Understanding Logistic Regression Models

Overview

Logistic regression fits an S-curve between probabilities and causes

S-curves have a standard mathematical form that is easy to estimate

Two equivalent methods of fitting S-curves are commonly used

One of these methods cleverly utilises linear regression in logistic regression

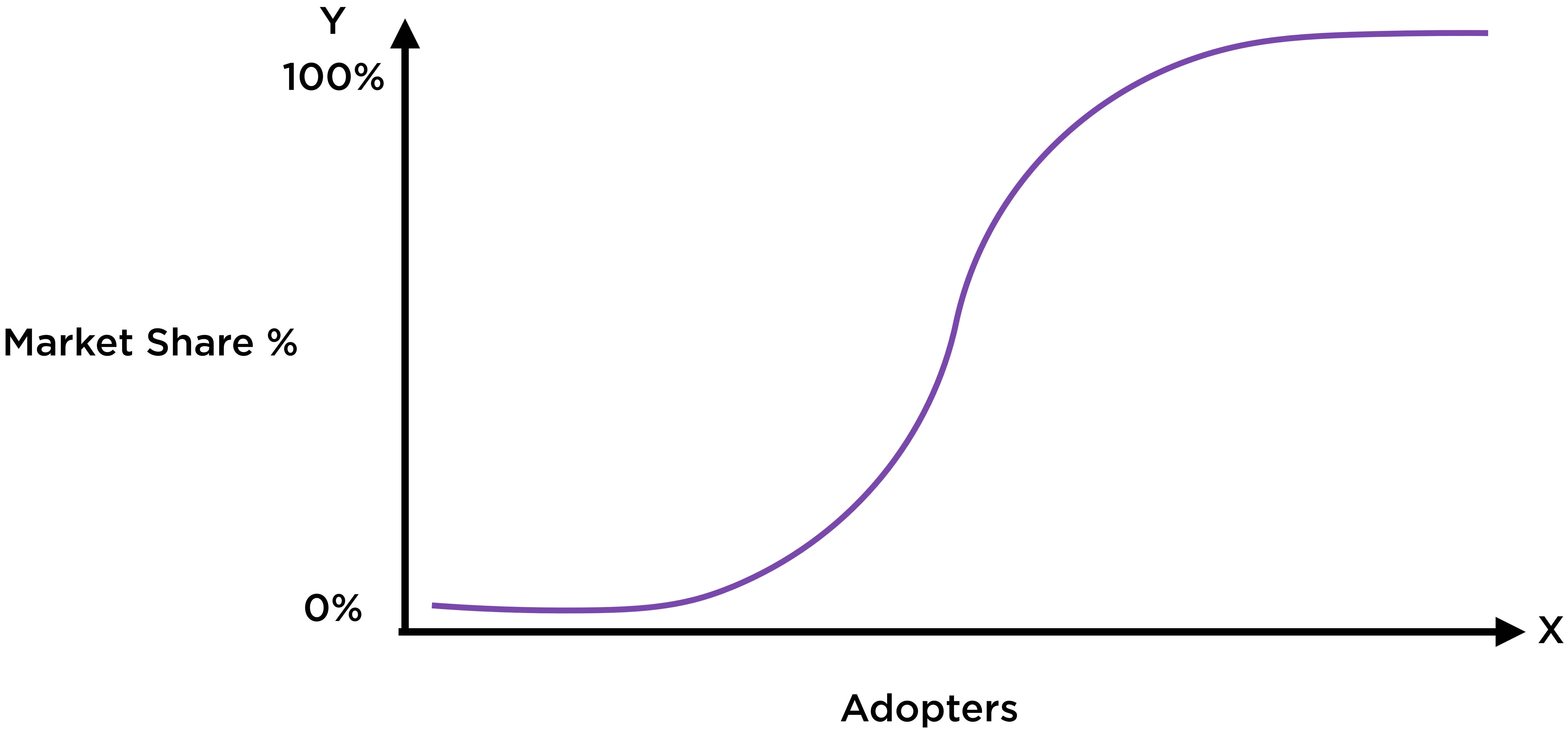
Logistic regression can be easily extended to more than 2 categories in the effect

The Intuition Behind Logistic Regression

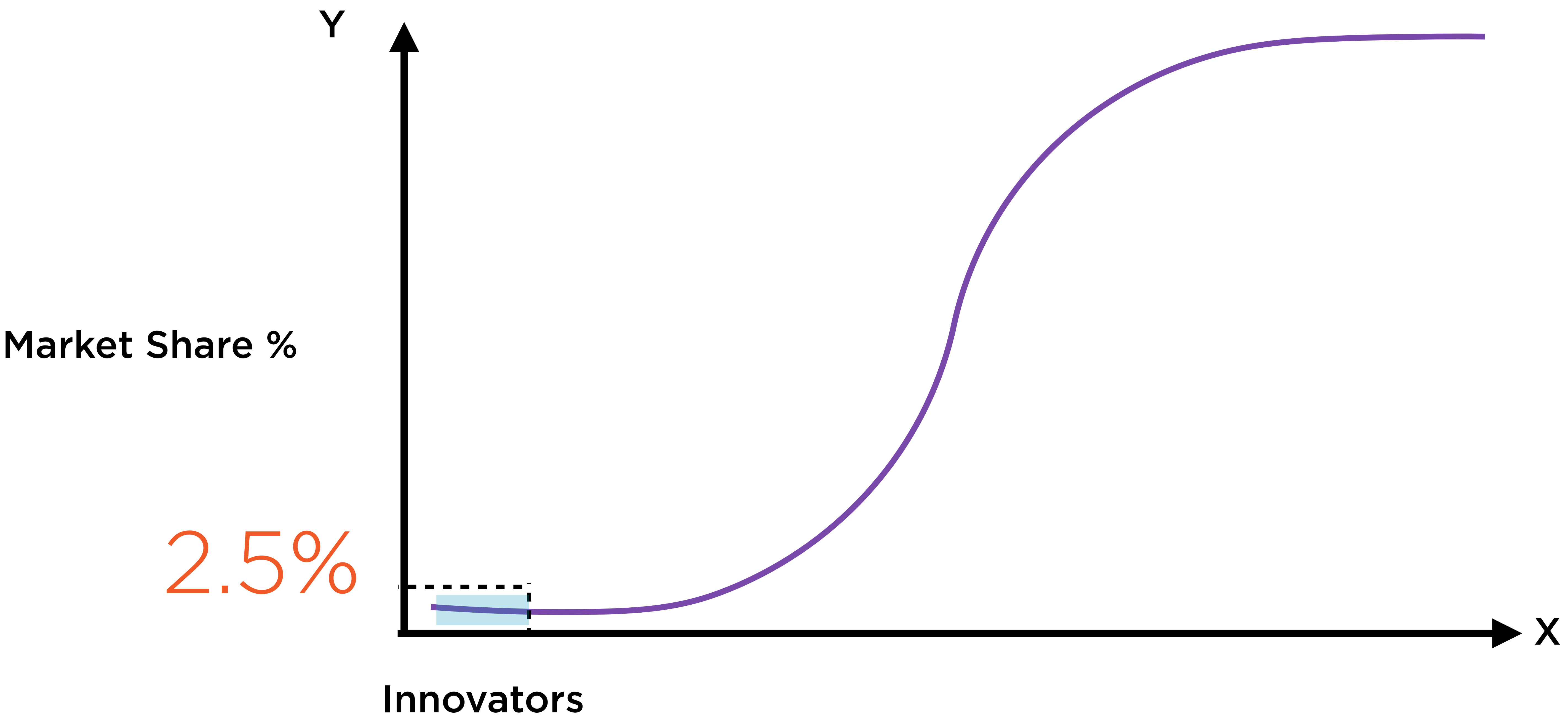
Tipping Point

A point in time when a group—or a large number of group members—rapidly and dramatically changes its behavior by widely adopting a previously rare practice

Diffusion of Innovation



Diffusion of Innovation



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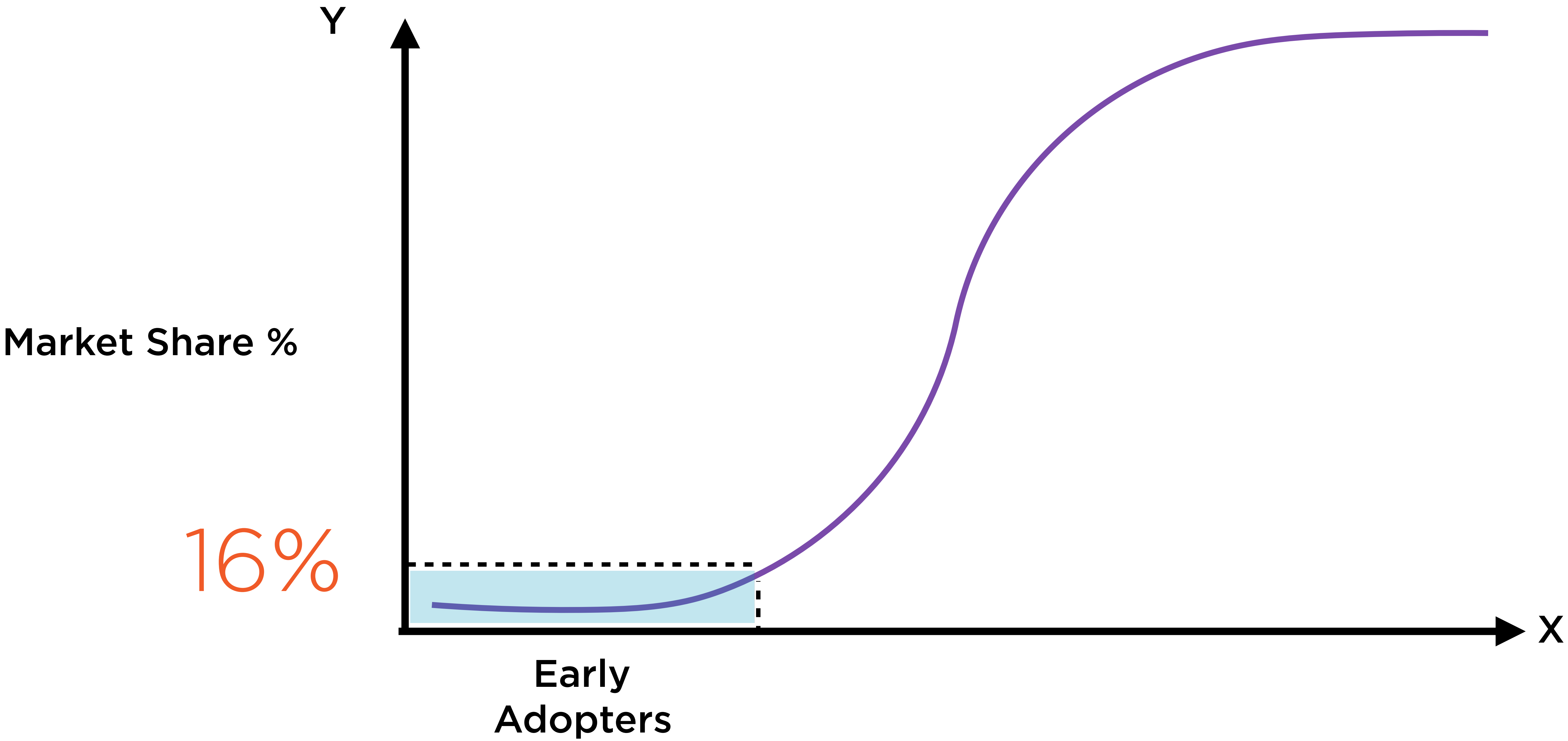

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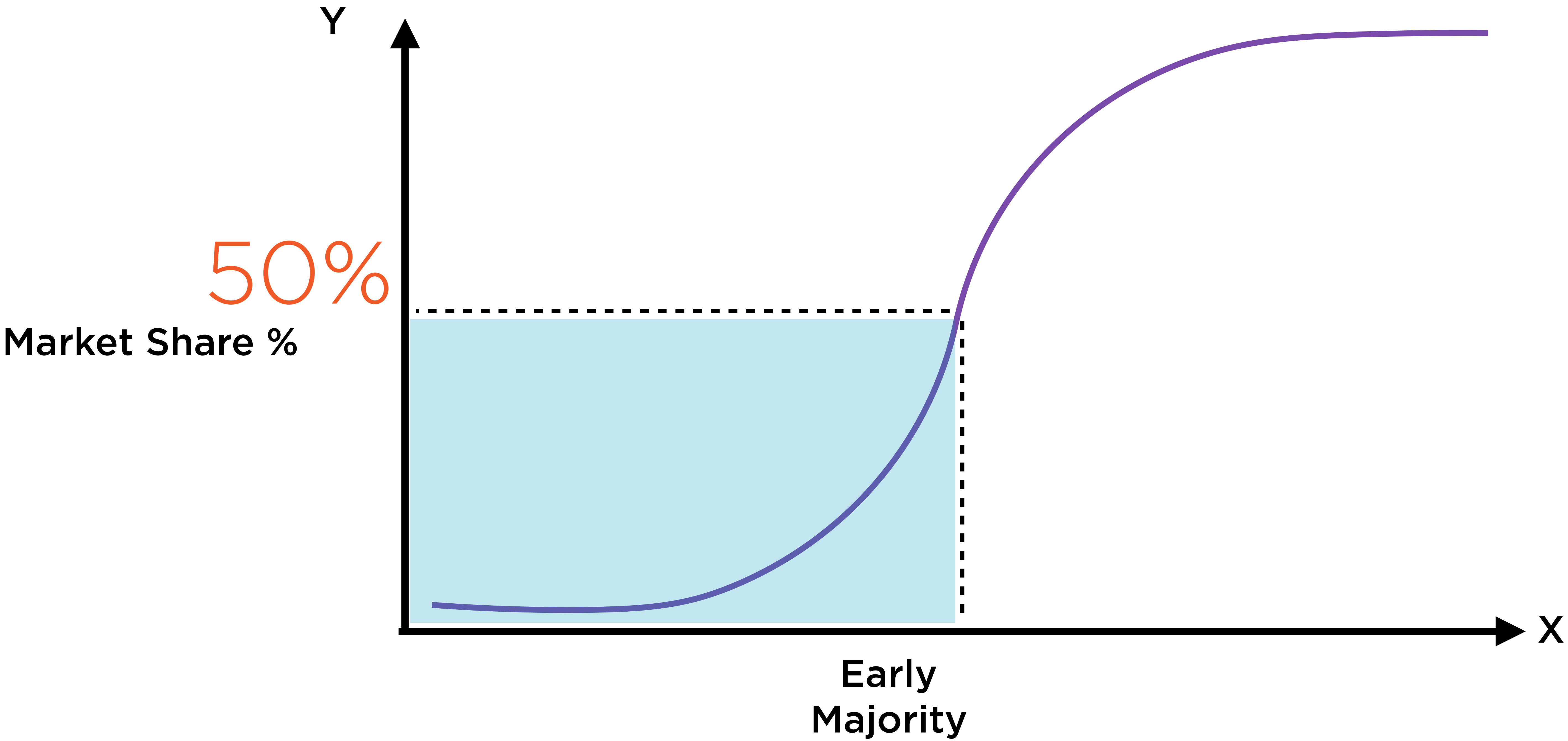

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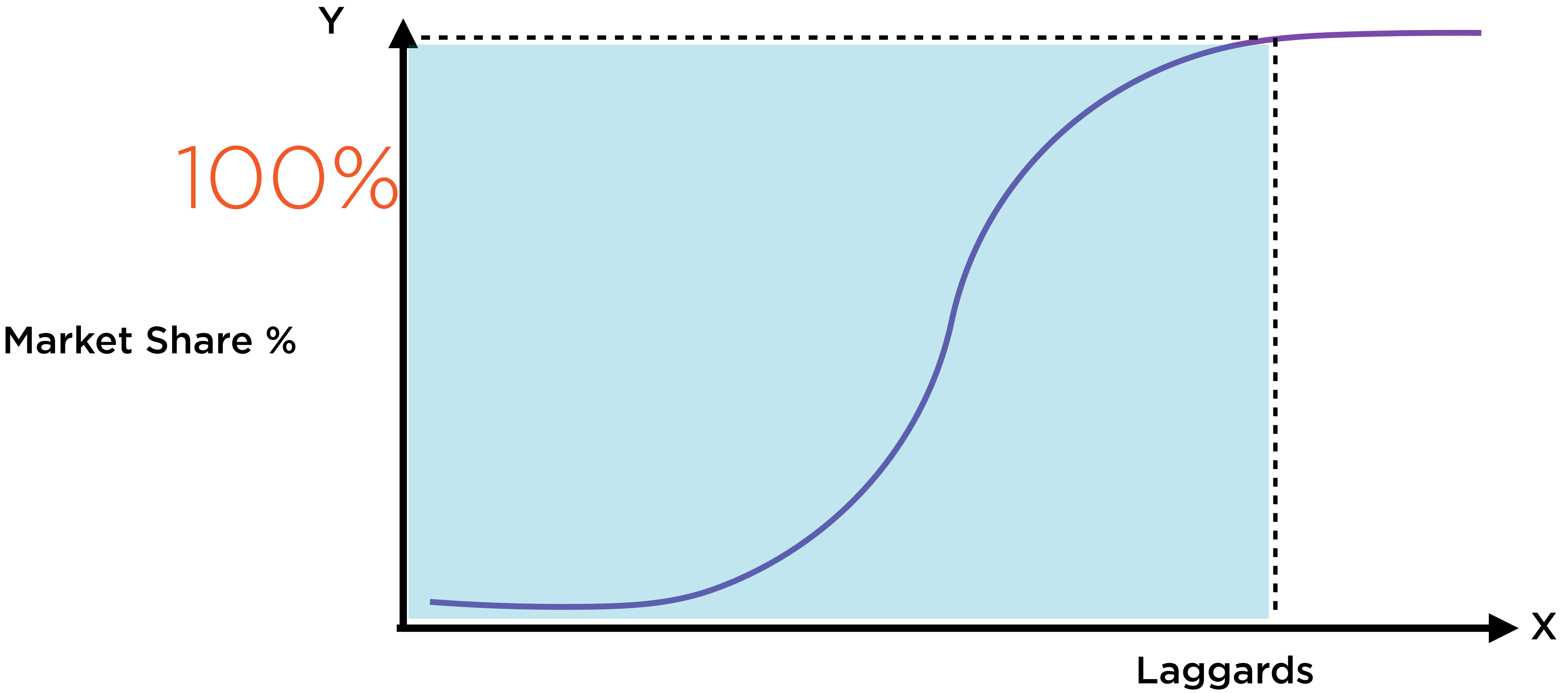

