Graph Theory and Combinatorics

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Graph Theory and Combinatorics

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Graph Theory and Combinatorics

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Module - I

Unit- I: Introduction to Graph Theory

Introduction:

In the field of mathematics, graph theory plays a vital role in engineering and computer science. It deals with the study of graphs that concerns with the relationship among vertices and edges. It is an important subject having many more applications in computer science, information technology, biomathematics, etc. It is also used an extensively in the circuit connections. Also, it is used to facilitate the study of algorithms like Kruskal's algorithm, Prim's algorithm, Dijkstra's algorithm etc. There is a main relationship between topology and graph theory, it is applied in the different types of topologies like star, bridge, series, parallel and network topologies etc. The interconnected computers in the network follows from the relationship among the principles of graph theory. The characterizations of graphs are depending on their structures.

Definition of a Graph: 1.1

A Graph is said to be a pair of vertices and edges which consisting a set of objects (V, E) where $V=\{v_1, v_2, v_3, \ldots, v_n\}$ are called **vertices** and $E=\{e_1, e_2, e_3, \ldots, e_n\}$ are called **edges** . The vertex v_1 is called the **initial vertex** and the vertex v_n is called the **terminal vertex**. Each edge is associated with an unordered pair of vertices. Let us see an example of the graph in the figure (1.1)

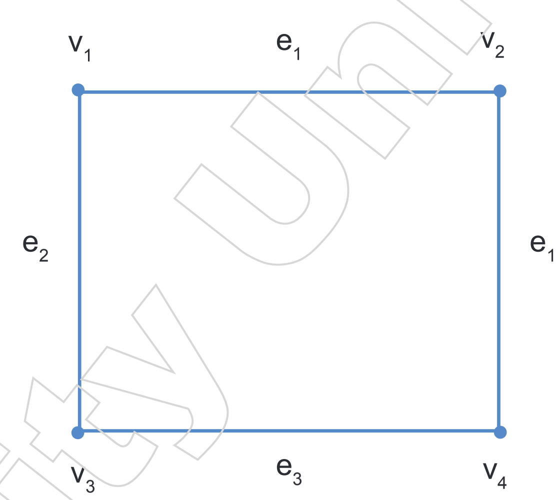


Fig (1.1)

In the above figure (1.1), there are four vertices and four edges namely $V=\{v_1, v_2, v_3, v_4\}$ and $E=\{e_1, e_2, e_3, e_4\}$

Definition: 1.2

A graph having without self loop and parallel edges is said to be a simple graph. [Equivalently a graph having neither self loop nor parallel edges is said to be a simple graph]. Let us see an example of simple graph

Notes

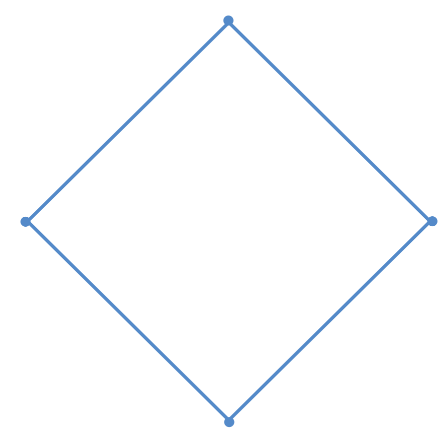


Fig (1.2)

The above Fig (1.2) shows the graph is a simple graph which has no self loop and parallel edges.

Definition: 1.3

Any two edges having the same initial vertex and the same end vertex is said to be Parallel edges. An edge having the same vertex as an initial vertex and end vertex then it is said be a Self loop. The below example having the graph with self loop and parallel edges.

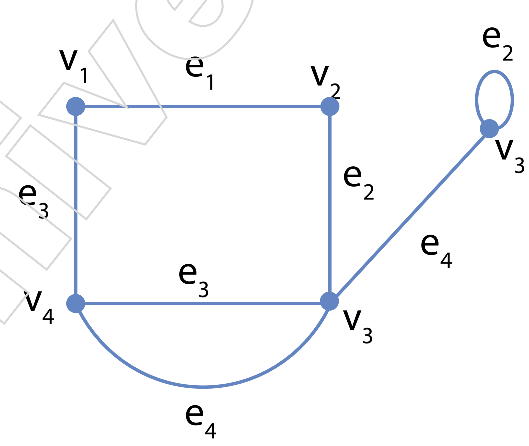


Fig (1.3)

In the Fig (1.3) the graph consists of five vertices and 7 edges.

1. In this graph V= {v_1, v_2, v_3, v_4, v_5} and E= {e_1, e_2, e_3, e_4, e_5, e_6, e_7}. Here the two edges 3 and 4 are parallel edges , the edges 3 and 4 having the same initial vertex 'c' and end vertex 'd'.
2. An edge 7 having same initial and end vertex namely 'e' and it is said be a self loop.

Definition: 1.4

A graph with finite number of vertices is called finite graph otherwise is called infinite graph.

Notes

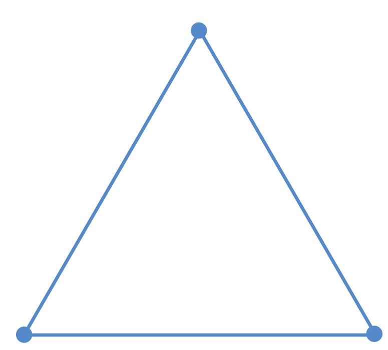


Fig (1.4a)

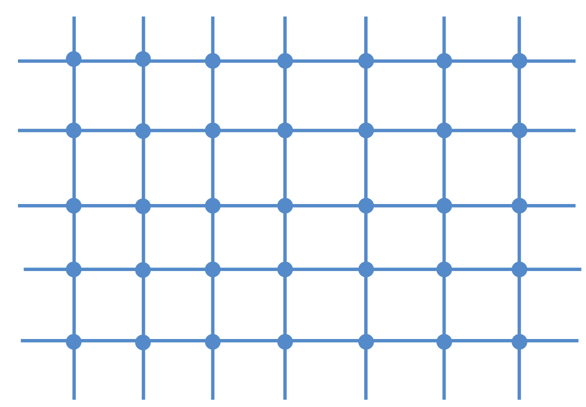


Fig (1.4b)

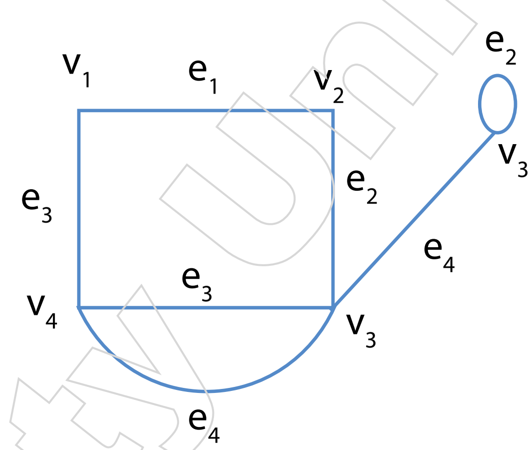
The above figure 1.4a is an example of finite graph which has three vertices and three edges and the figure 1.4b is an example of infinite graph which has an infinite number of vertices and infinite number of vertices.

Definition: 1.5

Let the two vertices are said to be adjacent if they are connecting by an edge. If a vertex is incident to an edge then the vertex is one of the two vertices that connects the edges .In the above figure (1.1) the vertex ‘a’ is adjacent to the vertex ‘b’ and also is adjacent to ‘d’.

Definition: 1.6

The degree of the vertex is said to be number of edges connecting in it. Let us see the below example



d(v1) = 2

d(v2) = 2

d(v3) = 4

d(v4) = 3

d(v5) = 3

Note:

- In the self loop the degree of the edges must be counted twice.
- The number of edges in the graph must be twice the sum of the degree of the vertices.

Notes

Theorem: 1.8 (Handshaking Lemma)

In any graph G(V,E) the sum of degrees of vertex is twice the number of edges. (i.e.)

$$\sum_{i=1}^{n} d(V_i) = 2e$$

Proof:

Let us consider the graph G with 'V' vertices and 'E' edges. Now we can prove this theorem with a clear example. For that we consider the above graph Fig (1.3)

The number of vertices in the graph is 5 and the number of edges in the graph is 7

$d(v_1) = 2$, $d(v_2) = 2$, $d(v_3) = 4$, $d(v_4) = 3$, $d(v_5) = 3$

$d(v_1) + d(v_2) + d(v_3) +4, d(v_4) +d(v_5) = 2+2+4+3+3$

=14 = twice the number of edges

Therefore $$\sum_{i=1}^{n} d(V_i) = 2e$$

Definition: 1.9

1. A vertex with degree one is called pendent vertex.
2. A vertex with degree zero is called isolated vertex.
3. A graph without any edges is called null graph

Fig (1.5a)

Fig (1.5b)

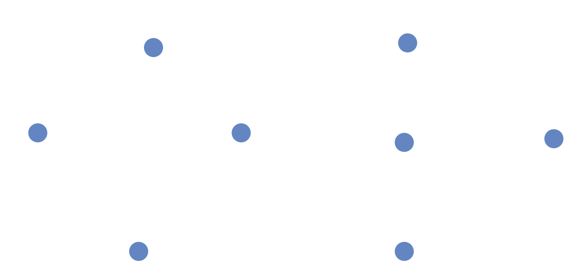


Fig (1.5c)

Fig (1.5a) represents the pendent vertex with degree one.

Fig (1.5b) represents the isolated vertex with degree zero.

Fig (1.5c) represents the null graph without any edges.

Notes

Theorem: 1.10

Prove that the number of vertices of odd degree in a graph is always even.

Proof:

Let G be graph with ‘n’ vertices and ‘e’ edges.

Let us split the vertices into two of odd degree and an even degree of vertices

Let ‘k’ be the odd degree vertices and ‘j’ be the even degree vertices

$$\sum_{i=1}^{n} d(V_i) = \sum_{k=1}^{n} d(V_k) + \sum_{j=1}^{n} d(V_j) \rightarrow (1)$$

We know that by theorem (1.8) the sum of number of all the edges is twice the sum of degree of all the vertices.

That is $\sum_{i=1}^{n} d(V_i) = 2e$

$$2e = \sum_{k=1}^{n} d(V_k) + \sum_{j=1}^{n} d(V_j) \rightarrow (2)$$

Therefore the R.H.S of (2) is an even number

Each d(vj)is even (since the sum of an even numbers is an even number)

Therefore eqn (2) is an even number

The total number of terms in the sum must be even

Hence the number of vertices of odd degree in a graph is always even.

Definition: 1.11

1. In a graph a walk is an alternating sequence of vertices and edges and the element which are incident, that starts and ends with a same vertex

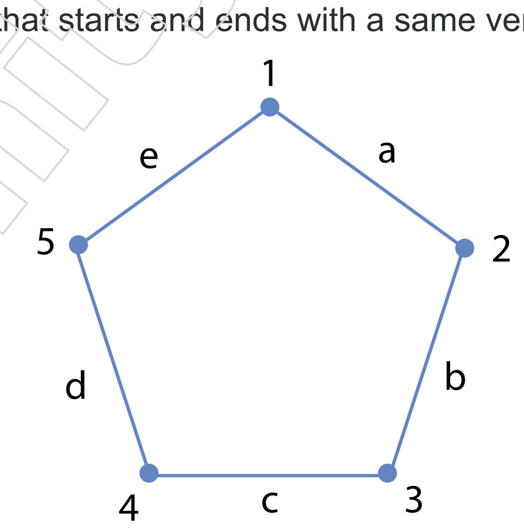


Fig (1.6)

Notes

In the above fig (1.6) 1a2b3c4d5e1 is a walk.

2. A trail is a walk without repeating any edges.

3. A path is a walk without repeating any vertices. In the above Fig (1.6) 1a2b3c4d4e is a path.

Definition: 1.12

1. A walk is said to be open if the initial and terminal vertices are different.

2. A walk is said to be closed if the initial and terminal vertices are same.

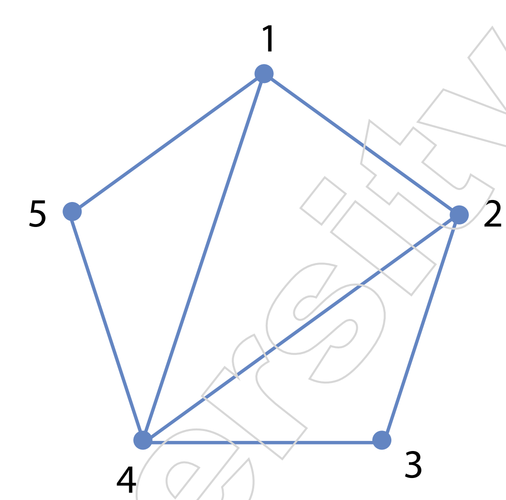


Fig (1.7)

In the above figure (1.7) $1 \to 2 \to 3 \to 4 \to 1 \to 5$ is an open walk.

In the above figure (1.6) 1a2b3c4d5e1 is a closed walk

Definition: 1.13

In a graph a path is said to be a finite or infinite sequence of edges which joins a sequence of vertices and the vertices are all distinct.

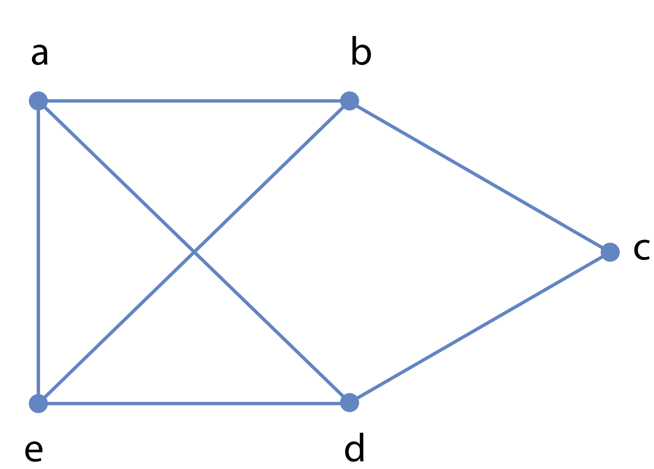


Fig (1.8)

The above figure (1.8) shows that a,b,c,d,e is a path

Notes

Definition: 1.14

A graph is said to be a circuit such that no one edge is repeated and it is also closed

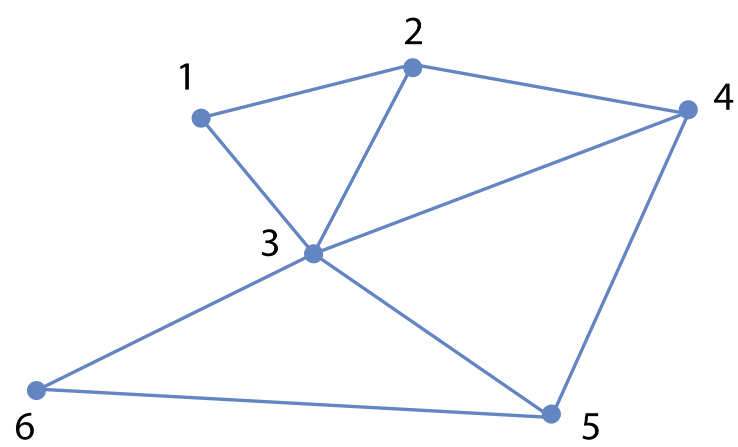


Fig (1.9)

In the above figure (1.9) $1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 6 \rightarrow 5 \rightarrow 3 \rightarrow 1$ is a circuit.

Remark:

- In a circuit vertex can be repeated and edge not repeated.
- In a cycle the vertex and edge are not repeated.

Definition: 1.15

A graph is said to be connected if every pair of vertices in the graph is connected. [Equivalently, there is a path between every pair of vertex and should be some path traverse. It is also called the connectivity of a graph]. Otherwise it is said to be disconnected.

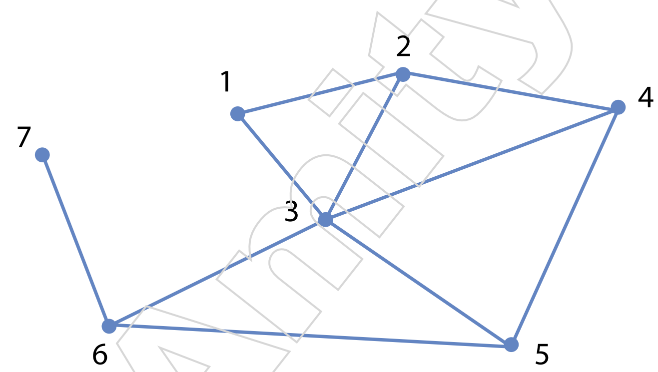


Fig (1.10a)

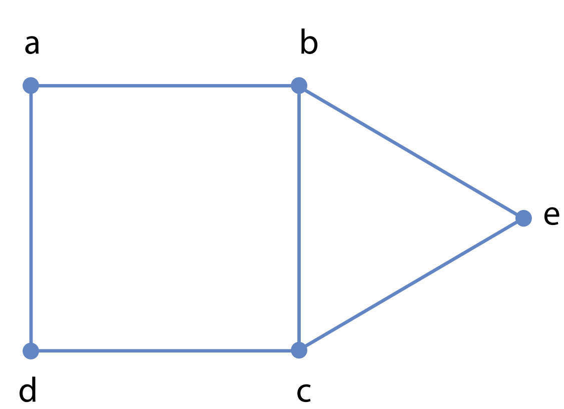


Fig (1.10b)

In the above figure (1.10a) represents the graph is connected.

In the above figure (1.10b) represents the graph is disconnected.

Notes

Definition: 1.16

A directed graph is a graph which has a set of vertices connected by the edges and the edges have a direction with them. The below figure represents the directed graph

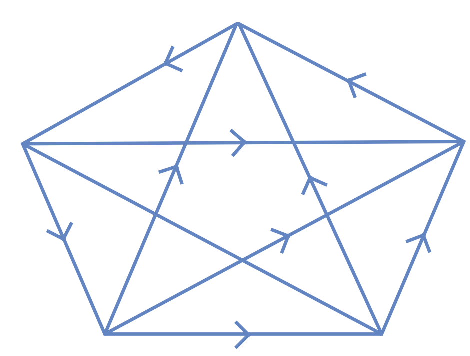


Fig (1.11)

Definition: 1.17

A graph 'g' is said to be a subgraph of a graph 'G', if all the vertices and all the edges of 'g' are in 'G' and each edge of 'g' has the same end vertices in 'g' as in 'G'.

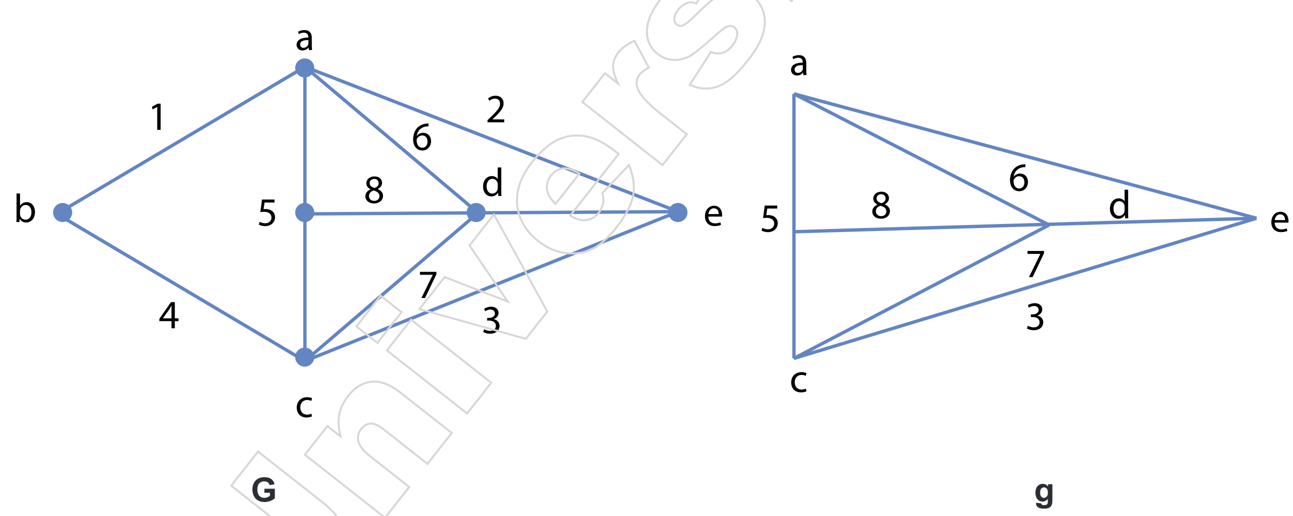


Fig (1.12)

In the above figure 'G' is a graph and 'g' is a subgraph of a graph 'G' where 'g' contains all the edges in the graph 'G'

Note:

1. Every graph has its own subgraph
2. A subgraph of a graph 'G' is a subgraph of 'G'
3. A single vertex in a graph 'G' is also a subgraph of 'G'
4. A single edge in 'G' together which its end vertices is also a subgraph of 'G'

Definition: 1.18

If two or more subgraphs 'g_1' and 'g_2' of G are said to be edge disjoint of 'G', if both 'g_1' and 'g_2' do not have any edges in common.

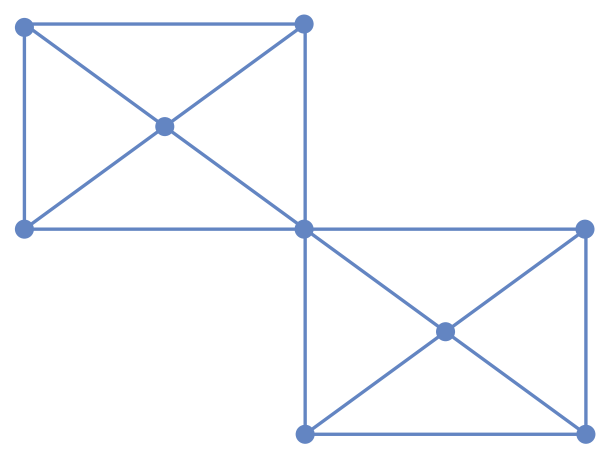


Fig (1.13a) [G]

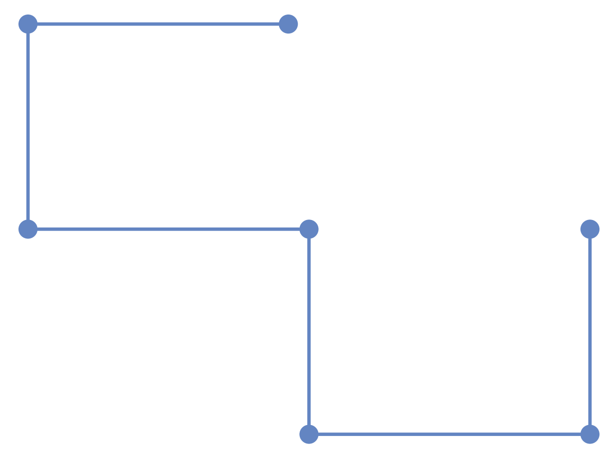


Fig (1.13b) [g_1]

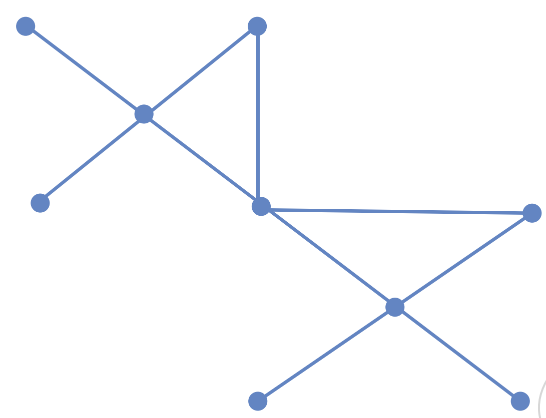
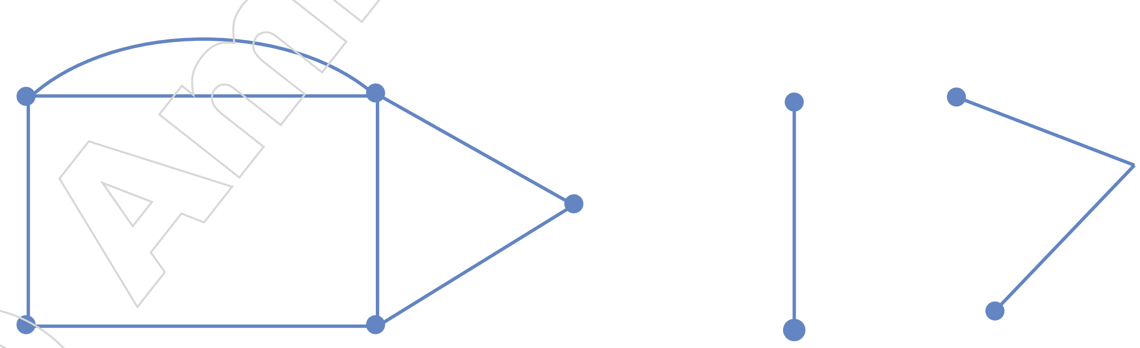


Fig (1.13c) [g_2]

The above figure represents the edge disjoint subgraphs

Remark:

1. Edge disjoint graphs do not have any edge in common, but they may have vertices in common
2. Subgraphs that do not have vertices in common is said to be vertex disjoint subgraphs. The below figure (1.14) represents the vertex disjoint subgraphs where H_1 and H_2 do not have any vertices in common.



G H_1 H_2

Fig (1.14)

Notes

Theorem: 1.18

Prove that a graph G is disconnected if and only if its vertes set can be partitioned into two non-empty disjoint subsets,v_1, and v_2 such that there exist no edge in G whose one end vertex is in subset v1 and the other in subset v_2.

Proof:

Let G be a disconnected graph and its vertex set can be partitioned into two non-empty subsets v_1, and v_2. Suppose that such a partition exist,

To prove that G is disconnected, Consider two arbitrary vertices ‘a’ and ‘b’of ‘G’ such that ‘a’ in v_1 and ‘b’ in v_2 , no path can exist between vertices ‘a’ and ‘b’, otherwise their would be at least one edge whose one end vertex would be in v_1 and the other in v_2. Hence if partition exist ’G’ is connected.

Conversely,

Let G be a disconnected graph.

Consider a vertex ‘a’ in G. Let v_1 be the set of all vertices that are joined by path to ‘a’ .Since G is disconnected ‘v’ does not include all vertices of G. Then the remaining vertices will form a set v_2.

Therefore no vertices in v_1 is joined to only in v_2 by an edge.

Theorem: 1.19

Prove that a graph G is (connected or disconnected) has exactly two vertices of odd degree, there must be a path joining two vertices.

Proof:

Case (i)

Let G be a connected graph with exactly two vertices of odd degree say v_1 and v_2. There is at least one path joining v_1 and v_2 because G is connected.

Case (ii)

Let G be a disconnected graph with exactly two vertices of odd degree say v_1 and v_2. Since G is disconnected, their must be at least two components, each of which is a connected subgraph.

If v1 and v_2 are in two different components say g_1 and g_2 respectively

Therefore the number of odd degree vertices in g1 is one and also in g_2, but g_1 is a graph which is connected.

We know that, in a graph the number of odd degree is always even, which is a contradiction

Therefore v_1 and v_2 must be the same component.

Hence there is a path between v_1 and v_2.

Remark:

The maximum number of edges in a simple graph with 'n' vertices is n(n-1)/2

Theorem: 1.20

Prove that a simple graph that is a graph without parallel edges or self loop with 'n' vertices and 'k' components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let G be a graph with 'n' vertices and 'k' components

To Prove:

G has atmost $\frac{(n-k)(n-k+1)}{2}$ edges

Let n_1, n_2, n_3, ……nk be the number of vertices in 1st , 2nd,…..kth components respectively.

$n_1 + n_2 + n_3, + \ldots\ldots + nk = n$

$$\Rightarrow \sum_{i=1}^{k} n_i - n, n_i \geq 1$$

The maximum number of edges in the i[th] component of G is $\frac{n_i(n_i-1)}{2}$

Therefore the maximum number of possible edges in a graph G is

$$\Rightarrow \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \rightarrow (1)$$

$$\Rightarrow \sum_{i=1}^{k} \frac{(n_i^2-1)}{2}$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} (n_i^2 - n_i)$$

Now consider $\sum^{k} n_i = n, \sum^{k} n_i^2 - k = n$

$(n_1 + n_2 + n_3, + \ldots\ldots + nk) - (1+1+1+\ldots\ldots+1) = n-k$

$(n1 -1) + (n_2 -1) + (n_3 -1) + \ldots\ldots + (nk -1) = n-k$

$$\sum_{i=1}^{k} n_i = (n_i - 1) = n - k$$

$$\sum_{i=1}^{k} n_i = (n_i - 1)^2 = (n-k)^2$$

Notes

$(n_1 -1)^2 +(n_2 -1)^2+(n_3-1)^2+......+(n_k -1)^2$ + some cross terms = $(n-k)^2$

$n_1^2 -2n_1 + n_2^2 -2n_2 +1+...... +n_k^2 -2n_k +1+$ some cross terms = $(n-k)^2$

Then $(n_1^2 +n_2^2 +....+ n_k^2) - 2(n_1+n_2 +...... +n_k)+(1+1+1+....+1)$

+ some cross terms = $(n-k)^2$

Therefore $(n_1^2 +n_2^2 +....+ n_k^2) - 2(n_1+n_2 +...... +n_k)+k$

+ some cross terms = $(n-k)^2$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2\sum_{j=1}^{k}(n_i + k) \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 - 2n + k \le (n-k)^2$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le (n-k)^2 + 2n - k$$

$$\Rightarrow n^2 + k^2(k-2n) - (k-2n)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 \le n^2 + (k-2n)(k-1)$$

$$\Rightarrow \sum_{i=1}^{k} n_i^2 = n^2 - (-2n-k)(k-1)$$

Eqn (1) $\Rightarrow \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$$\le \frac{1}{2}[n^2 - (2n-k)(k-1) - n]$$

$$\frac{1}{2}[n^2 - 2nk + 2n + k^2 - k - n]$$

$$\frac{1}{2}[n^2 - 2n - k + k^2 + 2n - k - n]$$

$$\Rightarrow \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} \le \frac{1}{2}[(n-k)^2 + (n-k)]$$

$$\Rightarrow \frac{1}{2}(n-k)(n-k+1)$$

Notes

Therefore the maximum number of edges is $\Rightarrow \frac{1}{2}(n-k)(n-k+1)$

Remark:

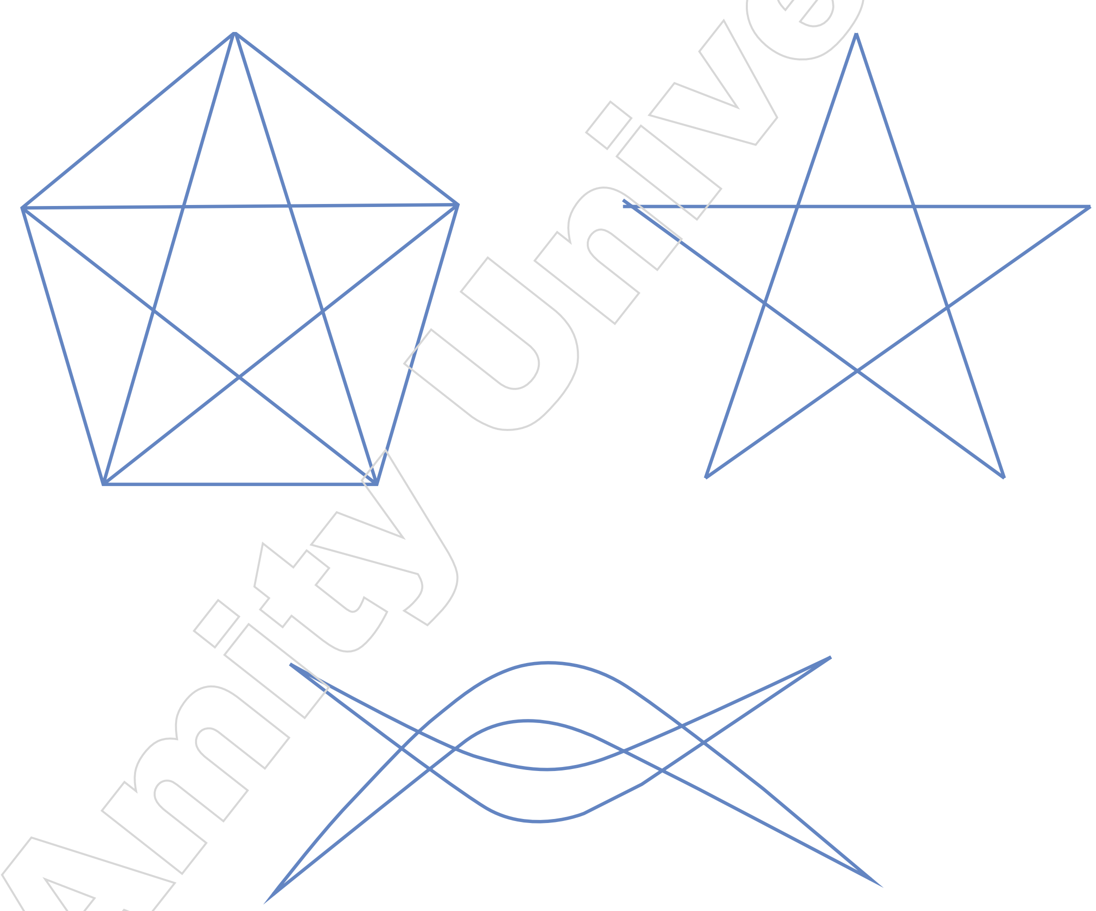
1. A path is also called simple path or an elementary path
2. A path does not intersect itself.
3. A self loop can be include in a walk but not in a path.
4. The number of edges in a path is called the length of the path

Definition: 1.21

A disconnected graph consists of two or more subgroups. Each of the connected subgraph is called a component.

Definition: 1.22

If some closed walk in a graph contains all the edges of the graph then the walk is called an Euler line and this is called as Euler graph

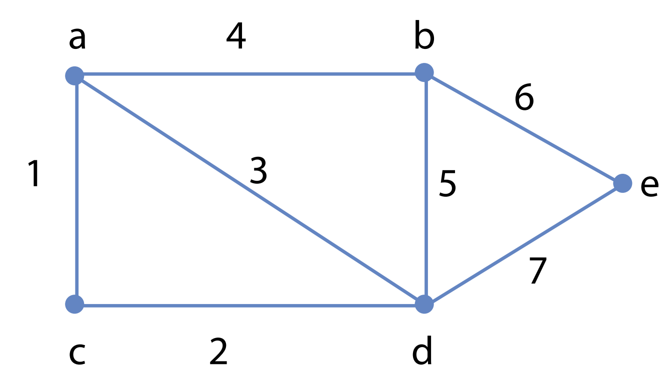


The above diagram represents the Euler graph

Definition: 1.23

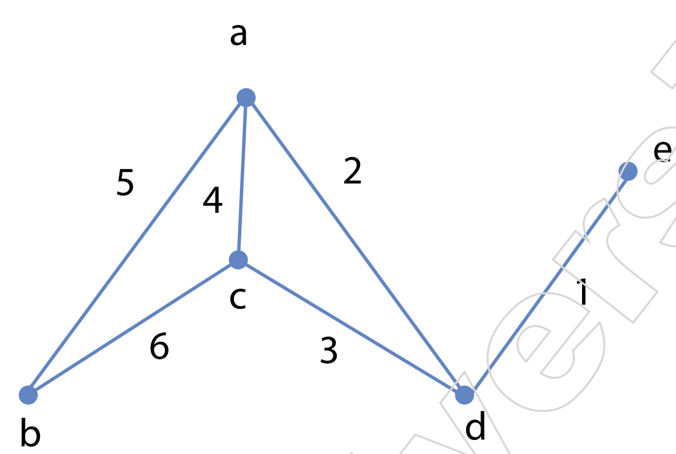
An open walk that includes or traces or covers all the edges of a graph without retracting any edge is called an unicursal line it is also said to be an open euler line. A connected graph that has a unicursal line is called unicursal graph

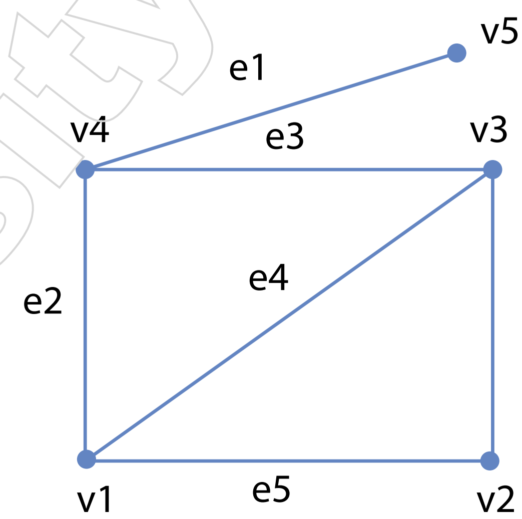
Notes



Definition: 1.24

If two graphs G and G' are said to be isomorphic to each other, if there is a one-to-one correspondence between the vertices and between their edges such that the incidence relationship is preserved.

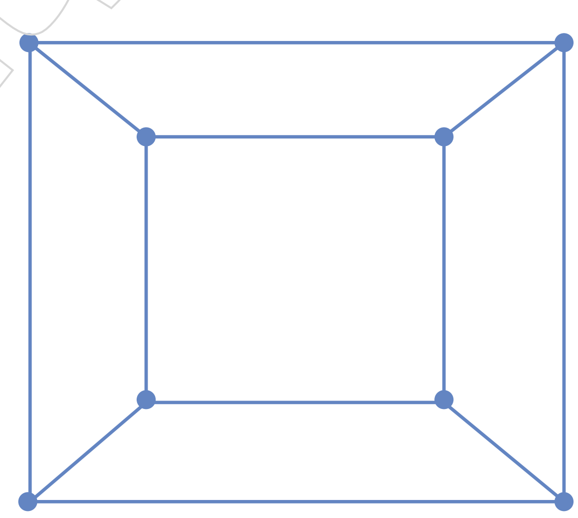


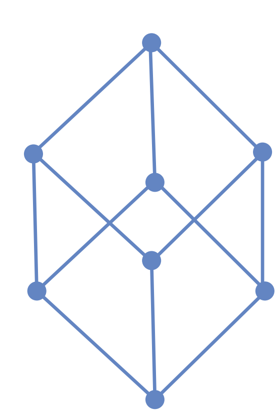


- The above two graphs are isomorphic.
- In the above graph the vertices a,b,c,d and e correspond to v_1, v_2,v_3,v_4, and v_5 respectively. The edges 1,2,3,4,5 and 6 correspond to e1, e_2,e_3,e_4,e_5 and e_6 respectively . Therefore G And G' are isomorphism

Note:

Without the labels of their vertices and edges the isomorphic graphs are the same graph that perhaps drawn differently

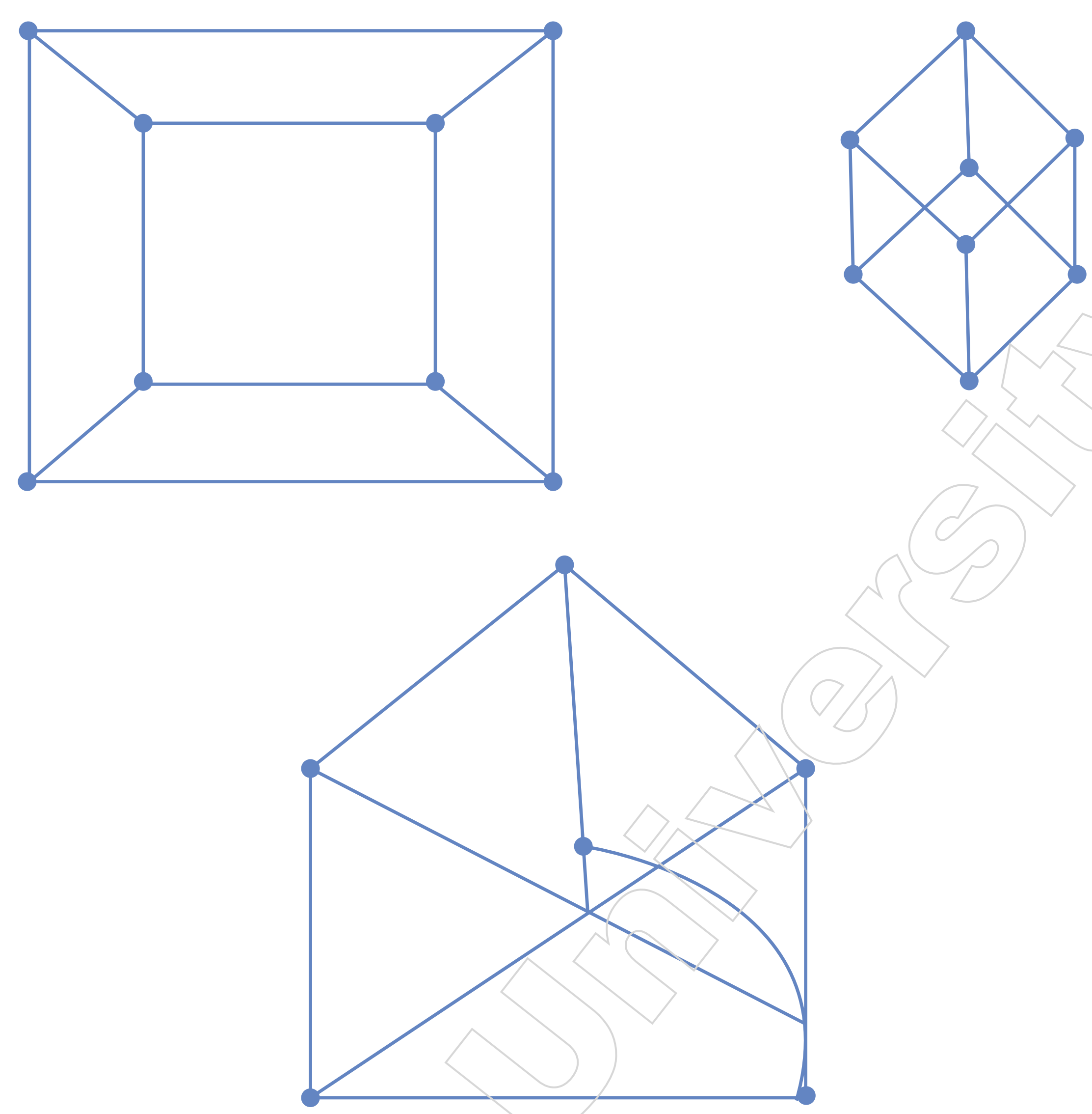




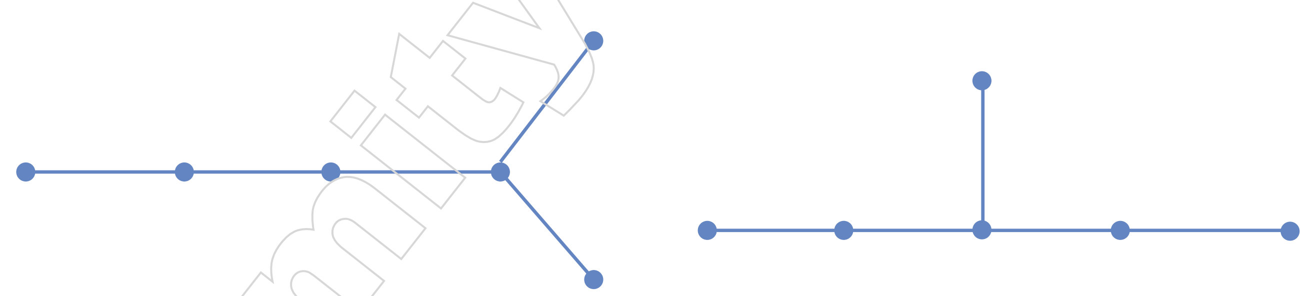
Notes

Remark:

1. The isomorphic graphs must have the same number of vertices and the same number of edges.
2. An equal number of vertices with a given degree



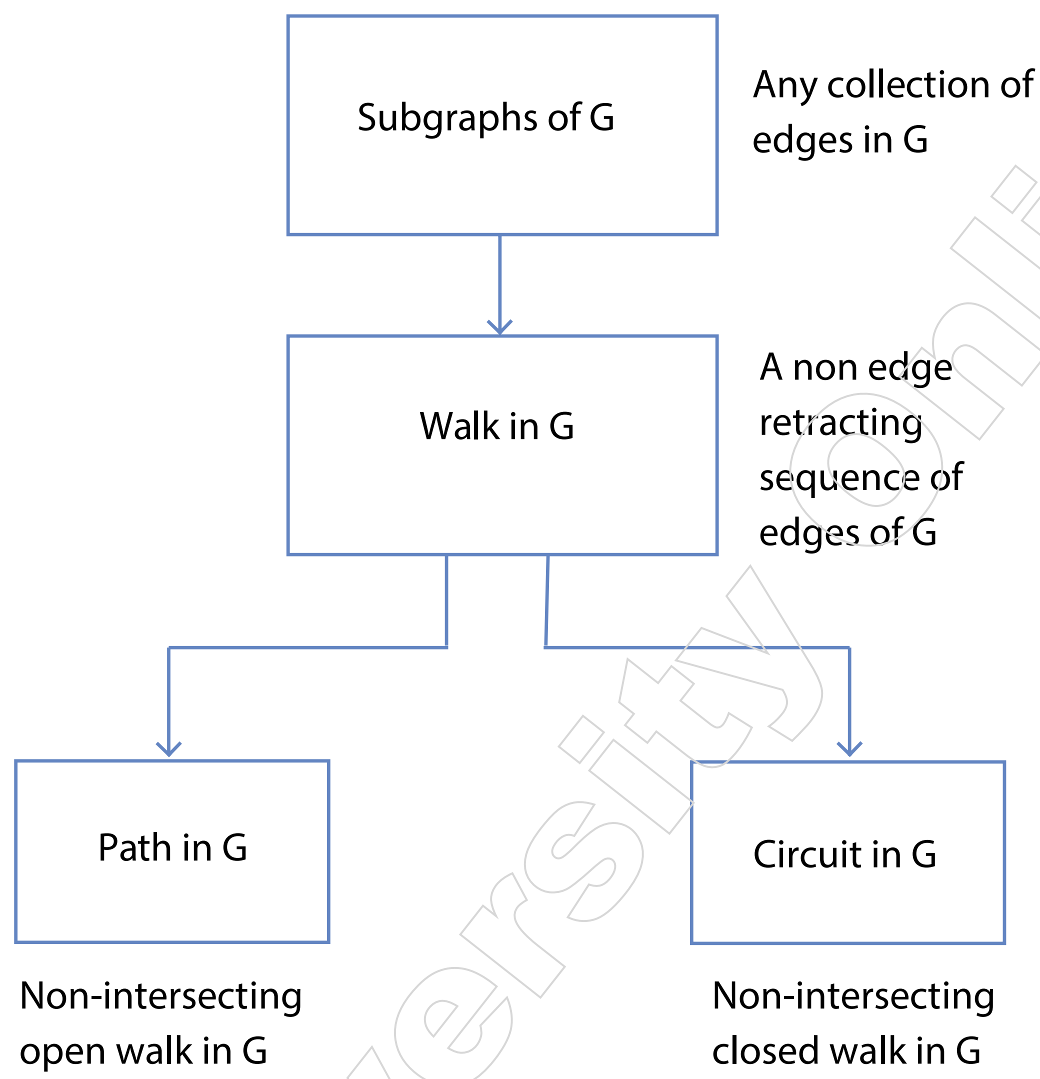
Isomorphic Graphs



Non-Isomorphic graphs

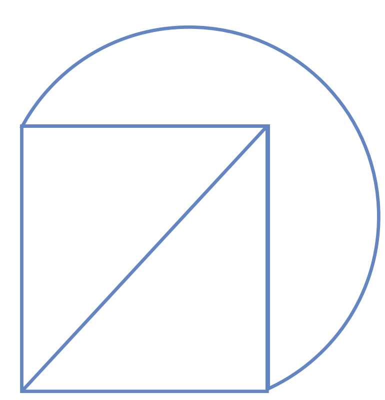
Relationship among subgraphs, walks, paths, and circuits:

Notes



Theorem: 1.25

A given connected graph G is an euler graph if and only if all vertices of a graph G are of even degree



Proof:

Necessary condition:

Suppose that G is an euler graph

To Prove:

All vertices in G are of even degree

Notes

G is an euler graph. It contains an euler line (which is a closed walk)

In tracing this closed walk, we know that every time the walk meets a vertex ‘v’ it goes through two new incident on v which one we entered and with the other existed

Thus is true not only of all intermediate vertices of the walk but also of the terminal vertex because we existed and entered to the same vertex at the beginning and end of the walk respectively.

