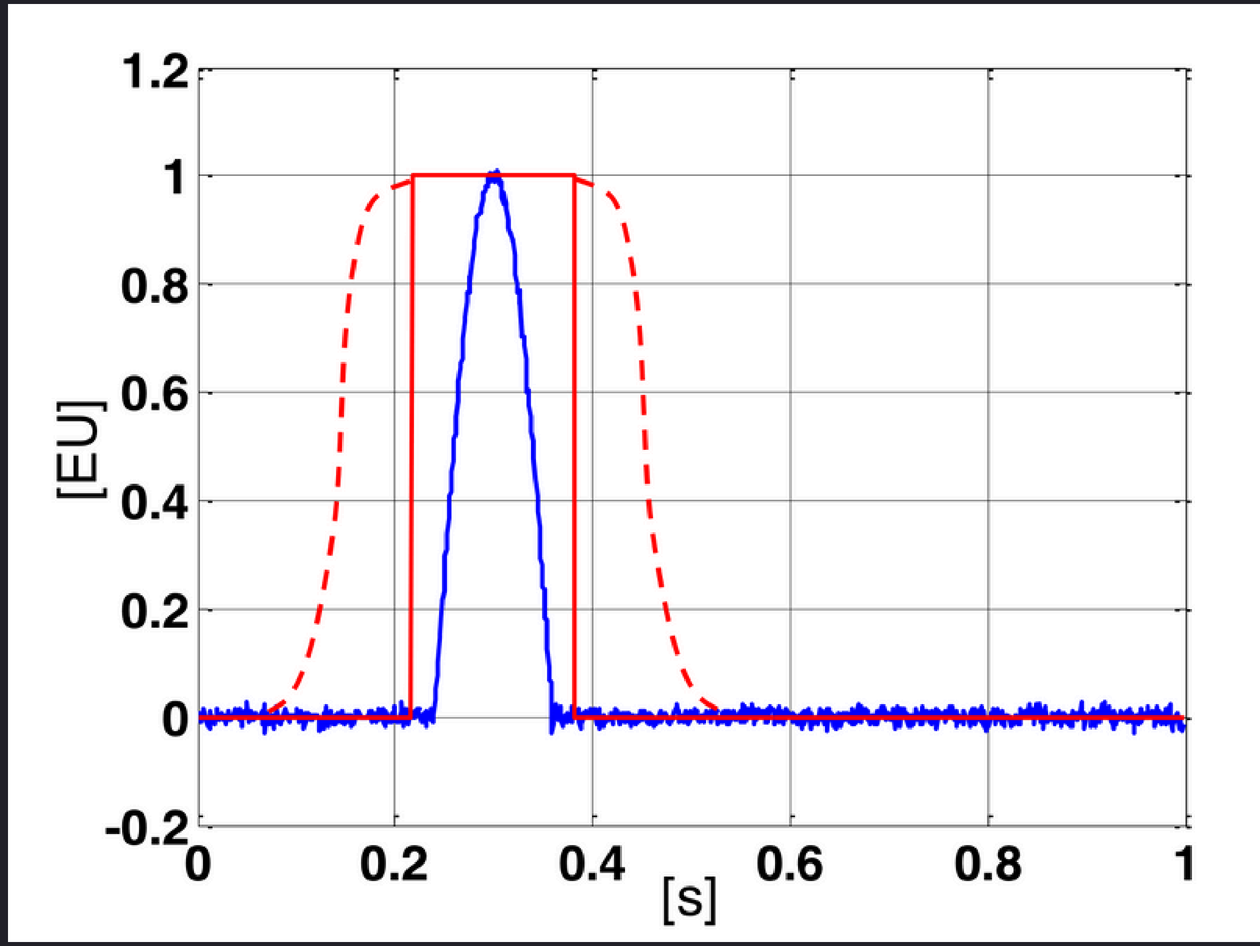
- Looked for the decay time, to decide about exponential window and acquisition time → T = 25 s
- 10 impacts for each sensor position, needed for the averages

DATA PREPROCESSING



MDOF
SDOF
Estimator choice
Frf estimators Coherence
Pre-Trigger Tukey window
Power-spectrum Cross-spectrum