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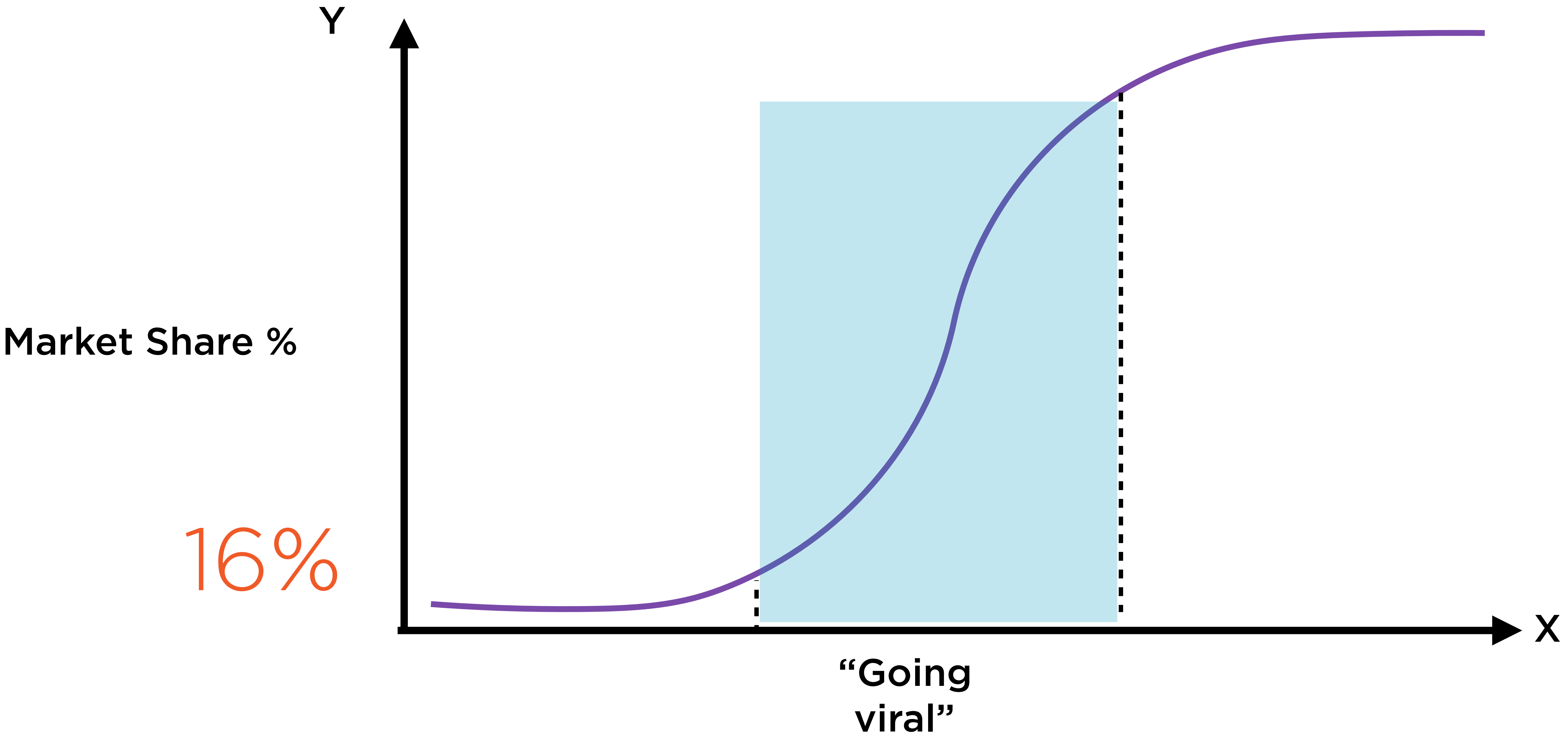
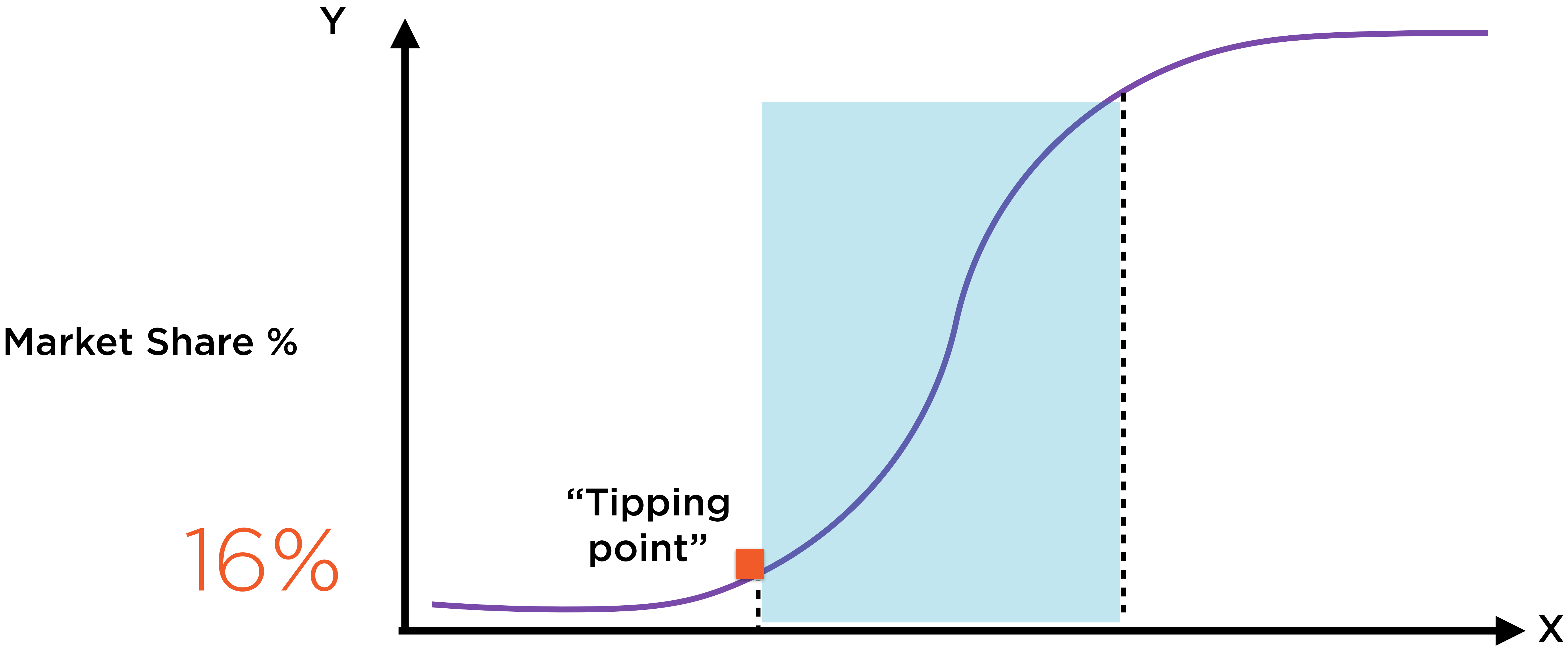
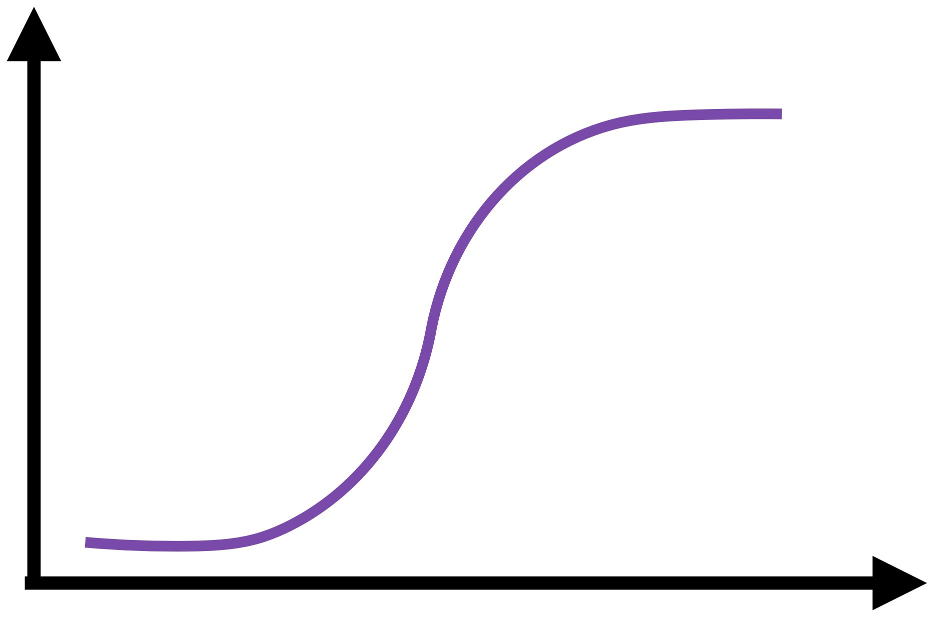


Diffusion of Innovation





S-curves are widely studied, well understood

$$y = \frac{1}{1 + e^{-(A+Bx)}}$$

Logistic regression uses S-curve to estimate probabilities

$$p(y) = \frac{1}{1 + e^{-(A+Bx)}}$$

Logistic Regression

Represent all n points as
(x_i, y_i), where i = 1 to n

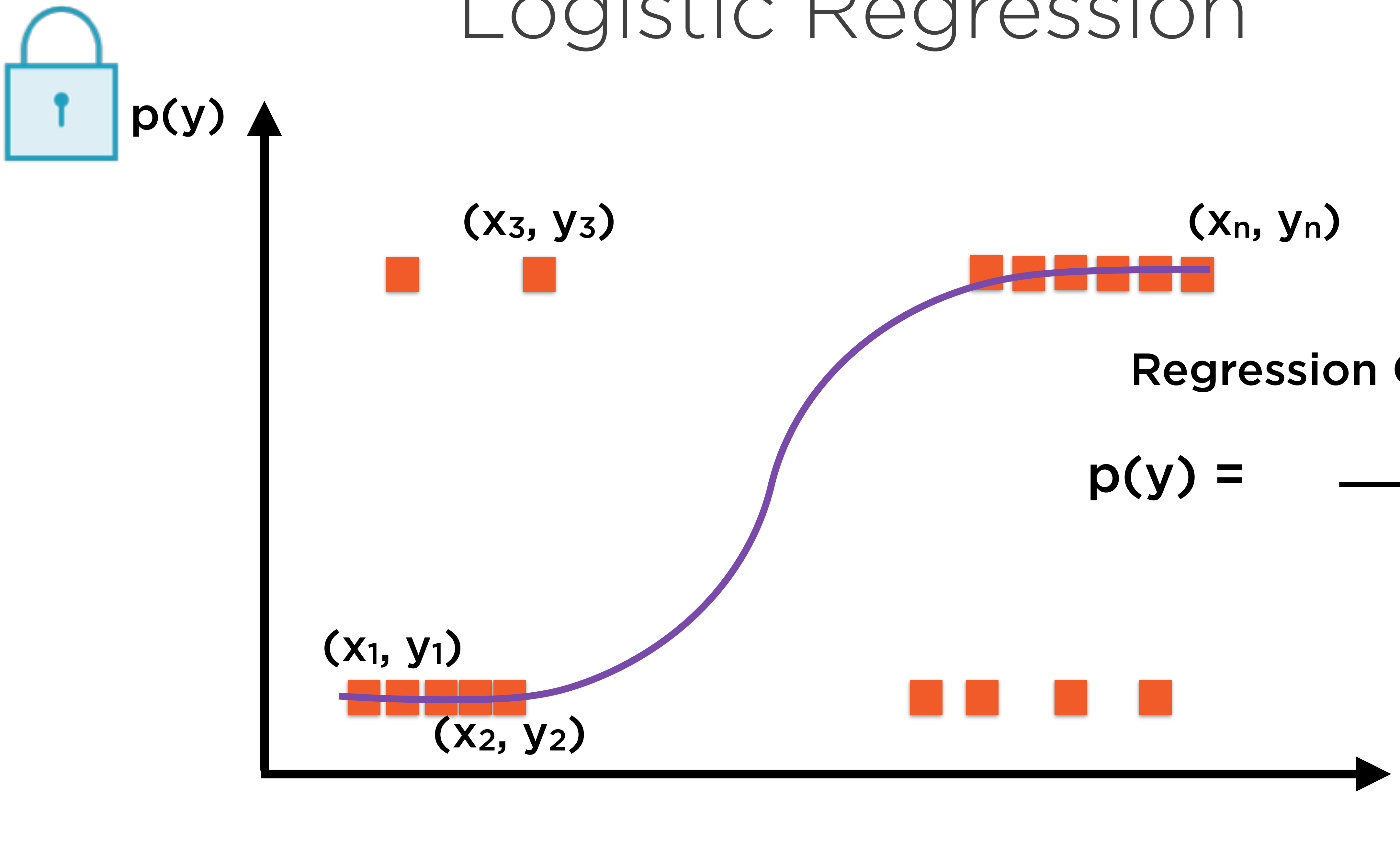
Logistic Regression

Regression Equation:

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

Given a set of points where x "predicts" probability of success in y, use logistic regression