Thus G is an euler graph, the degree of every vertex is even.

Sufficient condition:

Suppose that all vertices of a graph G are of even degree.

Now we construct a walk starting at an arbitrary vertex ‘v’ and passing through the edges of G such that non edges is repeated or more than once. We continue tracing as far as possible.

Since every vertex is of even degree, we can exit from every vertex , we enter , the tracing cannot stop at any vertex ‘v’

Since ‘v’ is also of even degree we shall eventually reach ‘v’ when the tracing come to an end

Case: (i)

If this closed wallk ‘h’ , we just traced includes all the edges of G is an euler graph.

Case: (ii)

If ‘h does not contains all the edges ‘G’ , we remove from G all the edges in ‘h’ and obtain a subgraph ‘h’ of G formed by the remaining edges

Since both G and ‘h’ have all their vertices of even degree, the degrees of the vertices of ‘h’ are also even.

Moreover ‘h’ must touch ‘h’ atleast at one vertex ‘a’ because G is connected.

Starting from ‘a’ we can again construct a new walk in ‘h’. Since all the vertices of ‘h’ are all even degree . This walk in ‘h’ must terminate at vertex ‘a’. But this walk in ‘h’ can be combained with ‘h’ to form a new walk, which starts and ends at vertex ‘v’ and has more edges than ‘h’. This process can be repeated until we obtain a closed walk that traverses all the edges of G .

Thus G is an euler graph.

Theorem: 1.26

In a connected graph ‘G with exactly 2k odd vertices, there exists ‘k’ edge disjoint subgraph such that they together contain all edges of ‘G’ and that each is unicursal graph

Notes

Proof:

Let the odd vertices of the given graph G be named $v_1,v_2,\ldots\ldots vk$; $w_1,w_2,\ldots..wk$.

In any arbitrary order, Add 'k' edges to G between the vertex pairs (v_1,w_1), (v_2,w_2), (v_3,w_3),….. (vk,wk) to form a new graph G'

Since every vertex of G' is of even degree g" consists of all and the euler line.

Now if we remove from the euler line the 'k'-edges, we just added, the euler line and will be split into 'k' walks, each of which is a unicursal line.

The first removal will live a single unicursal line. The second removal will split that into the unicursal lines and each successive removal will split a unicursal line into two unicursal lines until there are 'k' edges of them.

Definition: 1.27

A graph is said to be reach ability, if a directed graph is the pair of objects G=(V,E) where each node v in V represents a reachable marking and each edge e in E represents a transition markings. The set of reachable markings can be infinite, even for a finite petri net .

Questions:

1. A graph which has no self loop and no parallel edge graph is__________

   a) simple

   b) finite

   c) infinite

   d) none

2. A graph is __________both its vertex set and the edge set are finite

   a) simple

   b) finite

   c) infinite

   d) none

3. A digraph that has no parallel edges is________ graph

   a) simple

   b) finite

   c) infinite

   d) none

4. A digraph that has atmost one directed edge between a pair of vertices is ______.

   a) asymmetric

   b) symmetric

   c) a or b

Notes

d) a and b

5. A digraph which is both simple and symmetric is___________

a) simple symmetric graph

b) simple asymmetric graph

c) a or b

d) a and b

6. A digraph which is both simple and asymmetric is____________

a) simple symmetric graph

b) simple asymmetric graph

c) a or b

d) a and b

7. A digraph is simple antisymmetric if and only if its underlying undirected graph is___________ .

a) simple

b) finite

c) infinite

d) none

8. Two vertices u and v in V(G) are said to be__________ if there is an edges e in E(G) such that (e) = (u; v).

a) adjacent

b) not a adjacent

c) vertex

d) number

9. Let v be a vertex in a graph G. The degree dV of the vertex v in G is the ___________of G that are incident with v.

a) number of vertex

b) number of edges

c) a or b

d) a and b

10. Let G be a graph. Then (G) = min{d(v)/v V (G)} is __________

a) minimum degree

b) maximum degree

c) in degree

d) out degree

Notes

Answer:

1. a
2. b
3. a
4. b
5. a
6. b
7. a
8. a
9. b
10. a

Exercises:

1. Define Graph and give an example
2. Explain finite and infinite graphs with a suitable example
3. Draw a graph of your own and find the degree of that graph
4. Explain incidence and degree of a graph.
5. Prove that, the number of vertices of odd degree in a graph is always even.
6. Prove that, if a graph has exactly two of odd degree there must be a path joining these two vertices.
7. Define Isolated vertex, Pendant vertex and Null graph.
8. Draw all simple graphs of one, two, three and four vertices.
9. Define Isomorphism and explain it by comparing two graphs.
10. Define Edge-Disjoint Subgraphs and Vertex - Disjoint Subgraphs.
11. Define Connected graphs, Disconnected graphs and Components

Sample Questions with Answers:

1. **Prove that the sum of the degrees of the vertices of any finite graph is even.**

Proof:

Each edge ends at two vertices. If we begin with just the vertices and no edges, every vertex has degree zero, so the sum of those degrees is zero, an even number. Now add edges one at a time, each of which connects one vertex to another, or connects a vertex to itself (if you allow that). Either the degree of two vertices is increased by one (for a total of two) or one vertex's degree is increased by two. In either case, the sum of the degrees is increased by two, so the sum remains even.

Notes

2. Show that every simple fifinite graph has two vertices of the same degree.

Proof:

This can be shown using the pigeon hole principle. Assume that the graph has n vertices. Each of those vertices is connected to either 0, 1, 2, . . . , n-1 other vertices.

If any of the vertices is connected to n-1 vertices, then it is connected to all the others, so there cannot be a vertex connected to 0 others. Thus it is impossible to have a graph with n vertices where one is vertex has degree 0 and another has degree n-1. Thus the vertices can have at most n-1 different degrees, but since there are n vertices, at least two must have the same degree.

3. Prove that a complete graph with n vertices contains n(n-1)/2 edges.

Proof:

This is easy to prove by induction. If n = 1, zero edges are required, and 1(1-1-0)/2 = 0. Assume that a complete graph with k vertices has k(k -1)/2. When we add the (k + 1)st vertex, we need to connect it to the k original vertices, requiring k additional edges. We will then have k(k -1)/2 + k = (k + 1)((k + 1)) 1)/2 vertices.

Module - II

Unit- II: Operations on Graph

Definition: 2.1

The **union** of two graphs $G_1=(v_1, e_1)$ and $G_2=(v_2, e_2)$ is another graph G_3 written as $G_3 = G_1 \bigcup G_2$ whose vertex set $V_3 = V_1 \bigcup V_2$ and the edges are $E_3 = E_1 \bigcup E_2$

Definition: 2.2

- The **intersection** of graphs G_1 and G_2 is a graph G_4 consisting only of those vertices and edges that are in both G_1 and G_2
- The **ring sum** of two graphs G_1 and G_2 (written as $G_1 \oplus G_2$ is a graph) consisting of the verex set v_1 union v_2 and edges that are either in G_1 or G_2 but not in both

Definition: 2.3

If G_1 and G_2 are said to be **commutative** then

$G_1 \bigcup G_2 = G_2 \bigcup G_1$

$G_1 \bigcap G_2 = G_2 \bigcap G_1$

$G_1 \oplus G_2 = G_2 \oplus G_1$

- If G_1 and G_2 are **edge disjoint** then G_1 intersection G_2 is a null graph
  $G_1 \oplus G_2 = G_1 \bigcup G_2$
- If G_1 and G_2 are **vertex disjoint** then G_1 intersection G_2 is empty
  $\Rightarrow G_1 \bigcap G_2 = \phi$
- For any graph ,
  $G \bigcup G = G \bigcap G = G$
  $G \oplus G = null\ \ graph$
  $G \oplus g = G - g\ \ whenever\ \ g \subseteq G$

Definition: 2.4

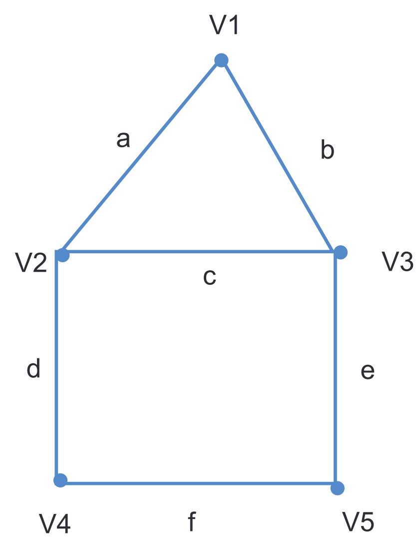
A graph G is said to have been **decomposed** into two subgraphs, if

$g_1 \bigcup g_2 = G$

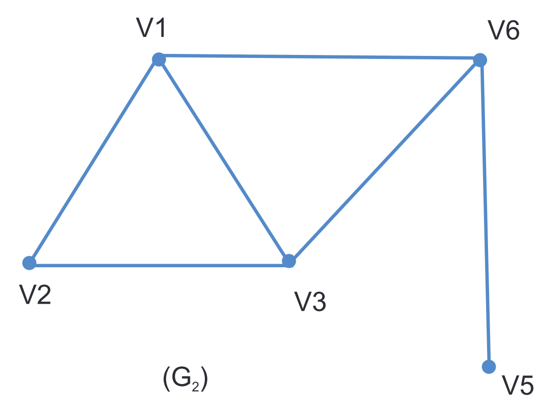
$g_1 \bigcap g_2 = a\ \ null\ \ graph$

The below diagram shows the union, intersection, and ring sum of the graphs

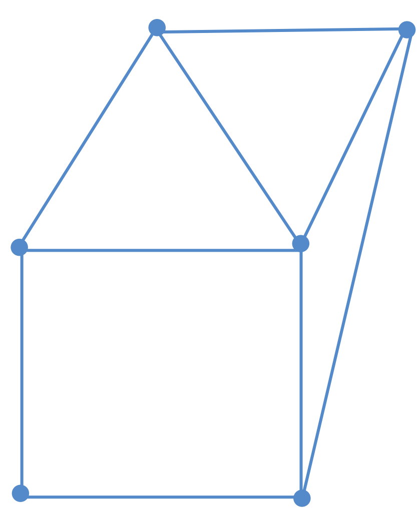
Notes



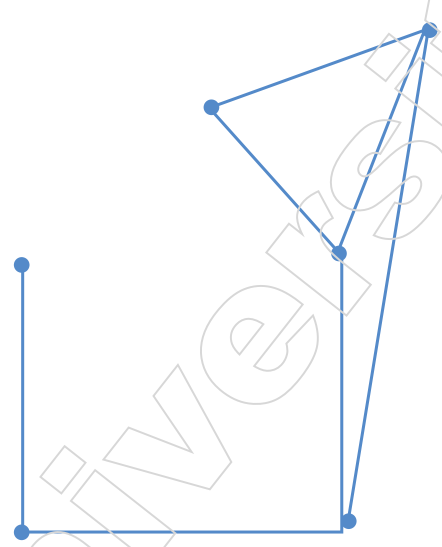
(G_1)



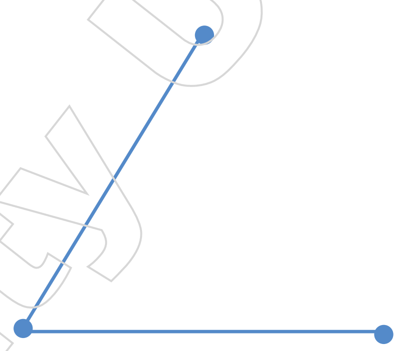
(G_2)



G_1 Union G_2



G_1 ringsum G_2

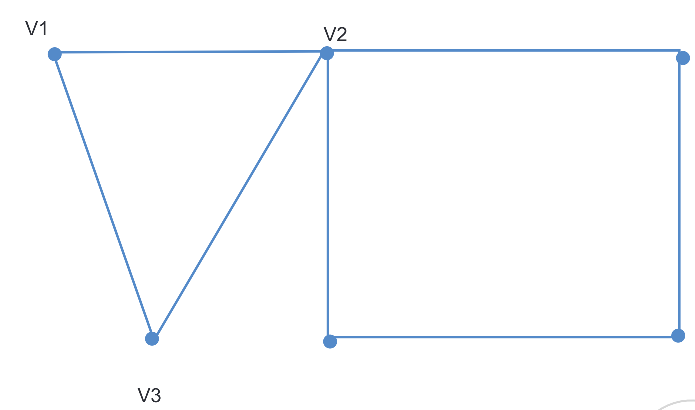


G_1 Intersection G_2

Theorem: 2.5

Prove that a connected graph G is an euler graph if and only if it can be decomposed into circuits.

Notes



Proof:

Suppose graph G can be decomposed into circuits (i.e.) G is a union of edge disjoint circuits

Since the degree of every vertex in a circuit is two, the degree of every vertex in G is even

Hence G is an euler graph.

Conversely,

Coinsider a vertex $v_{1,}$. There are atleast two edges incident at v_1

Let one of these edges between v_1 and v_2

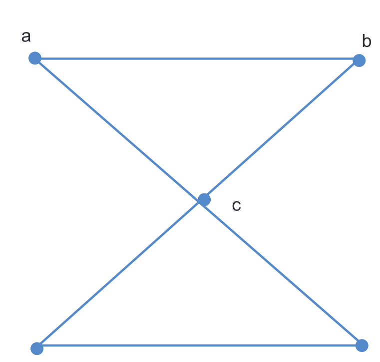
Since vertex v_2 is also of even degree, it must have atleast another edge say between v_2 and v_3

Proceeding in this way, we eventually arrive at a vertex that has previously been traversed thus forming a circuit. Let us remove the circuit form G. All the vertices in the remaining graph must also be of even degree. From the remaining graph remove another circuit in exactly the same way as be removed the circuit from G. Continuing this process until no edges are left.

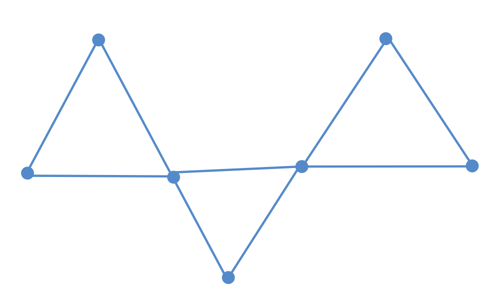
Hence a connected graph G is an euler graph if and only if it can be decomposed into circuits.

Definition: 2.6

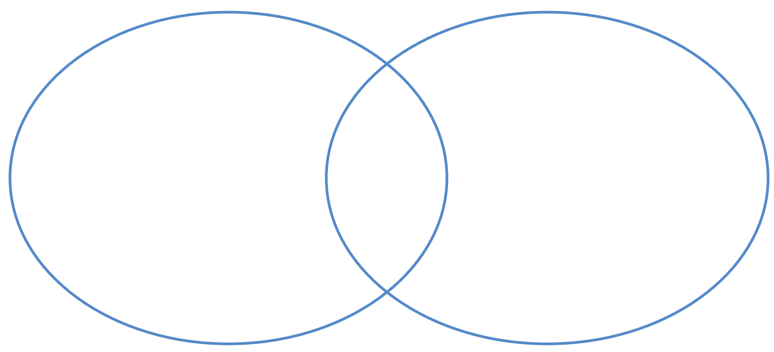
In an euler graph , it is an euler line is obtained when follows any closed walk from a vertex V such that one arrives at a vertex can select any edges which has not be previous traversed. Such a graph is called an **arbitrarily traceable graph** from the vertex



Arbitrarily traceable graph from G



Euler graph not arbitrarily traceable

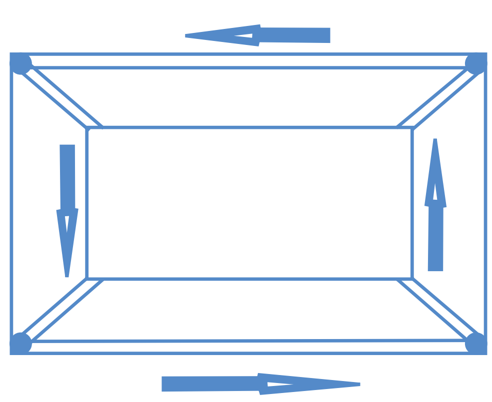


Arbitrarily
traceable graph
from all vertices

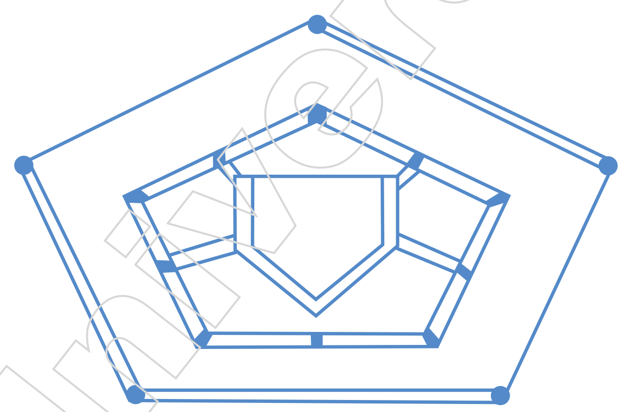
Definition: 2.7

Hamiltonian circuit in a connected graph is defined as a closed walk that traverses every vertex of G exactly once, except starting and final vertex

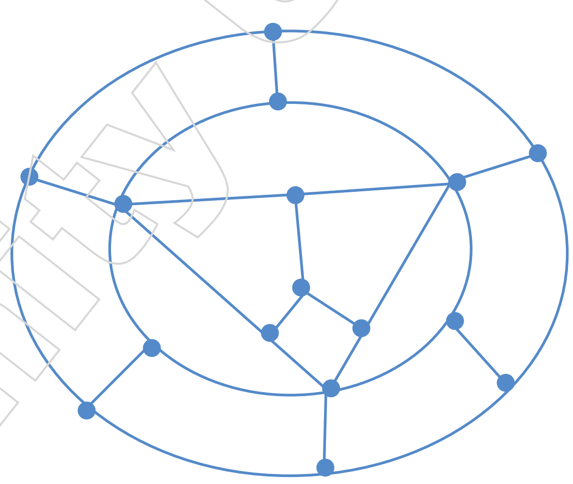
A circuit in a connected graph G is said to be Hamiltonian it includes every vertex of G. Hence a Hamiltonian circuit in a graph of n vertices of exactly n edges



Hamiltonian circuit



Graph of decahedron with Hamiltonian circuits



Graph without Hamiltonian circuits

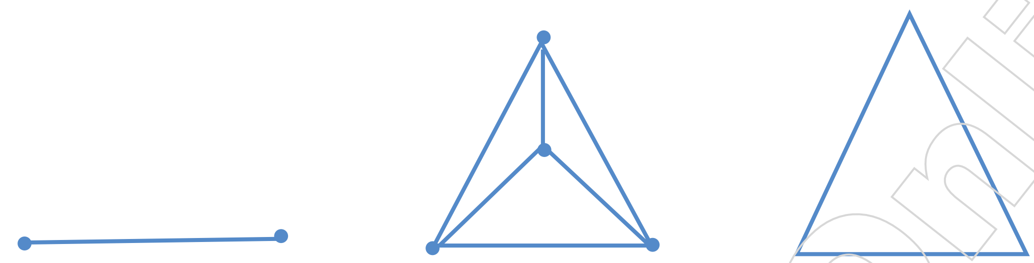
Definition: 2.8

If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a **Hamiltonian path**.

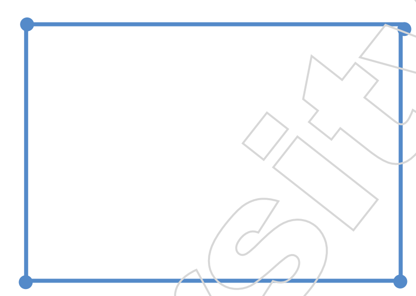
Notes

Definition: 2.9

A simple graph in which there exist an edge between every pair of vertices is called a **complete graph**



complete graphs



Not complete graph

Theorem: 2.10

In a complete graph with 'n' vertices there are n-1/2 edges disjoint Hamiltonian circuits if 'n' is an odd number greater than or equal to 3

Proof:

A complete graph G of 'n' vertices has n(n-1)/2 edges, and a Hamiltonian circuit in G consist of 'n' edges

Therefore the number of edge disjoint Hamiltonian circuit, when n is odd it can be shown as follows.

The subgraph (of complete graph of 'n' vertices) in figure is a Hamiltonian circuit

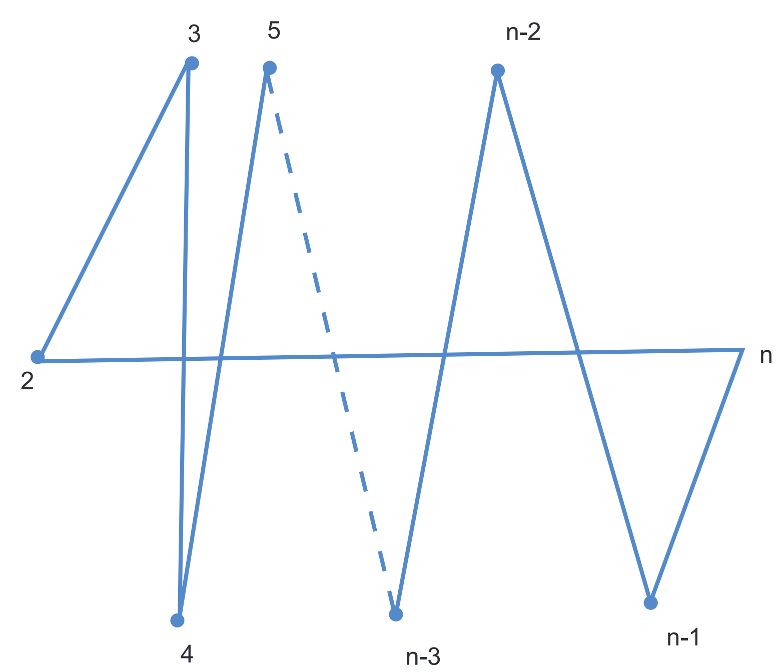
Keeping the vertices on a circle rotate the polygonal pattern clockwise by

$$\frac{360}{n-1}, 2.\frac{360}{n-1}, 3.\frac{360}{n-1}, \ldots\ldots \frac{n-3}{2}, \frac{360}{n-1} \text{ degrees}$$

Observe that each rotation produces a Hamiltonian circuit that has no edges in common with any of the previous one .

Thus n-3/2 new Hamiltonian circuits, all edge disjoint from the one in figure and also edge disjoint among themselves.

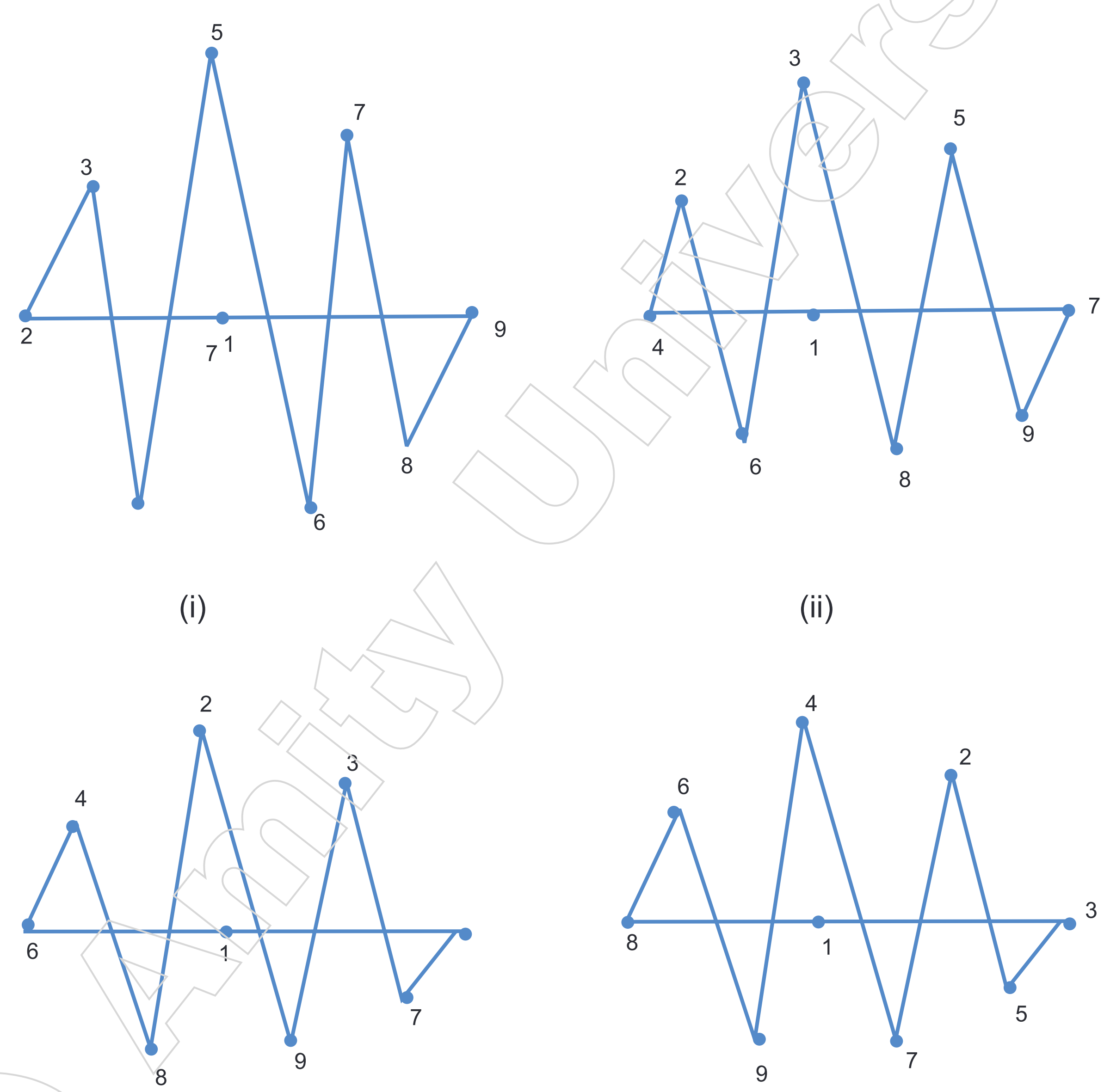
Hence there are n-1/2 edge disjoint Hamiltonian circuit.



Problem: 1

Find the 4 edge disjoint Hamiltonian cycles in K_9.

Solution:



(i) (ii)

(iii) (iv)

Edge disjoint hamiltonian cycles in K_9

Notes

Dirac's Theorem:

A vertex in G be at least sufficient (but by no means necessary condition) for a simple graph G to have a Hamiltonian circuit is that the degrees of every n/2, where 'n' is the number of vertices in G

Proof:

Assume that G is non-hamiltonian circuit

Let G be the maximal non-hamiltonian with $n \geq 3$

$d(v_i) \geq \frac{n}{2}$, i=1 to n

We know that,

Any complete graph is Hamiltonian.

Therefore G cannot be complete.

There exist a pair of vertices u and v in G, which are not adjacent

Consider G'=G+uv by choice of G. G' becomes Hamiltonian

As G is non-hamiltonian. this mean that every hamiltonian cycle of G must contain the edge uv (i.e.) Removal of an edge uv from any Hamiltonian cycle of G, results in a(u–v) Hamiltonian path in G

Let p=$v_1 v_2 v_3 \ldots\ldots, v_n$ with v_1 = u as original and v_n =v as terminous

By definition,

$A = \{v_i \,/\, uv_i \in E(G)\}$

$B = \{v_i \,/\, v_i u \in E(G)\}$

Now, $uv \notin E(G)$

$v \notin A\ u \notin B$ and $v \notin B$, $v \notin A \cup B$ and $|A \cup B| < n$

$\Rightarrow vk \in A \cap B$

$v_1 v_2 v_3 \ldots\ldots, v_k\, v_n v_{k-1}, v_1$ is a Hamiltonian cycle of G

Which is a contradiction to the fact that G is non-hamiltonian

Therefore $A \cap B = \phi$

(i.e.) $|A \cap B| = 0$

Now the definition of A and B,

|A|=dG(u) and |B|=dG(V)

Therefore dG(u) +dG(V)=|A| + |B|

$\leq |A \cup B| + |A \cap B|$

< n+0

dG(u) +dG(V) < n

Notes

By the hypothesis,

$\Rightarrow dg(u) \geq \frac{n}{2}$

$\Rightarrow dg(v) \geq \frac{n}{2}$

$\Rightarrow$ dG(u) +dG(V) $\geq \frac{n}{2} + \frac{n}{2}$

$\Rightarrow$ dG(u) +dG(V) $\geq n$

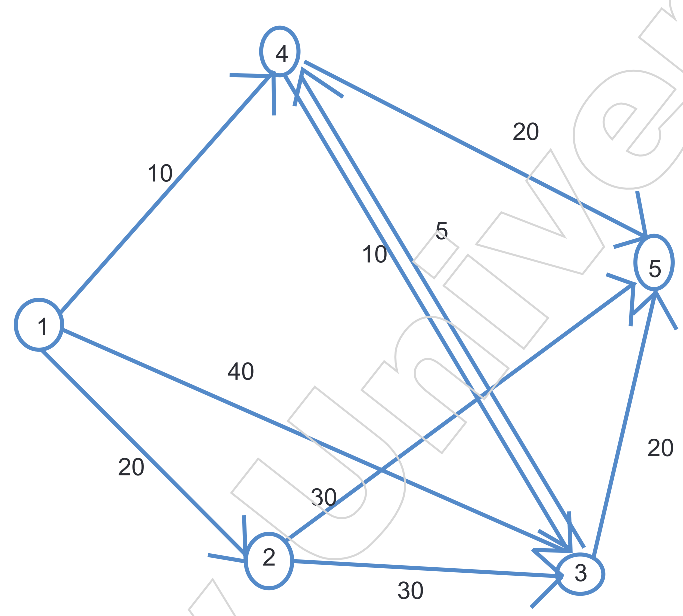
Which is a contradiction

Therefore our assumption is wrong

Therefore G must be Hamiltonian.

Shortest Path Problem:

Let us given a weighted network (*V,E,C*) with vertex set *V*, edge set *E*, and the weight set C specifying weight sc_{ij} for the edges (*i,j*) ∈*E*. Weare also given a starting node*s* ∈*V*. The **one-to-all shortest path** problem is the problem of determining the shortest path from node*s* to all the other nodes in the network.



Algorithms Solving the Problem

Dijkstra's algorithm:

Solves only the problems with non-negative costs, i.e.,

$c_{ij} \geq 0$ for all $(i,j) \in E$

Properties of Dijkstra's algorithm

Many more problems than you might at first think can be cast as shortest path problems, making Dijkstra's algorithm a powerful and general tool.

For example:

- Dijkstra's algorithm is applied to automatically find directions between physical locations, such as driving directions on websites like Map quest or Google Maps.

Notes

- In a networking or telecommunication applications, Dijkstra's algorithm has been used for solving the min-delay path problem (which is the shortest path problem).
- It is also used for solving a variety of shortest path problems arising in plant and facility layout, robotics and transportation.
- Dijkstra's algorithm solves such a problem and finds the shortest path from a given node*s* to all other nodes in the network where node *s* is called a starting node or an initial node
- Dijkstra's algorithm starts by assigning some initial values for the distances from node*s* and to every other node in the network.
- It operates in steps, where a teach step the algorithm improves the distance values.
- A teach step, the shortest distance from node*s* to another node is determined.

Characterizations:

- The algorithm characterizes each node by its state. The state of a node consists of two features
  (i) Distance value
  (ii) Status label
- Distance value of a node is a scalar representing an estimate of its distance from node*s*.
- Status label is an attribute specifying whether the distance value of a node is equal to the shortest distance to node *s* or not.
- The status label of a node is Permanent if its distance value is equal to the shortest distance from node*s.* Otherwise, the status label of a node is Temporary.
- The algorithm maintains and step-by-step updates the states of the nodes. At each step one node is designated as current

Algorithm Steps

Step 1. Initial Stage

- As sign the zero distance value to nodes, and label it as Permanent. [The state of node s is (0,p).]
- Assign to every node a distance value of ∞ and
  Label them as Temporary. [The state of every other node is (∞,t).].
- Designate the nodes as the current node

Step 2.

- Distance Value Update and Current Node Designation Update Let i be the index of the current node.

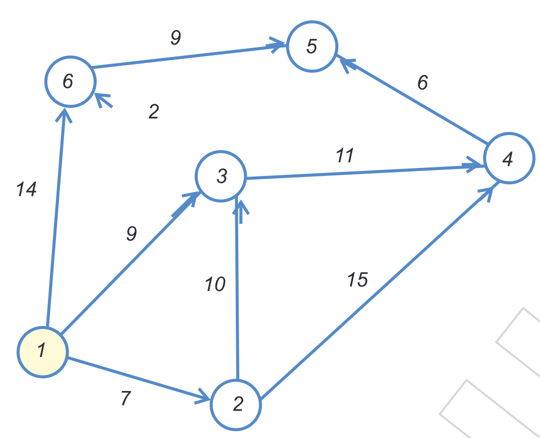
- Find the set J of nodes with temporary labels that can be reached from the current node i by a link (i, j). Update the distance values of these nodes.
- For each j∈J, the distance value d_j of node j is updated as follows new $d_j=\min\{d_j, d_i+c_{ij}\}$ where c_{ij} is the cost of link (i,j), as given in the network problem.
- Determine a node j that has the smallest distance value d_j among all nodes
- j ∈ J, find **j** such that $\min d_j = d_{j^*}$ j∈J.
- Change the label of node **j** permanent and designate this node as the current node.

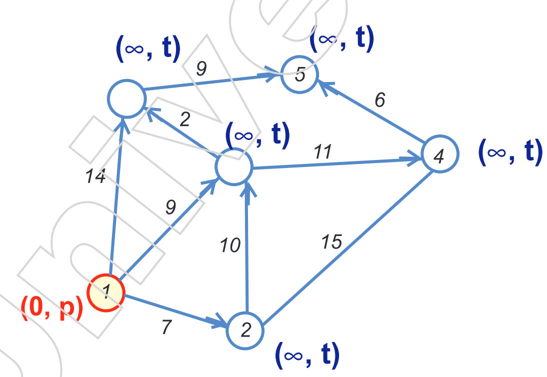
Step 3. Terminal Stage

- If all nodes that can be reached from nodes have been permanently labeled, then stop - we are done.
- If we cannot reach any temporary labeled node from the current node, then all the temporary labels become permanent
- Otherwise, go to **Step 2**.

Problem: 3

To find the shortest path from node 1 to all other nodes using Dijkstra's algorithm.

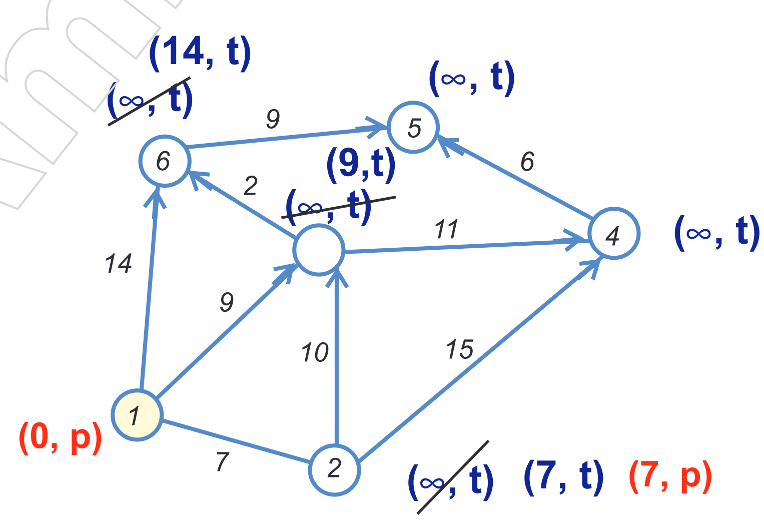




Step1: Initial Stage

- Node 1 is designated as the current state.
- The state of node 1 is $(0,p)$.
- Every other node has state (∞,t)

Step: 2

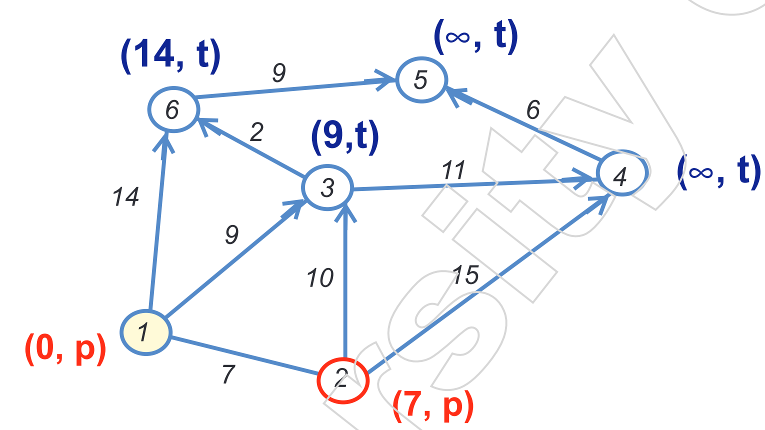


Notes

- Nodes 2,3, and 6 can be reached from the current node 1.
- Update distance values for these nodes $d_2 = \min\{\infty, 0 + 7\} = 7$
- $d_3 = \min\{\infty, 0 + 9\} = 9$ $d_6 = \min\{\infty, 0+14\} = 14$. Now, among the nodes 2,3, and6, node 2 has the smallest distance value.
- The status label of node 2 changes to permanent, so its state is $(7,p)$, while the status of 3 and 6 remains temporary.
- Node 2 becomes the current node

Step: 3

So graph is end at the step 2



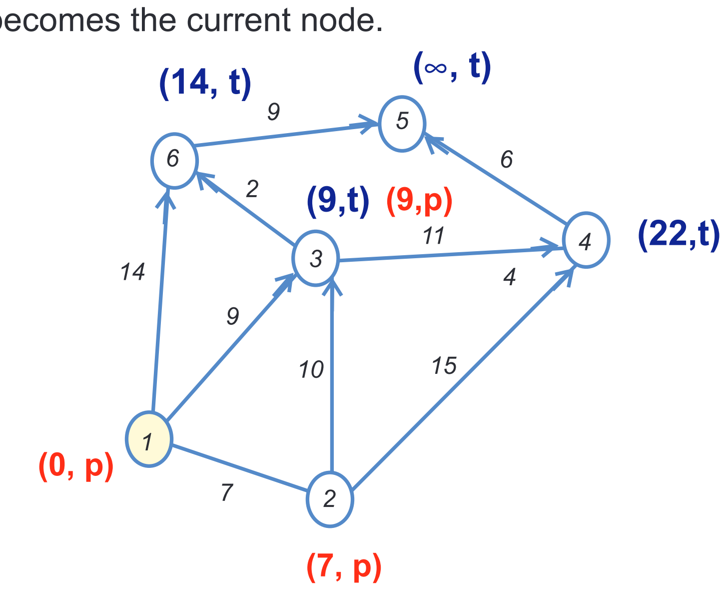
- We are not done, not all nodes have been reached from node1, so we will repeat again step: 2
- Node 2 becomes the current node

Another Implementation of Step: 2

- Nodes 3 and 4 can be reached form the current node 2.
- Update the distance values for theses nodes

  $d_3 = \min\{9, 7+10\} = 9$

  $d_6 = \min\{\infty, 7+15\} = 22$
- Now, between the nodes 3 and 4 node 3 has the smallest distance value.
- The status label of node 3 changes to permanent, while the status of 6 remains temporary.
- Node 3 becomes the current node.



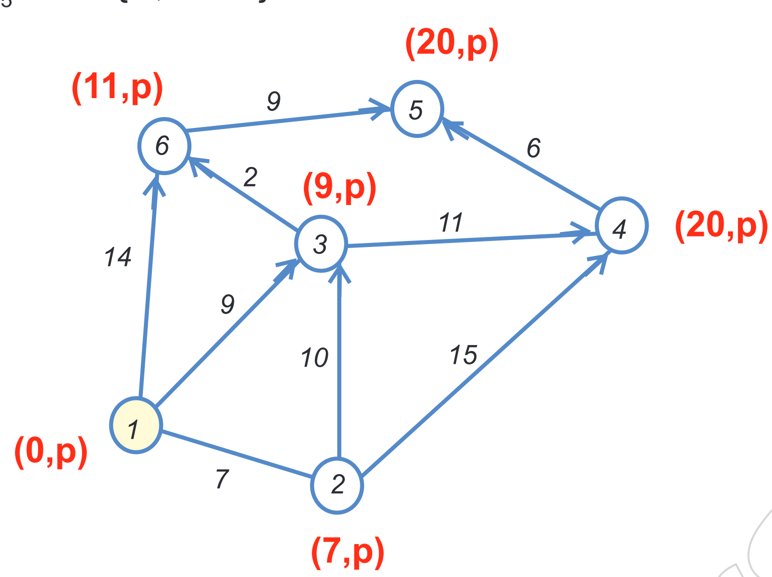
Notes

- We are not done (Step 3 fails), so we perform another Step 2

Another Step: 2

- Node 5 can be reached from the current node 6.
- Update distance value for node 5.

$$d_5 = \min\{\infty, 11+9\} = 20$$

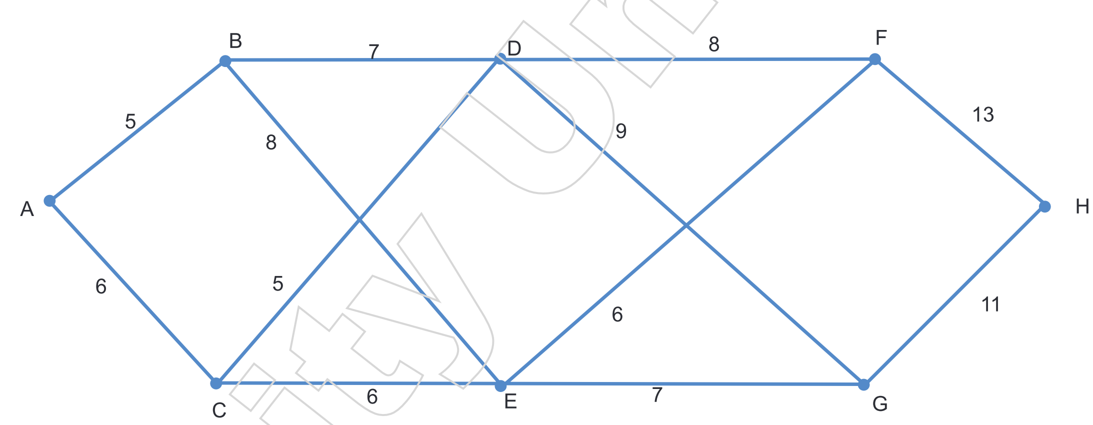


- Now, node 5 is the only candidate, so its status changes to permanent.
- Node 5 becomes the current node
- From node 5 we cannot reach any other node.

  Hence, node 4 gets permanently labeled

Problem: 4

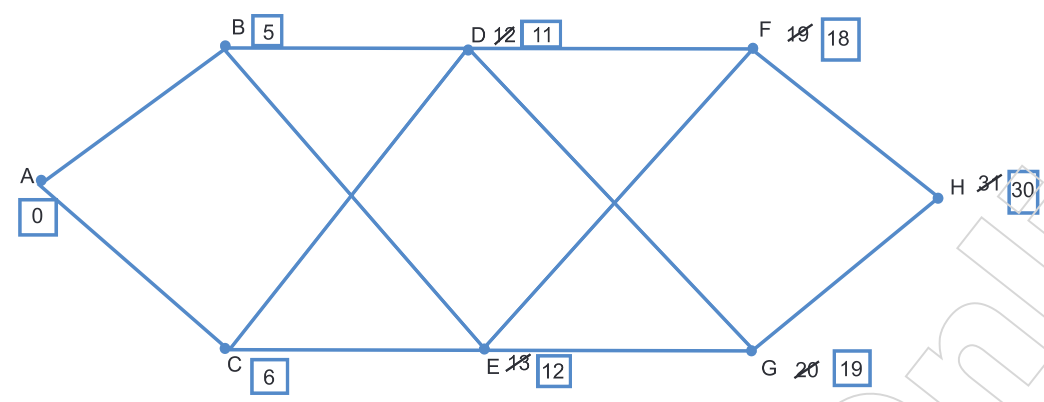
Find the shortest distance from A to H on the network below



Solution:

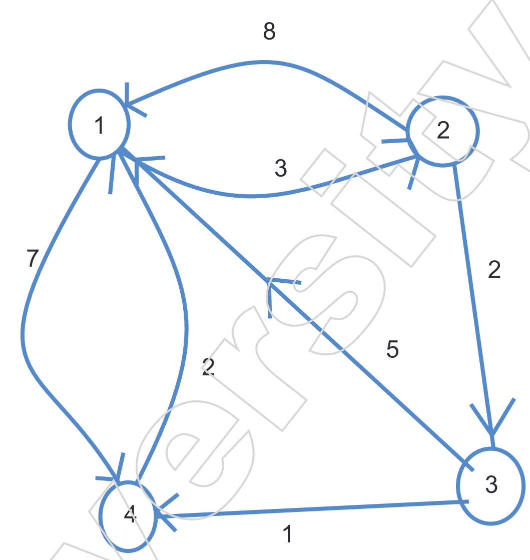
The fully labeled diagram below shows that the values, both temporary and permanent at each vertex

Notes



Problem: 6

Find all pairs of shortest path using dynamic programming



Solution:

Let us start 1→2,1→3,1→4

2→1,2→3,2→4

3→1,3→2,3→4

4→1,2→2,4→3

Now by solving dynamic programming, it can be solved to taking sequence of decisions. In each we have to take a decision.

Now we have to preparing a matrix using the problem

$$A^0 = \begin{matrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & \infty \\ 5 & \infty & 0 & 1 \\ 2 & \infty & \infty & 0 \end{matrix}$$

If there is no edge or an absence of an edge then it takes to be infinity

Step: 1

Let us consider the First row and first column it remains the same also the diagonals should be remains the same

Notes

$A^k[i,j\}=\min\{A^{k-1}[i,j],A^{k-1}[i,k]+A^{k-1}[k,j]\}$

(i) $A^0[2,3]= A^0[2,1]+ A^0[1,3]$

$2 < 8 + \infty$

(ii) $A^0[2,1]= A^0[2,1]+ A^0[1,4]$

$\infty > 8 + 7$

(iii) $A^0[2,1]= A^0[2,1]+ A^0[1,4]$

$\infty > 5 + 3$

In the same we have to prepare the values and form a matrix A^1

$$A^1 = \begin{matrix} 0 & 3 & \infty & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 8 & \infty & 0 \end{matrix}$$

Step: 2

Here also we consider the diagonals remains same and we prepare this matrix A^2 by the above matrix A^1 so that we have to kept the second row and second column must be constant.

(i) $A^1[1,3]= A^1[1,2]+ A^1[2,3]$

$2 > 3 + 2$

(ii) $A^1[1,4]= A^1[1,2]+ A^1[2,4]$

$7 < 3 + 15$

In the same we have to prepare the values and form a matrix A^1

$$A^2 = \begin{matrix} 0 & 3 & 5 & 7 \\ 8 & 0 & 2 & 15 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{matrix}$$

Step: 3

Here also we consider the diagonals remains same and we prepare this matrix A^2 by the above matrix A^1 so that we have to kept the third row and third column must be constant .

