Lecture 02

Wednesday, 21 December 2022 3:00 PM

Stochastic Process : Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be any ordered index set. A stochastic process is a collection of random variables $(X_\tau : \tau \in T)$

- The common range, S is called the 'State space'. We will restrict our discussion to at most countable S.
- The index set usually represent time. We will consider $T = \mathbb{Z}_+$ in this course.
- So we will be dealing with finite state, discrete time stochastic processes of "special kind."

Sample path : A sample path is a realization of the stochastic process. If the random experiment realizes $\omega \in \Omega$, then we realize the sample path

$$(X_\tau(\omega) = x_\tau : \tau \in T)$$

Discrete time Markov chain : A stochastic process $(X_n : n \in \mathbb{Z}_+)$ on probability space $(\Omega, \mathcal{F}, P)$ is called a DTMC if the state space, S is at most countable and satisfy the following property : $\forall\, i, j \in S$, $\forall\, n \in \mathbb{Z}_+$

$$\boxed{P(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0) = P(X_{n+1} = j \mid X_n = i)}$$
$$= P_{ij}(n)$$

This property is called the 'Markov' property.

- The stochastic process is independent of past given the present.

Time homogeneous DTMC: The DTMC $(X_n : n \in \mathbb{Z}_+)$ is called time homogeneous if $\forall\ k \in \mathbb{Z}$

$$P(X_{n+k+1} = j \mid X_{n+k} = i) = P(X_{n+1} = j \mid X_n = i)$$

i.e., transition probabilities are independent of time. or, $p_{ij}(n) = p_{ij}$.

- We will confine our discussion to homogeneous DTMCs.

Transition Probability Matrix (TPM): Consider homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$. p_{ij} is the transition probability from state i to next state j. The matrix $P_{|S|\times|S|}$ $P_{ij} = p_{ij}$ is called the transition probability matrix of the DTMC $(X_n : n \in \mathbb{Z}_+)$

Example: $|S| = 3$

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.6 & 0 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

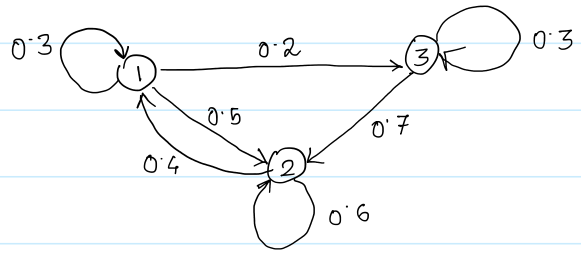
Property: Row sum is 1 for all rows. Such matrices are called Stochastic matrices.

Transition Graph: Consider the homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$. It can be represented by a directed weighted graph $G = (S, E, w)$ where the set of nodes is the state space S, the set of directed edges is the set of possible one-step transitions indicated by the initial and final state as

$$E = \{ [i,j\rangle \in S \times S : p_{ij} > 0 \}$$

In addition the graph has a weigh $w_e = p_{ij}$ where $e = [i,j\rangle \in E$.

* Draw the transition graph for P.



* N step transition probability:

Consider a homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$ with $|S| = m$.

WLOG consider $S = \{1, 2, \ldots, m\}$.

Transition probability from state i to k is given by p_{ik}, i.e.,

$$P(X_{n+1} = k \mid X_n = i) = p_{ik}$$

We want to find $P(X_{n+2} = j \mid X_n = i)$ in terms of p_{ij}'s.

- $\bigcup_{k=1}^{m} \{X_{n+1} = k\} = \Omega$

Let $A = \{X_{n+2} = j\}$, $B_k = \{X_{n+1} = k\}$, $C = \{X_n = i\}$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P\left(\bigcup_{k=1}^{m} (A \cap B_k \cap C)\right)}{P(C)}$$

$$= \sum_{k=1}^{m} \frac{P(A \cap B_k \cap C)}{P(C)}$$

$$= \sum_{k=1}^{m} \frac{P(A \cap B_k \cap C)}{P(B_k \cap C)} \cdot \frac{P(B_k \cap C)}{P(C)}$$

$$= \sum_{k=1}^{m} P(A \mid B_k \cap C) \cdot P(B_k \mid C)$$

$$= \sum_{k=1}^{m} P(X_{n+2} = j \mid X_{n+1} = k, X_n = i) \cdot P(X_{n+1} = k \mid X_n = i)$$

$$= \sum_{k=1}^{m} P(X_{n+2} = j \mid X_{n+1} = k) \cdot P(X_{n+1} = k \mid X_n = i)$$

$$= \sum_{k=1}^{m} p_{kj} \cdot p_{ik}$$

Consider the matrix P^2 now.

$$P^2_{i,j} = \sum_{k=1}^{n} p_{ik} \cdot p_{kj} = P(X_{n+2} = j \mid X_n = i)$$

So, P^2 is the 2-step transition probability matrix.

