

Lecture 02

Wednesday, 21 December 2022 3:00 PM

Stochastic Process: Let (Ω, \mathcal{F}, P) be a probability space and T be any ordered index set. A stochastic process is a collection of random variables $(X_\tau : \tau \in T)$

- The common range, S is called the 'State Space'. We will restrict our discussion to at most countable S .
- The index set usually represent time. We will consider $T = \mathbb{Z}_+$ in this course.
- So we will be dealing with finite state, discrete time stochastic processes of "special kind."

Sample path: A sample path is a realization of the stochastic process. If the random experiment realizes $\omega \in \Omega$, then we realize the sample path

$$(X_\tau(\omega) = x_\tau : \tau \in T)$$

Discrete time Markov chain: A stochastic process $(X_n : n \in \mathbb{Z}_+)$ on probability space (Ω, \mathcal{F}, P) is called a DTMC if the state space, S is at most countable and satisfy the following property: $\forall i, j \in S, \forall n \in \mathbb{Z}_+$

$$\begin{aligned} P(X_{n+1} = j | X_n = i, X_{n-1}, \dots, X_0) &= P(X_{n+1} = j | X_n = i) \\ &= P_{ij}(n) \end{aligned}$$

↙ this property is called the 'Markov' property.

- The stochastic process is independent of past given the present.

Time homogeneous DTMC : The DTMC $(X_n : n \in \mathbb{Z}_+)$ is called time homogeneous if $\forall k \in \mathbb{Z}$

$$P(X_{n+k} = j | X_n = i) = P(X_{n+k} = j | X_n = i)$$

i.e., transition probabilities are independent of time or.

$$P_{ij}(n) = P_{ij}.$$

- We will confine our discussion to homogeneous DTMCs.

Transition Probability Matrix (TPM) : Consider homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$. P_{ij} is the transition probability from state i to next state j . The matrix $P_{|S| \times |S|}$ $P_{ij} = P_{ij}$ is called the transition probability matrix of the DTMC $(X_n : n \in \mathbb{Z}_+)$

Example: $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.4 & 0.6 & 0 \\ 0 & 0.7 & 0.3 \end{bmatrix}$

Property : Row sum is 1 for all rows. Such matrices are called Stochastic matrices.

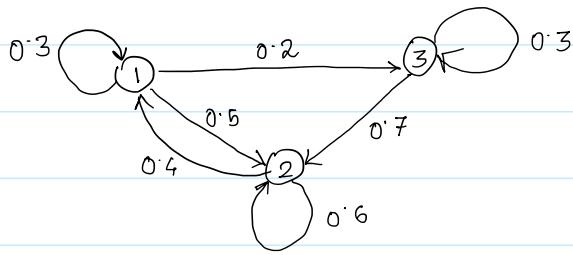
Transition Graph : Consider the homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$.

It can be represented by a directed weighted graph $G = (S, E, w)$ where the set of nodes is the state space S , the set of directed edges is the set of possible one-step transitions indicated by the initial and final state as

$$E = \{(i, j) \in S \times S : P_{ij} > 0\}$$

In addition the graph has a weight $w_e = P_{ij}$ where $e = (i, j) \in E$.

- * Draw the transition graph for P .



* N step transition Probability:

Consider a homogeneous DTMC $(X_n : n \in \mathbb{Z}_+)$ with $|S| = m$.

WLOG consider $S = \{1, 2, \dots, m\}$.

Transition probability from state i to k is given by p_{ik} , i.e.,

$$P(X_{n+1} = k | X_n = i) = p_{ik}$$

We want to find $P(X_{n+2} = j | X_n = i)$ in terms of p_{ij} 's.

- $\bigcup_{k=1}^m \{X_{n+1} = k\} = \Omega$

Let $A = \{X_{n+2} = j\}$, $B_k = \{X_{n+1} = k\}$, $C = \{X_n = i\}$

$$\begin{aligned} P(A|C) &= \frac{P(A \cap C)}{P(C)} = \frac{P\left(\bigcup_{k=1}^m (A \cap B_k \cap C)\right)}{P(C)} \\ &= \sum_{k=1}^m \frac{P(A \cap B_k \cap C)}{P(C)} \\ &= \sum_{k=1}^m \frac{P(A \cap B_k \cap C)}{P(B_k \cap C)} \cdot \frac{P(B_k \cap C)}{P(C)} \\ &= \sum_{k=1}^m P(A|B_k \cap C) \cdot P(B_k|C) \\ &= \sum_{k=1}^m P(X_{n+2} = j | X_{n+1} = k, X_n = i) \cdot P(X_{n+1} = k | X_n = i) \\ &= \sum_{k=1}^m P(X_{n+2} = j | X_{n+1} = k) \cdot P(X_{n+1} = k | X_n = i) \\ &= \sum_{k=1}^m p_{kj} \cdot p_{ik} \end{aligned}$$

Consider the matrix P^2 now.

$$P_{i,j}^2 = \sum_{k=1}^m p_{ik} \cdot p_{kj} = P(X_{n+2} = j | X_n = i)$$

So, P^2 is the 2-step transition probability matrix.

