Numerical implementation of nonlinear time integration scheme

using generalised-α method

Studienarbeit

von

Avinash Bapu Sreenivas

Institut für Angewandte Mechanik
Technische Universität Braunschweig
www.tu-braunschweig.de/iam

Month 2016

Declaration

I, Avinash Bapu Sreenivas , declare that this student project titled, “Numerical implementation of nonlinear time integration scheme” and the work presented in it are my own. I confirm that this work was done without any unauthorized outside help, and no other sources other than those specified were used.

Avinash ,Bapu Sreenivas
Braunschweig 29.02.2016

Abstract

In field of engineering and mainly in structural engineering the consideration of the transient behavior of the system emerges to be crucial and an important attribute.The branch of science which deals with these attributes is known as elastodynamics.The widely used technique in this field of science is the dynamic finite element method .Dynamic finite element method provides a strong platform that facilitates a better understanding of how a structure or domain performs under various dynamic factors (for instance,dynamic loading).Dynamic loading of a structure or domain not only causes displacement in the domain but also velocity and acceleration distribution,as velocity and acceleration are first order and second order time-derivatives of displacement respectively.

This work aims to provide insights into the formulation and numerical implementation of the constitutive equations of elastodynamics by employing a nonlinear time-integration scheme knowns as generalized-alpha method.This work is further extended to different cases for examination and to get a concrete understanding of the method.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In today's world,technological advancement is at its peak.This poses various challenges to engineers to carry out increasingly complex and expensive projects, which are subjected to severe reliability and safety constraints.These projects cover various domains concerning numerous industries such as aerospace & aeronautical industry,nuclear power industry ,automotive industry and so on.

Engineering sciences such as mechanics of solids & fluids,thermodynamics describe the behavior of physical systems (or structures) in the form of partial differential equations. Hence finite element analysis has become one of the most widely used techniques for solving the various forms of partial differential equations.

Finite element analysis (especially non-linear) is an essential component of computer aided design.Testing prototypes is increasingly being replaced by simulation with non-linear finite element methods as it is more rapid and less expensive method for evaluation of design concepts.

Application of the principles of finite element analysis in dynamic situations plays an even important role to understand and minimize the various factors (eg:dynamic load) affecting numerous varieties of structures. Minimization of the dynamic factors increases the reliability and hence proportionately improves the safety of these structure.

In the field of aerospace industry,dynamic finite element analysis are considered for both evaluation of early design concepts and details of final design in place of full scale tests. One of the critical aims of employing dynamic finite element analysis in the field of aerospace industry is to understand the effects of discrete factors such as gust of wind,maneuvering situations etc on the response of aircraft structures like an aircraft wing .This helps to facilitate the improvisation of survivability of future aircraft to adverse damage events.

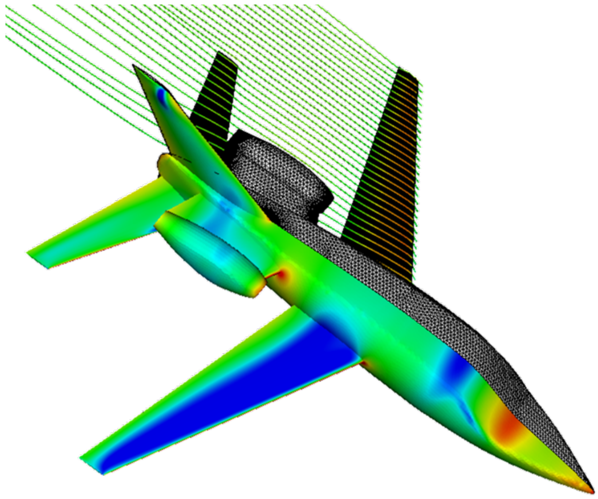


FIGURE 1.1: Dynamic finite element analysis of an aircraft wing due to gust of wind

In automotive industry,dynamic finite element analysis is useful to understand the behavior of a model under various load condition, such as crash analysis, crack propagation and etc.

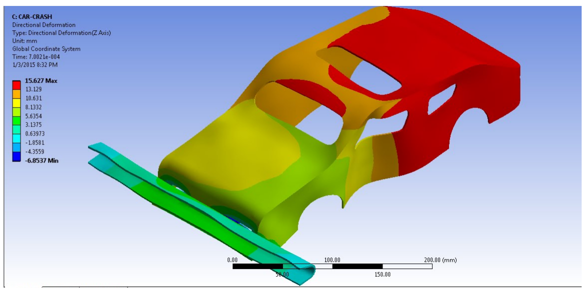


FIGURE 1.2: Crash analysis of an automobile

1.1 Framework

The main content of this work begins in chapter 2 with the clear understanding of the concepts behind linear elasticity and the derivation of the governing equations.Some light is thrown on the understanding the various time dependent methods and their differences.

In chapter 3 emphasis is laid is on detailed employment of the finite element discretization using the principle of virtual work (weak form) as derived from chapter 2.Using the same principle of linear elasticity,implementation of dynamic case using implicit time integration scheme is carried out.Here, we utilize the generalized-α method and calculate the values of the variables such as displacement,velocity and acceleration for different operating conditions (external load applied and so on) .

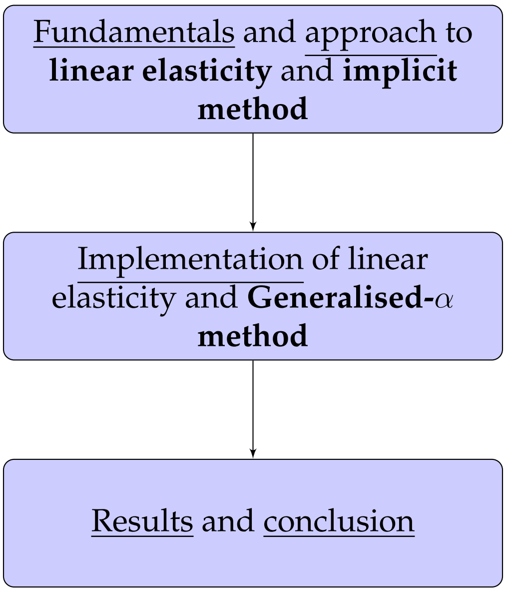


FIGURE 1.3: Flowchart showing the course of this project.