$k=1$

So, P^2 is the 2-step transition probability matrix.

· In the same way P^N is the N-step TPM.

Now let distribution of state at time n is given by π_n i.e., $\pi_n(i) = P(X_n = i)$. We want to find π_{n+1}.

$$\pi_{n+1}(j) = P(X_{n+1} = j) = \sum_{i=1}^{n} P(X_{n+1} = j \mid X_n = i) \cdot P(X_n = i)$$

$$= \sum_{i=1}^{n} p_{ij} \cdot \pi_n(i)$$

$\therefore \quad \pi_{n+1} = \pi_n P$. Similarly $\pi_{n+\ell} = \pi_n P^\ell$

Random Mapping Theorem:

Let $(Z_n : n \in \mathbb{N})$ is an i.i.d random process and $f : S \times \Lambda \to S$ be a function with S at most countable.

A random process $(X_n : n \in \mathbb{Z}_+)$ such that $X_{n+1} = f(X_n, Z)$ is a DTMC.

$$P(X_{n+1} = y \mid X_n = x, X_{n-1}, \ldots, X_1, X_0)$$

$$= P(f(X_n, Z_n) = y \mid X_n = x, X_{n-1}, \ldots, X_0)$$

$$= P(f(X_n, Z_n) = y \mid X_n = x)$$

$$= P(X_{n+1} = y \mid X_n = x)$$

Consider a parking lot with 2 parking slots. Parking events in two slots are independent of each other. Suppose that a slot is occupied at time n, then probability that it will remain be occupied at time $n+1$ is q. Likewise if a slot is empty at time n then probability that it will be empty at time $n+1$ is p.

Consider the number of empty slots as state, find the transition probability matrix of the DTMC. Draw the transition graph.

Definitions: Consider a random process $(X_n : n \in \mathbb{Z}_+)$ with initial state $X_0 = i \in S$.

1. First hitting time to state j is denoted by τ_j^i and is defined as

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$$\tau_j^i = \inf\{ n \in \mathbb{Z}_+ : X_n = j, X_0 = i \}$$

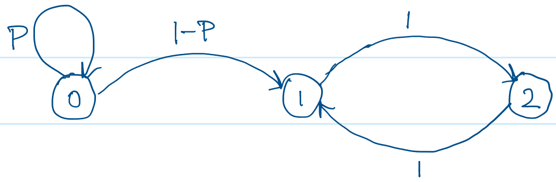
2. The probability of hitting state j eventually is denoted by $f_{ij} = P(\tau_j^i < \infty)$

3. The probability of first visit to state j at time $n \in \mathbb{N}$ is denoted by $f_{ij}^{(n)} = P(\tau_j^i = n)$

4. The first passage time distribution is $((f_{ij}^{(n)} : n \in \mathbb{N}), 1 - f_{ij})$

5. The first recurrence time distribution is $((f_{ii}^{(n)} : n \in \mathbb{N}), 1 - f_{ii})$

<u>Example 1</u> : Consider a DTMC with the following transition graph



Find first recurrence distribution of each state.

$f_{00}^{(1)} = P \quad f_{00}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1\}$. $\therefore f_{00} = P \Rightarrow 1 - f_{00} = 1 - P$

$f_{11}^{(1)} = 0, \quad f_{11}^{(2)} = 1, \quad f_{11}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1,2\}$.

$\therefore f_{11} = 1$.

<u>Definitions</u> :

1. A state i is called recurrent if $f_{ii} = 1$, and is called transient if $f_{ii} < 1$.

2. Mean recurrence time of a state i is denoted by

2. Mean recurrence time of a state i is denoted by

$$m_i = \mathbb{E}[\tau_i^i] = \sum_{n \in \mathbb{N}} n \cdot f_{ii}^{(n)} \cdot + \infty \cdot \mathbb{1}\{f_{ii} < 1\}$$

3. For a recurrent state $i \in S$, i is null recurrent state if $m_i = \infty$, otherwise i is called positive recurrent.