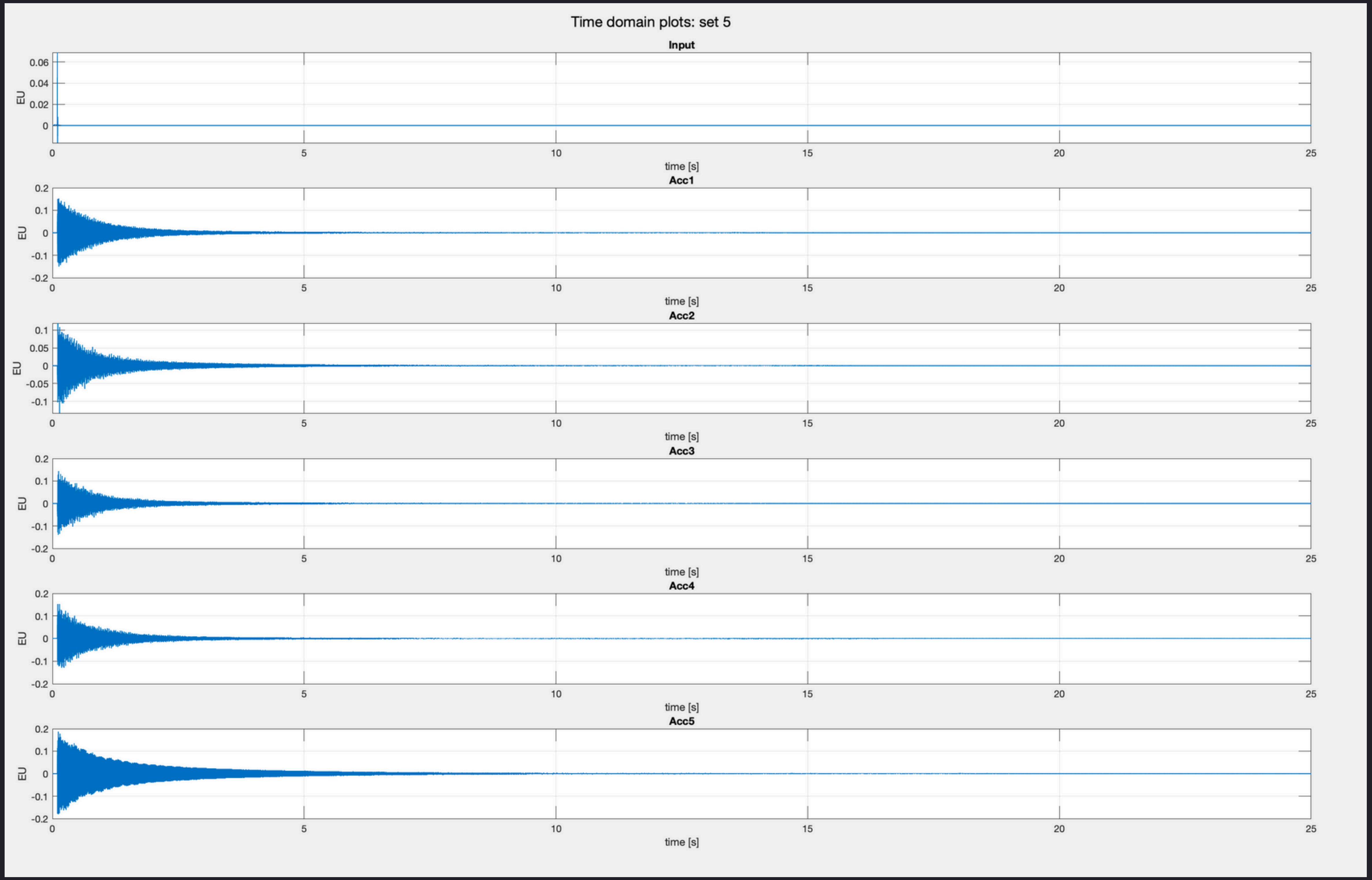


Pre Trigger: 0.1 s

STEP 01

SENSITIVITY - PRE TRIGGER - TUKEY WINDOW

Tukey Window

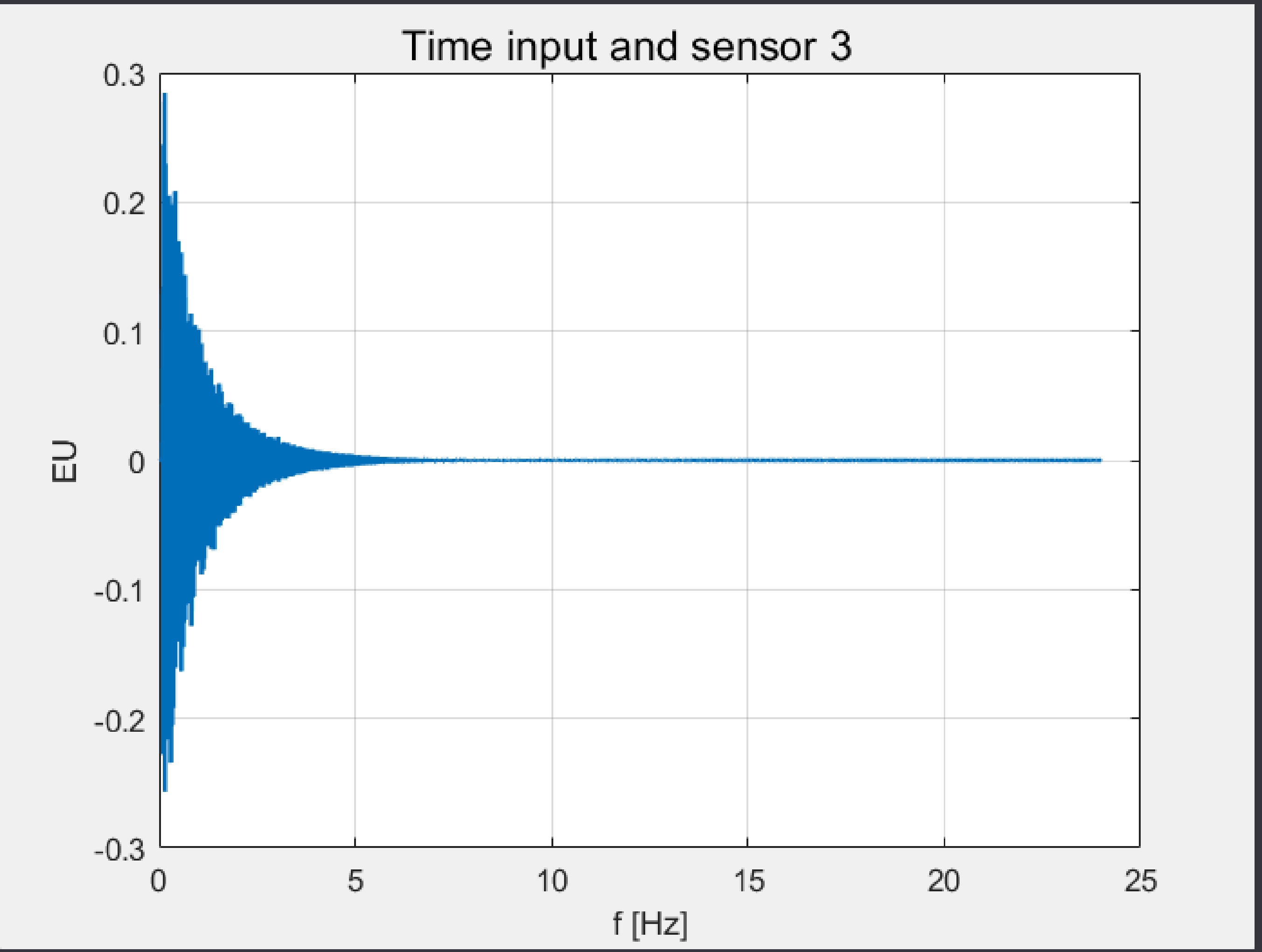


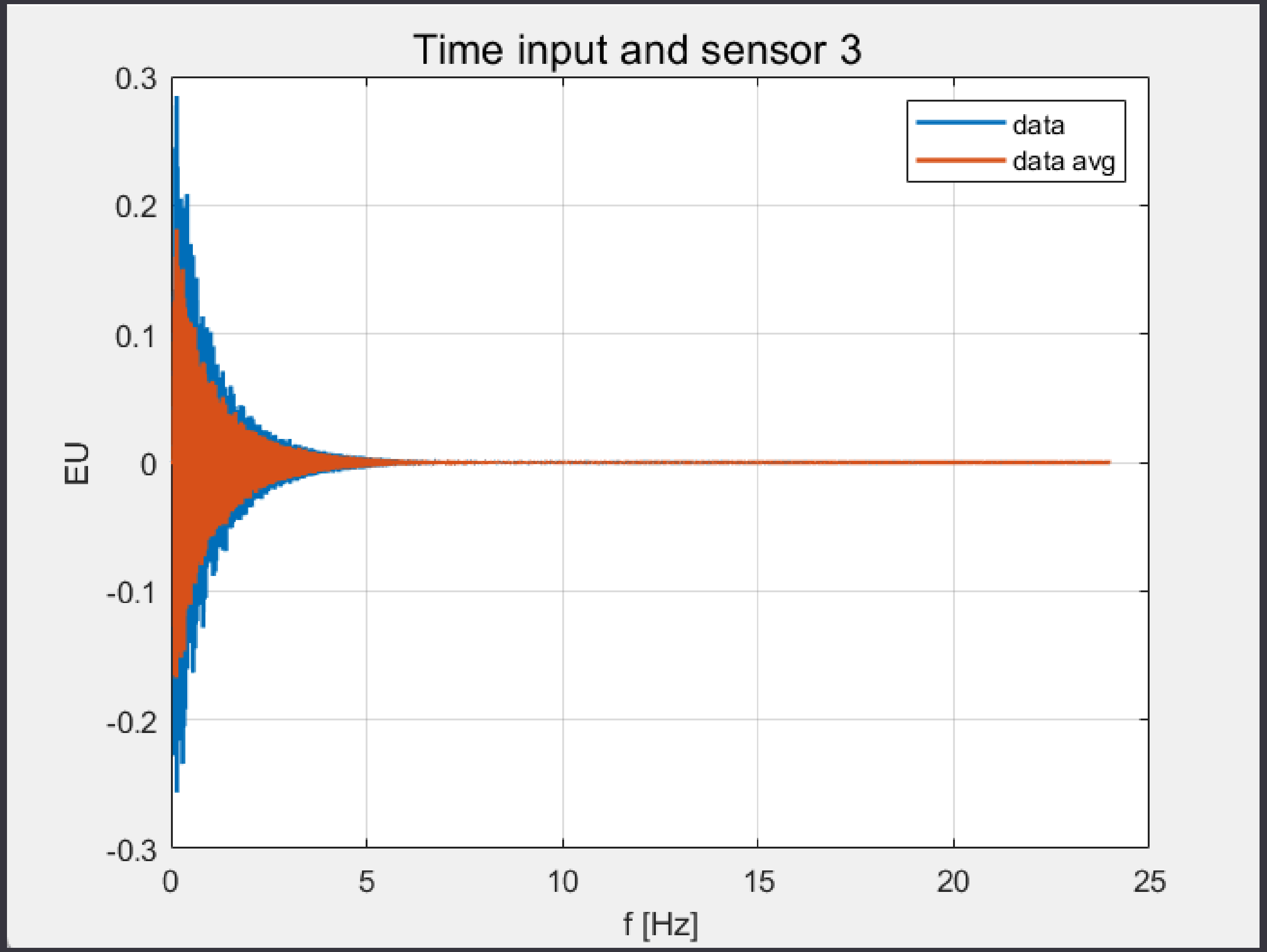
STEP 01

- Outputs from the sensors
- 25 sec time acquisition

TIME AVERAGING

10 averages for each test to reduce noise

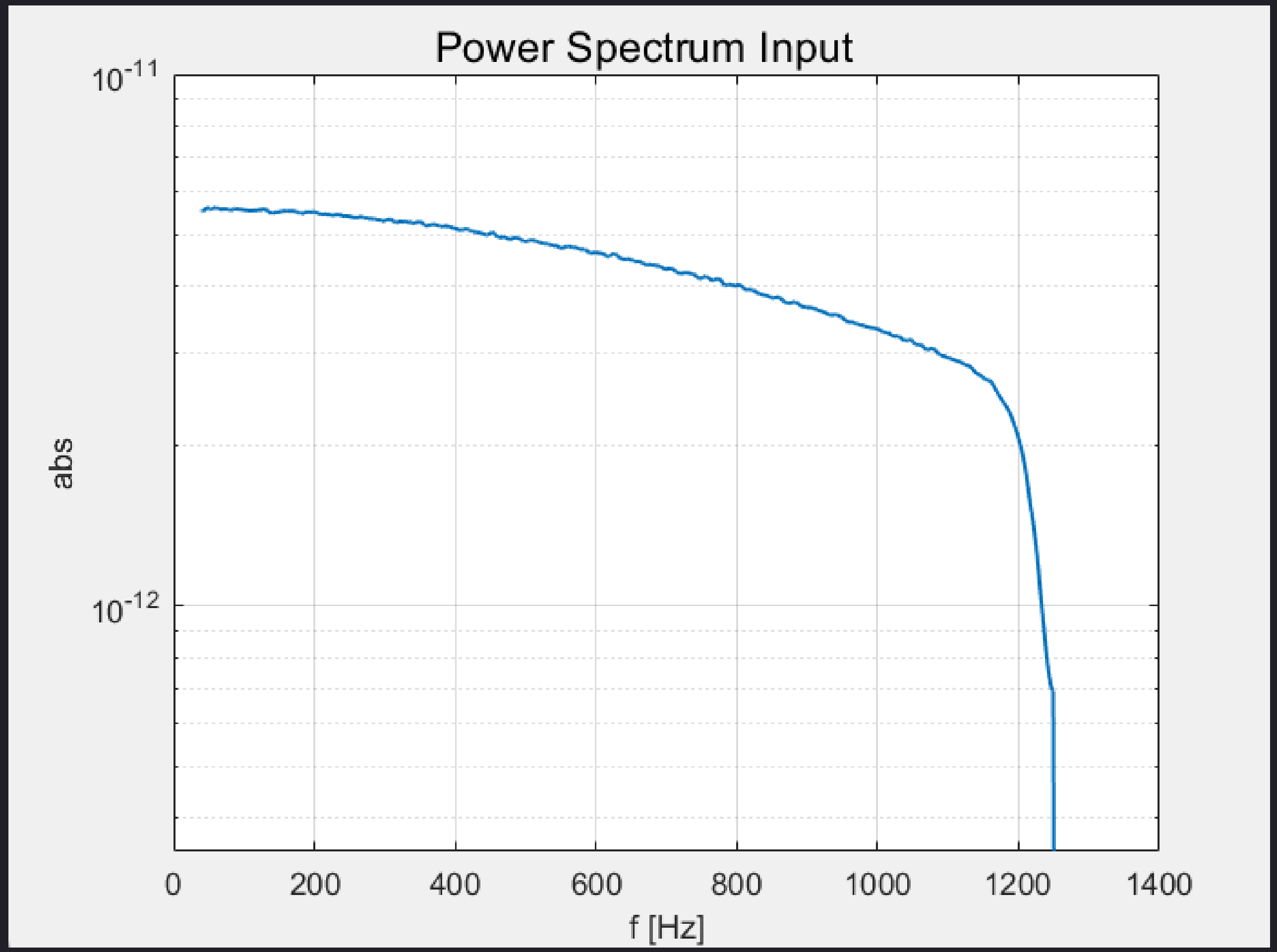




STEP 02

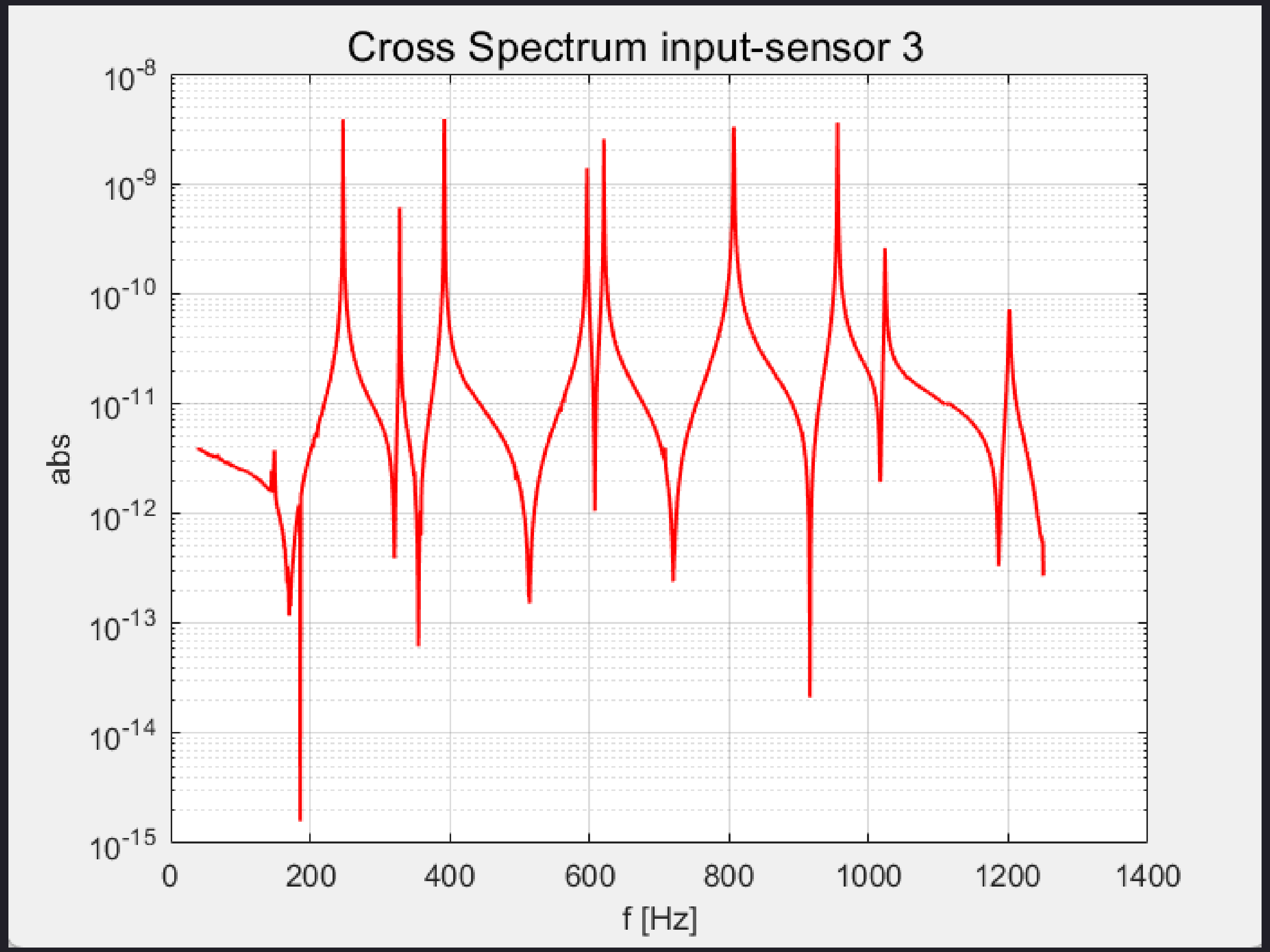
Power-spectrum

$$S_{AA}(f) = A^*(f)\,A(f)$$



Cross-spectrum

$$S_{AB}(f) = A^*(f)\,B(f)$$



STEP 03

FRF Estimators and Coherence

H1

$$H(f) = \frac{Y(f)}{X(f)} \equiv \frac{G_{xy}(f)}{G_{xx}(f)} = H_1(f)$$

H2

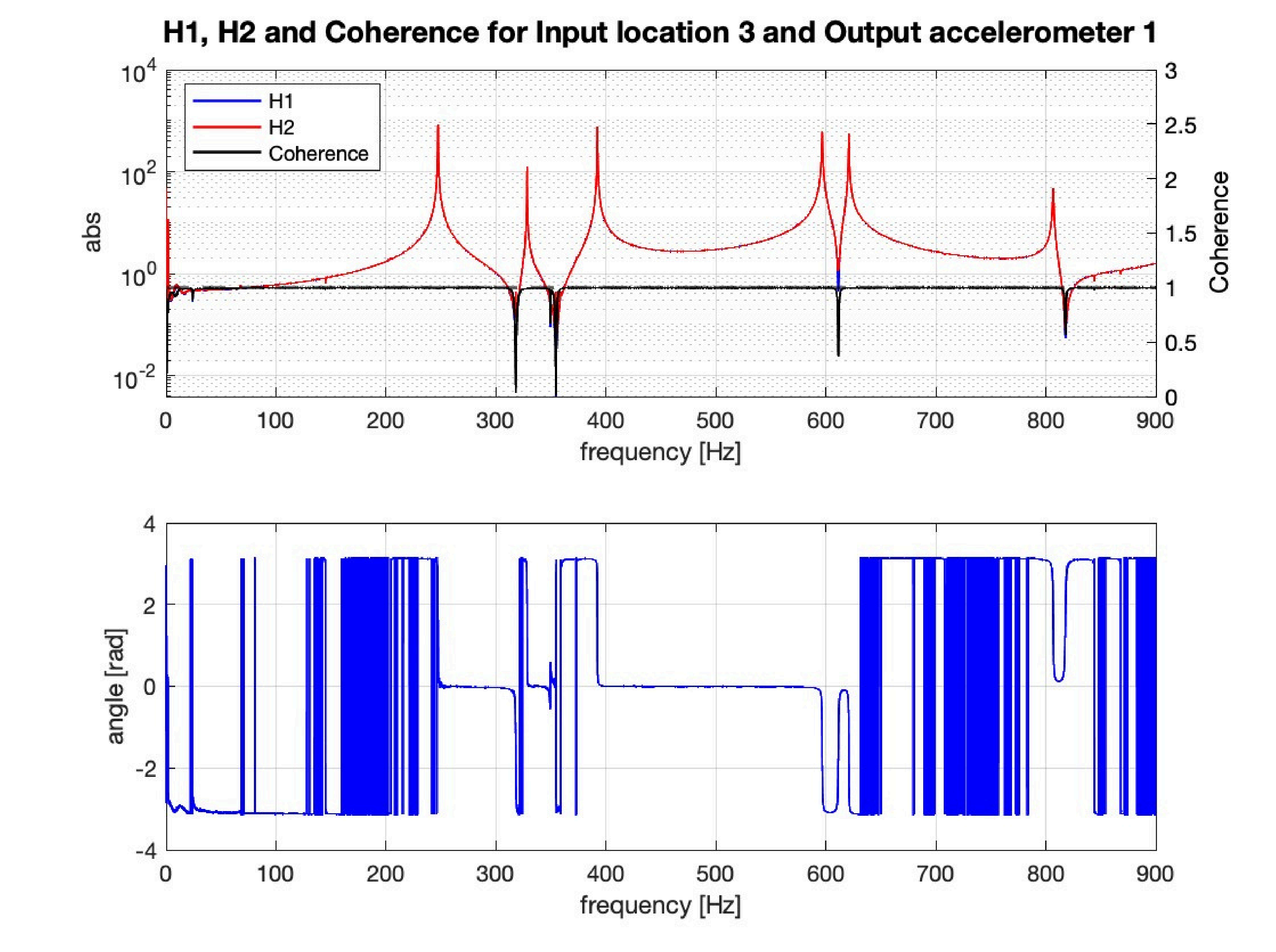
$$H(f) = \frac{Y(f)}{X(f)} \equiv \frac{G_{yy}(f)}{G_{yx}(f)} = H_2(f)$$

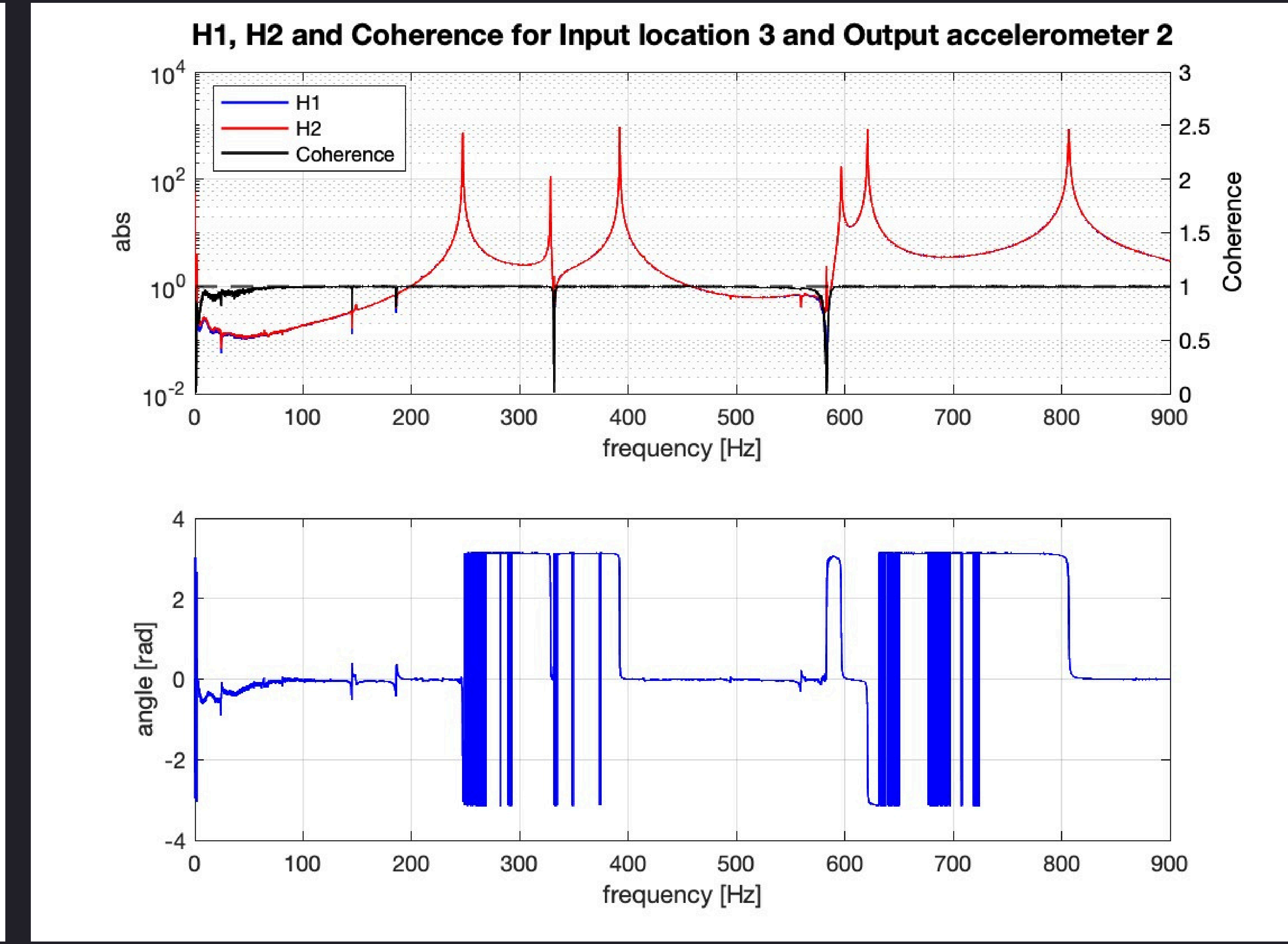
Coherence

$$\gamma_{AB}^2(f_k) = \frac{\left|G_{AB}(f_k)\right|^2}{G_{AA}(f_k)\, G_{BB}(f_k)} = \frac{\left|S_{AB}(f_k)\right|^2}{S_{AA}(f_k)\, S_{BB}(f_k)}$$

Non co-located FRFs (Input Location 3)

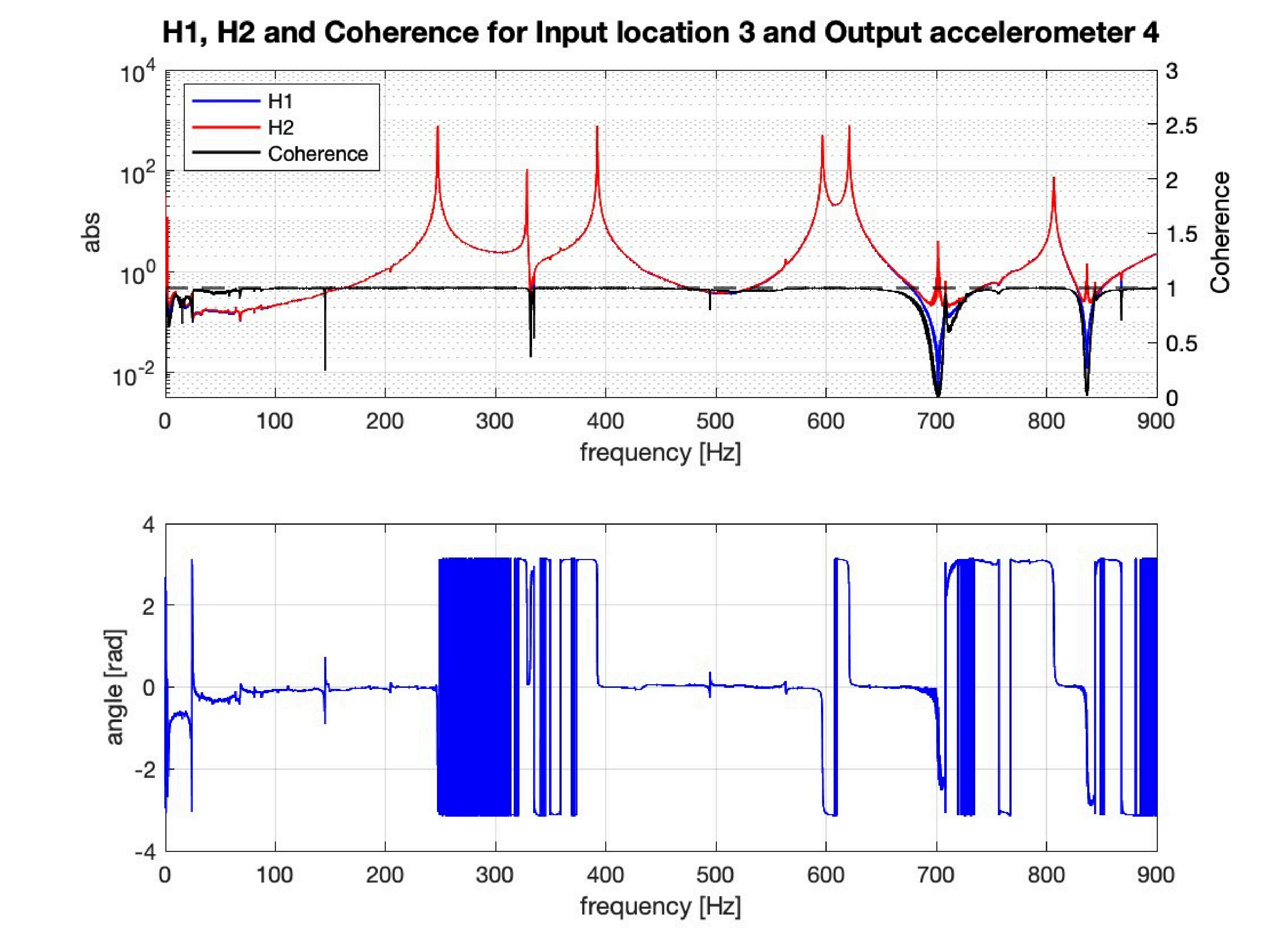
STEP 03

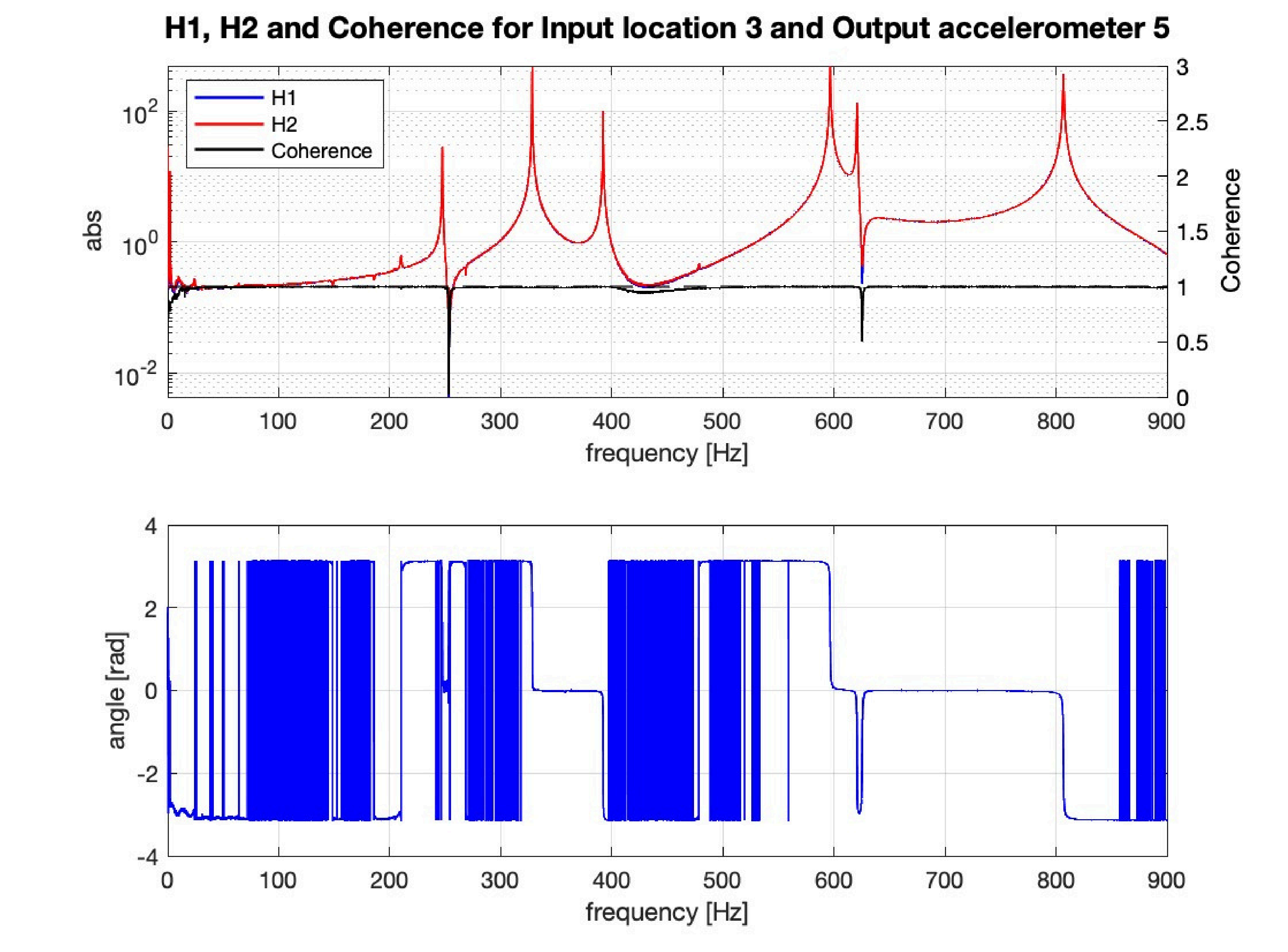


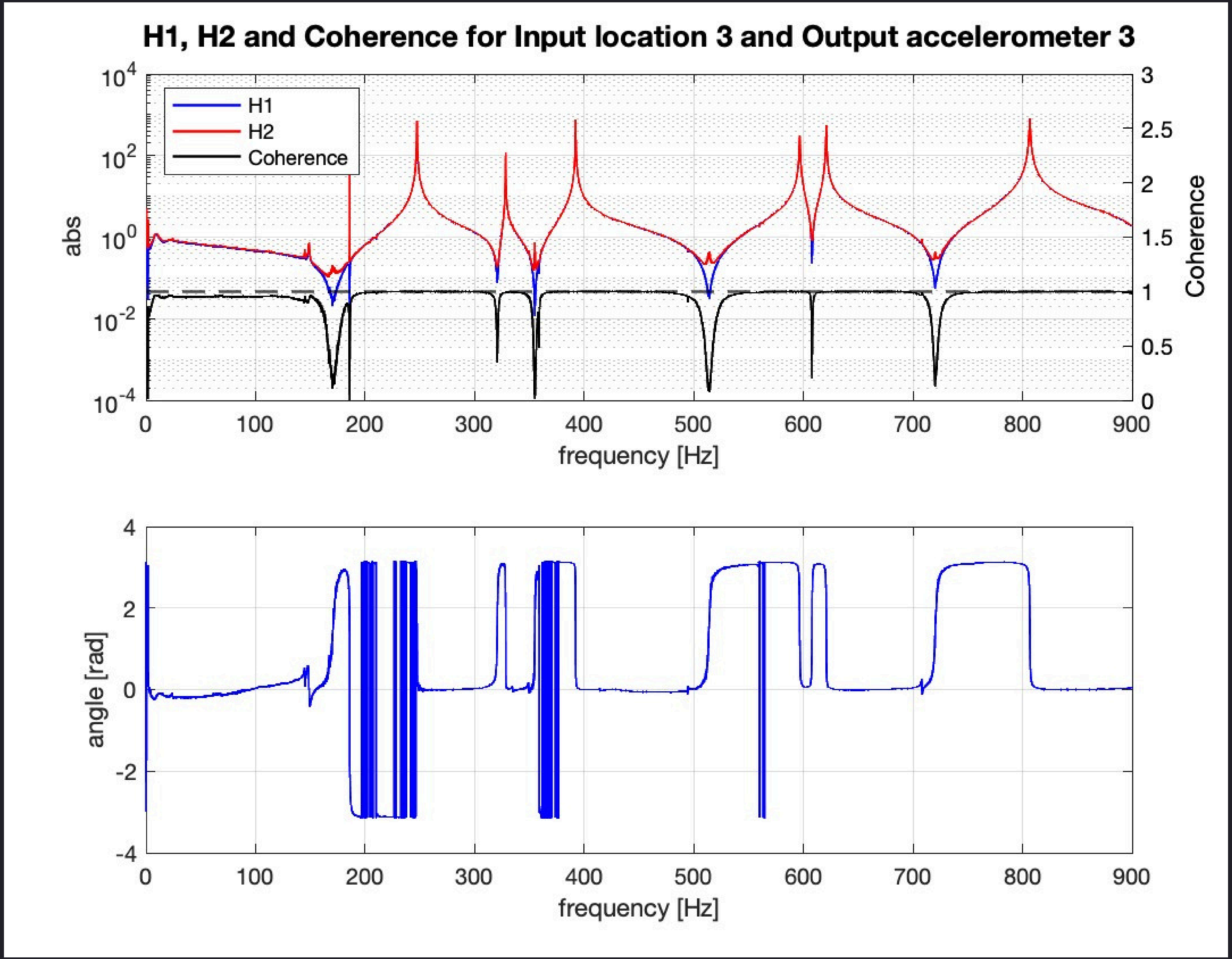


Non co-located FRFs (Input Location 3)

STEP 03

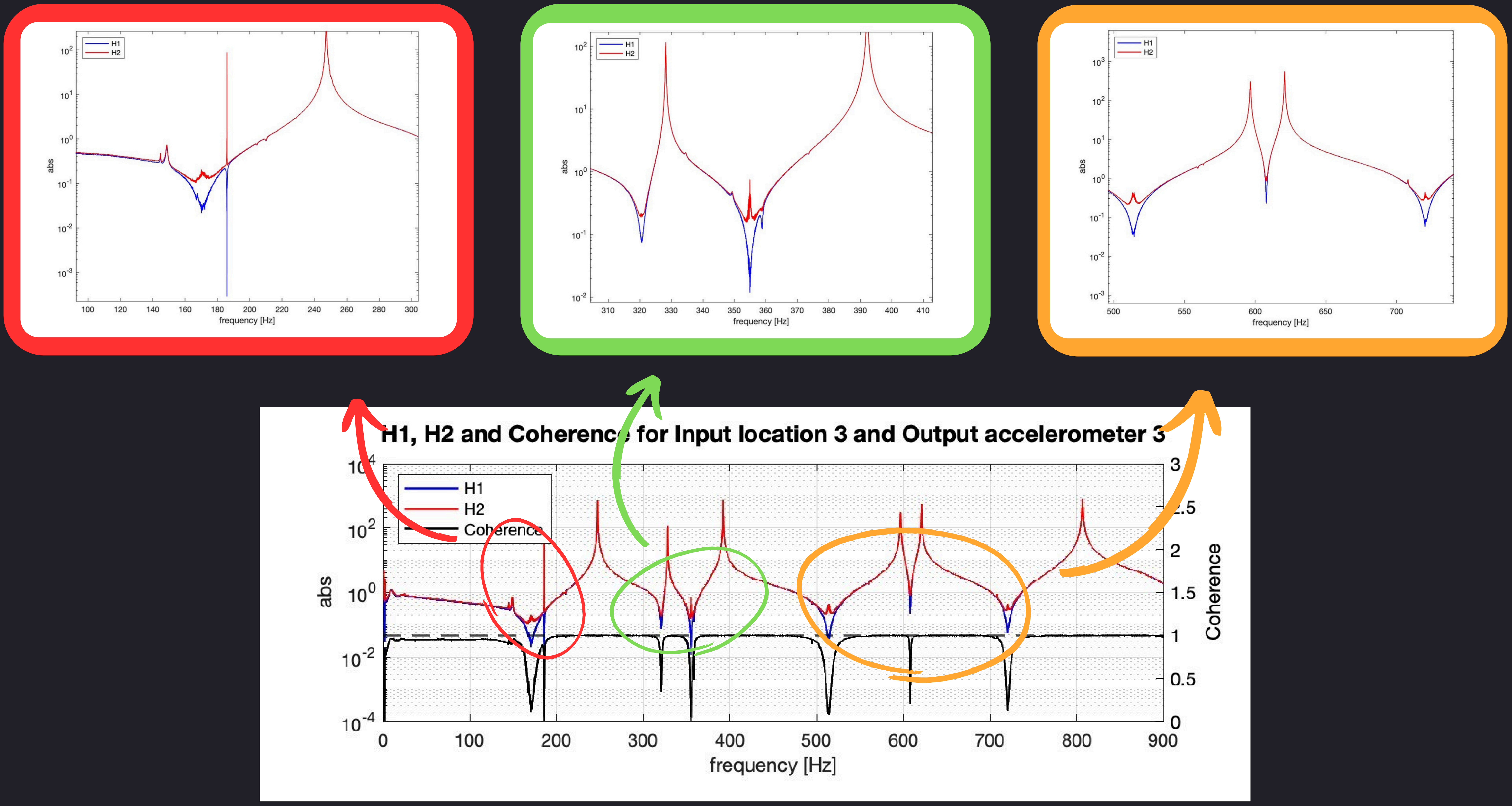






STEP 03

Co-located FRF (Input Location 3)