Logistic Regression



Regression Curve

$$p(y) = \frac{1}{1 + e^{-(A+Bx)}}$$

Represent all n points as (x_i, y_i), where i = 1 to n

Two Approaches to Deadlines

Start 5 minutes before deadline

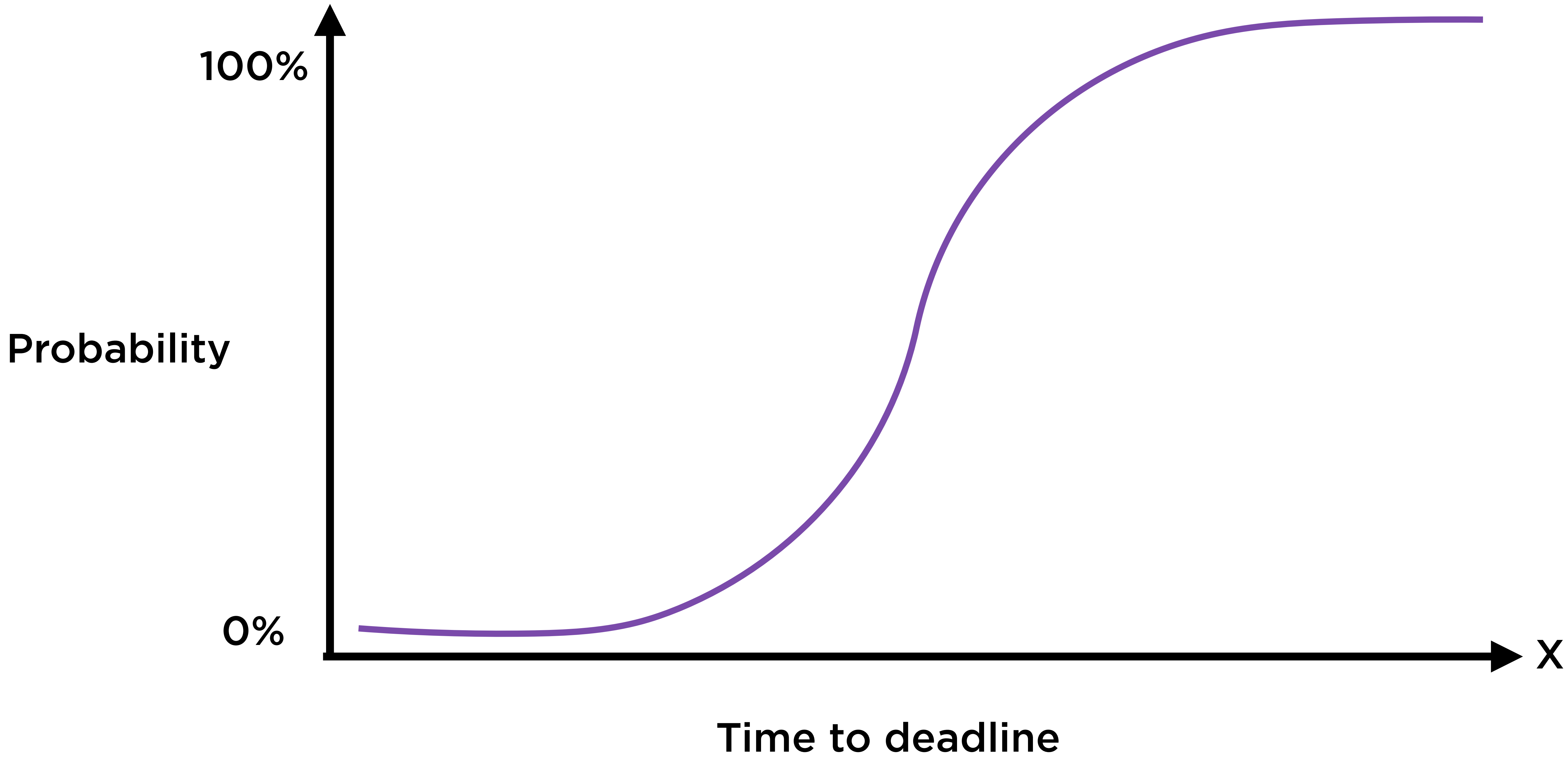
Good luck with that

Start 1 year before deadline

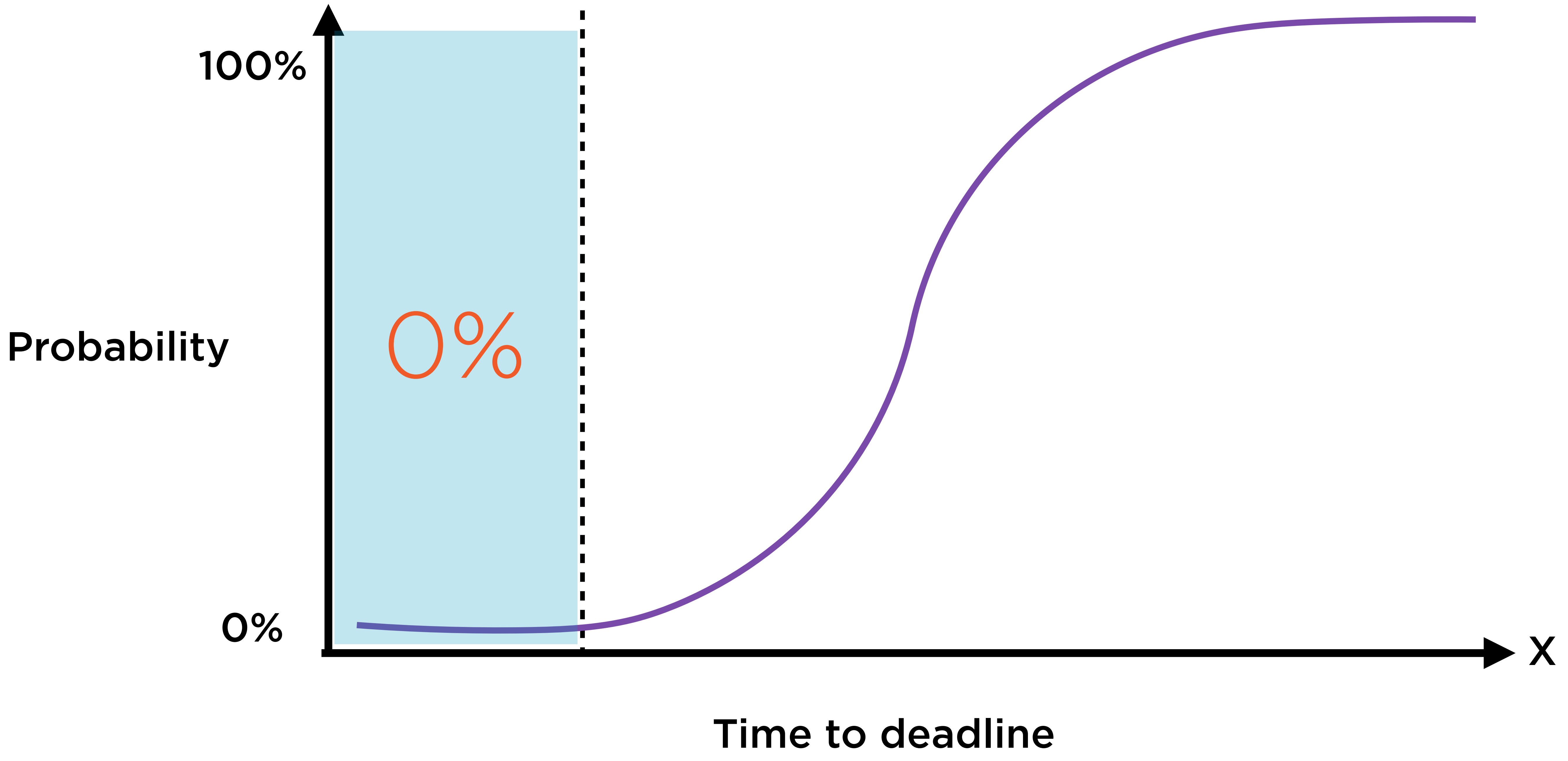
Maybe overkill

Neither approach is optimal

Working Smart with Logistic Regression

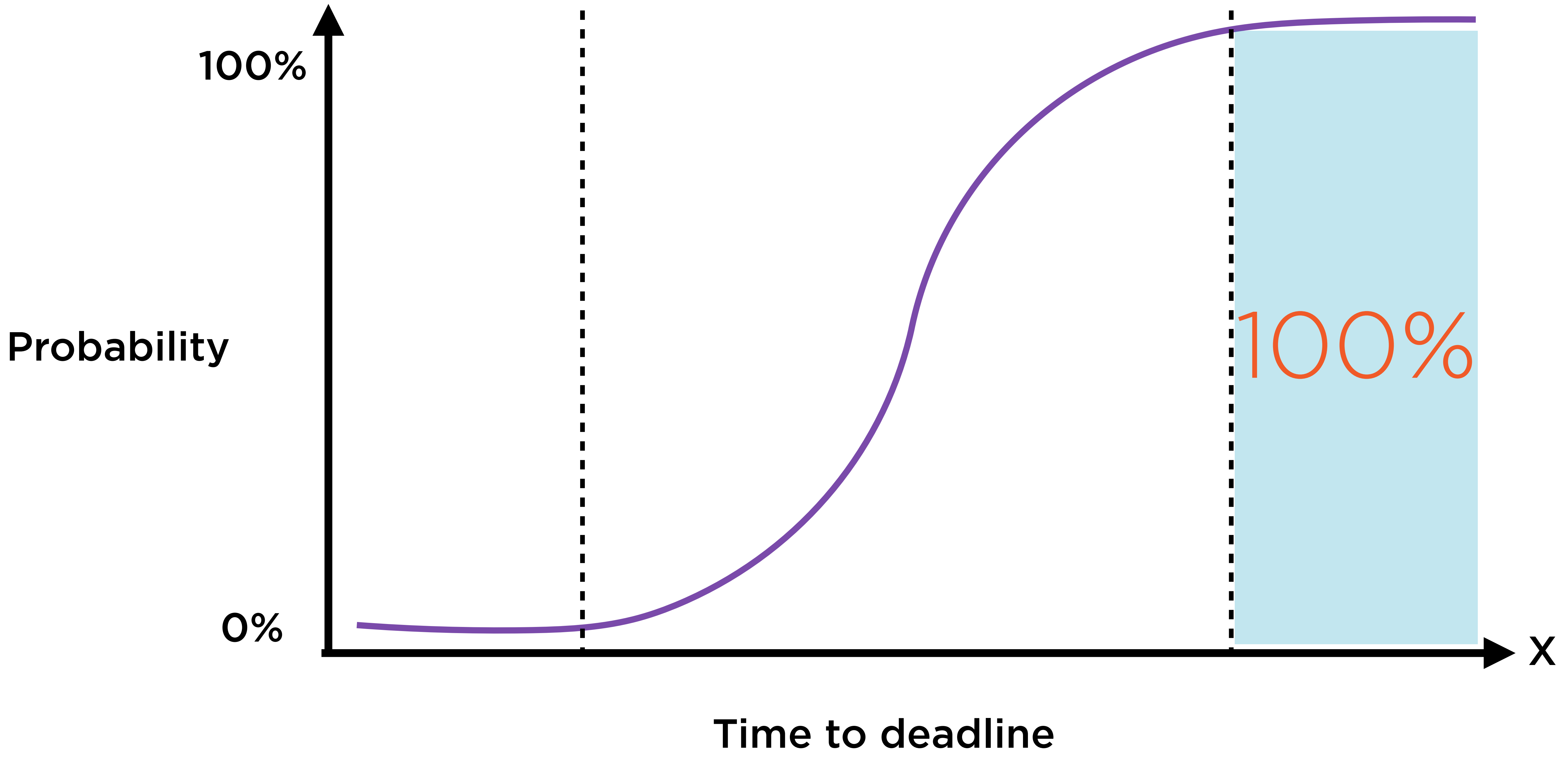


Working Smart with Logistic Regression



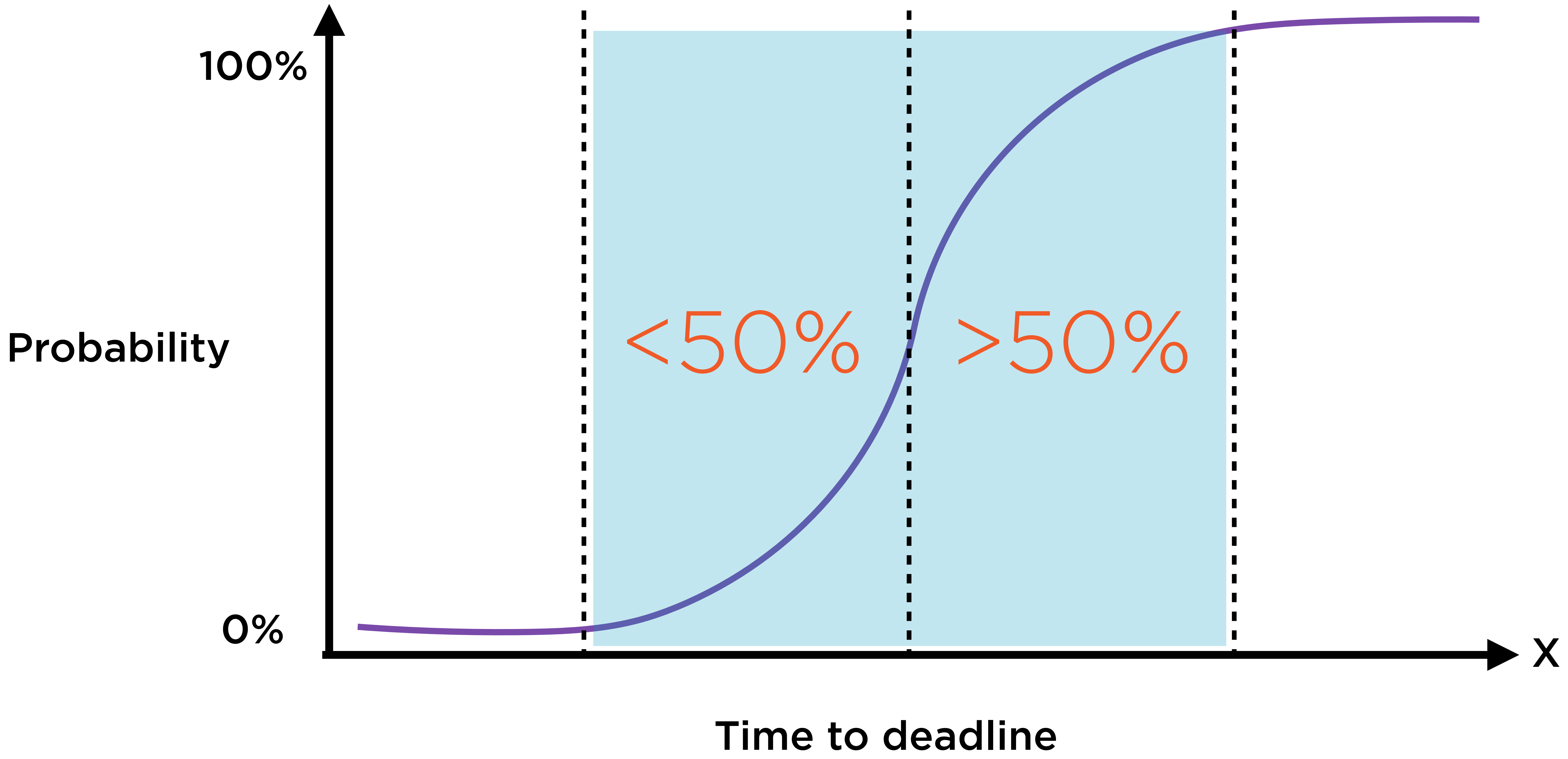
Start too late, and you'll definitely miss

Working Smart with Logistic Regression

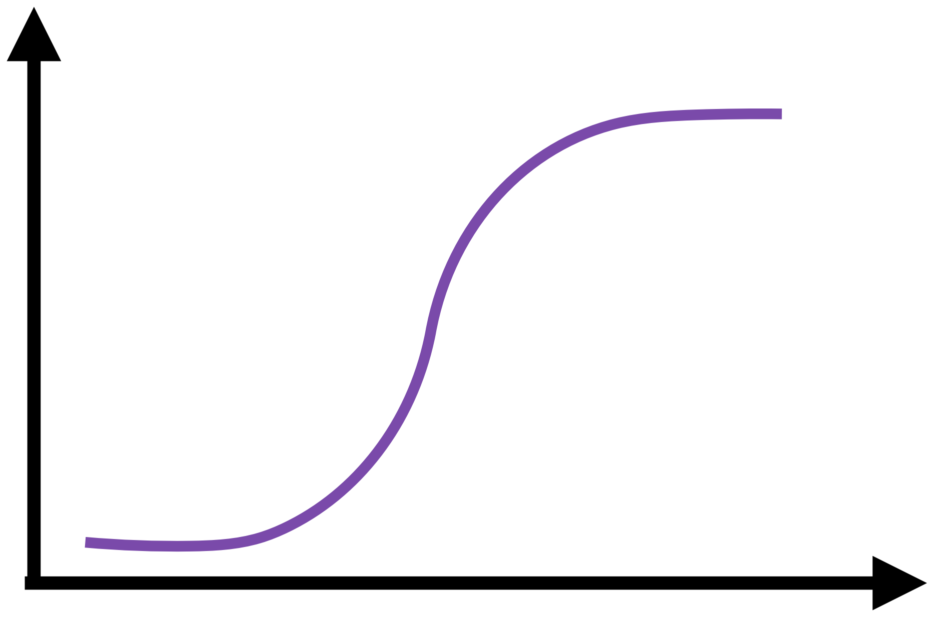


Start too early, and you'll definitely make it

Working Smart with Logistic Regression



Working smart is knowing when to start



Y-axis: probability of meeting deadline

X-axis: time to deadline

Meeting or missing deadline is binary

Probability curve flattens at ends

- floor of 0
- ceiling of 1

Working Smart with Logistic Regression

Probabilities	Categorical Variables	Causes
p(y)	y	x

Logistic Regression helps estimate how probabilities of categorical variables are influenced by causes

Hitting Deadlines

Probability of hitting deadline	Deadline: Hit or miss?	Time of starting work
$p(y)$	$y = 1$ or 0	x

Logistic Regression helps estimate how probabilities of categorical variables are influenced by causes

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

$y_i = 1$ or 0 (hit or miss)

x_i = time spent working on deadline

$p(y_i)$ = probability that $y_i = 1$

$1 - p(y_i)$ = probability that $y_i = 0$

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

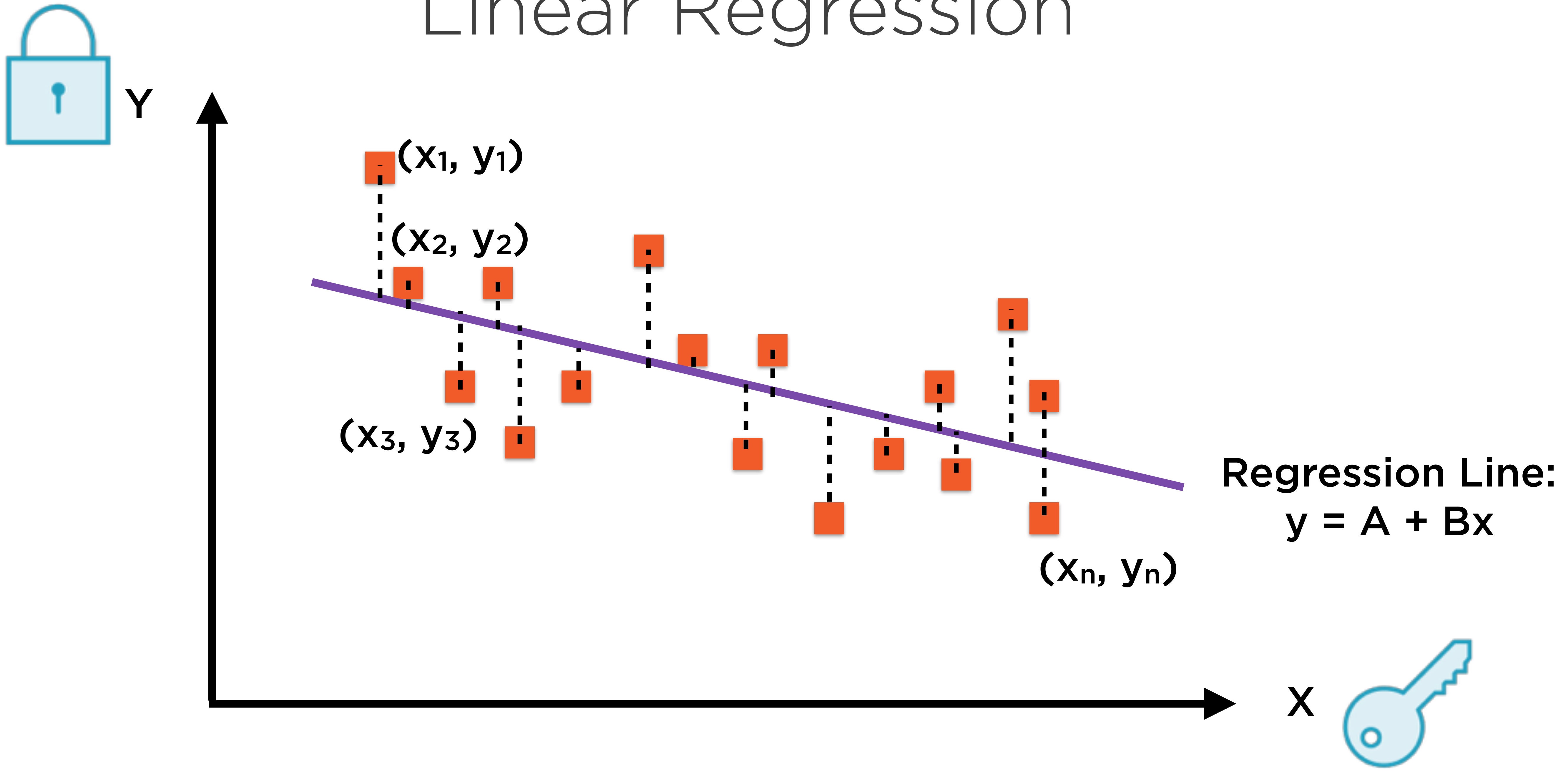
Logistic regression involves finding the "best fit" such curve

- A is the intercept

- B is the regression coefficient

(e is the constant 2.71828)

Linear Regression



Represent all n points as (x_i, y_i), where i = 1 to n

Similar, yet Different

Linear Regression

$$y_i = A + Bx_i$$

Objective of regression is to find A, B that "best fit" the data

Logistic Regression

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

Objective of regression is to find A, B that "best fit" the data

Similar, yet Different

Linear Regression

$$y_i = A + Bx_i$$

Relationship is already linear (by assumption)

Logistic Regression

$$\ln\left(\frac{p(y_i)}{1 - p(y_i)}\right) = A + Bx_i$$

Relationship can be made linear (by log transformation)

Similar, yet Different

Linear Regression

$$y_i = A + Bx_i$$

Solve regression problem using cookie-cutter solvers

Logistic Regression

$$\text{logit}(p) = A + Bx_i$$

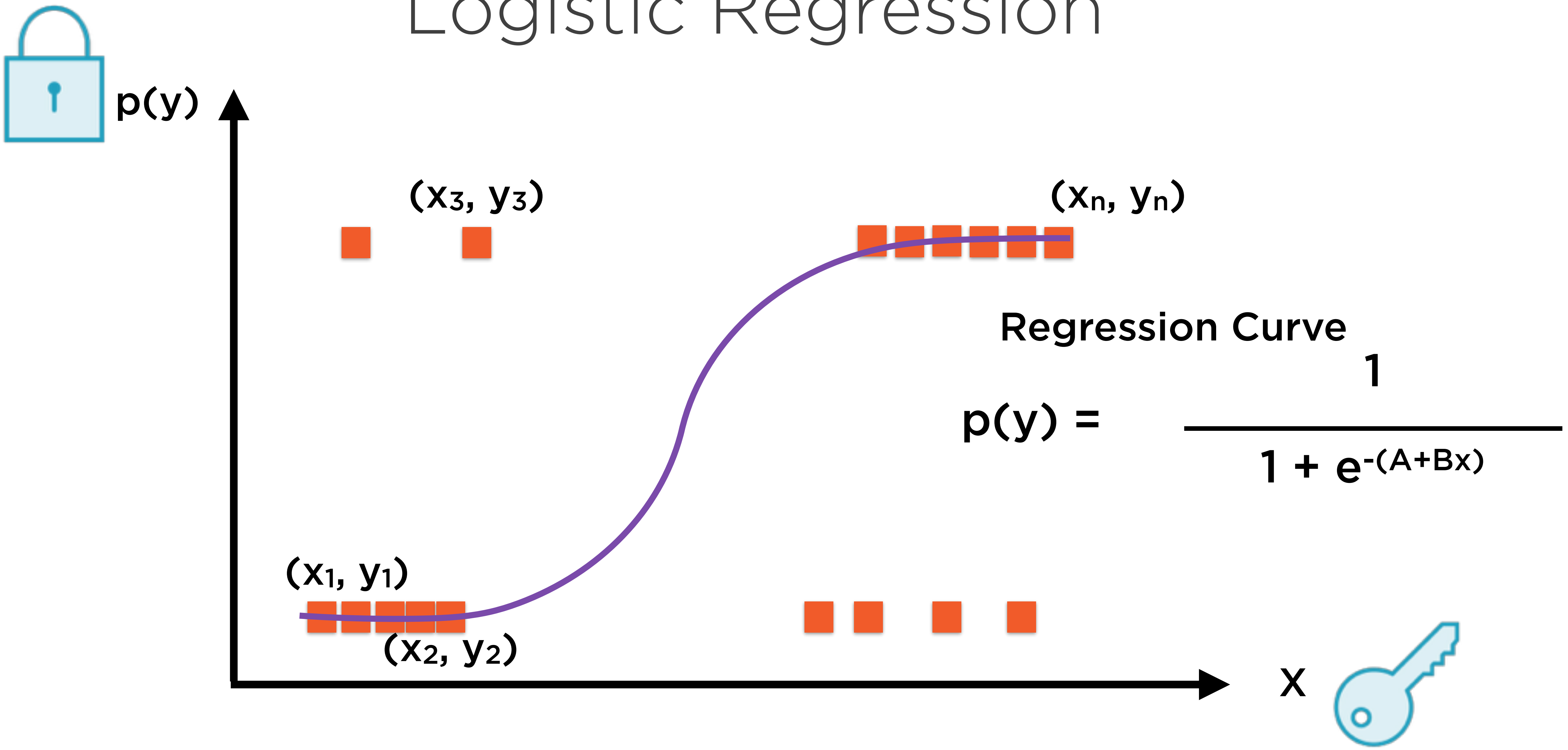
$$\text{logit}(p) = \ln\left(\frac{p}{1 - p}\right)$$

Solve regression problem using cookie-cutter solvers

Logistic Regression

Represent all n points as (x_i, y_i), where i = 1 to n

Logistic Regression



Represent all n points as
(x_i,y_i), where i = 1 to n

Linear Regression

$$y = A + Bx$$

$$\begin{aligned}
y_1 &= A + Bx_1 \\
y_2 &= A + Bx_2 \\
y_3 &= A + Bx_3 \\
&\dots \\
y_n &= A + Bx_n
\end{aligned}$$

Logistic Regression

$$p(y) = \frac{1}{1 + e^{-(A+Bx)}}$$

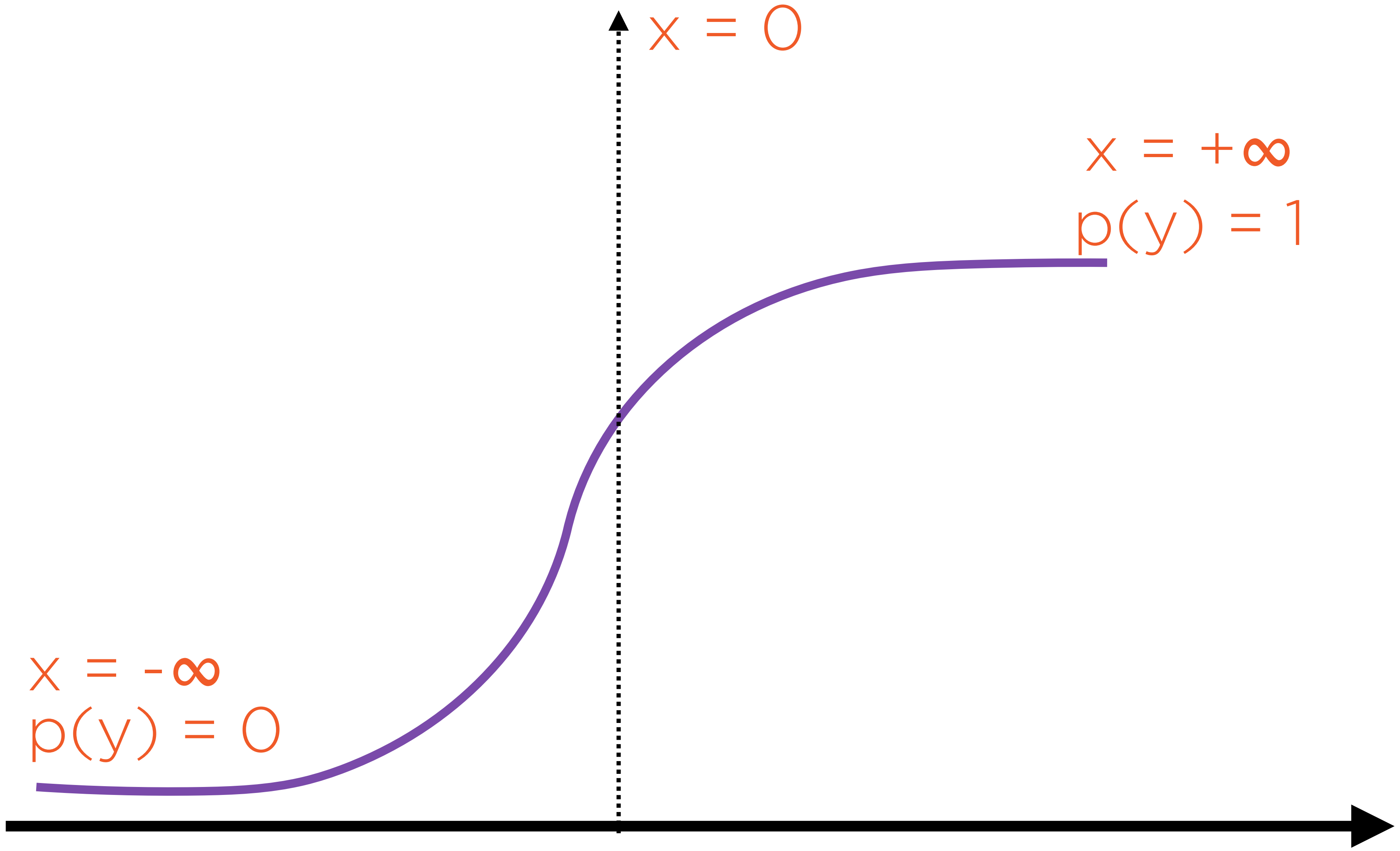
$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

$$p(y_1) = \frac{1}{1 + e^{-(A+Bx_1)}}$$

...

