Design And Analysis Of Algorithms
Assignment : 4

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1 Ques. : 1

Proof of : **A graph G and integer k, does $G(V, E)$ have a cycle, with no repeated nodes, of length at least k** - is a NP Complete problem.

- **The given problem is in NP Class** :-
  - If any problem is in NP, then, given a 'certificate', which is a solution to the problem and an instance of the problem, we will be able to verify whether the solution given is correct or not in polynomial time.
  - **Certificate :** A sequence of vertices A from the graph
  - **Certifier :-**
    - First verify the length of the sequence is at least k
    - Then verify vertices in the sequence belong to the graph $[\forall v \in A \implies v \in V]$
    - Now verify for each pair of vertices $u, v \in A$, which are adjacent according to the sequence, have an edge between them in our graph $[(u, v) \in E]$ (Because that will make sure the cycle exists)
    - This Certifier algorithm is a polynomial time algorithm $[O(k) : k = \text{length of the sequence}]$. And also the Certificate is in the order of the no. of vertices (Polynomial size). So the problem is in NP.
- **The Hamilton cycle problem [known NP-Complete problem] is polynomial time reducible to the given problem : HAMILTON-CYCLE $\leq_p$ Given-Problem**
  - Given a graph $G(V, E)$, we will construct an instance of Given problem, that has a simple cycle of length at least k iff the graph has a Hamilton Cycle.
  - **Construction :** assume $|V|$ = number of vertices in the graph. Now consider the $k = |V|$. That means we have find a cycle, with no vertices repeating, of length at least $|V|$
  - **G has a simple cycle of length $\geq k \implies G$ has a Hamilton Cycle**
    G has a cycle of length $\geq k$
    $\implies$ G has a cycle of length $\geq |V|$ [as $k = |V|$]
    $\implies$ The cycle covers all the vertices [As no vertices are repeating]
    $\implies$ The cycle is a Hamilton Cycle
  - **G has a Hamilton Cycle $\implies G$ has a simple cycle of length $\geq k$**
    G has a Hamilton Cycle
    $\implies$ There is a simple cycle that visits all the vertices in the graph
    $\implies$ The cycle covers $|V|$ vertices in the graph and none of the vertices are repeated
    $\implies$ There is a cycle (no vertices repeated) with length at least $(k = |V|)$
  - **HAMILTON-CYCLE $\leq_p$ Given-Problem**
- So, The problem is in NP-Class and Hamilton Cycle Problem is polynomial time reducible to the given problem. So, the problem is **NP-Complete**. [proved]

2 Ques. : 2

Proof of : **HITTING SET Problem -** Given a family of sets $\{S_1, S_2, ..., S_n\}$ and an integer b, is there a set H with b or fewer elements such that H intersects all the sets in the family - is a NP-Complete problem.

- **HITTING SET problem is in NP Class** :-
  - If any problem is in NP, then, given a 'certificate', which is a solution to the problem and an instance of the problem, we will be able to verify whether the solution given is correct or not in polynomial time.
  - **Certificate :** A set of elements representing the set H
  - **Certifier :-**
    - First verify the number of elements in H is at most b, $|H| \leq b$
    - Then verify $\forall S_i \in \{S_1, S_2, ..., S_n\} \implies (H \cap S_i) \neq \phi$
    - As we know that checking for intersection takes at most $O(PQ)$ time, for sets of size P and Q, the whole process would take at most $O(NPQ) \approx O(N^3)$ time
    - This Certifier algorithm is a polynomial time algorithm. And also the Certificate size is polynomial to the order of no. of elements ($O(N)$). So the problem is in NP.
- **The VERTEX COVER problem [known NP-Complete problem] is polynomial time reducible to the HITTING SET Problem : VERTEX COVER $\leq_p$ HITTING SET**
  - Consider a graph $G(V, E), k^{'}$ be an instance of VERTEX COVER. We construct an instance of HITTING SET as follows :
  - **Construction :** For every edge $e(u, v) \in E$, we will construct a set $S_e = \{u, v\}$. In total we will have $|E|$ sets, and we set $k = k^{'}$. We claim that G has a Vertex Cover of size at max $k^{'}$ iff $\{S_e | e \in E\}$ has a Hitting Set H of size $k = k^{'}$ or less
  - G **has Vertex Cover of size** $\leq k^{'} \implies \{S_e | e \in E\}$ **has a Hitting Set** H **of size** $\leq k^{'}$
    G has a Vertex Cover (C) of size $\leq k$
    $\implies \forall$ edges $(u, v) \in E$, either $u \in C$ or $v \in C$
    $\implies \forall S_e = \{u, v\}$, either $u \in C$ or $v \in C$ [$\forall e(u, v) \in E \implies \exists S_e = \{u, v\}$]
    $\implies \forall S_e : e \in E, (C \cap S_e) \neq \phi$, so C intersects with each S_e in the set family
    $\implies$ Hitting Set of size $\leq k^{'}, H = C$ [As C has size $\leq k^{'}$ and C intersects with each S_e]
  - $\{S_e | e \in E\}$ **has a Hitting Set** H **of size** $\leq k^{'} \implies G$ **has Vertex Cover of size** $\leq k^{'}$
    $\{S_e | e \in E\}$ has a Hitting Set H of size $\leq k^{'}$
    $\implies$ H intersects with each S_e [$\forall e \in E, (H \cap S_e) \neq \phi$]
    $\implies \forall S_e = \{u, v\}$, either $u \in H$ or $v \in H$ (or both)
    $\implies \forall e(u, v) \in E$, either $u \in H$ or $v \in H$ (or both) [as $S_e = \{u, v\} \implies e(u, v) \in E$]
    $\implies$ So, H is a Vertex Cover of graph G [with size $\leq k^{'}$]
  - **VERTEX COVER Problem $\leq_p$ HITTING SET Problem**
- So, The problem is in NP-Class and Vertex Cover Problem is polynomial time reducible to the given Hitting Set problem. So, the problem is **NP-Complete**. [proved]