(i) $A^2[1,2]= A^2[1,3]+ A^2[3,2]$

$3 > 5 + 8$

In the same we have to prepare the values and form a matrix A^1

Notes

$$A^3 = \begin{matrix} 0 & 3 & 5 & 6 \\ 7 & 0 & 2 & 3 \\ 5 & 8 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{matrix}$$

Step: 4

Like wise we prepared the matrix A^4

$$A^4 = \begin{matrix} 0 & 3 & 5 & 6 \\ 5 & 0 & 2 & 3 \\ 3 & 6 & 0 & 1 \\ 2 & 5 & 7 & 0 \end{matrix}$$

$A^k [i,j\}=\min\{A^{k-1} [i,j],A^{k-1} [i,k]+A^{k-1} [k,j]\}$

Travelling salesman problem:

- There are number of cities a salesman must visit e.g.: A,B,C&D
- The distance / time/cost between every pair of cities is given
- The salesman starts from his home city, he must visit every city exactly once and return to his home city
- The problem is to find the route shortest distance / time/ cost

Phase –I

- TSP can be first solved as Assignment problem (AP) by using Hungarian method to find optimum solution.
- Then check the TSP condition
- If the condition is satisfied, then the AP solution will be the optimum solution even for TSP.

If not go to Phase –II

- I - the solution can be adjusted by inspection
- II-form a single circuit
- III- The iterative procedure –Branch and Bound method

Problem: 6

A travelling salesman has planned to visit from a particular city, visit each city only once and return to the starting city. The travelling cost in rupees is given in the table below. Find the least cost route.

Notes

To city

From city

	A	B	C	D
A	0	25	75	45
B	35	0	150	25
C	35	40	0	15
D	65	75	130	0

Solution:

Step: 1 (Row Reduction Method)

First assign the ∞ for the diagonal values

	A	B	C	D
A	∞	0	50	20
B	10	∞	125	0
C	20	25	∞	0
D	0	10	65	∞

Step: 2 (Column Reduction Method)

	A	B	C	D
A	∞	0	0	20
B	10	∞	75	0
C	20	25	∞	0
D	0	10	15	∞

Notes

Step: 3

	A	B	C	D
A	∞	$\boxed{0}$	0	20
B	10	∞	75	$\boxed{0}$
C	20	25	∞	0
D	$\boxed{0}$	10	15	∞

Step: 4

	A	B	C	D
A	∞	0	0	30
B	10	∞	65	0
C	20	15	∞	0
D	0	0	5	∞

Step: 5

	A	B	C	D
A	∞	$\boxed{0}$	0	30
B	10	∞	65	$\boxed{0}$
C	20	15	∞	0
D	$\boxed{0}$	0	5	∞

Notes

Step: 6

	A	B	C	D
A	∞	0	$\boxed{0}$	40
B	$\boxed{0}$	∞	55	0
C	10	5	∞	$\boxed{0}$
D	0	$\boxed{0}$	5	∞

A→C→D→B→A

75+15+75+35

=Rs. 200

Bipartite Graphs

Definition: 2.11

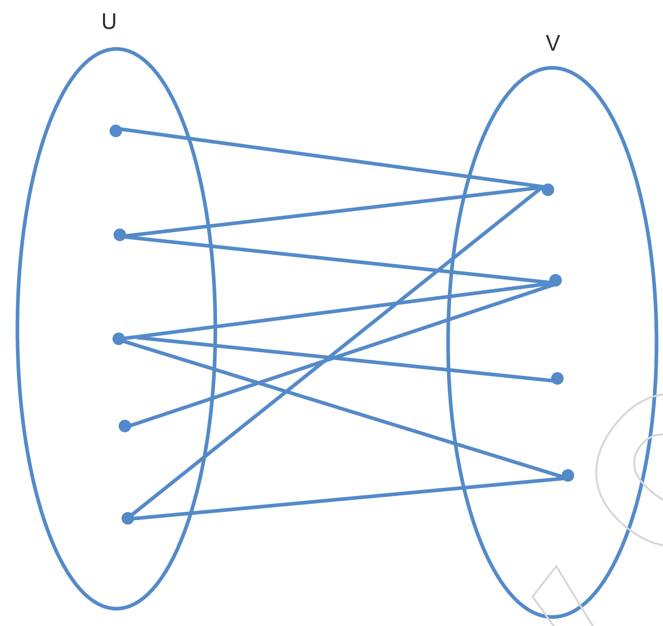
A **Bipartite graph** is a graph whose vertices can be divided into two disjoint and independent setsU {\displaystyle U} U and V {\displaystyle V} V such that every edge connects in a vertex in U {\displaystyle U} U to a vertex V. It is called **Bigraph.** Equivalently each edge is either (u, v) which connects edge a vertex from set U to vertex from set V or (v, u) which connects edge a vertex from set V to vertex from set U.

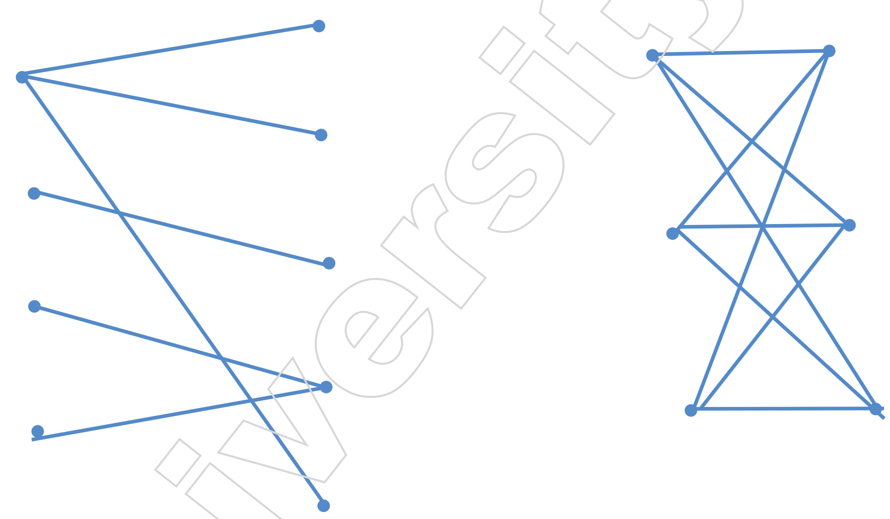
If both sets have equal cardinality (means both sets have an equal number of vertices), then it is called **balanced bipartite graph** and if each vertex in set U has an edge to all the vertices in another set V and vice versa then that bipartite graph is called **complete bipartite graph**.

Some of the properties are:

- Every tree is bipartite.
- Cycle graphs with an even number of vertices are bipartite.
- Every planar graph whose faces all have even length is bipartite.
- The complete bipartite graph on *m* and *n* vertices, denoted by $K_{n,m}$ is the bipartite graph G = (U , V , E) {\displaystyle G=(U,V,E)}, where *U* and *V* are disjoint sets of size *m* and *n*, respectively, and *E* connects every vertex in *U* with all vertices in *V*. It follows that $K_{m,n}$ has *mn* edges.
- Hypercube graphs, partial cubes, and median graphs are bipartite.

Notes

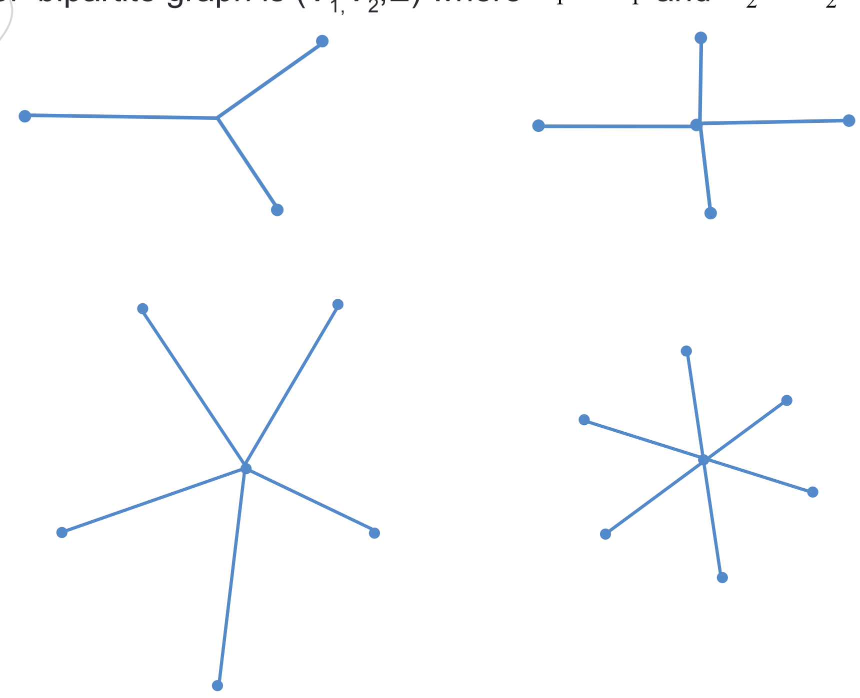




The above graphs are called Bipartite graphs

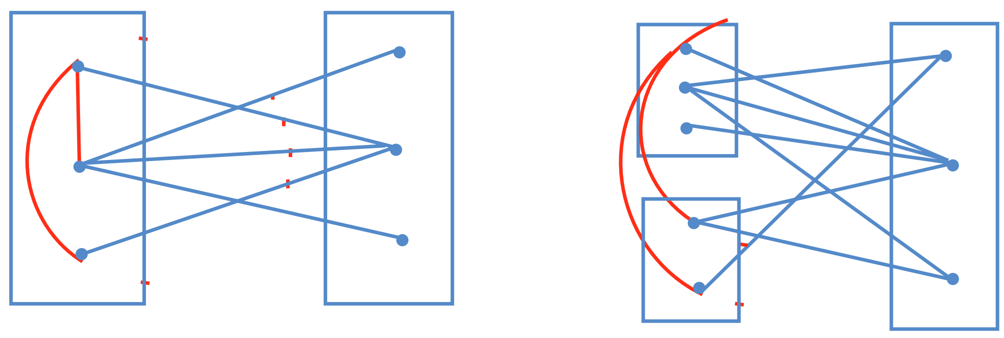
Definition: 2.12

A graph is said to **complete bipartite graph** whose vertices can be partitioned into two subsets v_1 and v_2 such that no edge has both endpoints in the same subset, and every possible edge that could relate vertices in different subsets is part f the graph. The structure of bipartite graph is (V_1, V_2,E) where $v_1 \in V_1$ and $v_2 \in V_2$



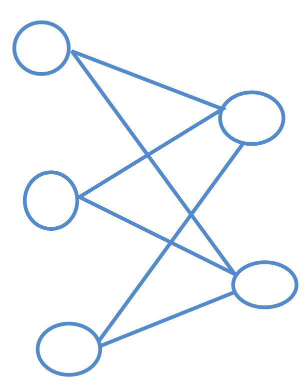
Notes

The above graphs are called complete bipartite graphs

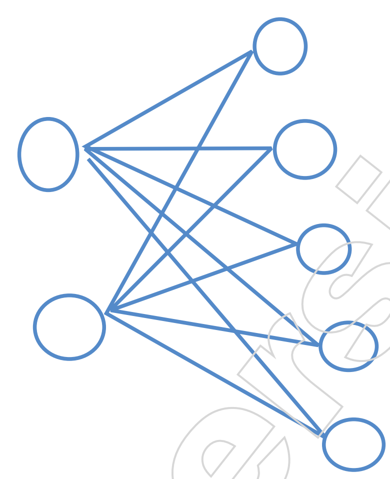


Non Bipartite graphs due to shared features

Examples of complete bipartite graphs:



$K_{3,2}$



$K_{2,5}$

Questions:

1. A simple digraph in which there is exactly one edge directed from every vertex to every other vertex is a___________ digraph.
   a) complete symmetric
   b) complete asymmetric
   c) symmetric
   d) asymmetric.
2. The vertices and edges in a __________need not be distinct.
   a) walk
   b) not a walk
   c) a or b
   d) a and b.
3. A graph is said to___________ whose vertices can be partitioned into two subsets
   a) bipartite graph
   b) complete graph
   c) graph
   d) none of the above.

Notes

4. Cycle graphs with an ______ number of vertices are bipartite.

   a) Odd

   b) even

   c) both a and b

   d) none.

5. In Dijkstra's algorithm characterizes each node by its state. The state of a node consists of two features are Distance value __________.

   a) Nodes

   b) vertices and edges

   c) Status label

   d) none .

6. If both sets have equal cardinality (means both sets have an equal number of vertices, then it is called ____________.

   a) Bipartite graph

   b) complete graph

   c) balanced bipartite graph

   d) digraph

7. If we remove any one edge from a Hamiltonian circuit, we are left with a path. This path is called a ___________

   a) Euler walk

   b) Hamiltonian path

   c) fundamental circuit

   d) none of the above.

8. A simple graph in which there exist an edge between every pair of vertices is called a ____________.

   a) complete graph

   b) simple graph

   c) bipartite graph

   d) none of the above .

9. If G_1 and G_2 are said to be __________ then

   $G_1 \cup G_2 = G_2 \cup G_1$

   $G_1 \cap G_2 = G_2 \cap G_1$

   $G_1 \oplus G_2 = G_2 \oplus G_1$

   a) Commutative

   b) Associative

c) additive

d) none of the above

10. If G_1 and G_2 are __________ then G_1 intersection G_2 is a null graph

$G_1 \oplus G_2 = G_1 \cup G_2$

a) Vertex disjoint

b) edge disjoint

c) both a and b

d) none of the above

Notes

Answer:

1. a
2. a
3. a
4. b
5. c
6. c
7. b
8. a
9. a
10. b

Problems:

1. If a set S={1,2,3...,n}, among these we have selected m elements, for which the sum of these elements is k. What will be the best logic for this?
2. How to arrange the vertices (which have different lengths) of a graph on a straight line such that the bandwidth of the graph is minimized?
3. What are the ways to transfer a graph from one Relation space to a Euclidean space with less time complexity?
4. Derive the relationship between vertex connectivity and edge connectivity.

Exercises:

1. Explain Shortest Path Problem
2. Write algorithm for shortest path problem
3. Define Vertex connectivity and Edge Connectivity
4. Define Hamiltonian cycle and Hamiltonian Circuit

Notes

1. For what values of n does the graph Kn contain an Euler trail? An Euler tour? A Hamilton path? A Hamilton cycle?

   Solution:

   Euler trail: K1, K2, and Kn for all odd n -3.

   Euler tour: Kn for all odd n -3.

   Hamilton path: Kn for all n -1.

   Hamilton cycle: Kn for all n- 3

2.

   (a) For what values of m and n does the complete bipartite graph Km;n contain an Euler tour?

   (b) Determine the length of the longest path and the longest cycle in Km;n, for all m; n.

   Solution:

   (a) Since for connected graphs the necessary and sufficient condition is that the degree of each vertex is even, m and n must be even positive integers.

   (b) The length of the longest cycle is 2. Any cycle must be even, and it must alternate vertices from the two sides. Thus it cannot be longer than twice the smaller side (and such a cycle clearly exists). By a similar reasoning, we get that if m = n, the longest path contains all the 2m vertices, so its length is 2m -1, and if m 6= n, the length of the longest path is 2 and starting and ending in the larger class.

3.

   (a) Find a graph such that every vertex has even degree but there is no Euler tour.

   (b) Find a disconnected graph that has an Euler tour.

   Solution:

   (a) Take a graph that is the vertex-disjoint union of two cycles. It is not connected, so there is no Euler tour.

   (b) The empty graph on at least 2 vertices is an example. Or one can take any connected graph with an Euler tour and add some isolated vertices.

4. Determine the girth and circumference of the following graphs.

   Solution:

   The graph on the left has girth 4; it's easy to _nd a 4-cycle and see that there is no 3-cycle.

   It has circumference 11, since below is an 11-cycle (a Hamilton cycle).

   The graph on the right also has girth 4. It also has circumference 11, since below is an 11-cycle.

   Actually, one can check that the two graphs are isomorphic.

Module - III

Key Learning Objectives:

At the end of this module, you will be able to:

- Discuss basic information on trees
- Describe rooted trees
- Determine the path length in rooted trees and spanning trees
- Perform a study of fundamental circuits
- Interpret spanning trees of a weighted graph
- Implement the concepts of cut sets and cut vertices
- Discuss basic concepts of minimum spanning tree

Notes

Unit- III

Definition: 3.1

A **Tree** is a connected graph without any circuits.

Theorem: 3.2

Prove that there is one and only one path between every pair of vertices in a tree T.

Proof:

Since T is connected, there must exist atleast one path between any pair of vertices in T.

Now suppose that between two vertices and ‘a‘ and ‘b’ of T. There are two distinct Paths.

The union of these two paths will contain a circuit which is impossible in a tree.

Hence there is a unique path between any pair of vertices.

Theorem: 3.3

Prove that in a graph G there is one and only one path between every pair of vertices G is a tree.

Proof:

In a graph G, there is one and only one path between every pair of vertices.

$\Rightarrow$ G is connected

Suppose, G has a circuit.

$\Rightarrow$ There is atleast one pair of vertices a, b such that, there are two distinct paths between a and b.

Since G has one and only one path between every pair of vertices, G can have no circuit.

Hence G is a tree.

Theorem: 3.4

Prove that a tree with ‘n‘ vertices has n-1 edges

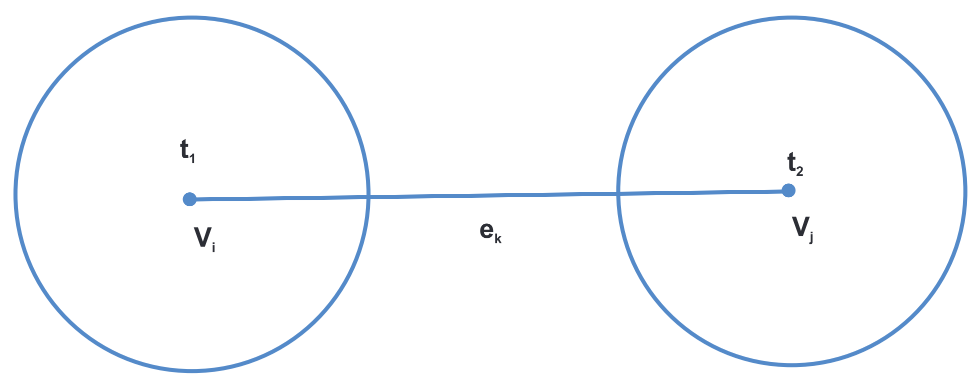
Proof:

The Theorem will be proved by induction on the number of vertices.

It is easy to see that the theorem is true for n = 1,2,3.

Assume that the theorem for all trees with f ever than n vertices.

Let us now consider a tree T with n vertices



T

In T let e_k be an edge with end vertices V_i and V_j. According to theorem, there is no other path between V_i and V_j except e_k.

Deletion of e_k from T will disconnect the graph as shown in figure.

Further more T-e_k consist of exactly two components, and since there were no circuits in T to begin with, each of these component is a tree.

Let the number of vertices of tree t_1 and t_2 be n_1 and n_2 respectively such that

$n_1+n_2=n$

Also $n_1<n$ and $n_2<n$

By induction hypothesis, the number of edges in t_1 and t_2 are n_1-1 and n_2 -1 respectively.

Thus T-e_k consist of

$n_1-1+n_2-1=n_1+n_2-2$

$\Rightarrow$ n-2 edges

Therefore T has exactly n-1 edges.

Theorem: 3.5

Prove that a connected graph with ‘n’ vertices and ‘n-1’ edges is a tree

Proof:

Let G be a connected graph with ‘n’ vertices and ’n-1’ edges

We show that G contains no circuits

Assume to the contrary that G contain circuit. Remove an edge from a circuit so that, the resulting graph is again connected.

Continue this process of removing one edge from one circuit at a time till the resulting graph ‘H’ is a tree

As H has ‘n’ vertices so number of edges in H is n-1. Now the number of edges in G is greater than the number of edges in H.

So, n-1 > n-1, which is not possible

Hence G has no circuits and therefore is a tree.

Notes

Definition: 3.6

A graph G is said to be **minimally connected** if removal of anyone edge from it disconnects the graph. Clearly a minimally connected graph has no circuits.

Theorem: 3.7

Prove that a graph is a tree if it is minimally connected.

Proof:

Let the graph G be minimally connected.

Then G has no circuits and therefore is a tree.

Conversely,

Let G be a tree.

Then G contains no circuits and deletion of any edge from G disconnect the graph.

Hence **G** is minimally connected.

Theorem: 3.8

Prove that a Graph **G** with n vertices, n-1 edges no circuits is connected.

Proof:

Suppose that there exist a circuitless graph G with `n' vertices and (n-1) edges which is disconnected.

In that case, **G** will consist of two or more circuitless component. Without loss of generality, Let G consist of two components g_1 and g_2.

Add an edge **e** between a vertex v_1 in g_1 and v_2 in g_2.

Since there was no path between v_1 and v_2 in **G,** adding **e** did not create a circuit This $G \cup e$ is a circuitless connected graph (i.e., a tree) of n vertices and n edges which is not possible. Hence **G** is connected.

Note:

- G is connected and is circuitless (or)
- G is connected and has (n-1) edges (or)
- G is circuitless and has (n-1) edges (or)
- There is exactly one path between every pair of vertices in G (or)
- G is minimally connected graph.

Definition: 3.9

In a tree, the vertex is said to be **pendent** if its degree is one.

Theorem: 3.10

Prove that in any tree (with two or more vertices) there are atleast two pendent vertices.

Proof:

Suppose that the tree T has `n' vertices.

Case: (i)

$n = 2$

T is of the form

Hence we have two pendent vertices.

Case: (ii)

$n>2$

Suppose T has n-1 edges.

We know that,

Every edge contribute two degrees. There are 2(n-1) degrees in the graph.

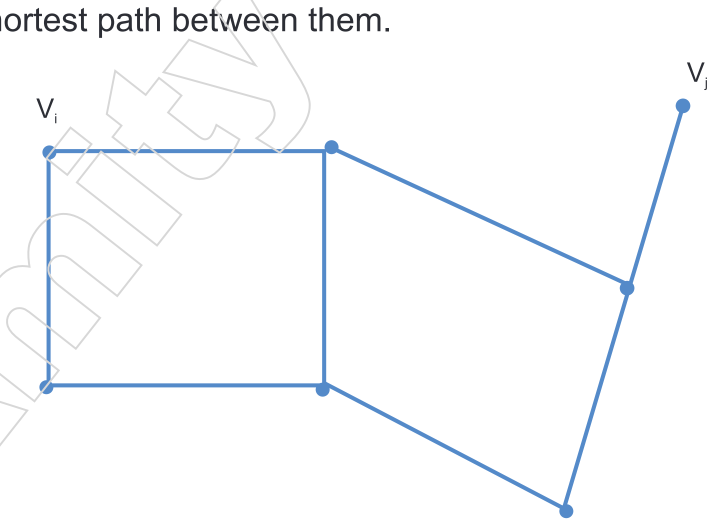
There 2(n-1) degree assigned to n vertices. In a tree we cannot have any vertex of degree zero.

Therefore atleast two vertices having degree one.

Therefore atleast two pendent vertices in a tree.

Definition: 3.11

In a connected graph G, the **distance** $d(v_i, v_j)$ between two of its vertices v_i and v_j is the length of the shortest path between them.



$d(V_i,V_j)=3$

Definition: 3.12

Let x be a nonempty set. The function 't' is said to be **metric** of an 'x'. If it satisfies the following condition.

Notes

(i) Non-negativity $f(x,y) \geq 0$ and $f(x,y)=0$ iff $x=y$.

(ii) Symmetry : $f(x,y) = f(y,x)$

(iii) Triangle inequality: $f(x,y) \leq f(x,z)$ if (z,y) for some z.

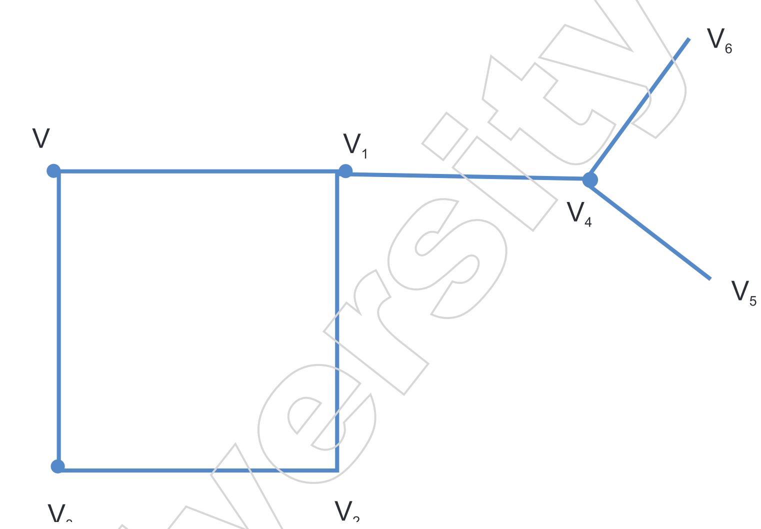
Definition: 3.13

The **eccentricity** E(v) of a vector V in a graph G in the distance from V to the vertex farthest from V in G

(i.e,) $E(v) = \max d(v,v_i)$

$V_i \in G$

Example:

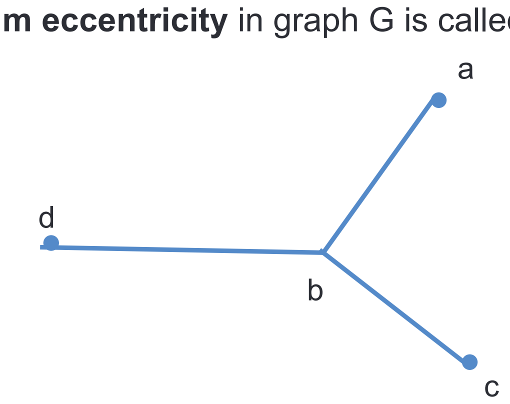


$d(v,v_1)=1$, $d(v,v_3)=1$, $d(v,v_2)=2$, $d(v,v_4)=2$, $d(v,v_5)=3$

Therefore $E(v)=3$

Definition: 3.14

A vertex with **minimum eccentricity** in graph G is called a center of G.



E(d) =3 E(b) =1

E(c) =2 E(a) =2

Therefore ‘b’ is the center.

Definition: 3.15

If a graph G has two center we refer to such centers as **bicenters**.

Theorem: 3.16

Prove that every tree has either one or two centers.

Proof:

Case: (i)

T has one vertex.

The vertex itself its center.

Case: (ii)

T has two vertices.

T is of the form

Therefore 'T' has two centers.

Case: (iii)

T has more than two vertices by theorem,

There are atleast two pendent vertices.

Let v be any vertex in T.

The maximum distance, max $d(v,v_i)$ from given vertex v to any other vertex v_i occurs only when v_i is a pendent vertex.

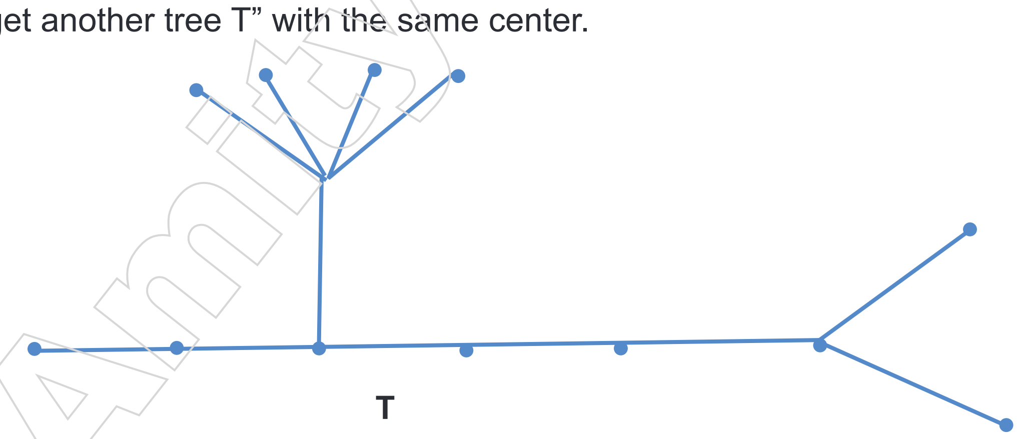
Delete all the pendent vertices from T.

The remaining graph T' is still a tree.

T' is also true with eccentricity of each vertex in one less than of T.

The center of T' are the center of T, only by deleting all the pendent vertices in T'.

We get another tree T" with the same center.



T

T'

Notes

Proceeding in this way until there is left a vertex or an edge.

If we have one vertex then the tree has one center. If we have one edge then the tree has two center the end vertices of the edge.

Therefore the tree has one or two centers.

Remark:

If a tree T has two centers, the two centers must be adjacent.

Definition: 3.17

An eccentricity of a center in a tree defined as the **radius of tree**.

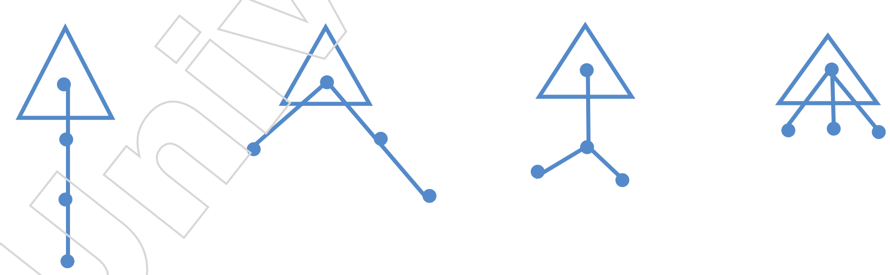
Definition: 3.18

The **diameter** of a tree T is defined as the length of longest path in tree.

Definition: 3.19

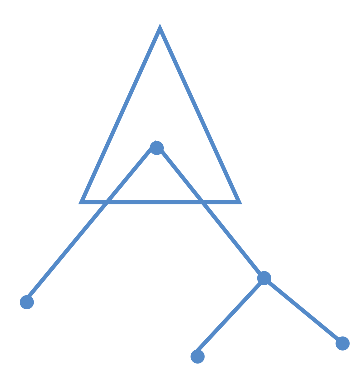
A tree in which one vertex called the root is distinguished from all the others is called a **rooted trees**.

Non-rooted trees are called **free** tree.



Definition: 3.20

A **binary tree** is defined as a tree in which is exactly one vertex of degree two, and each of the remaining vertices is of tree one or three.



Note:

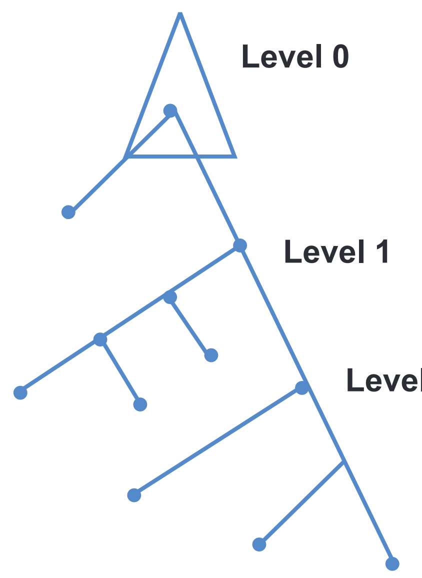
The number of vertices is a binary tree is always odd.

Definition: 3.21

A non-pendent vertex is a tree is called **internal vertex**.

Definition: 3.22

In a binary tree a vertex v_i is said to be at **level** l_i, if v_i at a distance of l_i from the root.



Definition: 3.23

The maximum level l_{max} of any vertex in binary tree is called the height of a tree.

Note:

Minimum possible height of an n-vertex binary tree is min

$l_{max} = \lceil \log_2(n+1)-1 \rceil$ where $\lceil n \rceil$ denotes the smallest integer ≥ n.

Max l_{max} =n-1 /2.

Definition: 3.24

Weighted path length is defined as every pendent vertex v_j of a binary tree has associated with it a positive real number w_j.

Given $w_1, w_2, \ldots, w_n$ the problem is to construct a binary tree that minimize.

$\sum w_j l_j$ where l_j is the level of pendent vertex v_j, and the sum is taken over all pendent vertices.

Definition: 3.25

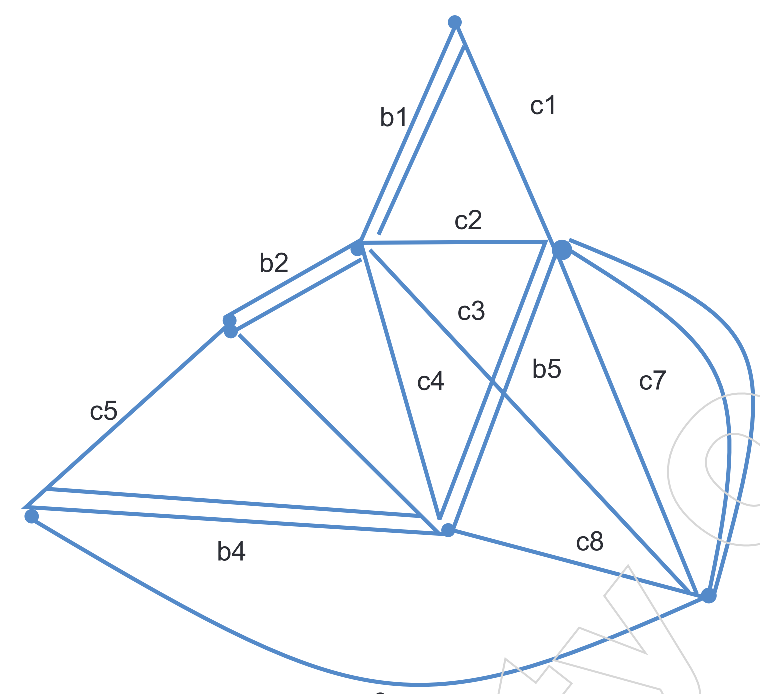
A tree t is said to be a **spanning tree** of a connected graph G if T is a subgraph of G and T contains all vertices of G.

Spanning tree is sometimes referred to as a skeleton or scaffolding of G.

Definition: 3.26

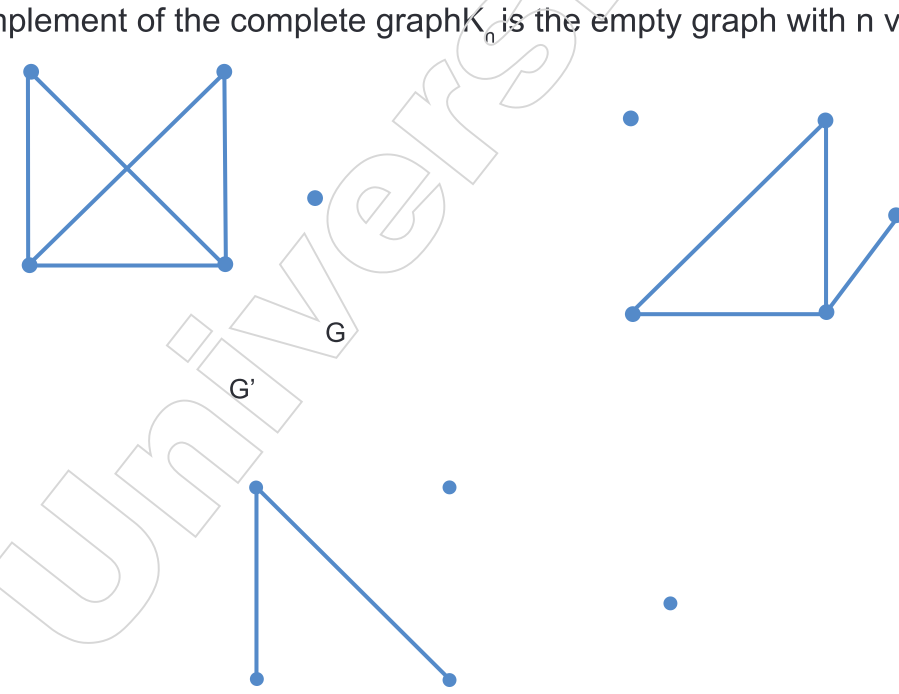
A collection of tree is called a **forest.**

Notes



Remark:

The complement of the complete graph K_n is the empty graph with n vertices.



G

G'

G-G'

Theorem: 3.27

Every connected graph has atleast one spanning tree.

Proof :

Let G be a connected graph.

Case: (i)

G has no circuits.

Since G is connected .G is a tree

Therefore G itself is a spanning tree.

Case: (ii)

G has circuits.

Delete an edge from a circuit in G, the remaining graph is also a connected graph.

If there are more circuits in the remaining graph repeat the produce till on edge from the last circuit from graph that contain all the vertices of G.

Hence it is spanning tree.

Definition: 3.28

An edge in a spanning tree T is called **branch** of T.

Definition: 3.29

An edge of G that is not in a given spanning tree T is called a **chord.**

Chord is sometimes referred to a tie or a link.

Note:

The subgraph T is a collection of chords it is referred to as the chord said the (or tie set or cotree).

Theorem: 3.30

With respect to any of its spanning trees, a connected graph of n vertices and **e** edges has n-1 branches and e-n+1 chords.

Proof:

Let G be a graph with `n' vertices and e edges, suppose that T is any spanning tree in G.

$\Rightarrow$ T is a tree with n vertices .

$\Rightarrow$ T has n-1 edges.

The number of branches in T is n-1.

Therefore the remaining e-(n-1) are chords of T.

There are e-n+1 chords of T.

Definition: 3.31

Let G be a graph with 'k' -components 'n' vertices, 'e' edges

Rank(r)=n-k

Nullity(μ)=e-n+k

Note:

Rank of G = number of branches in any spanning tree of G.

Notes

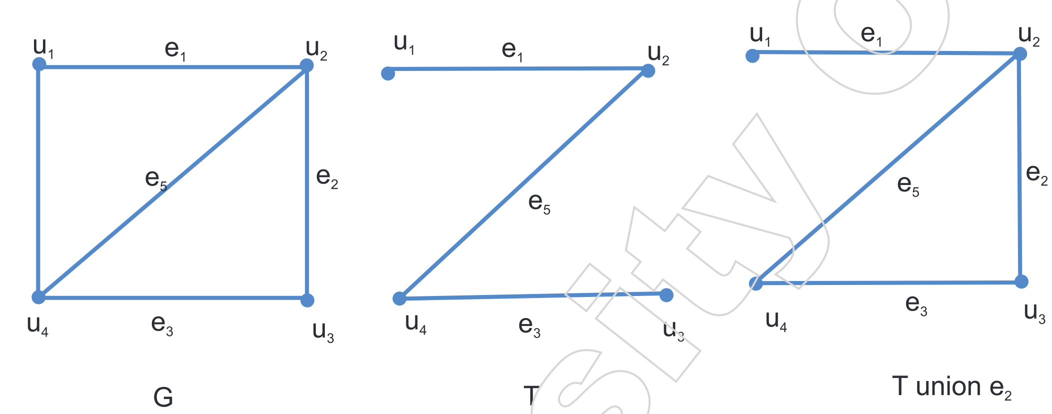
Nullity of G = number of chords in G.

Rank+Nullity = number of edges in G.

Nullity of a graph is also referred to as its cyclomatic number or first betti number.

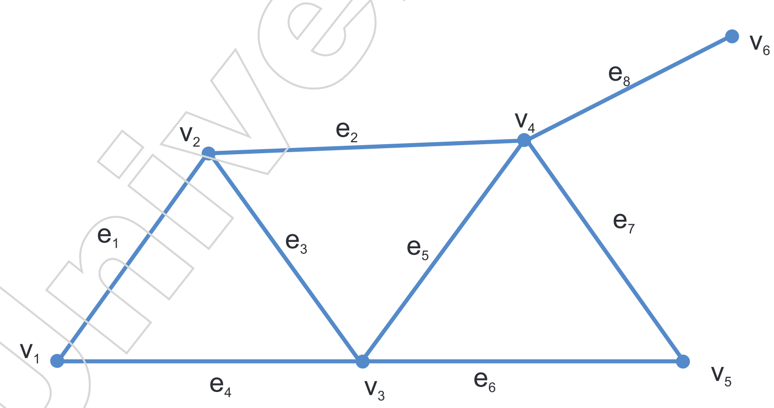
Definition: 3.31

Let T be any spanning tree in a connected graph G. Adding any one chord to T will create exactly one circuit such a circuit, formed by adding a chord to a spanning tree is called a **fundamental circuit.**



Problem: 1

In the graph



$\{e_1, e_4\}$, $\{e_6, e_7\}$, $\{e_1, e_2, e_3\}$, $\{e_8\}$, $\{e_3, e_4, e_5, e_6\}$, $\{e_2, e_5, e_7\}$, $\{e_2, e_5, e_6\}$ and $\{e_2, e_3, e_4\}$ arecut sets. Will find that there are any other cut sets?

Solution:

In a graph G= (V, E), a pair of subsets V_1 and V_2 of V satisfying

$V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $V_1 = \emptyset$, $V_2 = \emptyset$,

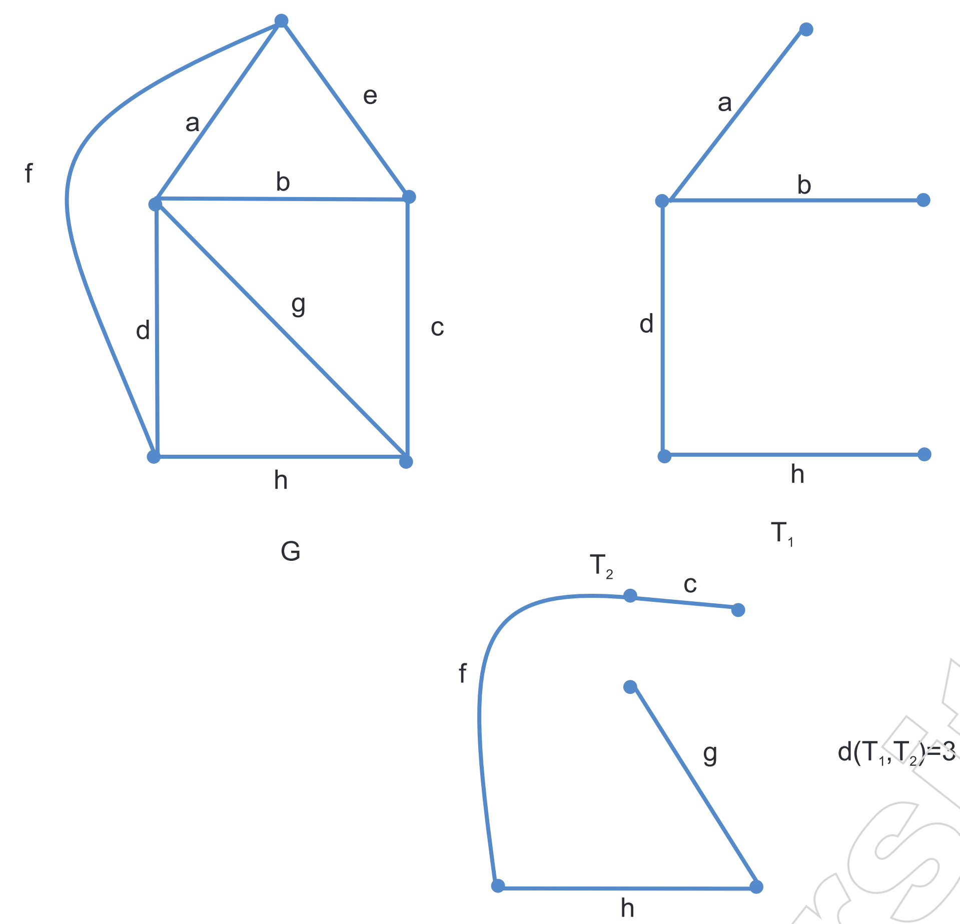
is called a cut (or a partition) of G, denoted V_1, V_2 .

Usually, the cuts V_1, V_2 and V_2, V_1 are considered to be the same.

Definition: 3.32

The distance between two spanning tree T_i and T_j of a graph G is defined G as the no of edges of G present in one tree but not in other.

The distance may written as $d(T_i, T_j)$



Remark:

The distance between the spanning tree of a graph is a metric.

ie.,) i) $d(T_i,T_j) \geq 0$ $d(T_i,T_j) = 0$ iff $T_i = T_j$

ii) $d(T_i, T_j) = d(T_i,T_j)$

iii) $d(T_i,T_j) \leq d(T_i,T_e) + d(T_k,T_j)$

Remark:

Max $d(T_i,T_j)$ =1/2 max $N(T_i \oplus T_j)$

$\leq r$, the rank of G

Max $d(T_i,T_j) \leq \mu$

Max $d(T_i,T_j) \leq \min(\mu,r)$

Definition: 3.33

For a spanning tree T_0 of a graph G, let max $d(T_i,T_j)$denote the maximal distance between T_0 and any other spanning tree of tree. Then T_0 is called a **central tree** of G if max $d(T_o,T_i) \leq d(T,T_j)$ for every tree T of G.

Definiton: 3.34

The **tree graph** of a given graph G is defined as a graph in which each vertex corresponds to a spanning tree of G, and each edge corresponds to a cyclic interchange between the spanning tree is of G representing by the two and vertices of the edge.

Notes

Definition: 3.35

A graph G is called a **labeled graph** if its edges or vertices are assigned data of one kind or another. In particular, G is called a weighted graph if each edge e of G is assigned a nonnegative number w(e) called the weight or length of v.

The below figure shows a weighted graph where the weight of each edge. The weight (or length) of a path in such a **weighted graph**G is defined to be the sum of the weights of the edges in the path. In the below figure the length of a shortest path between P and Q is 14;

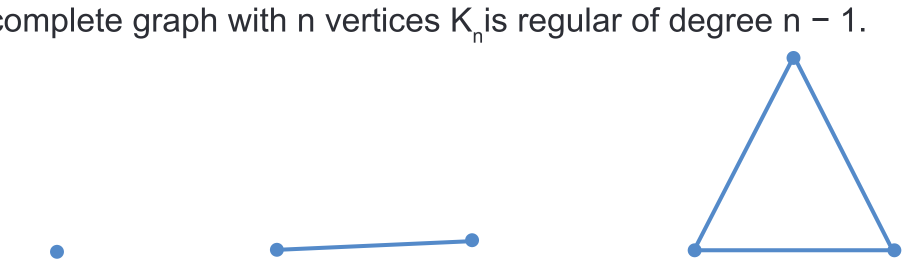
One such path is $(P,A_1,A_2,A_5,A_3,A_6,Q)$

Definition: 3.36

A graph G is said to be **complete** if every vertex in G is connected to every other vertex in G. Thus a complete graph G must be connected. The complete graph with n vertices is denoted by Kn.

Definition: 3.37

- A graph G is said to be **regular** of degree k or k-regular if every vertex has degree k. In other words, a graph is regular if every vertex has the same degree.
- The connected regular graphs of degrees 0, 1, or 2 are easily described.
- The connected 0-regular graph is the trivial graph with one vertex and no edges.
- The connected 1-regular graph is the graph with two vertices and one edge connecting them.
- The connected 2-regular graph with n vertices is the graph which consists of a single n-cycle.
- The 3-regular graphs must have an even number of vertices since the sum of the degrees of the vertices is an even number.
- In general regular graphs can be quite complicated. For example, there are nineteen 3-regular graphs with ten vertices.
- The complete graph with n vertices K_n is regular of degree n − 1.