Chapter 2

Fundamentals and approach

2.1 Linear elasticity

The main objective is to obtain the principle of virtual work for the formulation of the finite element method.The underlining factor to derive the weak formulation is characterized by the description of deformation of a material body by the following means:

- displacement field and corresponding strains(Kinematics)
- force equilibrium of stresses on a differential volume element(Kinetics)
- Formulation of geometric and static boundary conditions
- constitutive relationship between stresses and strain(Material Law)

In Initial boundary value problem of elastostatics,displacements are considered as the primary variables.The reason behind the above consideration is due to the fact that the stresses can be described by the means of constitutive law as a function of stress.For structures in motion(Dynamics),the displacements(primary variable) along with its second time derivative ,which is known as acceleration are considered.
The transformation from the strong form of the PDE along with the boundary conditions to the weak form marks the end of the principle of virtual work.The weak form which is the result of the above mentioned transformation is used for the implementation of finite element discretization. Therefore,the significance of the integral formulation(weak formulation) is that it allows the exact solution to be replaced by an approximated solution,where it satisfies the integral but not the local form of the corresponding differential equation.

2.1.1 Governing Equations

The three governing equations involved in finite element method are

- Kinematics
- Kinetics

- Balance of Momentum

The detail explanation of the three governing equations are described below:

2.1.1.1 Kinematics

The Kinematics describes the geometry of a body ,its motion in space and the deformation during motion.The basis behind this description is the assumption of a body as a group of material points with the representation of their initial and current position by the means of position and their displacement vectors. The position vector of a material point 'X' and its change of position due to deformation in 3 dimension is shown below,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \tag{2.1}$$

The motion of this material point from an undeformed state to a deformed state is described with the help of a displacement vector 'u'

$$u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} ; \vec{u} = \vec{u}_x i + \vec{u}_y j + \vec{u}_z k \tag{2.2}$$

Similarly,the position and displacement vector of a material point in 2 dimension is depicted below,

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{2.3}$$

$$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix} ; \vec{u} = \vec{u}_x i + \vec{u}_y j \tag{2.4}$$

The figure below describes the the strain components on a 3D element,both shear and normal strain.

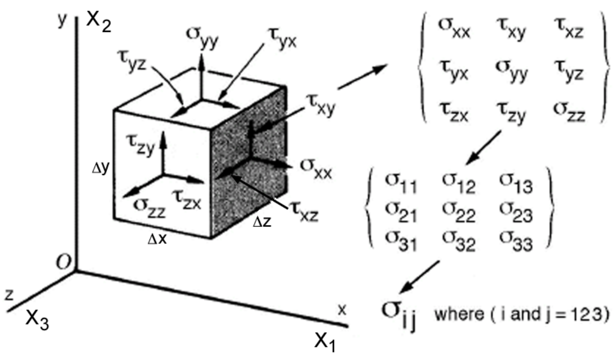


FIGURE 2.1: Strain components in 3D.[1]

The strain state also known as green Lagrange strain tensor ε is mathematically represented as,

$$E = \frac{1}{2}[\nabla u + \nabla^T u + \nabla^T u \cdot \nabla u] \tag{2.5}$$

note:The displacements in a rigid body (∇u) are free of strain.

Eq 2.5 can be simplified in such a way that the displacement gradient(∇u) is decomposed into a symmetrical and skew-symmetrical part.

$$\nabla u = \nabla^{sym} u + \nabla^{skw} u = \frac{1}{2}[\nabla u + \nabla^T u] + \frac{1}{2}[\nabla u - \nabla^T u] \tag{2.6}$$

Therefore,green Lagrange strain tensor can be written as ,

$$E = \nabla^{sym} u + \frac{1}{2}\nabla^T u \cdot \nabla u \qquad where, \nabla^{sym} u = \frac{1}{2}[\nabla u + \nabla^T u] \tag{2.7}$$

Considering the case of geometrically linear theory,the strain measure is defined by the symmetric part of the displacement gradient ∇u

$$\varepsilon = \nabla^{sym} u \tag{2.8}$$

In order to define a linear strain measure,the non-linear term of the strain tensor can be neglected for very small deformation.This is known as theory of small deformation or geometrically linear theory. So,the revised definition of strain measure for theory of small strains is given by,

$$\varepsilon = \begin{bmatrix} u_{1,1} & \frac{1}{2}(u_{1,2}+u_{2,1}) & \frac{1}{2}(u_{1,3}+u_{3,1}) \\ \frac{1}{2}(u_{1,2}+u_{2,1}) & u_{2,2} & \frac{1}{2}(u_{2,3}+u_{3,2}) \\ \frac{1}{2}(u_{1,3}+u_{3,1}) & \frac{1}{2}(u_{2,3}+u_{3,2}) & u_{3,3} \end{bmatrix} = \frac{1}{2}(u_{i,j}+u_{j,i}) e_i \otimes e_j \tag{2.9}$$

As shown in figure 2.1,the 1st index defines the strain direction and the 2nd index defines the normal to the distorted surface of the volume element.

Now,moving into the field of finite element method,the strain state is characterized by means of strain vector ε,which is represented mathematically below,

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{12} & 2\varepsilon_{23} & 2\varepsilon_{13} \end{bmatrix} \tag{2.10}$$

The factor 2 which is coupled with the shear strains has two advantages.