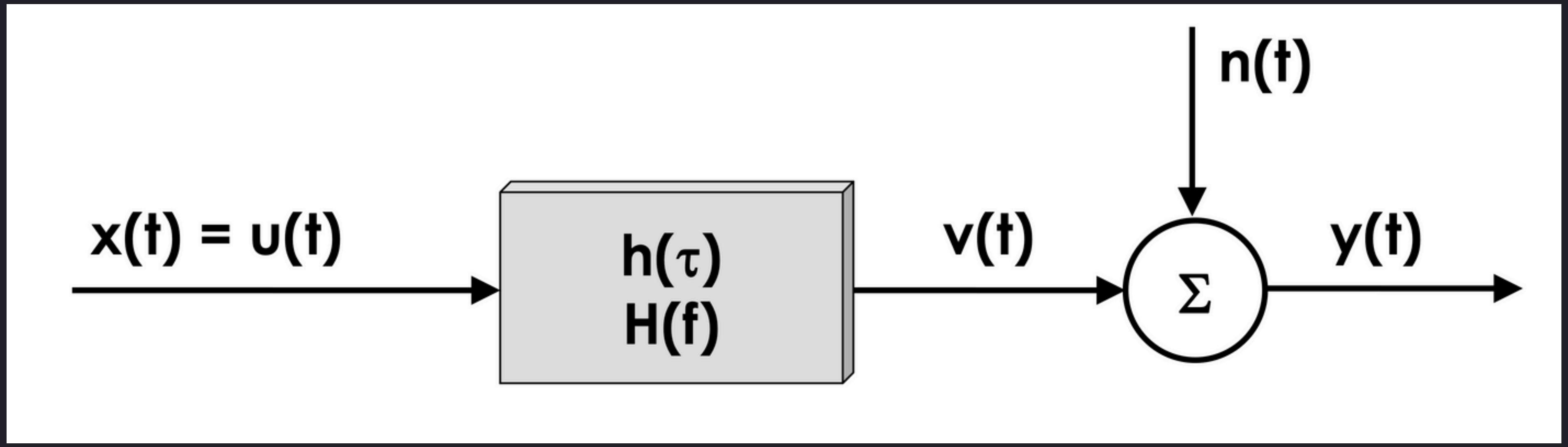
H1, H2 and Coherence for Input location 3 and Output accelerometer 3
H1
H2
Coherence
abs
frequency [Hz]
Coherence

STEP 04

Choice of the estimator

<u>Which one is better?</u>

Noise at the output:

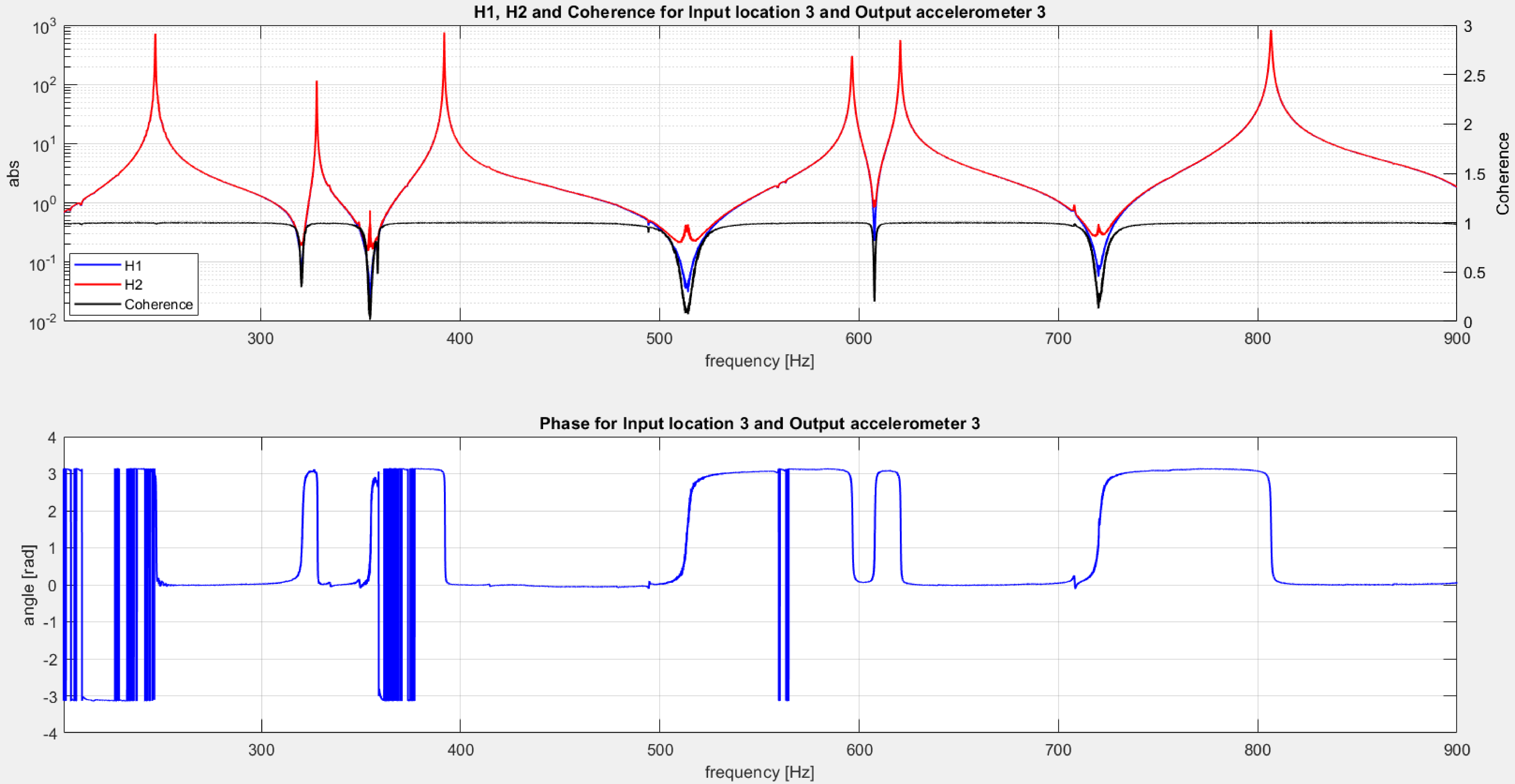


$H_1(f)$ allows to get a correct estimate of the FRFs

SDOF

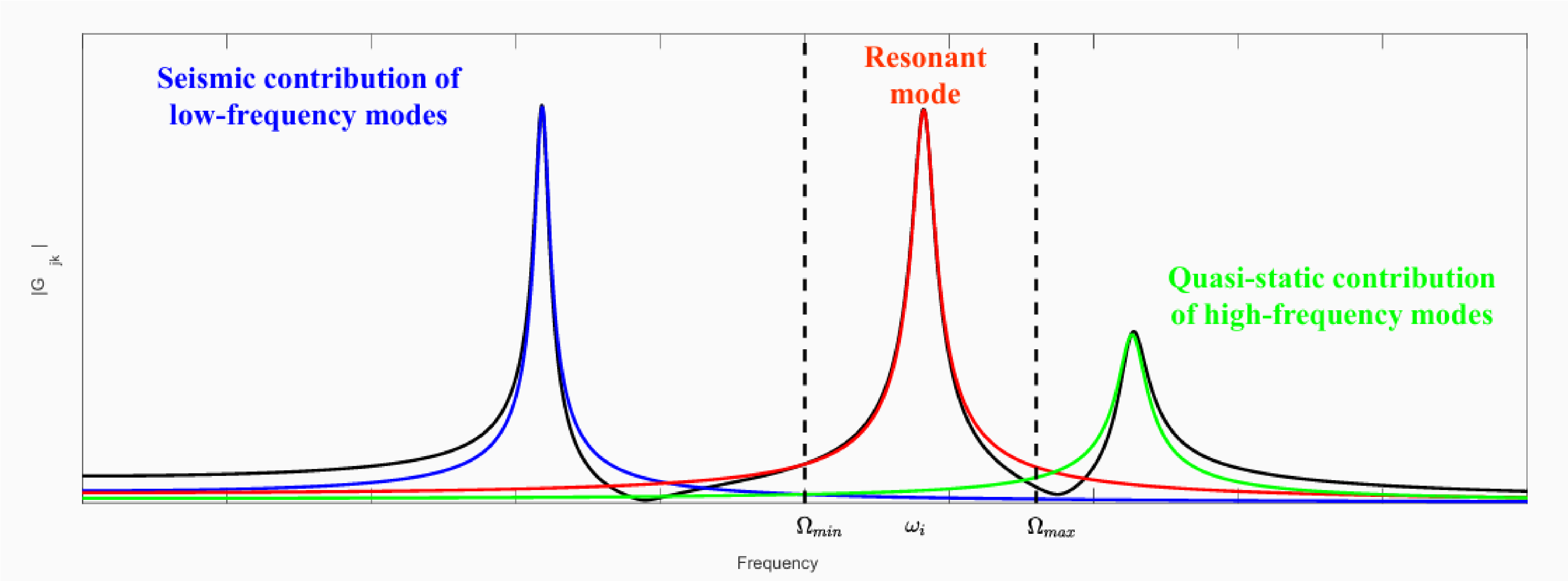
Single degree of freedom method for modal analysis

FRF of the 3rd accelerometer



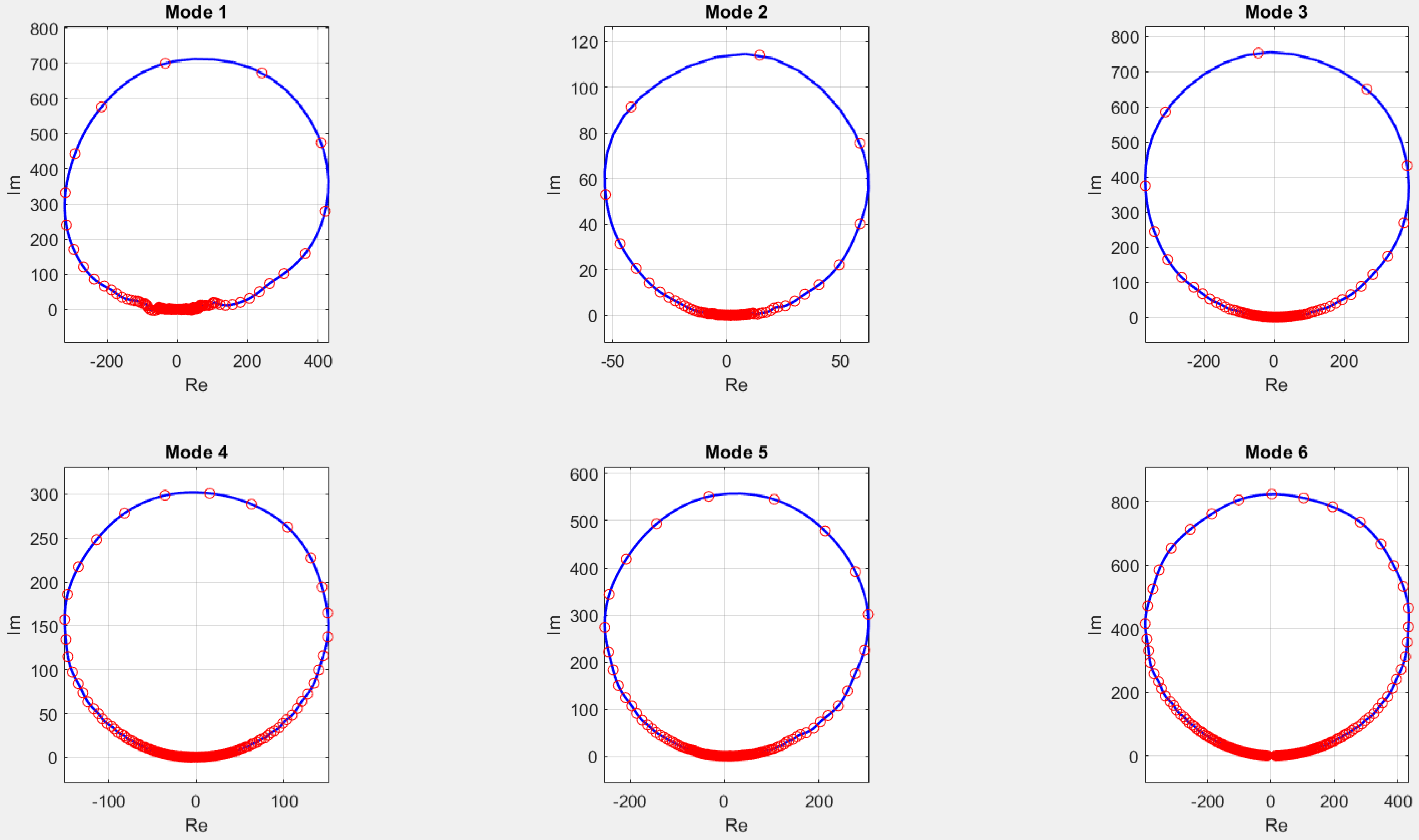
$$G_{jk}(\Omega) = \sum_{i=1}^{N} \frac{-\Omega^2\, \phi_i(x_j)\phi_i(x_k) / m_i}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2}$$

$$G_{jk}^{NUM}(\Omega) = \frac{A_{jk}^{(i)}}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2} + \frac{R_{jk}^{L}}{\Omega^2} + R_{jk}^{H}$$



APPROXIMATION FORMULA

- Reconstruction of FRF considering for each peak one resonant mode at time
- Extraction of modal parameters from optimization function



REQUIREMENTS TO USE SDOF MODAL ANALYSIS

- **FRF peaks far from each other in frequency**
- **Low damping structure**

INITIAL PARAMETERS

- <u>**Natural Frequencies**</u>

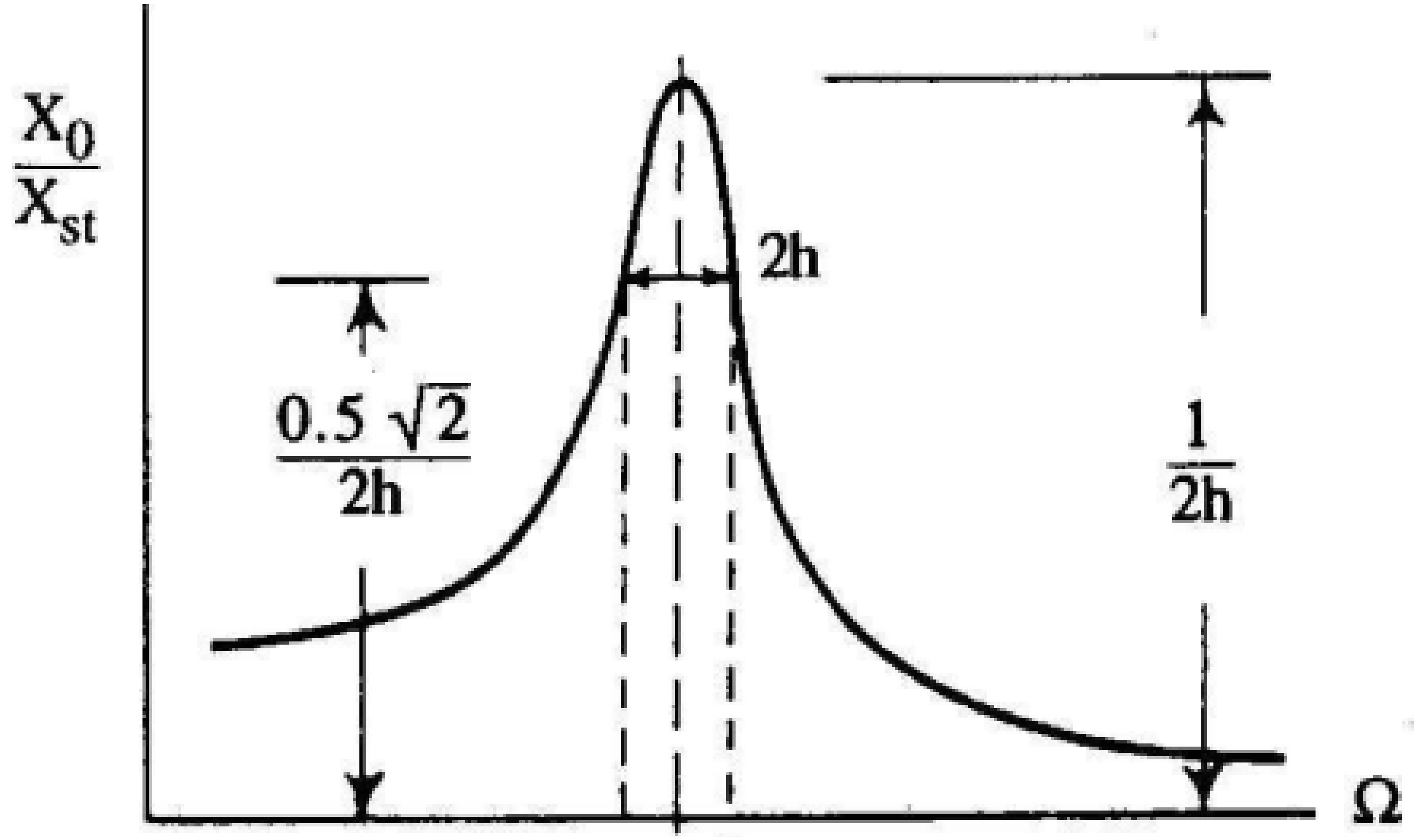
Computed as:

- **Maximum value of the FRF magnitude**
- **Zero of Real part of the FRF**
- **Maximum of the amplitude of the imaginary part of the FRF**

	frequency [Hz]
1st mode	247.32
2nd mode	328.20
3rd mode	392.10
4th mode	596.60
5th mode	620.84
6th mode	806.64

- **Damping Ratios**
  - Half power method
  - Phase derivative technique

	Csi
1st mode	3.80e-4
2nd mode	1.22e-4
3rd mode	1.53e-4
4th mode	3.53e-4
5th mode	2.40e-4
6th mode	7.15e-4



- <u>**A constants and residuals**</u>

Computation of A constants different from co-located FRF to non co-located FRF:

$$G_{jk}(\Omega) = \sum_{i=1}^{N} \frac{-\Omega^2 \phi_i(x_j)\phi_i(x_k)/m_i}{-\Omega^2 + j2\xi_i\omega_i\Omega + \omega_i^2}$$

in resonances:

$$G_{jk}(\omega_i) = \frac{\varphi_i(x_j)\varphi_i(x_k)}{j2\zeta_i}$$

- Co-located accelerometer:

$$A_{ik} = \varphi_i^2(x_k)$$

- Non co-located accelerometers:

$$A_{ij} = \varphi_i(x_j)\varphi_i(x_k)$$

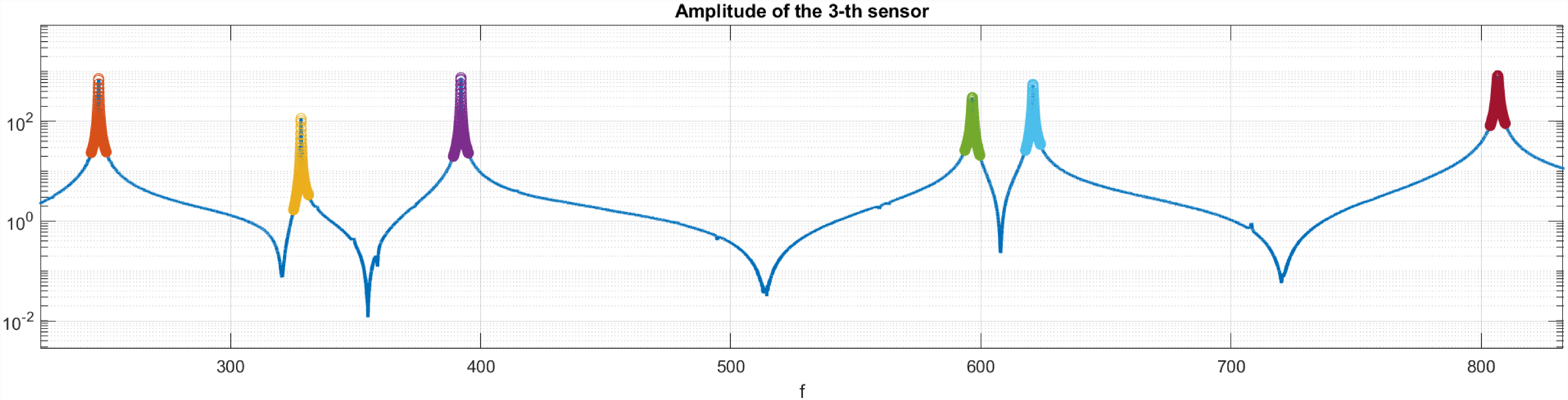
Residuals:

$$R_l = 0$$
$$R_h = 0$$

OPTIMIZATION METHOD

lsqnonlin MatLab function

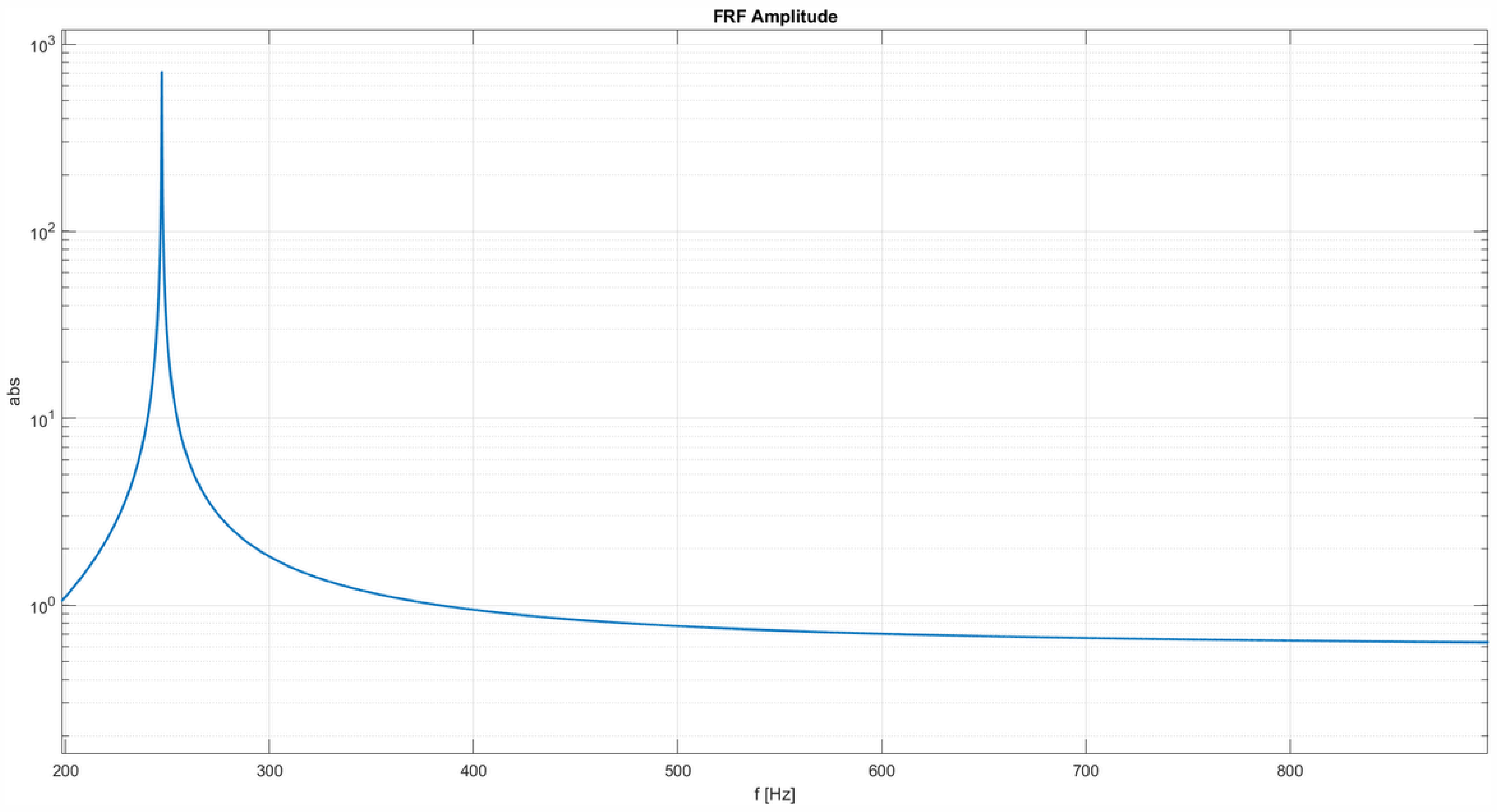
$$\epsilon = \sum_{r=1}^{N} \sum_{s=1}^{M} Re(G_r^{EXP}(\Omega_s) - G_r^{NUM}(\Omega_s))^2 + Im(G_r^{EXP}(\Omega_s) - G_r^{NUM}(\Omega_s))^2$$