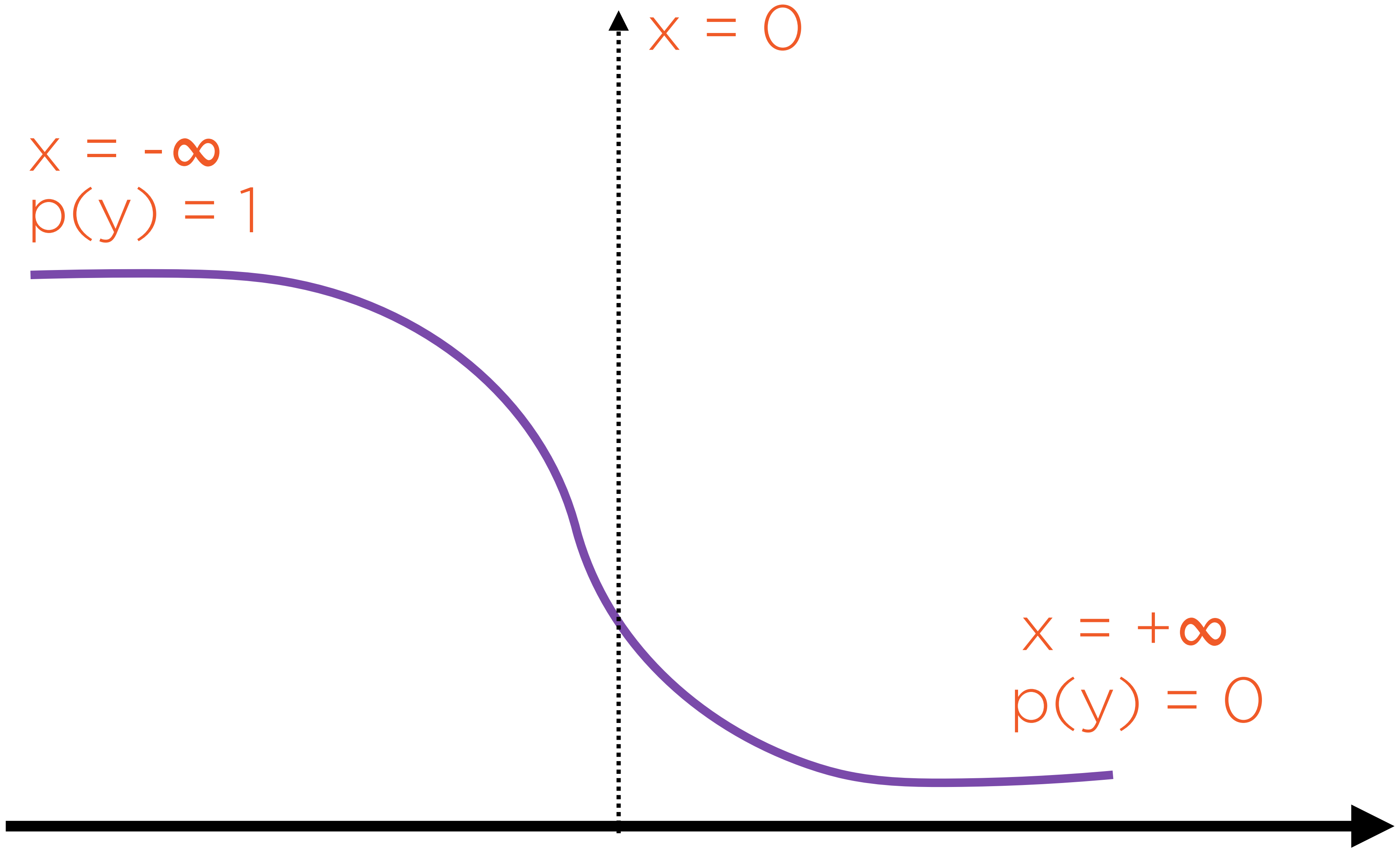
$$p(y_n) = \frac{1}{1 + e^{-(A+Bx_n)}}$$

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$



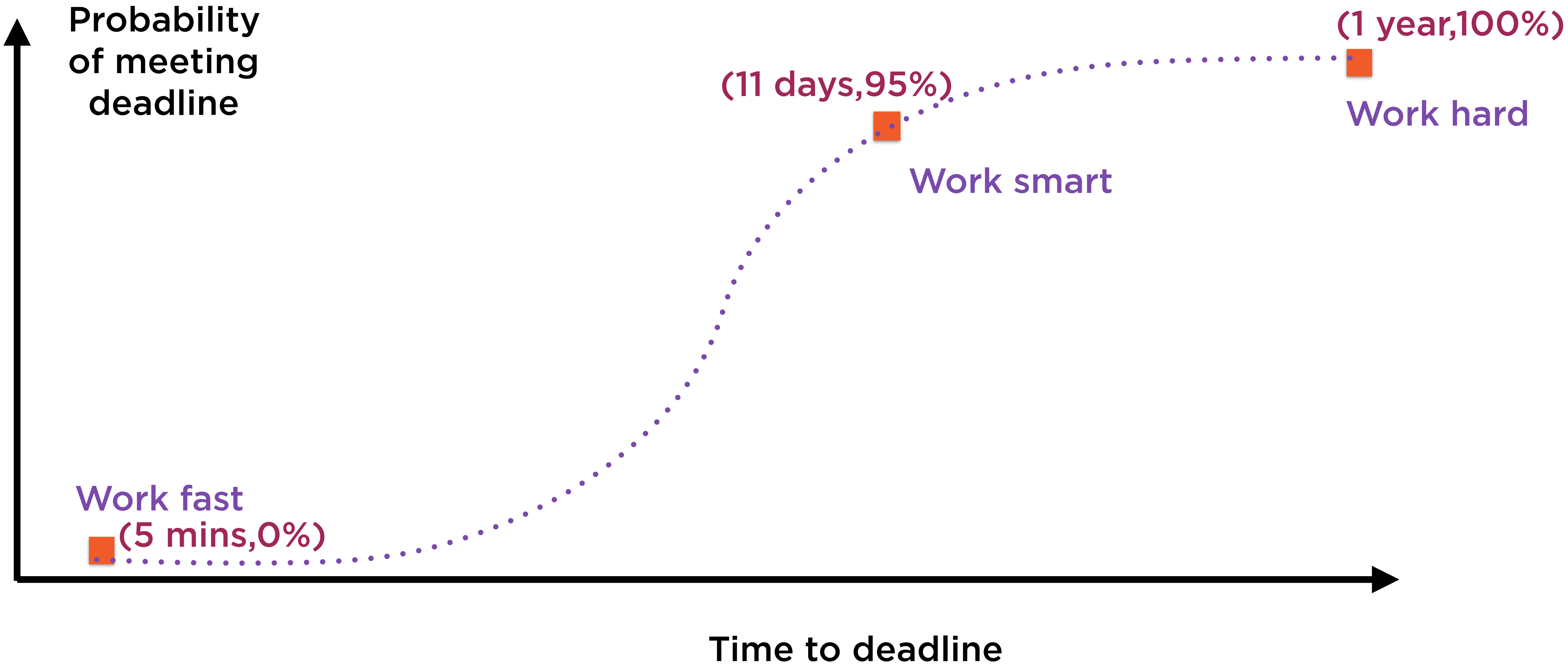
If A and B are **positive**

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

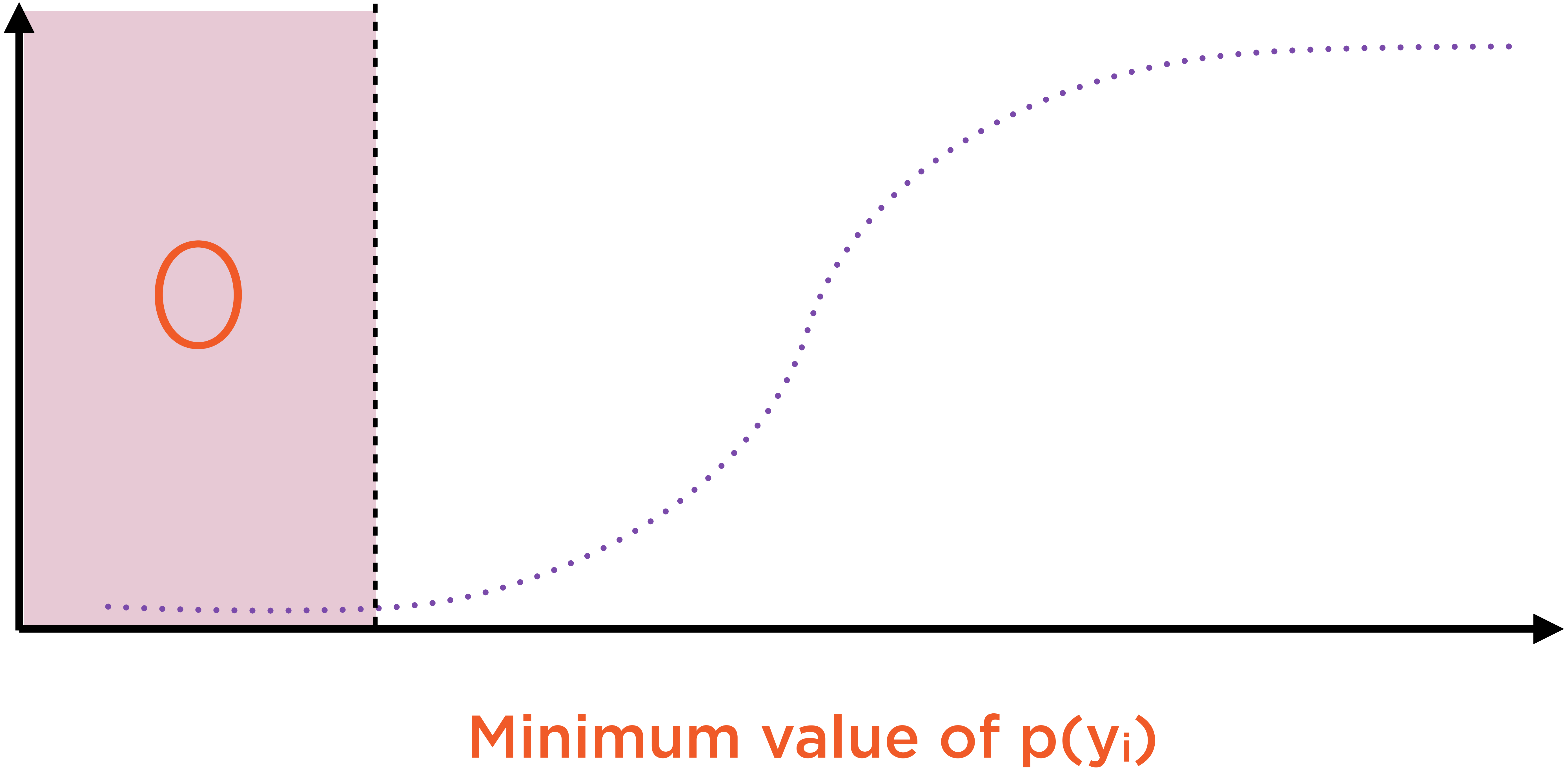


If A and B are **negative**

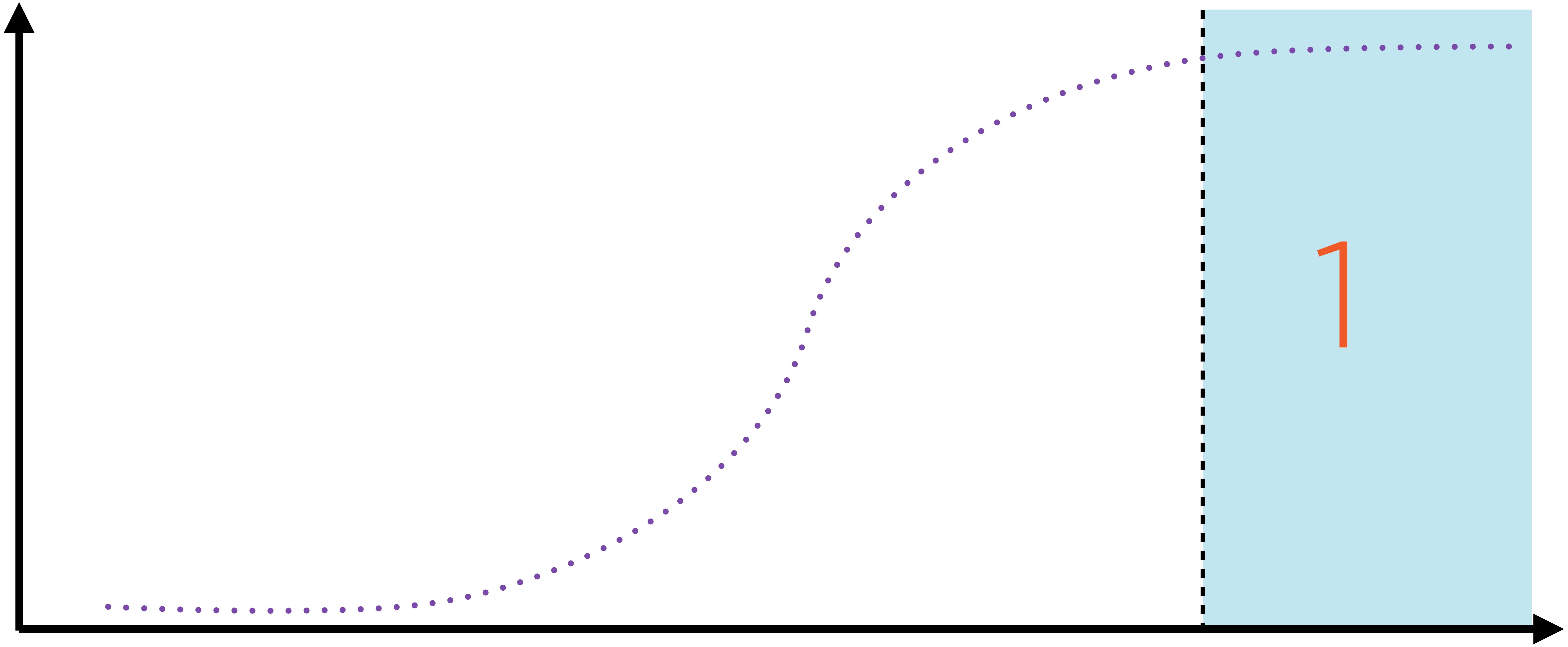
Working Hard, Fast, Smart



Working Hard, Fast, Smart

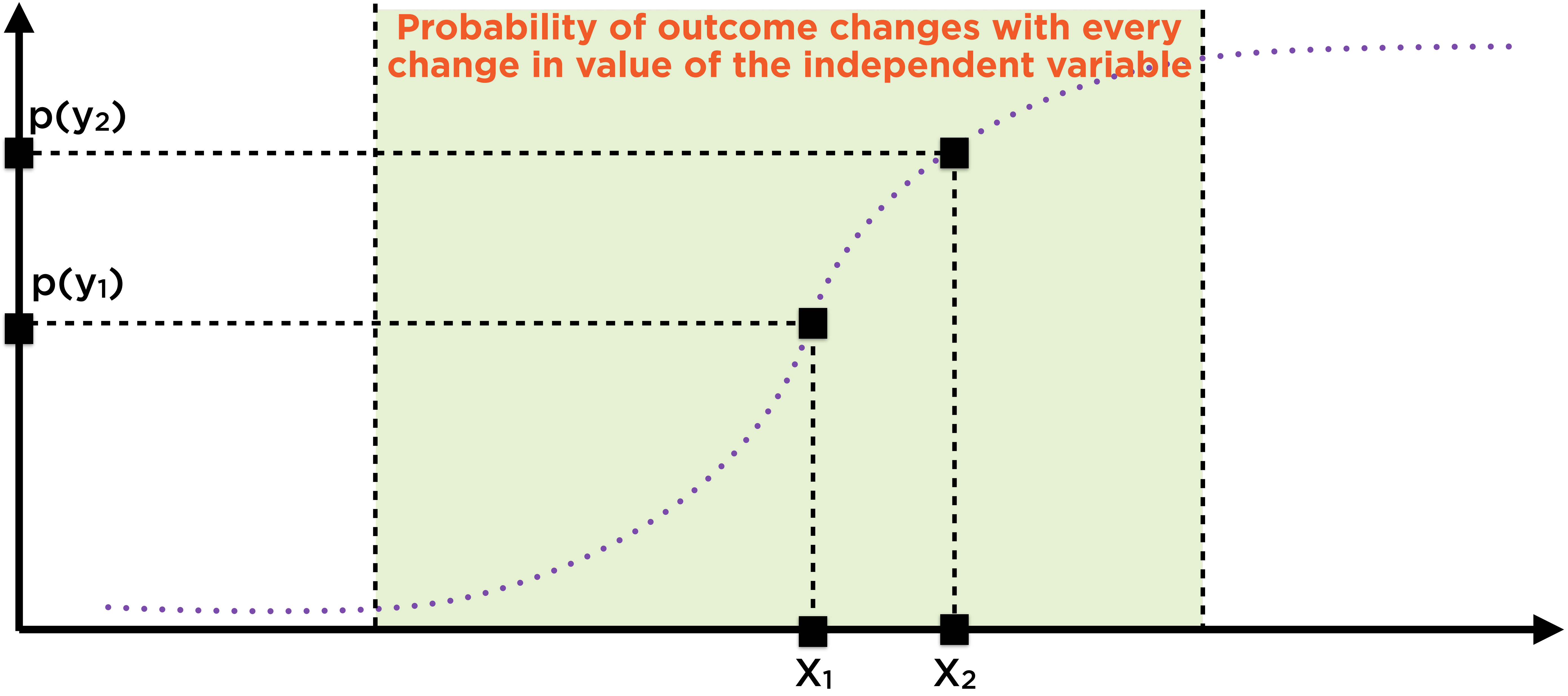


Working Hard, Fast, Smart



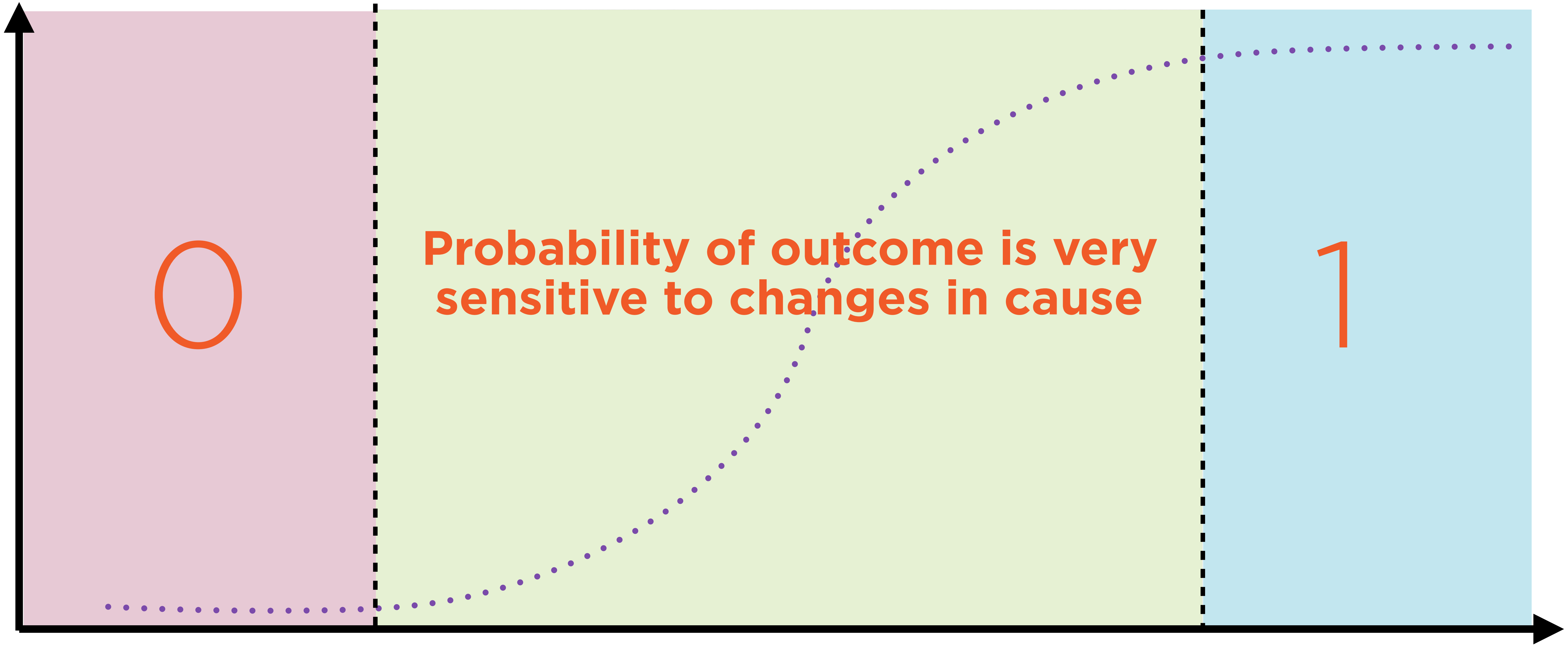
Maximum value of $p(y_i)$

Working Hard, Fast, Smart



Between maximum and minimum values of $p(y_i)$

Logistic Regression



$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

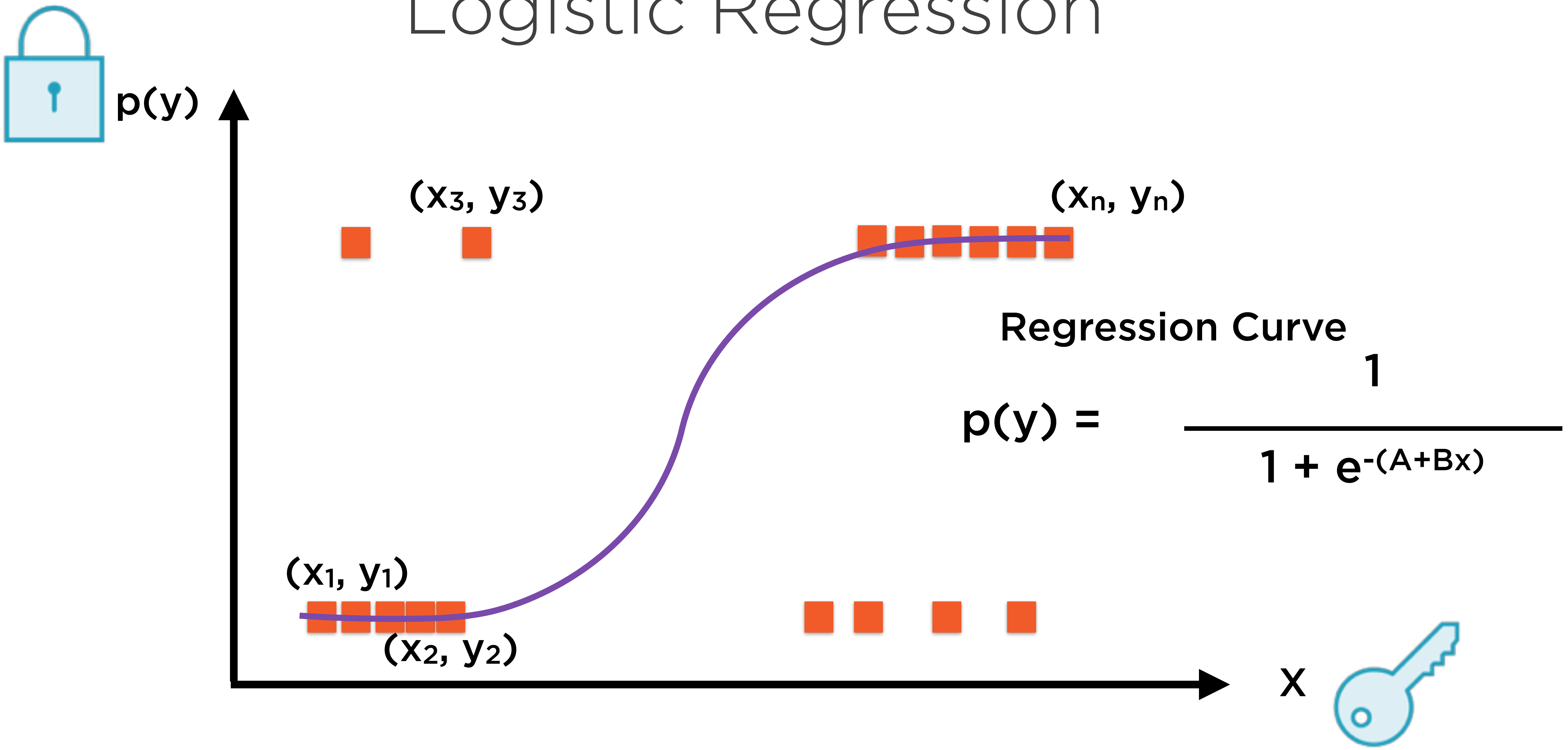
Logistic Regression fits an **S-curve** to estimate how probabilities of categorical variables are influenced by causes