P^2 is the 2-step transition probability matrix.
 In the same way P^N is the N-step TPM.

Now let distribution of state at time n is given by π_n i.e.,

$\pi_n(i) = P(X_n = i)$. We want to find π_{n+1} .

$$\begin{aligned}\pi_{n+1}(j) &= P(X_{n+1} = j) = \sum_{i=1}^n P(X_{n+1} = j | X_n = i) \cdot P(X_n = i) \\ &= \sum_{i=1}^n p_{ij} \cdot \pi_n(i)\end{aligned}$$

$\therefore \pi_{n+1} = \pi_n P$. Similarly $\pi_{n+l} = \pi_n P^l$

Random Mapping Theorem:

Let $(Z_n : n \in \mathbb{N})$ is an i.i.d random process and $f: S \times A \rightarrow S$ be a function with S at most countable.

A random process $(X_n : n \in \mathbb{Z}_+)$ such that $X_{n+1} = f(X_n, Z)$ is a DTMC.

$$\begin{aligned}P(X_{n+1} = y | X_n = x, X_{n-1}, \dots, X_1, X_0) \\ &= P(f(X_n, Z_n) = y | X_n = x, X_{n-1}, \dots, X_0) \\ &= P(f(X_n, Z_n) = y | X_n = x) \\ &= P(X_{n+1} = y | X_n = x)\end{aligned}$$

Consider a parking lot with 2 parking slots. Parking events in two slots are independent of each other. Suppose that a slot is occupied at time n , then probability that it will remain be occupied at time $n+1$ is q . Likewise if a slot is empty at time n then probability that it will be empty at time $n+1$ is p .

Consider the number of empty slots as state, find the transition probability matrix of the DTMC. Draw the transition graph.

Definitions: Consider a random process $(X_n : n \in \mathbb{Z}_+)$ with initial state $X_0 = i \in S$.

i. First hitting time to state j is denoted by τ_j^i and is defined as

is defined as

$$z_j^i = \inf \{ n \in \mathbb{Z}_+ : X_n = j, X_0 = i \}$$

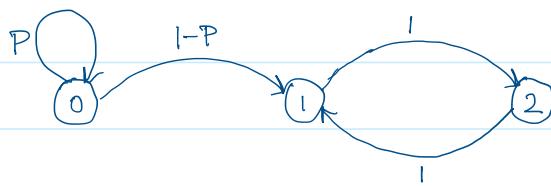
2. The probability of hitting state j eventually is denoted by $f_{ij} = P(z_j^i < \infty)$

3. The probability of first visit to state j at time $n \in \mathbb{N}$ is denoted by $f_{ij}^{(n)} = P(z_j^i = n)$

4. The first passage time distribution is $((f_{ij}^{(n)} : n \in \mathbb{N}), 1 - f_{ij})$

5. The first recurrence time distribution is $((f_{ii}^{(n)} : n \in \mathbb{N}), 1 - f_{ii})$

Example 1 : Consider a DTM C with the following transition graph



Find first recurrence distribution of each state.

$$f_{00}^{(1)} = P \quad f_{00}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1\}. \quad \therefore f_{00} = P \Rightarrow 1 - f_{00} = 1 - P$$

$$f_{11}^{(1)} = 0, \quad f_{11}^{(2)} = 1, \quad f_{11}^{(n)} = 0 \quad \forall n \in \mathbb{N} \setminus \{1, 2\}.$$

$$\therefore f_{11} = 1.$$

Definitions :

1. A state i is called recurrent if $f_{ii} = 1$, and is called transient if $f_{ii} < 1$.

2. Mean recurrence time of a state i is denoted by

$$\text{---} \quad \# [i] \quad \sum_m r^{(n)} \quad . \quad \text{in 11 S P 113}$$

2. Mean recurrence time of a state i is denoted by

$$m_i = \mathbb{E}[\tau_i^i] = \sum_{n \in \mathbb{N}} n \cdot f_{ii}^{(n)} + \infty \cdot \mathbb{I}\{f_{ii} < 1\}$$

3. For a recurrent state $i \in S$, i is null recurrent state if $m_i = \infty$, otherwise i is called positive recurrent.