OPTIMIZED PARAMETERS

1st mode:

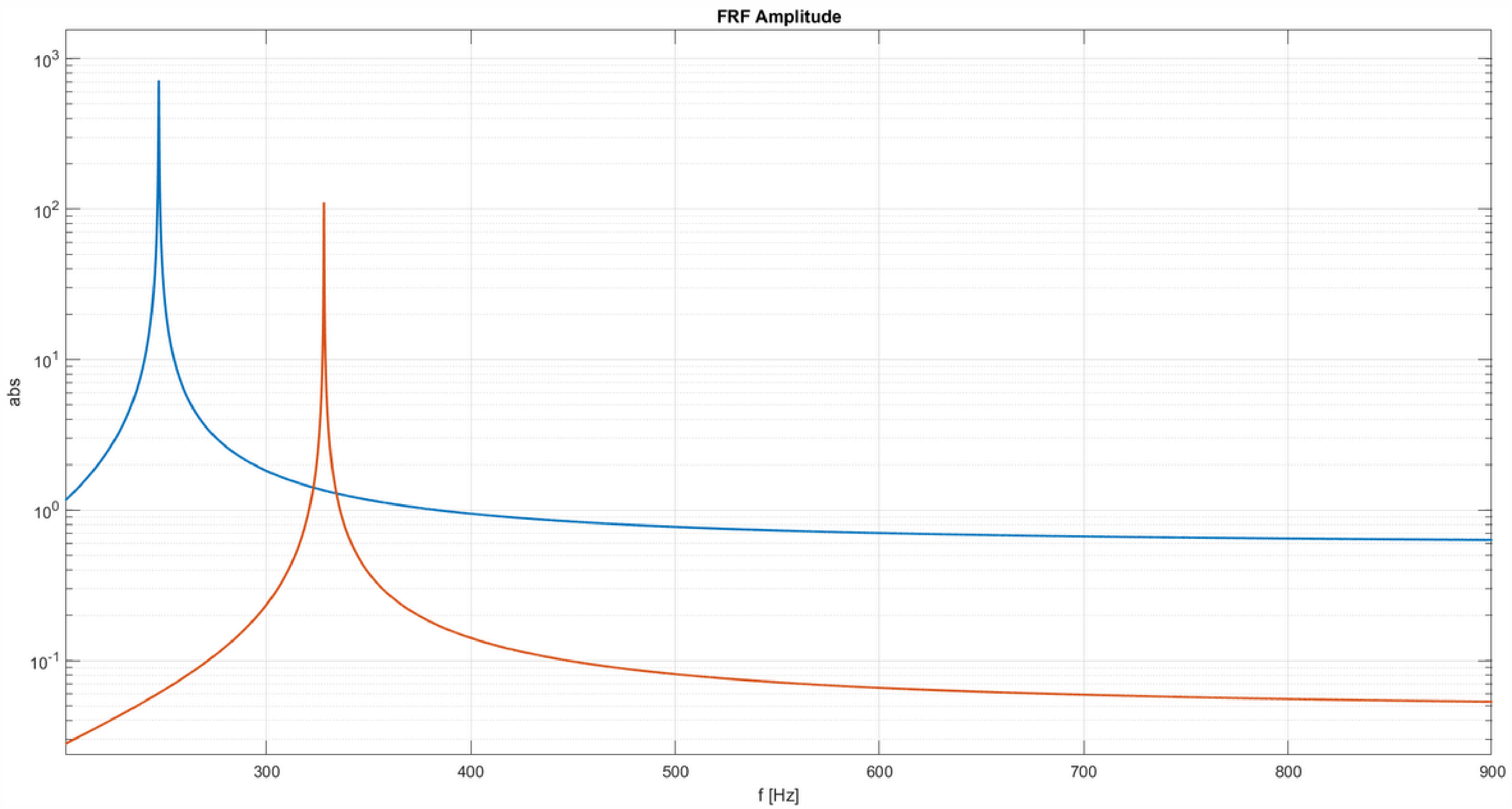
	VALUE
frequency	**247.32 Hz**
csi	**4.047e-4**
A	**3.57e+4**
RI	**-6+2.41i**
Rh	**0.32-1.167i**



OPTIMIZED PARAMETERS

2nd mode:

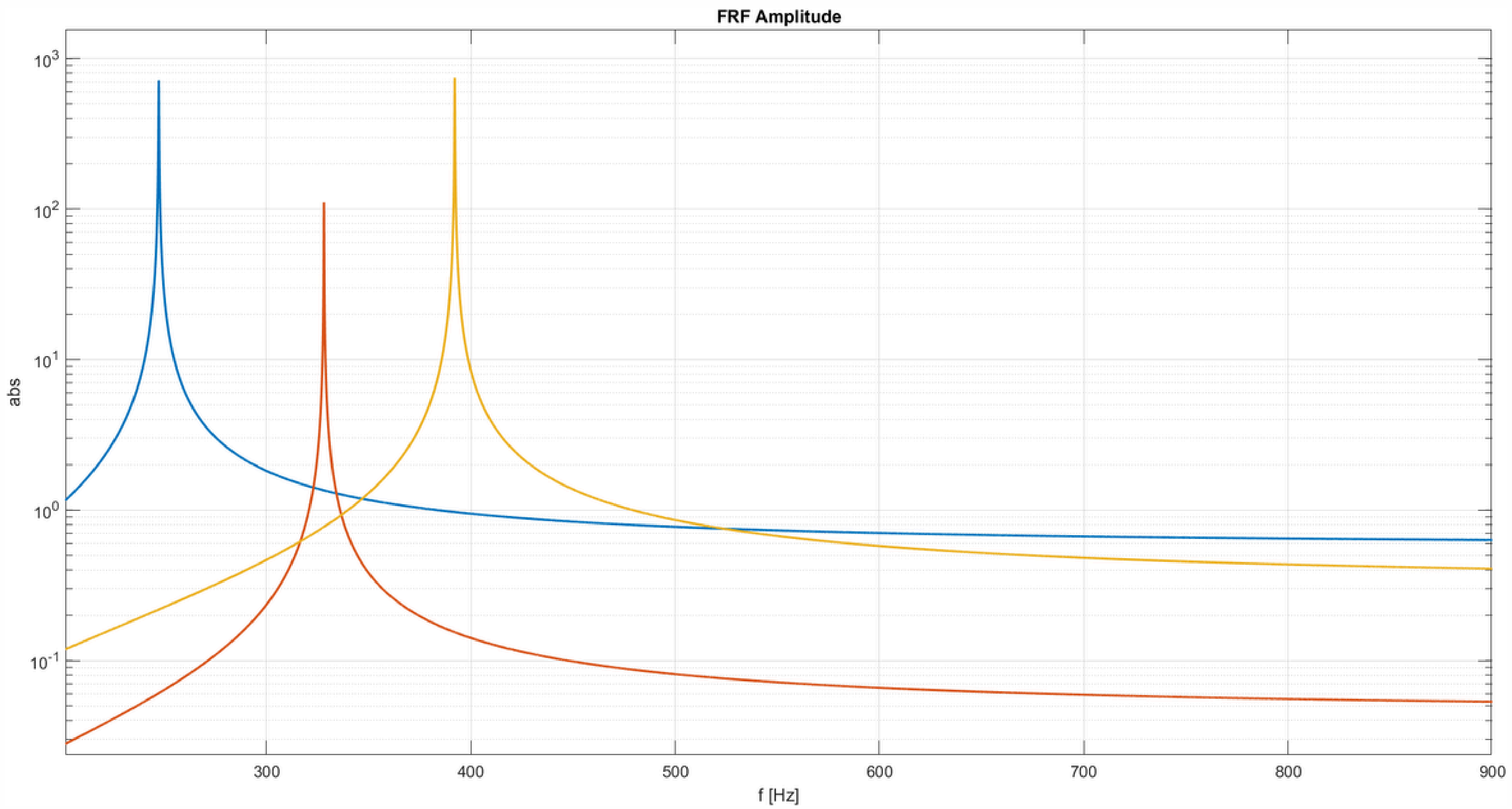
	VALUE
omega	**328.20 Hz**
csi	**2.01e-4**
A	**4.98e+3**
RI	**0.0045+0.0024i**
Rh	**0.0081+0.0031i**



OPTIMIZED PARAMETERS

3rd mode:

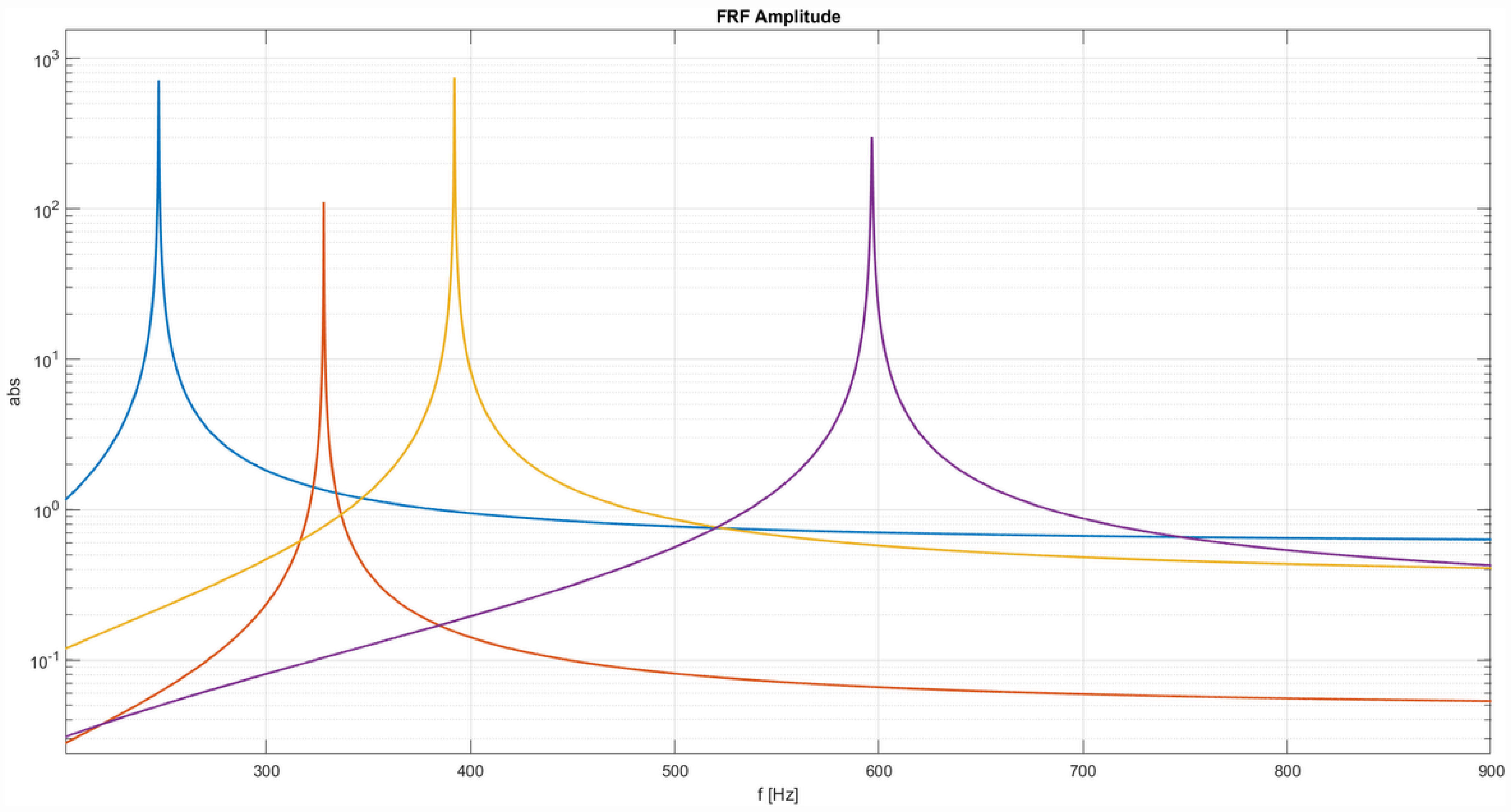
	VALUE
omega	**392.12 Hz**
csi	**2.18e-4**
A	**5.07e+4**
Rl	**-3.45+0.66i**
Rh	**1.35+0.009i**



OPTIMIZED PARAMETERS

4th mode:

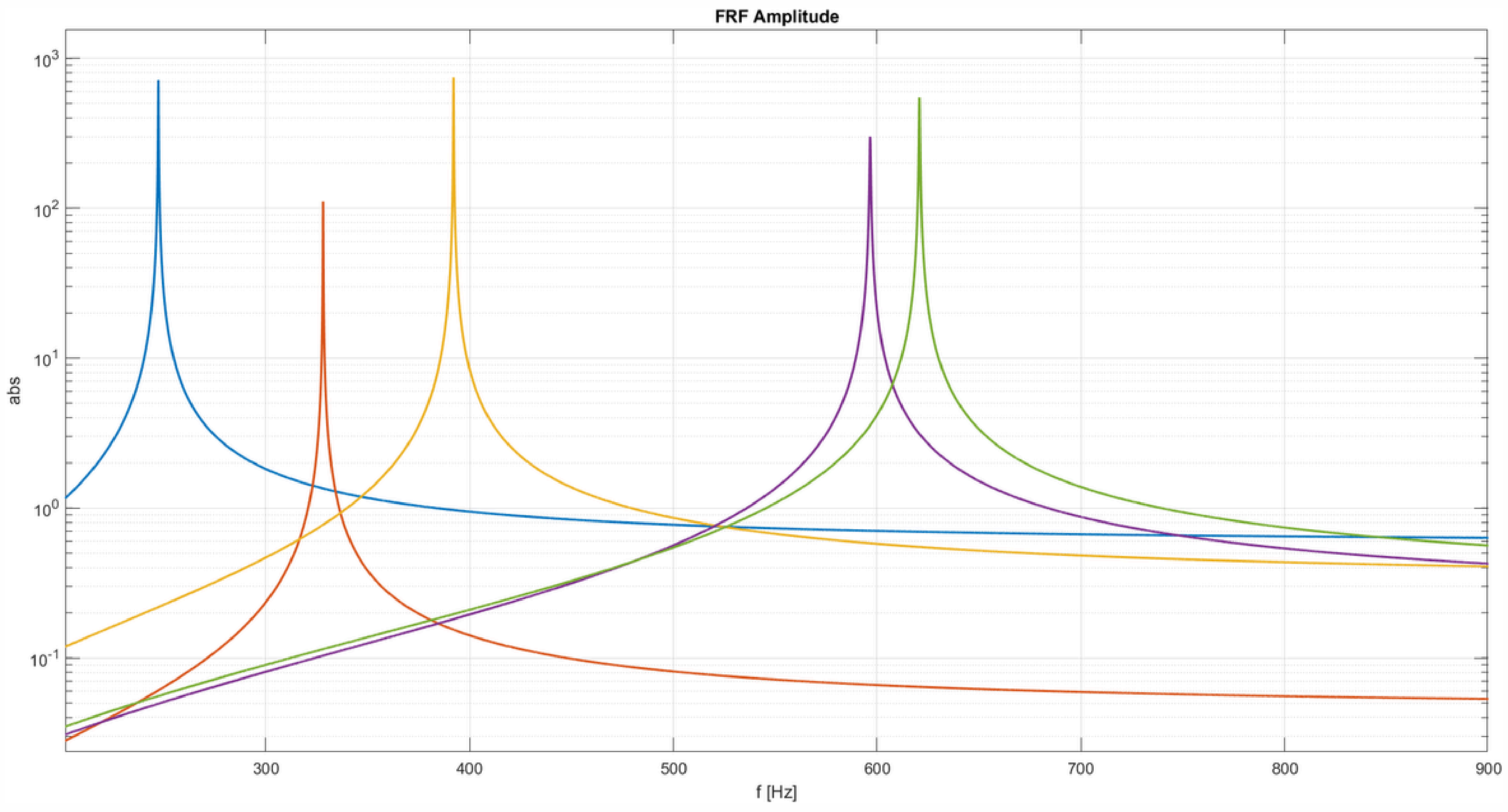
	VALUE
omega	**596.60 Hz**
csi	**3.96e-4**
A	**8.49e+4**
RI	**0+1.8i**
Rh	**-2.76+0.28i**



OPTIMIZED PARAMETERS

5th mode:

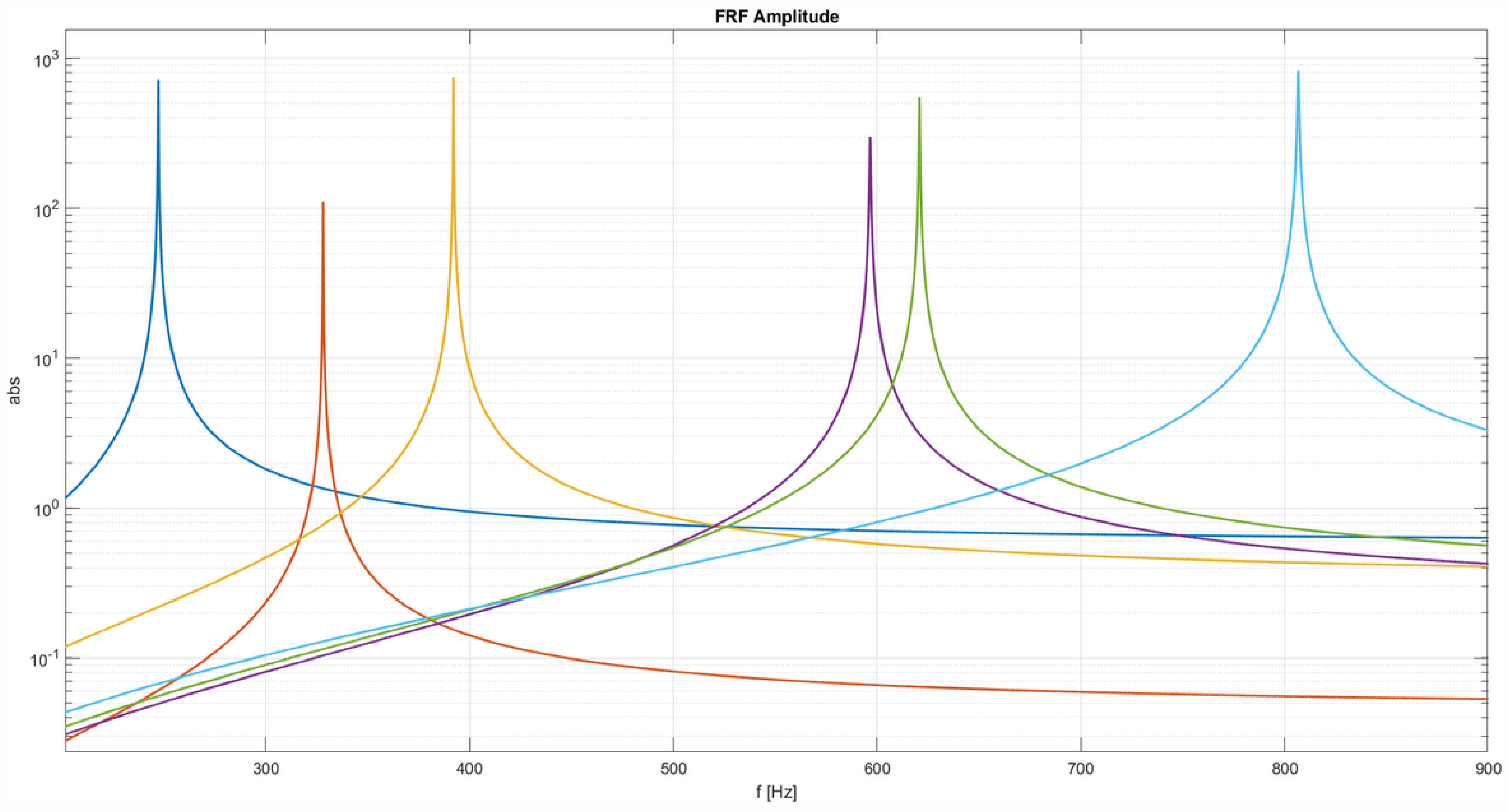
	VALUE
omega	**620.81 Hz**
csi	**2.66e-4**
A	**1.11e+5**
RI	**0+17i**
Rh	**4.22+0.53i**



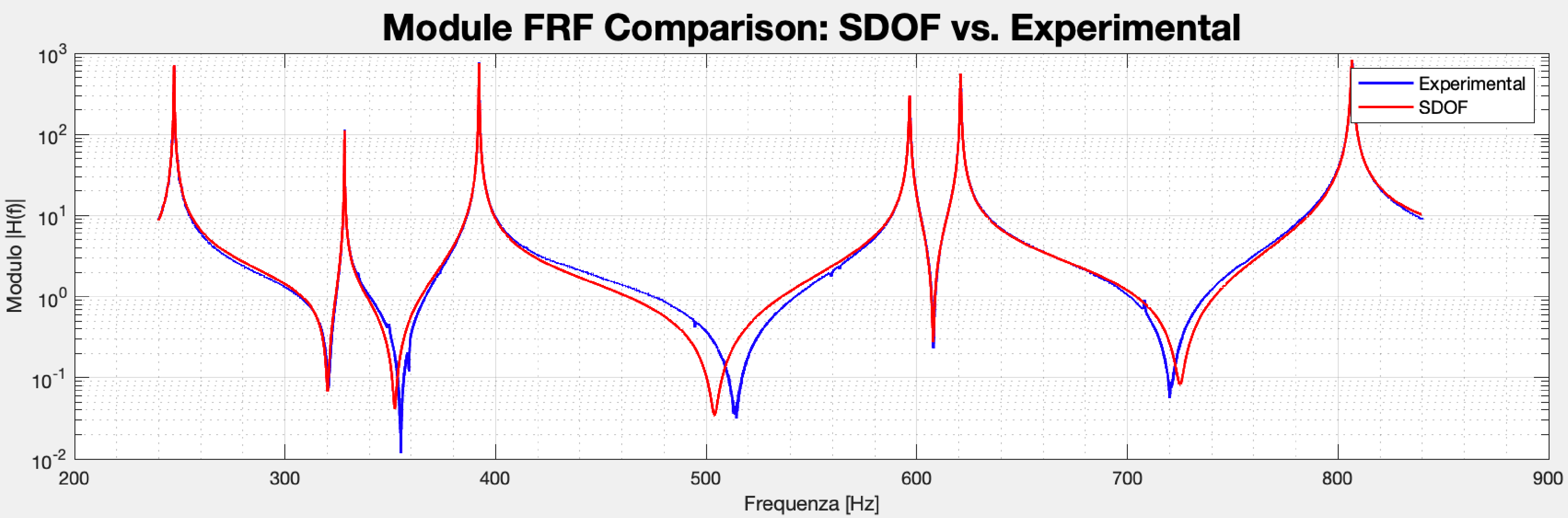
OPTIMIZED PARAMETERS

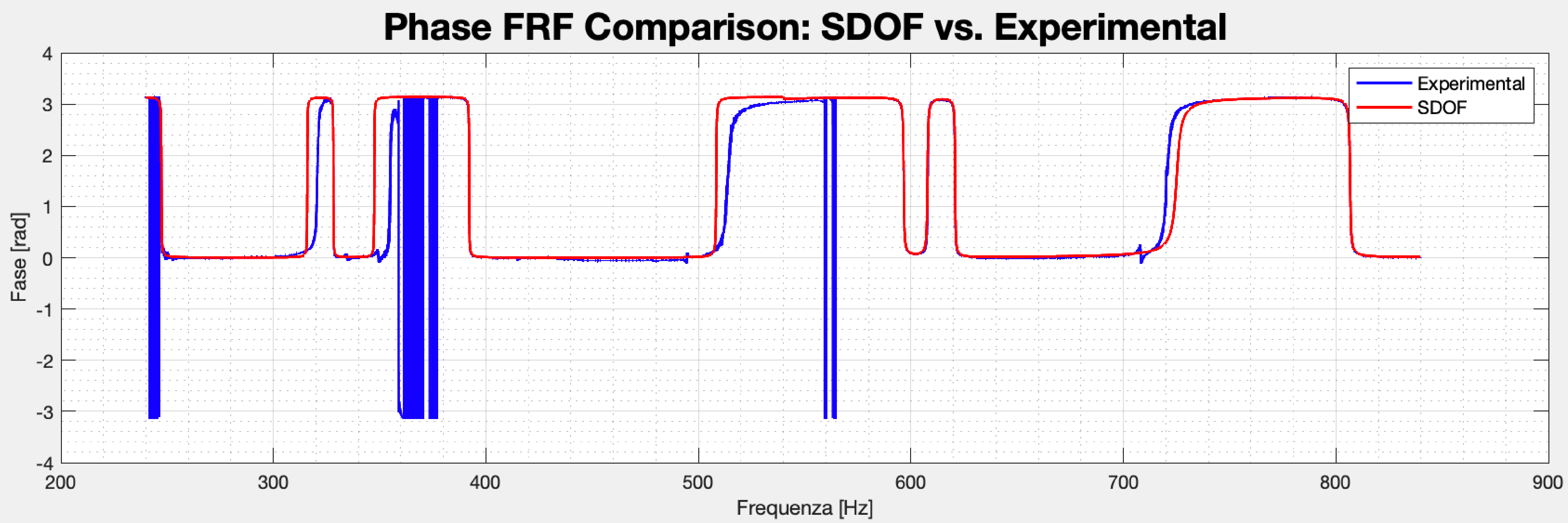
6th mode:

	VALUE
omega	**806.68 Hz**
csi	**3.94e-4**
A	**4.22e+5**
RI	**-2.38+0.04i**
Rh	**3.89+1.1i**



FRF RECONSTRUCTION - SDOF





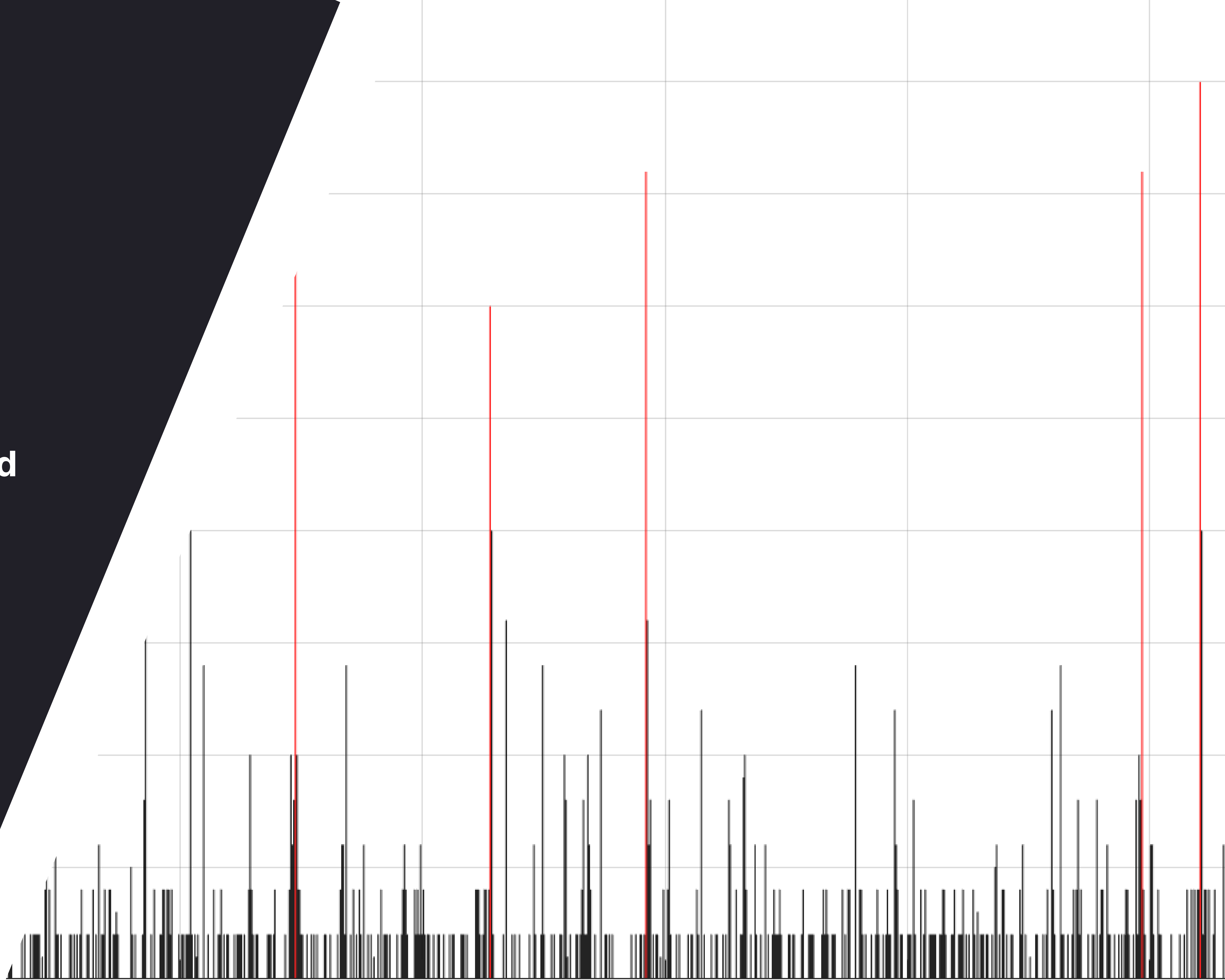
Data Analysis Assignement a.a. 2024/2025

MDOF

Multi degree of freedom method for modal analysis

MDOF PRONY

Complex exponential Method (Prony Method)



STARTING POINT

Impulse Response Functions

$$h_{pq}(t) = \sum_{r=1}^{N} \frac{\psi_{p,r}\psi_{q,r}}{2\mathrm{j}m_r\omega_r\sqrt{1-\xi_r^2}} e^{s_r t} + \left(\frac{\psi_{p,r}\psi_{q,r}}{2\mathrm{j}m_r\omega_r\sqrt{1-\xi_r^2}}\right)^* e^{s_r^* t}$$

$$h_0, h_1, h_2, \ldots, h_m = h(0), h(\Delta t), h(2\Delta t), \ldots, h(m\Delta t)$$

SOME NOTATION

$$h(t) = \sum_{r=1}^{2N} A_r e^{s_r t}$$

FROM CONTINUOUS TO SAMPLED TIME

$$e^{s_r \Delta t} \rightarrow V_r$$

$$h_\ell = \sum_{r=1}^{2N} A_r V_r^{\ \ell}$$

PRONY'S IDEA

INTRODUCE COEFFICIENTS BETAS

$$\begin{cases} h_0 = A_1 + A_2 + \ldots + A_{2N} \\ h_1 = V_1 A_1 + V_2 A_2 + \ldots + V_{2N} A_{2N} \\ h_2 = {V_1}^2 A_1 + {V_2}^2 A_2 + \ldots + V_{2N}^2 A_{2N} \\ \vdots \\ h_m = {V_1}^m A_1 + {V_2}^m A_2 + \ldots + V_{2N}^m A_{2N} \end{cases}$$

ILL CONDITIONED

↓

$$\begin{cases} \beta_0 h_0 = \beta_0 A_1 + \beta_0 A_2 + \ldots + \beta_0 A_{2N} \\ \beta_1 h_1 = \beta_1 V_1 A_1 + \beta_1 V_2 A_2 + \ldots + \beta_1 V_{2N} A_{2N} \\ \beta_2 h_2 = \beta_2 {V_1}^2 A_1 + \beta_2 {V_2}^2 A_2 + \ldots + \beta_2 V_{2N}^2 A_{2N} \\ \vdots \\ \beta_m h_m = \beta_m {V_1}^m A_1 + \beta_m {V_2}^m A_2 + \ldots + \beta_m V_{2N}^m A_{2N} \end{cases}$$

DECOUPLING EIGEN VALUES AND VECTORS

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \ldots + \beta_m V^m = 0$$

CHARACTERISTIC POLYNOMIAL

$$\sum_{i=0}^{m} \beta_i h_i = \sum_{j=1}^{2N} \left(A_j \underbrace{\cancel{\sum_{i=0}^{m} \beta_i V_j^i}}_{=0} \right)$$

SOLVING BETAS?

$$\sum_{i=0}^{m} \beta_i h_i = 0$$

FIX ORDER TO 2N

$$\begin{bmatrix} h_0 & \cdots & \cdots & h_{2N-1} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ h_m & \cdots & \cdots & h_{m+2N-1} \end{bmatrix} \begin{Bmatrix} \beta_0 \\ \vdots \\ \vdots \\ \beta_{2N-1} \end{Bmatrix} = - \begin{Bmatrix} h_{2N} \\ \vdots \\ \vdots \\ h_{m+2N} \end{Bmatrix}$$

WITH m > 2N : LEAST SQUARE PROBLEM

$$\{\beta\} = -[h]^{+}\left\{\tilde{h}\right\}$$

SISO TO SIMO

WE CHOSE THE 5 IRFs, WITH INPUT 3

$$\begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{2N-1} \\ h_1 & h_2 & h_3 & \cdots & h_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_m & \cdots & \cdots & \cdots & h_{m+2N-1} \end{bmatrix} = R_{i3} \qquad \text{i=1,2,...,5}$$

$$\begin{bmatrix} R_{13} \\ R_{23} \\ \vdots \\ R_{53} \end{bmatrix} = R \longrightarrow R \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{2N-1} \end{Bmatrix} = - \begin{Bmatrix} h_{2N} \\ h_{2N+1} \\ \vdots \\ \vdots \\ h_{5m} \end{Bmatrix}$$

WITH 5m >> 2N : LEAST SQUARE PROBLEM

$$\{\beta\} = -[R]^{+} \left\{\tilde{h}\right\}$$

DETERMINING POLES

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \ldots + \beta_m V^m = 0$$

ROOTS OF THE POLYNOMIAL

$$V_r = e^{s_r \Delta t}$$

$$s_r = -\omega_r \xi_r \pm j\omega_r \sqrt{1 - \xi_r^2}$$

$$\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ V_1 & V_2 & \cdots & \cdots & V_{2N} \\ V_1^2 & V_2^2 & \cdots & \cdots & V_{2N}^2 \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ V_1^{2N-1} & V_2^{2N-1} & \cdots & \cdots & V_{2N}^{2N-1} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_{2N} \end{Bmatrix} = \begin{Bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{2N-1} \end{Bmatrix}$$

SOLVE THIS SYSTEM FOR EACH IRF

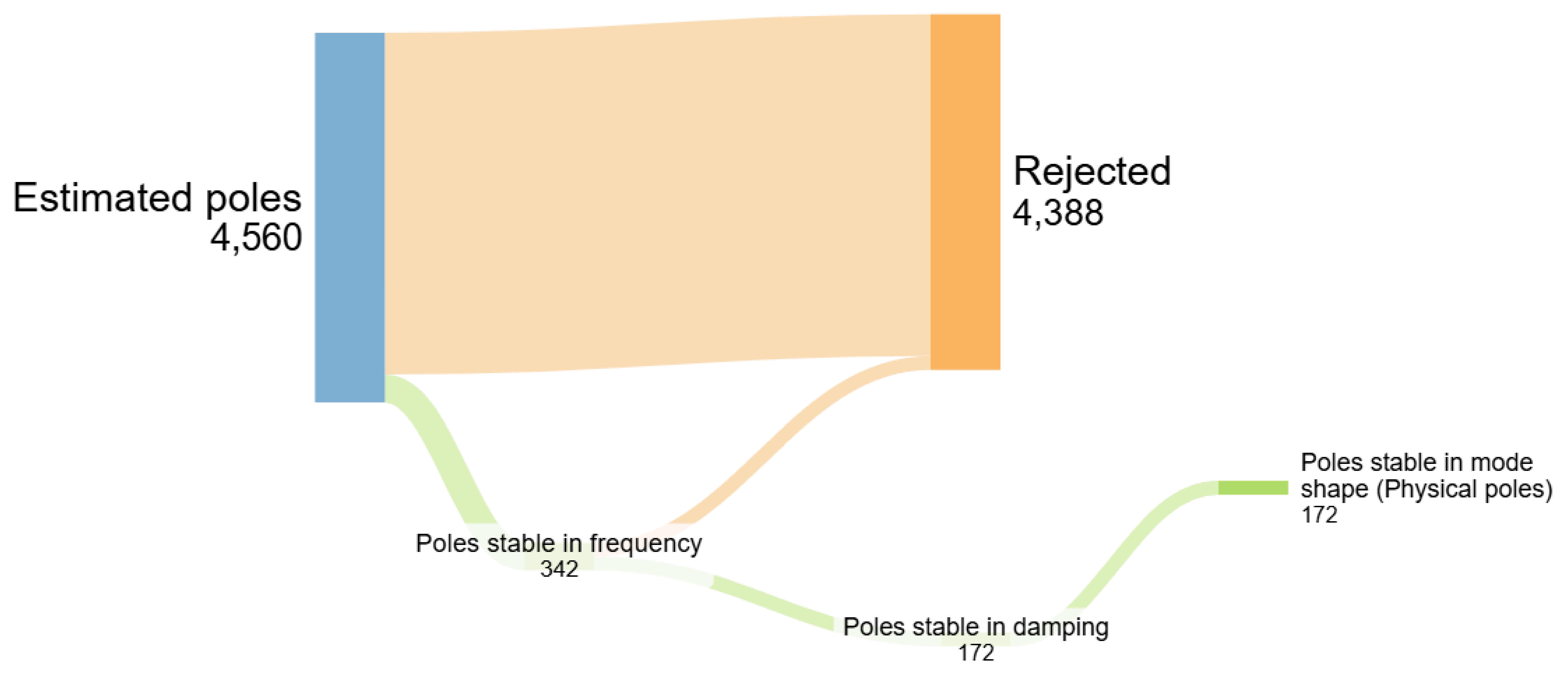
$$[A_{1,i3}\ A_{2,i3}\ \cdots\ A_{2N,i3}]^T \quad \text{i=1,2,...,5}$$

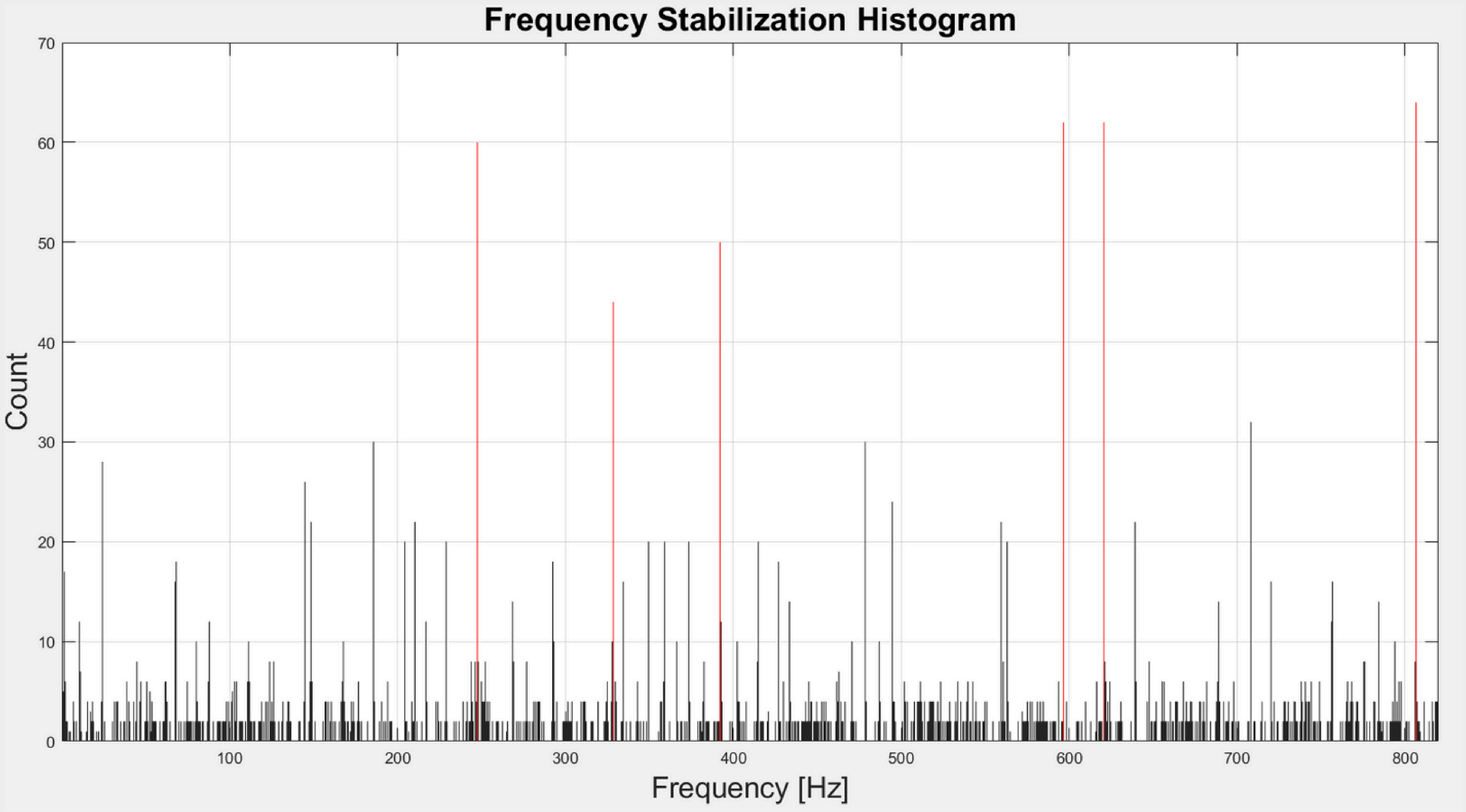
$$A_{r,i3} = \frac{\psi_{i,r}\,\psi_{3,r}}{2jm_r\omega_r\sqrt{1-\xi_r^2}}$$

DETERMINING "MODES"