- It facilitates the equivalent formulation of the specific internal energy in the tensor and vector ($\sigma : \varepsilon = \sigma \cdot \varepsilon$).
- Due to the above representation,it becomes unambiguously clear during the formulation of the differential operator and its transpose. Then,the differential operator is used in the representation of the strain and balance of momentum.

The aim of the kinematics is to directly calculate the strain from the displacement vector and this aim can be achieved by developing the differential operator.The desired kinematic relation of the strain and displacement is obtained from the eq 2.9

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial X1} & 0 & 0 \\ 0 & \frac{\partial}{\partial X2} & 0 \\ 0 & 0 & \frac{\partial}{\partial X3} \\ \frac{\partial}{\partial X2} & \frac{\partial}{\partial X1} & 0 \\ 0 & \frac{\partial}{\partial X3} & \frac{\partial}{\partial X2} \\ \frac{\partial}{\partial X3} & 0 & \frac{\partial}{\partial X1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow \varepsilon = D_\varepsilon u \tag{2.11}$$

where,

$$\varepsilon_{11} = \frac{\partial}{\partial X1} u_1 = u_{1,1} \tag{2.12}$$

and

$$2\varepsilon_{12} = \frac{\partial}{\partial X2} u_1 + \frac{\partial}{\partial X1} u_2 \tag{2.13}$$

2.1.1.2 Kinetics

Kinetics describes the relation between external and internal forces acting on the material of the body.The stress tensor (σ) exists in a material body as a consequence of external forces.
Mathematical representation of stress is defined as the force acting per unit area

to the normal surface.

$stress = \frac{force}{area}$

In the field of mechanics,the stress tensor (second order tensor) consists of nine components,each defining the state of stress at a point inside a material. According to Cauchy's theorem ,the stress vector "t"on arbitrary cross section of material is the ratio of force acting on a surface to the cross-sectional area ,when the area approaches zero.

$$t = \lim_{\Delta A \to 0} \frac{\Delta f}{\Delta A} \tag{2.14}$$

Now according to Cauchy's lemma ,traction is the relation between stress and its orientation of the surface which depends on its normal vector n.

$$t = \sigma \cdot n \tag{2.15}$$

where σ is as symmetrical stress tensor also known as Cauchy's stress tensor.

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \sigma_{ij} e_i \otimes e_j \tag{2.16}$$

The components of σ_{ij} are shown in the figure below,

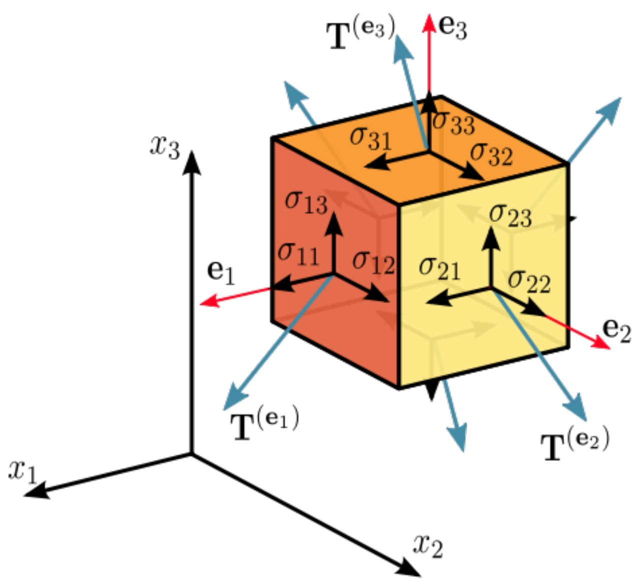


FIGURE 2.2: Stress components in 3D [2]

Similar to the strain definition as explained in section 2.1.1.1,the first index indicates the stress direction and the second indicates the surface with the corresponding normal.

2.1.1.3 Balance of Momentum

Balance of momentum equation represents the state of equilibrium between the internal forces and the stresses.The forces acting on the body can be categorized as:

- Deformation-independent and volume specific loads $\rho b = \rho \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$ (physical units $\frac{N}{m^3}$)
- volume-specific inertial forces,which are opposite in direction to the acceleration $-\rho \ddot{u} = -\rho \begin{bmatrix} \ddot{u}_1 & \ddot{u}_2 & \ddot{u}_3 \end{bmatrix}^T$(physical units $\frac{kg}{m^3}\frac{m}{s^2} = \frac{N}{m^3}$),according to Newton's law
- Forces resulting from stresses.

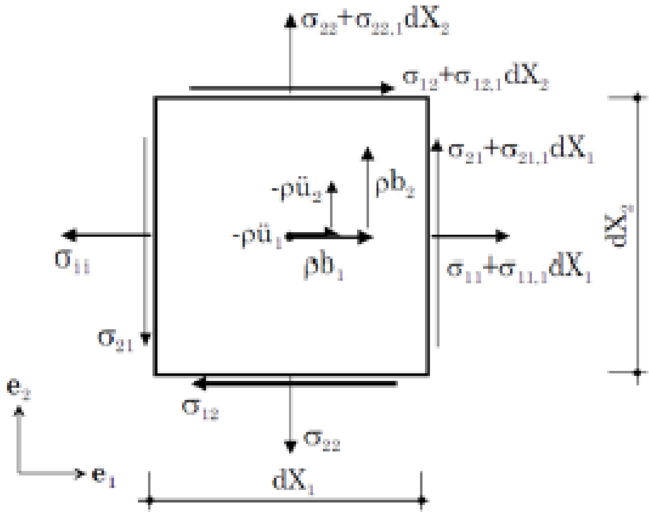


FIGURE 2.3: Figure Balance of momentum on a volume element.[3]

The Cauchy's equation of motion also known as force equilibrium equation or local form of the momentum balance in tensor form is represented below,

$$\boxed{\rho \ddot{u} = div\sigma + \rho b = (\sigma_{ij,j} + \rho b_i)\, e_i} \tag{2.17}$$

where,
$div\ \sigma$=divergence of the Cauchy's stress tensor.