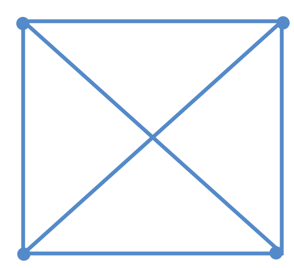


K_1 isolated vertex

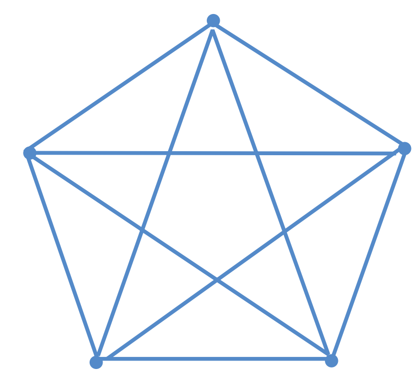
K_2 line segment

K_3 Triangle

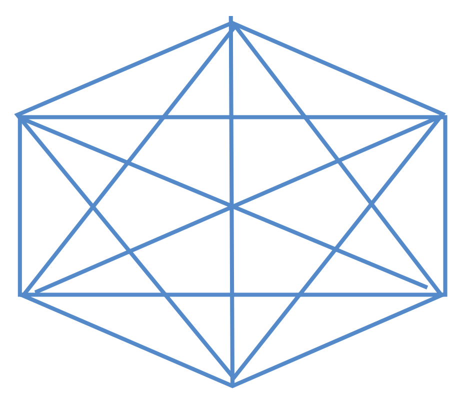
Notes



K_4



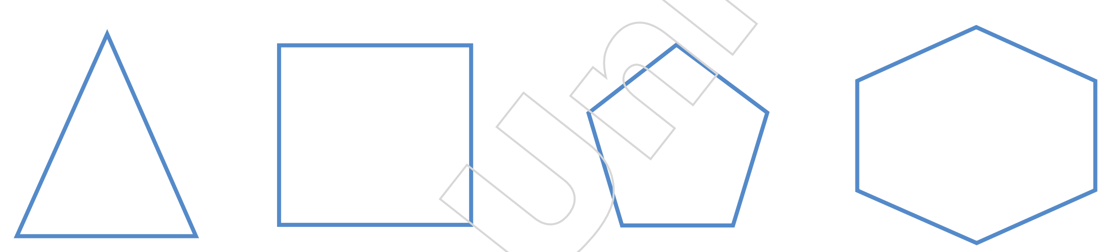
K_5



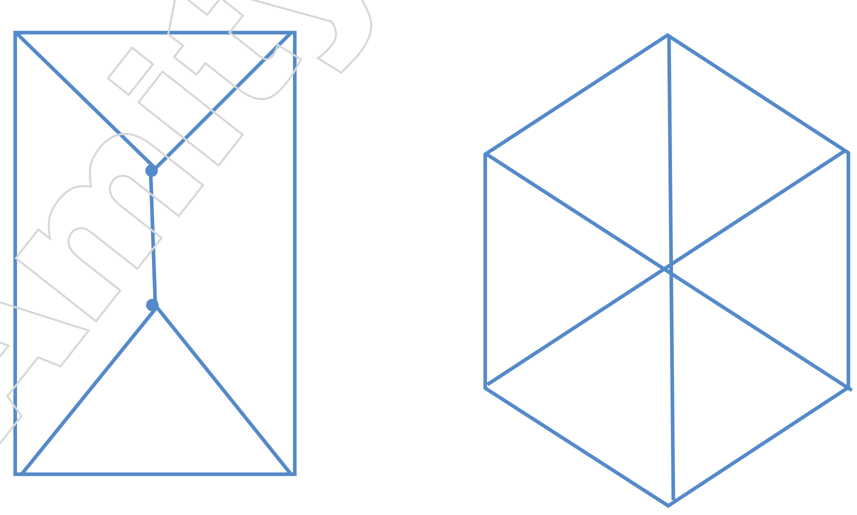
K_6

0-Regular

2-Regular



2-Regular



3-Regular

Notes

Remark:

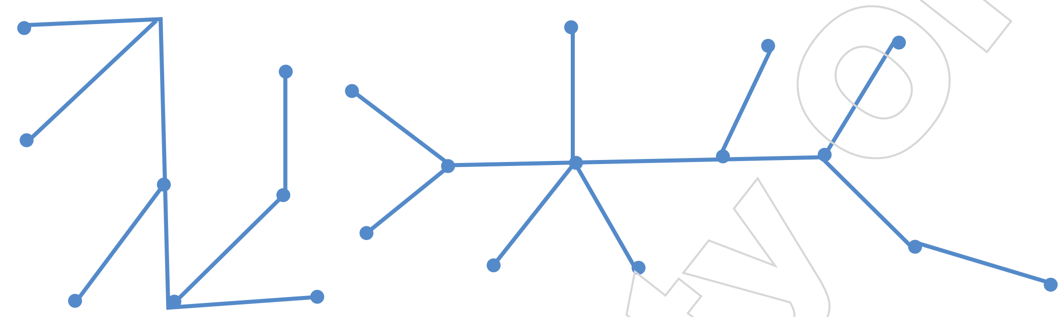
Let G be a graph with $n > 1$ vertices. Then the following are equivalent:

(i) G is a tree.

(ii) G is a cycle-free and has $n - 1$ edges.

(iii) G is connected and has $n - 1$ edges.

It shows that a finite tree T with n vertices must have $n-1$ edges.



Definition: 3.38

If a graph G is a weighted graph (i.e.if there is a real number associated with each edge of G), then the **weight of a spanning tree** T of G is defined as the sum of the weight of all the branches in T.

Definition: 3.39

A spanning tree with the smallest weight in a weighted graph is called a **shortest spanning tree** (or) shortest distance spanning tree (or) minimal spanning tree.

Questions:

1. A tree is a __________connected graph.

   a) cyclic

   b) acyclic

   c) a or b

   d) a and b

2. Every tree is a __________graph.

   a) simple

   b) cyclic

   c) loop

   d) parallel

3. A tree with n vertices has___________edges.

   a) n

   b) n-1

   c) n-2

   d) n+1

Notes

4. Every nontrivial tree has atleast two vertices of degree________.
   a) zero
   b) one
   c) two
   d) none
5. Every connected graph contains a________tree.
   a) cyclic
   b) spanning
   c) minimal
   d) all
6. A vertex with minimum eccenricity is the_______of tree.
   a) center
   b) length
   c) a or b
   d) a and b
7. Every tree has either_________or two centers.
   a) zero
   b) one
   c) a or b
   d) a and b
8. A minimum-weight spanning tree is an_______tree.
   a) optimal
   b) minimum
   c) maximum
   d) none
9. The shortest path problem is to the shortest paths from one vertex to all other vertices in a_______ graph.
   a) simple
   b) tree
   c) weighted
   d) none
10. The tree T is __________.
   a) a binary tree
   b) a full binary tree

Notes

c) a ternary tree

d) a full ternary tree

Answer:

1. b
2. a
3. b
4. b
5. b
6. a
7. b
8. a
9. c
10. b

Exercises:

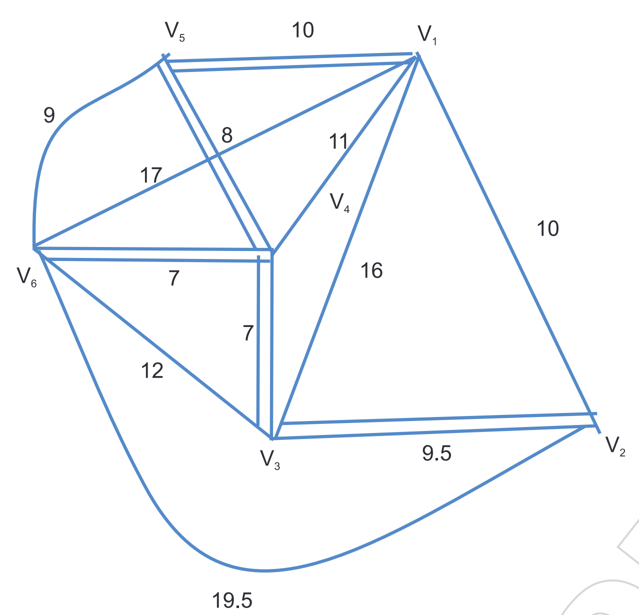
1. Define Decomposition graph with give an example and justify your answer.
2. Prove that, a connected graph G is an Euler graph if and only if it can be decomposed into circuits.
3. Explain complete graph.
4. Show that, there is one and only one path between every pair of vertices in a tree T.
5. In a graph G, there is one and only one path between every pair of vertices, prove that G is a tree.
6. Show that, a graph G with n vertices, n-1 edges and no circuits is connected.
7. Prove that a graph is a tree if and only if it is minimally connected.

Unit- IV

Notes

Algorithm for shortest spanning tree:

Start from v_1 and connect it to its nearest neighbour say v_k. Now consider v_1 and v_k as once a graph and connect this subgraph to its closest neighbour, let this new vertex be v_i.we continue this process until all vertices have been connected by (n-1) edges. Let us now illustrate this method of finding a shortest spanning tree.



Shortest spanning tree in a weighted graph

–	10	16	11	10	17
10	–	9.5	∞	∞	19.5
16	9.5	–	7	∞	12
11	∞	7	–	8	7
10	∞	∞	8	–	9
17	19.5	12	7	9	–

Theorem: 4.1

A spanning tree (of a given of a weighted connected graph G) is a shortest spanning tree of G if there exists no of the spanning tree of G at a distance of one from T whose weight is smaller than of T.

Proof:

Suppose that T_1 is shortest spanning tree.

There exist no spanning tree at a distance of one from T, whose weight is smaller than that of T.

Conversely,

Suppose that T_1 is a spanning tree in G such that there exist no other spanning tree in G such that there exist no other spanning tree at a distance of one from T_1 whose weight is smaller than T_1.

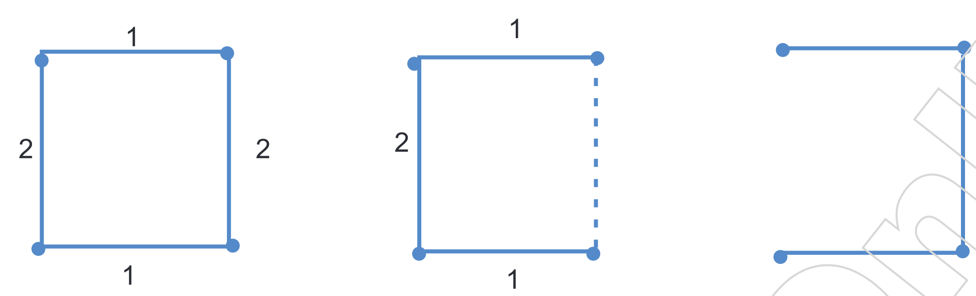
To prove: T_1 is a shortest distance spanning tree.

Suppose, T_2 is also shortest distance spanning tree.

Notes

Case: (i)

If $T_1 = T_2$

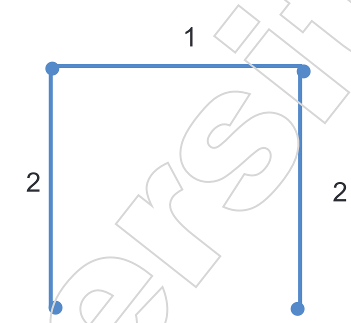


$T_{1'}$ T_2

Then T_1 is the shortest spanning tree.

Case: (ii)

If $T_1 \neq T_2$



There exist atleast one edge which is in T_2 but not in T_1.

Let e be an edge in T_2 but not in T_1.construct a fundamental circuit in T_1 by adding the edge e to T_1.

But not all of the branch in T_1 that forms the fundamental circuit which may be in T_2.

Each branch in T_1 that lies in the fundamental circuit has a weight smaller or equal to that of e.

(i.e.) Weight of e_k < weight of e $\longrightarrow$ (1)

For all edge e_k in the fundamental circuit in T_1 among all these edges in the circuit there exist one edge which is not in T_2 say b_j from a fundamental circuit in T_2 containing e by adding b_j.

T_2 is the shortest spanning tree.

ie.,)Weight of $b_j \geq$ weight of e $\longrightarrow$ (2)

Eqn (1) is tree for all e_k is in the fundamental circuit of T_1.

b_j is also one of the edges in the fundamental circuit.

weight of $b_j \leq$ weight of e $\longrightarrow$ (3)

From (2) and (3)

Weight of e_j = weight of e

Construct a new spanning tree $T_1 = T_1 \cup e - b_j$

Weight of T'_1 = weight of T_1

Notes

Repeat this producer by taking and soon and producing a series of tree of equal weight $T_1, T_1, T_1, \ldots\ldots.$ each a unit distance closure to T_2 and we get T_2 itself.

weight of T_1 = weight of T_2

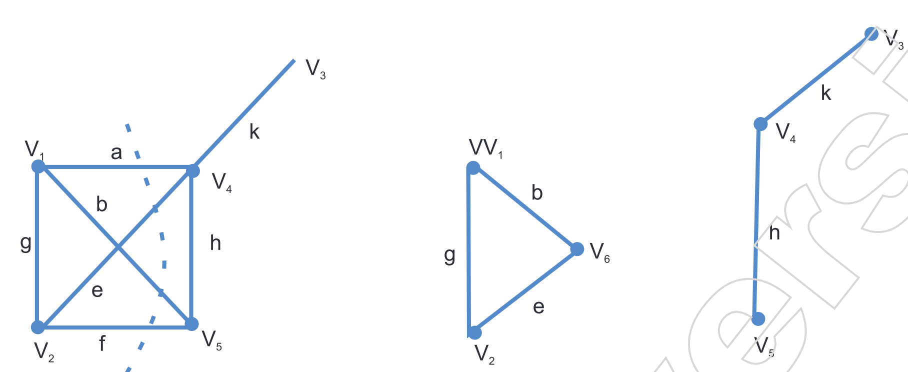
Therefore T_1 is the shortest spanning tree.

Definition: 4.2

In a connected graph G, a **cutset** is a set of edges whose removal from G leaves G disconnected, provided removal of these edges disconnects G.

The set of edge {a,c,d,f} is a cutset. There are many other cutset such as {a,b,g},{a,b,e,f} and {d,h,f} edge {k} alone as also a cutset .

The set of edges {a,c,h,d} is not a cutset . Because one of its proper subset {a,c,h} is a cutset.



Remark:

The another name of cutsets are minimal cutsets or proper cutsets or simple cutsets or cocycle.

Every edge of a tree is cutset.

Theorem: 4.3

Prove that the ringsum of any two cutsets in a graph is either a third cutsets or an edge disjoint union of cutsets.

Proof:

Let G be a connected graph graph and s_1 and s_2 be two cutset in the given connected graph.

Let v_1 and v_2 be the unique and disjoint partitioning of the vertex set V of G corresponding to s_1.

Let v_3 and v_4 be the partitioning corresponding to $s_{2,}$ clearly,

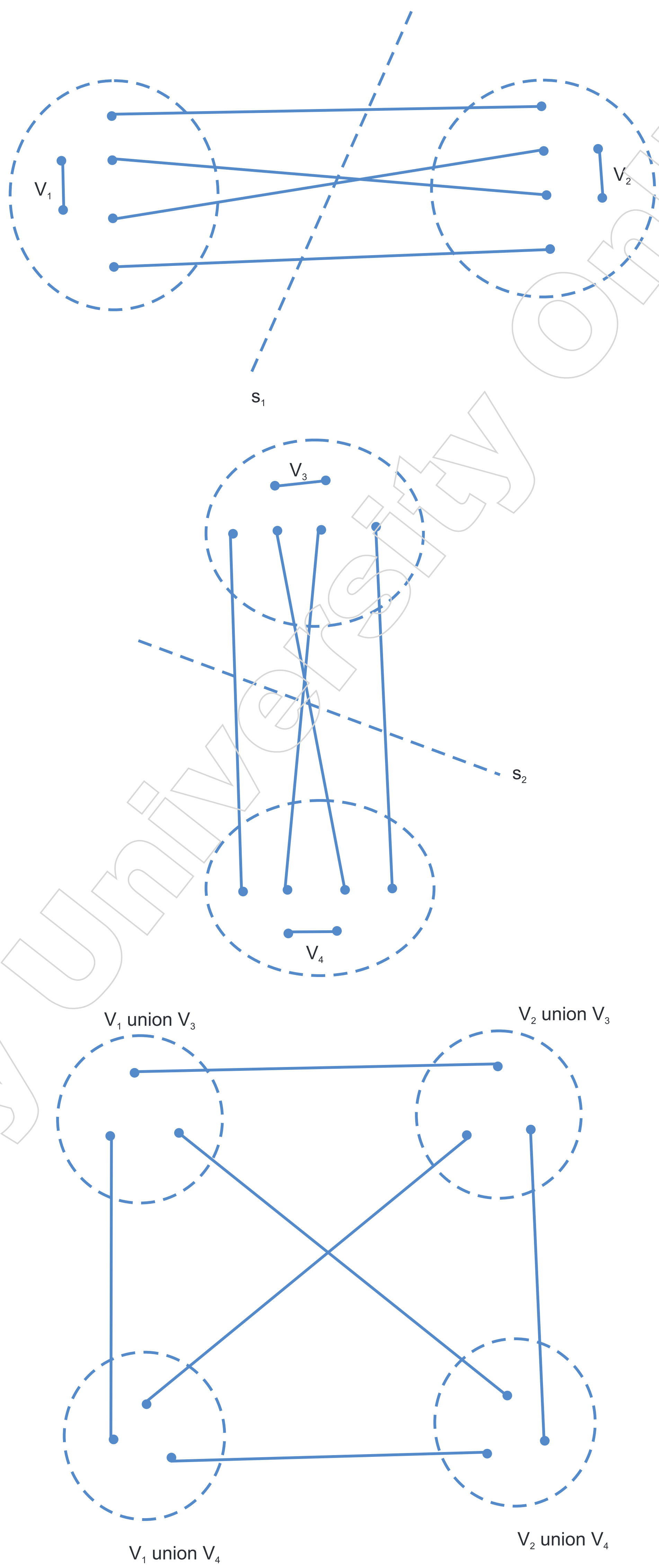
$V_1 \bigcup V_2 = V$ &

$V_1 \bigcap V_2 = \varphi$

$V_3 \bigcup V_4 = V$ &

$V_3 \bigcap V_4 = \varphi$

Notes



Consider $v_5 = (v_1 \cap v_4) \cup (v_2 \cap v_3)$

$= v_1 \oplus v_3$

$V_6 = (v_1 \cap v_3) \cup (v_2 \cap v_4)$

$= v_2 \oplus v_4$

The ringsum of two cutsets $s_1 \oplus s_2$ can be seen to consist only of edges that join vertices in v_5 to those in v_6.

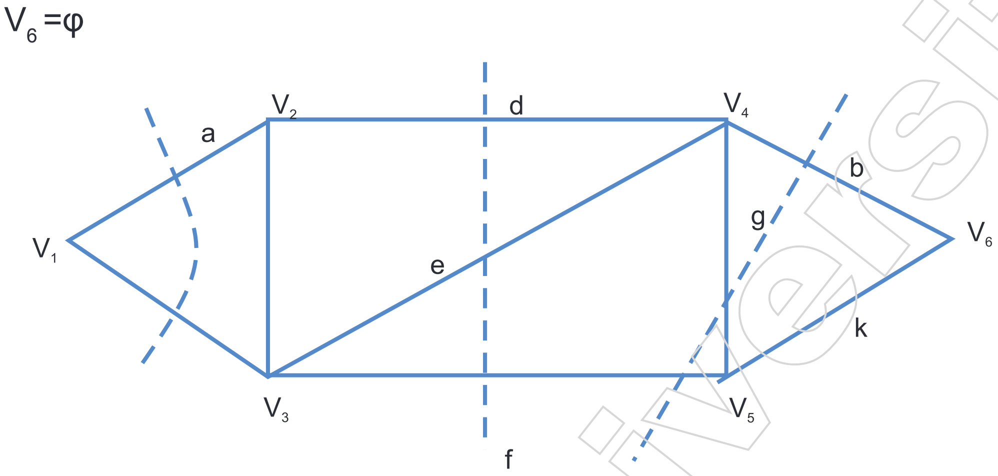
Also there are no edges outside $s_1 \oplus s_2$ that joint vertices in v_5 to those in v_6.

Thus the set of edges $s_1 \oplus s_2$ produces a partitioning of V into v_5 and v_6.

Such that,

$V_5 \cup V_6 = V$ &

$V_5 \cap V_6 = \varphi$



Hence $s_1 \oplus s_2$ is a cutset if the subgraphs containing v_5 and v_6 each remain connected. After $s_1 \oplus s_2$ is removed from G. otherwise $s_1 \oplus s_2$ is an edge disjont union of cutsets.

Definition:4.4

Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T, and let the **fundamental circuit** made by c_i be called Γ consisting of k - branches $b_1, b_2, b_3, \ldots\ldots b_k$ in addition to the chord c_i.

i.e.,) $\Gamma = \{c_i, b_1, b_2 \ldots b_k\}$ is a fundamental circuit with respect to T.

Theorem: 4.5

With respect to a given spanning tree T, a chord c_i that determine a fundamental circuit $^{v_i \in V_i}$ occurs in every fundamental cutset associated with the branches in Γ and in no other.

Proof:

Consider a spanning tree T of the given connected graph G.

Let c_i be the chord with respect to T and Γ be the fundamental circuit consisting of k. branches of T together with c_i.

Notes

i.e.,) Γ =[c_i,b_1,b_2,.....b_k]

Where b_1,b_2,.....b_k are branches of T. We know that every branches of a spanning tree has a fundamental cutset associated with it.

Let s_1 be the fundamental cutset associated with the branch b_1.

Let s_1={b_1, c_1,c_2,....c_q}

Where c_1,c_2, ...c_q are chords of T.

$b_1 \in s_1 \cap \Gamma$

We know that N($s_1 \cap \Gamma$) is even

s_1 and Γ must have exactly one more common edges and that should be c_i

$\Rightarrow$ c_i =s_1

By a similar argument we can prove that $c_i \in$ the cutsets s_2,s_3,. . . . s_k respectively.

The chord c_i is in every fundamental cutset.

Now let s' be the fundamental cutset other than s_1,s_2,. . . s_k.

Claim: $c_i \notin s' \Rightarrow c_i \in s_1$

Assume the contrary

i.e) $c_i \in s'$

s' = {c_i,b'_1,b'_2,. . . b'_k}

Now Γ n s' ={c_i}

Which is a contrary to fact that there is an even no of edges in common to a circuit and a cutset.

$C_i \notin s'$

Theorem: 4.6

With respect to a given spanning tree T, a branch b_i that determine a fundamental cutset S is contained in every fundamental circuit associated with the chords in S and in no others.

Proof:

Let G be a given connected graph. Let S be a fundamental cutset associated with a branch b_i of T in G.

Ie) S = {b_i,c_1,c_2,. . . c_k}

Let Γ_1 be a fundamental circuit associated with the chord c_1.

Ie) Γ_1 = {c_1,b_1,b_2,. . . b_m}

Since S is a cutset and Γ_1 is a circuit. There will be an even no of edges in common to S and Γ_1.

Notes

Obviously c_1 belongs to both ε & Γ_1.

Therefore b_i must belong to Γ_1.

By a similar argument we can prove that b_1 belongs to every fundamental circuit determine by the chords

$C_2, C_3, \ldots C_k$.

Claim: b_i does not occurs in any other fundamental circuit.

Let Γ' be a fundamental circuit other than $\Gamma_1, \Gamma_2, \ldots \Gamma_k$.

Assume $b_i \in \Gamma$.

$\Gamma' = \{c_1, b_1, \ldots b_i, b_i+1, \ldots b_m\}$

$\Gamma' \cap S = \{b_i\}$

Which is a contradiction to that $\Gamma' \cap S$ must have an even no of edges.

Then $b_i \in \Gamma'$.

Therefore b_i dose not occur in any other fundamental circuit.

Definition: 4.7

The number of edges in the smallest cutset is defined as the **edge connectivity** of G.

Remark:

The edge connectivity of a tree is one.

Definition: 4.8

The **vertex connectivity** of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph disconnected.

Definition: 4.9

A connected graph is said to be **seperable**. If its vertex connectivity is one. All other connected graph are called non-seperable.

Definition: 4.10

In a seperable graph a vertex whose removal disconnects the graph is called a **cut-vertex** a cut-node or and articulation point.

Remark:

The edge connectivity and vertex connectivity of a disconnected graph is zero.

Every tree is seperable.

Notes

Theorem: 4.11

A vertex V in a connected graph G is cut vertex if there exist two vertices x and y in G such that every path between x and y passes through V.

Proof:

Let v be cut vertex of G. Assume the contrary that is for every pare of vertices x and y there exist a path that not pass through V.

Therefore if v remove the vertex V the graph is connected.

$\Rightarrow$ we cannot a cut vertex.

Which is a contradiction.

Hence every path between x and y passes through V.

Conversely,

Assume that there exist vertices x and y such that every path between x and y passes through V.

If we remove the vertex then there is no path between x and y.

G-{v} is disconnected.

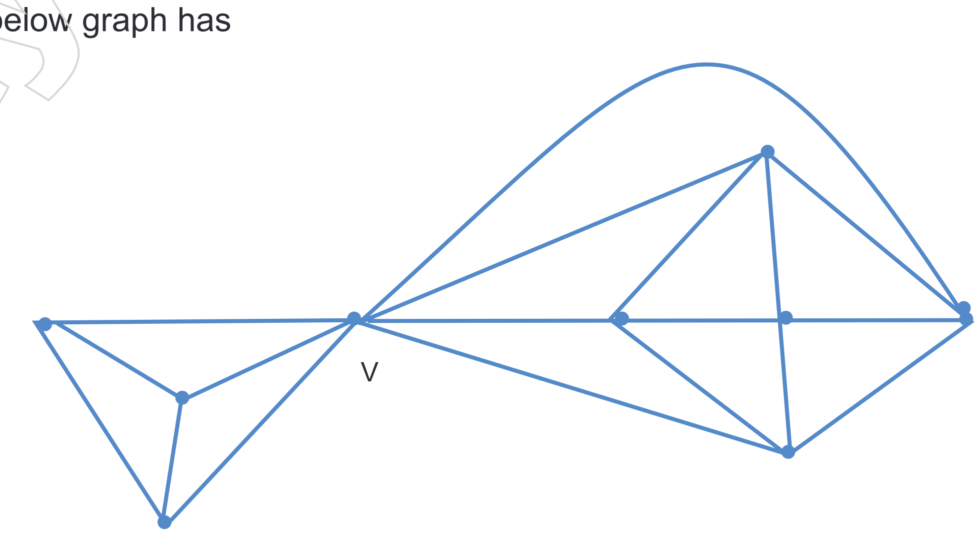
Therefore V is a cut vertex in G.

Applications:

Suppose we are given n stations that are to be connected by means of e lines (telephone lines, bridges, railroads, tunnels or highway), where e≥n-1. What is the best way of connecting by best we m can that the network should be as invulnerable to destruction are indidue station and individual line as possible. In other words, construct a graph with 'm' vertices and 'e' edges that has the maximum possible edge connectivity and vertex connectivity.

Example:

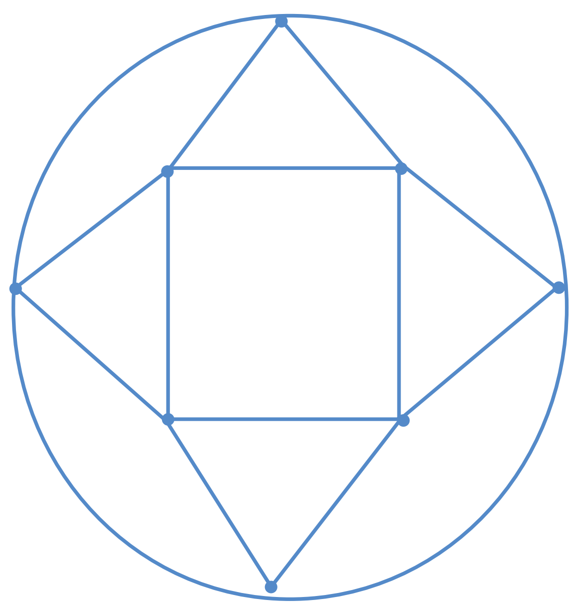
The below graph has



n=8, e=16 and has vertex connectivity of one and edge connectivity of three.

Another graph the same number of vertices and edges can be drawn as shown in the below figure.

Notes



It can be easily be seen that the edges connectivity as well as the vertex connectivity of this graph is four.

Consequently, even after any three station are bombed or any three lines destroyed, the remaining station can still continue to communicate with each other.

Theorem: 4.12

Prove that the Edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof:

Let V_i be the vertex with the smallest degree in G.

Let $d(V_i) = k$ (say)

There are k-edges incident on V_i and no other vertex will have number of edges incident on it.

If we remove k- edges, the vertex v will be isolated.

That is, the graph will be disconnected.

Edge connectivity $\leq k$

Edge connectivity $\leq d(v_i)$

Theorem: 4.13

The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof:

Let be the vertex connectivity and be the edge connectivity of G.

Since is the edge connectivity of G. There is a cutest S with edges

The removal of S from G partition S vertex set into two disjoint subsets.

By removing at most vertices from v_1 and v_2 on which the edges in a incidents, all the edges in S will be removed from G leaving it disconnected.

Notes

As the vertex connectivity of a connected graph is the smallest number of vertices whose removal disconnects the graph.

Therefore $\alpha \leq \beta$

Hence the theorem

Theorem: 4.14

Every cutest in a non seperable graph with more than two vertices contains atleast two edges

Proof:

Let G be a non – seperable graph with more than two vertices.

Therefore Vertex connectivity ≥ 2

We know that, vertex connectivity $\leq$ edge connectivity

Edge connectivity $\geq$ vertex connectivity ≥ 2

Therefore edge connectivity ≥ 2

Therefore every cutest in G must contain two or more edges.

Theorem: 4.15

The maximum vertex connectivity one can achieve with a graph G of ‘n‘ vertices and ‘e’ edges (e > n-1) is the integral part of the number 2e/n that is (2e/n)

Proof:

Let G be a graph with ‘n’ vertices and ‘e’ edges

We know that

$$\sum_{i=1}^{n} d(v_i) = 2e$$

The total 2e degrees is divided among n vertices

Therefore there must be atleast one vertex in G whose degree is equal to or less than the number 2e/n

The vertex connectivity of G cannot exceed this number.

Vertex connectivity $\leq$ smallest degree

Vertex connectivity $\leq \frac{2e}{n}$

Vertex connectivity $\leq$ edge connectivity <2e/n

To show that this value can actually be achieved, one can first construct an n vertex regular graph of (2e/n)

Definition: 4.16

A graph G is said to be k-connected if the vertex connectivity of G is k

Notes

Therefore a one connected graph is the same as separable graph.

Remark:

- A connected graph G is k-connected if every pair of vertices in G is joined by k or more paths do not interested and at least one pair of vertices is joined by exactly k-non intersecting paths.
- The edge connectivity of graph G is k if every pair of vertices in G is joined by k or more edge disjoined paths (ie, paths that may intersect but have no edges common) and at least one pair of vertices is joined by exactly k-edges disjoined paths.

Theorem: 4.17

Prove that the ring sum of two circuits in a graph G is either a circuit or an edge disjoined union of circuits

Proof:

Let circuit 1 and circuit 2 be any two circuits in a graph G.

If the two circuits have no edges or vertices common, their ring sum of circuit 1 $\oplus$ circuit 2 is a disconnected subgraph of G, and is obviously an edge disjoined union of circuits.

If, on the other hand circuit 1 and circuit 2 do have edges and vertices is common we have the following possible situation.

Since the degree of every vertex in a graph (i.e.), circuit is two, every vertex 'v' in subgraph has degree d(v) where d(v) =2 , 'v' is in circuit 1 only or circuit 2 only or if one of the edges formally incident on 'v' was in both and circuit 1 and circuit 2 or d(v)= 4 if circuit 1 and circuit 2 just intersect at 'v'

There is no other type of vertex in circuit 1 $\oplus$ circuit 2.Thus circuit 1 $\oplus$ circuit 2 is an Euler graph ,and Therefore consist of either a circuit or an edge disjoined union of circuits.

Theorem: 4.18

If the graph Ghasnvertices and m edges, then the following statements are equivalent:

(i) G is a tree.

(ii) There is exactly one path between any two vertices in G and G has no loops.

(iii) G is connected and $m = n - 1$.

(iv) G is circuitless and $m = n - 1$.

(v) G is circuitless and if we add any new edge to G, then we will get one and only one circuit.

Proof:

(i)⇒(ii)

Notes

Let G be a tree, then it is connected and circuitless.

Thus, there are no loops in G. There exists a path between any two vertices of G. we know that there is only one such path.

(ii)⇒(iii)

G is connected. Let us use induction on m.

By induction, If m= 0, G is trivial and the statement is obvious.

Induction Statement Proof:

Let e be an edge in G. Then G−e has ℓ edges. If G−e is connected, then there exist two different paths between the end vertices of e

Therefore so (ii) is false. Therefore, G−e has two components G_1 and G_2.

Let there be n_1 vertices and m_1 edges in G_1. Similarly, let there be n_2 vertices and m_2 vertices in G_2. Then,

$n = n_1 + n_2$ and $m = m_1 + m_2 + 1$.

The Induction Hypothesis states that

$m_1 = n_1 - 1$ and $m_2 = n_2 - 1$, so $m=n_1+n_2-1 =n-1$.

(iii)⇒(iv) ____________________

Consider the hypothesis: There is a circuit in G. Let e be some edge in that circuit. Thus, there are n vertices and n−2 edges in the connected graph G

(iv)⇒(v)

If G is circuitless, then there is at most one path between any two vertices. If G has more than one component, then we will not get a circuit when we can draw an edge between two different components. By adding edges, we can connect components without creating circuits

So G is connected.

When we add an edge between vertices that are not adjacent, we get only one circuit. Otherwise, we can remove an edge from one circuit so that other circuits will not be affected and the graph stays connected, in contradiction to (iii)⇒(iv).

Similarly, if we add a parallel edge or a loop, we get exactly one circuit.

(v)⇒(i)

G is not a tree

i.e. it is not connected. When we add edges as we did previously, we do not create any circuits

Since spanning trees are trees, and it is also true for spanning trees.

Theorem: 4.19

Prove that a connected graph has at least one spanning tree.

Proof:

Consider the connected graph G with n vertices and m edges.

If m=n−1, then G is a tree. Since G is connected, m≥n–1.

If m≥ n, where there is a circuit in G.

We remove an edge e from that circuit. G − e is now connected.

We repeat until there are n−1 edges.

Then, we are left with a spanning tree.

Remark.

- We can get a spanning tree of a connected graph by starting from an arbitrary sub-forest M (as we did previously). Since there is no circuit whose edges are all in M, we can remove those edges from the circuit which are not in M.
- The subgraph G_1 of G with n vertices is a spanning tree of G (thus G is connected) if any three of the following four conditions hold:
  (i) G_1 has n vertices.
  (ii) G_1 is connected.
  (iii) G_1 has n − 1 edges.
  (iv) G_1 is circuitless.

Theorem: 4.20

If a tree is not trivial, then prove that there are at least two pendant vertices.

Proof:

If a tree has n(≥2) vertices, then the sum of the degrees is 2(n−1).

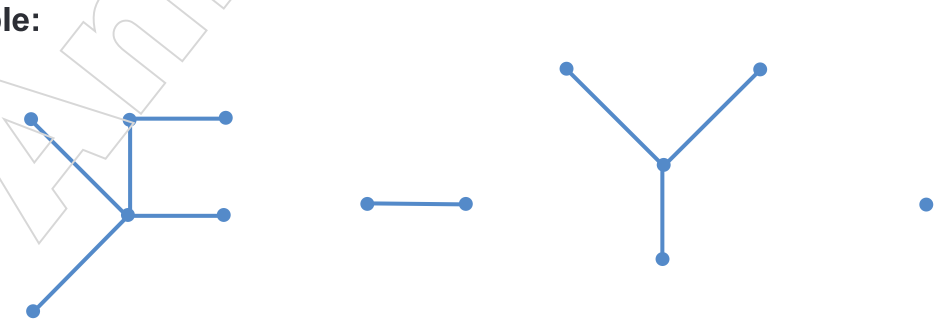
If every vertex has a degree ≥2, then the sum will be ≥2 n .

On the other hand, if all but one vertex have degree ≥2, then the sum would be ≥1 + 2(n−1) = 2n−1.

Remark:

A forest with k components is sometimes called a k-tree. (So a 1-tree is a tree.)

Example:



4-tree

Notes

Theorem: 4.21

The edge set F of the connected graph G is a cut set of Gif and only if

(i) F includes at least one branch from every spanning tree of G, and

(ii) $H \subset F$, then there is a spanning tree none of whose branches is in H.

Proof:

Let us first consider the case where F is a cut set.

Then, (i) is true.

If $H \subset F$ then $G-H$ is connected and has a spanning tree T. This T is also a spanning tree of G.

Hence, (ii) is true.

Let us next consider the case where both (i) and (ii) are true.

Then $G-F$ is disconnected. If $H \subset F$ there is a spanning tree T none of whose branches is in H.

Thus T is a subgraph of $G - H$ and $G - H$ is connected.

Hence, F is a cut set.

Theorem: 4.22

Prove that the subgraph C of the connected graph G is a circuit if and only if

(i) C includes at least one link from every cospanning tree of G, and

(ii) D is a subgraph of C and D=C, then there exists a cospanning tree none of whose links is in D.

Proof:

Let us first consider the case where C is a circuit.

Then, C includes at least one link from every cospanning tree so (i) is true. If D is a proper subgraph of C, it obviously does not contain circuits, i.e. it is a forest.

We can then supplement D so that it is a spanning tree of G

i.e. some spanning tree T of G includes D and D does not include any link of T^*.

Thus, (ii) is true.

Now we consider the case where (i) and (ii) are both true.

Then, there has to be at least one circuit in C because C is otherwise a forest and we can supplement it so that it is a spanning tree of G.

We take a circuit C' in C. Since (ii) is true, $C'=C$ is not true, because C' is a circuit and it includes a link from every cospanning tree.

Therefore, $C=C'$ is a circuit.

Theorem: 4.23

Prove that a circuit and a cut set of a connected graph have an even number of common edges.

Proof:

We choose a circuit C and a cut set F of the connected graph G.

G−F has two components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

If C is a subgraph of G_1 or G_2, then the theorem is obvious because they have no common edges.

Let us assume that C and F have common edges. We traverse around a circuit by starting at some vertex v of G_1.

Since we come back to v, there has to be an even number of edges of the cut V_1, V_2 in C.

Theorem: 4.24

A fundamental circuit corresponding to links of the cospanning tree T∗ of a connected graph is formed exactly by those branches of T whose corresponding fundamental cut set includes c.

Proof:

There exists a fundamental circuit C that corresponds to link c of T^*. The other edges $b_1, \ldots, b_k$ of C are branches of T.

We denote I_i as the fundamental cut set that corresponds to branch b_i.. Then, b_i is the only branch of T which is in both C and I_i.

On the other hand, c is the only link of T^* in C. By Theorem 2.6, we know that the common edges of C and I_i are b_i and c, equivalently c is an edge of I_i.

Then, we show that there is no c in the fundamental cut sets $I_{k+1}, \ldots, I_{n-1}$ that correspond to the branches $b_{k+1}, \ldots, b_{n-1}$ of T .

For instance, if c were in I_{k+1}, then the fundamental cut set I_{k+1} and the circuit C would have exactly one common edge.

So c is only in the fundamental cut sets $I_1, \ldots, I_k$.

Definition: 4.25

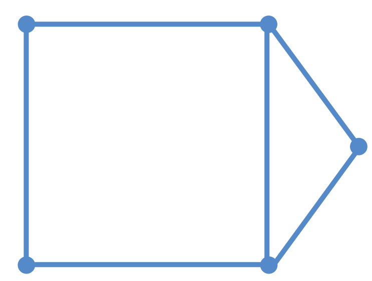
A spanning tree of a connected graph is a subtree that includes all the

vertices of that graph. If it is a spanning tree of the graph G, then

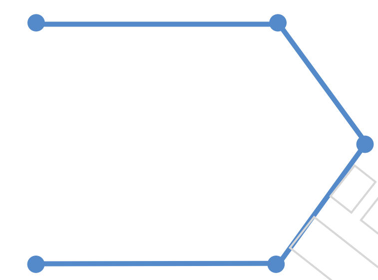
$G - T =_{def.} T^*$ is the cospanning tree.

Example:

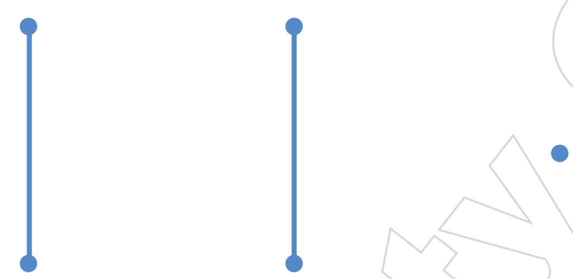
Notes



(G)



Spanning tree



Cospanning tree

Theorem: 4.26

The fundamental cut set corresponding to branch b of the spanning treeT of a connected graph consists exactly of those links of T^* whose corresponding fundamental circuit includes b.

Proof:

Let I be a fundamental cut set that corresponds to the branch b of T. Other edges $c_1, \ldots, c_k$ of I are links of $T*$.

Let C_i denote the fundamental circuit that corresponds to c_i. .Then, c_i is the only link of T^* in both I and C_i.

On the other hand, b is the only branch of T, the common edges of I and C_i are b and c_i, in other words, b is an edge of C_i.

Then, we show that the fundamental circuits $C_{k+1}, \ldots, C_{m-n+1}$corresponding to the links $c_{k+1}, \ldots, c_{m-n+1}$do not include b.

For example, if b were in C_{k+1}, then the fundamental circuit

$C_{k+1}$and the cut set I would have exactly one common edge.

Hence, the branch b is only in fundamental circuits $C_1, \ldots, C_k$. I

Minimum spanning trees of a weighted graph:

The minimum spanning tree of a weighted graph constructed using

- Prim's algorithm (using vertex)
- Kruskal's algorithm (using edges)

Algorithm for prim's algorithm:

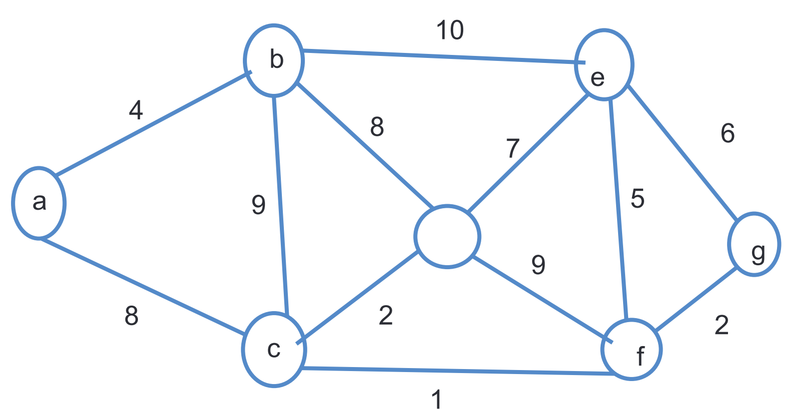
- Step:1 Select any connected vertices with minimum weight

Notes

- Step: 2 Select the unvisited vertex which is adjacent of the visited vertices with minimum weight
- Step: 3 Repeat step:2 until all vertices are visited

Problem: 1

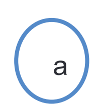
Find the minimum spanning tree using prim's algorithm



Solution:

Step: 1

Let us take the vertex 'a'



S={a}

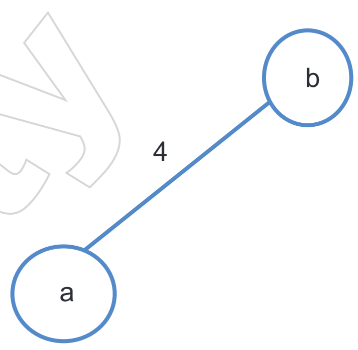
The remaining vertices={b,c,d,e,f,g}

Starting vertex A={ ϕ }

The lighted edges={a,b}

Step: 2

Let S={a,b}



S={a,b}

The remaining vertices={c,d,e,f,g}

Starting vertex A={{a,b}}

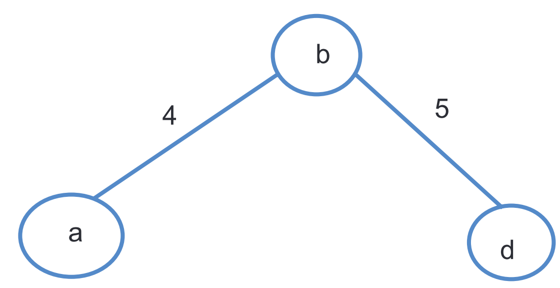
The lighted edges are b→d or a→c and both have the weight 8.