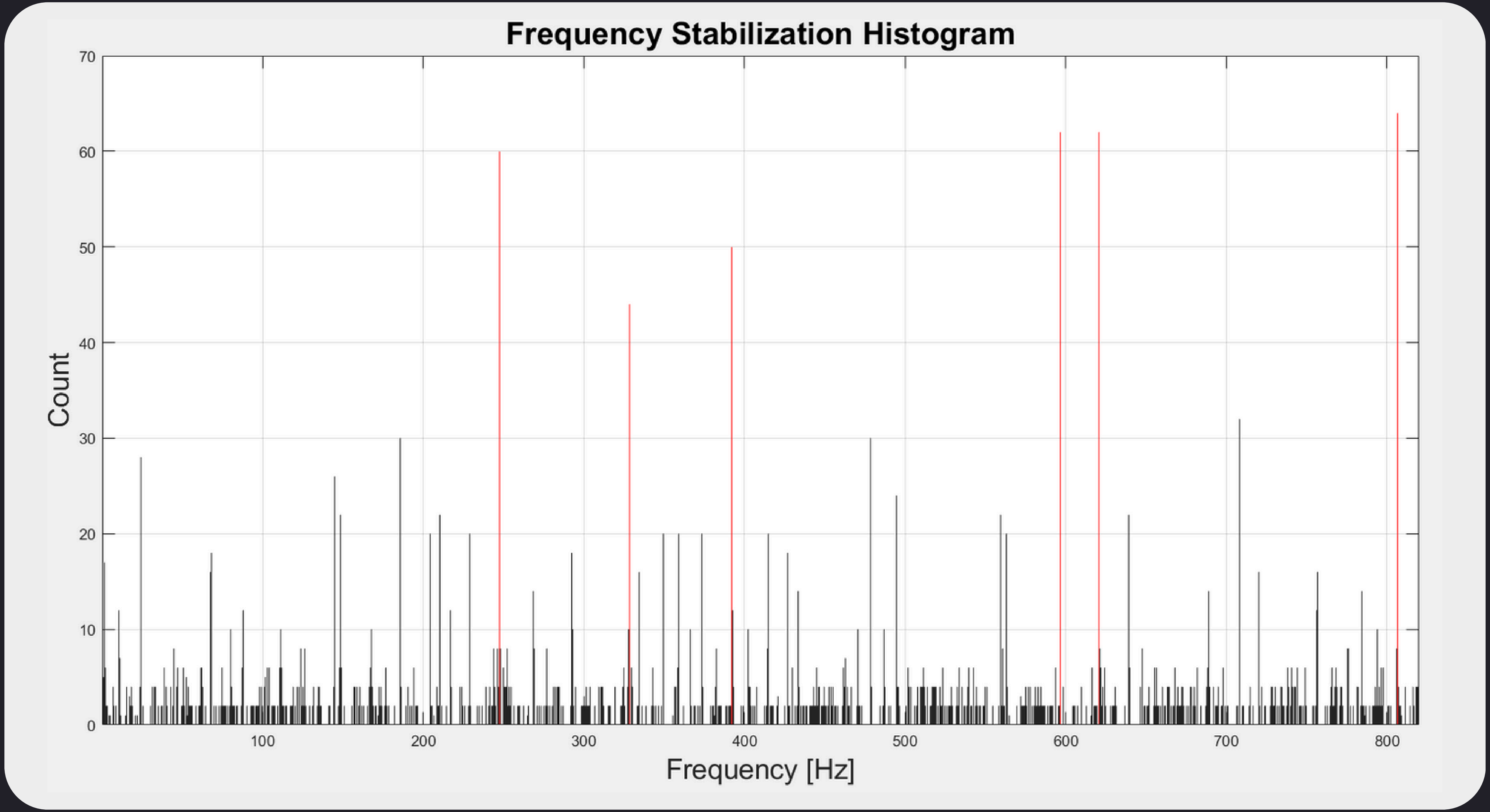
FILTERING PHYSICAL POLES

ONLY 3.8% SURVIVED





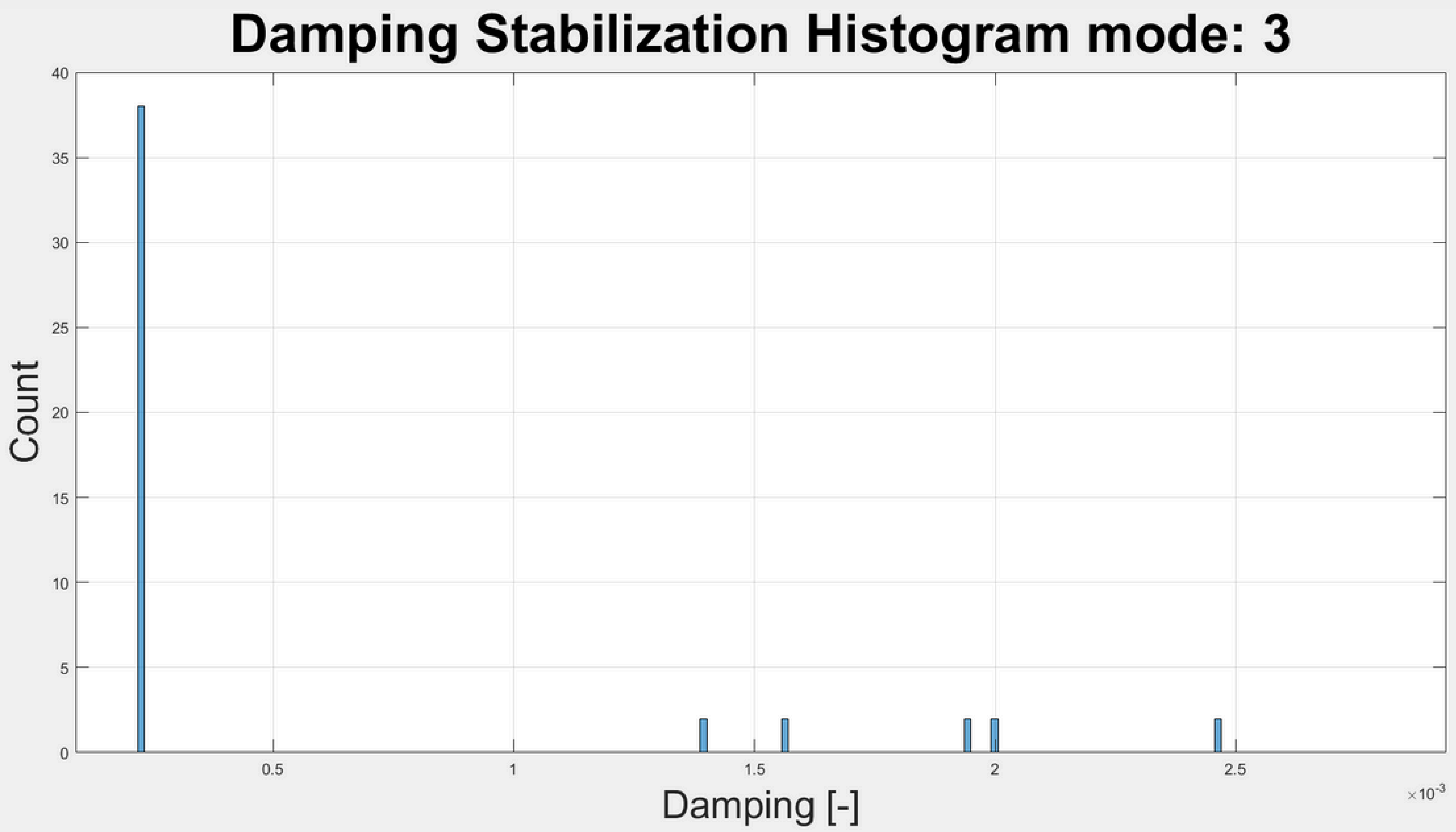
FREQUENCY STABLE POLES

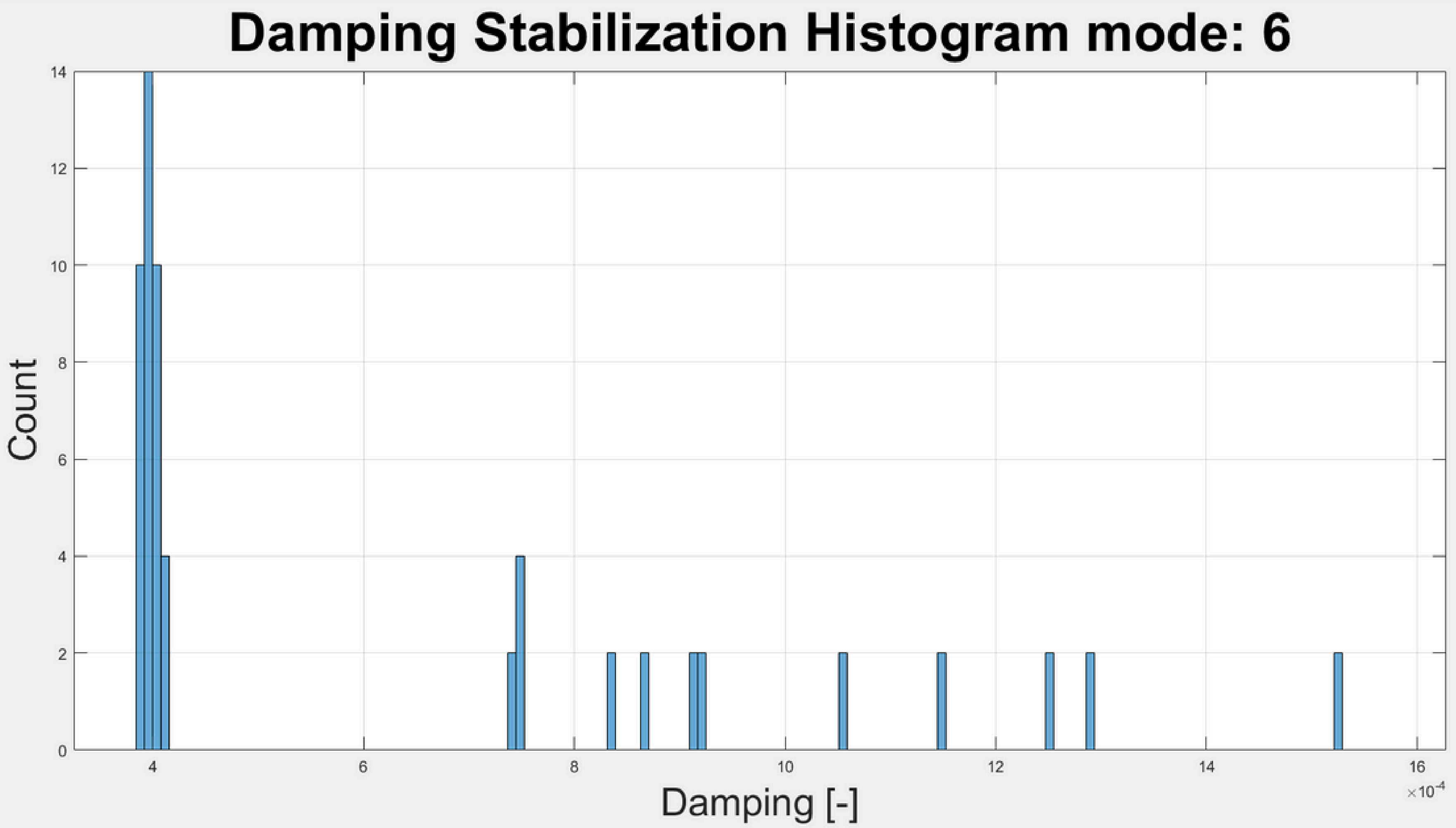


FREQUENCY STABLE POLES

- ≈247 Hz
- ≈328 Hz
- ≈392 Hz
- ≈596 Hz
- ≈621 Hz
- ≈807 Hz

DAMPING STABLE POLES





MODE STABLE POLES

MAC VALUES

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	0.9999	0.9999	0.9996	0.9996	0.9986	0.9986	0.9997	0.9997	0.9995	0.9995	0.9990
2	1	1	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9999	0.9999	0.9998	0.9998	0.9997
3	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	1	1	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
5	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
6	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999

	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0.9990	0.9994	0.9994	0.9998	0.9998	0.9994	0.9994	0.9995	0.9995	0.9980	0.9980	0.9973	0.9973
2	0.9997	0.9997	0.9997	0.9998	0.9998	0	0	0	0	0	0	0	0
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	1.0000	0.9999	0.9999	0.9998	0.9998	0.9998	0.9998	0.9997	0.9997	0.9996	0.9996	0.9996	0.9996
5	0.9999	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998
6	0.9999	0	0	0	0	0	0	0	0	0	0	0	0

	27	28	29	30	31	32	33	34	35	36	37	38	39
1	0.9994	0.9994	0.9984	0.9984	0.9994	0.9994	0.9990	0.9990	0.9972	0.9972	0.9984	0.9984	
2	0	0	0	0	0	0	0	0	0	0	0	0	
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
4	0	0	0	0	0	0	0	0	0	0	0	0	
5	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	
6	0	0	0	0	0	0	0	0	0	0	0	0	

MODAL MASSES ESTIMATION

$$H_{pq}(\omega) = \sum_{r=1}^{2*6} \frac{\psi_{pr}\psi_{qr}}{j\omega - s_r} Q_r + R_U + \frac{R_L}{\omega^2}$$

SETUP A LEAST SQUARE PROBLEM

$$\begin{Bmatrix} H_{pq}(\omega_1) \\ H_{pq}(\omega_2) \\ \vdots \\ H_{pq}(\omega_{N_f}) \end{Bmatrix} = \begin{bmatrix} \frac{\psi_{p1}\psi_{q1}}{j\omega_1 - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_1 - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_1 - s_{2*6}} & 1 & \frac{1}{\omega_1^2} \\ \frac{\psi_{p1}\psi_{q1}}{j\omega_2 - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_2 - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_2 - s_{2*6}} & 1 & \frac{1}{\omega_2^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\psi_{p1}\psi_{q1}}{j\omega_{N_f} - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_{N_f} - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_{N_f} - s_{2*6}} & 1 & \frac{1}{\omega_{N_f}^2} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{2*6} \\ R_U \\ R_L \end{Bmatrix}$$

MODAL MASS EXTRACTION

USING DEFINITION OF Q

$$Q_r = \frac{-\omega^2}{2j\, m_r\, \omega_r \sqrt{1-\xi_r^2}}$$

$$m_r = \frac{-\omega^2}{2j\, Q_r\, \omega_r \sqrt{1-\xi_r^2}}$$

LEADS TO MODAL MASSES

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Res.Frequencies [Hz]	247.29	328.17	392.12	596.57	620.81	806.65
Damping [-]	0.00040	0.00019	0.00022	0.00040	0.00027	0.00041
Norm. Mode shapes	-1	-0.2735	-0.8235	1	0.6604	-0.0584
	0.9313	-0.2929	-1	-0.2372	1	-1
	-0.8302	-0.2178	-0.7586	-0.4473	-0.5542	-0.8085
	0.9234	-0.2075	-0.8791	0.8707	-0.9111	-0.0870
	-0.0319	-1	0.1290	-0.8208	0.1669	0.4433

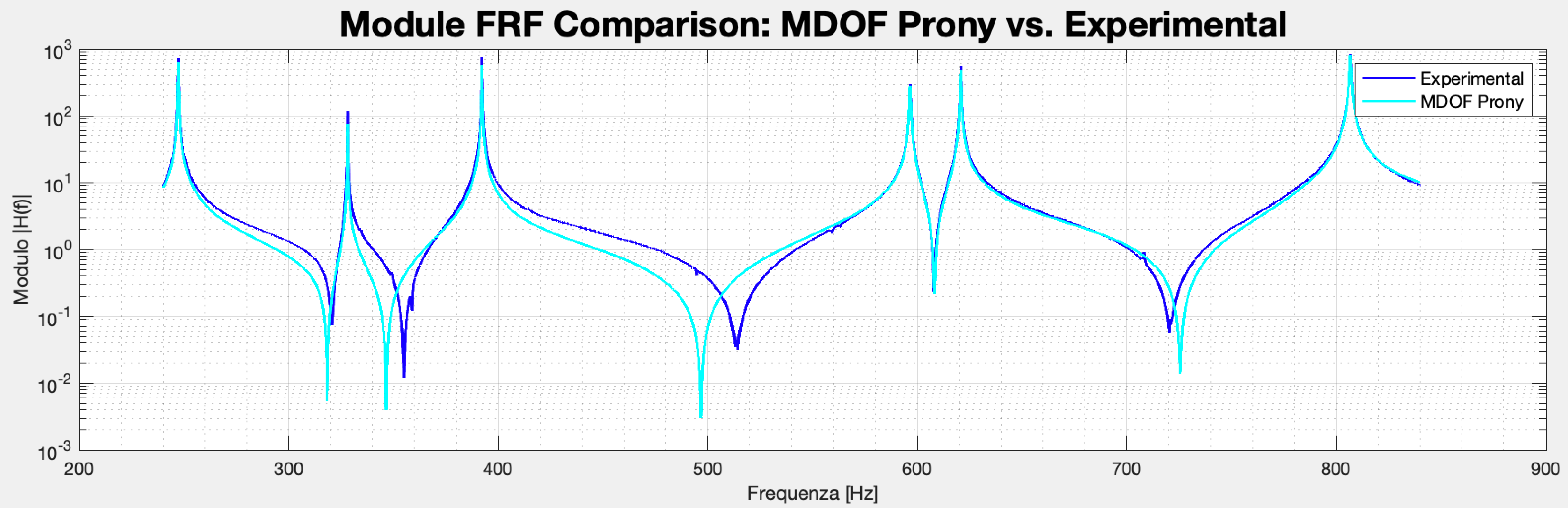
MODAL PARAMETERS SUMMARY

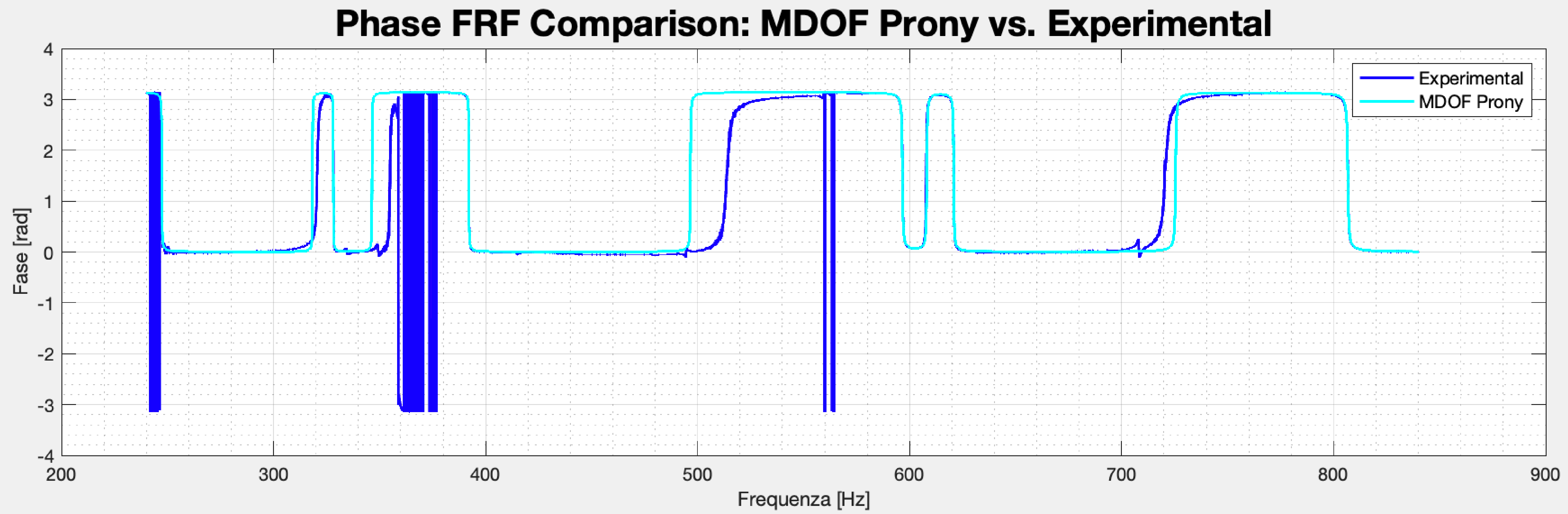
FREQUENCIES: CONCORDING WITH FEM

DAMPING: LOW -> THIN SOLID STEEL PLATE

MODE SHAPE: CONCORDING WITH FEM

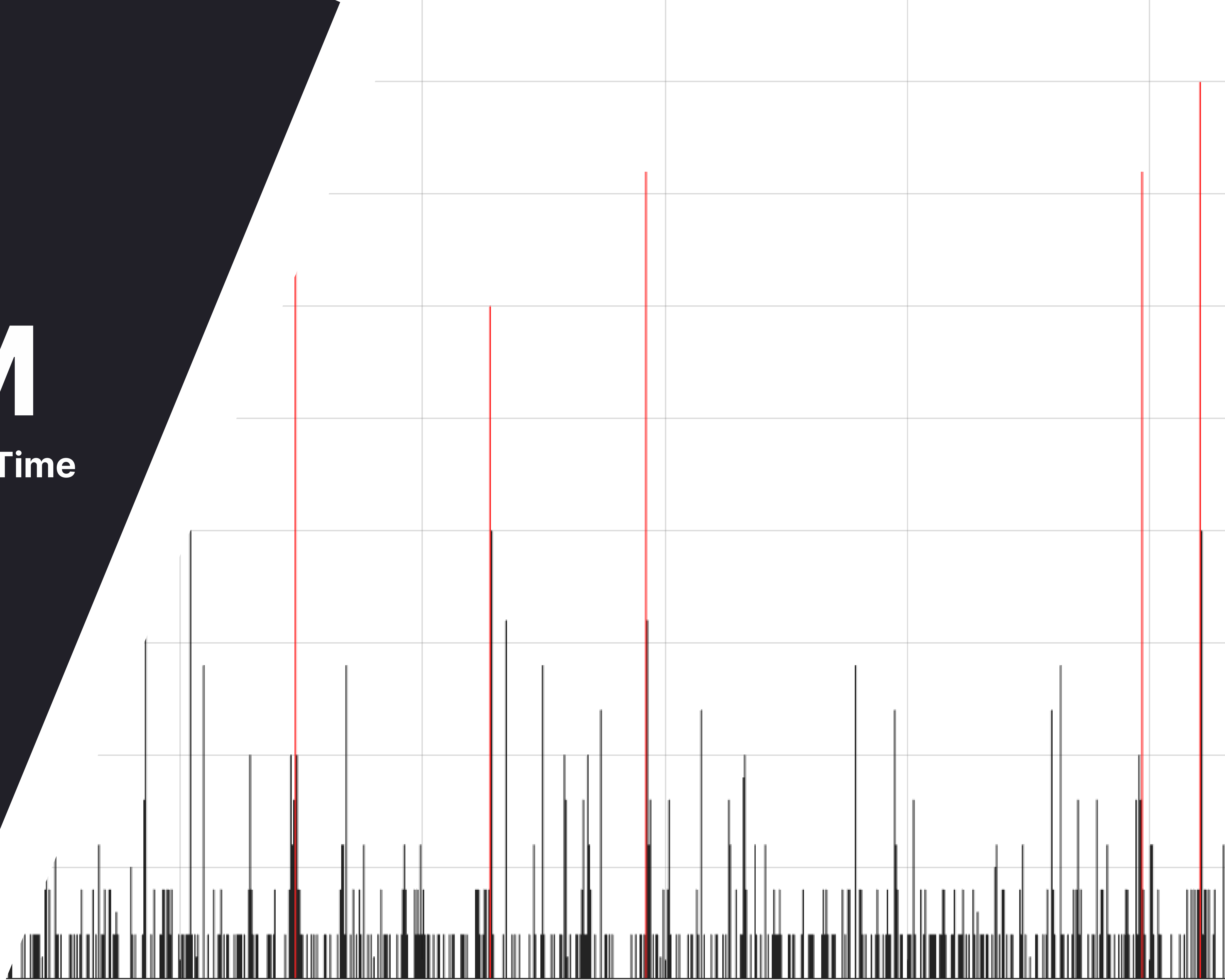
FRF RECONSTRUCTION - MDOF PRONY





MDOF IBRAHIM

Multiple-Reference Ibrahim Time Domain Method (MITD)



FUNDAMENTAL IDEA

Impulse Response Functions

$$h_{pq}(t) = \sum_{r=1}^{N} \frac{\psi_{p,r}\psi_{q,r}}{2\mathrm{j}m_r\omega_r\sqrt{1-\xi_r^2}} e^{s_r t} + \left(\frac{\psi_{p,r}\psi_{q,r}}{2\mathrm{j}m_r\omega_r\sqrt{1-\xi_r^2}}\right)^* e^{s_r^* t}$$

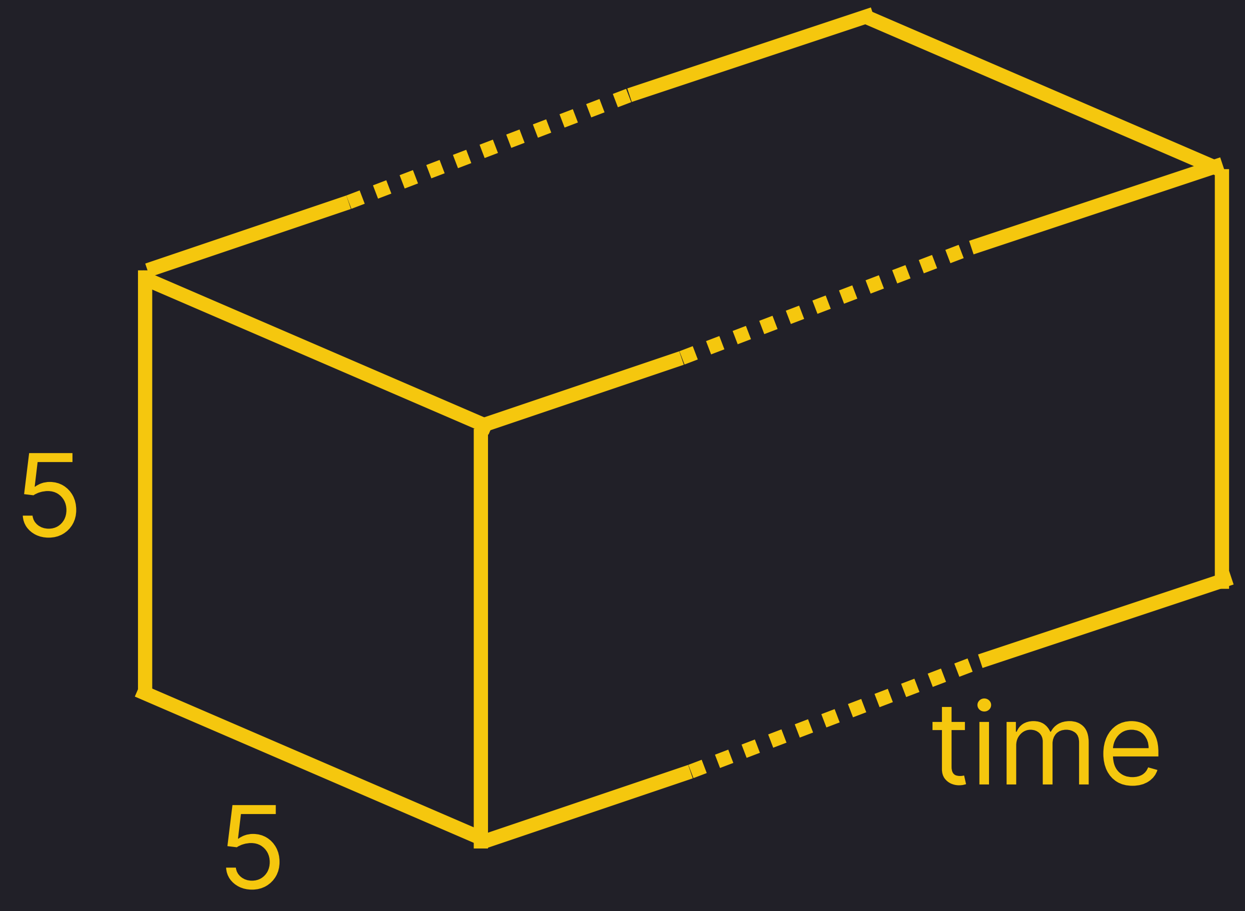
Matrix formulation:

$$[h(t)] = [\Psi][e^{s_r t}][L]^T$$

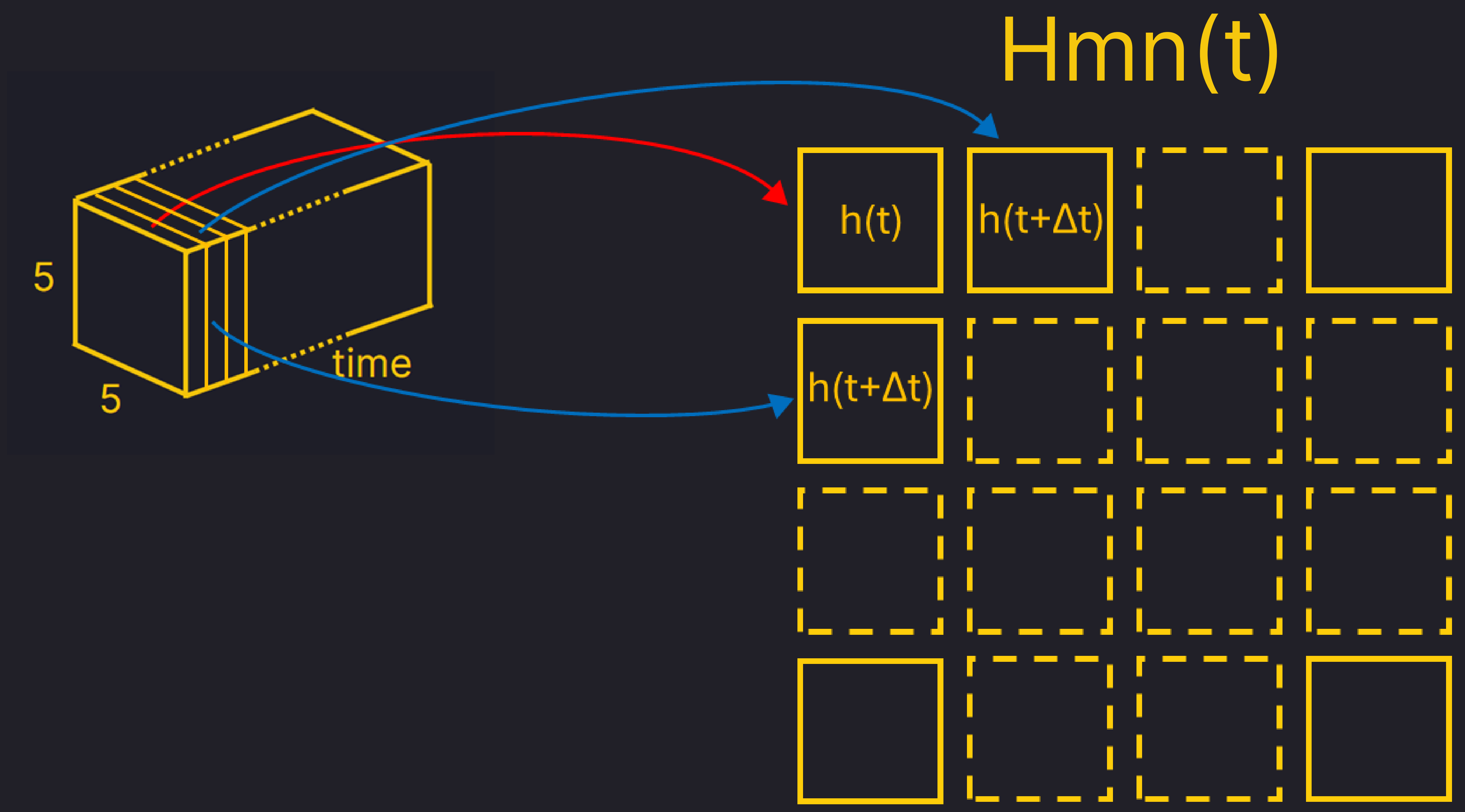
IRFs
MATRICES

$$\begin{cases} n_l = 5 & \text{number of outputs} \\ n_s = 5 & \text{number of inputs} \end{cases}$$