Solving the Logistic Regression Problem via Maximum Likelihood Estimation (MLE)

Logistic Regression

Represent all n points as (x_i, y_i), where i = 1 to n

Logistic Regression



Represent all n points as (x_i, y_i), where i = 1 to n

Logistic Regression

Regression Equation:

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

Solve for A and B that “best fit” the data

A Thought Experiment

Toss n coins

Head: $y_i = 1$

Tail: $y_i = 0$

Probability of Head = p_i

Probability of Tail = $1-p_i$

A Thought Experiment

Coin i	Result	y_i	Probability
1	Heads	1	p_1
2	Tails	0	$1-p_2$
3	Heads	1	p_3
4	Heads	1	p_4
5	Tails	0	$1-p_5$
6	Tails	0	$1-p_6$
7	Heads	1	p_7
8	Heads	1	p_8
9	Heads	1	p_9
…	…	…	…
n	Tails	0	$1-p_n$

A Thought Experiment

Probability of independent events = product of individual probabilities

A Thought Experiment

Overall likelihood of getting these results

$$L = (p_1)*(1-p_2)*(p_3)*(p_4)*(1-p_5)...*(1-p_n)$$

A Thought Experiment

Conveniently combine probabilities of head or tail into one expression

Outcome of coin i = $p_i^{y_i}(1-p_i)^{1-y_i}$

If outcome = Head

$y_i = 1$

$$p_i^{y_i}(1-p_i)^{1-y_i} = p_i^1(1-p_i)^0 = p_i$$

If outcome = Tail

$y_i = 0$

$$p_i^{y_i}(1-p_i)^{1-y_i} = p_i^0(1-p_i)^1 = 1 - p_i$$

Tossing n Coins

$$L = (p_1)*(1-p_2)*(p_3)*(p_4)*(1-p_5)...*(1-p_n)$$

$$p_i^{y_i}(1-p_i)^{1-y_i}$$

$$L = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$$

π denotes product of multiple terms

Tossing n Coins

$$L = \prod_{i=1}^{n} p_i^{y_i}(1-p_i)^{1-y_i}$$

Transform equation by taking natural log (ln)

$$LL = \ln L = \sum_{i=1}^{n} [y_i \ln(p_i) + (1-y_i)\ln(1-p_i)]$$

$\sum$ denotes sum of multiple terms

Logistic Regression

$$p(y_i) = \frac{1}{1 + e^{-(A+Bx_i)}}$$

Solve for A and B that "best fit" the data

Tossing n Coins

$$LL = \ln L = \sum_{i=1}^{n} [y_i \ln(p_i) + (1-y_i)\ln(1-p_i)]$$

The “best fit” values of A and B are those that maximise this likelihood

Maximum Likelihood Estimation (MLE)

Solving the Logistic Regression Problem via Linear Regression

A Thought Experiment

Toss n coins

Head: $y_i = 1$

Tail: $y_i = 0$

Probability of Head = p_i

Probability of Tail = $1-p_i$

A Thought Experiment

Coin i	Result	y_i	Probability
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7	Heads	1	p_7
8	Heads	1	p_8
9	Heads	1	p_9
…	…	…	…
n	Tails	0	$1-p_n$

A Thought Experiment

Coin i	Result	y_i	x_i	Probability
1	Heads	1	x_1	p_1
2	Tails	0	x_2	$1-p_2$
3	Heads	1	x_3	p_3
4	Heads	1	x_4	p_4
5	Tails	0	x_5	$1-p_5$
6	Tails	0	...	$1-p_6$
7	Heads	1	...	p_7
8	Heads	1	...	p_8
9	Heads	1	...	p_9
...	...	...	...	...
n	Tails	0	x_n	$1-p_n$

A Thought Experiment

Coin i	Result	y_i	x_i	Probability
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4	Heads	1	x_4	p_4
5	Tails	0	x_5	$1-p_5$
6	Tails	0	…	$1-p_6$
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8	Heads	1	…	p_8
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A Thought Experiment

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A Thought Experiment

Coin i	Result	y_i	x_i	Probability
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3	Heads	1	x_1	p_1
4	Heads	1	x_1	p_1
5	Tails	0	x_1	$1-p_1$

A Thought Experiment

Unique x_i	Frequency(y = 1)	Frequency(y = 0)	$p(y_i)$	$1 - p(y_i)$
x_1	3	2	$p_1 = 3/(3+2) = 3/5$	2/5

Collapse these 5 rows into a single row, where the probabilities "fit" the data

If x is continuous, we will need to create ranges of x-values

Frequency Table

Unique x_i	Frequency(y = 1)	Frequency(y = 0)	$p(y_i)$	$1 - p(y_i)$
x_1	3	2	$p_1 = 3/(3+2) = 3/5$	2/5
x_2	8	12	$p_2 = 8/(8+12) = 2/5$	3/5
...	...	...	...	...

Create a frequency table with 1 row for each unique value of x

Frequency Table

Unique x_i	Frequency(y = 1)	Frequency(y = 0)	$p(y_i)$	$1 - p(y_i)$
x_1	3	2	$p_1 = 3/(3+2) = 3/5$	2/5
x_2	8	12	$p_2 = 8/(8+12) = 2/5$	3/5
...	...	...	...	...

Now, unlike with the MLE approach, each p_i is a continuous variable

Odds from Probabilities

$$\text{Odds}(p) = \frac{p}{1-p}$$

Odds of an Event

$$p = \frac{1}{1 + e^{-(A+Bx)}}$$

$$p = \frac{e^{A + Bx}}{1 + e^{A + Bx}}$$

$$1 - p = 1 - \frac{e^{A + Bx}}{1 + e^{A + Bx}}$$

Odds of an Event

$$1 - p = 1 - \frac{e^{A + Bx}}{1 + e^{A + Bx}}$$

$$1 - p = \frac{1 + e^{A + Bx} - e^{A + Bx}}{1 + e^{A + Bx}}$$

$$1 - p = \frac{1}{1 + e^{A + Bx}}$$

Odds of an Event

$$p = \frac{e^{A + Bx}}{1 + e^{A + Bx}}$$

$$1 - p = \frac{1}{1 + e^{A + Bx}}$$

$$\text{Odds}(p) = \frac{p}{1 - p} = e^{A + Bx}$$

Logit Is Linear

$$\text{Odds}(p) = \frac{p}{1 - p} = e^{A + Bx}$$

$$\text{logit}(p) = A + Bx$$

ln(Odds(p)) is called the logit function

Logit Is Linear

$$\ln \text{Odds}(p) = \ln(p) - \ln(1-p)$$

$$p = \frac{1}{1 + e^{-(A+Bx)}}$$

$$\text{logit}(p) = \ln \text{Odds}(p) = A + Bx$$

This is a linear function!

Logit Is Linear

$$\text{logit}(p) = A + Bx$$

$$\begin{aligned}
\text{logit}(p_1) &= A + Bx_1 \\
\text{logit}(p_2) &= A + Bx_2 \\
\text{logit}(p_3) &= A + Bx_3 \\
&\ldots \\
\text{logit}(p_n) &= A + Bx_n
\end{aligned}$$

Tossing n Coins - Linear

$$\text{logit}(p) = A + Bx$$

$$\text{logit}(p_1) = A + Bx_1 + \varepsilon_1$$

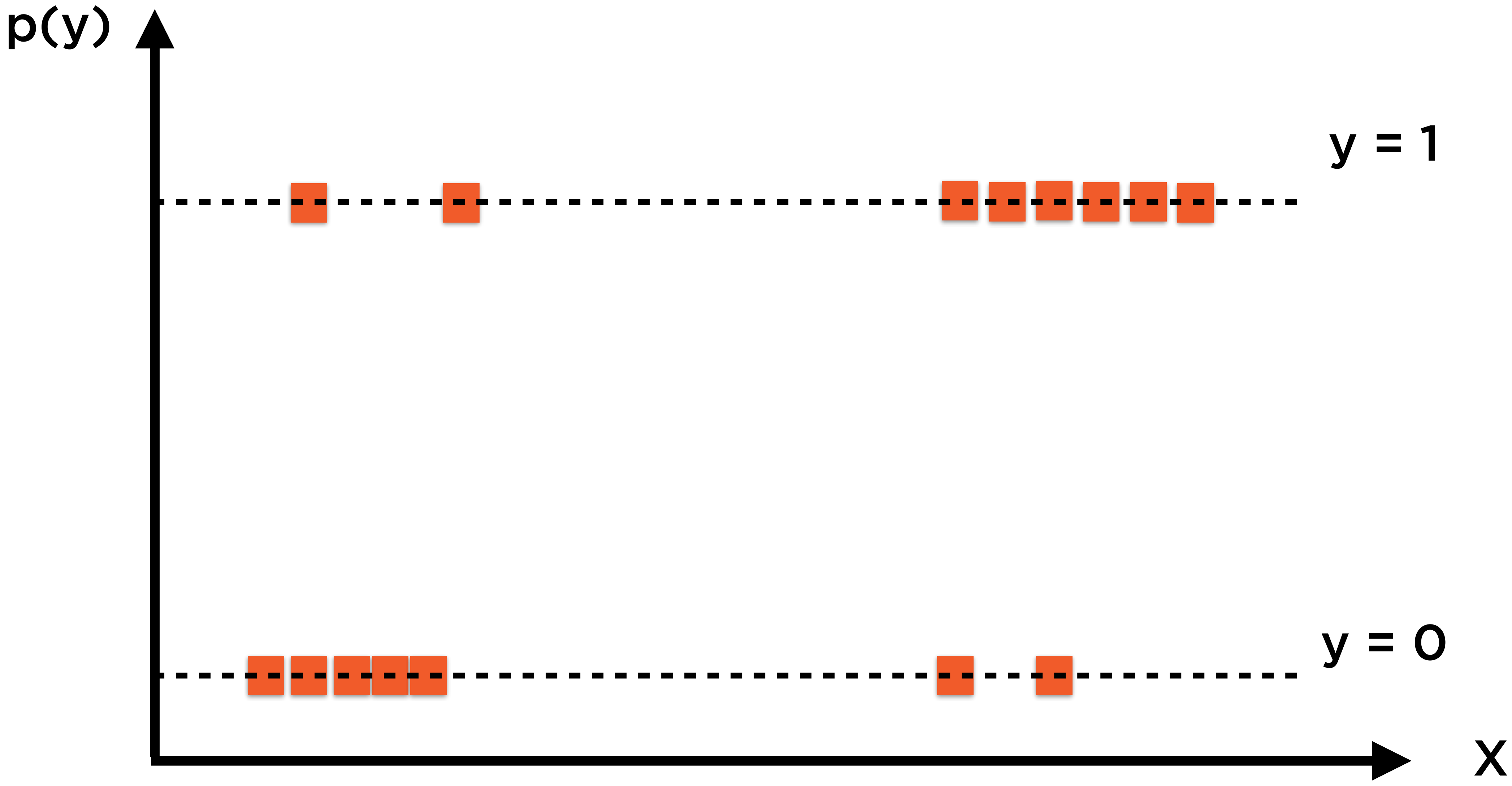
$$\text{logit}(p_2) = A + Bx_2 + \varepsilon_2$$

$$\text{logit}(p_3) = A + Bx_3 + \varepsilon_3$$

$$\ldots \qquad \ldots \qquad \ldots$$

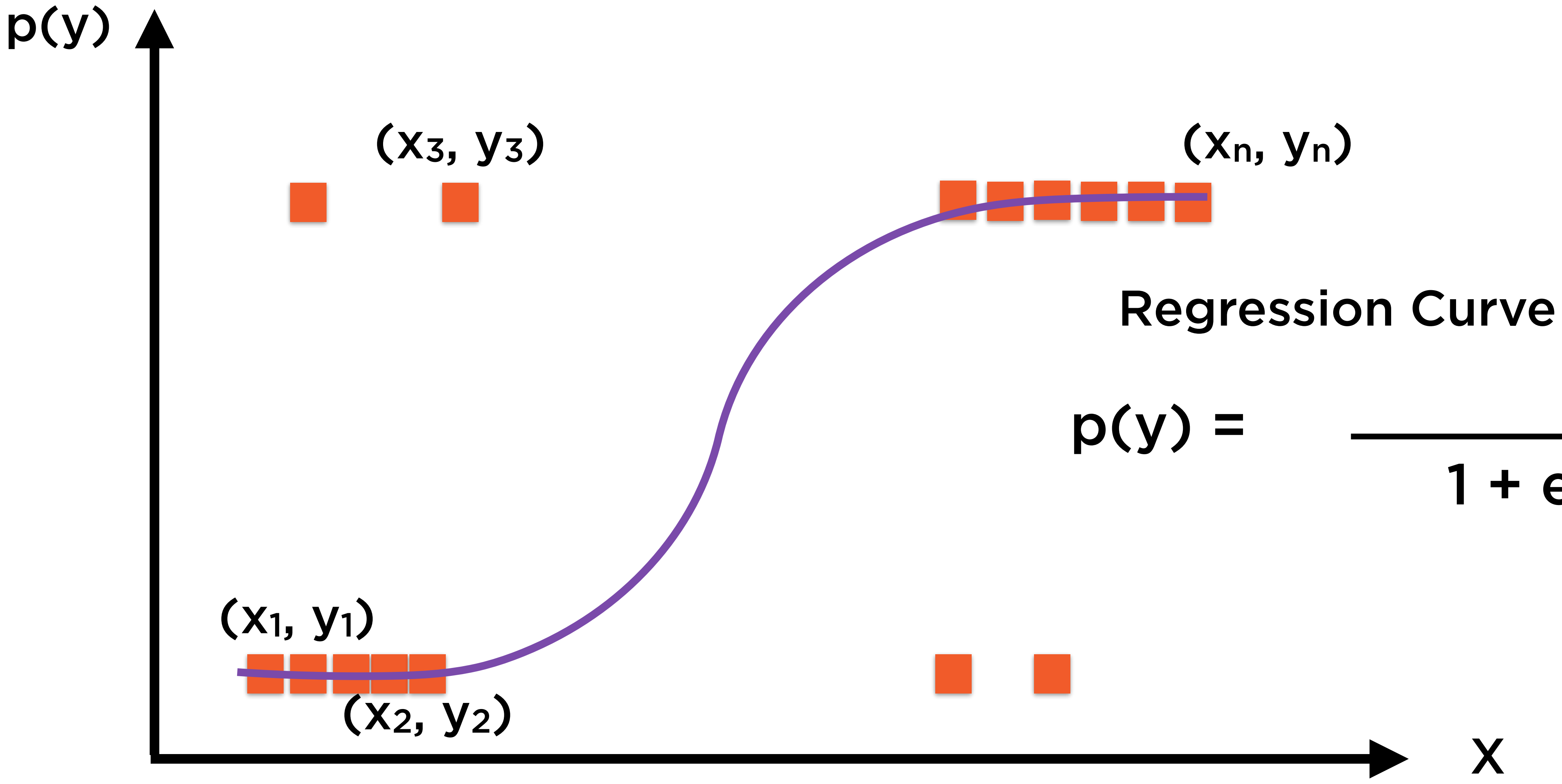
$$\text{logit}(p_n) = A + Bx_n + \varepsilon_n$$

Logistic Regression



Represent all n points as
(x_i, y_i), where i = 1 to n

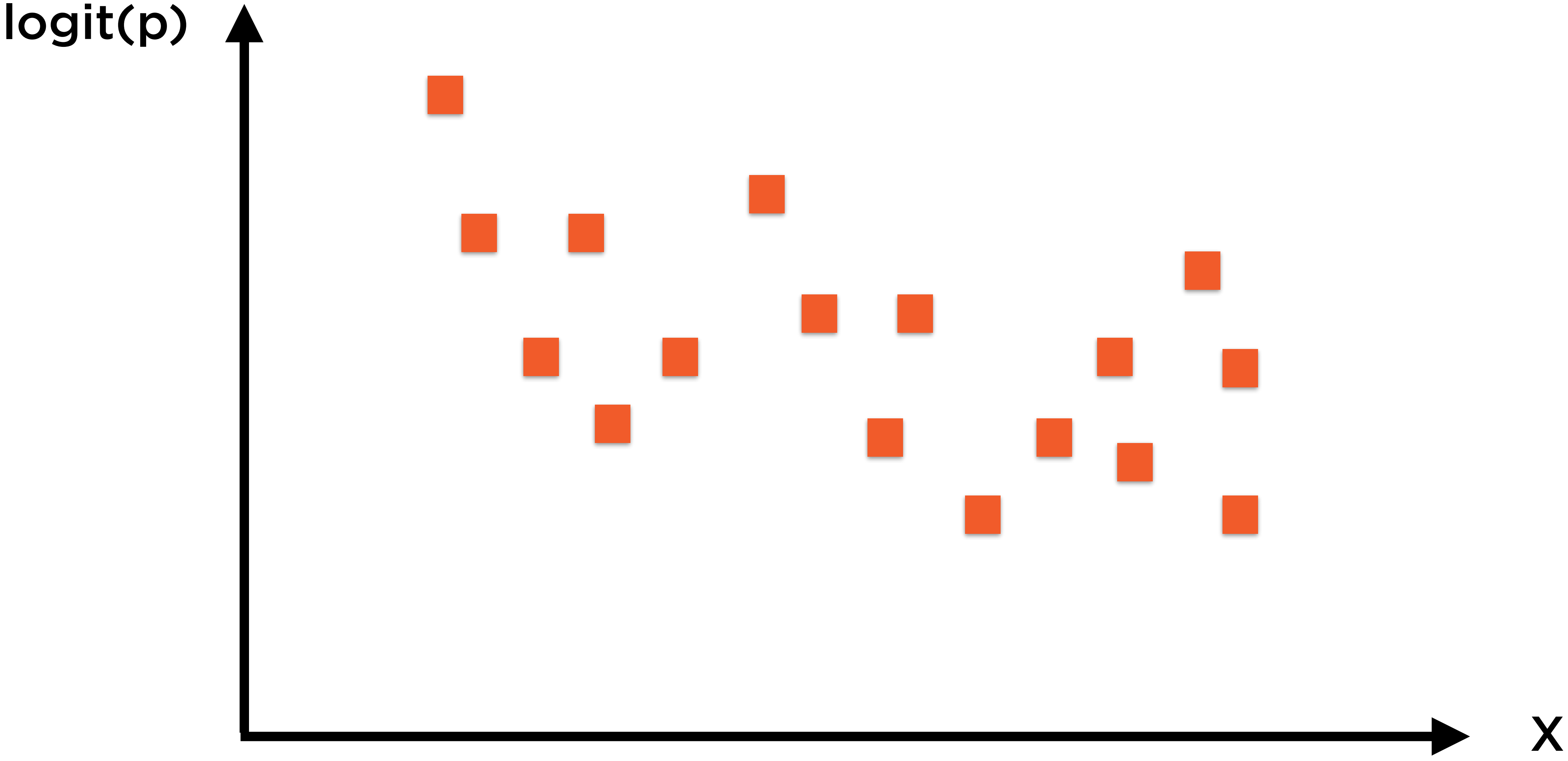
Logistic Regression



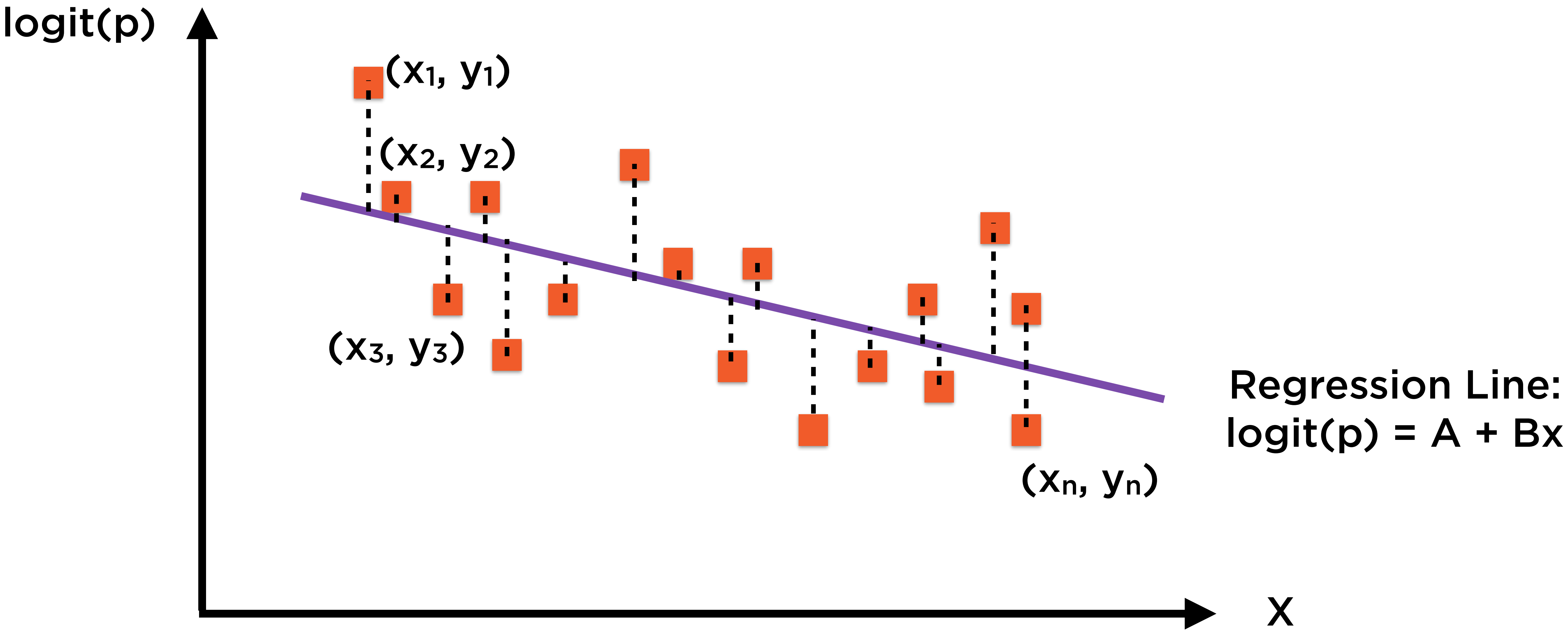
$$p(y) = \frac{1}{1 + e^{-(A+Bx)}}$$

Represent all n points as
(x_i, y_i), where i = 1 to n

Linear Regression



Linear Regression



Represent all n points as
(x_i,y_i), where i = 1 to n

Logistic Regression can be solved via **linear regression on the logit function** (log of the odds function)

