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Design and Analysis of Algorithms
Homework: Building Familiarity With Proofs

1. **Prove that: If x is an odd integer, then x + 1 is even.**
    a. The definition of an odd integer is an integer that can be represented as $2k + 1$, where k is an integer. Thus, adding 1 to an odd integer equals $(2k + 1) + 1$, which equals $2k + 2$. We can factor 2 out of this expression to give us $x = 2(k + 1)$. Let $k + 1 = n$. Since both k and 1 are integers, n is also an integer. Thus, we now have $x = (2k + 1) + 1 = 2k + 2 = 2(k + 1) = 2(n)$. Since the definition of an even number is any number that can be written as 2k, where k is an integer, this number is even. (Alternatively, since $2k + 2$ is the sum of two even numbers, it must also be even.)

2. **Theorem: $\forall n \in N, 3 \mid (n^3 - n)$. Prove the theorem using induction.**
    a. To prove using induction, we begin with a base case. Take $n = 2$. Thus, we have $(2^3 - 2) = 8 - 2 = 6$. Since 6 is divisible by 3, we have proven the base case.

        Before we begin, we can first factor the original expression to give us: $\mathbf{n(n + 1)(n - 1)}$.

        Next, we must prove that assuming the theorem is true for any natural number n (what we will call the **original expression**), it remains true for n + 1 (hereafter called the **"n + 1" expression**).

        Plugging in n + 1 for n in the original expression, we get: $[(n + 1)^3 - (n + 1)]$. This expression can be simplified to give us: $n^3 + 3n^2 + 3n + 1 - n - 1$, which can be further reduced to: $n^3 + 3n^2 + 2n$. Factoring out an n, we get: $n(n^2 + 3n + 2)$, and fully factored, we get: $\mathbf{n(n + 1)(n + 2)}$.

        At this point, the original expression – $\mathbf{n(n + 1)(n - 1)}$ – is very similar to our "n + 1" expression – $\mathbf{n(n + 1)(n + 2)}$. In fact, they are both a multiple of three integers, the first two of which are identical between them: $n(n + 1)$. To simplify things, we may set $n(n + 1) = x$, where x is an integer. Our original expression now becomes $x(n - 1)$, and our "n + 1" expression becomes $x(n + 2)$.

        If we let $(n – 1) = m$, we see that our original expression can be written as **mx**, a product of two integers. Since $[(n - 1) + 3] = (n + 2)$, substituting m for $(n - 1)$, our "n + 1" expression becomes $x(n + 2) = x(m + 3) = \mathbf{mx + 3x}$.

        Since we know that **mx** is divisible by 3 (since it is equivalent to the original proposition, which we are presuming to be true), and **3x** must be divisible by 3, as it is the product of an integer x and 3, we have proven that if the theorem holds true for any natural number n, it must also be true for n + 1.

        Thus, by induction, the theorem must be true.