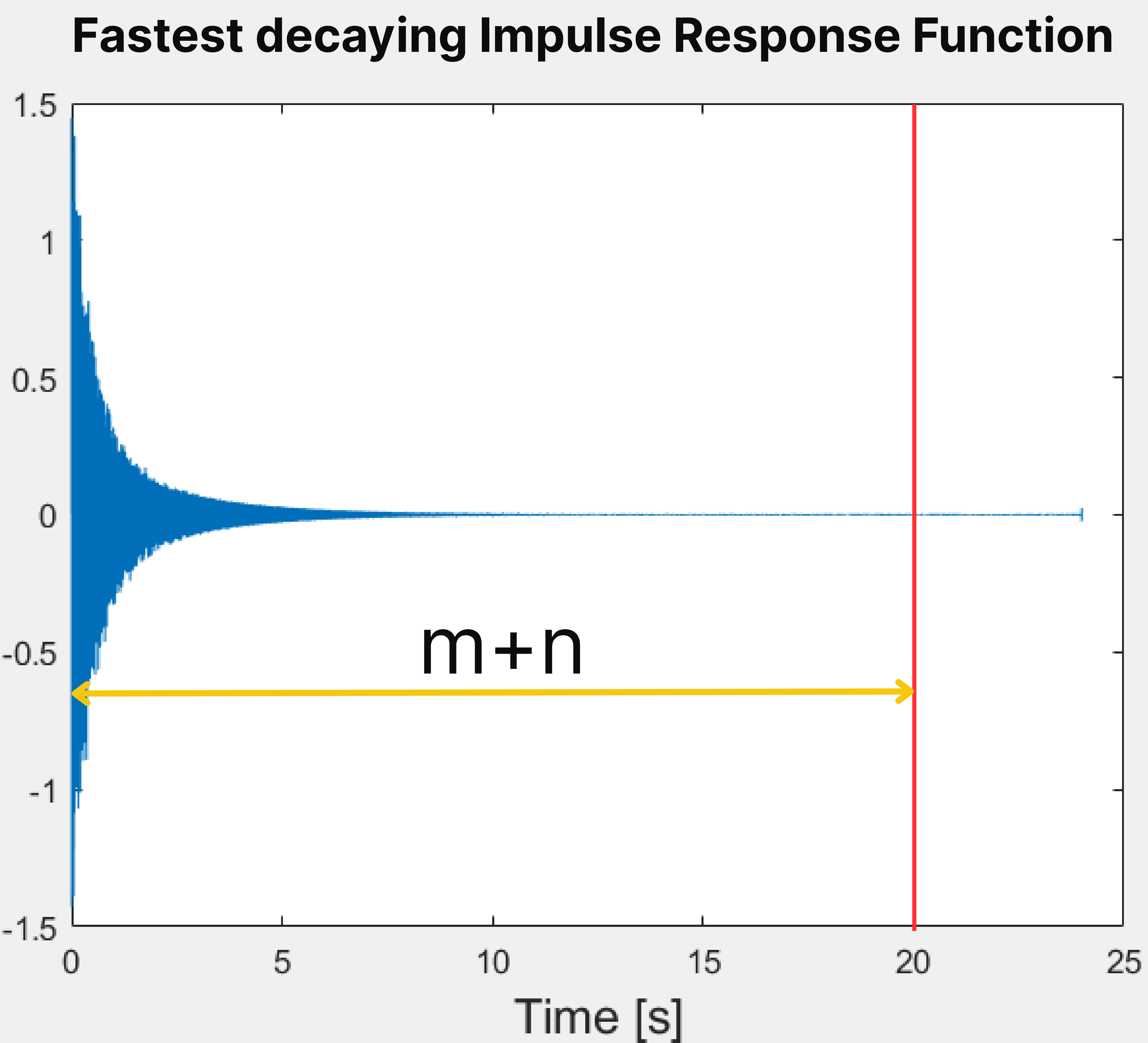
$$[h(t)] = \begin{bmatrix} h_{11}(t) & h_{12}(t) & h_{13}(t) & h_{14}(t) & h_{15}(t) \\ h_{21}(t) & h_{22}(t) & h_{23}(t) & h_{24}(t) & h_{25}(t) \\ h_{31}(t) & h_{32}(t) & h_{33}(t) & h_{34}(t) & h_{35}(t) \\ h_{41}(t) & h_{42}(t) & h_{43}(t) & h_{44}(t) & h_{45}(t) \\ h_{51}(t) & h_{52}(t) & h_{53}(t) & h_{54}(t) & h_{55}(t) \end{bmatrix}$$



FROM 3D TO 2D



$$[H_{mn}(t)] = \begin{bmatrix} h(t) & h(t+\Delta t) & \dots & h(t+(n-1)\Delta \mathrm{t}) \\ h(t+\Delta t) & h(t+2\Delta t) & \dots & h(t+n\Delta t) \\ \vdots & \vdots & \vdots & \vdots \\ h(t+(m-1)\Delta \mathrm{t}) & h(t+m\Delta t) & \dots & h(t+(m+n-2)\Delta \mathrm{t}) \end{bmatrix}$$



IS ALL THE DATA NECESSARY?

- Time series is cropped to limit noise introduction
- Length of time series fixes m+n = constant

$$[H_{mn}(t)] = [\tilde{\Psi}]\left[e^{s_r t}\right][\tilde{L}]$$

Rectangular Matrix!

- $[H_{mn}(t)]$ is $m * 5$ x $n * 5$
- $[\tilde{\Psi}]$ is $m * 5$ x $2N_{modes}$
- $[e^{s_r t}]$ is $2N_{modes}$ x $2N_{modes}$
- $[\tilde{L}]$ is $2N_{modes}$ x $n * 5$

DIMENSIONS MATTER!

$$[H_{mn}(t)] = [\tilde{\Psi}]\,[e^{s_r t}]\,[\tilde{L}]$$

Rectangular Matrix!

- $[H_{mn}(t)]$ is $m * 5$ x $n * 5$
- $[\tilde{\Psi}]$ is $m * 5$ x $2N_{modes}$
- $[e^{s_r t}]$ is $2N_{modes}$ x $2N_{modes}$
- $[\tilde{L}]$ is $2N_{modes}$ x $n * 5$

TIME SHIFTED Hmn

$$[H_{mn}(t+\Delta t)] = [\widetilde{\Psi}][e^{s_r \Delta t}][e^{s_r t}][\tilde{L}]^T = [\widetilde{\Psi}][e^{s_r \Delta t}][\widetilde{\Psi}]^+[H_{mn}(t)]$$

NOTICE:

$$[H_{mn}(t)] = [\widetilde{\Psi}][e^{s_r t}][\tilde{L}]^T \Rightarrow [\widetilde{\Psi}]^+[H_{mn}(t)] = [e^{s_r t}][L]^T$$

↓

$$[H_{mn}(t+\Delta t)]\ [H_{mn}(t)]^+ = [\widetilde{\Psi}][e^{s_r \Delta t}][\widetilde{\Psi}]^+$$

SQUARE MATRIX 5m x 5m

SQUARE MATRIX

EIGENVALUE-EIGENVECTOR APPROACH

$$[H_{mn}(t+\Delta t)]\ [H_{mn}(t)]^{+} = [\tilde{\Psi}][e^{s_r \Delta t}][\tilde{\Psi}]^{+}$$

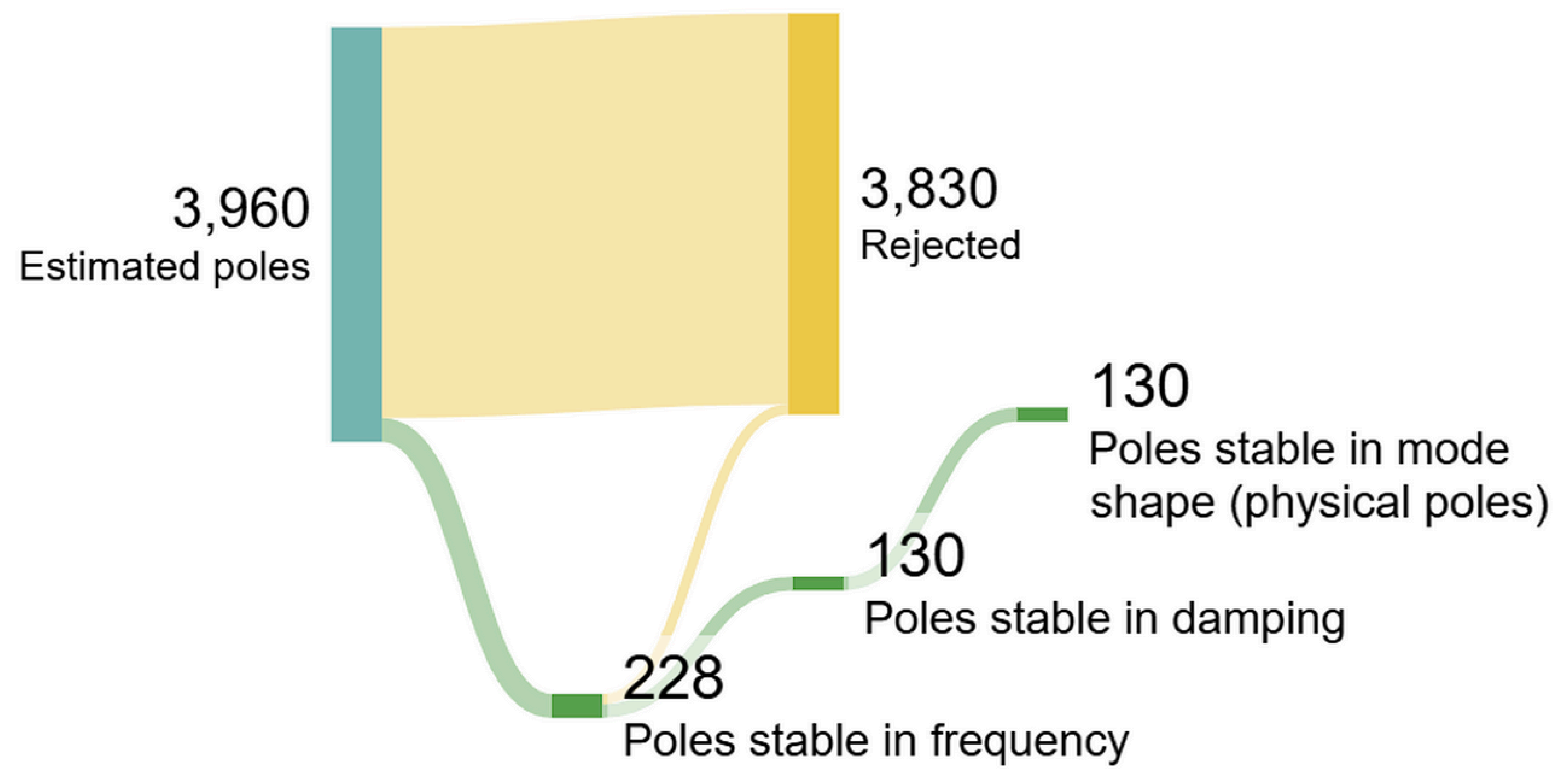
- $[H_{mn}(t+\Delta t)][H_{mn}(t)]^{+}$ is $m*5$ x $m*5$
- $[\tilde{\Psi}]$ is $m*5$ x $m*5$
- $[e^{s_r \Delta t}]$ is $m*5$ x $5*m$
- $[\tilde{\Psi}]^{+}$ is $m*5$ x $m*5$

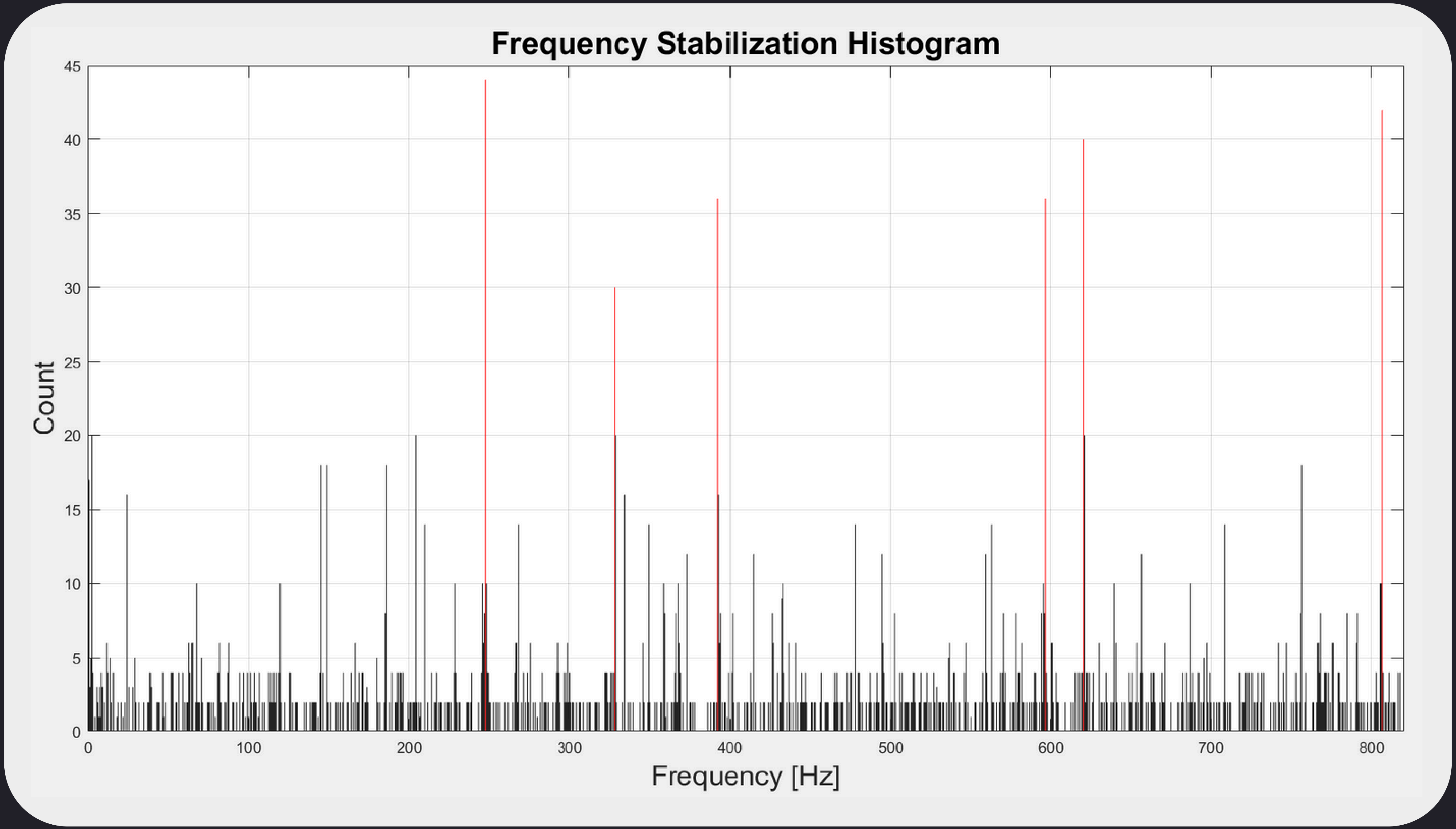
SYSTEM'S ORDER

2N = 5m

FILTERING PHYSICAL POLES

ONLY 3.3% SURVIVED

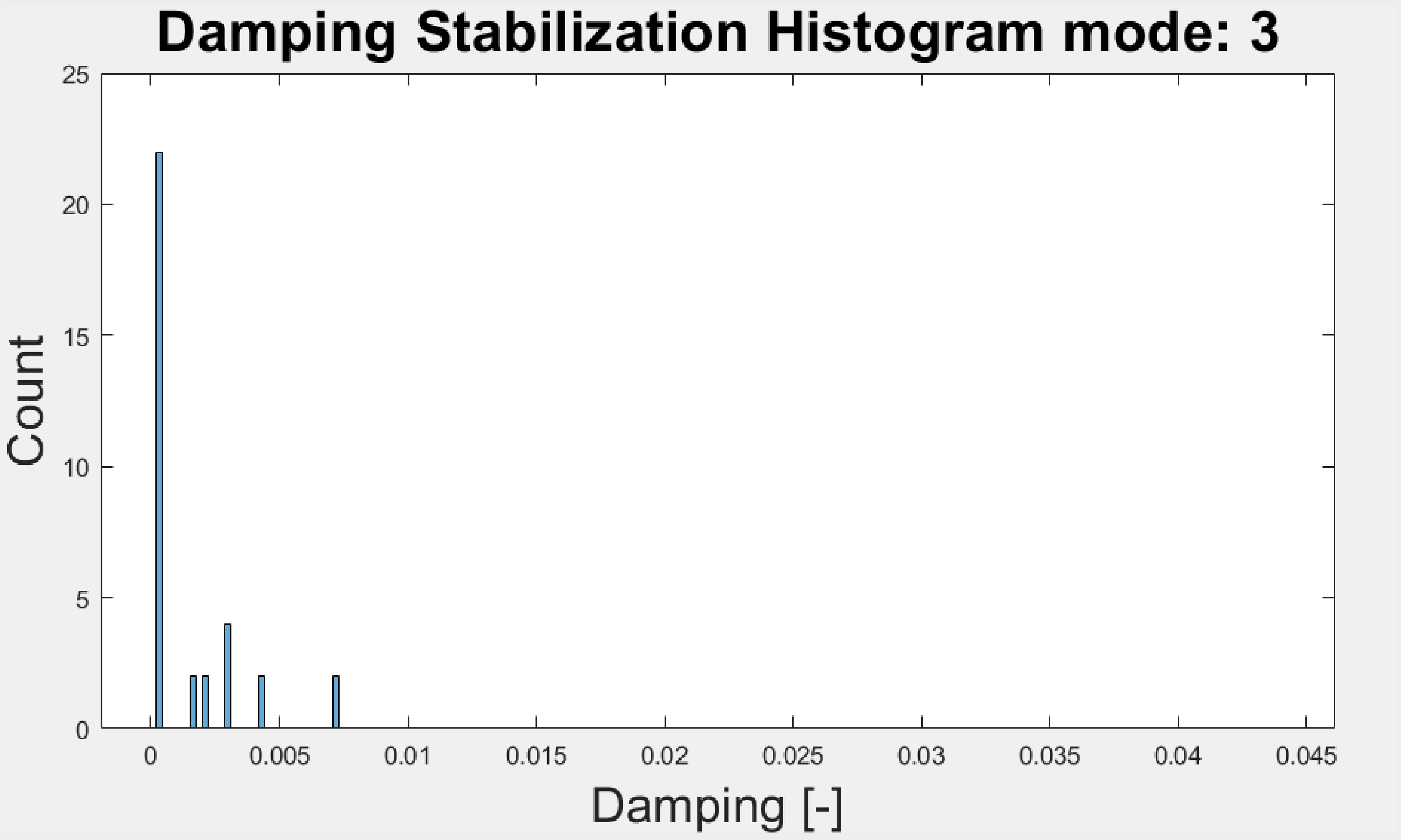


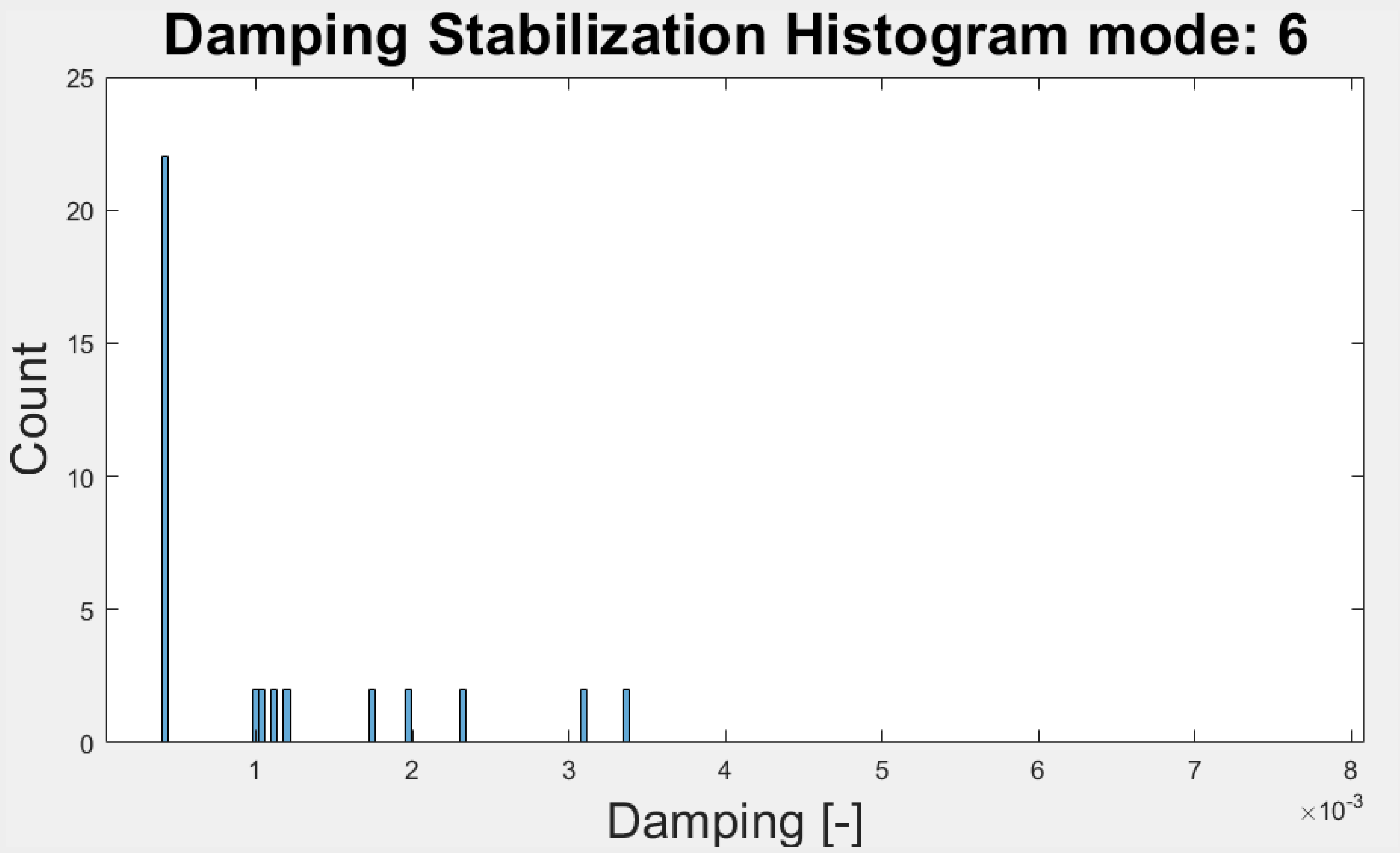


FREQUENCY STABLE POLES

- ≈247 Hz
- ≈328 Hz
- ≈392 Hz
- ≈596 Hz
- ≈621 Hz
- ≈807 Hz

DAMPING STABLE POLES





MODE STABLE POLES

MAC Good enough...