Linear Regression Estimation Methods

Method of moments

Method of least squares

Maximum likelihood estimation

Cookie cutter techniques to determine the values of A and B (regression coefficients)

Binomial and Multinomial Logistic Regression

Binomial and Multinomial

Binomial

Two categorical outcomes

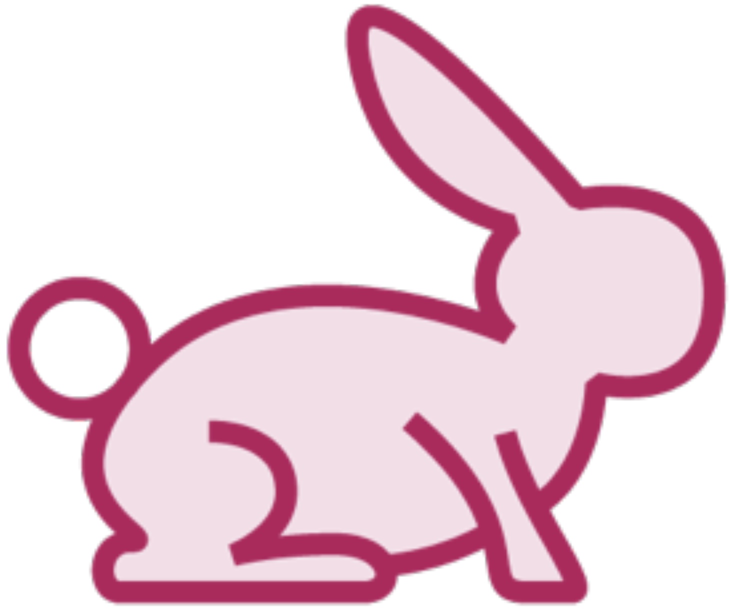
(Head/Tail; True/False)

Multinomial

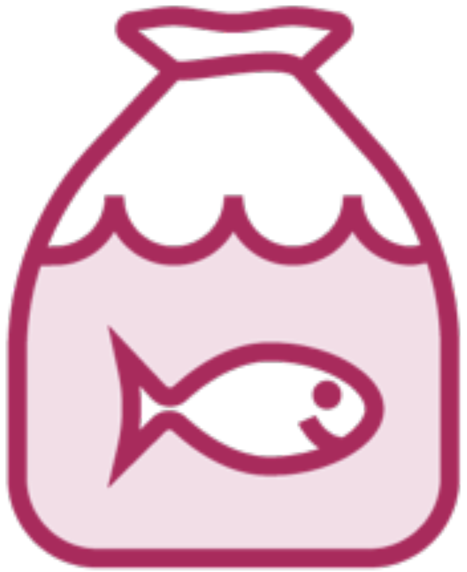
>Two categorical outcomes

(Days in a week; Months in a year)

Binomial Is Binary



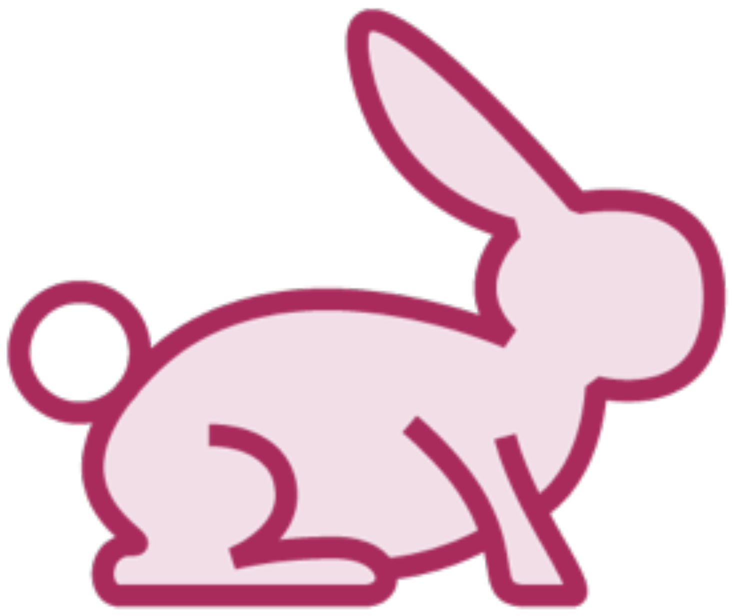
Mammals



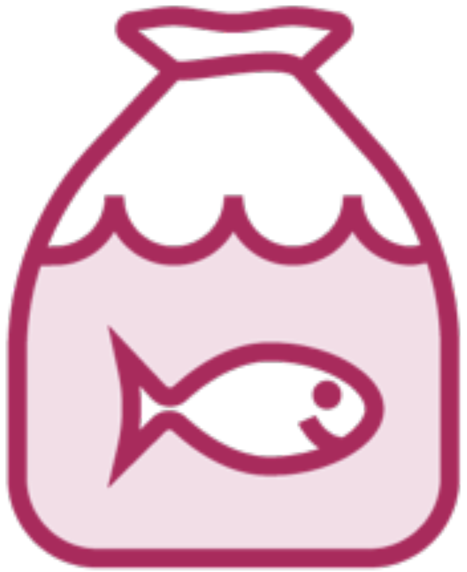
Fish

Whales: Mammals or Fish?

Binomial Is Binary



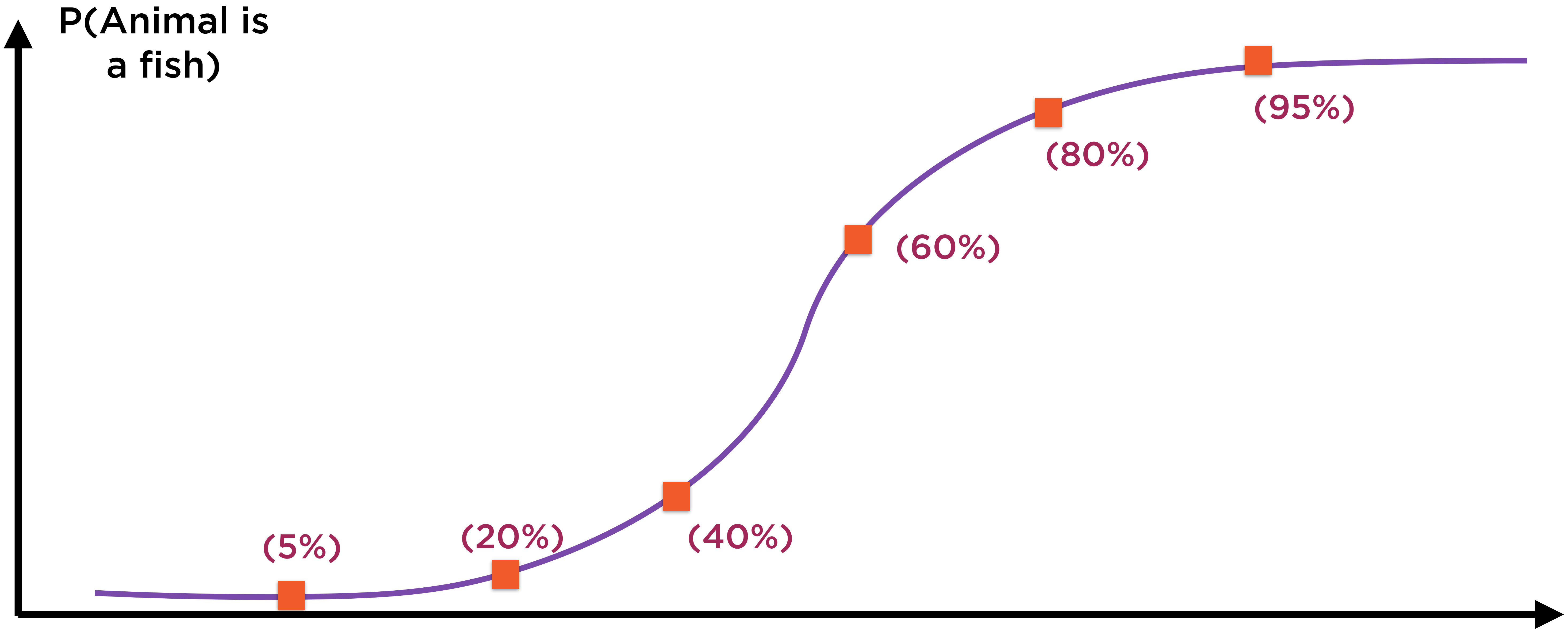
Mammals



Fish

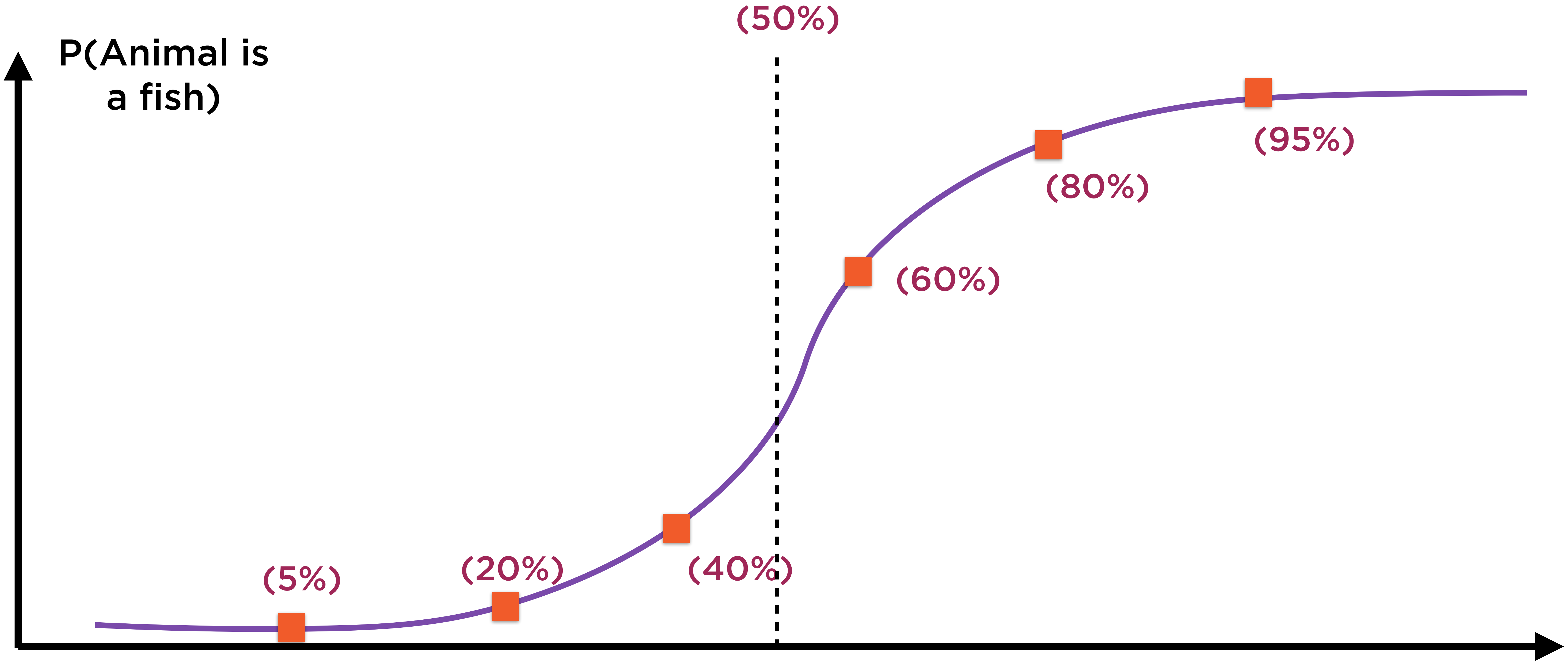
Result	Mammals	Fish
Label (y_i)	0	1
Probability ($p(y_i)$)	p	1-p

Binomial Is Binary



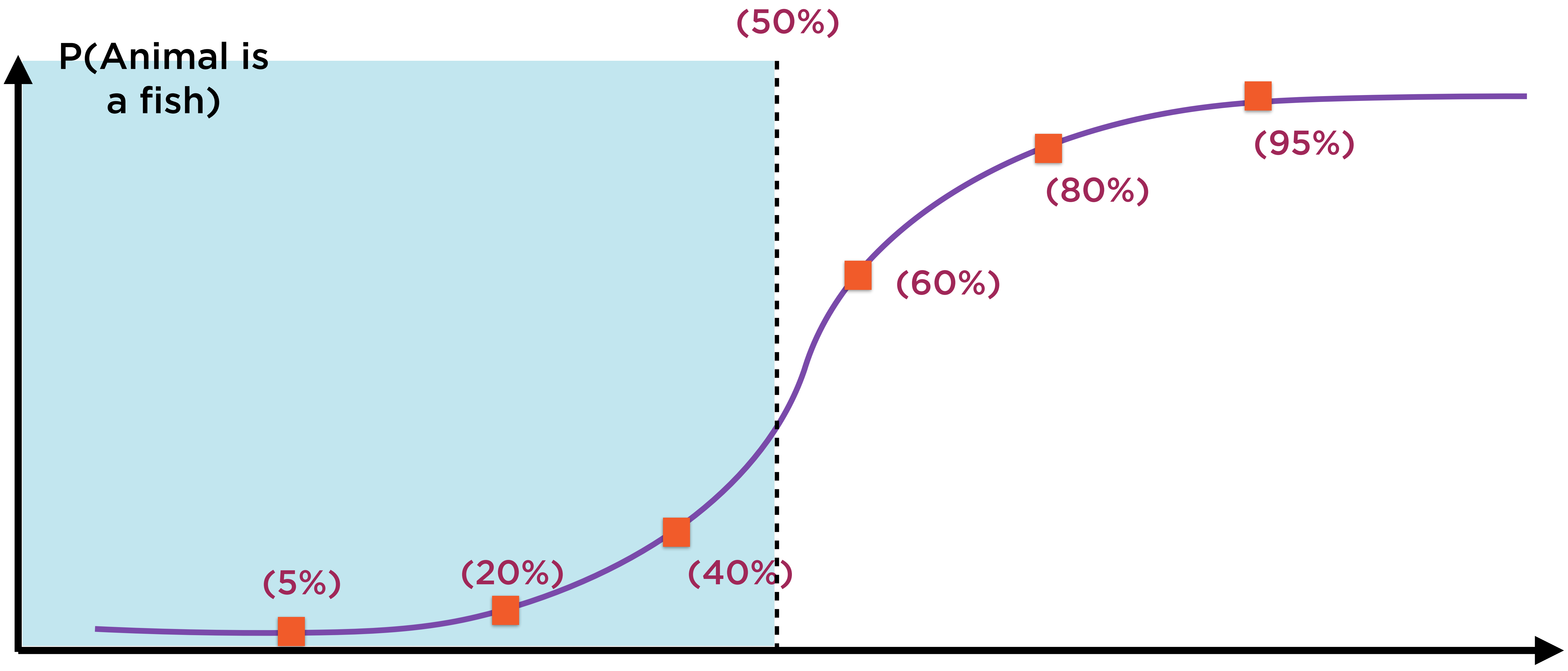
Whales: Mammals or Fish?

Binomial Is Binary



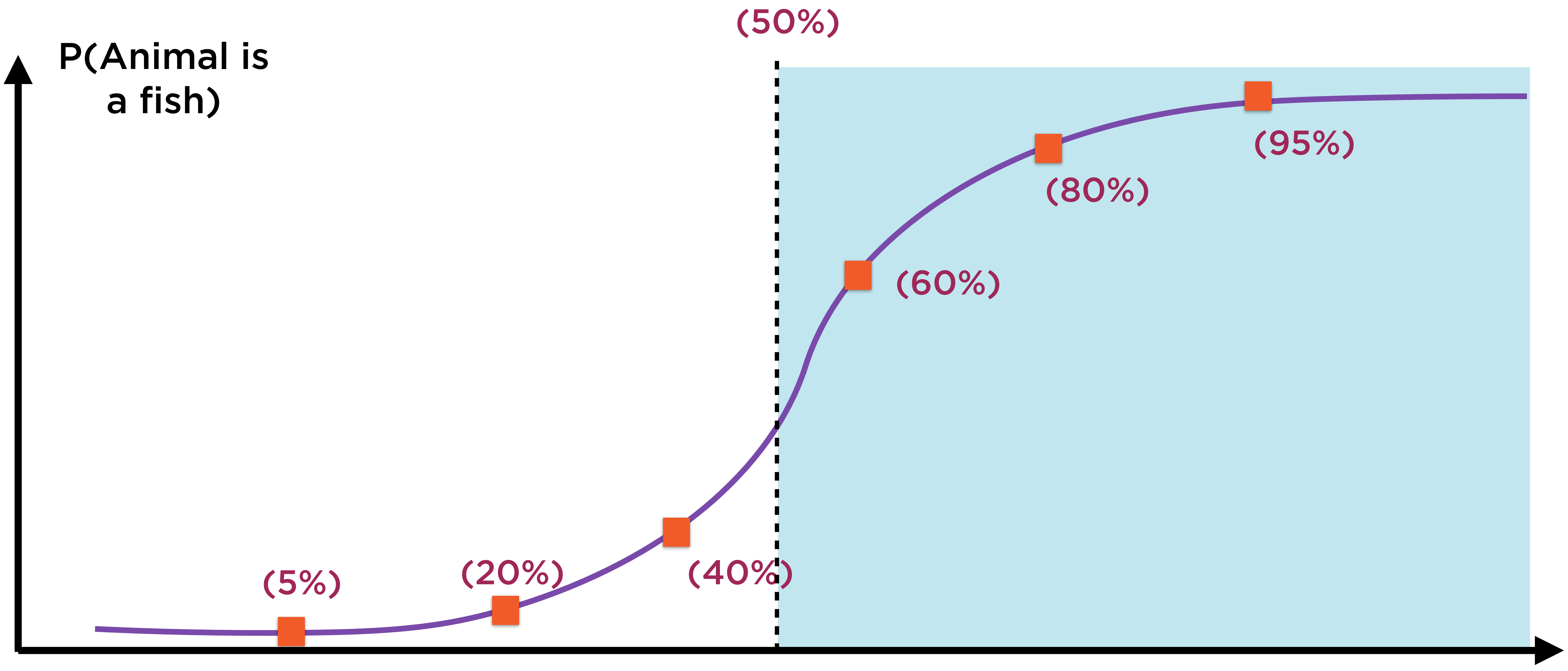
Rule of 50%

Binomial Is Binary



If probability < 50%, it's a mammal

Binomial Is Binary

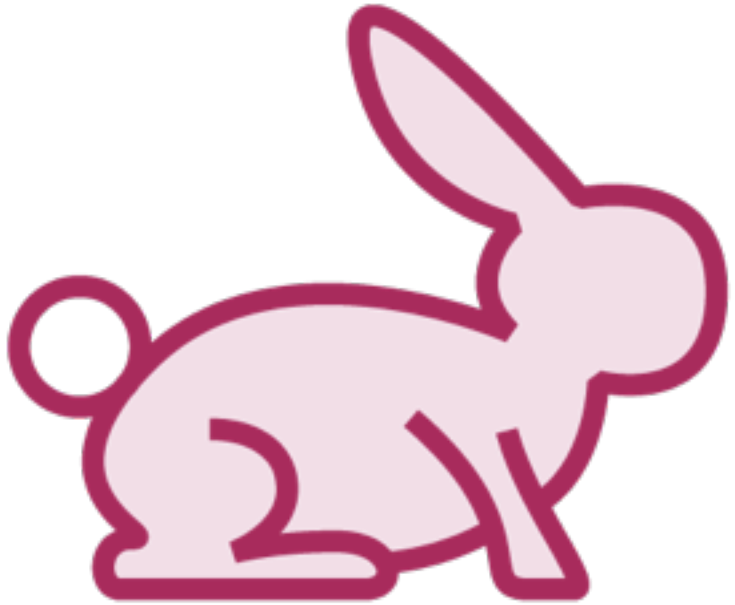


If probability > 50%, it’s a fish

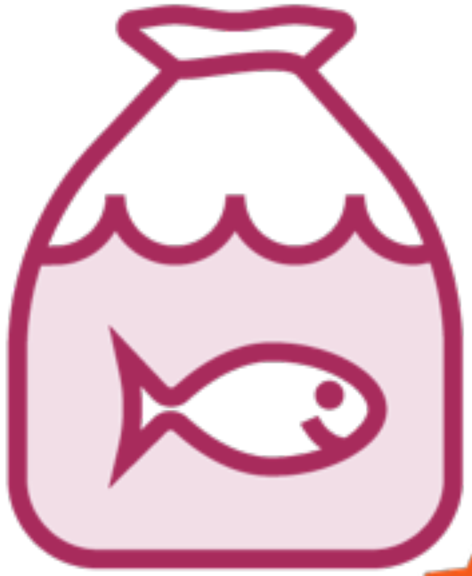
Binomial Is Binary

Probability of whales being Fish < 50%

Binomial Is Binary



Mammals



Fish

Probability of whales being Fish > 50%

Binomial and Multinomial

Binomial

Two categorical outcomes

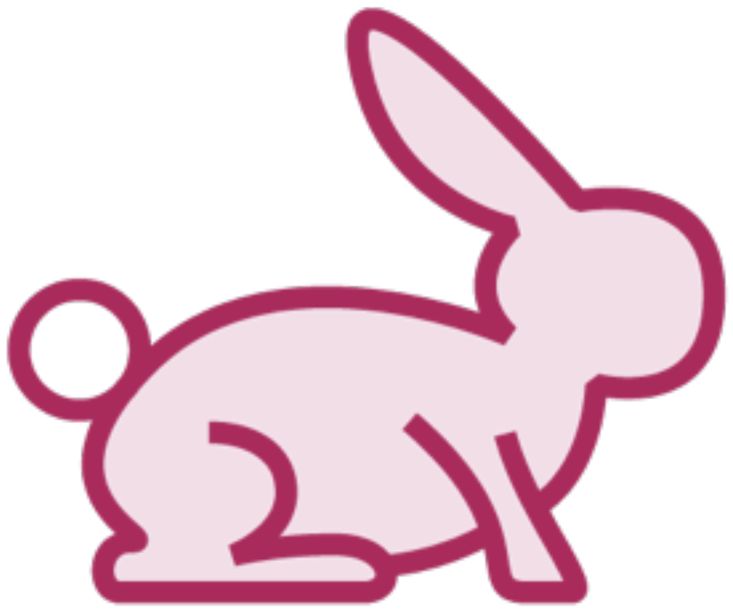
(Head/Tail; True/False)