Note:When divergence is applied to a second order tensor,the resultant yields a Vector.

Hence the Divergence of the Cauchy's stress tensor yields a volume-specific force vector. Mathematical representation of divergence of a Cauchy's stress tensor is shown below,

$$div\sigma = \begin{bmatrix} \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} \\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} \\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} \end{bmatrix} = \sigma_{ij,j} e_i \tag{2.18}$$

Similar to the strain vector definition in section 2.1.1.1,the Cauchy's stress tensor can be expressed in a vector form. This vector form consists of the normal stress components and shear stress components.But the only difference between the strain and stress vector is that,the shear stress components can not be factorized.

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{23} & \sigma_{13} \end{bmatrix}^T \tag{2.19}$$

Based on the stress vector and the balance of momentum(eq 2.17),the differential operator (D_σ) can be constructed as,

$$\rho \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial X_1} & 0 & 0 & \frac{\partial}{\partial X_2} & 0 & \frac{\partial}{\partial X_3} \\ 0 & \frac{\partial}{\partial X_2} & 0 & \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial X_3} & 0 & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} + \rho \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = D_\sigma \sigma + \rho b \tag{2.20}$$

now,comparing the eq 2.11 and eq 2.20 ,we arrive at a relation between D_ε and D_σ.

$$\boxed{D_\varepsilon = D_\sigma^T} \tag{2.21}$$

2.1.2 Insights into Boundary conditions

- Dirichlet Boundary condition
- Neumann Boundary condition

Consider a domain Ω which is limited by the boundary Γ. This boundary Γ is divided into non-overlapping Dirichlet Γ_u and Neumann Boundary conditions Γ_σ as shown in fig 2.4.

Rule related to Boundary conditions:

$$\Gamma = \Gamma_u \cup \Gamma_\sigma \qquad \Gamma_u \cap \Gamma_\sigma = \varnothing \tag{2.22}$$

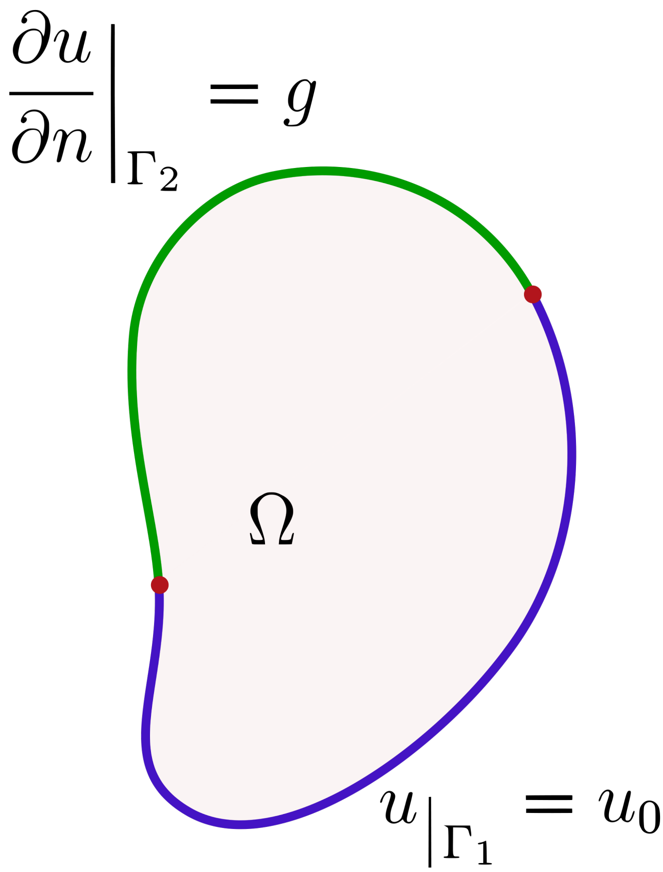


FIGURE 2.4: Dirichlet and Neumann Boundary Condition.
[4]

The primary variable(displacement) is prescribed by Dirichlet boundary condition and the dependent quantities(velocity & acceleration) are prescribed by Neumann boundary condition.

2.1.2.1 Dirichlet Boundary condition

When applied on a partial or an ordinary differential equation,it quantifies the value that a solution needs to take along the boundary of the domain as shown in fig 2.4. In elastomechanics,Dirichlet boundary conditions are the prescribed displacements at a given time for a region Γ_u of the boundary Γ.

$$\boxed{u(X,t) = u^*(X,t)} \qquad \forall X \in \Gamma_u \tag{2.23}$$

2.1.2.2 Neumann Boundary condition

When applied on a partial or an ordinary differential equation,it quantifies the value that the derivative of the solution needs to take along the boundary of the domain as shown in fig 2.4. In elastomechanics,Neumann boundary conditions are the stress vector t at a given time for a region Γ_σ of the boundary Γ.

$$\boxed{\sigma(X,t) \cdot n = t(X,t)} \qquad \forall X \in \Gamma_\sigma \tag{2.24}$$

2.1.3 Constitutive law

Constitutive equations acts as a link between the governing equations and provides various relation, to name a few, between stress-strain, forces-deformation.But they are constrained locally.Local constrain here,means at a particular material point.
Based on the definition of strain and stress vector explained in section 2.1.1.1 and section 2.1.1.2 respectively,a linear relation between Kinematics and Kinetics(also known as generalized Hooke's law) is obtained ,as shown below.

$$\sigma = C : \varepsilon \rightarrow \quad in\ tensor\ notation \tag{2.25}$$

where, C=Constitutive (elasticity)Tensor

The constitutive matrix for an anisotropic material is given by,

case1:Plane stress
It generally occurs in thin flat plates that are acted upon by load forces that are parallel to the surface.In this case,it is assumed that the stress components,the stress components along z-direction are zero.