But we have to visit the weight from an adjacent to the unvisited vertex

So we have to select the vertex 'b'

Notes

Step: 3



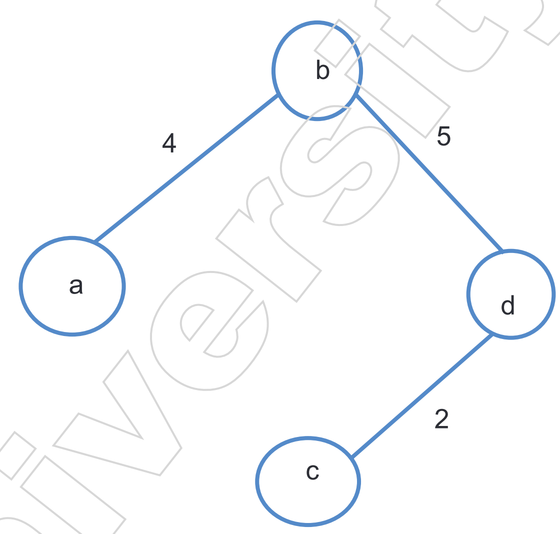
S={a,b,d}

The remaining vertices={c,e,f,g}

Starting vertex A={{a,b},{b,d}}

The lighted edges are d→c

Step: 4



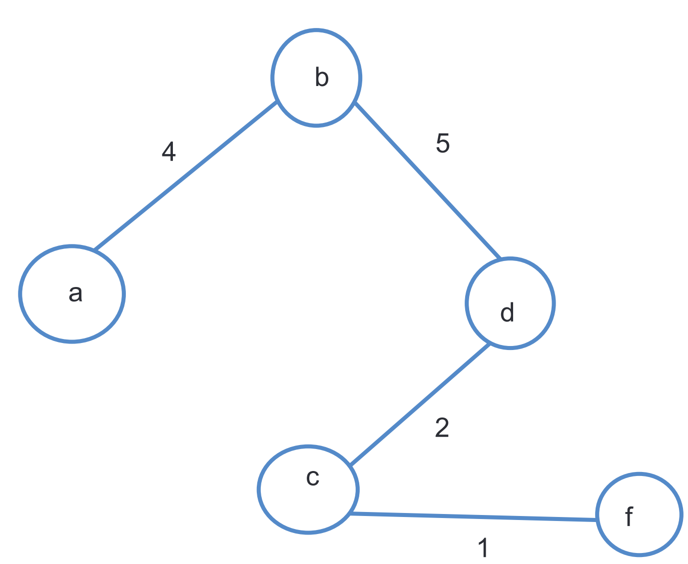
S={a,b,c,d}

The remaining vertices={e,f,g}

Starting vertex A={{a,b},{b,d},{d,c}}

The lighted edges are c→f

Step: 5



S={a,b,c,d,f}

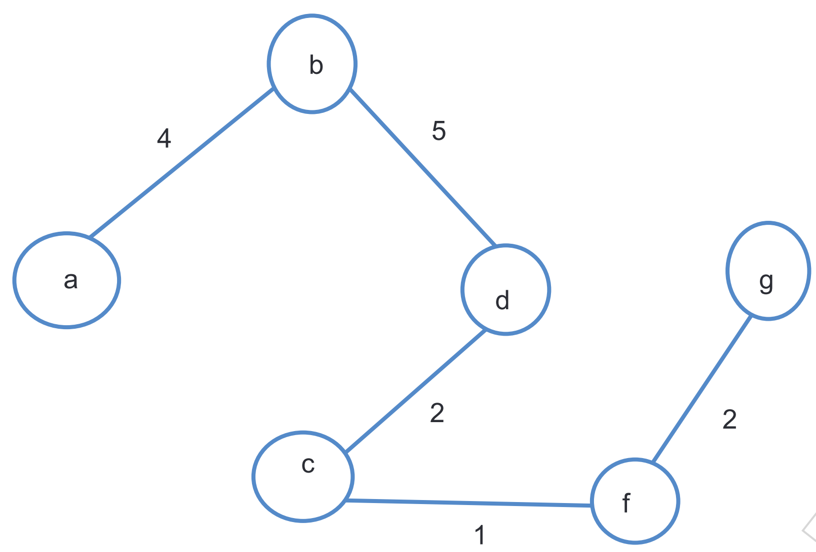
The remaining vertices={e,g}

Notes

Starting vertex A={{a,b},{b,d},{d,c},{c,f}}

The lighted edges are f→g

Step: 6



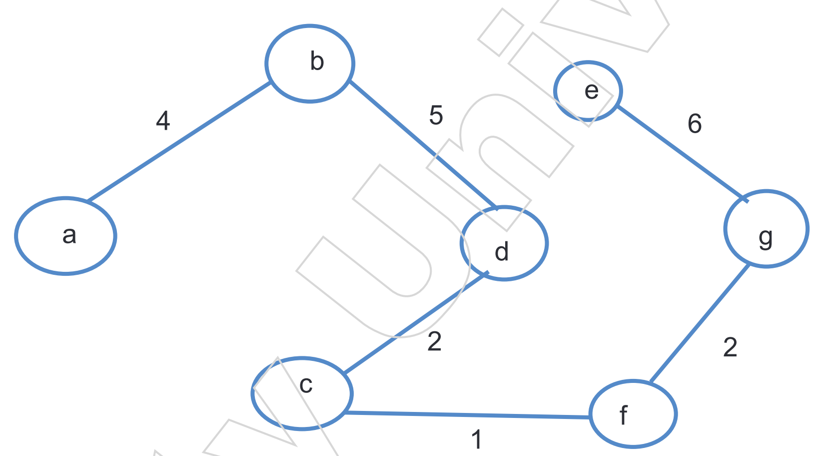
S={a,b,c,d,f,g}

The remaining vertices={e}

Starting vertex A={{a,b},{b,d},{d,c},{c,f},{f,g}}

The lighted edges are g→e

Step: 7



S={a,b,c,d,e,f,g}

The remaining vertices={ ϕ }

Starting vertex A={{a,b},{b,d},{d,c},{c,f},{f,g},{g,e}}

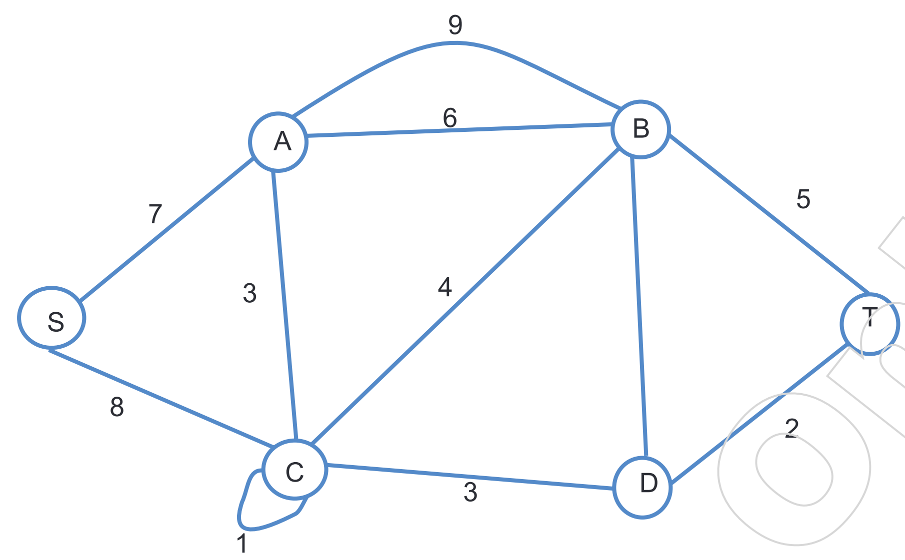
The lighted edges are g→e

Now the minimum spanning tree is completed

Problem: 2

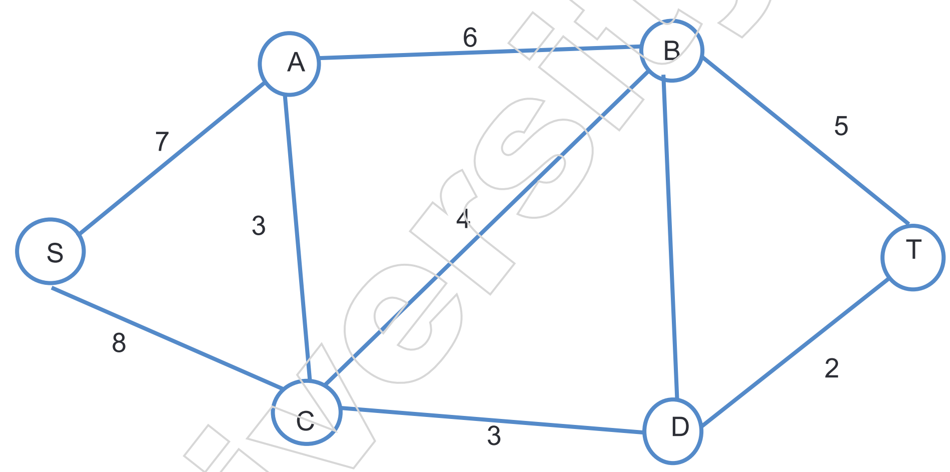
Find the minimum spanning tree using kruskal's algorithm

Notes



Step: 1

Remove all loops and parallel edges



Step: 2

Arrange all edges in their increasing order of weight

(i) B→D=2

(ii) D →T=2

(iii) A→C=3

(iv) C→D=3

(v) C→B=4

(vi) B→T=5

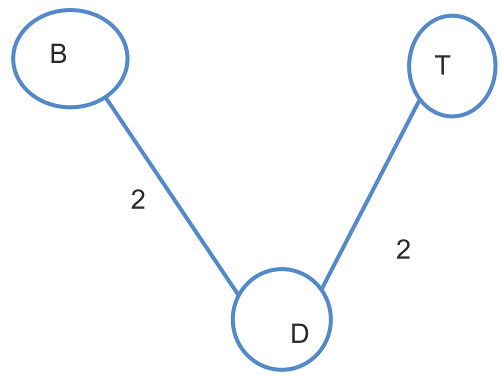
(vii) A→B=6

(viii) S→A=7

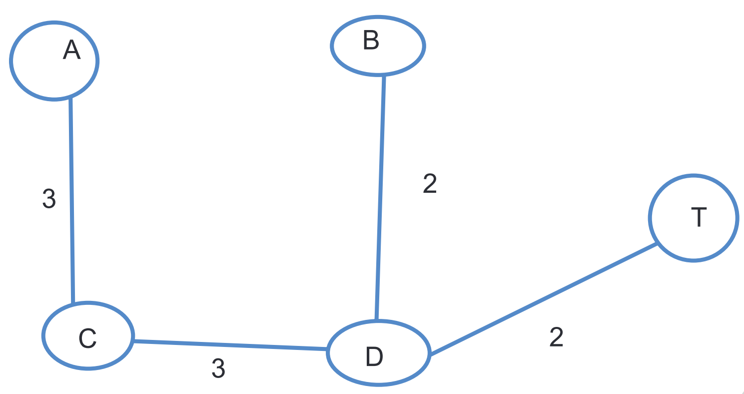
(ix) S→C=8

Step: 3

Add the edge which has least weightage



Next least cost is 3, so the edges are A→C or C→D

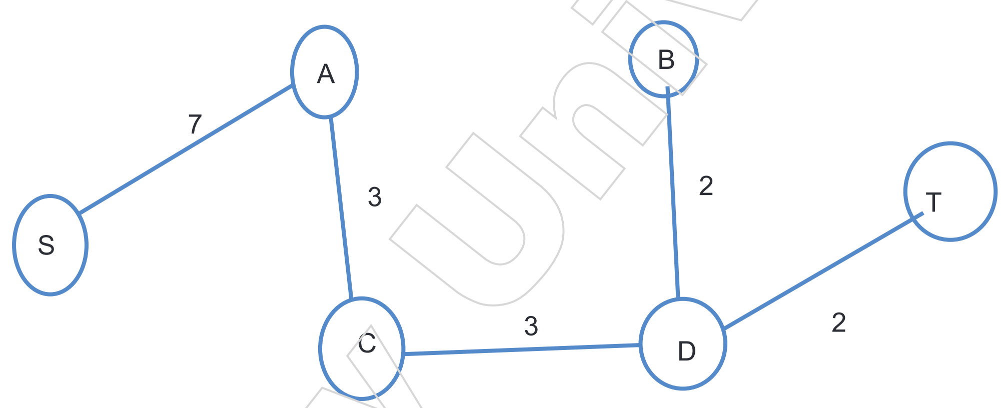


Next least cost is 4, but if we take 4 it becomes closed circuit so we discard to avoid circuit.

So next least cost is 5, and it becomes also closed circuit. So discard it

And next least cost is 6, and it becomes also closed circuit. So discard it

Next least cost is 7 and 8



Therefore the minimal spanning tree is completed.

Questions:

1. The number of internal vertices of T is________.
   a) 6
   b) 7
   c) 8
   d) 9.
2. The number of vertices of T at level 1 or 2 is_________
   a) 2
   b) 4

Notes

c) 6

d) 8

3. The number of binary trees with 5 vertices is ______
   a) 2
   b) 3
   c) 4
   d) 5
4. The number of rooted trees with three vertices is________
   a) 2
   b) 3
   c) 4
   d) 5
5. The number of rooted binary trees with 5 vertices is_________
   a) C(10, 5)
   b) C(10, 5)/6
   c) C(10,5)/5
   d) C(10,5)/10
6. The number of chords of a spanning tree T of a connected graph G with n vertices and e edges is__________.
   a) e-1
   b) n-1
   c) n-e+1
   d) e-n+1
7. The number of spanning trees of a connected graph with n vertices and n edges is_________.
   a) n
   b) n-1
   c) n+1
   d) none of these
8. The number of chords of a spanning tree T of a connected graph G with n vertices and e edges is__________.
   a) e-1
   b) n-1
   c) n-e+1
   d) e-n+1

Notes

9. The number of spanning trees and minimal spanning trees of the complete graph with 4 vertices with distinct weight is_________.
    a) 16 and 16
    b) 16 and 1
    c) C(6,3) and 1
    d) C(6,3) and C(6,3)
10. The chromatic number of a tree with n vertices is________.
    a) n
    b) n-1
    c) 3
    d) 2
11. A tree with 3 or more vertices remains a tree when ________
    a) any vertex is removed
    b) any edge is removed
    c) a pendant vertex is removed
    d) none of these
12. A tree with 5 vertices can be ________.
    a) D_5
    b) P_5
    c) C_5
    d) W_5
13. Every tree is__________
    a) bipartite
    b) not a bipartite
    c) a or b
    d) a and b
14. A full binary tree with n vertices has________ leaves.
    a) n
    b) (n+1)/2
    c) n-1
    d) (n-1)/2
15. The number of vertices n in a full binary tree is always_______.
    a) odd
    b) even

Notes

c) a or b

d) a and b

Answer:

1. b
2. c
3. d
4. b
5. b
6. d
7. d
8. d
9. b
10. d
11. c
12. b
13. a
14. b
15. a

Problems:

1. Show that There exists a least number $f(s,t)$ such that the vertices of any simple digraph with minimum outdegree at least $f(s,t)$ can be partitioned into two sets inducing subdigraphs with minimum outdegree at least s and at least t, respectively. [It is known that $f(1,1)=3$.
2. Prove that every tournament of order $2n-2$ contains every oriented tree of order n)
3. What is the maximum cyclic edge-connectivity of a 5-connected planar graph?
4. Does every planar subgraph have an induced forest with at least half of the vertices?

Module IV

Key Learning Objectives:

At the end of this module, you will be able to:

- Describe directed graphs and directed trees
- Discuss the concept of connectedness
- Practise studying network flows
- Interpret the max flow - min cut theorem
- Prepare the matrix representation of a graph
- Describe planar graphs
- Compare combinational and geometric duals
- Prepare Kuratowski's graphs
- Detect planarity
- Study thickness and crossing

Notes

Unit- V: Directed graphs

Definition: 5.1

A graph G is said to be a **directed graph or digraph** it consists of vertex set and the edge set $V=\{v_1, v_2, \ldots\}$ and $E=\{e_1, e_2, \ldots\}$ such that every pair of vertices representing any edge by an ordered or a direct pair (v_1, v_2) where v_1 is the tail and v_2 is the head of the edge

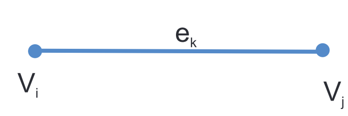


Fig 3.1a

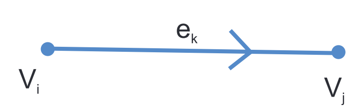
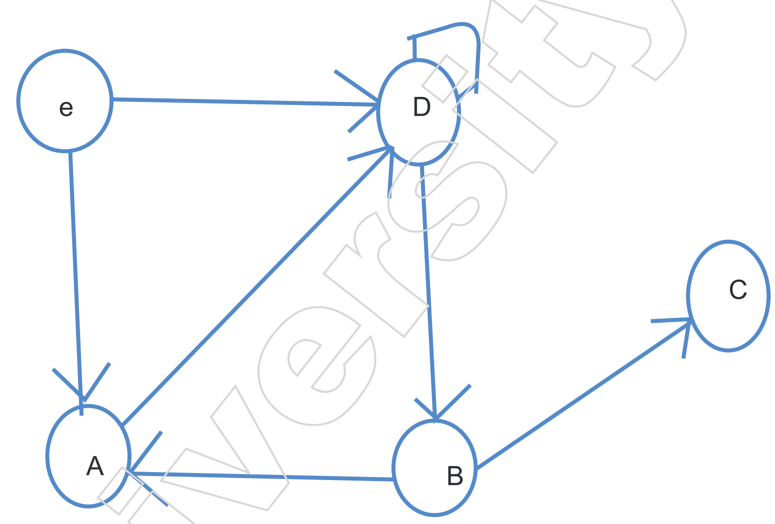


Fig 3.1b

Fig 3.1c

Fig 3.1 b and 3.1c represents the directed graph

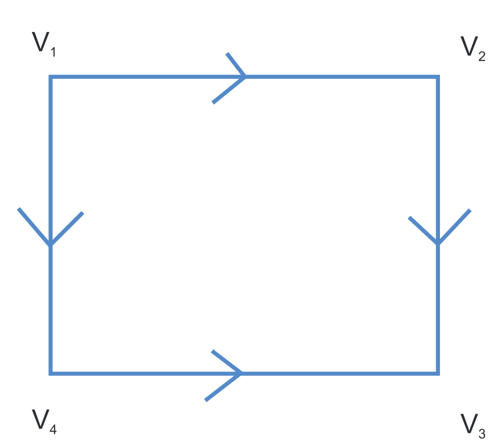


Directed graph G

Properties:

(i) Number of edges incident out of vertex v_i, $d^+(v_i)$ is said to be **out degree** or out valene

(ii) Number of edges incident into of vertex v_i, $d^-(v_i)$ is said to be **in degree** or in valene

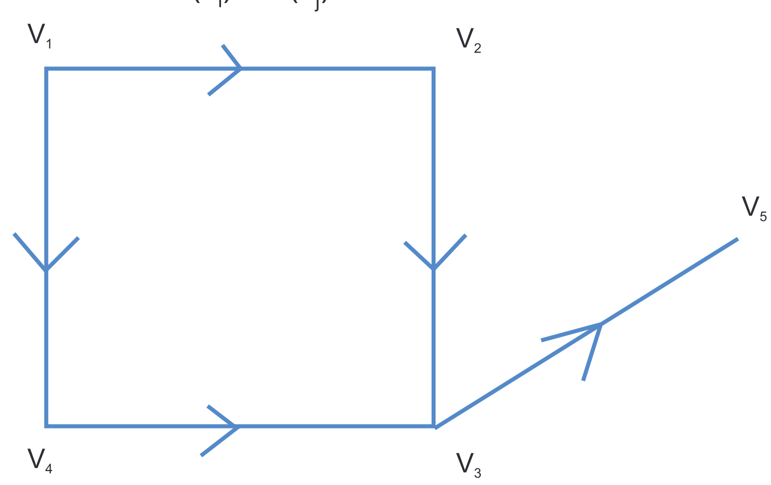


$d^+(v_1) = 2$	$d^-(v_1) = 0$
$d^+(v_2) = 1$	$d^-(v_2) = 1$
$d^+(v_3) = 0$	$d^-(v_3) = 2$
$d^+(v_4) = 1$	$d^-(v_4) = 1$

Notes

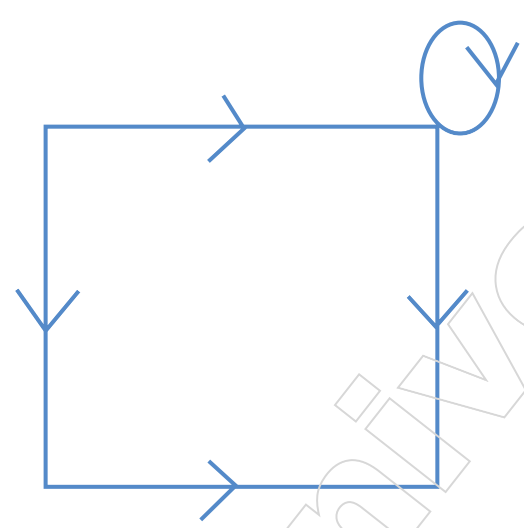
(iii) For an isolated vertex in degree = out degree = 0

(iv) In Pendent vertex $d^{+}(v_i) + d^{-}(v_j) = 1$

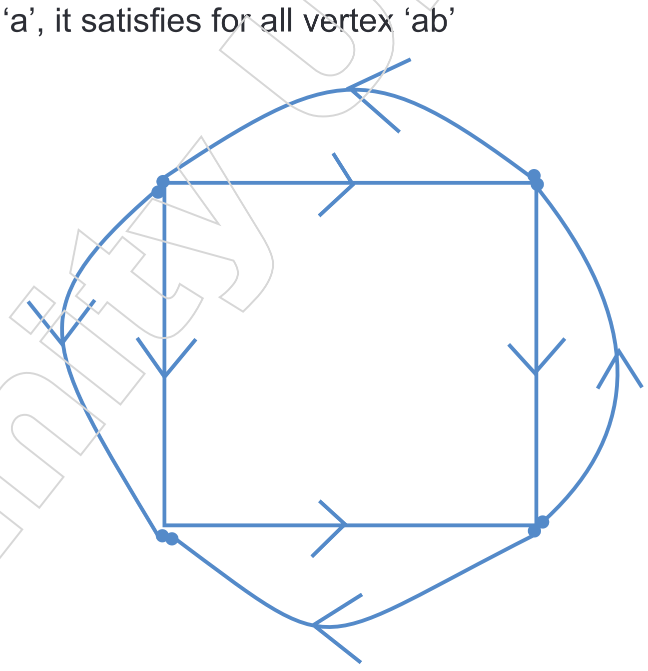


(v) In a simple digraph there are no self loops and no parallel edges

(vi) In a Asymmetric digraph that have atmost one directed edge between the pair of vertices and that are allowed to have self loops

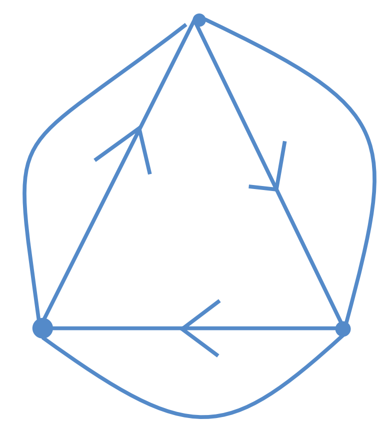


(vii) In a symmetric digraph if there is a vertex from ‘a’ to ‘b’ and the same for vertex ‘b’ to vertex ‘a’, it satisfies for all vertex ‘ab’

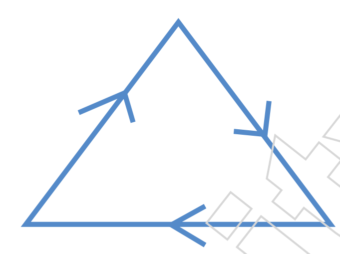


(viii) In complete symmetric digraph is a simple digraph in which there is exactly one edge directed from each vertex to every other vertex

Notes



(ix) In a complete asymmetric digraph is also a simple graph there is an exactly one edge between a pair of vertices. It is also called as a tournament.



(x) A digraph is balanced for all v_i indegree = outdegree .

Definition: 5.2

A directed walk from a vertex v_i to an v_j is an alternating sequence of vertices and edges, beginning with v_i and ending with v_j such that each edge is oriented form the vertex preceding it to the vertex following it (no edge retracting)

Definition: 5.3

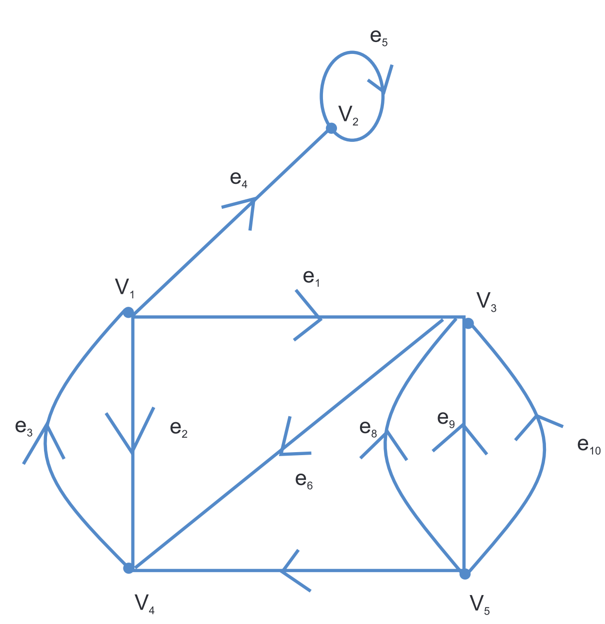
A **semi walk** in a directed graph is a walk in the corresponding undirected graph

Definition: 5.4

A directed circuit is a **closed circuit** starting from v_i and ending on v_i such that each edge is oriented from the vertex preceding it to the vertex following it

Definition: 5.5

A **semi circuit** is a closed walk which the orientation is not considered



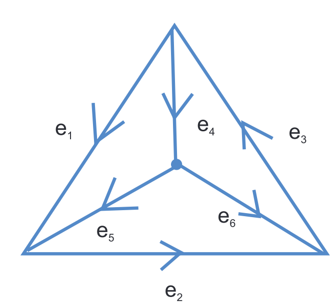
In the above figure,

- Directed walk: $v_1\ e_2\ v_4\ e_3\ v_1\ e_4\ v_2$
- Semi walk: $v_1\ e_2\ v_4\ e_7\ v_5$
- Directed circuit: $v_1\ e_1\ v_4\ e_6\ v_4\ e_3\ v_1$
- Semi circuit: $v_1\ e_2\ v_4\ e_7\ v_5\ e_9\ v_3\ e_1\ v_1$

Types of connectedness:

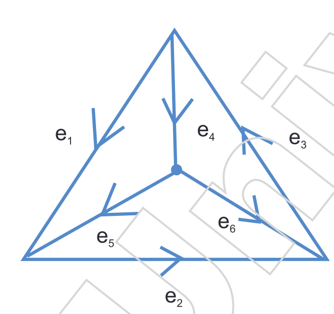
Strongly connected graph:

A graph is said to be strongly connected if there is at least one directed path from one vertex to every other vertex



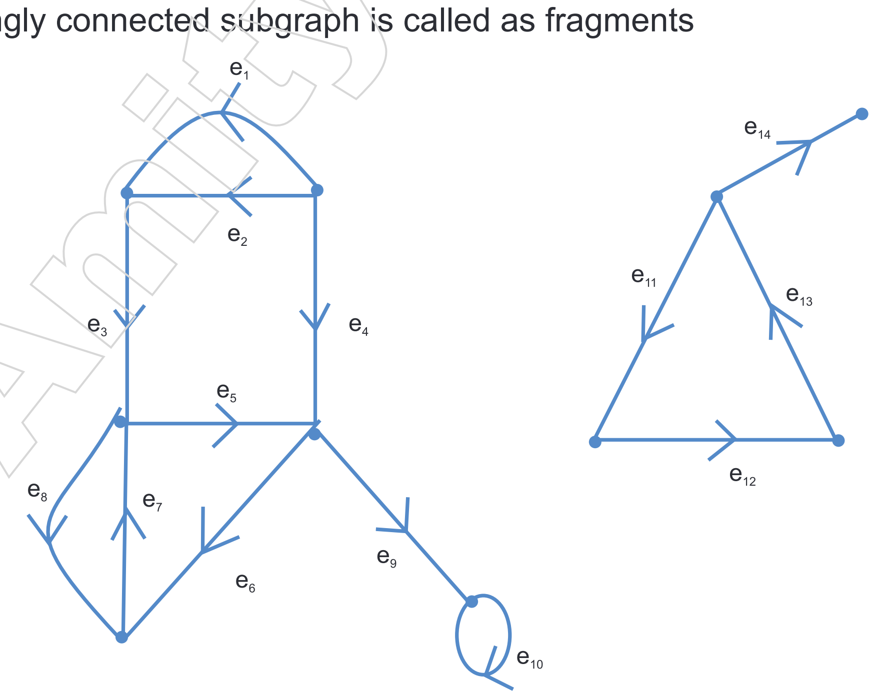
Weakly connected graph:

A graph is said to be weakly connected if its corresponding undirected graph is connected



Definition: 5.6

A graph is said to be strongly connected fragment, within each component of G, the maximal strongly connected subgraph is called as fragments



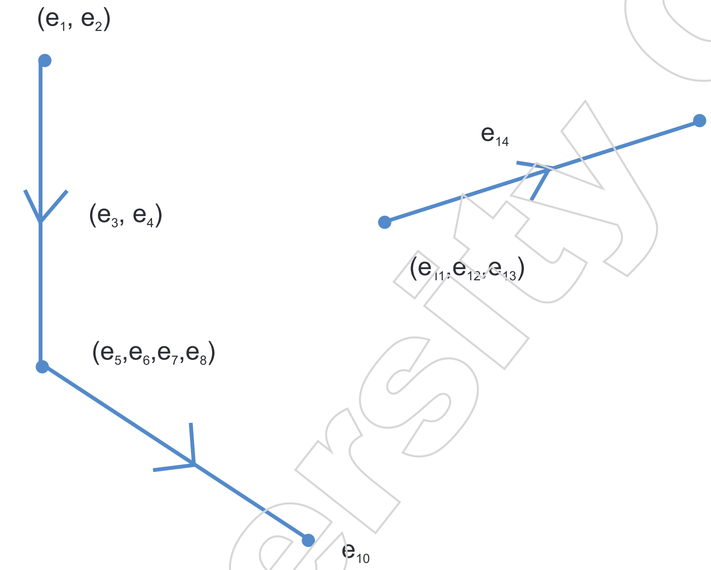
Notes

Definition: 5.7

The condensation G of a digraph G is a digraph in which each strongly connected fragment is replaced by a vertex and all directed edge from one strongly connected component to another are replaced by a single directed edge

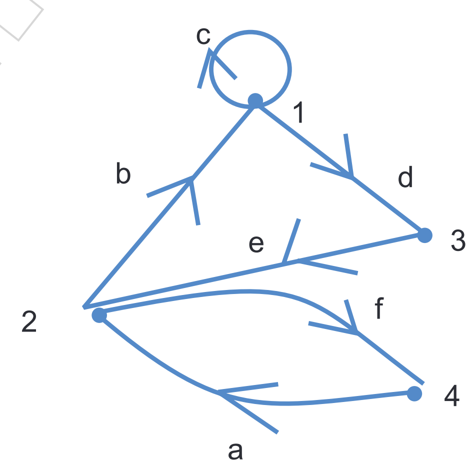
Remark:

- The condensation of strongly connected graph is simply a vertex.
- The condensation of a digraph has no directed circuit.



Definition: 5.8

A diagraph D is said to be euler if it contains a closed walk which traverses every edge of D exactly once, such a walk is called a euler walk. A digraph D is said to be unicursal if it contains an open euler walk.



In the above figure the euler line is a b c d e f

Network Flows

Definition: 5.9

A directed network D, with each arc (i, j) assigned a positive real number c_{ij} called the capacity of the arc, and two distinguished vertices called a sink [t] and a source (s), a second set of non-negative real numbers, x_{ij}, assigned to the arcs is called an (s, t) -feasible flow if $0 <= x_{ij} <= c_{ij}$ for all arcs, and $\sum_{j=1}^{n} x_{ij} \sum_{j=1}^{n} = x_{ji}$ for all vertices i, except s and t, if there is no arc (i, j) then $x_{ij} = 0$.

Definition: 5.10

A partition of the vertices of the directed network D into two sets, one containing s, called S, and the other containing t, called T, is called a cut. A cut can be determined from, and is usually identified with, the set of arcs starting in S and ending in T. The capacity of a cut is the sum of the capacities on these arcs. A minimum cut is a cut with the minimum capacity.

Remark:

- The value of any feasible flow is less than or equal to the capacity of any cut.
- If the value of some feasible flow equals the capacity of some cut, then the flow is a maximum flow and the cut is a minimum cut.
- For any arc (i, j) of the directed network, the slack, $s_{ij} = c_{ij} - x_{ij}$.

Algorithm:

- A simple algorithm for finding a larger flow is to find a path from s to t such that all the arcs on the path have positive slack.
- We can add the smallest slack on these arcs to each x_{ij} of the path to obtain a larger flow. Unfortunately, this procedure does not always lead to a maximum flow.
- Now consider the underlying graph of the directed network. Given a chain from s to t in this graph, orient the all the edges of the chain from s to t.
- Then compare this orientation with the orientation that these arcs have in the original network. If the orientation is the same for an arc, it will be called a forward arc, if the orientation is opposite, the arc is called a backward arc.
- A flow augmenting chain in the directed network, is a chain from s to t in the underlying graph, so that each forward arc has positive slack and each backward arc has a positive flow.
- Given a flow augmenting chain, a larger flow is obtained by taking the smallest value of the positve slacks on forward arcs and the positve flows on backward arcs and adding this value to the flow on all forward arcs and subtracting it from the flow on all backward arcs of the chain.

Note:

- ♦ A feasible flow is maximum if and only if there are no flow augmenting chains in the network.
- ♦ The value of the maximum flow of a directed network is the capacity of a minimum cut.

Labeling Algorithm:

- Some of the arcs of the directed network will be considered as members of two sets, F and B.
- F consists of all arcs that have positive slack and B consists of all arcs that have a positive flow. Notice that these sets are not disjoint.
- The algorithm proceeds by scanning vertices and labeling some of them in the scanning process.

Notes

- When all the labeled vertices have been scanned, the algorithm stops. If t has been labeled, then an augmenting chain can be found, otherwise there is no augmenting chain.
- Initially, the vertex s is labeled with the label
- A labeled, but unscanned vertex, call it i.
- Scan i, meaning
  1. for all arcs (i,j), if (i,j) is in F and j is unlabeled, then label j with (i+).
  2. for all arcs (j,i), if (j,i) is in B and j is unlabeled, then label j with (i-).
- Repeat until all labeled vertices have been scanned, or t has been labeled.
- If t is labeled, then the augmenting chain is determined by starting with t and following the labels back until you get to s.
- The +'s and -'s show whether the arc is a forward or backward arc.

Flows and Matchings in Bipartite Graphs

In a bipartite graph (X, Y) we can form a directed network as follows:

- Add a vertex s, and join it with arcs going from s to each vertex in X. Give all of these arcs a capacity of 1.
- Connect all the edges in the bipartite graph from X to Y.
- Give all of these arcs an infinite capacity.
- Finally, add a vertex t, and join it with arcs from each vertex in Y to t. Note all of these arcs a capacity of 1and say this directed network N.
- Notice that any feasible flow of N where all the flow values are integers (necessarily either 0 or 1), corresponds to a matching of the original graph (the matching consists of the edges which correspond to arcs between X and Y which have positive flow), and vice versa, any matching of the graph gives rise to a feasible flow.
- The value of the flow is the number of edges in the matching.

Direct relation between flows and matchings:

A covering of the graph corresponds to in the directed network.

Let A be a subset of X and B a subset of Y. We will consider the set K = A U B. Let S = {s} U (X-A) U B and T = {t} U (Y-B) U A, which is clearly a cut in the network.

Theorem: 5.11

'K' is a covering of the graph if and only if (S,T) is a cut of the network of finite capacity.

Proof:

Let 'K' be a covering of the graph.

Suppose (S,T) is a cut of finite capacity.

Then no arcs from X to Y are in this cut, in particular there are none from X-A to Y-B in the cut.

So, all arcs from A go to Y-B, and all arcs from X-A go to B.

Therefore, every edge of the graph has one vertex in A or one vertex in B, and so, K is a covering.

The arcs in the cut that start at s all go to A, and those that end at t all start in B, each having capacity 1, so the capacity of the cut is |A U B|.

Now, suppose that K is a covering of the graph. Let A be the intersection of K with X and B the intersection of K with Y.

Then Form S and T as before.

Consider an arc from a vertex in S to one in T. If this arc starts at s, it must go to A. If it starts in X-A, it must go to B since K is a covering, so such an arc does not go to T. If it starts in B, then it must go to t, and so, has capacity 1.

Thus, all arcs from S to T have finite capacity

So the cut (S, T) has finite capacity.

Finally, for a bipartite graph, the size of a maximum matching equals the size of a minimum cover.

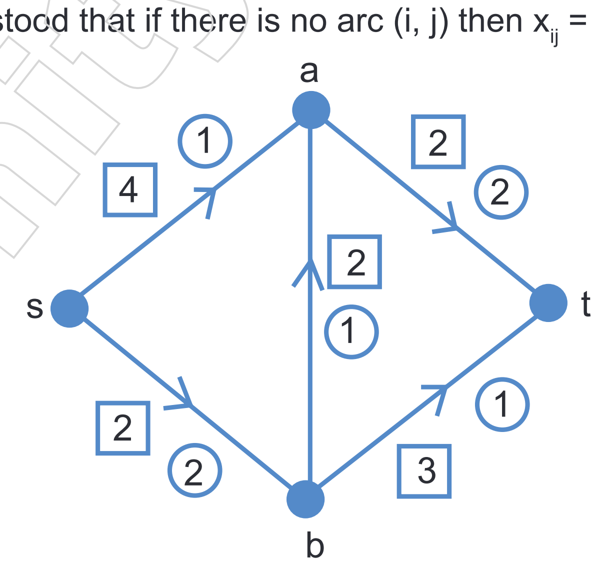
Flows and Cuts in Networks

A **capacitated single source-single sink network** is a directed graph D, with each arc (i, j) assigned a positive real number c_{ij} called the capacity of the arc, and two distinguished vertices called a sink (t) and a source (s). The source has out-degree of at least one and the sink has in-degree of at least one. In almost all of the examples we consider the source will have in-degree of 0 and the sink will have out-degree of 0, but this is not required in the definition. We shall refer to these as **(s,t)-networks.**

Given an (s,t)-network, a second set of non-negative real numbers, x_{ij}, assigned to the arcs is called an **(s,t) - feasible flow** if

1. $0 \leq x_{ij} \leq c_{ij}$ for all arcs, and
2. $\sum_j x_{ij} = \sum_j x_{ji}$ for all vertices i, except s and t

(Where it is understood that if there is no arc (i, j) then $x_{ij} = 0$.).



At s and t, the conservation law (the second condition above) does not hold, and we define

Notes

$v_t = \sum_j x_{jt} - \sum_j x_{tj}$ and $v_s = \sum_j x_{sj} - \sum_j x_{js}$.

Now, if we sum over all the vertices i, we get:

$\sum_i \left(\sum_j x_{ij} - \sum_j x_{ji} \right) = v_s - v_t$.

But the left hand side of this equation is 0, so $v_s = v_t$ and we call this common value v, and define it to be the **value** of the flow.

For the above figure, the value of that flow is 3.

Definition: 5.12

A partition of the vertices of the (s,t)-network D into two sets, one containing s, called S, and the other containing t, called T, is called a **cut**. A cut can also be determined from, and is usually identified with, the set of arcs starting in S and ending in T. The **capacity of a cut** is the sum of the capacities on these arcs. A **minimum cut** is a cut with the minimum capacity.

Remark:

- The value of any feasible flow is less than or equal to the capacity of any cut.
- If the value of some feasible flow equals the capacity of some cut, then the flow is a maximum flow and the cut is a minimum cut.
- For any arc (i,j) of the (s,t)-network, its **slack** is defined as, $s_{ij} = c_{ij} - x_{ij}$.
- A simple algorithm for finding a larger flow is to find a path from s to t such that all the arcs on the path have positive slack.
- We can add the smallest slack on these arcs to each x_{ij} of the path to obtain a larger flow. Unfortunately, this procedure does not always lead to a maximum flow.
- Consider the underlying graph of the directed network. Given a path from s to t in this graph, orient the all the edges of the path from s to t.
- Now, compare this orientation with the orientation that these arcs have in the original network. If the orientation is the same for an arc, it will be called a *forward arc*, if the orientation is opposite, the arc is called a *backward arc*.
- A **flow augmenting path** in the (s,t)-network, is a path from s to t in the underlying graph, so that each forward arc has positive slack and each backward arc has a positive flow.
- Given a flow augmenting path, a larger flow is obtained by taking the smallest value of the positive slacks on forward arcs and the positive flows on backward arcs and adding this value to the flow on all forward arcs and subtracting it from the flow on all backward arcs of the path.

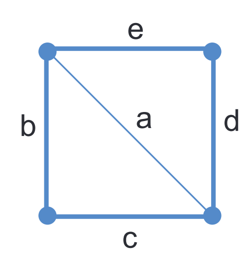
Labeling Algorithm for Finding Flow Augmenting Paths:

- Some of the arcs of the (s,t)-network will be considered as members of two sets, F and B.
- F consists of all arcs that have positive slack and B consists of all arcs that have a positive flow. Notice that these sets are not disjoint. A forward arc in F or a backward arc in B is called a *usable* arc.

Notes

Matchings, Transversals and Vertex Covers:

A set M of edges in a graph G is called a *matching* of G if no two edges in set M have an endpoint in common. A *maximum matching* of graph G is a matching of G with the greatest number of edges while a *maximal matching* is a matching which is not contained in any larger matching. While any maximum matching is certainly maximal, the reverse is not generally true.



In the above graph, {a}, {b,d}, and {c,e} are all maximal matchings, but only the last two are maximum matchings.

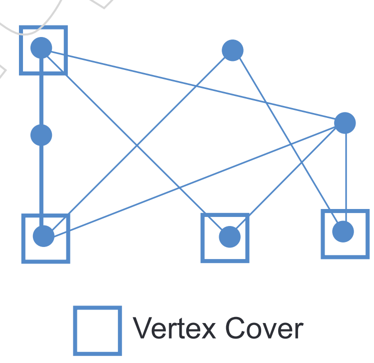
Job Assignment Problem revisited:

Given a bipartite graph (X, Y) we can form an (s,t)-network as follows:

- Add a vertex s, and join it with arcs going from s to each vertex in X. Orient all the edges in the bipartite graph from X to Y.
- Finally, add a vertex t, and join it with arcs from each vertex in Y to t. Give all of the arcs leaving s or entering t a capacity of 1, but the arcs between X and Y will get infinite capacity. Call this network N.
- Note that any feasible flow of N where all the flow values are integers (necessarily either 0 or 1), corresponds to a matching of the original graph (the matching consists of the edges which correspond to arcs between X and Y which have positive flow), and vice versa, any matching of the graph gives rise to a feasible flow.
- The value of the flow is the number of edges in the matching.

Definition: 5.13

Let G be a graph and C a subset of the vertices of G. The set C is a *vertex cover* of graph G if every edge of G is incident with at least one vertex in C. A minimum vertex cover is a vertex cover with the least number of vertices.



Theorem: 5.14

K is a vertex covering of the graph if and only if (S,T) is a cut of the network of finite capacity.

Notes

Proof:

Suppose (S,T) is a cut of finite capacity.

Then no arcs from X to Y are in this cut, in particular there are none from X-A to Y-B in the cut.

So, all arcs from A go to Y-B, and all arcs from X-A go to B. Therefore, every edge of the graph has one vertex in A or one vertex in B, and so, K is a covering.

The arcs in the cut that start at s all go to A, and those that end at t, all start in B, each having capacity 1, so the capacity of the cut is $|A \cup B|$.

Suppose that K is a vertex covering of the graph. Let A be the intersection of K with X and B the intersection of K with Y.

Form S and T as before. Consider an arc from a vertex in S to one in T. If this arc starts at s, it must go to A (and so, has capacity 1). If it starts in X-A, it must go to B since K is a vertex covering, so such an arc does not go to T.

If it starts in B, then it must go to t, and so, has capacity 1. Thus, all arcs from S to T have finite capacity, so the cut (S, T) has finite capacity.

Theorem: 5.15

Let G be a bipartite graph. Then the size of a maximum matching equals the size of a minimum vertex cover.

Proof:

Let G be a bipartite graph.

Consider the associated network of the bipartite graph.

Change the infinite capacities to 2's, this will not change any flows.

All capacities are integers, so the maximum flow exists and will have integer values.

The value of this flow is the number of edges in a maximum matching, call it n.

Let 'n' is the minimum capacity of any (S, T) cut.

The trivial cut, S = {s}, has finite capacity, so the minimum will have to be finite. Any cut with the minimum capacity corresponds to a covering with $|A \cup B| = n$. But now, there exists a cover with n vertices and a matching with n edges

So the cover must be a minimum cover.

Matrix Representation of graphs

Introduction:

A matrix is a convenient and useful way of representing a graph to a computer. Matrices lend themselves easily to mechanical manipulations. Many known results of matrix algebra can be applied to the study of structural properties of graph.

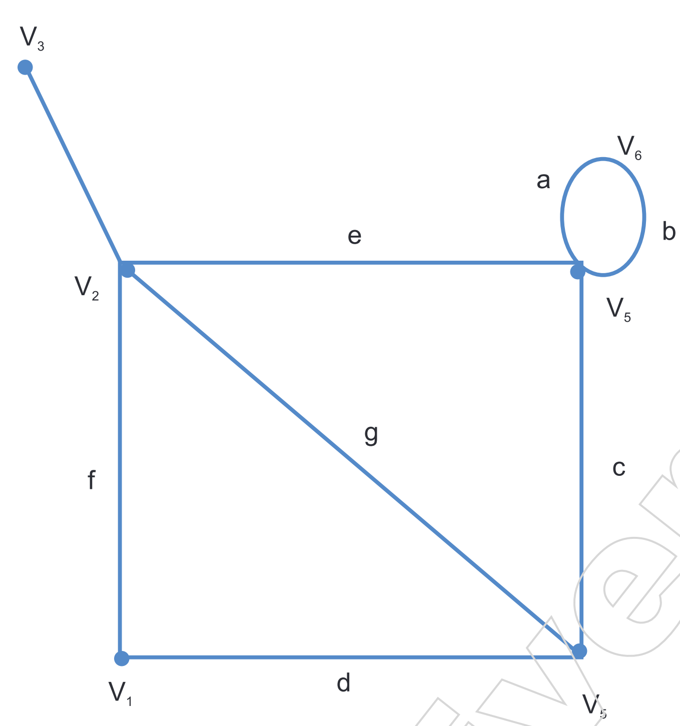
Definition:5.16

Let G be a graph with 'n' vertices, 'e' edges and no self loops. Define an n by 'e' matrix $A=[a_{ij}]$ whose 'n' rows correspond to the 'n' vertices and the 'e' columns correspond to the 'e' edges .

The matrix element

$a_{ij} = 1$, if the 'j'th edge e_j is incident on the 'i'th vertex v_i and

$a_{ij} = 0$, otherwise



$$\begin{array}{cc} & \text{Edges} \\ \text{Vertices} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Graph and incidence matrix

- In the above graph, such a matrix is called the vertex –edge incidence matrix or simply matrix.
- The incidence matrix contains only two elements 0 and 1.
- Such a matrix is called binary matrix or a (0,1) matrix.

Observations:

- Each edge is incident on exactly two vertices, each column of 'A' has exactly two 1's

Notes

- The number of 1's in each row is equals the degree of the corresponding vertex
- A row with all 0's therefore represents an isolated vertex.
- Parallel edges in a graphg produce identical columns in its incidence matrix

Definition: 5.17

Each row in an incidence matrix A(G) may be regarded as a vector over in the vector space of a graph G

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ . \\ . \\ A_n \end{bmatrix}$$

Definition: 5.18

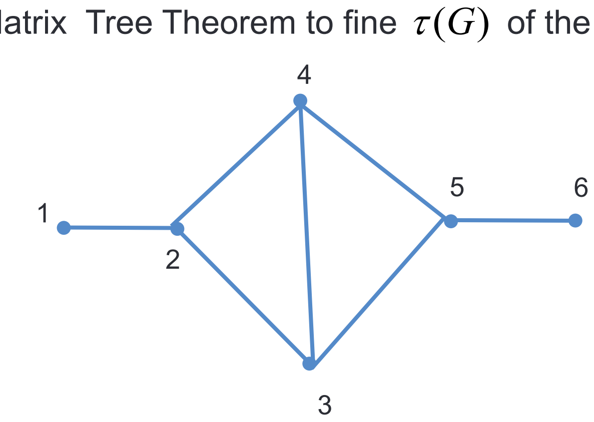
Let 'g' be a subgraph of a graph 'G' and let A(g) and A(G) be the incidence matrices of 'g' and 'G' respectively. A(g) is a submatrix of A(G). There is a non-to-one correspondence between each 'n' by 'k' submatrix of A(G) and a subgraph of G with 'k' edges of vertices in G.

Theorem: 5.19 (Matrix Tree Theorem)

Given a loopless graph G with vertex set $v_1, v_2, ...v_n$, let a_{ij} be the number of edges having v_i and v_j as their end vertices. Let Q^* be the matrix with entry *(i, j)* being $-a_{ij}$ where $i \neq j$ and $d(v_i)$ when $i = j$. If Q^* is the matrix obtained by deleting any one row, say s^{th} row and any one column, say t^{th} column of Q, then $\tau(G) = (-1)^{s+t}$ det Q^*. $[i, e, \tau(G)$ is a confactor A_s of $G]$.

Example:

Now we use the Matrix Tree Theorem to fine $\tau(G)$ of the graph G given in Figure



$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

matrix Q

Notes

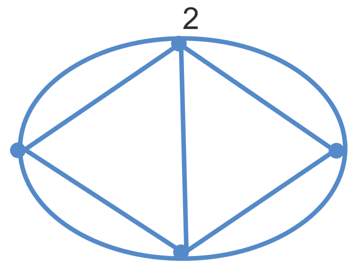
We omit the second row and the second column to find the confactor A_{22}

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{vmatrix}$$

$$1\begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & -1 \end{vmatrix} + 0 + 0 + 0 + 0 = 8$$

Problem: 1

Find $\tau(G)$ for the graph G given to figure



Solution:

The matrix Q is

$$\begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 5 & -1 & -2 \\ -2 & -1 & 5 & -2 \\ 0 & -2 & -2 & 4 \end{pmatrix}$$

$\tau(G)$ = the cofactor A_{11} = $(-1)^{2}\begin{vmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 4 \end{vmatrix}$

= 5(20-4)+1(-4-4)-2(2+10)

= 80-8-24

= 48

Theorem: 5.20 (Cayley's Formula)

Prove that if $n \geq 2, \tau(k) = n^{n-2}$

Proof:

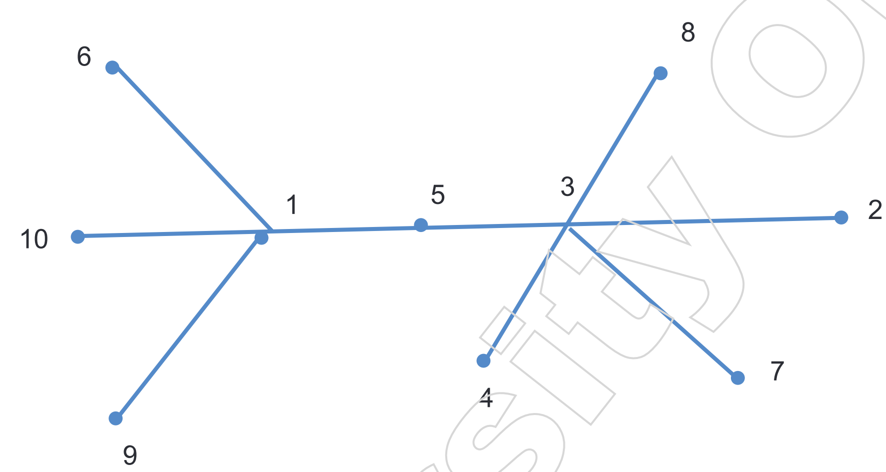
Let the vertices of the complete graph k_n be labeled with integers $1,2,......n.$

We note that there are n^{n-2} sequences of length n-2 with entries from $(1,2,......n.)$. Denote the set of all such sequence by S.

