# Design And Analysis Of Algorithms
# Assignment : 1

**Name : Avik Samanta**
**Roll no. : 204101016**

**Submission Date : September 20, 2020**

## 1 Ques. : 1(A)

Suppose $S$ is a stable matching for a given instance $I$ of the Stable Matching algorithm, not necessarily the one produced by the $Gale - Shapley\ Algorithm$.

(a) Does the matching $S$ necessarily become unstable if we reverse the preference list of all the women (and keep the men's preferences intact)?

**Claim :** If we only reverse the preference list of each Women, the Stable Matching $S$ won't **necessarily** become Unstable.

**Proof (By Counter-Example) :**

- Let's assume, $I$ is an instance of Stable Matching Algorithm, where **Each Men** has **Unique first preference**.
  - Man $M_i$ has first preference Woman $W_x$,
  - Man $M_j$ has first preference Woman $W_y$
  - Then, $i \neq j \implies W_x \neq W_y$
- And $S$ is a **Stable Matching** for instance $I$, where **Each Men** is **matched** (paired) with his **First Preference**.

| Men | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| M1 | W1 | W2 | W3 |
| M2 | W2 | W3 | W1 |
| M3 | W3 | W1 | W2 |

| Women | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| W1 | M2 | M1 | M3 |
| W2 | M1 | M3 | M2 |
| W3 | M3 | M2 | M1 |

- Now let's reverse the women's preference list.

| Women | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| W1 | M3 | M1 | M2 |
| W2 | M2 | M3 | M1 |
| W3 | M1 | M2 | M3 |

- Now let's assume the pair $M_i - W_j$ is an **unstable pair**
  - So $M_i$ and $W_j$ are **not matched** in Matching $S$
  - So $W_j$ is **not the first preference** for $M_i$ (As in $S$ each man matched with his first preference)

- As $M_i$ is a Man, and his preference list has not been reversed, So definitely his **current partner is still his first preference**.
- So $M_i$ **prefers his current partner to $W_j$ in $S$**
- Definitely the pair $M_i - W_j$ is a **stable pair** in $S$, which yields a **contradiction**. Means, there's no such unstable pair.
- Even after reversing the women's preference list, the matching is still stable

- For the Stable Matching $S$, that we have taken, we have shown that, the matching is stable even after reversing the preference list of each of the Women. That proves **reversing the Women preference list, won't necessarily make the matching Unstable**.[hence proved]

## 2 Ques. : 1(B)

Suppose $S$ is a stable matching for a given instance $I$ of the Stable Matching algorithm, not necessarily the one produced by the $Gale - Shapley\ Algorithm$.

(b) What happens when we reverse the preference lists for all the men as well as all the women – does $S$ necessarily become unstable?

**Claim :** Even if we reverse the preference list of each Men and Women, the Stable Matching $S$ won't **necessarily** become Unstable.

**Proof (By Counter-Example) :**

- Let's assume, $I$ is an instance of Stable Matching Algorithm, where **Each Men** has **Unique first preference**, and **women** at their first preference **ranked them the worst/last**.

| Men | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| M1 | W1 | W2 | W3 |
| M2 | W2 | W3 | W1 |
| M3 | W3 | W1 | W2 |

| Women | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| W1 | M2 | M3 | M1 |
| W2 | M3 | M1 | M2 |
| W3 | M1 | M2 | M3 |

- And $S$ is a **Stable Matching** for instance $I$, where **Each Men** is **matched** (paired) with his **First Preference**. That also means, **each women** is **matched** with her **last preference**.

- Now let's reverse both the men's and women's preference list.

| Men | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| M1 | W3 | W2 | W1 |
| M2 | W1 | W3 | W2 |
| M3 | W2 | W1 | W3 |

| Women | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| W1 | M1 | M3 | M2 |
| W2 | M2 | M1 | M3 |
| W3 | M3 | M2 | M1 |

- Now what happened due to this is that, each women's preference list has been reversed. So now their last preference becomes the first preference. And for the original instance, in $S$ they were matched with the last preference. Now after reversing they are now paired with their first preference.

- Now if we consider $(M_i, W_j)$ an unstable pair.

  - $W_j$ is not paired with $M_i$
  - That means $M_i$ is not her first preference (in the reversed preference list)
  - so, $W_j$ prefers her current partner than $M_i$ (as her current partner is her first preference)
  - $(M_i, W_j)$ a stable pair, which yields a contradiction

- For the Stable Matching $S$, that we have taken, we have shown that, there is no Unstable Pair even after reversing the preference list of each of the Men and Women. That proves **reversing the preference list of both men and women, won't necessarily make the matching Unstable**.[hence proved]

# 3 Ques. : 2

Let $I$ be an instance of the stable matching problem, where $M$ and $W$ denotes the set of men and women respectively. Let $S$ be any perfect matching, not necessarily stable.For a person $p$ in $I$, where $p$ could be either a man or a woman, define the quantity $S(p)$, the cost of $S$ for $p$ to be the ranking of $p$'s partner $q$ in $p$'s preference list. For example, if woman $w$'s preference list is $m3, m2, m1, m4$ and her partner in $S$ is $m1$ then $c_S(w) = 3$ since $m1$ is ranked third in $w$'s preference list. Then define the regret $r(S)$ of the matching $S$ to be $r(S) = \Sigma(p \in M \cup W)c_S(p)$ , i.e., the sum of all the costs of $S$ for each person. Suppose $S$ is a perfect matching with minimum regret. Is $S$ necessarily a stable matching?

**Claim :** Perfect Matching with minimum regret does not necessarily be a Stable Matching.

**Proof (By Counter-Example) :**

- Let's consider the following instance $I$ for the stable matching algorithm.

| Men | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| M1 | W1 | W2 | W3 |
| M2 | W2 | W1 | W3 |
| M3 | W2 | W3 | W1 |

| Women | Pref 1 | Pref 2 | Pref 3 |
|---|---|---|---|
| W1 | M1 | M3 | M2 |
| W2 | M3 | M2 | M1 |
| W3 | M3 | M1 | M2 |

- Now let's consider two perfect matching, $S1$ and $S2$.
  $S1 = (M1, W1), (M2, W2), (M3, W3)$
  $S2 = (M1, W1), (M2, W3), (M3, W2)$

- If we analyse, we can see $S2$ is a stable matching. As there are no unstable pair for $S2$ in $I$.

- Whereas $S1$ is a perfect matching but not stable.
  - Consider the pair $(M3, W2)$ for $S1$.
  - $M3$ prefers $W2$ to his current partner $W3$, as well as $W2$ prefers $M3$ to her current partner $M2$
  - $(M3, W2)$ is an unstable pair for $S1$
  - $S1$ is not a stable matching

- Now for $S2$, $c(M1) = 1$, $c(M2) = 3$, $c(M3) = 1$, $c(W1) = 1$, $c(W2) = 1$, $c(W3) = 3$
  which leads to total regret $r(S2) = (1 + 3 + 1 + 1 + 1 + 3) = 10$ .

- Now for $S1$, $c(M1) = 1$, $c(M2) = 1$, $c(M3) = 2$, $c(W1) = 1$, $c(W2) = 2$, $c(W3) = 1$
  which leads to total regret $r(S1) = (1 + 1 + 2 + 1 + 2 + 1) = 8$
  which for the given instance $I$ is the minimum regret .

- So we can see $S1$ which leads to the minimum regret, is not actually a Stable Matching. So perfect matching with minimum regret, is not necessarily stable. [hence proved]