Multinomial

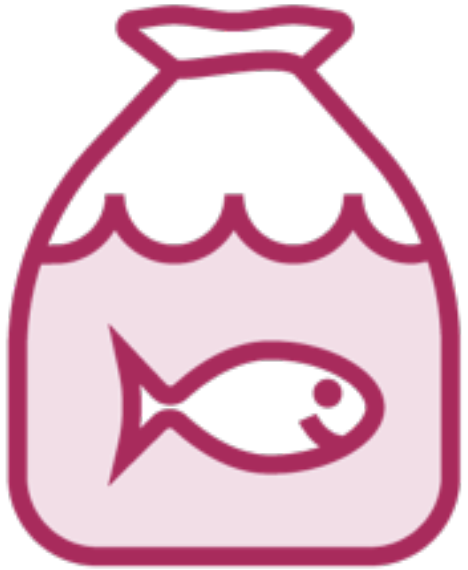
>Two categorical outcomes

(Days in a week; Months in a year)

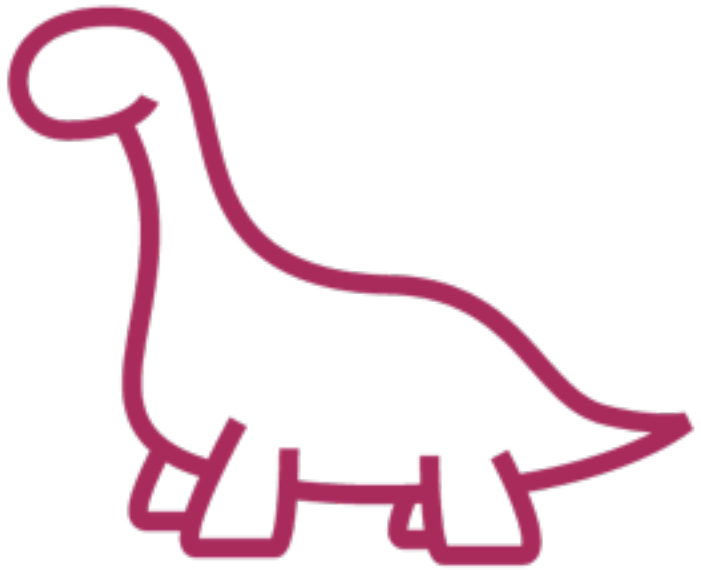
Multinomial Is Non-binary



Mammals



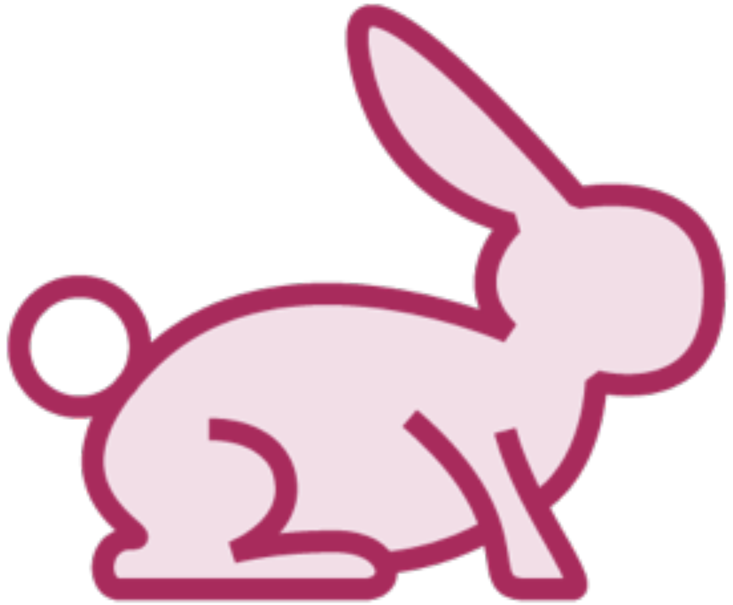
Fish



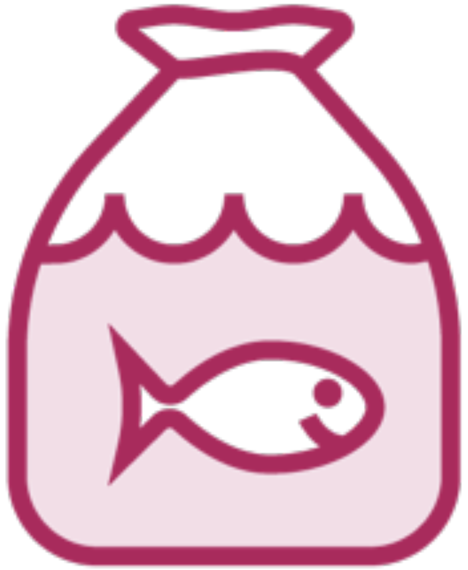
Reptiles

Whales: Mammals or Fish or Reptiles?

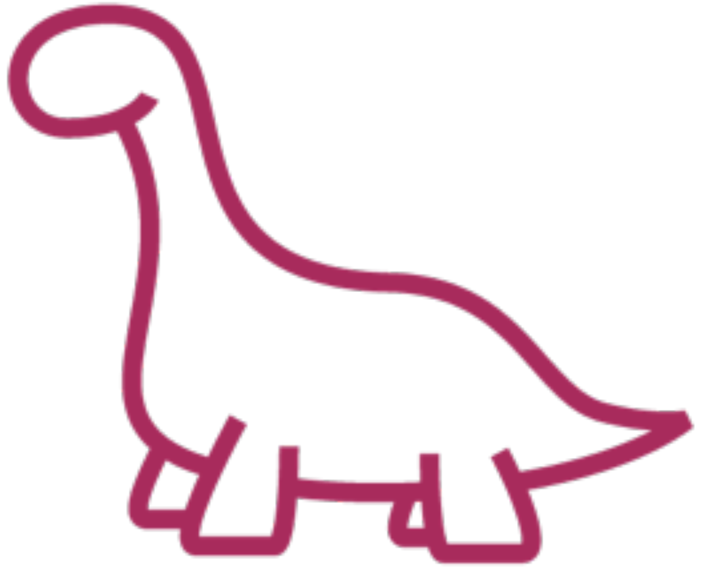
Multinomial Is Non-binary



Mammals



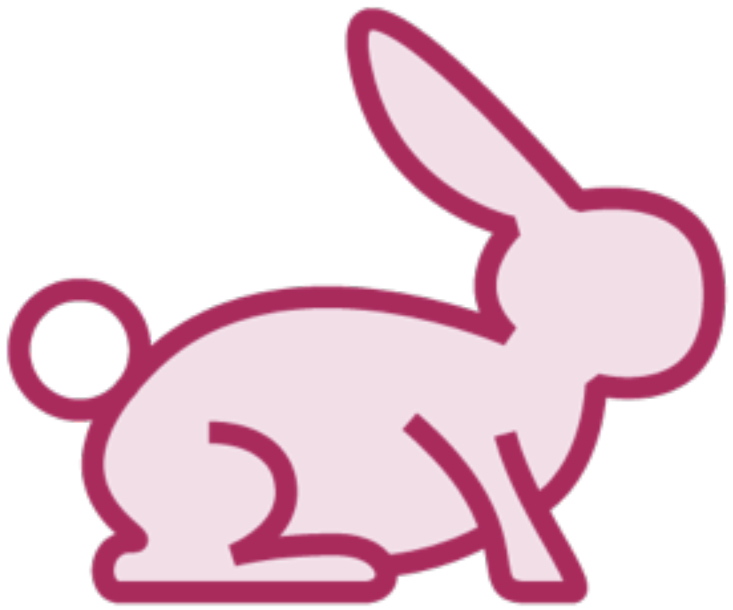
Fish



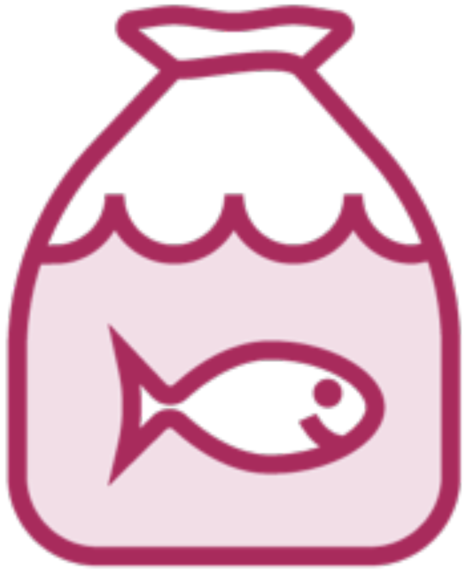
Reptiles

Result	Mammals	Fish	Reptiles
Label (y_i)	0	1	2
Probability ($p(y_i)$)	p_0	p_1	p_2
Probability ($p(y_i')$)	$1-p_0$	$1-p_1$	$1-p_2$

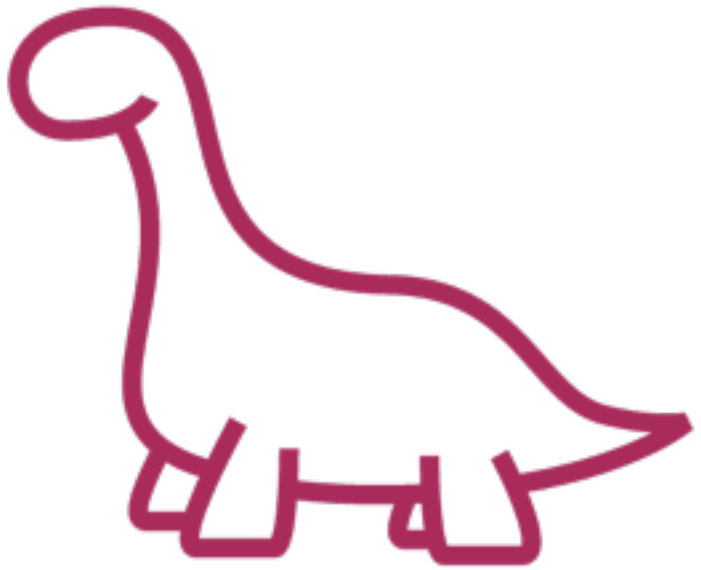
Multinomial Is Non-binary



Mammals



Fish

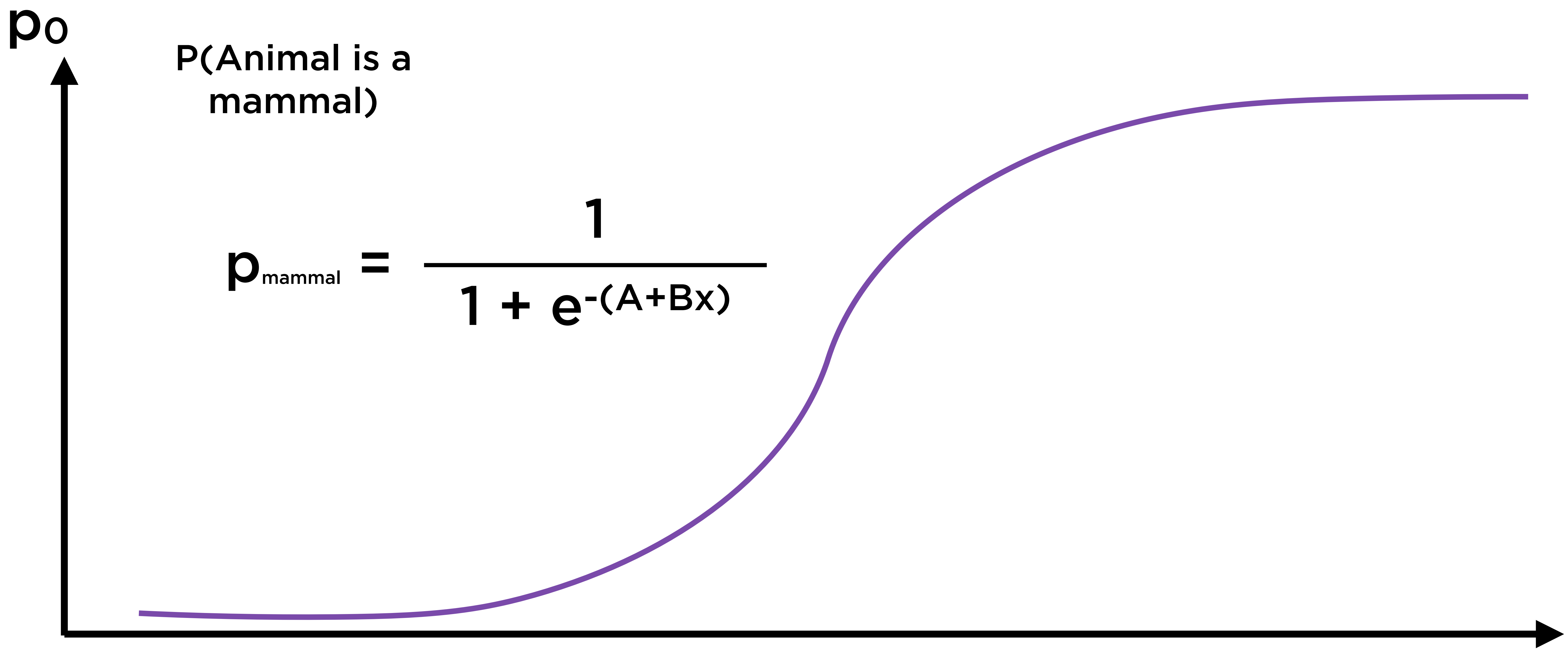


Reptiles

Whales: Mammals or Fish or Reptiles?

Run three logistic regressions

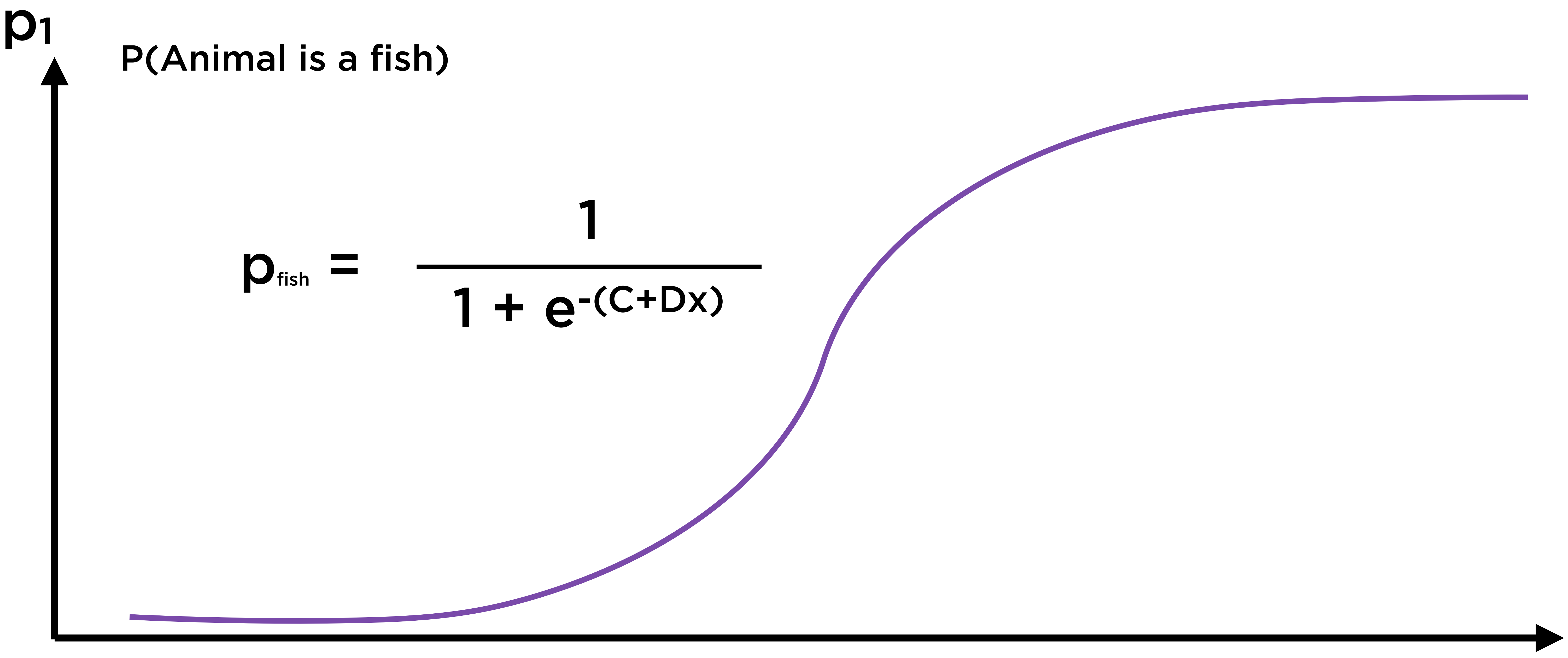
Multinomial Is Non-binary



$$p_{mammal} = \frac{1}{1 + e^{-(A+Bx)}}$$

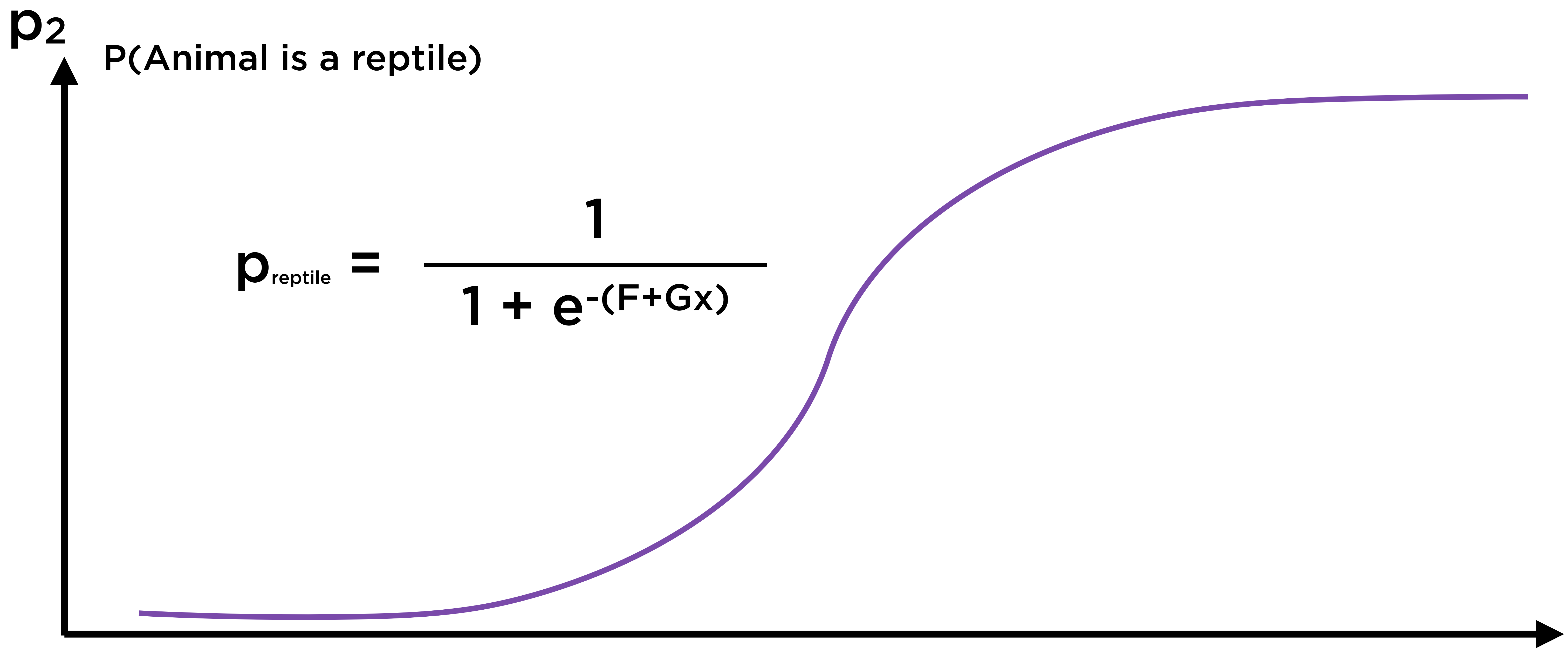
Mammals or Not?

Multinomial Is Non-binary



Fish or Not?