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \tag{2.26}$$

and

$$C = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{2.27}$$

case2:Plain strain
They are stresses acting perpendicular to its length. Therefore the strain or displacement along the length is zero. The strain components along z-direction are zero.

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0 \tag{2.28}$$

and

$$C = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \tag{2.29}$$

The two cases of plane stress and plane strain are clearly represented in the figure below,

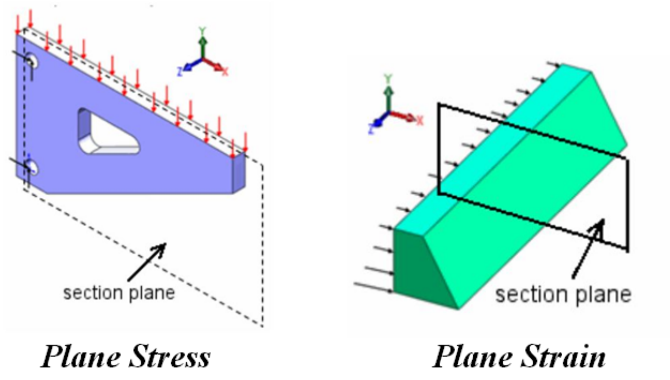


FIGURE 2.5: Plane stress and strain [5]

2.1.4 Strong and weak formulation (Principle of Virtual Work) of boundary value problems

2.1.4.1 Strong Form

The principle and crucial components for small and linear deformation are the relation between displacement and strain field [from equation (from 2.8)],the equilibrium of forces [equation (from 2.17)] and the constitutive equation relating stresses and strains.The aforementioned components collectively in their tensor notation result in the strong form (second order partial differential equation).

$$\rho\ddot{u} = div\sigma + \rho b$$
$$\sigma = C : \varepsilon$$
$$\varepsilon = \nabla^{sym} u$$

2.1.4.2 Principle of virtual Work

For the formulation of principle of virtual work ,the strong form of the partial differential equation corresponding to the balance of momentum and the static boundary conditions are multiplied(scalar multiplication) by a test function (in vector form) and then integrated over a volume and respectively over the Neumann boundary condition.

Here, the test function is considered as virtual displacements (δu)

The test function (δu) has the following properties:

- δu satisfies the geometrical boundary condition.

$$\delta u = 0 \qquad \forall \;\; X \in \Gamma_u \tag{2.30}$$

- δu satisfies the field conditions

$$\nabla^{sym} \delta u = \delta \varepsilon \tag{2.31}$$

- δu is infinitesimal
- δu is arbitrary

  The weak formulation of the balance of momentum [from equation (2.17)] and the static boundary condition [from equation (2.24)] are rearranged into the following equations:

$$\rho \ddot{u} - div\sigma - \rho b = 0 \qquad \sigma \cdot n - t = 0 \tag{2.32}$$

next multiply the strong form with test function δu and simultaneously integrate over the volume and Neumann boundary respectively *Note*:The integration term are added as well.

$$\int_\Omega \delta u \cdot (\rho \ddot{u} - \rho b)\, dV - \int_\Omega \delta u \cdot div\sigma \, dV + \int_{\Gamma_\sigma} \delta u \cdot (\sigma \cdot u - t)\, dA = 0 \tag{2.33}$$

Equation (2.33) can be simplified further by applying the product rule for divergence to term $\delta u \cdot div\sigma$,which is then transformed into $div(\delta u \cdot \sigma)$.

Then,the Gauss theorem for the divergence of the first order tensor(vector) is applied to the volume integral term

$$\int_\Omega div(\delta u \cdot \sigma)\, dV = \int_\Gamma \delta u \cdot \sigma \cdot n \, dA = \int_{\Gamma_\sigma} \delta u \cdot \sigma \cdot n \, dA \tag{2.34}$$

Now,by substituting Γ with Γ_σ we can arrive at the principle of virtual work or the weak form as the test function is zero at the Dirichlet boundary Γ_u (based on the properties of test function).

$$\int_\Omega \delta u \cdot \ddot{u}\rho \, dV + \int_\Omega \delta\varepsilon : \sigma \, dV = \int_\Omega \delta u \cdot b\rho \, dV + \int_{\Gamma_\sigma} \delta u \cdot t \, dA \tag{2.35}$$

The individual terms in Equation (2.35) represent the virtual work of inertial forces δW_{dyn}, internal virtual work δW_{int} and external forces virtual work δW_{ext}.The respective formula of the individual terms are shows below:

$$\delta W_{dyn} + \delta W_{int} = \delta W_{ext} \tag{2.36}$$

where,

$$\boxed{\delta W_{dyn} = \int_{\Omega} \delta u \cdot \ddot{u}\rho \, dV} \tag{2.37}$$

$$\boxed{\delta W_{int} = \int_{\Omega} \delta \varepsilon : \sigma \, dV} \tag{2.38}$$

$$\boxed{\delta W_{ext} = \int_{\Omega} \delta u \cdot b\rho \, dV + \int_{\Gamma_\sigma} \delta u \cdot t \, dA} \tag{2.39}$$

The principle of virtual work is solved by the means of approximation functions.In the case of the finite element method,when the approximate solution for displacement are introduced into the strong form of momentum equation,it leads to errors.These errors are known as residuum.

The basis for the development of finite element method is that,

- The integral form of the equilibrium equation and the Neumann boundary condition facilitates local errors.