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	1	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

	12	13	14	15	16	17	18	19	20	21	22
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0	0
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

ONLY 20 POLES FOR MODE 1

MODAL MASSES ESTIMATION

$$H_{pq}(\omega) = \sum_{r=1}^{2*6} \frac{\psi_{pr}\psi_{qr}}{j\omega - s_r} Q_r + R_U + \frac{R_L}{\omega^2}$$

SETUP A LEAST SQUARE PROBLEM

$$\left\{ \begin{array}{c} H_{pq}(\omega_1) \\ H_{pq}(\omega_2) \\ \vdots \\ H_{pq}(\omega_{N_f}) \end{array} \right\} = \begin{bmatrix} \frac{\psi_{p1}\psi_{q1}}{j\omega_1 - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_1 - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_1 - s_{2*6}} & 1 & \frac{1}{\omega_1^2} \\ \frac{\psi_{p1}\psi_{q1}}{j\omega_2 - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_2 - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_2 - s_{2*6}} & 1 & \frac{1}{\omega_2^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{\psi_{p1}\psi_{q1}}{j\omega_{N_f} - s_1} & \frac{\psi_{p2}\psi_{q2}}{j\omega_{N_f} - s_2} & \cdots & \frac{\psi_{p2*6}\psi_{q2*6}}{j\omega_{N_f} - s_{2*6}} & 1 & \frac{1}{\omega_{N_f}^2} \end{bmatrix} \left\{ \begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{2*6} \\ R_U \\ R_L \end{array} \right\}$$

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Res.Frequencies [Hz]	247.29	328.17	392.12	596.57	620.81	806.63
Damping [-]	0.00041	0.00019	0.00022	0.00040	0.00027	0.00041
Norm. Mode shapes	-1	0.26	0.8	1	0.66	-0.05
	0.91	0.24	1	-0.28	1	-1
	-0.89	0.25	0.80	-0.49	-0.66	-0.95
	0.92	0.23	0.81	0.82	-0.92	-0.09
	-0.03	1	-0.11	-0.79	0.16	0.43

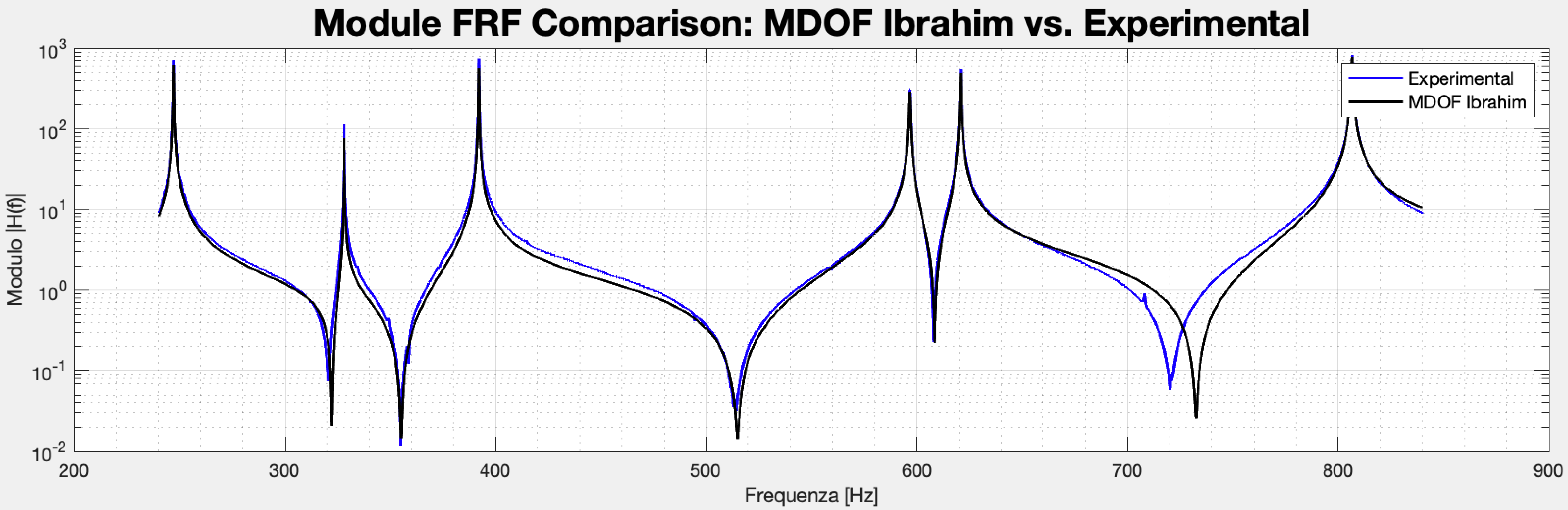
MODAL PARAMETERS SUMMARY

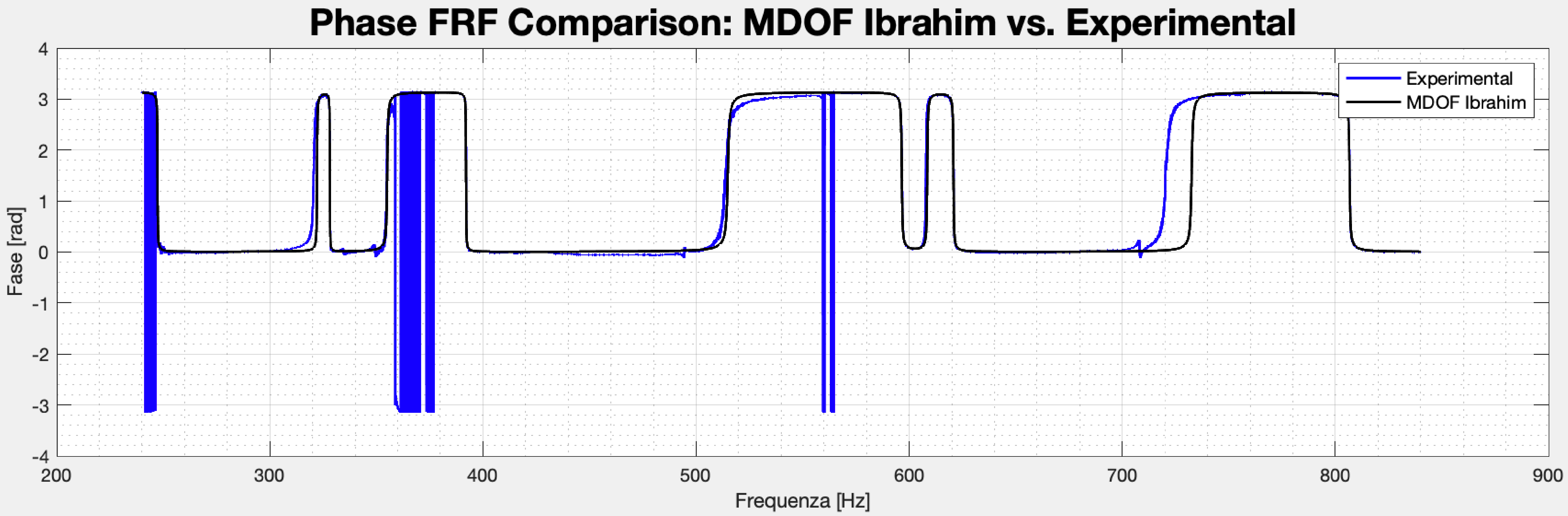
FREQUENCIES: CONCORDING WITH FEM

DAMPING: LOW -> THIN SOLID STEEL PLATE

MODE SHAPE: CONCORDING WITH FEM

FRF RECONSTRUCTION - MDOF IBRAHIM

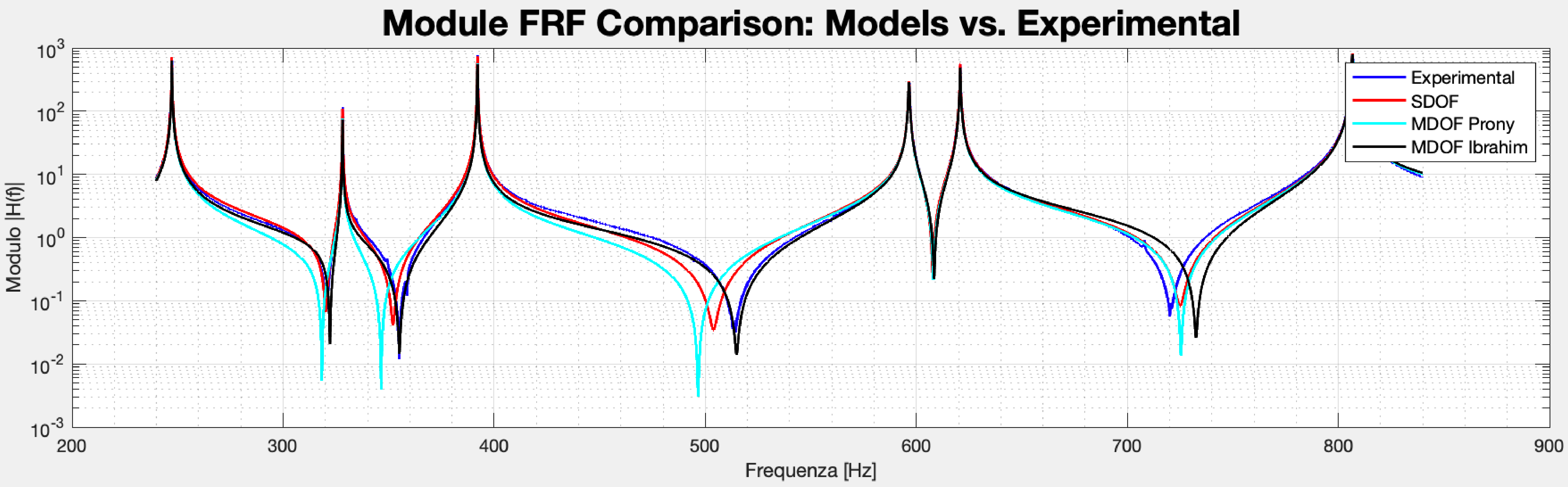


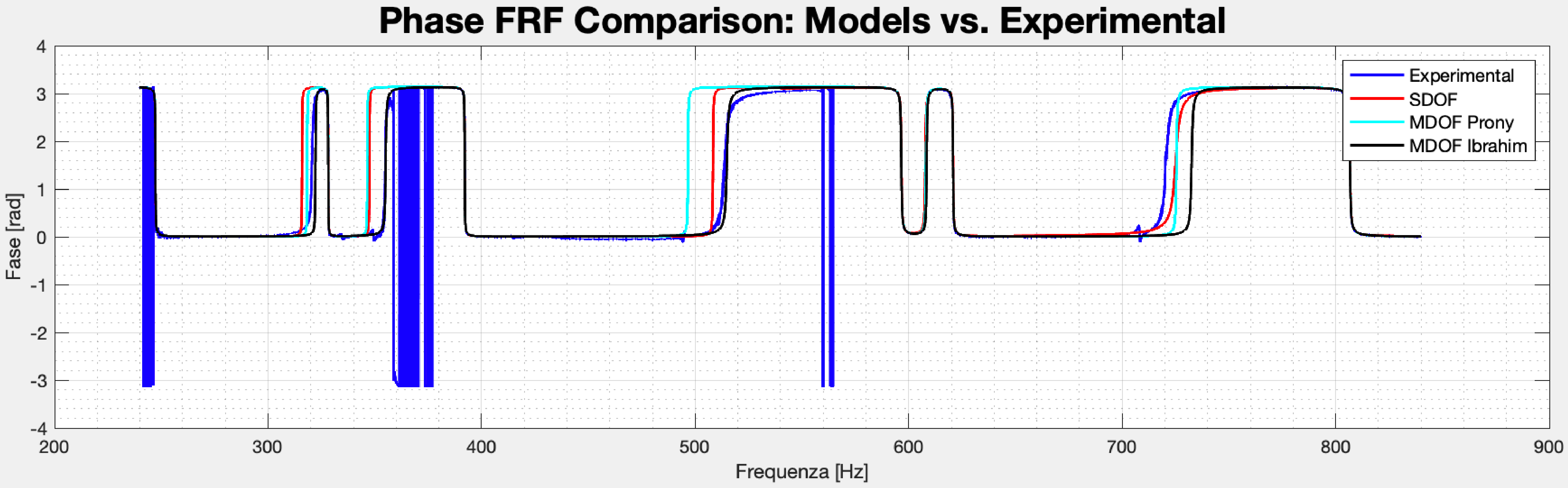


RESULTS

MAC computation between experimental and numerical mode shapes

FRF RECONSTRUCTION - FINAL COMPARISON



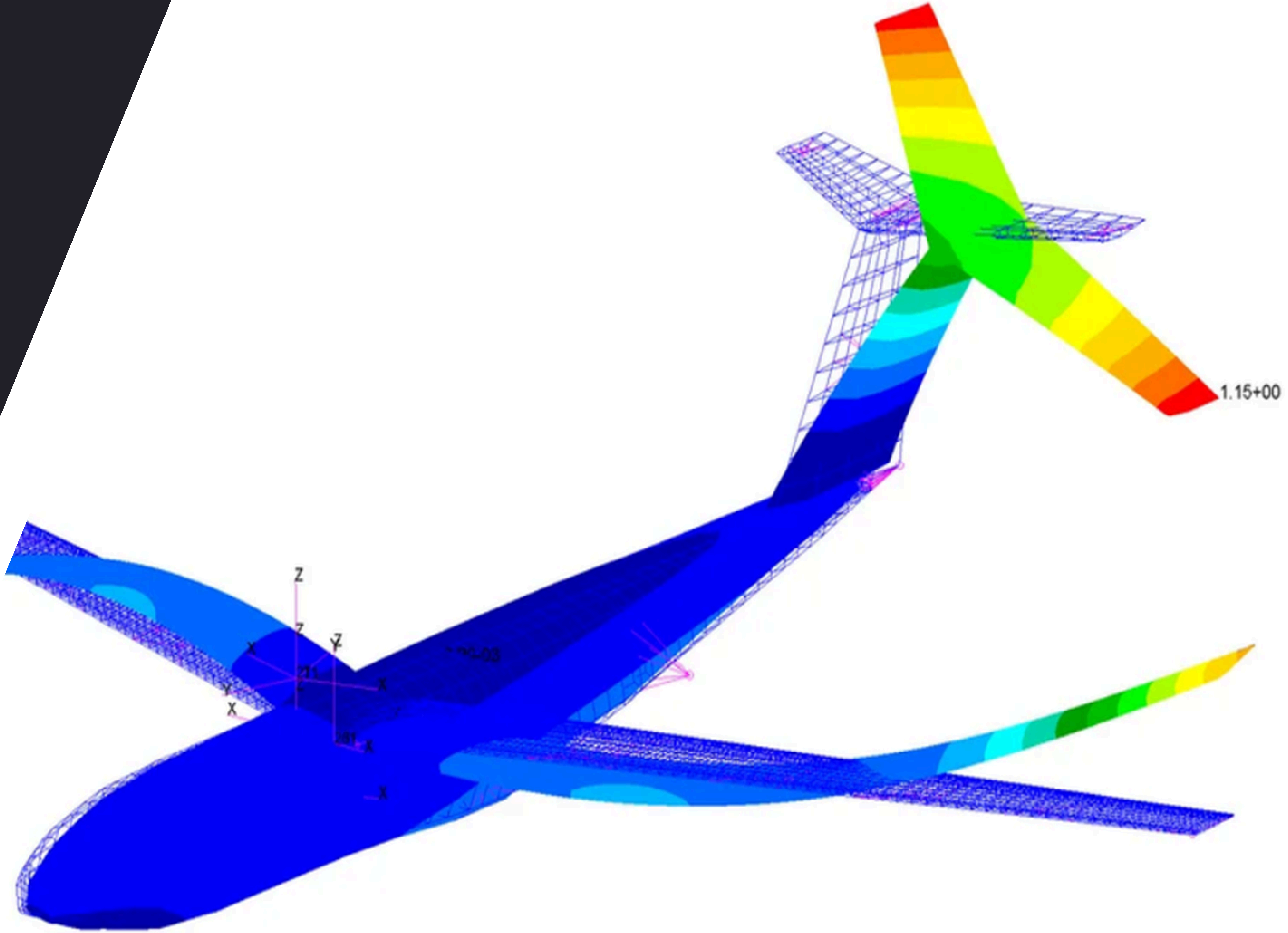


MAC COMPARISON

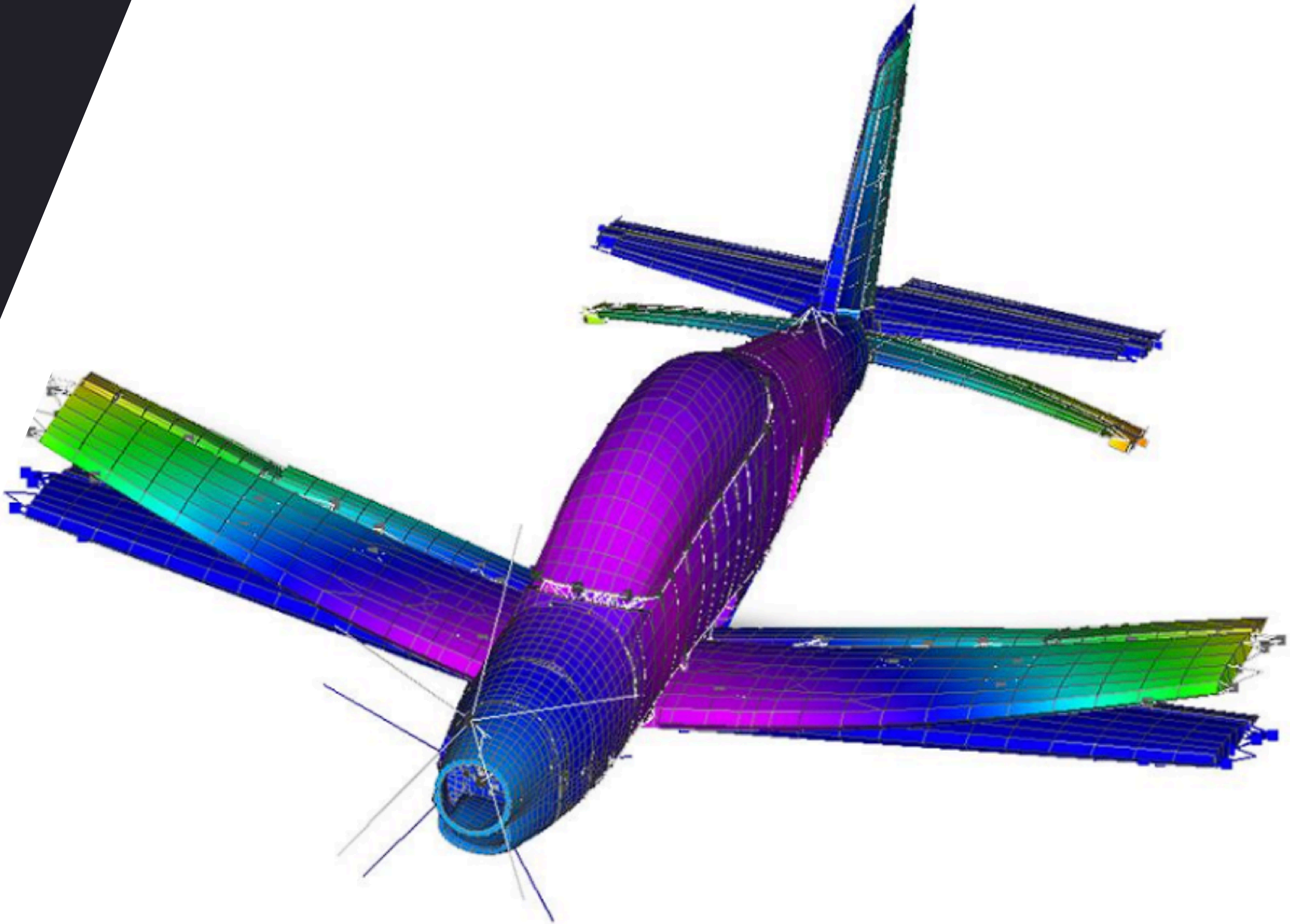
$$MAC(A,X)=\frac{\left|\sum_{j=1}^{n}(\psi_X)_j(\psi_A)_j^*\right|^2}{\left(\sum_{j=1}^{n}(\psi_X)_j(\psi_X)_j^*\right)\cdot\left(\sum_{j=1}^{n}(\psi_A)_j(\psi_A)_j^*\right)}=\frac{\left|\{\psi_X\}^T\{\psi_A\}\right|^2}{\left(\{\psi_X\}^T\{\psi_X\}\right)\cdot\left(\{\psi_A\}^T\{\psi_A\}\right)}$$

MODE	SDOF	PRONY	IBRAHIM
1st	0.9996	0.9979	0.9996
2nd	0.9973	0.9939	0.9972
3rd	0.9980	0.9938	0.9980
4th	0.9986	0.9954	0.9983
5th	0.9968	0.9922	0.9969
6th	0.9987	0.9902	0.9988

WHY ALL THIS?



WHY ALL THIS?



Data Analysis Assignement
a.a. 2024/2025

THANK YOU

FOR YOUR ATTENTION

Group 8