We establish a bijection between the set of all spanning trees of k_n and the set S. Using the labels of the vertices, we consider the vertex set $v(k_n)$ as an ordered set.

Notes

To each spanning tree T, we associate a sequence $f(T) = (t_1, t_2, ...t_{n-2})$ as follows. Let s_1 be the first vertex of degree one in T and t_1 be the unique vertex which is adjacent to s_1 in T. Now $T - s_1$ is a tree. Let s_2 be the first vertex of degree one in $T - s_1$ and t_2 be the unique vertex which is adjacent to s_2 in $T - s_1$. Let s_3 be the first vertex of degree one in $T - \{s_1, s_2\}$. After $n-2$ iterations a single edge remains. Thus we have produced a sequence $f(T) = (t_1, t_2, ...t_{n-2})$ of length $n-2$. For example for the spanning tree T given in the figure 40, the associated sequence f(T) is (3,3,1,3,3,5,1,1).



Next we define a function of the produces a tree from each sequence x.

Let $x = (t_1, t_2, ..., t_{n-2})$ be the given sequence.

First we begin with n isolated vertices (and edge set E = Φ).

Let s_1 be the least number that does not appear in $(t_1, t_2, ..., t_n)$.

Join the vertices s_1 and t_1 by an edge and set $E = \{s_1, t_1\}$

Let $s_2 \neq s_1$ be the least number that is not in $(t_2, t_3, ..., t_n)$.

Join the vertices s_2 and t_2 by an edge and let $E = \{s_1 t_1, s_2 t_2\}$

Let s_3 be the least number such that $s_3 \notin \{s_1, s_2\}$ and s_3 is not in .

Join s_3 and t_3 by an edge and set $E = \{s_1 t_1, s_2 t_2, s_3 t_3\}$

Continue in this way until we get $n-2$ edges $s_1 t_1, s_2 t_2, ..., s_{n-2} t_{n-2}$.

T is now obtained by adding the edge joining the remaining two vertices of $\{1, 2, ..., n\} - \{s_1, s_2, ..., s_{n-2}\}$

The resulting graph is denoted by $g(t_1, t_2, ...t_{n-2})$

At each step we add one edge, there are n-1 edges including one that is added at the last step.

After the i th step in construction of g(x)

we have n-i components.

So after adding the edge at the last step.

We get a connected graph so g(x) is a connected graph with n vertices and n-1 edges and hence it is a spanning tree.

Notes

Questions:

1. The Konigsberg bride problem was a long-standing problem until solved by euler in__________by means of a graph.
   a) 1930
   b) 1936
   c) 1940
   d) 1946
2. A graph G is said to be eulerian if it contains an ____________.
   a) euler tour
   b) euler circuit
   c) a or b
   d) a and b
3. If degree of every vertex is even in graph G, then 2(G) and hence G contains atleast____________cycle.
   a) zero
   b) one
   c) two
   d) three
4. If a closed trail C contains an edge e, then it contains a___________containing e.
   a) cycle
   b) not a cycle
   c) a or b
   d) a and b
5. An open trial that contains every edge of G exactly___________is an open euler trial.
   a) once
   b) two
   c) three
   d) none
6. A cycle that contains every vertex of G is a Hamilton___________of G.
   a) path
   b) walk
   c) cycle
   d) none

Notes

7. The complete graph K_n on n vertices is hamiltonian.
   a) 3n
   b) 2 < n
   c) 2n
   d) all
8. The complete bipartie graphs $K_{m;n}$ (2 m; n) are hamiltonian if and only if m is equal to__________.
   a) n-1
   b) n
   c) n+1
   d) none
9. The graph $K_{3;3}$ is __________.
   a) planar
   b) non-planar
   c) a or b
   d) a and b
10. The Euler's formula for any connected plane graph is_________.
   a) n+e+f=2
   b) n-e-f=2
   c) n-e+f=2
   d) all

Answer:

1. b
2. a
3. b
4. a
5. a
6. c
7. a
8. b
9. b
10. c

Problems:

Notes

1. Write down the properties of Incidence Matrix.
2. Prove that, the reduced incidence matrix of a tree is non-singular.
3. If B is a circuit matrix of a connected graph G with e edges and n vertices, prove that rank of $B = e\text{-}n + 1$.
4. Define Cut-set Matrix with give an example.
5. Explain path matrix.

Notes

Unit-VI: Planar and Dual Graphs

We studied properties of subgraphs, such as paths, circuits, spanning trees, and cut-sets, in a given connected graph G.

In this chapter we shall subject the entire graph G to the following important question: It is possible to draw G in a plane without its edges crossing over

The question of planarity is of great significance from a theoretical point of view. In addition, planarity and other related concepts are useful in any practical situations. For instance, in the design of a printed-circuit board, the electrical engineer must know if he can make the required connections without an extra layer of insulation. The solution to the puzzle if three utilities, posed in Chapter 1, requires the knowledge of whether or not the corresponding graph can be drawn in a plane.

Combinatorial Versus Geometric Graphs

As mentioned in Chapter 1, a graph exists as an abstract object, devoid of any geometric connotation of its ability of being drawn in three-dimensional Euclidean space. For example, an abstract graph G_1 can be defined as

$$G_1 = (V, E, \psi),$$

Where the set V consists of the five objects named a, b, c, d, and e, that is,

$$V=\{a,b,c,d,e\}$$

And the set E consists of seven objects (none of which is in set) named 1,2,3,4,5,6, and 7, that is,

$$E=\{1,2,3,4,5,6,7\},$$

And the relationship between the two sets is defined by the mapping ψ, which consists of

$$\psi = \left[\begin{array}{l} 1 \longrightarrow (a,c) \\ 2 \longrightarrow (c,d) \\ 3 \longrightarrow (a,d) \\ 4 \longrightarrow (a,b) \\ 5 \longrightarrow (b,d) \\ 6 \longrightarrow (d,e) \\ 7 \longrightarrow (b,e) \end{array}\right.$$

Here, he symbol 1 → (a,c) says that object 1 from set E is mapped onto the (unordered) pair (a,c) of objects from set V.

Now it so happens that this combinatorial abstract object G_1 can also be represented by means of a geometric figure. In fact, the sketch in Fig 2-13 is one such geometric representation of this graph. Moreover, it is also true that any graph can be represented by means of such a configuration in three-dimensional Euclidean space.

It is important to realize that what is sketched in Fig 2-13 is merely one (out of infinitely many) representation of the graph G_1 and not the graph G_1 itself. We could have, for instance, twisted some of the edges or could have placed e within the triangle a,d,b and thereby obtained a different figure representing G_1. However, when there is no chance of confusion, a pictorial representation of the graph has been and will be regarded as the graph itself.

This convenient slurring over is done deliberately for the sake of simplicity and clarity. Learning graph theory for the first time without any diagrams would be extremely difficult and little fun. Graph and a geometric representation of a graph.

Definition: 6.1

A graph G is said to be **planar** if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called **nonplanar.**

A drawing of a geometric representation of a graph on any surface such that no edges intersect is called **embedding**. Thus, to declare that a graph G is nonplanar. We have to show that of all possible geometric representation of G none can be embedded in a plane. Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar. An embedding of a planar graph G on a plane is called a plane representation of G.

For instance, consider the graph represented by. The geometric representation clearly is not embedded in a plane, because the edges e and f are interesting. But if we redraw edge foutside the quadrilateral, leaving the other edges unchanged, we have embedded the new geometric graph in the plane, thus showing that the graph which is being represented is planar. As another example, the two isomorphic diagrams in are different geometric representations of one and the same graph. One of the diagram is a plane representation; the other one is not. The graph, of course, is planar. On the other hand, you will not be able to draw any of the three configurations in on a plane without edges intersecting. The reason is that the graph which these three different diagrams in represent is nonplanar.

A graph G [which may be given by an abstract notation G = (V, E, Γ) or by one of its geometric representation] is planar or nonplanar

Kuratowski's Two Graphs

Theorem: 6.2

Prove that the complete graph of five vertices is nonplanar.

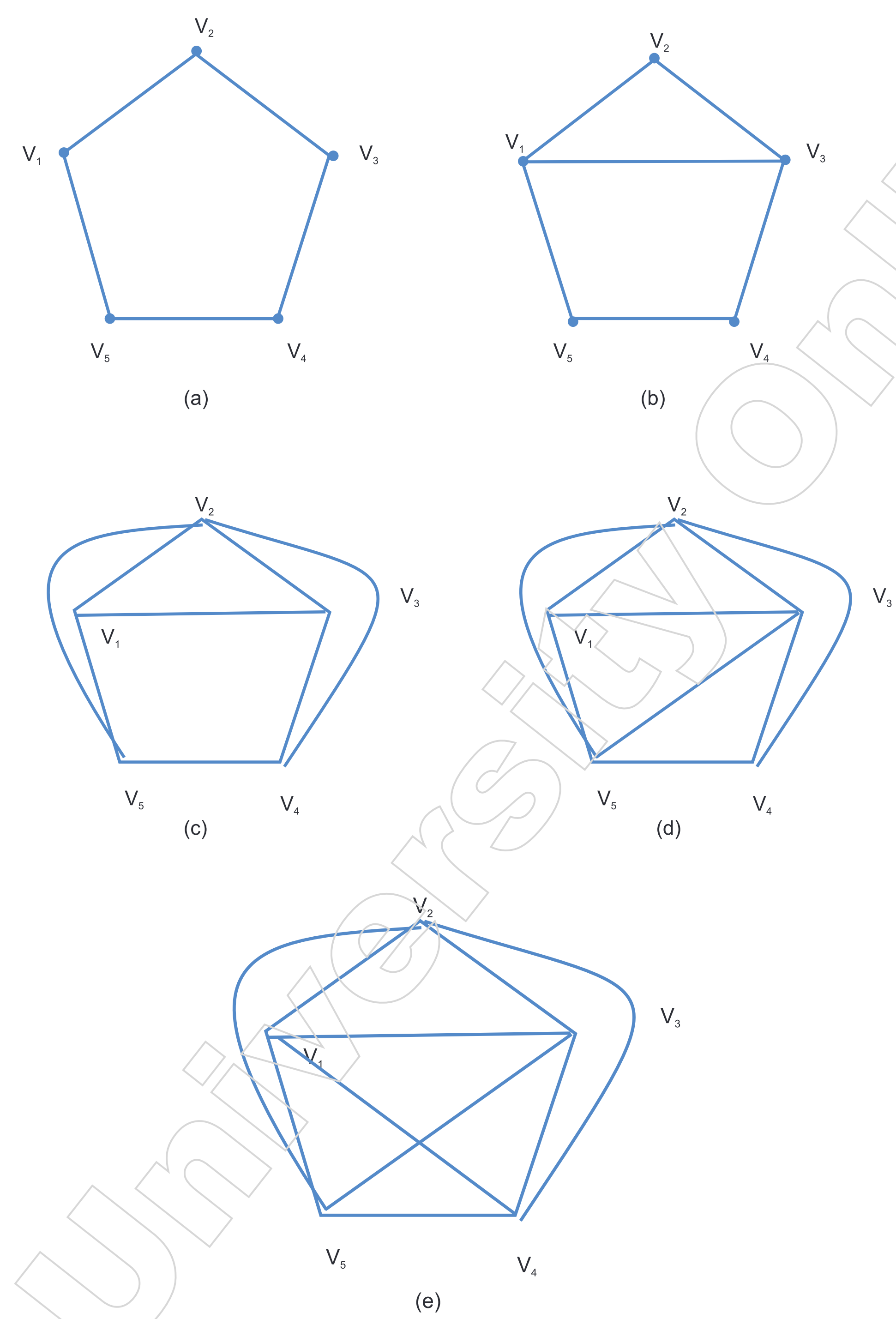
Proof

Let the five vertices in the complete graph be named v_1,v_2,v_3,v_4 and v_5.

A complete graph, as you may recall, is a simple graph in which every vertex is joined to every other vertex by means of an edge.

This being the case, the "meeting" of edges at a vertex is not considered an intersection.

Notes



The five vertices of complete graph

A circuit going from v_1 to v_2 to v_3 to v_4 to v_5 to v_1—that is, a pentagon. See Fig 5-1(a). This pentagon must divide the plane of the paper into two regions, one inside and the other outside (Jordan Curve Theorem).

Since vertex v_1 is to be connected to v_3 by means of an edge, this edge may be drawn inside or outside the pentagon (without intersecting the five edges drawn previously).

Suppose that we choose to draw a line from v_1 to v_3 inside the pentagon. (If we choose outside, we end up with the same argument.)

Now we have to draw and edge from v_2 to v_4 and another one from v_2 to v_5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon. The edge connecting v_3 and v_5 cannot be drawn outside the pentagon without crossing the edge between v_2 and v_4.

Therefore, v_3 and v_5 have to be connected with an edge inside the pentagon.

Notes

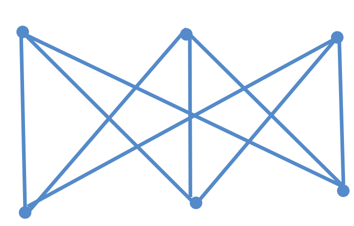
Now we have yet to draw and edge between v_1 and v_4.

This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane.

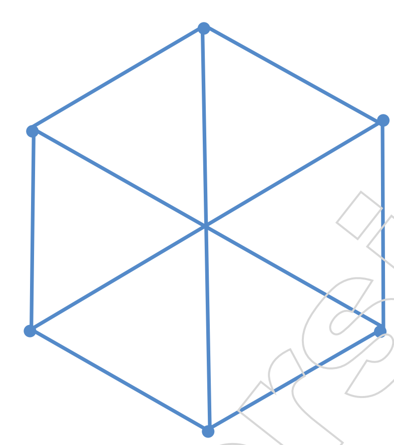
A complete graph with five vertices is the first of the two graphs of Kuratowski.

The second graph of Kuratowski is a regular connected graph with six vertices and nine edges, shown in its two common geometric representation in where it is fairly easy to see that the graphs are isomorphic.

Employing visual geometric arguments similar to those used in proving, it can be shown that the second graph of Kuratowskiis also nonplanar.



(a)



(b)

Kuratowski's second graph

Remark:

- Kuratowski's second graph is also nonplanar.
- Both are regular graphs.
- Both are nonplanar.
- Removal of one edge or a vertex makes each a planar graph.
- Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges.
- Thus the both are the simplest nonplanar graphs.

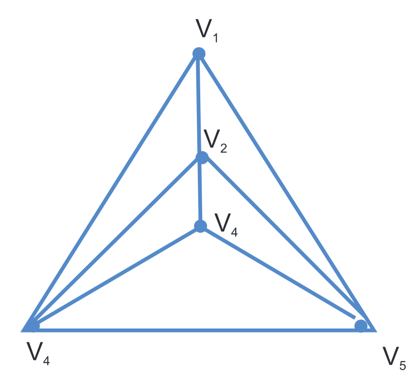
Different Representations of a Planar Graph

In following the proof of the Theorem 5-1, it may have appeared that one's ability to draw a planar graph in a plane depended on his ability to draw many crooked lines through devious routes. This is not the case. The following important and somewhat surprising result, due to Fary, tells us that there is no need to bend edges in drawing a planar graph to avoid edge intersections. Any simple planar graph can be embedded in plane such that every edge is drawn as a straight line segment.

Definition: 6.3

A plane representation of a graph divides the plane into regions also called windows, faces, or meshes) as shown in Figure A region is characterised by set of

Notes



Straight line representation of the graph

Theorem: 6.4

A planar graph may be embedded in a plane such that any specified region (i.e., specified by the edges forming it) can be made the infinite region.

Proof:

In terms of the regions on the sphere, we see that there is no real difference between the infinite region and the finite regions on the plane.

Therefore, when we talk of the regions in the plane representation of a graph, we include the infinite region.

Also, since there is no essential difference between an embedding of a planar graph on a plane or on a sphere (a plane may be regarded as the surface of a sphere or infinitely large radius), the term "plane representation" of a graph is often used to include spherical as well as planar embedding.

Theorem: 6.5

Prove that a connected planar graph with n vertices and e edges has e-n+2 regions.

Proof

It will suffice to prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one.

We can also remove all edges that do nor form boundaries of any region. Three such edges are shown in Fig. 5-4. Addition (or removal) of any such edge increases (or decreases) e by one and increases (or decreases) n by one, keeping the quality e-n unaltered.

Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon (a polygon net). Let the polygonal net representing the given graph consist of f regions or faces, and let k_p be the number if p-sided regions. Since each edge is on the boundary of exactly two regions,

$$3.k_3 + 4.k_4 + 5.k_5 + \ldots + r.k_r = 2.e.$$

Notes

Where k is the number of polygons, with maximum edges.

Also,

$$k_3 + k_4 + k_5 + \ldots + k_r = f.$$

The sum of all angles subtended at each vertex in the polygonal net is

$$2\pi n.$$

Recalling that the sum of all interior angles of a p-sided polygon is π (p-2), and the sum of the exterior angles is π (p+2), let us compute the expression as the grand sum of all interior andlesf-1 finite regions plus the sum of the exterior angles of the polygon defining the infinite region. This sum is

$$\pi(3-2).k_3 + \pi(4-2).k_4 + \ldots + \pi(r-2).k_r + 4\pi$$

$$= \pi(2e-2f) + 4\pi.$$

Equating we get

$$2\pi(e-f) + 4\pi = 2\pi n.$$

or $$e-f+2 = n.$$

Therefore, the number of regions is

$$f = e-n+2.$$

Theorem: 6.6

In any simple, connected planar graph with f regions, n vertices, and e edges (e>2), the following inequalities must hold:

1. $e \geq \frac{3}{2} f$
2. $e \leq 3n - 6$

Proof:

Since each region is bounded by at least three edges and each edge belongs to exactly two regions,

$$2e \geq 3f,$$

or $$e \geq 3/2 f.$$

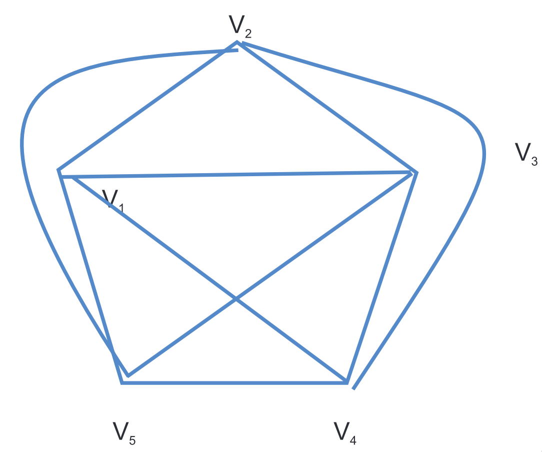
Substituting for f from Euler's formula is inequality

$$e \geq 3/2\,(e-n+2)$$

or $$e \leq 3n-6.$$

It is often useful in finding out if graph is nonplanar. For, example, in the case of K_3, the complete graph of five vertices.

Notes



(e)

$n = 5, \quad e = 10, \quad 3n - 6 = 9 < e.$

Thus the graph violates and hence it is not planar.

Incidentally, this is an alternative and independent proof of the non-planarity of Kuratowski's first graph.

Every simple planar graph must satisfy the above equation the mere satisfaction of this inequality does not guarantee the planarity of a graph.

For example, Kuratowski's second graph, $K_{3,3}$, satisfies because

$$e = 9,$$

$$3n - 6 = 3.6-6 = 12.$$

Yet the graph is non-planar.

To prove the non-planarity of Kuratowski's second graph, we make use of the additional fact that no region in the graph can be bounded with fewer than four edges.

Hence, if this graph were planar, we would have

$$2e \geq 4f,$$

And substituting for f from Euler's formula,

$$2e \geq 4(e-n+2),$$

or $$2.9 \geq 4(9-6+2),$$

or $$18 \geq 20, \quad \text{a contradiction.}$$

Hence the graph cannot be planar.

Plane Representation and Connectivity:

In a disconnected graph the embedding of each component can be considered independently. Therefore, it is clear that a disconnected graph is planar if and only if its components is planar. Similarly, in a separable (or 1-connected) graph the embedding of each block (i.e., maximal non-separable subgraph) can be considered independently. Hence a separable graph is planar if and only if each of its blocks is planar.

Therefore, in questions of embedding or planarity, one need consider only nonseparable graphs.

Notes

We must define the meaning of unique embedding can be made to coincide by suitably rotating one sphere with respect to the other and possibly distorting regions (without letting a vertex cross an edge). If of all possible embeddings on a sphere no two are distinct, the graph is said to have a unique embedding on a sphere (or a unique plane representations).

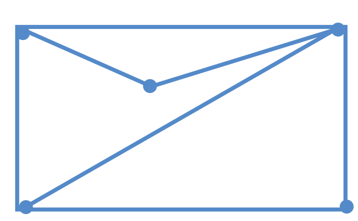
For example, consider two embeddings of the same graph.

The embedding (b) has a region bounded with five edges, but embedding (a) has no region with five edges. Thus, rotating the two spheres on which (a) and (b) are embedded will not make them coincide. Hence the two embeddings are distinct, and the graph has no unique plane representation.

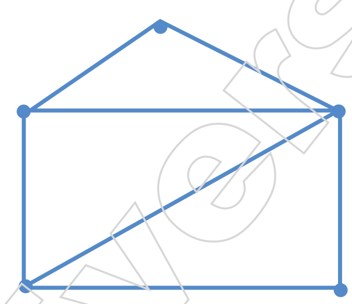
On the other hand, the embeddings when considered on a sphere, can be made to coincide. (Remember that edges can be bent, and in a spherical embedding there is no infinite region), when a graph is uniquely embeddable in a sphere.

Remark:

The spherical embedding of every planar 3-connected graph is unique.



(a)



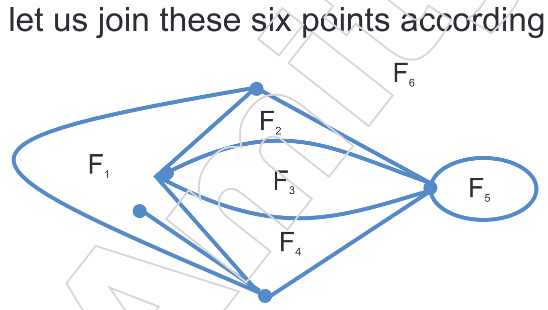
(b)

Two distinct plane representation of the same graph

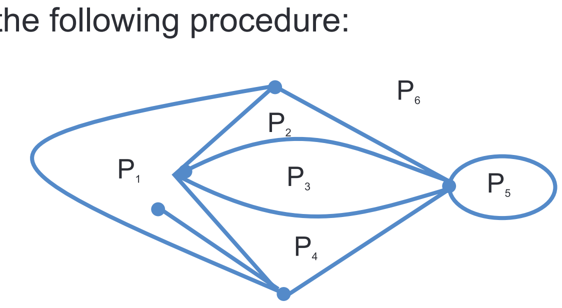
Geometric Dual

Consider the plane representation of a graph in figure with six regions or faces F_1, F_2, F3, F4, F_5, and F_6.

Let us place six points $p_1, p_2, \ldots, p_6$. One in each of the regions, as shown in figure. Next let us join these six points according to the following procedure:

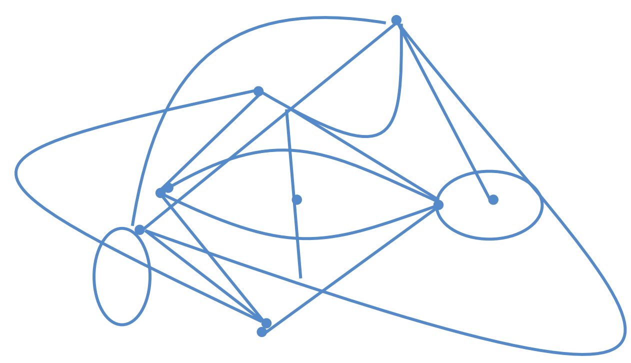


(a)



(b)

Notes



Construction of a dual graph

If two regions F_i and F_j are adjacent (i.e., have a common edge), draw a line joining points p_i and p_j that intersects the common edge between F_i and F_j exactly once.

If there is more than one edge common between F_i and F_j, draw one line between points p_i and p_j for each of the common edges.

For an edge e lying entirely in one region, say F_k, draw a self-loop at point p_k intersecting e exactly once.

By this procedure we obtain a new graph G* consisting of six vertices, p_1, p_2, . . , p_6 and of edges joining these vertices. Such a graph G* is called a dual (or strictly speaking, a geometric dual) of G.

Clearly, there is a one -to-one correspondence between the edges of graph G and its dual G* - one edge of G* intersecting one edge of G.

Observations that can be made about the relationship between a planar graph G and its dual G* are

1. An edge forming a self-loop in G yields a pendant edge in G*.
2. A pendant edge in G yields a self-loop in G*.
3. Edges that are in series in G produce parallel edges in G*.
4. Parallel edges in G produce edges in series in G*.
5. The result of the general observation that the number of edges constituting the boundary of a region F_i in G is equal to the degree of the corresponding vertex p_i in G*, and vice versa.
6. Graph G* is also embedded in the plane and is therefore planar.
7. Considering the process of drawing a dual G* from G, it is evident that G is a dual of G* [see figure]. Therefore, instead of calling G* a dual of G, we usually say that G and G* are dual graphs.
8. If n, e, f, r, and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n*, e*, f*, r*, and μ* are the corresponding numbers in dual graph G*, then

$$n^* = f,$$

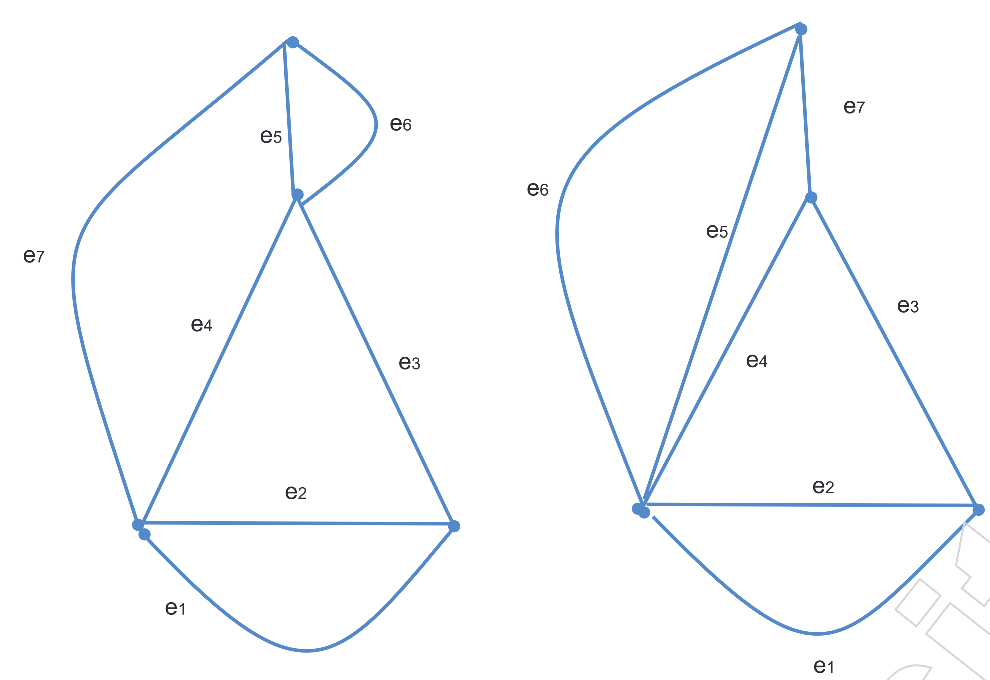
$$e^* = e,$$

$$f^* = n.$$

Using the above relationship, one can immediately get

$$r^* = \mu,$$

$$\mu^* = r.$$



Dual of graphs

Unique the above Dual Graphs:

In an incident on a pendant vertex is called a pendant edge.

Other words, are all duals of a given graph isomorphic. From the methods of constructing a dual, it is reasonable to except that a planar graph G will have a unique dual if and only of it has a unique plane representation or unique embedding on a sphere.

For instance, in figure the same graph (isomorphic) had two distinct embeddings, (a) and (b). Consequently, the duals of these isomorphic graphs are non-isomorphic, as shown in figure

The graphs in figure, however, are 2-isomorphic. This stated without proof, is a generalization of this example.

Theorem: 6.7

All duals of a planar graph G are 2-isomorphic; and every graph 2-isomorphic; to a dual of G is also a dual of G.

Proof:

It is quite appropriate to refer to a dual as the dual of a planar graph.

Since a 3-connected planar graph has a unique embedding on a sphere, its dual must also be unique. In other words, all duals of a 3-connected graph are isomorphic.

Combinatorial Dual:

We have defined and discussed duality of planar graphs in a purely geometric sense. The following provides us with an equivalent definition of duality independent of geometric notions.

Notes

Theorem: 6.8:

A necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows: There is a one-to-one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a circuit if and only if the corresponding set in G_2 forms a cut-set.

Proof:

Let us consider a plane representation of a planar graph G. Let us also draw (geometrically) a dual G* of G. Then consider an arbitrary circuit Γ in G. Clearly, Γ will form some closed simple curve in the plane representation of G dividing the plane into two areas. (Jordan Curve Theorem).

Thus the vertices of G* are partitioned into two nonempty, mutually exclusive subsets one inside Γ and the other outside.

In other words, the set f edges Γ^* in G* corresponding to the set Γ in G is a cut-set in G*. (No proper subset of Γ^* will be a cut-set in G*).

Likewise it is apparent that corresponding to a cut-set S in G such that S is a circuit. This proves the necessity portion

To prove the sufficiency, let G be a planar graph and let G' be a graph for which there is a one-to-one correspondence between the cut-sets of G and circuits of G', and vice versa.

Let G* be a dual graph of G. There is a one-to-one correspondence between the circuits of G' and cut-sets of G, and also between the cut-sets of G and circuits of G*.

Therefore there is a one-to-one correspondence between the circuits of G' and G*, implying that G' and G* are 2-isomorphic G' must be a dual of G.

Dual of a Subgraph:

Let G be a planar graph and G* be its dual. Let a be an edge in G, and the corresponding edge in G* be a*. Suppose that we delete edge a from G and then try to find the dual of G-a. If edge a was on the boundary of two regions, removal of a would merge these two regions into one. Thus the dual (G - a)* can be obtained from G* by deleting the corresponding edge a* and them fusing the two end vertices of a* in G*-a*. On the other hand, if edge a is not on the boundary, a* forms a self-loop. In that case G*-a* is the same as (G-a)*. Thus if a graph G has a dual G*, the dual of any subgraph of G can be obtained by successive application of this procedure.

Dual of a Homeomorphic Graph:

Let G be a planar graph and G* be its dual. Let a be an edge in G, and the corresponding edge in G* be a*. Suppose that we create an additional vertex in G by introducing a vertex of degree two in edge a (i.e., a now becomes two edges in series). It will simply add an edge parallel to a* in G*. Likewise, the reverse process of merging two edges i series will simply eliminate one of the corresponding parallel edges in G*. Thus if a graph G has a dual of any graph homeomorphic to G can be obtained from G* by the above procedure.

Notes

So far we have been studying duality for planar graphs only. This was forced upon us because the very definition of duality depended on the graph being embedded in a plane. However, now that provides us with an equivalent abstract definition of duality (namely, the correspondence between circuits and cut-sets), which does not depend on a plane representation of a graph, we will see if the concept of duality can extended to nonplanar graphs also. In other words, given a non-planar graph G, can we find another graph G' with one-to-one correspondence between their edges such that every circuit in G corresponds to a unique cut-set in G', and vice versa. The answer to this question is no, as shown in the following important theorem, due to Whitney.

Theorem: 6.9

Prove that a graph has a dual if and only is it is planar.

Proof:

We need prove just the "only if" part. Modu That is, we have only to prove that a nonplanar graph does not have a dual. Let G be a nonplanar graph. Then according to Kuratowski's theorem, G contains K_5 or $K_{3,3}$ or a graph homeomorphic to either of these. We have already seen that a graph G can have a dual only if every subgraph g of G and every graph homeomorphic to g has a dual. This if we can shoe that neither K_5 nor $K_{3,3}$ has a dual, we have proved the theorem.

This we shall prove by contradiction as follows:

(a) Suppose that $K_{3,3}$ has a dual D. Observe that the cut-sets in $K_{3,3,}$ correspond to circuits in D and vice versa. Since $K_{3,3}$ has no cut-set consisting of two edges, D has no circuit consisting of two edges. That is, D contains no part of parallel edges. Since every circuit in $K_{3,3}$ is of length four or six, D has no cut-set with less than four edges. Therefore, the degree of every vertex in D is at least four. As D has no parallel edges and the degree of every vertex is at least four, D must have at least five vertices each of degree four or more. That is D must have at least (5 x 4)/2 =10 edges. This is a contradiction, because $K_{3,3}$ has nine edges and so must its dual. Thus $K_{3,3}$ cannot have a dual. Likewise,

(b) Suppose that the graph K_5 has a dual H. Note that K_5 has (1) 10 edges, (2) no pair of parallel edges, (3) no cut-set with two edges, and (4) cut-sets with only four or six edges. Consequently, graph H must have (1) 10 edges, (2) no vertex with degree less than three, (3) no pair of parallel edges, and (4) circuits of length four and six only. Now graph H contains a hexagon (a circuit of length six), and length three or a pair of parallel edges [see figure]. Since both of these are forbidden in H and H has 10 edges, there must be at least seven vertices in H. The degree of each of these vertices is at least three. This leads to H having at least 11 edges. Which is a contradiction.

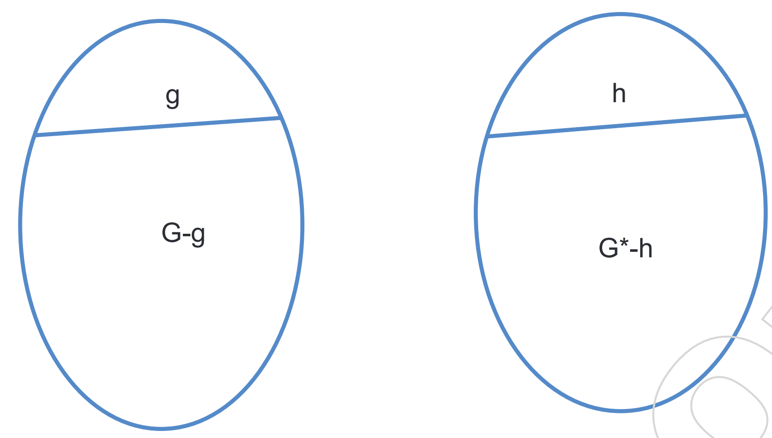
There is yet another equivalent combinatorial definition of duality, and also given

Two planar graphs G and G* are said to e duals (or combinatorial duals) of each other if there is a one-to-one correspondence between the edges of G and G* such that if g is any subgraph of G and h is the corresponding subgraph of G*, then

Rank of (G*-h) = rank of G* - nullity of g.

Notes

This relationship is shown diagrammatically in figure



Rank of (G*-h)=Rank of G*-Nullity of g

Combinatorial duals

As an example, consider the graph in figure and its dual in figure 5-11. Take the subgraph $\{e_4, e_5, e_6, e_7\}$ in figure and the corresponding subgraph $\{e^*_4, e^*_5, e^*_6, e^*_7\}$ in figure

Rank of $(G^* - \{e^*_4, e^*_5, e^*_6, e^*_7\})$ = rank of $\{e^*_1, e^*_2, e^*_3\} = 2$,

Rank of $G^* = 3$,

nullity of $\{e_4, e_5, e_6, e_7\} = 1$,

and

$2 = 3-1$.

Clearly, this definition is also independent of the geometric connotation. It is therefore often preferred for proving results in purely algebraic fashion. However, in deciding whether or not two given graphs are dual the combinatorial definitions are difficult to use.

The proof of equivalence of combinatorial and geometric duals is quite involved. Since the geometric and combinatorial duals are one and the same, we simply refer to them as the dual, rather than the geometric or combinatorial dual.

Self- Dual Graphs:

If a planar graph G is isomorphic to its own dual, it is called a self-dual graph. It can be easily shown that the four-vertex complete graph is a self-dual graph. Self-dual graphs have interesting properties and pose some unsolved problems.

More on Criteria of Planarity

Set of Basic Circuits:

A set C of circuits in a graph is said to be a complete set of basic circuits if (i) every circuit in the graph can be expressed as a ring sum of some or all circuits in C, and (ii) no circuit in C can be expressed as a ring sum of others in C.

The significance of complete sets of basic circuits will be clearer in Chapter 6, in relation to the vector space of a graph. It may, however, be mentioned here that whereas a set of fundamental circuits (as defined in Chapter 3 with respect to a spanning tree always constitutes a complete set of basic circuits, the converse does not hold for all graphs

In a planar graph a complete set of basic circuits has an additional property

In a plane representation of a planar, connected graph G the set of circuits forming the interior regions constitutes a complete set of basic circuits. For any circuit Γ in G can be expressed as the ring sum of the circuits defining the regions contained in Γ. Observe that every edge appears in at most two of these basic circuits. Thus for every planar graph G we can find a complete set of basic circuits such that no edge appears in more than two of these basic circuits. This result and its converse (Proof of which can be found in [5-6]) lead to another well-known characterization of planar graphs.

Theorem: 6.10:

A graph G is planar if and only if there exists a complete set of basic circuits (i.e., all μ of them, μ being the nullity of G) such that no edge appears in more than two of these circuits.

Proof:

A three of these classic characterizations suffer from two shortcomings. First, they are extremely difficult to implement for a large graph. Second in case the graph is planar they do not give a plane representation of the graph.

These drawbacks have prompted recent discoveries of several map-construction methods, where the testing of planarity itself is based on several other construction methods, some of them quite similar, have been implemented on digital computers. In most of these methods, the given graph is first reduced to one or more simple, non-separable graphs with every vertex of degree three or more and with $e \leq 3n - 6$. Then the construction algorithm is applied such that either one succeeds in will be said on such algorithms.

Some algorithms are better than others, but all are laborious and time- consuming. The search for a simple, elegant, and practical characterization of a planar graph is far from over.

Thickness and Crossing:

When a graph has a pair of edges that cross, it's known as a crossing on the graph. Counting up all such crossings gives you the total number for that drawing of the graph. Therefore, one of the main problems is to minimize the number of crossings by adjusting the positions of the vertices.

This is usually done to improve the readability of graphs or to solve some real-world application such circuit board design or road or railroad crossings. The minimum number of crossings in any drawing of a graph is known as the **crossing number.**

Computing the crossing number is an NP-hard problem, which means there is no algorithm that can find the minimum number of crossings of any graph in deterministic polynomial time. For rectilinear complete graphs, we know the crossing number for graphs up to 27 vertices, the rectilinear crossing number.

Notes

Since this problem is NP-hard, it would be at least as hard to have software minimize or draw the graph with the minimum crossing, except for graphs where we already know the crossing number. In all other cases, it is best to have a triangular convex hull. You can discover why by manually drawing complete graphs with a low number of vertices.

Local Crossing Number:

Though there is a lot of study around the crossing number, one area that has received less attention is the *local crossing number*. The local crossing number of a drawing of a graph is the largest number of crossings on a single edge. The *minimum local crossing* in any drawing of a graph is the local crossing number for that graph.

Questions:

1. If a plane graph has k components, than n-e+f is equal to_________
   a) k-1
   b) k
   c) k+1
   d) none
2. If K_5 were planar graph, then e is less then or equal to_________
   a) n-6
   b) 2n-6
   c) 3n-6
   d) all.
3. The graph k_n is planar if and only if________
   a) n>5
   b) n = 3
   c) n<4
   d) n = 4
4. A 3-regular graph with four or more vertices is__________
   a) an euler graph
   b) not an euler graph
   c) a wheel
   d) a cycle
5. $K_{m;n}$ is planar if and only if
   a) m 2 or n 2
   b) m 2 and n 2
   c) m = 2 or n = 2
   d) m = 2 and n = 2

Notes

6. A tree with 3 or more vertices ___________
   a) has an euler line and a hamiltonian circuit
   b) has an euler line but not a hamiltonian circuit
   c) has neither an euler nor a hamiltonian circuit
   d) has a hamiltonian circuit but not an euler line.
7. The graph Q_3 has______________
   a) has an euler line and a hamiltonian circuit
   b) has an euler line but not a hamiltonian circuit
   c) has neither an euler nor a hamiltonian circuit
   d) has a hamiltonian circuit but not an euler line
8. The graph Q_4 has______________
   a) has an euler line and a hamiltonian circuit
   b) has an euler line but not a hamiltonian circuit
   c) has neither an euler nor a hamiltonian circuit
   d) has a hamiltonian circuit but not an euler line
9. The graph K_5 has ___________
   a) has an euler line and a hamiltonian circuit
   b) has an euler line but not a hamiltonian circuit
   c) has neither an euler nor a hamiltonian circuit
   d) has a hamiltonian circuit but not an euler line
10. The graph K_5 has______________
    a) is planner
    b) is nonplanar after the removal of any edge
    c) is planar after the removal of any edge
    d) none of these

Answer:

1. c
2. c
3. c
4. b
5. a
6. c
7. d
8. a

Notes

9. a
10. c
11. d
12. c
13. b

Questions:

1. Define planar graphs with an example
2. Explain Hamiltonian cycle and hamiltonain circuit
3. Relationship between Hamiltonian cycle and Hamiltonian circuit

Module V

Key Learning Objectives:

At the end of this module, you will be able to:

- Implement the concept of partitions and counting functions
- Divide the number of partitions into odd or unequal parts
- Discuss the concepts of necklaces, Euler's function, set of symmetries, and enumeration in the odd and even cases

Notes

Unit- VII

Partitions:

In number theory and combinatorics, a **partition** of a positive integer, also called an **integer partition**, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. (if order matters, the sum becomes a composition.) For example, 4 can be partitioned in five distinct ways:

4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

The order-dependent composition 1 + 3 is the same partition as 3 + 1, while the two distinct compositions 1 + 2 + 1 and 1 + 1 + 2 represent the same partition 2 + 1 + 1.

A summand in a partition is also called a **part**. The number of partitions of n is given by the partition function p(n). So p(4) = 5. The notation $\lambda \vdash n$ means that λ is a partition of n.

Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in a number of branches of mathematics and physics, including the study of symmetric polynomials and of the symmetric group and in group representation theory in general

Example:

The seven partitions of 5 are:

- 5
- 4 + 1
- 3 + 2
- 3 + 1 + 1
- 2 + 2 + 1
- 2 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1

In some sources partitions are treated as the sequence of summands, rather than as an expression with plus signs. For example, the partition 2 + 2 + 1 might instead be written as the tuple (2, 2, 1) or in the even more compact form $(2^2, 1)$ where the superscript indicates the number of repetitions of a term.

Odd parts and distinct parts:

Among the 22 partitions of the number 8, there are 6 that contain only odd parts:

- 7 + 1
- 5 + 3

Notes

- 5 + 1 + 1 + 1
- 3 + 3 + 1 + 1
- 3 + 1 + 1 + 1 + 1 + 1
- 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

Alternatively, we could count partitions in which no number occurs more than once. Such a partition is called a **partition with distinct parts.** If we count the partitions of 8 with distinct parts, we also obtain 6:

- 8
- 7 + 1
- 6 + 2
- 5 + 3
- 5 + 2 + 1
- 4 + 3 + 1

This is a general property. For each positive number, the number of partitions with odd parts equals the number of partitions with distinct parts, denoted by $q(n)$.

For every type of restricted partition there is a corresponding function for the number of partitions satisfying the given restriction. An important example is $q(n)$.

Partition of a set:

A partition of a set X is a set of non-empty subsets of X such that every element x in X is in exactly one of these subsets (i.e., X is a disjoint union of the subsets).

Equivalently, a family of setsP is a partition of X if and only if all of the following conditions hold

- The family P does not contain the empty set (that is $\emptyset \notin P$).
- The union of the sets in P is equal to X. The sets in P are said to coverX.
- The intersection of any two distinct sets in P is empty. The elements of P are said to be pairwise disjoint.

The sets in P are called the blocks, parts or cells of the partition

The **rank** of P is |X| − |P|, if X is finite.

Examples:

- The empty set has exactly one partition, namely for any non-empty set X, P = {X} is a partition of X, called the trivial partition.
- Particularly, every singleton set {x} has exactly one partition, namely { {x} }.
- For any non-empty proper subsetA of a set U, the set A together with its complement form a partition of U.
- The set { 1, 2, 3 } has these five partitions (one partition per item):
  - o { {1}, {2}, {3} }, sometimes written 1|2|3.
  - o { {1, 2}, {3} }, or 12|3.
  - o { {1, 3}, {2} }, or 13|2.

Notes

  - { {1}, {2, 3} }, or 1|23.
  - { {1, 2, 3} }, or 123 (in contexts where there will be no confusion with the number).
- The following are not partitions of { 1, 2, 3 }:
  - { {}, {1, 3}, {2} } is not a partition (of any set) because one of its elements is the empty set.
  - { {1, 2}, {2, 3} } is not a partition (of any set) because the element 2 is contained in more than one block.
  - { {1}, {2} } is not a partition of {1, 2, 3} because none of its blocks contains 3; however, it is a partition of {1, 2}.

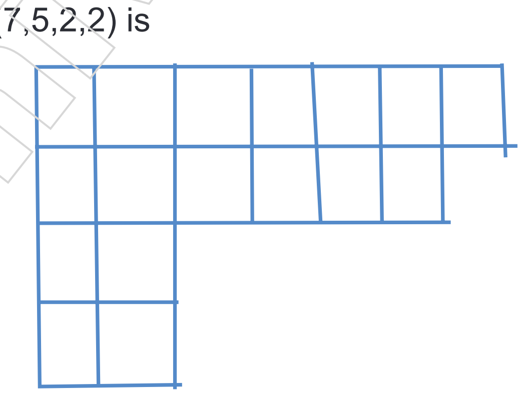
Partitions of n:

A partition of n is a combination (unordered, with repetitions allowed) of positive integers, called the parts, that add up to n

In other words, a partition is a multiset of positive integers, and it is a partition of n if the sum of the integers in the multiset is n. It is conventional to write the parts of a partition in descending order,

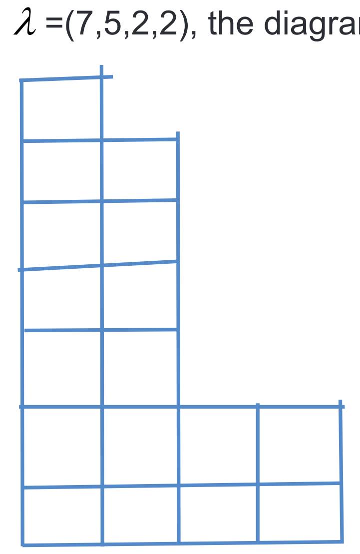
Ferrers diagram and conjugate partition:

The Ferrers diagram, also called Young diagram, of a partition $\lambda \vdash n$ is a rectangular array of n boxes, or cells, with one row of length j for each part j of λ. For example, the diagram of (7,5,2,2) is



The conjugate of a partition $\lambda \vdash n$ is the partition of n whose diagram you get by reacting the diagram of λ about the diagonal so that rows become columns and columns become rows. We use the notation for the conjugate of λ.

