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Design and Analysis of Algorithms
Homework: Building Familiarity with Proofs

1. **Prove that: If x is an odd integer, then x + 1 is even.**
    a. The definition of an odd integer is an integer that can be represented as $2k + 1$, where k is an integer. Thus, adding 1 to an odd integer equals $(2k + 1) + 1$, which equals $2k + 2$. We can factor 2 out of this expression to give us $x = 2(k + 1)$. Let $k + 1 = n$. Since both k and 1 are integers, n is also an integer. Thus, we now have $x = (2k + 1) + 1 = 2k + 2 = 2(k + 1) = 2(n)$. Since the definition of an even number is any number that can be written as 2k, where k is an integer, this number is even. (Alternatively, since $2k + 2$ is the sum of two even numbers, it must also be even.)

2. **Theorem: $\forall n \in N, 3 \mid (n^3 - n)$. Prove the theorem using induction.**
    a. To prove using induction, we begin with a base case. Take n = 0 or 1. Thus, we have $(0^3 - 0) = 0$, or $(1^3 - 1) = 0$. Since 0 is divisible by 3 (0/3 = 0), we have proven the base case.

       Next, we must prove that assuming the theorem is true for any natural number i (what we will call the **original expression**), it remains true for i + 1 (hereafter called the **"i + 1" expression**).

       Substituting i for n in the expression and factoring, our original expression becomes: $(i^3 - i) = i(i^2 - 1) =$ **$i(i + 1)(i - 1)$.**

       Plugging in i + 1 for i in this expression, we get: $(i + 1)[(i + 1) + 1][(i + 1) - 1]$. This expression can be simplified to give us: $(i + 1)(i + 2)(i) =$ **$i(i + 1)(i + 2)$**.

       At this point, the original expression – **$i(i + 1)(i - 1)$** – is very similar to our "i + 1" expression – **$i(i + 1)(i + 2)$**. In fact, they are both a multiple of three integers, the first two of which are identical between them: $i(i + 1)$. To simplify things, we may set $i(i + 1) = x$, where x is an integer. Our original expression now becomes $x(i - 1)$, and our "i + 1" expression becomes $x(i + 2)$.

       If we let $(i - 1) = m$, we see that our original expression can be written as **mx**, a product of two integers. Since $[(i - 1) + 3] = (i + 2)$, substituting m for $(i - 1)$, our "i + 1" expression becomes: $x(i + 2) = x(m + 3) =$ **$mx + 3x$**.

       Since we know that **mx** is divisible by 3 (since it is equivalent to the original proposition, which we are presuming to be true), and **3x** must be divisible by 3, as it is the product of an integer x and 3, we have proven that if the theorem holds true for any natural number i, it must also be true for i + 1.

       Thus, by induction, the theorem must be true.