Editorial: Exam in MAC - 1935D

Avighna

June 2024

To begin with, let me state an obvious boolean logic fact: $\neg(a \land b) = \neg a \lor \neg b$. One way to solve this problem is to find the number of pairs that **don't satisfy** the given condition, and subtract this from the total number of pairs.

The negated problem statement is this: given an array of unique integers s, find the number of pairs of integers (x, y) such that $0 \le x \le y \le c$, and either $x + y \in s$ or $x - y \in s$, or both.

The **total** number of these (x, y) pairs is $\frac{(c+1)(c+2)}{2}$. This example with $c = 3$ should clear things up:

$(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$,
$(1, 1)$, $(1, 2)$, $(1, 3)$,
$(2, 2)$, $(2, 3)$,
$(3, 3)$

Now, for a pair (a, b), let e being 'present on the left' mean that $a + b = e$, and let e being 'present on the right' mean that $b - a = e$. Let $\mathrm{lp}(x)$ be a function that returns the number of times x is present on the left across all $\mathcal{O}(n^2)$ pairs, and let $\mathrm{rp}(x)$ be a similar function that returns the number of times x is present on the right across all $\mathcal{O}(n^2)$ pairs. Along with these two functions, we need to define another additional function: Let $\mathrm{pe}(x, y)$ be 1 if there exists a pair (a, b) with $a, b \le c$ such that both $a + b = x$ and $b - a = y$, and 0 otherwise (all of this may seem a little bit arbitrary, but stick with me for a second).

We'll now try to find a way to count the number of pairs satisfying our negated problem by iterating over s (that's really the only thing we can iterate over). With some PIE, we get that the number of pairs (x) is:

$$x = \sum_{e \in s} \mathrm{lp}(e) + \sum_{e \in s} \mathrm{rp}(e) - \sum_{u \in s} \sum_{v \in s} \mathrm{pe}(u, v)$$

Let us first try to find $v_3 := \sum_{u \in s} \sum_{v \in s} \mathrm{pe}(u, v)$.

$$\begin{cases} a + b = x \\ b - a = y \end{cases}$$

is a system of two equations with two unknowns. Solving for a and b, we get $a = \dfrac{x - y}{2}$ and $b = \dfrac{x + y}{2}$. For (a, b) to be a valid pair, both a and b need to be integers between $[0, c]$. In other words: $0 \le a \le c$ and $0 \le b \le c$
$\implies 0 \le \dfrac{x - y}{2} \le c$ and $0 \le \dfrac{x + y}{2} \le c$
$\implies 0 \le x - y \le 2c$ and $0 \le x + y \le 2c$

From this we infer that if $\mathrm{pe}(x, y) = 1 \implies y \le x$. If we iterate over each element s_i and find the first index j such that $s_i + s_j > 2c$ (if there's no such element, $j = n$), $j - i$ pairs of the form (s_u, s_i) with $u \in [i, j)$ will be valid, existing pairs: we simply add $j - i$ to v_3 (initialized to 0) on every iteration.

But wait, we're not done yet! We also need to make sure that $s_u + s_i$ is even, so that a and b are integers. This means that we want the parities of s_u and s_i to match. Here's an easy way to do this: if s_i is even, subtract from v_3 a number equal to how many odd numbers s_k there are with $k \in [i, j)$, and if s_i is odd, do the same thing for even numbers. Two prefix-sum arrays can be used to quickly count how many odd/even numbers there are within a range.

This will take $\mathcal{O}(n \log n)$ time with binary search.

Now let us determine an expression for $\text{lp}(e)$. How many pairs have e present on the left is equivalent to asking how many ways there are to write e as a sum of two integers $x, y \leq c$, without double counting (that is, if (x, y) has been counted as a solution, (y, x) should not). Let a pair (x, y) summing up to e mean that $x + y = e$.

$e = 0$ is a special case. If $e = 0$, there is only 1 pair that sums up to e: $(0, 0)$. Similarly, since $x, y \leq c$, if $e > 2c$, it's impossible to have any pairs that sum up to e. Now let us consider two cases

If $e \leq c$, then we can write all of the pairs that sum up to e as:

$(0, e)$
$(1, e-1)$
$(2, e-2)$
...
$(e, 0)$

We will now find the first overcounted pair in this list. Let the n^{th} pair be $(n-1, e-n+1)$. Then we get the inequality: $n - 1 > e - n + 1$
$\implies 2n > e + 2$
$\implies n > \dfrac{e+2}{2}$

Let us define another function:

$$\text{gtf}(e) = \begin{cases} \dfrac{e}{2} + 2 & \text{if } e \text{ is even} \\ \lceil \dfrac{e+2}{2} \rceil & \text{otherwise} \end{cases}$$

('gtf' is short for 'greater function')

Since the first overcounted pair is $\text{gtf}(e)$, the number of pairs in this case is $\text{lp}(e) = \text{gtf}(e) - 1$.

When $e > c$, our pairs still remain the same, but some at the beginning (for example, the pair $(0, e)$ in all cases) and some at the end will be invalid. The valid pairs will lie somewhere in the middle. We have three conditions to abide by: 1) $n - 1 \leq c$, the first term of the pair
2) $e - n + 1 \leq c$, the second term of the pair
3) $n \leq \text{gtf}(e) - 1$, to prevent overcounting

Solving these three inequalities gives us $e - c + 1 \leq n \leq \min(c + 1, \text{gtf}(e) - 1)$, which means $n \in [e - c + 1, \min(c + 1, \text{gtf}(e) - 1)]$ (it can be checked that the start of this range is only bigger than its end when $e > 2c$). Thus the number of valid pairs in this case will be $\text{lp}(e) = \min(c+1, \text{gtf}(e)-1) - e + c$.

We now have:

$$\mathrm{lp}(e) = \begin{cases} 1 & \text{if } e = 0 \\ 0 & \text{if } e > 2c \\ \mathrm{gtf}(e) - 1 & \text{if } e \leq c \\ \min(c+1, \mathrm{gtf}(e) - 1) - e + c & \text{otherwise} \end{cases}$$

Let us finally determine an expression for $\mathrm{rp}(e)$. How many pairs have e present on the right is equivalent to asking how many ways there are to write e as a difference of two integers $x, y \leq c$, without double counting. Let a pair (x, y) differing to e mean that $y - x = e$.

If $e > c$, the answer is 0 since a difference of two non-negative integers $(y - x)$ cannot equal e for any $y < e$. If $e \leq c$, then the pairs differing to e will be:

$(0, e)$
$(1, e+1)$
$(2, e+2)$
...
$(c-e, c)$

There are $c - e + 1$ such pairs. So:

$$\mathrm{rp}(e) = \begin{cases} 0 & \text{if } e > c \\ c - e + 1 & \text{otherwise} \end{cases}$$

We now have all the tools necessary to find the answer, which is $\dfrac{(c+1)(c+2)}{2} - x$.