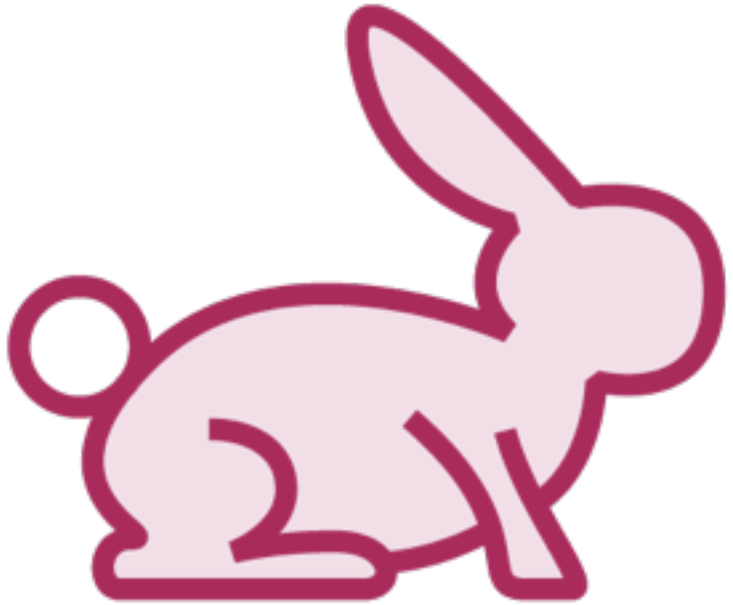
Multinomial Is Non-binary



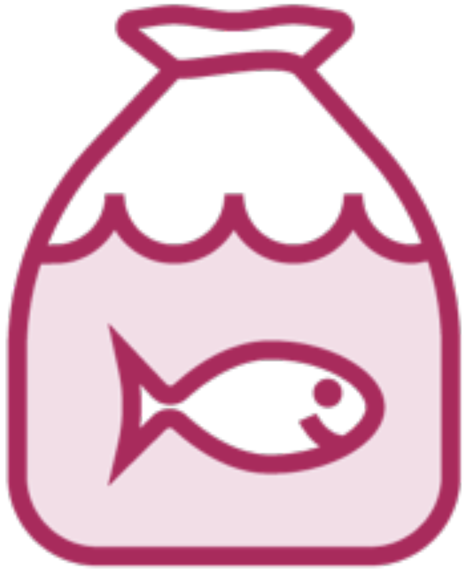
$$p_{reptile} = \frac{1}{1 + e^{-(F+Gx)}}$$

Reptiles or Not?

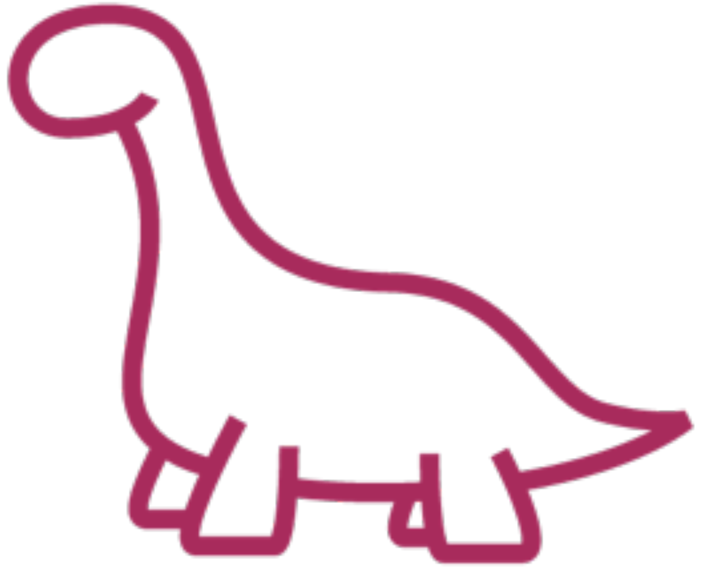
Multinomial Is Non-binary



Mammals



Fish

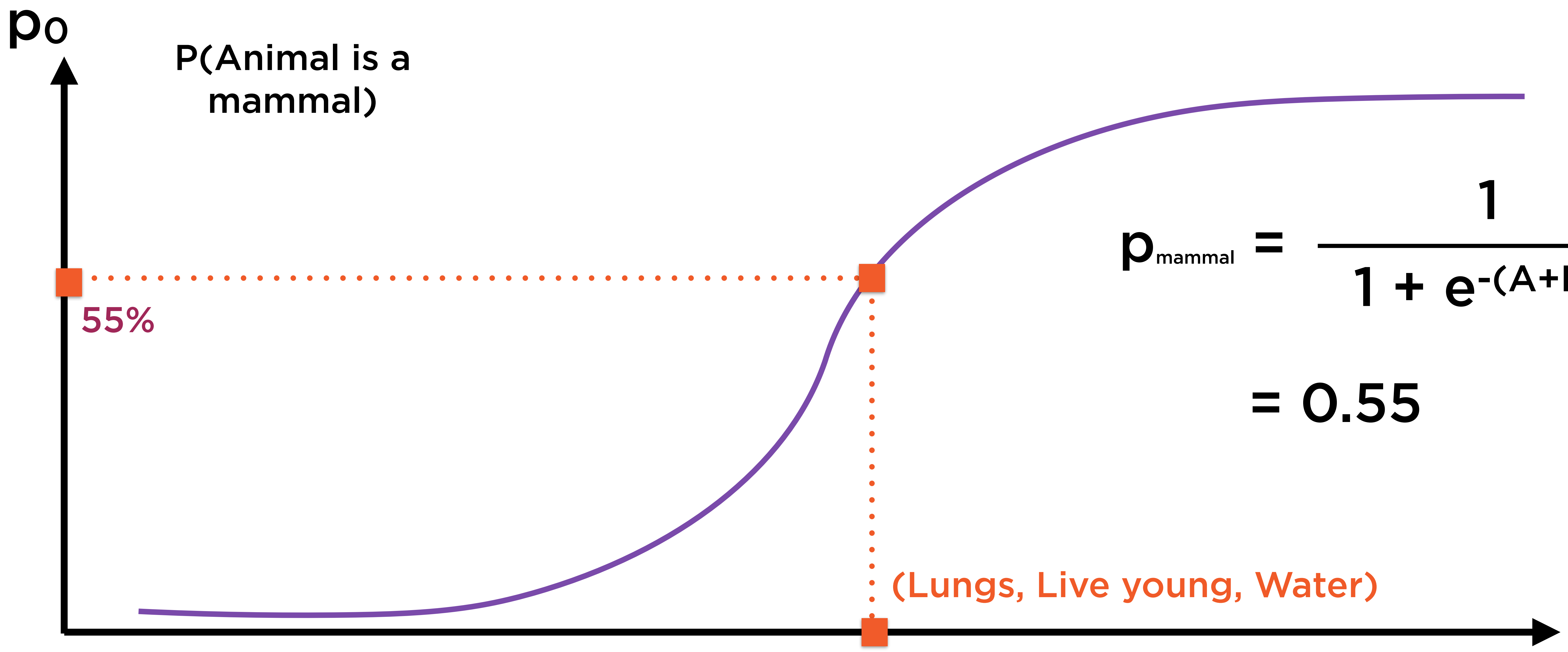


Reptiles

Whales: Mammals or Fish or Reptiles?

Choose the highest probability

Multinomial Is Non-binary

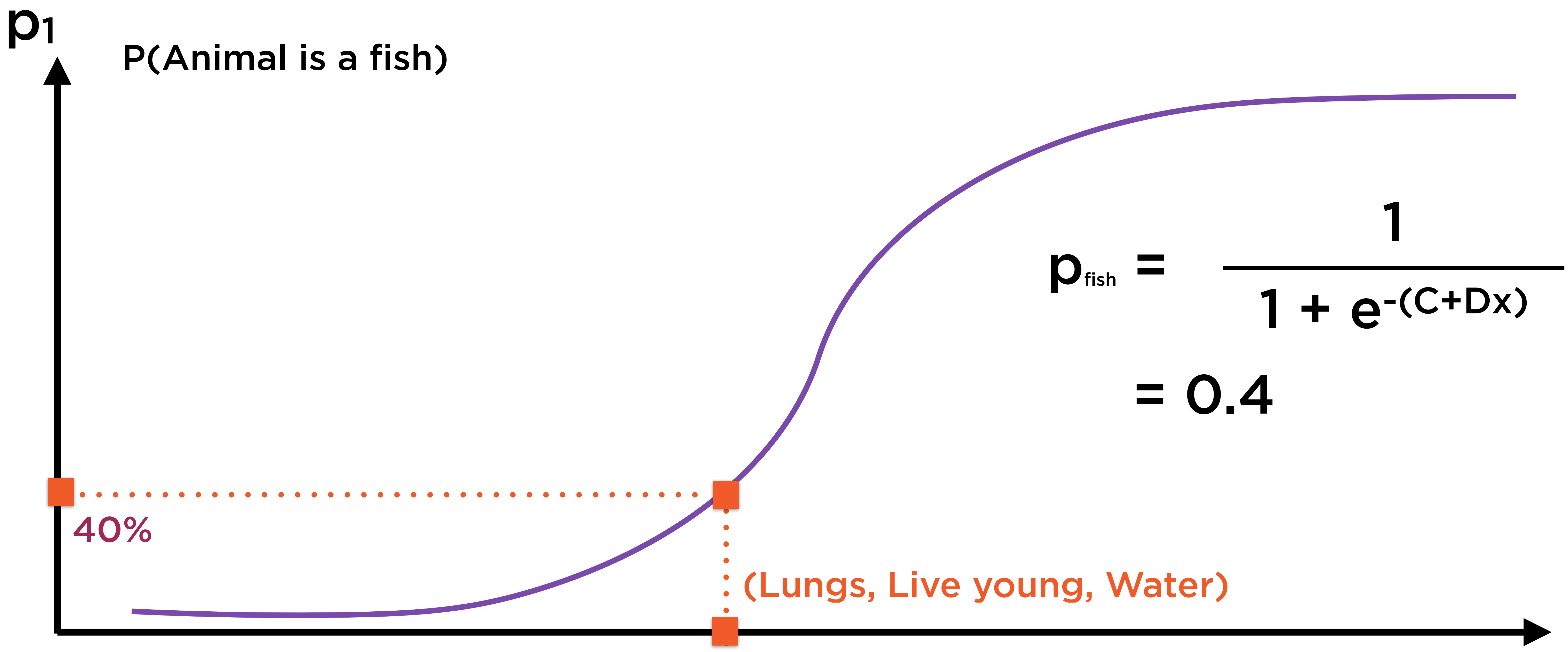


$$p_{mammal} = \frac{1}{1 + e^{-(A+Bx)}}$$

$$= 0.55$$

Mammals or Not?

Multinomial Is Non-binary

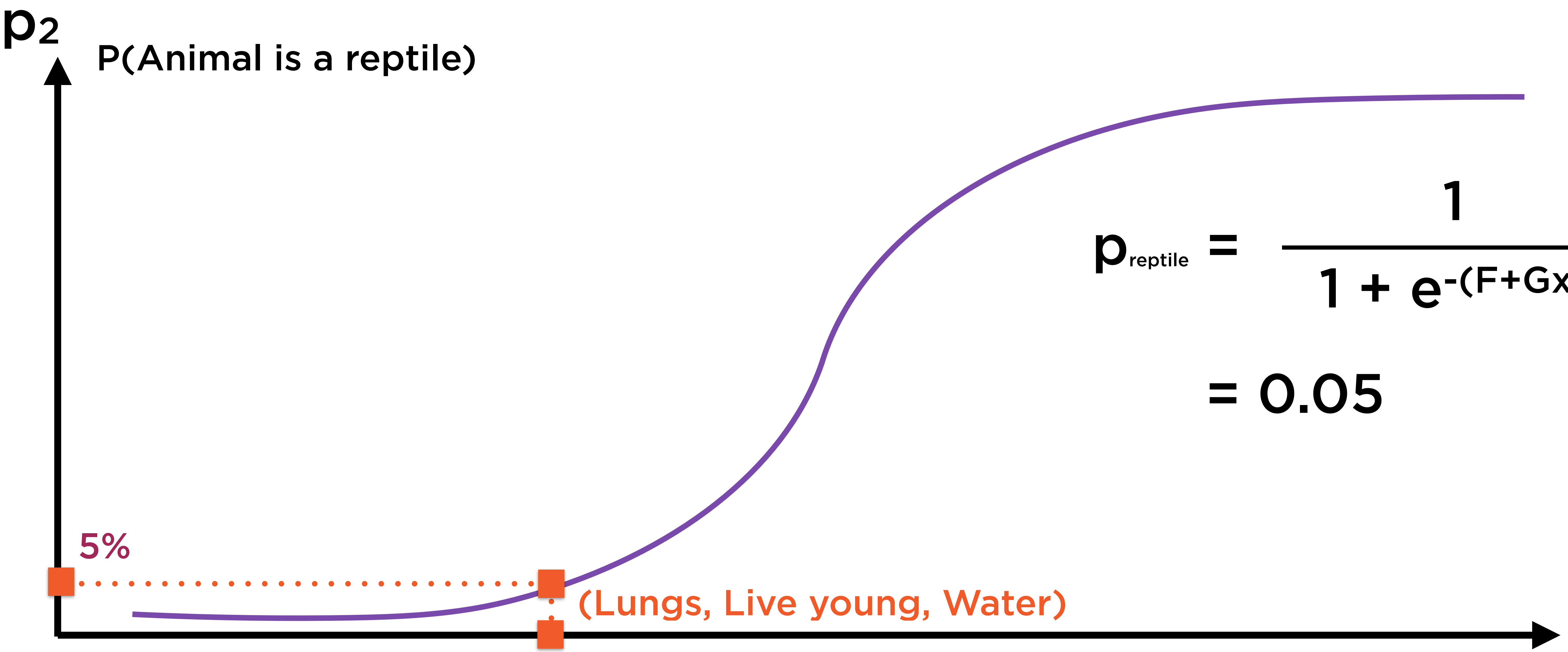


$$p_{fish} = \frac{1}{1 + e^{-(C+Dx)}}$$

$$= 0.4$$

Fish or Not?

Multinomial Is Non-binary



$$p_{reptile} = \frac{1}{1 + e^{-(F+Gx)}}$$

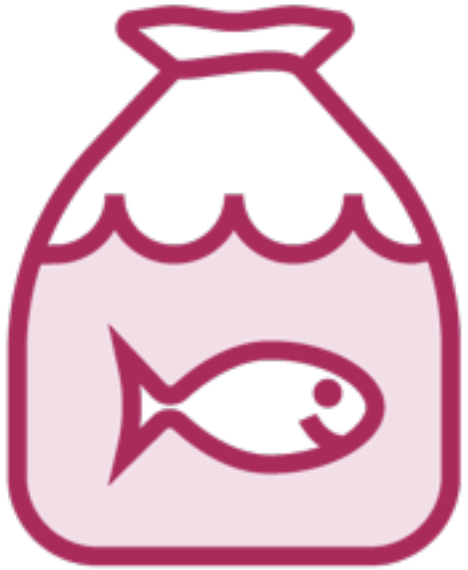
$$= 0.05$$

Reptiles or Not?

Multinomial Is Non-binary

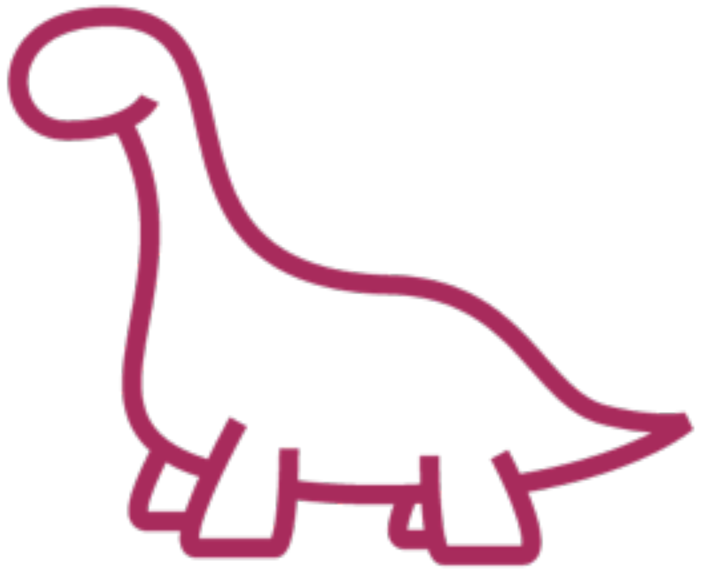
Mammals

$p_{mammal} = 0.55$



Fish

$p_{fish} = 0.4$

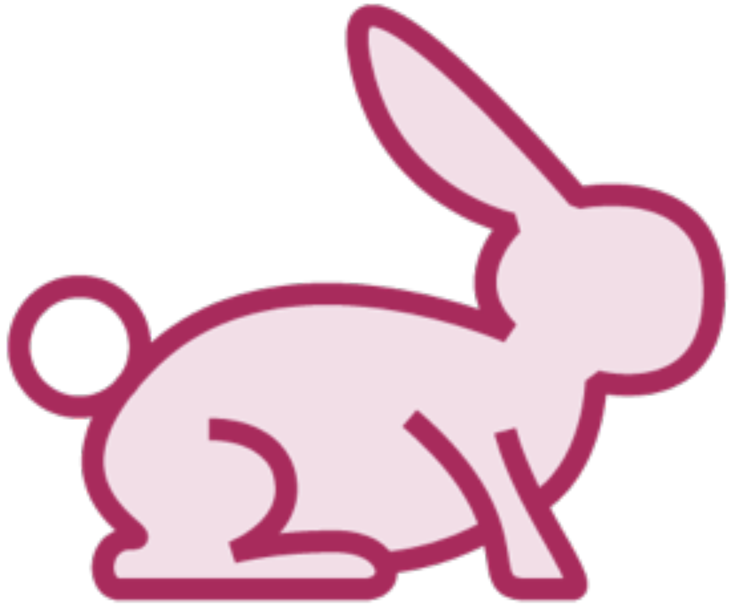


Reptiles

$p_{reptile} = 0.05$

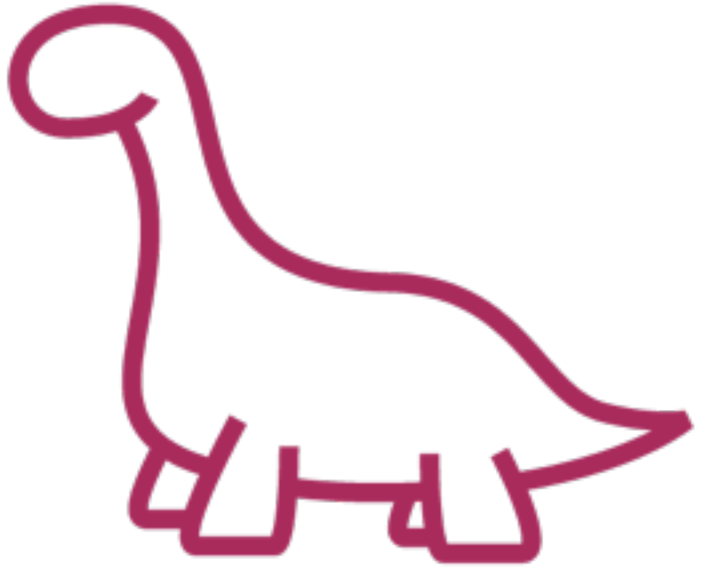
$$p_{mammal} > p_{fish} > p_{reptile}$$

Multinomial Is Non-binary



Mammals

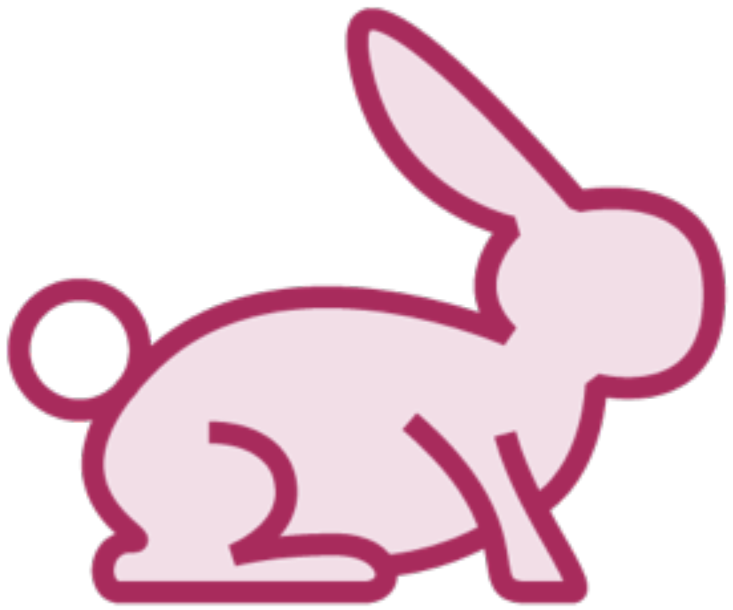
Fish



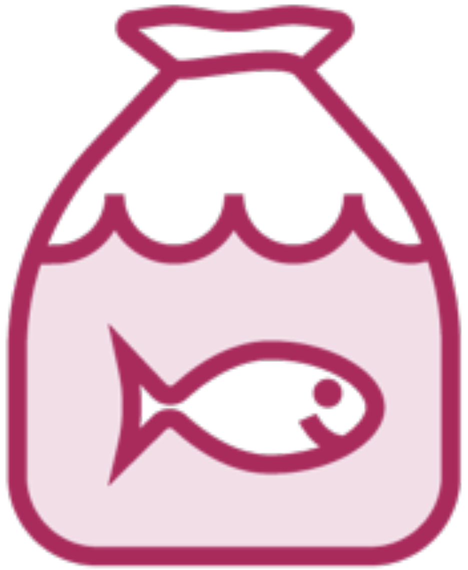
Reptiles

$$p_{mammal} < p_{fish} > p_{reptile}$$

Multinomial Is Non-binary



Mammals



Fish

Reptiles

$$p_{mammal} < p_{fish} < p_{reptile}$$

Binomial and Multinomial

Binomial

Two categorical outcomes

One logistic regression

Multinomial

N categorical outcomes

N logistic regressions

Regression: Excel, R or Python

Excel

Create a regression slide for an important presentation

R

Create a regression case study for a seminar

Python

Build trading model that scrapes websites, combines sentiment analysis and regression

A simple multinomial logistic regression technique uses N logistic models for N categories

Summary

Logistic regression fits an S-curve between probabilities and causes

S-curves have a standard mathematical form that is easy to estimate

Two equivalent methods of fitting S-curves are commonly used

One of these methods cleverly utilises linear regression in logistic regression

Logistic regression can be easily extended to non-binary categorical variables