In our example above, with λ =(7,5,2,2), the diagram of



Notes

Partition function and Recurrence relation:

P(n) sometimes also denoted $P(n)$ gives the number of ways of writing the integer n as a sum of positive integers, where the order of addends is not considered significant.

By convention, partitions are usually ordered from largest to smallest For example, since 4 can be written

$$4 = 4 \quad (1)$$
$$= 3 + 1 \quad (2)$$
$$= 2 + 2 \quad (3)$$
$$= 2 + 1 + 1 \quad (4)$$
$$= 1 + 1 + 1 + 1, \quad (5)$$

it follows that $P(4) = 5$. $P(n)$ is sometimes called the number of unrestricted partitions, and is implemented Partitions P[n].

The values of $P(n)$ for $n = 1, 2, \ldots$, are 1, 2, 3, 5, 7, 11, 15, 22, 30,

The first few prime values of $P(n)$ are 2, 3, 5, 7, 11, 101, 17977, 10619863, ….., corresponding to indices 2, 3, 4, 5, 6, 13, 36, 77, 132, ...The largest known n giving a probable prime is 1000007396 with 35219 decimal digits, while the largest known n giving a proven prime is 221444161 with 16569 decimal digits

6

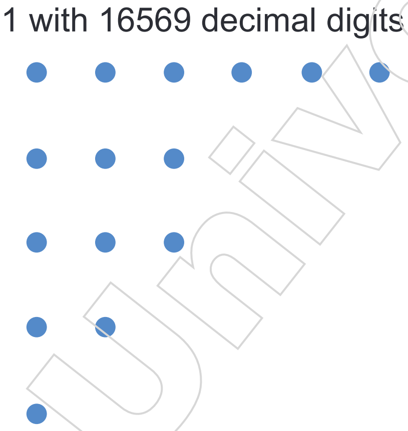
+3

+3

+2

+1

= 15



When explicitly listing the partitions of a number n, the simplest form is the so-called natural representation which simply gives the sequence of numbers in the representation (e.g., (2, 1, 1) for the number 4 = 2+1+1). The multiplicity representation instead gives the number of times each number occurs together with that number (e.g., (2, 1), (1, 2) for 4=2.1+1.2). The Ferrers diagram is a pictorial representation of a partition. For example, the diagram above illustrates the Ferrers diagram of the partition 6+3+3+2+1=15 .

Euler gave a generating function for $P(n)$ using the q-series

$$(q)_\infty \equiv \prod_{m=1}^{\infty}(1-q^m) \quad (6)$$

$$= \sum_{-\infty}^{\infty}(-1)^n q^{n(3n+1)/2} \quad (7)$$

$$= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}+\ldots. \quad (8)$$

Notes

Here, the exponents are generalized pentagonal numbers 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, ... and the sign of the kth term (counting 0 as the 0th term) is$(-1)^{\lfloor (k+1)/2 \rfloor}$ (with $\lfloor x \rfloor$ the floor function). Then the partition numbers p(n) are given by the generating function

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} P(n)q^n \qquad (9)$$

$$= 1 + q + 2q^2 + 3q^3 + 5q^4 + ... \qquad (10)$$

The number of partitions of a number n into m parts is equal to the number of partitions into parts of which the largest is m, and the number of partitions into at most m parts is equal to the number of partitions into parts which do not exceed m. Both these results follow immediately from noting that a Ferrers diagram can be read either row-wise or column-wise

For example, if $a_n = 1$ for all n, then the Euler transform b_n is the number of partitions of n into integer parts.

Euler invented a generating function which gives rise to a recurrence equation in $P(n)$,

$$P(n)=\sum_{k=1}^{n}(-1)^{k+1}[P(n-\frac{1}{2}k(3k-1))+P(n-\frac{1}{2}k(3+1))] \qquad (11)$$

Other recurrence equations include

$$P(2n+1)=P(n)+\sum_{k=1}^{\infty}[P(n-4k^2-3k)+P(n-4k^2+3k)]- \sum_{k=1}^{\infty}(-1)^k[P(2n+1-3k^2+k)+P(2n+1-3k^2-K)] \qquad (12)$$

and

$$P(n)=\frac{1}{n}\sum_{k=0}^{n-1}\sigma_1(n-k)P(k) \qquad (13)$$

where $\sigma_1(n)$ is the divisor function as well as the identity

$$\sum_{k=[-(\sqrt{24n+1}+1)/6]}^{[(\sqrt{24n+1}-1)/6]} (-1)^k P(n-\frac{1}{2}k(3k+1))=0 \qquad (14)$$

where $\lfloor x \rfloor$ is the floor function and $\lceil x \rceil$ is the ceiling function.

A recurrence relation involving the partition function Q is given by

$$P(n)=\sum_{k=0}^{[n/2]}Q(n-2k)P(k) \qquad (15)$$

$$\sum_{n=0}^{\infty}P(5n+1)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{5n})}{(1-q^{5n-4})(1-q^{5n-1})} \pmod 5 \qquad (16)$$

$$\sum_{n=0}^{\infty}P(5n+1)q^n = 2\prod_{n=1}^{\infty}\frac{(1-q^{5n})}{(1-q^{5n-3})(1-q^{5n-2})} \pmod 5 \qquad (17)$$

$$\sum_{n=0}^{\infty}P(5n+3)q^n=3\prod_{n=1}^{\infty}\frac{(1-q^{5n-4})(1-q^{5n-1})(1-q^{5n})}{(1-q^{5n-3})^2(1-q^{5n-2})^2} \pmod 5 \qquad (18)$$

$$P(n)-P(n-1)-P(n-2)+P(n-5)+P(n-7)-P(n-12)-P(n-15)+...=0 \qquad (20)$$

Notes

where the sum is over generalized pentagonal numbers $\le n$ and the sign of the kth term is $(-1)^{\lfloor (k+1)/2 \rfloor}$, as above. Ramanujan stated without proof the remarkable identities

$$\sum_{k=0}^{\infty} P(5k+4)q^k = 5\frac{(q^5)_\infty^5}{(q)_\infty^6} \tag{21}$$

and

$$\sum_{k=0}^{\infty} P(7k+5)q^k = 7\frac{(q^7)_\infty^3}{(q)_\infty^4} + 49q\frac{(q^7)_\infty^7}{(q)_\infty^8} \tag{22}$$

To obtain the asymptotic solution

$$P(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \tag{23}$$

$$P(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\, \frac{d}{dn}\left[\frac{\sinh\left(\frac{\pi}{k}\sqrt{n-\frac{1}{24}}\right)}{\sqrt{n-\frac{1}{24}}}\right] \tag{24}$$

where

$$A_k(n) = \sum_{h=1}^{k} \delta GCD(h,k),1 \exp\left[\pi i \sum_{j=1}^{k-1} \frac{j}{k}\left(\frac{hj}{k} - \left\lfloor\frac{hj}{k}\right\rfloor - \frac{1}{2}\right) - \frac{2\pi i h n}{k}\right] \tag{25}$$

$\delta_{...}$ is the Kronecker delta, and $\lfloor x \rfloor$ is the floor function. The remainder after N terms is

$$R(N) < CN^{-1/2} + D\sqrt{\frac{N}{n}} \sinh\left(\frac{K\sqrt{n}}{N}\right) \tag{26}$$

where C and D are fixed constants

To found an algebraic formula for the partition function P(n) as a finite sum of algebraic numbers as follows. Define the weight-2 meromorphic modular form F(z) by

$$F(z) \quad \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{2 \quad \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^3(6z)} \tag{27}$$

were $q = e^{2\pi i z}$, $E_2(q)$ is an Eisenstein series, and $\eta(q)$ is a Dedekind eta function. Now define

$$R(z) = -\left(\frac{1}{2\pi i}\frac{d}{dz} + \frac{1}{2\pi y}\right)F(z) \tag{28}$$

where $z = x + iy$. Additionally let Q_n be any set of representatives of the equivalence classes of the integral binary quadratic form $Q(x,y) = ax^2 + bxy + cy^2$ such that 6|a with a>0 and $b \equiv 1 \pmod{12}$, and for each $Q(x,y)$, let α_Q be the so-called CM point in the upper half-plane, for which $Q(\alpha_Q, 1) = 0$. Then

$$P(n) = \frac{Tr(n)}{24n - 1} \tag{29}$$

where the trace is defined as

$$Tr(n) = \sum_{Q \in Q_n} R(\alpha_Q) \tag{30}$$

Notes

Let $f_O(x)$ be the generating function for the number of partitions $P_O(n)$ of ^{n} containing odd numbers only and $f_D(x)$ be the generating function for the number of partitions $P_D(n)$ of ^{n}without duplication, then

$$f_O(x) = f_D(x) \quad (31)$$

$$= \prod_{k=1,3,..}^{\infty} \sum_{i=0}^{\infty} x^{ik} \quad (32)$$

$$= \frac{1}{\prod_{k=1,3,..}^{\infty} 1-x^k} \quad (33)$$

$$= \prod_{k=1}^{\infty}\left(1+x^k\right) \quad (34)$$

$$= \frac{1}{2}(-q;x)_\infty \quad (35)$$

$$= 1+x+x^2+2x^3+2x^4+3x^5+..... \quad (36)$$

The identity

$$\prod_{k=1}^{\infty}(1+z^k)=\prod_{k=1}^{\infty}(1-z^{2k-1})^{-1} \quad (37)$$

is known as the Euler identity

The generating function for the difference between the number of partitions into an even number of unequal parts and the number of partitions in an odd number of unequal parts is given by

$$\prod_{k=1}^{\infty}(1-z^k)=1-z-z^2+z^5+z^7-z^{12}-z^{15}+... \quad (38)$$

where

$$ck = \begin{cases} (-1)^n & \text{for } k \text{ of the form } \frac{1}{2}n\ (3n\pm1) \\ 0 & \text{otherwise} \end{cases} \quad (40)$$

Let $P_E(n)$ be the number of partitions of even numbers only, and let $P_{EO}(n)(P_{DO}(n))$ be the number of partitions in which the parts are all even (odd) and all different. Then the generating function of $P_{DO}(n)$ is given by

$$f_{DO}(x) = \prod_{k=1,3,..}^{\infty} 1+x^k \quad (41)$$

$$= (-x;x^2)_\infty \quad (42)$$

The number of partitions with no even part repeated is the same as the number in which no part occurs more than three times and the same as the number in which no part is divisible by 4, all of which share the generating functions

$$P_3(n) = \prod_{k=1}^{\infty} \frac{1+x^{2k}}{1-x^{2k-1}} \quad (43)$$

$$= \prod_{k=1}^{\infty}(1+x^k+x^{2k}+x^{3k}) \quad (44)$$

Notes

$$= \prod_{k=1}^{\infty} \frac{1-x^{4k}}{1-x^k} \qquad (45)$$

$$= \frac{(x^4)_\infty}{(x)_\infty} \qquad (46)$$

In general, the generating function for the number of partitions in which no part occurs more than d times is

$$P_d(n) = \prod_{k=1}^{\infty} \sum_{i=0}^{d} x^{ik} \qquad (47)$$

$$= \prod_{k=1}^{\infty} \frac{1-x^{(d+1)k}}{1-x^k} \qquad (48)$$

The generating function for the number of partitions in which every part occurs 2, 3, or 5 times is

$$P_{2,3,5}(n) = \prod_{k=1}^{\infty} (1+x^{2k}+x^{3k}+x^{5k}) \qquad (49)$$

$$= \prod_{k=1}^{\infty} (1+x^{2k})(1+x^{3k}) \qquad (50)$$

$$= \prod_{k=1}^{\infty} \frac{1-x^{4k}}{1-x^{2k}} \frac{1-x^{6k}}{1-x^{3k}} \qquad (51)$$

$$= \frac{(x^4)_\infty (x^6)_\infty}{(x^2)_\infty (x^3)_\infty} \qquad (52)$$

The first few values are 0, 1, 1, 1, 1, 3, 1, 3, 4, 4, 4, 8, 5, 9, 11, 11, 12, 20, 15, 23, ...

The number of partitions in which no part occurs exactly once is

$$P_1(n) = \prod_{k=1}^{\infty} (1+x^{2k}+x^{3k}+\ldots) \qquad (53)$$

$$= \prod_{k=1}^{\infty} \frac{1-x^k+x^{2k}}{1-x^k} \qquad (54)$$

$$= \prod_{k=1}^{\infty} \frac{1+x^{3k}}{1-x^{2k}} \qquad (55)$$

$$= \prod_{k=1}^{\infty} \frac{1-x^{6k}}{(1-x^{2k})(1-x^{3k})} \qquad (56)$$

Remark:

1. The number of partitions of n in which no even part is repeated is the same as the number of partitions of n in which no part occurs more than three times and also the same as the number of partitions in which no part is divisible by four.
2. The number of partitions of n in which no part occurs more often than d times is the same as the number of partitions in which no term is a multiple of d+1.

Notes

3. The number of partitions of n in which each part appears either 2, 3, or 5 times is the same as the number of partitions in which each part is congruent mod 12 to either 2, 3, 6, 9, or 10.
4. The number of partitions of n in which no part appears exactly once is the same as the number of partitions of n in which no part is congruent to 1 or 5 mod 6.
5. The number of partitions in which the parts are all even and different is equal to the absolute difference of the number of partitions with odd and even parts. P(n) satisfies the inequality

$$P(n) \leq \frac{1}{2}[P(n+1) + P(n-1)] \qquad (58)$$

P(n,k) denotes the number of ways of writing n as a sum of exactly k terms or, equivalently, the number of partitions into parts of which the largest is exactly k. (Note that if "exactly k" is changed to "k or fewer" and "largest is exactly k," is changed to "no element greater than k," then the partition function q is obtained.) For example, P(5,3)=2, since the partitions of 5 of length 3 are {3,1,1}and {2,2,1}, and the partitions of 5 with maximum element 3 are {3,2}and {3,1,1}.

The P(n,k) such partitions can be enumerated in the Integer Partitions

[n, {k}].

P(n,k) can be computed from the recurrence relation

$$P(n,k) = P(n-1,k-1) + P(n-k,k) \qquad (59)$$

with $P(n,k) = 0$ for $k > n, P(n,n) = 1$ and $P(n,0) = 0$. The triangle of $P(k,n)$ is given by

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 3 & 3 & 2 & 1 & 1 \end{array} \qquad (60)$$

The number of partitions of n with largest part k is the same as P(n,k).

The recurrence relation can be solved exactly to give

$$P(n,1) = 1 \qquad (61)$$

$$P(n,2) = \frac{1}{4}\left[2n-1+(-1)^n\right] \qquad (62)$$

$$P(n,3) = \frac{1}{72}\left[6n^2-7-9(-1)^n+16\cos\left(\frac{2}{3}\pi n\right)\right] \qquad (63)$$

where P(n,k)=0for $n < k$. The functions P(n,k) can also be given explicitly for the first few values of k in the simple forms

Notes

$$P(n,2) = \left[\frac{1}{2}n\right] \quad (65)$$

$$P(n,3) = \left[\frac{1}{12}n^2\right] \quad (66)$$

where $\lfloor x \rfloor$ is the floor function and $[x]$ is the nearest integer function defines

$$t_k(n) = n + \frac{1}{4}k(k-3) \quad (67)$$

and then yields

$$P(n,2) = \left[\frac{1}{2}t_2(n)\right] \quad (68)$$

$$P(n,3) = \left[\frac{1}{12}t_3^2(n)\right] \quad (69)$$

$$P(n,4) = \begin{cases} \left[\frac{1}{144}t_4^3(n) - \frac{1}{48}t_4(n)\right] \\ \left[\frac{1}{144}t_4^3(n) - \frac{1}{12}t_4(n)\right] \end{cases} \quad (70)$$

Then obtained the exact asymptotic formula

$$P(n) = \sum_{k<\alpha\sqrt{n}} P_k(n) + O(n^{-1/4}) \quad (71)$$

where α is a constant. However, the sum diverges

$$\sum_{k=1}^{\infty} P_k(n) \quad (72)$$

Counting Functions:

Combinatorics can be traced back more than 3000 years to India and China. For many centuries, it primarily comprised the solving of problems relating to the permutations and combinations of objects. The use of the word combinatorial" can be traced back to Leibniz in his dissertation on the art of combinatorial in 1666. Over the centuries, combinatorics evolved in recreational pastimes.

These include the Konigsberg bridges problem, the four-colour map problem, the Tower of Hanoi, the birthday paradox and Fibonacci's 'rabbits' problem. In the modern era, the subject has developed both in depth and variety and has cemented its position as an integral part of modern mathematics. Undoubtedly part of the reason for this importance has arisen from the growth of computer science and the increasing use of algorithmic methods for solving real-world practical problems. These have led to combinatorial applications in a wide range of subject areas, both within and outside mathematics, including network analysis, coding theory, and probability.

Addition and multiplication rules:

1) How many possible crossword puzzles are there?

Suppose we have to select 4 balls from a bag of 20 balls numbered 1 to 20.

Notes

2) How often do two of the selected balls have consecutive numbers?

How many ways are there of rearranging the letters in the word ALPHABET?

We observe various things about the above problems. A priori, unlike many problems in mathematics, there is hardly any abstract or technical language. Despite the initial simplicity, some of these problems will be frustratingly di cult to solve. Further, we notice that despite these problems appearing to being diverse and unrelated, they principally involve selecting, arranging, and counting objects of various types. We will address the problem of counting. Clearly, we would like to be able to count without actually counting.

In other words, can we find out how many things there are with a given property without actually enumerating each of them. Quite often this entails deep mathematical insight. We now introduce two standard techniques which are very useful for counting without actually counting. These techniques can easily be motivated through the following examples.

Example:

Let the cars in New Delhi have license plates containing 2 alphabets followed by two numbers. What is the total number of license plates possible?

Solution:

Here, we observe that there are 26 choices for the first alphabet and another 26 choices for the second alphabet.

After this, there are two choices for each of the two numbers in the license plate. Hence, we have a maximum of $26\times 26 \times 10 \times 10 = 67600$ license plates.

Example:

Let the cars in New Delhi have license plates containing 2 alphabets followed by two numbers with the added condition that \in the license plates that start with a vowel the sum of numbers should always be even". What is the total number of license plates possible?

Solution:

Here, we need to consider two cases.

Case 1:

The license plate doesn't start with a vowel. Then using the previous example, the number of license plates equals

$21\times 26\times 10\times 10 = 54600$.

Case 2:

The license plate starts with a vowel. Then the number of license plates equals

$5 \times 26 \times (5\times 5+5 \times 5)=6500$.

Hence, we have a maximum of $54600 + 6500 = 61100$ license plates.

Generalization of the first example leads to what is referred to as the rule of

Notes

product and that of the second leads to the rule of addition. To understand these rules, we explain the involved ideas.

Suppose we have a task to complete and that the task has some parts. Assume that each of the parts can be completed on their own and completion of one part does not result in the completion of any other part. We say the parts are compulsory to mean that each of the parts must be completed to complete the task. We say the parts are alternative to mean that exactly one of the parts must be completed to complete the task. With this setting we state the two basic rules of combinatorics.

Basic counting rule:

Let n, $m_1, \ldots m_n \in N$.

Multiplication/Product rule:

If a task consists of n compulsory parts and the i-th part can be completed in m_i ways, i = 1, 2,…..n, then the task can be completed in $m_1 m_2 \ldots\ldots m_n$ ways.

Addition rule:

If a task consists of n alternative parts, and the i-th part can be completed in m_i ways, i = 1; : : : ; n, then the task can be completed in

$m_1 + m_2 + \ldots + m_n$ ways.

Let us consider the following examples.

1. How many three digit natural numbers can be formed using digits

   0,1,….., 9?

   Identify the number of parts in the task and the type of the parts (compulsory or alternative).

Solution:

The task of forming a three digit number can be viewed as lining three boxes kept in a horizontal row. It has three compulsory parts.

Part 1:

Choose a digit for the left most place.

Part 2:

Choose a digit for the middle place.

Part 3:

Choose a digit for the rightmost place.

Apply multiplication rule. $\boxed{}\ \boxed{}\ \boxed{}$

$9 \times 10 \times 10 = 900$

Notes

Prime Counting Function:

The prime counting function is the function $\pi(x)$ giving the number of primes less than or equal to a given number ‘x’. For example, there are no primes ≤ 1, so $\pi(1) = 0$. There is a single prime (2) ≤ 2, so $\pi(2) = 1$. There are two primes (2 and 3) ≤ 3, so $\pi(3) = 2$. And so on.

The notation $\pi(n)$ for the prime counting function is slightly unfortunate because it has nothing whatsoever to do with the constant $\pi = 3.1415\ldots$. This notation was introduced by number theorist Edmund Landau in 1909 and has now become standard. In the words of Derbyshire (2004, p. 38), “I am sorry about this; it is not my fault. You’ll just have to put up with it.”

Letting p_n denote the ‘n’th prime, p_n is a right inverse of $\pi(n)$ since

$$\pi(Pn) = n \quad (1)$$

for all positive integers. Also,

$$p_{\pi(n)} = n \quad (2)$$

iff ‘n’is a prime number.

The first few values of $\pi(n)$ for $n = 1$, 2, ... are 0, 1, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 6, ... The Wolfram Language command giving the prime counting function for a number x is PrimePi[x], which works up to a maximum value of $x \approx 8 \times 10^{13}$.

The notation $\pi_{a,b}(x)$ is used to denote the modular prime counting function, i.e., the number of primesof the form $ak + b$ less than or equal to x.

One of the most fundamental and important results in number theory is the asymptotic form of $\pi(n)$ as n becomes large. This is given by the prime number theorem, which states that

$$\pi(n) \sim \mathrm{Li}(n), \quad (3)$$

where $\mathrm{Li}(x)$ is the logarithmic integral and $\sim$ is asymptotic notation. This relation was first postulated by Gauss in 1792.

Prime Counting Function:

The prime counting function can be expressed by Legendre’s formula, Lehmer’s formula, Mapes’ method, or Meissel’s formula. A brief history of attempts to calculate $\pi(n)$ is given by Berndt (1994). The following table is taken from Riesel (1994), where $O(x)$ is asymptotic notation.

An approximate formula due to illustrated above, is given by

$$\pi(n) \approx \frac{n}{h_n} \quad (4)$$

where h_n is related to the harmonic number H_n by $h_n = H_n - 3/2$. This formula is within ≈ 2 of the actual value for $50 \leq n \leq 1000$. The values for which $\pi(n) - n/h_n > 0$ are 1, 109, 113, 114, 199, 200, 201, ... This quantity is positive for all $n \geq 1429$.

An upper bound for $\pi(n)$ is given by

Notes

$$\pi(n) < \frac{1.25506\, n}{1n_n} \tag{5}$$

for $n > 1$, and a lower bound by

$$\frac{n}{1n_n} < \pi(n) \tag{6}$$

for $n \geq 17$.

It gives the formula

$$\pi(n) = -1 + \sum_{j=3}^{n}\left[(j-2)! - j\left[\frac{(j-2)!}{j}\right]\right] \tag{7}$$

for $n > 3$, where $\lfloor x \rfloor$ is the floor function.

A modified version of the prime counting function is given by

$$\pi_0(p) \equiv \begin{cases} \pi(p) & \text{for } p \text{ composite} \\ \pi(p) - \frac{1}{2} & \text{for } p \text{ rime} \end{cases} \tag{8}$$

$$\pi_0(p) \equiv \sum_{n=1}^{\infty} \frac{\mu(x) f(x^{1/n})}{n} \tag{9}$$

where $\mu(n)$ is the Möbius function and $f(x)$ is the Riemann prime counting function.

It also shows that

$$\frac{d\pi(x)}{dx} \sim \frac{1}{x1nx} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} x1/n \tag{10}$$

where $\mu(n)$ is the Möbius function.

The smallest x such that $x \geq n\,\pi(x)$ for $n = 2$, 3, ... are 2, 27, 96, 330, 1008, ..., and the corresponding $\pi(x)$ are 1, 9, 24, 66, 168, 437, ... The number of solutions of $x = n\,\pi(x)$ for $n = 2$, 3, ... are 4, 3, 3, 6, 7, 6, ...

It shows that for sufficiently large x,

$$\pi^2(x) < \frac{ex}{1nx}\pi\left(\frac{x}{e}\right) \tag{11}$$

This holds for $x = 6$, 9, 10, 12, 14, 15, 16, 18, ... (OEIS A091886). The largest known prime for which the inequality fails is 38358837677. The related inequality

$$[1i(x)]^2 < \frac{ex}{1nx} 1i\left(\frac{x}{e}\right) \tag{12}$$

where

$$li(x) \int_0^x \frac{dt}{\ln t} \tag{13}$$

is true for $x \geq 2418$ and no smaller number.

Notes

Riemann Counting function:

Riemann defined the function $f(x)$ by

$$f(x) \equiv \sum_{p^v sx} \frac{1}{v} \quad (1)$$

p prime

$$= \sum_{n=1}^{[1gx]} \frac{\pi\left(x^{1/n}\right)}{n} \quad (2)$$

$$= \pi(x)+\frac{1}{2}\pi\left(x^{1/2}\right)+\frac{1}{3}\pi\left(x^{1/3}\right)+\ldots \quad (3)$$

Amazingly, the prime counting function $\pi(x)$ is related to $f(x)$ by the Möbius transform

$$\pi(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}f(x^{1/n}) \quad (4)$$

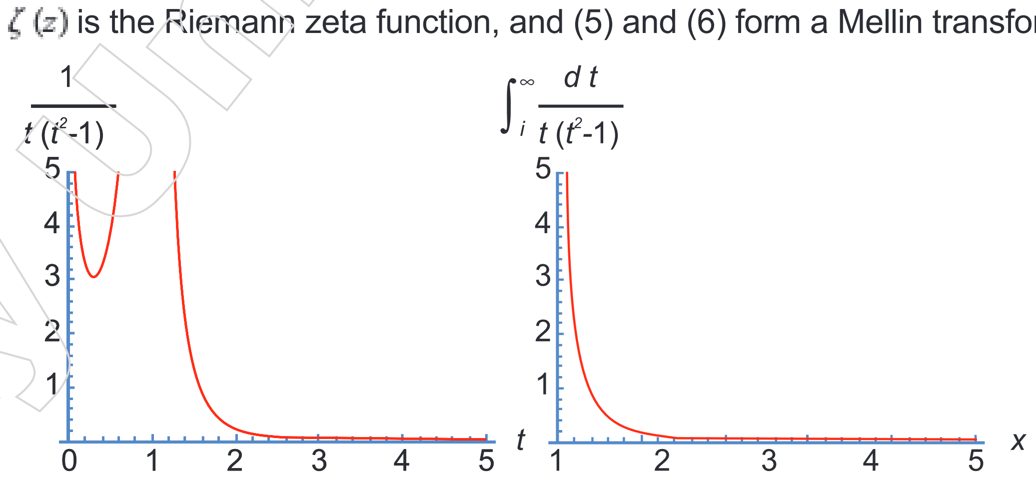
where $\mu(n)$ is the Möbius function. More amazingly still, $f(x)$ is connected with the Riemann zeta function $\zeta(s)$ by

$$\frac{1n[\zeta(s)]}{s}=\int_0^{\infty} f(x)x^{-s-1}dx \quad (5)$$

$f(x)$ is also given by

$$f(x)=\lim_{t\to\infty}\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^s}{x}1n\zeta(s)ds \quad (6)$$

where $\zeta(z)$ is the Riemann zeta function, and (5) and (6) form a Mellin transform pair.



Riemann (1859) proposed that

$$f(x)=1i(x)-\sum_{\rho}1i(x^{p})-1n2+\int_x^{\infty}\frac{dt}{t(t^2-1)1nt^5} \quad (7)$$

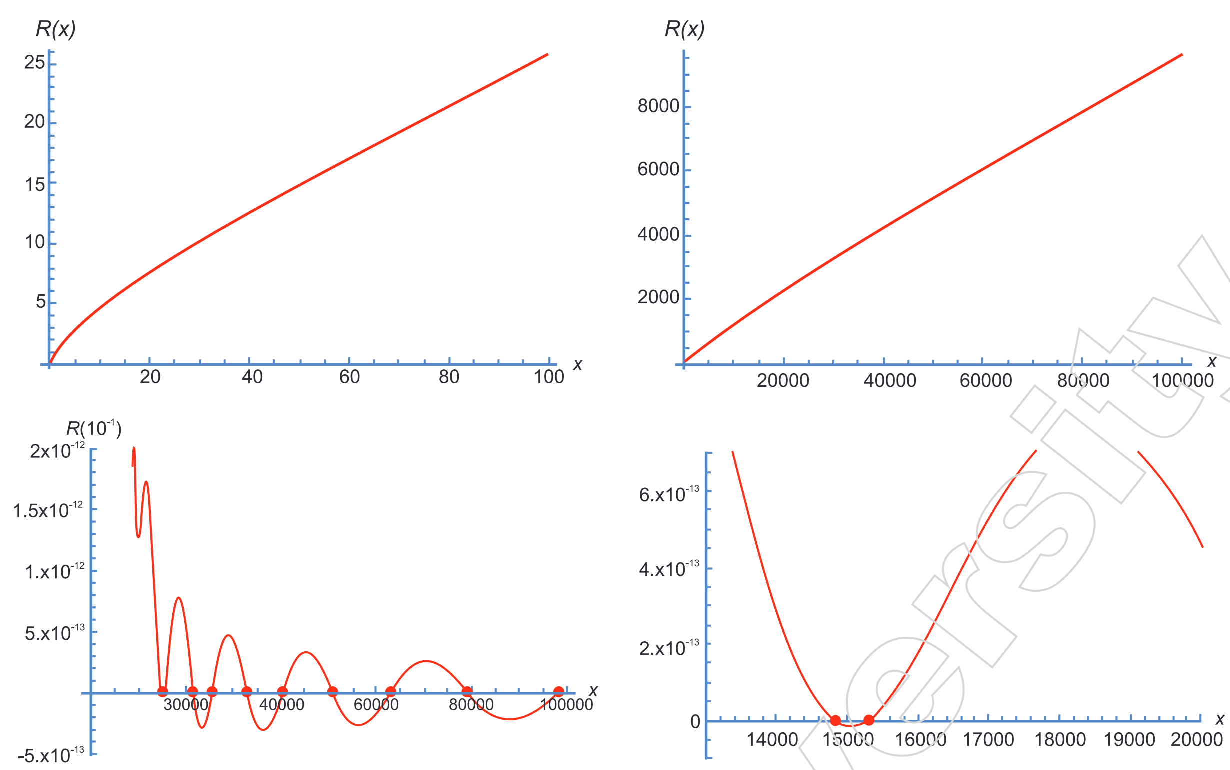
where $\text{li}(x)$ is the logarithmic integral and the sum is over all nontrivial zeros ρ of the Riemann zeta function $\zeta(z)$.

Actually, since the sum of roots is only conditionally convergent, it must be summed in order of increasing $\text{I}[\rho]$ even when pairing terms ρ with their “twins”$1-\rho$, so

$$\sum_{\rho}1i(x^{p})=\sum_{1(\rho)>0}\left[Li(x^{p})+Li(x^{1-\rho})\right] \quad (8)$$

This formula was subsequently proved. The integral on the right-hand side converges only for $x > 1$, but since there are no primes less than 2, the only values of interest are for $x \geq 2$. Since it is monotonic decreasing, the maximum therefore occurs at $x = 2$, which has value

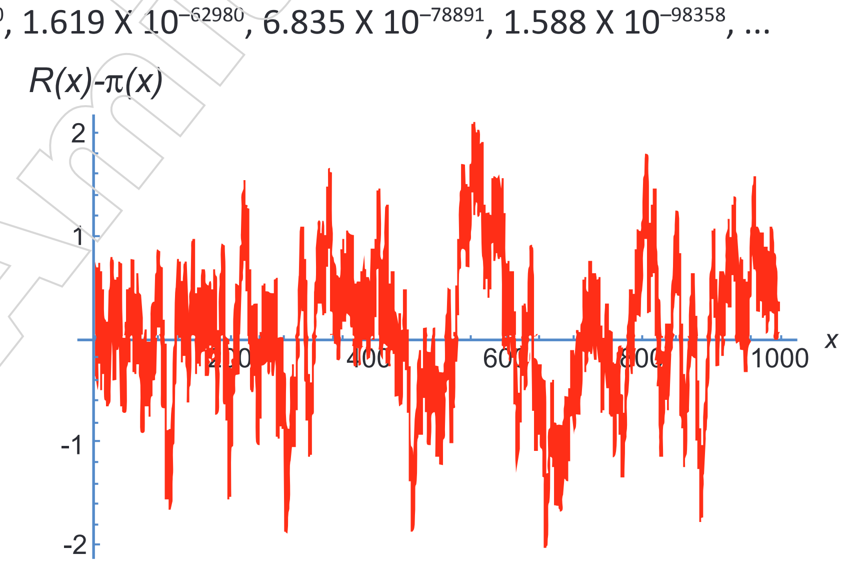
$$\int_2^\infty \frac{dt}{t\,\mathrm{ln}\,t(t^2-1)} = 0.14001010114328692668... \quad (9)$$



Riemann also considered the function

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \mathrm{li}(x^{1/n}) \quad (10)$$

Sometimes also denoted $\mathrm{Ri}(x)$ obtained by replacing $f(x^{1/n})$ in the Riemann function with the logarithmic integral $\mathrm{li}(x^{1/n})$, where ζ (z) is the Riemann zeta function and μ(n) is the Möbius function which illustrate the fact that *R*(x) has a series of zeros near the origin. These occur at 10^{-x} for x=14827.7, 15300.7, 21381.5, 25461.7, 32711.9, 40219.6, 50689.8, 62979.8, 78890.2, 98357.8, ..., corresponding to x = 1.829×10^{-14828}, 2.040×10^{-15301}, 3.289×10^{-21382}, 2.001×10^{-25462}, 1.374×10^{-32712}, 2.378×10^{-40220}, 1.420×10^{-50690}, 1.619×10^{-62980}, 6.835×10^{-78891}, 1.588×10^{-98358}, ...

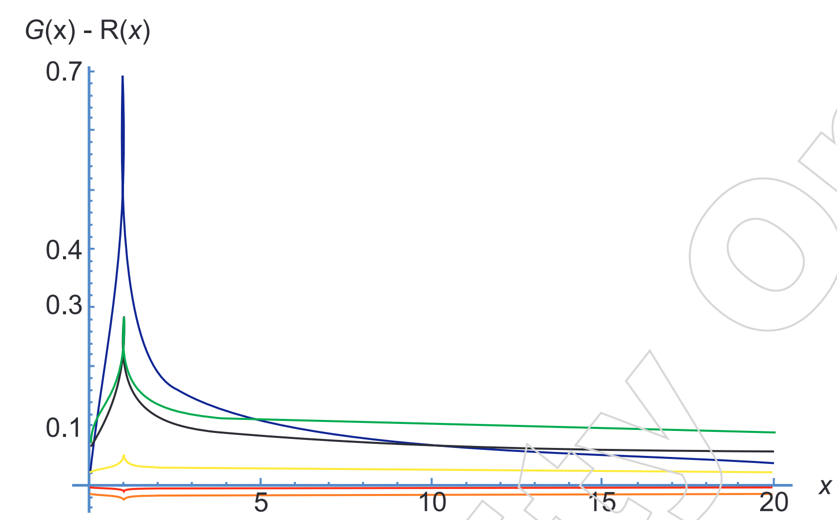


The quantity $R(x) - \pi(x)$ is plotted above.

Notes

This function is implemented in the Wolfram Language as Riemann R[x].

Ramanujan independently derived the formula for $R(n)$, but non rigorously. The following table compares $\pi(10^n)$ and $R(10^n)$ for small n. Riemann conjectured that $R(n) = \pi(n)$, but this was disproved by Littlewood in 1914.



The Riemann prime counting function is identical to the Gram series

$$G(x) = 1 + \sum_{k=1}^{\infty} \frac{(1nx)^k}{kk!\zeta(k+1)} \tag{11}$$

where $\zeta(z)$ is the Riemann zeta function, but the Gram series is much more tractable for numeric computations. For example, the plots above show the difference $G(x) - R(x)$ where $R(x)$ is computed.

Questions:

1. A partition of a positive integer, also called an ___________ partition.
   a) Rational
   b) integer
   c) both a and b
   d) none
2. Two sums that differ only in the order of their summands are considered the ________ partition.
   a) Same
   b) different
   c) both a and b
   d) none
3. A summand in a partition is also called a _________.
   a) Odd part
   b) even part
   c) part
   d) both a and b

Notes

4. Partitions can be graphically visualized with ____________.
   a) Young diagrams or Ferrers diagrams.
   b) Counting functions
   c) Necklaces
   d) Euler's function
5. If we count partitions in which no number occurs more than once. Such a partition is called a __________.
   a) Partition with distinct parts.
   b) Counting functions
   c) Necklaces
   d) Euler's function

Answer:

1. b
2. a
3. c
4. a
5. a

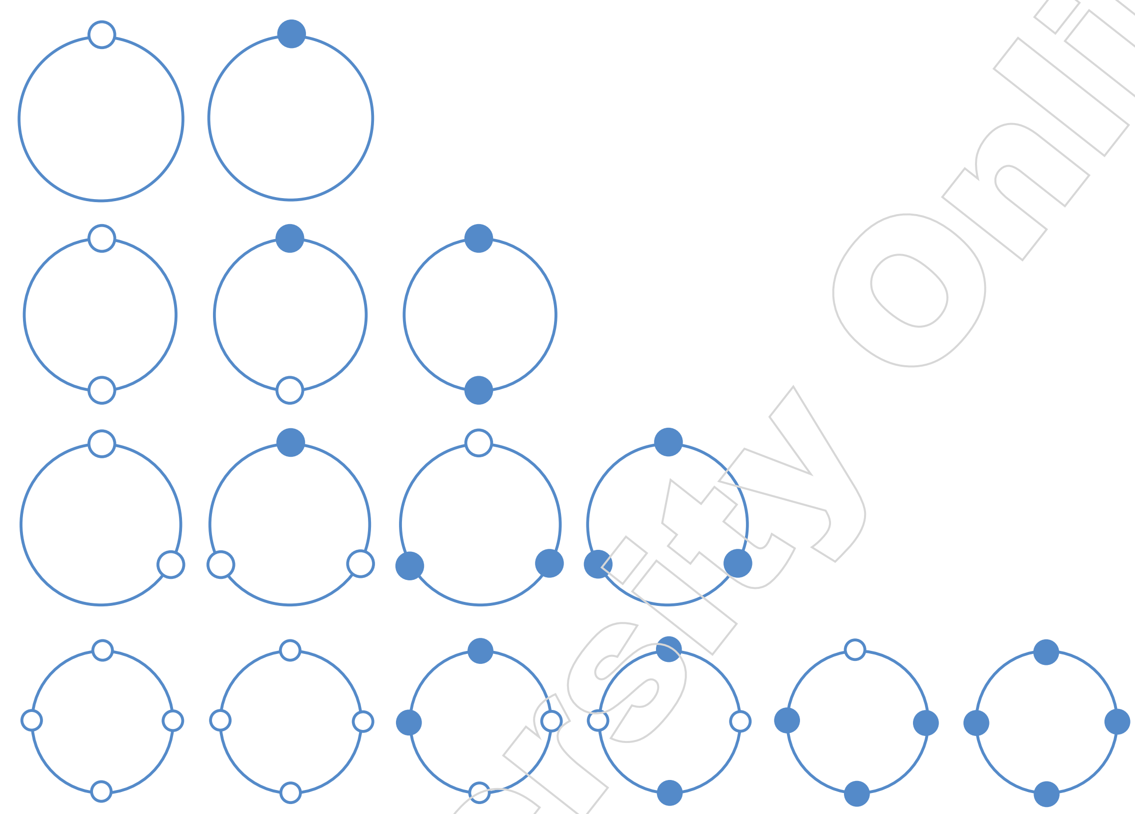
Exercises:

1. Define Partition
2. Explain Counting Functions
3. Explain generating functions with an example
4. Explain Euler's Function

Notes

Unit- VIII

Necklaces:



In the technical combinatorial sense, an a-ary necklace of length n is a string of 'n' characters, each of a possible types. Rotation is ignored, in the sense that $b_1\, b_2 \ldots b_n$ is equivalent to $b_k\, b_{k+1} \cdots b_n\, b_1\, b_2 \cdots b_{k-1}$ for any k.

In fixed necklaces, reversal of strings is respected, so they represent circular collections of beads in which the necklace may not be picked up out of the plane (i.e., opposite orientations are not considered equivalent). The number of fixed necklaces of length n composed of a types of beads $N(n, a)$ is given by

$$N(n,a) = \frac{1}{n}\sum_{i=1}^{v(n)} \phi(d_i) a\, n/d_i \qquad (1)$$

where d_i are the divisors of n with $d_1 \equiv 1,\ d_2,\ \ldots,\ d_v(n) \equiv n,\ v(n)$ is the number of divisors of n, and $\phi(x)$ is the totient function.

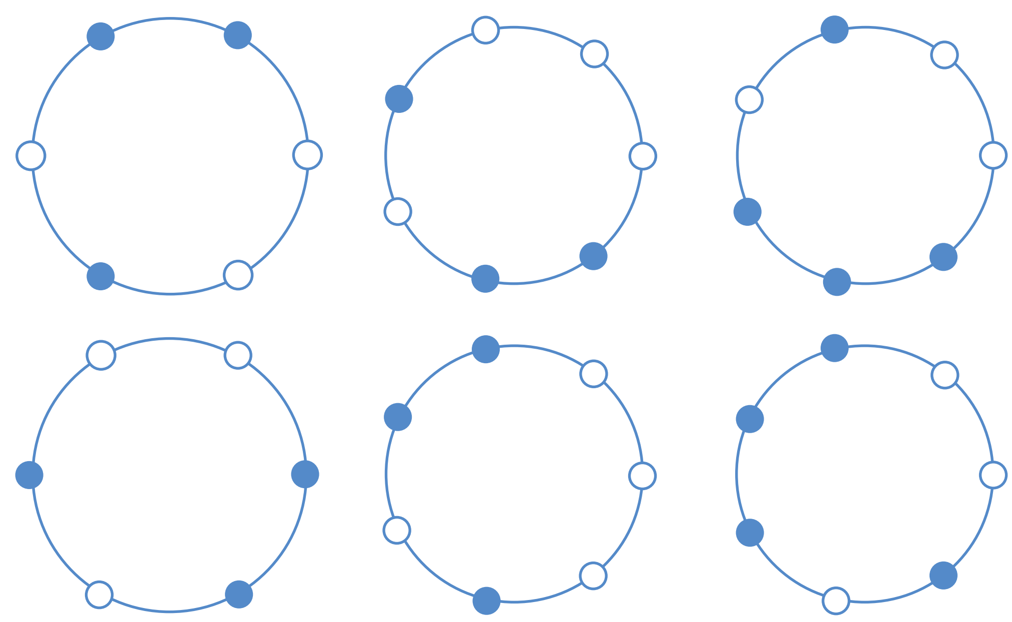
For free necklaces, opposite orientations (mirror images) are regarded as equivalent, so the necklace can be picked up out of the plane and flipped over. The number $N'(n, a)$ of such necklaces composed of n beads, each of a possible colors, is given by

$$N'(n,a) = \frac{1}{2n}\begin{cases}\sum_{i=1}^{v(n)} \phi(d_i) a^{n/d_i} + n a^{(n+1)/2} & \text{for } n \text{ odd} \\ \sum_{i=1}^{v(n)} \phi(d_i) a^{n/d_i} + \frac{1}{2} n(1+a) a^{n/2} & \text{for } n \text{ even}\end{cases} \qquad (2)$$

For $a = 2$ and $n = p$ an odd prime, this simplifies to

$$N'(\rho,2) = \frac{2^{\rho-1}-1}{\rho} + 2^{(\rho-1)/2} + 1 \qquad (3)$$

Notes



A table of the first few numbers of necklaces for $a = 2$ and $a = 3$ follows. Note that $N\ (n, 2)$ is larger than $N'\ (n, 2)$ for $n \geq 6$. For $n = 6$, the necklace 110100 is inequivalent to its mirror image 001011, accounting for the difference of 1 between $N\ (6, 2)$ and $N'\ (6, 2)$.

Similarly, the two necklaces 0010110 and 0101110 are inequivalent to their reversals, accounting for the difference of 2 between $N\ (7, 2)$ and $N'\ (7, 2)$.

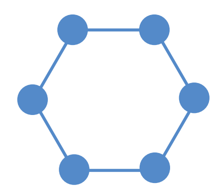
n	*N (n, 2)*	*N' (n, 2)*	*N' (n, 3)*
Sloane	**A000031**	**A000029**	**A027671**
1	2	2	3
2	3	3	6
3	4	4	10
4	6	6	21
5	8	8	39
6	14	13	92
7	60	46	1219
8	108	78	3210
9	188	126	8418
10	352	224	22913
11	632	380	62415
12	1182	687	173088
13	2192	1224	481598

Let us consider the problem of finding the number of distinct arrangements of n people in a ring such that no person has the same two neighbors two or more times. For 8 people, there are 21 such arrangements.

Example:

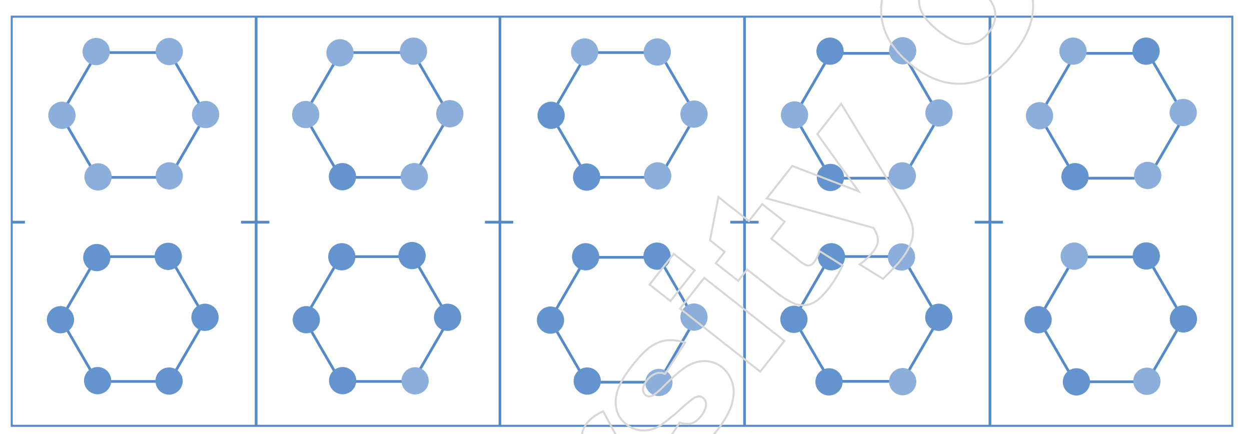
A jeweler sells six-beaded necklaces in his shop. Given that there are two different colors of beads, how many varieties of necklaces does he need to create in order to have every possible permutation of colors?

Notes



Given 6 beads we have 2 choices of color per bead, so there are 64 ways to color an unmovable beaded necklace.

However, if we are given the freedom to rotate and reflect it, there are only 13 distinct varieties. Furthermore, if the colors are interchangeable (i.e. a completely light necklace is equivalent to a completely dark necklace), then there are only 8 distinct arrangements.



The set of necklaces with 0, 1, or 2 beads of one color.

We might guess that it has something to do with symmetry. The hexagon has 12 symmetries: rotation by 0, 60, 120, 180, 240, or 300 degrees, three reactions through opposite vertices, and three reactions through opposite sides. There is no obvious relationship between the number of possibilities for a xed necklace, the number of symmetries, and the number of distinct varieties, but surely there is one. In this paper, we use combinatorics and group theory to work through the problem of the six-beaded necklace and others like it.