Significance of principle of virtual work:

- The Dirichlet boundary conditions are strongly fulfilled in the principle of virtual work.
- The Neumann boundary conditions and the equilibrium equation must be fulfilled only weakly in the principle of virtual work.
- The advantage of integral form over differential from is that allows the facilitation of the local errors that arise during approximation.
- Due to the above reason,weak formulation forms the basis of Finite element method.

2.2 Time Dependent Methods

As we move from the static finite element method into area of transient approximation of the system,we must employ some type of time integration techniques.
These techniques are described as semi-discrete or space-time methods.

Elastodynamics is an excellent example where we can introduce the time-integration methods as it is the study of the transient behavior of elastic solids.

As compared to the static case,the forces are imbalanced and this drives the acceleration(according to Newton's second law) in elastodynamics.

The strong from of initial boundary value problem:

$$
\begin{aligned}
&\rho u_{i,tt} = \sigma_{ij,j} + f_i \quad in\ \Omega \times (0,T) \\
&u_i = g_i \quad on\ \Gamma_{D_i} \times (0,T) \\
&\sigma_{ij,j} n_j = h_i \quad on\ \Gamma_{N_i} \times (0,T) \\
&u_i(x,0) = u_{0i}(x) \quad x \in \Omega \\
&u_{i,t}(x,0) = \dot{u_{0i}}(x) \quad x \in \Omega
\end{aligned}
\tag{2.40}
$$

where,
$f_i : \Omega \times (0,T) \to \mathbb{R}$
$g_i :: \Gamma_{D_i} \times (0,T) \to \mathbb{R}$
$h_i :: \Gamma_{N_i} \times (0,T) \to \mathbb{R}$

2.2.1 Different types of methods

1. Implicit & Explicit method
2. Single & Multi step method
3. Direct & indirect integration
4. Finite difference & Galerkin Method

To explain a few ,lets start with,

- Direct & Indirect method:
  Direct method is the integration of second oder differential equation.

$$M\ddot{u} + C\dot{u} + Ku = f \tag{2.41}$$

  Indirect method is the integration of first order differential equation.The first order differential equation is obtained by the transformation of the second order differential equation by the introduction of velocity as an independent variable.

$$v = \dot{u} \quad and \quad \ddot{u} = \dot{v} \tag{2.42}$$

Therefore,

$$M\dot{v} + Cv + Ku = f \qquad \Rightarrow \dot{v} = M^{-1}[F - Cv - Ku] \tag{2.43}$$

- Implicit & Explicit method:
  Explicit method is the direct computation of dependent variables in terms of known quantities.
  Explicit method calculates the state of a system at a later time from the state of the system at the current time

  $$Y(t + \Delta t) = F(Y(t)) \tag{2.44}$$

  Implicit Method is a numerical method in which the dependent variables are defined by coupled sets of equation and hence either a matrix or an iterative technique is employed to obtain the solution.
  Implicit method determines a solution by solving an equation involving both the current state of the system and the later one.

  $$G(Y(t), Y(t + \Delta t)) = 0 \tag{2.45}$$

In this work,we are going to employ the implicit method in order to arrive at the solution.Hence from here on,we are going to discuss in-depth regarding the implicit methods.

The Different Implicit Methods are:

1. Linear acceleration method
2. Newmark Method
3. Generalized-α method

Now ,let us consider the Newmark method and the generalized-α method and below is a brief introduction to both the methods,

- Newmark Method:
  Newmark method is a numerical integration used to solve differential equation.It is most commonly used in numerical evaluation in finite element analysis to model dynamical system.

The equation of motion is given by the form:

$$M\ddot{u} + C\dot{u} + Ku = f \tag{2.46}$$

Newmark method begins with the linear change in acceleration between $[t_n.t_{n+1}]$,

$$\ddot{u}_{n+1}(\tau) = \ddot{u}_n + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{\tau}{\Delta t}2\gamma \tag{2.47}$$

where, γ=Optimization parameter Next,integrating eq 2.47 in the time interval $[t_n, t_{n+1}]$ and $\tau = \Delta t$ leads to velocity function and integrating velocity function yields displacement,

$$\dot{u}_{n+1} = \dot{u}_n + [(1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}]\Delta t \tag{2.48}$$

$$u_{n+1} = u_n + \dot{u}_n\Delta t + [(1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}]\frac{\Delta t^2}{2} \tag{2.49}$$

where, $\beta = \frac{\gamma}{3}$ and, $\alpha\&\beta$ are optimization parameter of Newmark algorithm.
Now ,$\dot{u}_{n+1}$ from eq 2.48 and $\ddot{u}_{n+1}$ from eq 2.49 as a function of u_{n+1},we get,

$$\dot{u}_{n+1}(u_{n+1}) = \frac{\gamma}{\beta\Delta t}(u_{n+1} - u_n) - \frac{\gamma-\beta}{\beta}\dot{u}_n - \frac{\gamma-2\beta}{2\beta}\Delta t\ddot{u}_n \tag{2.50}$$

$$\ddot{u}_{n+1}(u_{n+1}) = \frac{1}{\beta\Delta t^2}(u_{n+1} - u_n) - \frac{1}{\beta\Delta t}\dot{u}_n - \frac{1-2\beta}{2\beta}\ddot{u}_n \tag{2.51}$$

