Lecture 03

Saturday, 14 January 2023 11:09 AM

Last time:

1. Stochastic processes
2. DTMC, homogeneous DTMC
3. TPM, Transition graph
4. N-step transition probability.
5. Transient & recurrent states; Null recurrent & positive recurrent states

Today

Communicating Classes

Definitions

1. For states $i, j \in S$, we say j is accessible from i if $P_{ij}^{(n)} > 0$ for some $n \in \mathbb{Z}_+$ and denoted by $i \to j$.

2. We say i, j communicate each other if $i \to j$ and $j \to i$, and denote by $i \to j$.

2.a. A set of states $C \subseteq S$ that communicate are called a communicating class.

2.b. A communicating class C is called closed if no edges leaves this class that is $i \in C$, $k \in C^c$, $\forall n \in \mathbb{N}$ $p_{ik}^{(n)} = 0$, otherwise called open.

Equivalence Relation: A relation R is called equivalence relation if it satisfies following 3 properties -

i. Symmetry: If iRj then jRi.

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ii. Reflexivity: iRi
iii. Transitivity: If iRj and jRk then iRk.

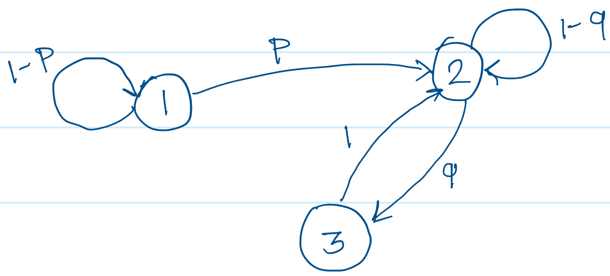
Proposition 1: Communication is an equivalence relation.

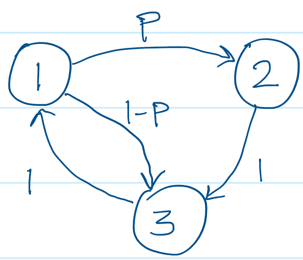
Proof:

- Equivalence relation partitions the space. So the state space is partitioned into number of communicating classes.
- Class property: A property of states is said to be a class property if for each communicating class C, either all state in C have the property or, none do.

Definitions

(Irreducibility) A DTMC with a single communicating class is called an irreducible DTMC. That is for any $i, j \in S$, $\exists\, n \in \mathbb{N}$ s.t. $P_{ij}^{(n)} > 0$.





$p, q \in (0,1)$

(Aperiodic) Let $\mathcal{T}(i) \triangleq \{n \in \mathbb{N} : p_{ii}^{(n)} > 0\}$ be the set of times when the chain can possibly return to the initial state i.
The period of any state $i \in S$ is defined as

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$$d_i = \gcd T(i) = \gcd\{n \in \mathbb{N} : p_{ii}^{(n)} > 0\}$$

Example (Continue from prev).

Proposition 2: Periodicity is class property, i.e., if $i \leftrightarrow j$. then $d_i = d_j$.

Proposition 3: Transience, positive recurrence and null recurrence are class properties.

Let $\mathbb{1} = (1 \ 1 \ \dots \ 1)^T \in \mathbb{R}^{|S| \times 1}$

What is $P \cdot \mathbb{1}$? $P \cdot \mathbb{1} = \mathbb{1}$.

So, $\mathbb{1}$ is an eigenvector of any stochastic matrix and corresponding eigen value is 1.

$A v = \lambda v$ Eigen value $u^T A = \lambda u^T$

right eigenvectors left eigenvectors

The eigen values are same.

So there must be an left eigenvector corresponding to eigenvalue 1, i.e., $\exists \ \mu \in \mathbb{R}^{|S| \times 1}$ s.t. $\mu^T P = \mu^T$.

Definition: A probability distribution μ on S is called invariant distribution for the DTMC $\{X_n\}_n$ if it satisfy the global balance equation

$$\mu = \mu P.$$

Definition: An irreducible, aperiodic, positive recurrent Markov.

Chain is called ergodic.

For ergodic Markov chains

$$\mu(j) = \lim_{n\to\infty} P_{ij}^{(n)} \rightarrow \text{steady state probability.}$$

Ex 1 : Let there are two urns and each urn has 5 balls. Out of total 10 balls 5 are white and 5 are black. At each time 1 ball is drawn from each of the urns and exchanged. Let X_n takes the number of white balls in urn 1.

a. Show that $\{X_n\}$ is a Markov chain.

b. Find the steady state probability that the number of white balls in the 1st urn be zero.

Sol^n State space, $\mathcal{X} = \{0, 1, \ldots, 5\}$. Let the current state be $x \in \mathcal{X}$. 4 cases are possible -

① White from urn 1, White from urn 2 : Prob - $\frac{x}{5} \cdot \frac{5-x}{5}$

② White from urn 1, Black from urn 2 : Prob - $\frac{x}{5} \cdot \frac{x}{5}$

③ Black from urn 1, White from urn 2 : Prob - $\frac{5-x}{5} \cdot \frac{5-x}{5}$

④ Black from urn 1, Black from urn 2 : Prob - $\frac{5-x}{5} \cdot \frac{x}{5}$

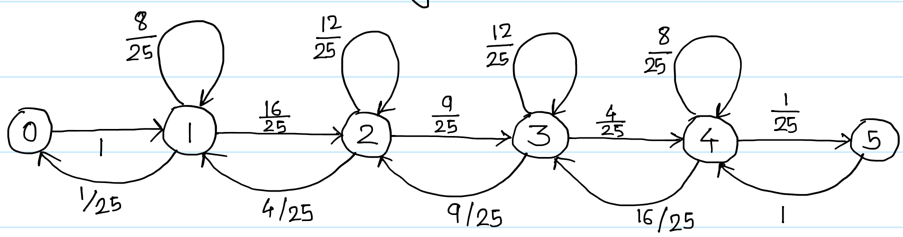
If 1 or 4 happens, $X_{n+1} = X_n$

If 2 happens, $X_{n+1} = X_n - 1$

If 3 happens, $X_{n+1} = X_n + 1$

$$X_{n+1} = \begin{cases} X_n & \text{w.p. } \frac{2(5-X_n)X_n}{25} \\ X_n + 1 & \text{w.p. } \frac{(5-X_n)^2}{25} \\ X_n - 1 & \text{w.p. } \frac{X_n^2}{25} \end{cases}$$

So, by random mapping theorem $\{X_n\}$ is a Markov chain.



So, the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{25} & \frac{8}{25} & \frac{16}{25} & 0 & 0 & 0 \\ 0 & \frac{4}{25} & \frac{12}{25} & \frac{9}{25} & 0 & 0 \\ 0 & 0 & \frac{9}{25} & \frac{12}{25} & \frac{4}{25} & 0 \\ 0 & 0 & 0 & \frac{16}{25} & \frac{8}{25} & \frac{1}{25} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

* Find π that satisfies the equation $\pi = \pi P$.

$\pi(0)$ is the steady state probability of 'urn 1 having no white ball'.