One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry. Many of us have an intuitive idea of symmetry, and we often think about certain shapes or patterns as being more or less symmetric than others. A square is in some sense "more symmetric" than a rectangle, which in turn is "more symmetric" than an arbitrary four-sided shape. Can we make these ideas precise? Group theory is the mathematical study of symmetry, and explores general ways of studying it in many distinct settings. Group theory ties together many of the diverse topics we have already explored – including sets, cardinality, number theory, isomorphism, and modular arithmetic.

Shapes and Symmetries:

Many people have an intuitive idea of symmetry. The shapes in Figure 38 appear symmetric, some perhaps more so than others. However, despite our general intuitions about symmetry, it may not be clear how to make this statement precise. Can it make sense to discuss "how much" symmetry a shape has? Is

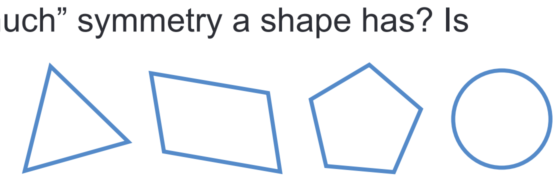
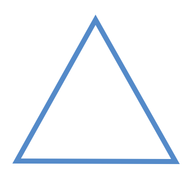


Figure 38: Some symmetric polygons.

Notes

there some way to make precise the idea that the regular pentagon is “more symmetric” than the equilateral triangle, or that the circle is “more symmetric” than any regular polygon? In this section we will explore symmetry and the way in which it arises in various contexts with which we are familiar, especially in the geometry of regular polygons (2D) and regular polyhedra (3D), such as the Platonic solids. The study of symmetry is a recurring theme in many disparate areas of modern mathematics, as well as chemistry, physics, and even economics.

To help us explore the idea of symmetry, we begin by considering a single concrete example, the equilateral triangle below.



Rotation symmetries

An equilateral triangle can be rotated by 120, 240, or 360 angles without really changing it. If you were to close your eyes, and a friend rotated the triangle by one of those angles, then after opening your eyes you would not notice that anything had changed. In contrast, if that friend rotated the triangle by 31 or 87, you would notice that the bottom edge of the triangle is no longer perfectly horizontal.

Many other shapes that are not regular polygons also have rotational symmetries. The shapes illustrated in Figure 39, for example, each have rotational symmetries. The first example can be rotated only 180, or else 360 or 0.

Figure 39: Several shapes with rotational symmetries.

The third shape can be rotated any integer multiple of 90. The fourth shape can be rotated any integer multiple of 72. The fifth shape can be rotated any integer multiple of 60.

More generally, we say that a shape has rotational symmetry of order n if it can be rotated by any multiple of 360 /n without changing its appearance. We can imagine constructing other shapes with rotational symmetries of arbitrary order. If the only rotations that leaves a shape unchanged are multiples of 360, then we say that the shape has only the trivial (order n = 1) symmetry.

Mirror reflection symmetries:

Another type of symmetry that we can find in two-dimensional geometric shapes is mirror reflection symmetry. More specifically, we can draw a line through some shapes and reflect the shape through this line without changing its appearance. This is called a mirror reflection symmetry.

Further consideration of the equilateral triangle (cf. Figure 40) shows that there are actually three distinct mirror lines through which we can reflect the shape without changing its appearance.

Notes

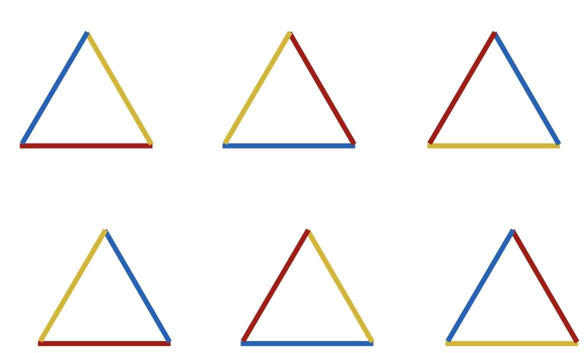
If we were to reflect the triangle through any other line, the shape as a whole would look different.

Counting symmetries:

One way in which we can quantify the “amount” of symmetry of an object is by counting its number of symmetries. For example, we might count the number of rotational symmetries of an object, along with its mirror reflection symmetries. However, counting the symmetries of a shape can be challenging. It is not immediately clear which symmetries we should count and which, if any, we should not count. To understand why we might not count certain symmetries, consider rotating the equilateral triangle by 120, 240 and 360. Of course the numbers by which we are rotating the triangle are different, and so we might be inclined to count each of them separately. But notice that we can also rotate the triangle by 480, 600 and 720. Should we count those as different symmetries? If we do count them, then what would stop us from counting an infinite number of rotational symmetries for a triangle, or for that matter, any shape?

One way to limit the number of symmetries we count involves coloring, or otherwise labeling, the shape. For example, we can color each edge of the equilateral triangle, as illustrated in Figure 42. Symmetries can then be captured as changes of colors that leave the uncolored shape fixed. Any triangle in either row can be obtained from any other triangle in that row through a rotation; triangles can be obtained from triangles in the other row through reflections.

Using this coloring allows us to count symmetries carefully. If changing the shape in two different ways results in the same coloring, then we should count those two symmetries as the same. For example, rotating the equilateral triangle by 120 or 480 results in the same coloring, so we count those as the same symmetry. Likewise, rotating the triangle by 0 and 360 also result in Figure 42: Equilateral triangle with edges colored.

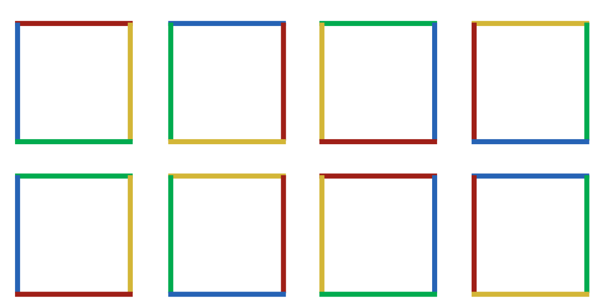


Any triangle in either row can be obtained from another triangle in the row through a rotation; triangles can be obtained from triangles in the other row through reflections. The same coloring, so we count those the same as well. To reduce confusion, we use a number between 0 and 360 (not including 360 itself) to describe the angle of a rotation; thus, we prefer 120 to 480, despite their equivalence rotations, or “doing nothing” to 360 rotations, despite their equivalence. We are therefore left with six symmetries of the triangle – the rotations (0, 120, and 240), and three reflections, one for each of the mirror planes passing through a corner and the center of the triangle. These symmetries can be pictured by how they transform the colored triangle.

Notes

Symmetries of the square:

A square is in some sense “more symmetric” than a triangle because it has more symmetries. Figure 43 below shows a square with colored edges arranged in different ways. Again you might notice that any two squares in the same row can be obtained from one another through rotations, whereas those in distinct rows can only be obtained from one another through a reflection. Some thought will show that there are no other rotations or mirror reflection symmetries, and so these figures represent all eight symmetries of the square.



Squares

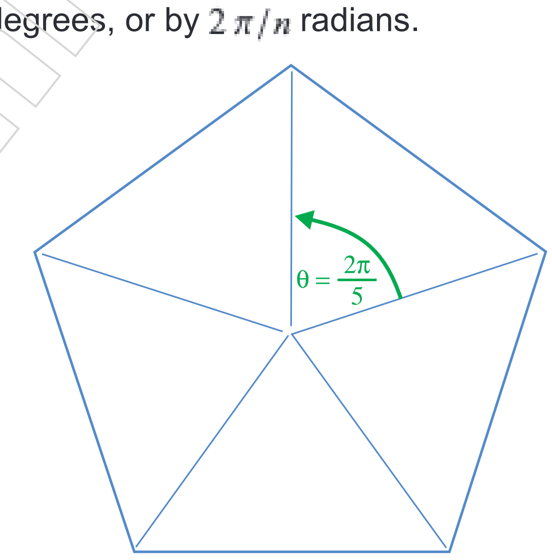
Although this section is concerned primarily with rotational and mirror re-flection symmetries of single objects in two dimensions, other types of symmetries arise in infinite systems and in higher dimensions.

Rotational Symmetry:

Abstractly, a spatial configuration $\mathcal{F}$ is said to possess rotational symmetry if $\mathcal{F}$ remains invariant under the group $C = C(\mathcal{F})$. Here, $C(\mathcal{F})$ denotes the group of rotations of F and is viewed as a subgroup of the automorphism group $\Gamma(\mathcal{F})$ of all automorphisms which leave $\mathcal{F}$ unchanged. A more intuitive definition of rotational symmetry comes from the case of planar figures in Cartesian space.

For ‘d’ arbitrary, a geometric object in $\mathbb{R}^d$ is said to possess rotational symmetry if there exists a point so that the object, when rotated a certain number of degrees (or radians) about said point, looks precisely the same as it did originally. This notion can be made more precise by counting the number of distinct ways the object can be rotated to look like itself; this number n is called the degree or the order of the symmetry.

Rotational symmetry of degree n corresponds to a plane figure being the same when rotated by $360/n$ degrees, or by $2\pi/n$ radians.



Notes

The regular pentagon in the figure above has a rotational symmetry of order 5 due to the fact that rotating it about the center point by $\alpha = 2\pi n/5$ radians, *n* = 0, 1, 2, 3, 4, yields the exact same figure. This is a particular example of a more general fact, namely that any regular planar n-gon has rotational symmetry of order n. In the case of the regular planar n-gon, the collection of all such symmetries is a group denoted by C_n, is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of integersmodulo$n$, and is a proper subgroup of the dihedral group D_n of all symmetries--rotational and otherwise--of the figure.

By the definition given above, rotational symmetry of degree 1 corresponds to an object having symmetry about a point only when rotated by $360/1 = 360$degrees; clearly, this condition is satisfied only by objects which have no symmetry, i.e., those objects whose rotational symmetry group is trivial. Therefore, the simplest possible rotational symmetry is of order 2 and is possessed, e.g., by planar parallelograms.

In some literature, rotational symmetry of order $n$is defined by classifying the results of rotating a figure about a lineℓrather than about a point (Weyl 1982). In particular, such sources define a figure to have rotational symmetry of order $n$if the figure which remains identical after a $2\pi/n$-radian rotation about ℓ(which is called the axis of rotation). These two perspectives yield the same result, however; for example, in the figure above, the $2\pi/5$-radian clockwise rotation of the pentagon about its center point can equivalently be viewed as a $2\pi/5$-radian clockwise rotation about the segment/line determined by the center point and the top right vertex.

In the $(d+1)$-dimensional Cartesian space $\mathbb{R}^{d+1}$, the d-sphereS^d has complete rotational symmetry in that its shape remains identical after any α-radian rotation about any line ℓ. Historically, this fact led some ancient civilizations to consider the circle and/or the sphere to be divine (Weyl 1982).

In addition to being a well-studied concept mathematically, rotational symmetry is also a far-reaching notion due to the prevalence of such symmetry among many naturally-occurring objects including snowflakes, crystals, and flowers.

Knot Symmetry:

A symmetry of a knot $K$is a homeomorphism of $\mathbb{R}^3$which maps Konto itself. More succinctly, a knot symmetry is a homeomorphism of the pair of spaces $(\mathbb{R}^3, K)$. Hoste et al. (1998) consider four types of symmetry based on whether the symmetry preserves or reverses orienting of $\mathbb{R}^3$and K,

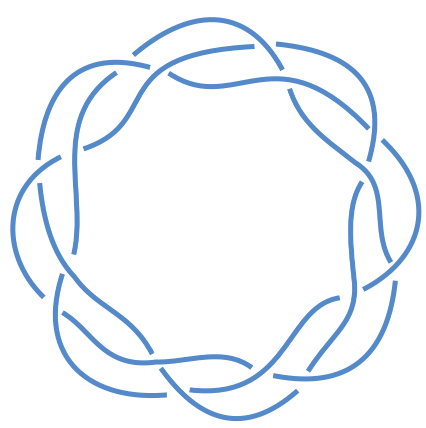
1. preserves $\mathbb{R}^3$, preserves K(identity operation),
2. preserves $\mathbb{R}^3$, reverses K,
3. reverses $\mathbb{R}^3$, preserves K,
4. reverses $\mathbb{R}^3$, reverses K.

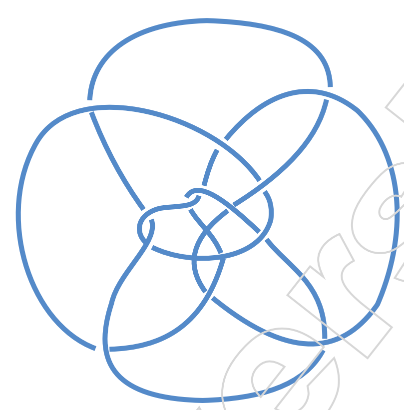
This then gives the five possible classes of symmetry summarized in the table below.

Notes

class	symmetries	knot symmetries
c	1	chiral, noninvertible
+	1, 3	+ amphichiral, noninvertible
–	1, 4	– amphichiral, noninvertible
i	1, 2	chiral, invertible
a	1, 2, 3, 4	+ and – amphichiral, invertible

In the case of hyperbolic knots, the symmetry group must be finite and either cyclic or dihedral (Riley 1979, Kodama and Sakuma 1992, Hoste et al. 1998). The classification is slightly more complicated for nonhyperbolic knots. Furthermore, all knots with ≤ 8 crossings are either amphichiral or invertible (Hoste et al. 1998). Any symmetry of a prime alternating link must be visible up to flypes in any alternating diagram of the link.


D_{16}


D_9

The following tables (Hoste et al. 1998) give the numbers of n-crossing knots belonging to cyclic symmetry groups Z_k (Sloane's A052411 for Z_1 and A052412 for) and dihedral symmetry groups D_k (Sloane's A052415 through A052422). Of knots with 16 or fewer crossings, there are only one each having symmetry groups Z_3, D_{14}, and D_{16} (above left). There are only two knots with symmetry group D_9, one hyperbolic (above right), and one a satellite knot. In addition, there are 2, 4, and 10 satellite knots having 14-, 15-, and 16-crossings, respectively, which belong to the dihedral group D_∞.

n	Z_1	Z_2	Z_3	Z_4
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	2	0	0	0
8	24	3	0	0
9	173	14	0	0
10	1047	57	0	0
11	6709	210	0	0
12	37177	712	0	2
13	224311	2268	1	0
14	1301492	7011	0	11

Notes

n	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{14}	D_{16}
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0	0	0	0	0
5	0	4	0	2	0	0	0	0	0	0	0	0
6	4	12	0	3	0	0	0	1	0	0	0	0
7	13	23	3	4	0	3	0	0	0	0	0	0
8	66	62	1	5	0	1	0	0	0	1	0	0
9	217	134	2	11	0	0	0	0	0	0	0	0
10	728	309	6	18	0	8	1	2	0	0	0	0
11	2391	647	1	21	2	3	1	2	0	0	0	0
12	7575	1463	4	31	2	2	0	0	0	0	1	0
15	23517	3065	50	53	3	12	0	2	1	4	0	0
16	73263	6791	15	89	0	10	1	8	1	1	0	1

Euler's Function:

Zn, it is frequently fruitful to ask whether something comparable applies to Un. Here we look at Un in the context of the previous section. To aid the investigation, we introduce a new quantity, the **Euler phi function**, written $\phi(n)$, for positive integers n

Definition:

$\phi(n)$ is the number of non-negative integers less than n that are relatively prime to n. In other words, if $n>1$ then $\phi(n)$ is the number of elements in Un, and $\phi(1)=1$

Example:

$\phi(2)=1$, $\phi(4)=2$, $\phi(12)=4$ and $\phi(15)=8$

Example:

If p is a prime, then $\phi(p)=p-1$, because 1, 2, …, $p-1$ are all relatively prime to p, and 0 is not.

For any number n, $\phi(n)$ turns out to have a remarkably simple form; that is, there is a simple formula that gives the value of $\phi(n)$. We've already seen how simple it is for primes. As is typical of many results in number theory, we will work our way gradually to any n,looking next at powers of a single prime.

Euler's φ function:

φ (n) is the number of integers m $\in$ [1,n] with m coprime to n.

Or, it is the order of the unit group of the ring **Z**/n**Z.**

Notes

Euler:

If a is coprime to n, then

$$a^{\varphi(n)} \equiv 1 (\mathrm{mod}\, n)$$

Euler's theorem is the basis of the RSA Cryptosystem:

If integers E,D satisfy $ED \equiv 1$ (mod φ(n)), then

$a^{ED} \equiv$ a (mod n),

For every integer a coprime to n, (In fact, this holds for all integers a if n is square free, such as the product of two different large primes.)

Encrypt message "a" : $b \equiv a^E$ (mod n),

Decrypt: $a = b^D$(mod n).

To encrypt, one should know E ,n. To decrypt, D as well.

Note that it is easy to compute a^E (mod n) given a, E, n and similarly it is easy to compute b^D(mod n).

Further, given φ (n), it is easy to come up with pairs E,D with

$ED \equiv 1$ (mod φ (n)). Indeed, keep choosing random numbers D until one is found that is coprime to φ (n), and then use Euclid's algorithm to find E.

As a public-key system, E, n are released to the public, but D is kept secret. Then anyone can send you encrypted messages that only you can read.

The security of the RSA Cryptosystem is connected to our ability to compute φ (n):

For p prime, $\varphi(p) = p-1$; more generally,

$\varphi(p) = p^k - p^{k-1} = p^k(1-1/p)$.

By the **Chinese Remainder Theorem,**

$$\varphi(n) = n \prod_{\substack{p\,prime \\ p / n}} (1 - 1/p).$$

So, knowing the prime factorization of n, one can compute φ (n) rapidly—in deterministic polynomial time.

What about the converse? Given n and φ (n) can one compute the prime factorization of n in deterministic polynomial time?

Yes, if $n=pq$, where p ,q are different primes, since

$\varphi(n) = (p-1)(q-1)$, so that $n-1-\varphi(n) = p + q$.

In general, the Extended Riemann Hypothesis implies that there is a deterministic, polynomial time algorithm to compute the prime factorization of n, given n and φ (n).

A random polynomial time algorithm to get a nontrivial factorization:

We may assume $\varphi(n) < n-1$ and that n and φ (n) are coprime.

Write $\varphi(n) = 2^s m$, where m is odd.

Notes

Choose a at random from [1,n-1]. We may assume a and n are coprime.

Note that,

$a^{\phi(n)} - 1 \equiv (a^{m} - 1)(a^{m} + 1)(a^{2m} + 1)...(a^{2s-1} + 1)$

Theorem:

If *p*is a prime and *a* is a positive integer, then

$\phi(p^a)=p^a-p^a-1$

Proof:

The number of non-negative integers less than $n=p^a$ that are relatively prime to *n*. As in many cases, it turns out to be easier to calculate the number that are *not* relatively prime to *n*, and subtract from the total. List the non-negative integers less than *pa*: 0, 1, 2, …, p^{a-1}; there are p^a of them.

The numbers that have a common factor with p^a (namely, the ones that are not relatively prime to *n*) are the multiples of *p*: 0, *p*, 2*p*, …, that is, every *p* th number. There are thus $p^{a/p=pa-1}$ numbers in this list,

so $\phi(p^a)=p^a-p^a-1$

Example:

$\phi(32)=32-16=16$,

$\phi(125)=125-25=100$

Example:

Since

U20={[1],[3],[7],[9],[11],[13],[17],[19]},

U4={[1],[3]},

U5={[1],[2],[3],[4]},

both U20 and U4×U5 have 8 elements.

In fact, the correspondence discussed in the Chinese Remainder Theorem between Z20 and Z4×Z5 is also a 1-1 correspondence between U20 and U4×U5:

[1] ↔ ([1],[1])

[3] ↔ ([3],[3])

[7] ↔ ([3],[2])

[9] ↔ ([1],[4])

[11] ↔ ([3],[1])

[13] ↔([1],[3])

[17] ↔([1],[2])

[19]↔ ([3],[4])

Notes

Theorem:

If a and b are relatively prime and $n=ab$, then $\phi(n)=\phi(a)\phi(b)$

Proof.

We want to prove that $|U_n|=|U_a|\cdot|U_b|$

As indicated in the example, we will actually prove more, by exhibiting a one to one correspondence between the elements of U_n and $U_a\times U_b$.

We already have a one to one correspondence between the elements of Z_n and $Z_a\times Z_b$. Again as indicated by the example, we just have to prove that this same correspondence works for U_n and $U_a\times U_b$.

That is, we already know how to associate any $[x]$ with a pair $([x],[x])$;

we just need to know that $[x]\in U_n$ if and only if $([x],[x])\in U_a\times U_b$.

$\Rightarrow [x]$ is in U_n if and only if $(x,n)=1$ if and only if $(x,a)=1$ and $(x,b)=1$ if and only if $([x],[x])\in U_a\times U_b$

Euler's function on average

The chance that two random integers are both divide by the prime p is $1/p^2$.

Should be

$$\alpha := \prod_{p prime}(1-1/p^2).$$

We have

$$1-1/p^2 = \left(\frac{p^2}{p^2-1}\right)^{-1} = \left(1+\frac{1}{p^2}+\frac{1}{p^2}+\ldots\right)^{-1}$$

So that

$$\alpha = \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Thus the probability that two random integers are coprime is $\frac{6}{\pi^2}$.

Another interpretation: choose a lattice point (m, n) at random, the probability that it is "visible" is $\frac{6}{\pi^2}$.

Using these thoughts it is easy to see that

$$\sum_{n\le x}\varphi(n) \approx \frac{3}{\pi^2}x^2$$

As $x \longrightarrow \infty$.

Extreme values of φ (n) are also fairly easy:

$$\limsup \frac{\varphi(n)}{n} = 1,$$

Notes

$$\liminf \frac{\varphi(\mathrm{n})}{\mathrm{n}} = 0,$$

in fact,

$$\liminf \frac{\varphi(n)}{n/\log\log n} = e^{-\gamma}$$

Clearly $\frac{\varphi(\mathrm{n})}{\mathrm{n}}$ jumps around a bit: if n has only large prime factors, the ratio is close to 1, but if n has many small prime factors, it is close to 0.

One can ask for a "distribution function". That is, let u be a real variable in [0,1] and consider.

$$\{\mathrm{n}: \frac{\varphi(\mathrm{n})}{\mathrm{n}} /\mathrm{n} \leq \mathrm{u}\}.$$

Euler's Totient Function:

The totient function $\phi(n)$, also called Euler's totient function, is defined as the number of positive integers $\leq n$ that are relatively prime to (i.e., do not contain any factor in common with) n, where 1 is counted as being relatively prime to all numbers.

Since a number less than or equal to and relatively prime to a given number is called a totative, the totient function $\phi(n)$ can be simply defined as the number of totatives of n. For example, there are eight totatives of 24 (1, 5, 7, 11, 13, 17, 19, and 23), so $\phi(24) = 8$.

The totient function is implemented in the Euler Phi[n].

The number $n - \phi(n)$ is called the cototient of n and gives the number of positive integers $\leq n$ that have at least one prime factor in common with n.

$\phi(n)$ is always even for $n \geq 3$. By convention, $\phi(0) = 1$, although the Euler Phi[0] equal to 0 for consistency with its Factor Integer[0] command. The first few values of $\phi(n)$ for $n = 1$, 2, ... are 1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, ... The totient function is given by the Möbius transform of 1, 2, 3, 4, ... $\phi(n)$ is plotted above for small n.

For a prime p,

$\phi(p) = p - 1,$

Since all numbers less than p are relatively prime to p. If $m = p^{\alpha}$ is a power of a prime, then the numbers that have a common factor with m are the multiples of p: p, $2p$, ..., $(p^{\alpha-1})p$. There are $p^{\alpha-1}$ of these multiples, so the number of factors relatively prime to p^{α} is

$$\phi(p^{\alpha}) = p^{\alpha} - p^{\alpha-1} \quad (2)$$

$$= p^{\alpha-1}(p-1) \quad (3)$$

$$= p^{\alpha}\left(1-\frac{1}{p}\right) \quad (4)$$

Notes

Now take a general m divisible by p. Let $\phi_p(m)$ be the number of positive integers $\leq m$ not divisible by p. As before, p, $2p$, ..., $(m/p)p$ have common factors, so

$$\phi_p(m) = m - \frac{m}{p} \tag{5}$$

$$= m\left(1 - \frac{1}{p}\right) \tag{6}$$

Now let q be some other prime dividing m. The integers divisible by q are q, $2q$, ..., $(m/q)q$. But these duplicate pq, $2pq$, ..., $(m/(pq))pq$.

So the number of terms that must be subtracted from ϕ_p to obtain ϕ_{pq} is

$$\Delta\phi_q(m) = \frac{m}{q} - \frac{m}{pq} \tag{7}$$

$$= \frac{m}{q}\left(1 - \frac{1}{p}\right) \tag{8}$$

and

$$\phi_{pq}(m) = \phi_p(m) - \Delta\phi_q(m) \tag{9}$$

$$= m\left(1 - \frac{1}{p}\right) - \frac{m}{q}\left(1 - \frac{1}{p}\right) \tag{10}$$

$$= m\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) \tag{11}$$

By induction, the general case is then

$$\phi(n) = n\prod_{p/n}\left(1 - \frac{1}{p}\right) \tag{12}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)....\left(1 - \frac{1}{p_r}\right) \tag{13}$$

where the product runs over all primes p dividing n. An interesting identity relating $\phi(n^k)$ to $\phi(n)$ is given by

$$\phi(n^k) = n^{k-1}\phi(n) \tag{14}$$

Another identity relates the divisors d of n to n via

$$\sum_{d/n}\phi(d) = n \tag{15}$$

The totient function is connected to the Möbius function $\mu(n)$ through the sum

$$\sum_{d} d\,\mu\left(\frac{n}{d}\right) = \phi(n) \tag{16}$$

Notes

where the sum is over the divisors of n, which can be proven by induction on n and the fact that $\mu(n)$ and $\phi(n)$ are multiplicative.

The totient function has the Dirichlet generating function

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \tag{17}$$

The totient function satisfies the inequality

$$\phi(n) \geq \sqrt{n} \tag{18}$$

for all n except $n = 2$ and $n = 6$. Therefore, the only values of n for which $\phi(n) = 2$ are $n = 3$, 4, and 6. In addition, for composite n,

$$\phi(n) \leq n - \sqrt{n} \tag{19}$$

also satisfies

$$\liminf_{n\to\infty} \phi(n) \frac{\ln \ln n}{n} = e^{-y} \tag{20}$$

where γ is the Euler-Mascheroni constant. The values of n for which $\phi(n) < e^{-\gamma} n / (\ln \ln n)$ are given by 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 22, ..

The divisor function satisfies the congruence

$$n\sigma(n) \quad = \quad 2(\operatorname{mod}\phi(n)) \tag{21}$$

$$= \begin{cases} 0(\operatorname{mod}\phi(n)) if\, \phi(n)=2 \\ 2(\operatorname{mod}\phi(n)) otherwise \end{cases} \tag{22}$$

for all primes $p \geq 5$ and no composite with the exception of 4, 6, and 22, where $\sigma(n)$ is the divisor function. No composite solution is currently known to

$$n - 1 \equiv 0(\operatorname{mod} \phi(n)) \tag{23}$$

A corollary of the Zsigmondy theorem leads to the following congruence,

$$\phi(a^n + b^n) \equiv 0(\operatorname{mod} n) \tag{24}$$

The first few 'n' for which

$$\phi(n) = \phi(n+1) \tag{25}$$

are given by 1, 3, 15, 104, 164, 194, 255, 495, 584, 975, ... which have common values $\phi(n) = 1$, 2, 8, 48, 80, 96, 128, 240, 288, 480, ...

The only $n < 10^{10}$ for which

$$\phi(n) = \phi(n+1) = \phi(n+2) \tag{26}$$

is n=5186, giving

$$\phi(5186) = \phi(5187) = \phi(5188) - 2^5 3^4 \tag{27}$$

Values of $\phi(n)$ shared among n that are close together include

$$\phi(25930) \quad = \quad \phi(25935) = \phi(25940) = \phi(25942) \tag{28}$$

$$= \quad 2^7 3^4 \tag{29}$$

Notes

$$\phi(404471) = \phi(404473) = \phi(404477) \quad (30)$$

$$= 2^8 3^2 5^2 7 \quad (31)$$

then for every positive integer m, there are primes p and q such that

$$\phi(p) + \phi(q) = 2m \quad = \quad (32)$$

$$\phi(\sigma(n)) = n \quad (33)$$

where $\sigma(n)$ is the divisor function.

Generating Functions:

A generating function $f(x)$ is a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (1)$$

Whose coefficients give the sequence $\{a_0, a_1, \ldots\}$.

Given a sequence of terms, Find Generating Function [{a1, a2, ...}, x] attempts to find a simple generating function in x whose nth coefficient is a_n.

Given a generating function, the analytic expression for the nth term in the corresponding series can be computing using Series Coefficient [expr, {x, x0, n}]. The generating function $f(x)$ is sometimes said to "enumerate".

Generating functions giving the first few powers of the nonnegative integers are given in the following table.

n^p	$f(x)$	series
1	$\frac{x}{1-x}$	$x + x^2 + x^3 + \ldots$
n	$\frac{x}{(1-x)^2}$	$x + 2x^2 + 3x^3 + 4x^4 + \ldots$
n^2	$n^2 \frac{x(x+1)}{(1-x)^3}$	$x + 4x^2 + 9x^3 + 16x^4 + \ldots$
n^3	$n^3 \frac{x(x^2+4x+1)}{(1-x)^4}$	$x + 8x^2 + 27x^3 + \ldots$
n^4	$n^4 \frac{x(x+1)(x^2+10x+1)}{(1-x)^5}$	$x + 16x^2 + 81x^3 + \ldots$

There are many beautiful generating functions for special functions in number theory. A few particularly nice examples are

$$f(x) = \frac{1}{(x)_\infty} \quad (2)$$

Notes

$$= \sum_{n=0}^{\infty} P(n)x^n \quad (3)$$

$$= 1 + x + 2x^2 + 3x^3 + ... \quad (4)$$

for the partition function P, where $(q)_\infty$ is a q-Pochhammer symbol, and

$$f(x) = \frac{x}{1 - x - x^2} \quad (5)$$

$$= \sum_{n=0}^{\infty} F_n x^n \quad (6)$$

$$= x + x^2 + 2x^3 + 3x^4 + ... \quad (7)$$

for the Fibonacci numbers F_n. Generating functions are very useful in combinatorial enumeration problems.

For example, the subset sum problem, which asks the number of ways $c_{m,s}$ to select m out of M given integers such that their sum equals s, can be solved using generating functions.

The generating function of $G(t)$ of a sequence of numbers $f(n)$ is given by the Z-transform of $f(n)$ in the variable $1/t$

Enumeration in the odd and even codes:

To enumerate a set of objects satisfying some set of properties means to explicitly produce a listing of all such objects. The problem of determining or counting all such solutions is known as the enumeration problem.

A generating function

$$F(x) = \sum_n a_n x^n$$

is said to enumerate

Even Codes:

An even number is an integer of the form $n = 2k$, where k is an integer. The even numbers are therefore ..., -4, -2, 0, 2, 4, 6, 8, 10, ...Since the even numbers are integrally divisible by two, the congruence $n \equiv 0 \pmod 2$ holds for even n. An even number n for which $n \equiv 2 \pmod 4$ also holds is called a singly even number, while an even number n for which $n \equiv 0 \pmod 4$ is called a doubly even number. An integer which is not even is called an odd number.

The oddness of a number is called its parity, so an odd number has parity 1, while an even number has parity 0.

The generating function of the even numbers is

$$\frac{2x}{(x-1)^2} = 2x + 4x^2 + 6x^3 + 8x^4 + \dots.$$

The product of an even number and an odd number is always even, as can be seen by writing

(2*k*)(2*l*+1) = 2 [*k*(2*l*+1)],

which is divisible by 2 and hence is even.

Parity:

The parity of an integer is its attribute of being even or odd. Thus, it can be said that 6 and 14 have the same parity (since both are even), whereas 7 and 12 have opposite parity (since 7 is odd and 12 is even).

A different type of parity of an integer n is defined as the sum $s_2(n)$ of the bits in binary representation, i.e., the digit count $N_1(n)$, computed modulo 2. So, for example, the number $10 = 1010_2$ has two 1s in its binary representation and hence has parity 2 (mod 2), or 0. The parities of the first few integers (starting with 0) are therefore 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, ... as summarized in the following table.

N	binary	parity	N	binary	parity
1	1	1	11	1011	1
2	10	1	12	1100	0
3	11	0	13	1101	1
4	100	1	14	1110	1
5	101	0	15	1111	0
6	110	0	16	10000	1
7	111	1	17	10001	0
8	1000	1	18	10010	0
9	1001	0	19	10011	1
10	1010	0	20	10100	0

A generating function for parity is given by

$$\frac{1}{2}\left(\frac{1}{1-x} - \prod_{k=0}^{\infty} 1 - x^{2^k}\right) = x + x^2 + x^4 + x^7 + \dots. \qquad (1)$$

The constant generated by interpreting the sequence of parity digits as a binary fraction $0.0110100011\ldots_2$ is called the Thue-Morse constant.

The parity function obeys the sum identity

$$\sum_{k=0}^{2^{n+1}} (-1)^{P(k)}(k+r)^n = 0 \qquad (2)$$

for any n. For example, for $n = 2$ and $r = 0$,

1 – 4 – 9 + 16 – 25 + 36 + 49 – 64 = 0 (3)

Notes

Odd Codes:

An odd number is an integerof the form $n = 2k + 1$, where k is an integer. The odd numbers are therefore ..., -3, -1, 1, 3, 5, 7, ... which are also the gnomonic numbers. Integers which are not odd are called even.

Odd numbers leave a remainder of 1 when divided by two, i.e., the congruence $n \equiv 1 \pmod 2$ holds for odd n. The oddness of a number is called its parity, so an odd number has parity 1, while an even number has parity 0.

The generating function for the odd numbers is

$$\frac{x(1+x)}{(x-1)^2} = x + 3x^2 + 5x^3 + 7x^4 + ...$$

The product of an even number and an odd number is always even, as can be seen by writing

(2 k) (2 / + 1) = 2 [k (2 / + 1)],

which is divisible by 2 and hence is even.

Parity check matrix:

Given a linear code C of length n and dimension k over the field F, a parity check matrix H of C is a $n \times (n-k)$ matrix whose rows generate the orthogonal complement of C, i.e., an element w of F^n is a code word of C if wH=0. The rows of H generate the null space of the generator matrix G.

Generator Matrix:

Given a linear code C, a generator matrix $\mathbf{G}$ of C is a matrix whose rows generate all the elements of C, i.e., if $\mathbf{G} = (g_1\ g_2\ \dots\ g_k)^{\mathrm{T}}$, then every code word w of C can be represented as

$w = c_1 g_1 + c_2 g_2 + ... + c_k g_k = cG$

in a unique way, where $\mathbf{c} = (c_1\ c_2\ \dots\ c_k)$.

An example of a generator matrix is the Golay code, which consists of all 2^{12} possible binary sums of the 11 rows.

Golay Code:

The Golay code is a perfect linear error-correcting code. There are two essentially distinct versions of the Golay code: a binary version and a ternary version.

The binary version G_{23} is a $(23, 12, 7)$ binary linear code consisting of $2^{12} = 4096$ code words of length 23 and minimum distance 7. The ternary version is a $(11, 6, 5)$ ternary linear code, consisting of $3^6 = 729$ code words of length 11 with minimum distance 5.

A parity check matrix for the binary Golay code is given by the matrix $\mathbf{H} = (\mathbf{M}\ \mathbf{I}_{11})$, where $\mathbf{I}_{11}$ is the 11×11 identity matrix and M is the 11×12 matrix.

Notes

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By adding a parity check bit to each code word in G_{23}, the extended Golay code G_{24}, which is a nearly perfect $[24, 12, 8]$ binary linear code, is obtained. The automorphism group of G_{24} is the Mathieu group M_{24}.

A second M_{24} generator is the adjacency matrix for the icosahedron, with $J_{12} - I_{12}$ appended, where J_{12} is a unit matrix and I_{12} is an identity matrix.

A third M_{24} generator begins a list with the 24-bit 0 word (000...000) and repeatedly appends first 24-bit word that has eight or more differences from all words in the list. Conway and Sloane list many further methods.

Amazingly, Golay's original paper was barely a half-page long but has proven to have deep connections to group theory, graph theory, number theory, combinatorics, game theory, multidimensional geometry, and even particle physics.

Hamming Code:

A binary Hamming code H_r of length $n = 2^r - 1$ (with $r \geq 2$) is a linear code with parity-check matrix H whose columns consist of all nonzero binary vectors of length r, each used once. H_r is an $(n = 2^r - 1, k = 2^r - 1 - r, d = 3)$ code. Hamming codes are perfect single error-correcting codes.

Triangular number :

The triangular number T_n is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one. This is illustrated above for $T_1 = 1$, $T_2 = 3$, ... The triangular numbers are therefore 1, $1 + 2$, $1 + 2 + 3$, $1 + 2 + 3 + 4$, ..., so for $n = 1$, 2, ..., the first few are 1, 3, 6, 10, 15, 21, ….

More formally, a triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer 'n', i.e.,

$$T_n \equiv \sum \quad (1)$$

$$= \frac{1}{2}n(n+1) \quad (2)$$

$$= \binom{n+1}{2} \quad (3)$$

Notes

where $\binom{n}{k}$ is a binomial coefficient. As a result, the number of distinct wine glass clinks that can be made among a group of n people (which is simply $\binom{n}{2}$) is given by the triangular number T_{n-1}.

The triangular number $T_n = n + (n-1) + \ldots + 2 + 1$ is therefore the additive analog of the factorial $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

A plot of the first few triangular numbers represented as a sequence of binary bits is shown above. The top portion shows T_1 to T_{255}, and the bottom shows the next 510 values.

The odd triangular numbers are given by 1, 3, 15, 21, 45, 55, ..., while the even triangular numbers are 6, 10, 28, 36, 66, 78, ...

$T_4 = 10$ gives the number and arrangement of the tetractys (which is also the arrangement of bowling pins), while $T_5 = 15$ gives the number and arrangement of balls in billiards. Triangular numbers satisfy the recurrence relation

$$T_{n+1}^2 - T_n^2 = (n+1)^3 \tag{4}$$

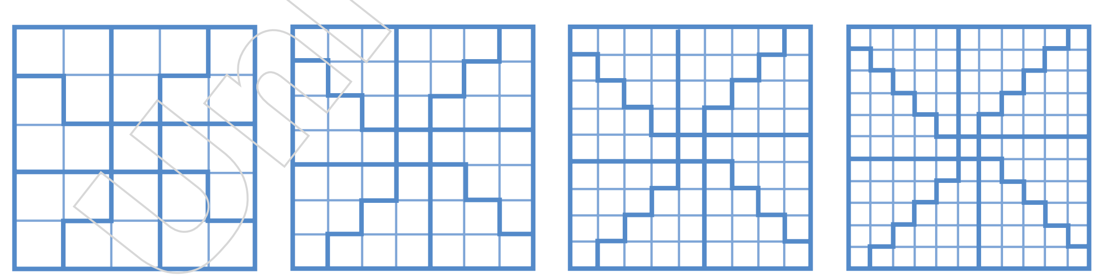
as well as

$$T_n^2 + T_{n-1}^2 = T_{n^2} \tag{5}$$

$$3T_n + T_{n-1} = T_{2n} \tag{6}$$

$$3T_n + T_{n+1} = T_{2n+1} \tag{7}$$

$$1+3+5+\ldots\ldots+(2n-1) = T_n + T_{n-1} \tag{8}$$



In addition, the triangle numbers can be related to the square numbers by

$$(2n+1)^2 = 8T_n + 1 \tag{9}$$

$$= T_{n-1} + 6T_n + T_{n+1} \tag{10}$$

The triangular numbers have the ordinary generating function

$$f(x) = \frac{x}{(1-x)^3} \tag{11}$$

$$= x + 3x^2 + 6x^3 + 10x^4 + 15x^5 + \ldots. \tag{12}$$

and exponential generating function

$$g(x) = (1 + 2x + \frac{1}{2}x^2)e^x \tag{13}$$

$$= 1 + 3x + 3x^2 + \frac{5}{3}x^3 + \frac{5}{8}x^4 + \ldots \tag{14}$$

Notes

$$= 1+3\frac{x}{1!}+6\frac{x^2}{2!}+10\frac{x^3}{3!}+15\frac{x^4}{4!}+..... \quad (15)$$

Every other triangular number T_n is a hexagonal number, with

$$H_n = T_{2n-1} \quad (16)$$

In addition, every pentagonal number is 1/3 of a triangular number, with

$$P_n = \frac{1}{3}T_{3n-1} \quad (17)$$

The sum of consecutive triangular numbers is a square number, since

$$T_r \quad T_r = \frac{1}{2}r(r+1)+\frac{1}{2}(r-1)r \quad (18)$$

$$= \frac{1}{2}r[(r+1)+(r-1)] \quad (19)$$

$$= r^2 \quad (20)$$

Interesting identities involving triangular, square, and cubic numbers are

$$\sum_{k=1}^{2n-1}(-1)^{k+1}T_k = n^2 \quad (21)$$

$$\sum_{k=1}^{n}k^3 = T_n^2 \quad (22)$$

$$= \frac{1}{4}n^2(n+1)^2 \quad (23)$$

$$\sum_{k=1}^{n}(2k-1)^3 = T_{2n^2-1} \quad (24)$$

$$= n^2(2n^2-1) \quad (25)$$

Triangular numbers also unexpectedly appear in integrals involving the absolute value of the form

$$\int_0^1\int_0^1|x-y|^n\,dxdy = \frac{2}{(n+1)(n+2)} \quad (26)$$

All evenperfect numbers are triangular T_P with prime P. Furthermore, ever evenperfect number $P > 6$ is of the form

$$P = 1+9T_n = T_{3n+1} \quad (27)$$

where T_n is a triangular number with $n = 8j+2$ (Eaton 1995, 1996). Therefore, the nested expression

$$9(0 \ldots (9(9(9\,T_n + 1)+1)+1)+1) \ldots + 1) + 1 \quad (28)$$

generates triangular numbers for any T_n. An integer k is a triangular number if $8k+1$ is a square number > 1.

The numbers 1, 36, 1225, 41616, 1413721, 48024900, ... are square triangular numbers, i.e., numbers which are simultaneously triangular and square (Pietenpol

Notes

1962). The corresponding square roots are 1, 6, 35, 204, 1189, 6930, ..., and the indices of the corresponding triangular numbers T_n are $n = 1$, 8, 49, 288, 1681, ...

Numbers which are simultaneously triangular and tetrahedral satisfy the binomial coefficient equation

$$T_n = \binom{n+1}{2} = \binom{m+2}{3} = Te_m \quad (29)$$

the only solutions of which are

$$Te_3 = T_4 = 10 \quad (30)$$

$$Te_8 = T_{15} = 120 \quad (31)$$

$$Te_{20} = T_{55} = 1540 \quad (32)$$

$$Te_{34} = T_{119} = 7140 \quad (33)$$

The following table gives triangular numbers T_p having prime indices p.

class	Sequence
T_n with prime indices	3, 6, 15, 28, 66, 91, 153, 190, 276, 435, 496, ...
odd T_n with prime indices	3, 15, 91, 153, 435, 703, 861, 1431, 1891, 2701, ...
ven T_n with prime indices	6, 28, 66, 190, 276, 496, 946, 1128, 1770, 2278, ...

The smallest of two integers for which $n^3 - 13$ is four times a triangular number is 5, as determine by the only Fibonacci numbers which are triangular are 1, 3, 21, and 55, and the only Peli number which is triangular is 1 (McDaniel 1996). The beast number 666 is triangular, since

$$T_{66} = T_{36} + 666 \quad (34)$$

The positive divisors of $4T(n)+1$ are all of the form $4k+1$, those of $6T(n)+1$ are all of the form $6k+1$, and those of $10T(n)+1$ are all of the form $10k \pm 1$; that is, they end in the decimal digit 1 or 9.

Fermat's polygonal number theorem states that every positive integer is a sum of at most three triangular numbers, four square numbers, five pentagonal numbers, and nn-polygonal numbers.

This case is equivalent to the statement that every number of the form $8m+3$ is a sum of three oddsquares . Dirichlet derived the number of ways in which an integer m can be expressed as the sum of three triangular numbers (Duke 1997). The result is particularly simple for a primeof the form $8m+3$, in which case it is the number of squares mod $8m+3$ minus the number of nonsquares mod $8m+3$ in the interval from 1 to $4m+1$

The only triangular numbers which are the product of three consecutive integers are 6, 120, 210, 990, 185136, 258474216

Notes

Questions:

1. The rank of P is $|X| - |P|$, if X is ___________.
   a) finite.
   b) infinite
   c) negative
   d) positive
2. Any non-empty set X, P = {X} is a partition of X, called the __________trivial partition.
   a) Non trivial
   b) trivial
   c) both a and b
   d) none
3. A partition is a ___________of positive integers, and it is a partition of n of the sum of the integer.
   a) even partition
   b) odd partition
   c) both a and b
   d) multiset
4. The number of partitions of n in which no part occurs more often than d times is the same as the number of partitions in which no term is a multiple of _____________.
   a) d-3
   b) d-2
   c) d-4
   d) d-1
5. The number of partitions in which the parts are all even and different is equal to the ______________ difference of the number of partitions with odd and even parts.
   a) Symmetric
   b) absolute
   c) both a and b
   d) none

Answers:

6. a
7. b
8. d
9. d
10. b

Notes

Questions:

1. Explain even codes
2. Explain odd codes
3. Relation between even and odd codes
4. Explain Enumeration

Sample Questions:

1. A coin is flipped 10 times. How many outcomes are possible?

   Solution:

   We have 2 multiplied by itself 10 times, or

   2^{10} =1,024 possible outcomes. The set itself is obviously too large to list.

2. An automobile's license plate consists of three letters (A-Z) followed by three numerical digits (0-9). How many plates are possible if

   a) there are no restrictions (repetition is allowed)?

   b) all letters and numbers must be different (repetition is not allowed)?

   Solution:

   In the first case (a) where there are no restrictions and repetition of the same letter or numeral is allowed, there are 26_3 =10_3 =17,576,000 possible license plates. We used exponential shorthand for the calculation. In the second case (b) where repetition is not allowed, we see that there are 26 x 25 x 24 x 10 x 9 x 8 = 11,232,000 possible plates.

3. A true-false exam has 8 questions, followed by 10 multiple-choice questions, each with four choices. How many ways can this test be filled?

   Solution:

   For the true-false portion, there are 2^8 = 256 ways to randomly answer the questions.

   For the multiple choice portion, there are 4^{10} = 1,048,576 ways to randomly answer the questions. Taken as a whole, we multiply the two results together: $2^8 . 4^{10}$ = 268,435,456 possible ways to fill in the exam.

4. Radio and television station call letters are a sequence of three or four letters, with the first letter being a W (for stations east of the Mississippi River) and K (for stations west of the Mississippi River). How many such call letter sequences are possible?

   Solution:

   For three letters sequences, there are two choices for the first position (W or K), and 26 choices each for the remaining two positions, for a total of 2 x 26 x 26 = 1,352 three letter sequences. For four letter sequences, the total number possible is 2 x 26 x 26 x 26 = 35,152. Because these form two distinct classes of possible call-letter sequences, the total number of 3-letter or 4-letter call sequences is the sum: We sum the two to obtain 36,504 possible three-letter or four-letter call sequences.

Notes

Text Books:

1) C.L. Liu, Elements of Discrete Mathematics, Tata McGraw Hill, 2nd Edition, 2000.

2) N. Deo, Graph Theory with Applications to Engineering and Computer Science, PHI publication, 3rd edition, 2009

3) Harikishan, ShivrajPundir and Sandeep Kumar, Discrete Mathematics, Pragati Publication, 7th Edition, 2010.

4) Colmun, Busby and Ross, Discrete Mathematical Structure, PHI Publication, 6th Edition, 2009