Finally,substituting eq 2.50 and eq 2.51 in eq 2.46 and then rearranging,we get,
The effective system of equation:

$$\begin{aligned}
K_{eff} &= M\frac{1}{\beta\Delta t^2} + C\frac{\gamma}{\beta\Delta t} + K \\
f_{eff} &= f_{n+1} \\
&+ C\left[\frac{\gamma}{\beta\Delta t}u_n + \left(\frac{\gamma}{\beta} - 1\right)\dot{u}_n + \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right)\ddot{u}_n\right] \\
&+ M\left[\frac{1}{\beta\Delta t^2}u_n + \frac{1}{\beta\Delta t}\dot{u}_n + \left(\frac{1}{2\beta} - 1\right)\ddot{u}_n\right]
\end{aligned} \tag{2.52}$$

where,
M=Mass Matrix
C=Damping Matrix
K=Stiffness Matrix
f=Force vector
$\ddot{u}_n$=acceleration at n^{th} time-step
$\dot{u}_n$=velocity at the n^{th} time-step

u_n=displacement at the n^{th} time-step

We must have the prior knowledge of the initial conditions,i.e:
$u_0, \dot{u}_0 \rightarrow$ must be known
$\ddot{u}_0 \rightarrow$ can be calculated from the equation of motion i.e,
$M\ddot{u}_0 + C\dot{u}_0 + Ku = f_0$

- Generalized-α Method
  The Generalized-α method is a one step implicit method for solving the transient problem which attempts to increase the amount of numerical damping present without degrading the order of accuracy.

  The equation of motion is evaluated within the time-interval $[t_n, t_{n+1}]$

$$M\ddot{u}_{n+1-\alpha_m} + C\dot{u}_{n+1-\alpha_d} + Ku_{n+1-\alpha_f} = f_{n+1-\alpha_f} \tag{2.53}$$

  where,
  α_m,α_d,α_f are the parameters that control the point within the interval $[t_n, t_{n+1}]$ which is required for the evaluation of equation of motion.

  Note:

  – when α_m=α_f,equation (2.53) becomes a Newmark equation of motion.

  – From hereon ,for generalized-α method,we are going to use the assumption $\alpha_d = \alpha_f$.

  The effective system of equation are obtained as shown below,

$$K_{eff}u_{n+1} = f_{eff} \tag{2.54}$$

  where,
  the effective stiffness matrix (K_{eff}) is given by,

$$K_{eff} = M\frac{1-\alpha_m}{\beta\Delta t^2} + C\frac{\gamma(1-\alpha_f)}{\beta\Delta t} + K(1-\alpha_f) \tag{2.55}$$

  and,the effective external force vector is given by,

$$\begin{aligned} f_{eff} = f_{n+1-\alpha_f} - K\alpha_f u_n \\ + C\left[\frac{\gamma(1-\alpha_f)}{\beta\Delta t}u_n + \frac{\gamma-\gamma\alpha_f-\beta}{\beta}\dot{u}_n + \frac{(\gamma-2\beta)(1-\alpha_f)}{2\beta}\Delta t\ddot{u}_n\right] \\ + M\left[\frac{1-\alpha_m}{\beta\Delta t^2}u_n + \frac{1-\alpha_m}{\beta\Delta t}\dot{u}_n + \frac{1-\alpha_m-2\beta}{2\beta}\ddot{u}_n\right] \end{aligned} \tag{2.56}$$

where,
M=Mass Matrix
C=Damping Matrix
K=Stiffness Matrix
f=Force vector
$\ddot{u}_n$=acceleration at nth time-step
$\dot{u}_n$=velocity at the nth time-step
u_n=displacement at the nth time-step

The derivation of the aforementioned equation of an effective system is clearly explained in the Chapter 3.2.1

Chapter 3

Process of implementation

Numerical implementation plays vital role in the development of constitutive models for engineering materials. The process of implementation include,

- a method to obtain the solutions of partial differential equations in solid mechanics
- and the incorporation of the constitutive models into the system of governing differential equations

In our study, the finite element method is employed for solving the boundary value problems in continuum mechanics. A suitable numerical scheme is needed for the integration of the linear constitutive equations.
For stability and convergence study of an implicit time-integration schemes,the generalized-α method and the Newton-Raphson Method are employed respectively.
In addition, due to the material non-linearity and the dynamical nature of the system(in consideration), the system of algebraic equations resulting from the finite element discretization also turns out to be non-linear with the stiffness matrix being dependent on the nodal displacements at every individual time-step.

First ,let us begin with the clear understanding of the various steps involved in Finite element method (Linear case).

3.1 Finite Element Method

The finite element method (FEM) is a numerical technique for solving a differential or integral equation to boundary value problems in order to arrive at approximate solutions . FEM subdivides a large problem into smaller and simpler parts called as finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem.FEM assumes a piecewise continuous function and then utilizes variational methods from the calculus to approximate a solution by minimizing an associated error in the solution.

Important features of FEM:

- By using piecewise approximation on finite elements,we can achieve good precision even by employing simple approximating functions.
- Very large number of nodal unknowns can be solved by using local approximation due to the fact that,it helps in creating a sparse system of equation for discretized problems.

From section 2.1.4.2,we have arrived at the weak form or the integral form from the strong form.This procedure in implemented using finite element method to obtain the desired results.

3.1.1 Implementation of Linear Elasticity

The finite element analysis and discretization consists of various steps:

- division of the domain under considerations into finite elements.
- approximation of continuously distributed physical quantities,such as displacements by discrete nodal degrees of freedom(DOF)
- assumption of nodal distribution over an element area.

The 2D domain (Ω) is subdivided into finite sub-domains(Ω^e),such that,

$$\Omega = \bigcup_{e=1}^{NE} \qquad with\ \ \Omega^i \cap \Omega^j = \qquad for\ \ i \neq j \tag{3.1}$$

The importance of the sub-domains (Ω) is that,the continuous field variables are approximated with the help of shape functions and discrete nodal degrees of freedom(DOF).
A basis is developed for plane finite element.The main requirement for this development is that,the principle of virtual work must be satisfied for every finite element.

As explained in sec 2.1.4.2 ,the equation for principle of virtual is,

$$\delta W_{ext} = \int_{\Omega} \delta u \cdot b\rho \, dV + \int_{\Gamma_\sigma} \delta u \cdot t \, dA \tag{3.2}$$

3.1.1.1 Shape Function and its importance

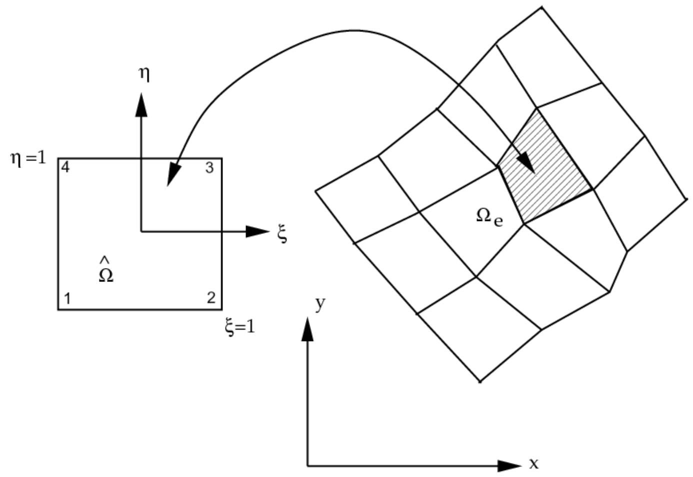


FIGURE 3.1: Discretization of 4-node elements with Physical(domain) and Natural(elemental) coordinates.**[linear FEM]**

In figure 3.1,the 2D domain is partitioned into quadrilateral finite elements.The facilitation of the elemental structure in the sud-domain is possible by finite element mesh.The process of its generation is called as meshing.

Requirement of mesh generation is to ensure that overall convergence is achieved by choosing certain shape functions which possess the following properties:

- Shape functions have to be smooth for every elemental interior Ω_e .
- Shape functions must be continuous along the boundary of the element $\partial\Omega_e$.
- Shape functions need to be complete polynomials.

The shape functions for quadrilateral element (shown in fig 3.1) are as follows:

$$\begin{aligned} N^1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N^2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N^3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N^4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

The geometry of a quadrilateral element in natural and physical coordinates is also shown in fig 3.1.

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T \qquad \xi = \begin{bmatrix} \xi & \eta \end{bmatrix}^T \tag{3.3}$$

$$X^e = \begin{bmatrix} u^{e1} & v^{e1} & u^{e2} & v^{e2} & u^{e3} & v^{e3} & u^{e4} & v^{e4} \end{bmatrix} \tag{3.4}$$

The approximation of physical coordinates as a function of natural coordinates is shown below,

$$\begin{bmatrix} u(\xi) \\ c(\xi) \end{bmatrix} = \begin{bmatrix} N^1(\xi) & 0 & N^2(\xi) & 0 & N^3(\xi) & 0 & N^4(\xi) & 0 \\ 0 & N^1(\xi) & 0 & N^2(\xi) & 0 & N^3(\xi) & 0 & N^4(\xi) \end{bmatrix} \begin{bmatrix} u_1^{e1} \\ v_1^{e1} \\ u_2^{e2} \\ v_2^{e2} \\ u_3^{e3} \\ v_3^{e3} \\ u_4^{e4} \\ v_4^{e4} \end{bmatrix} \tag{3.5}$$

3.1.1.2 Jacobi Transformation and its importance

Below are the importance of jacobi transformation:

- As the displacement components and the approximation of position vector are expressed as functions of natural coordinates,hence the necessary derivatives with respect to physical coordinates can be obtained with the help of Jacobi matrix $J(\eta)$.

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{bmatrix} \tag{3.6}$$

$$\Rightarrow \frac{\partial}{\partial \xi} = J(\xi) \frac{\partial}{\partial x} \tag{3.7}$$

Therefore,from eq 3.7,we obtain,

$$\frac{\partial}{\partial x} = J^-1(\xi) \frac{\partial}{\partial \xi} \tag{3.8}$$

- Another importance of jacobi transformation is that ,it helps in the generation of a relation of the surface element dA in natural and physical coordinates.

$$dA = du\, dv = |J(\xi)|\, d\xi\, d\eta \tag{3.9}$$

3.1.1.3 Approximation of Element quantities

1. Displacement

   The approximation of continuous displacements,variation and second time derivative of displacement(acceleration) are indicated below,

$$u(\xi) \approx \tilde{u}(\xi) = N(\xi)\, u^e \;\; ; \;\; u^e = \begin{bmatrix} u_1^{e1} & v_1^{e1} & u_2^{e2} & v_2^{e2} & u_3^{e3} & v_3^{e3} & u_4^{e4} & v_4^{e4} \end{bmatrix} \tag{3.10}$$

$$\partial u(\xi) \approx \partial \tilde{u}(\xi) = N(\xi)\, \partial u^e \;\; ; \;\; \partial u^e = \begin{bmatrix} \partial u_1^{e1} & \partial v_1^{e1} & \partial u_2^{e2} & \partial v_2^{e2} & \partial u_3^{e3} & \partial v_3^{e3} & \partial u_4^{e4} & \partial v_4^{e4} \end{bmatrix} \tag{3.11}$$

$$\ddot{u}(\xi) \approx \tilde{\ddot{u}}(\xi) = N(\xi)\, \ddot{u}^e \;\; ; \;\; \ddot{u}^e = \begin{bmatrix} \ddot{u}_1{}^{e1} & \ddot{v}_1{}^{e1} & \ddot{u}_2{}^{e2} & \ddot{v}_2{}^{e2} & \ddot{u}_3{}^{e3} & \ddot{v}_3{}^{e3} & \ddot{u}_4{}^{e4} & \ddot{v}_4{}^{e4} \end{bmatrix} \tag{3.12}$$

2. Approximation of strain vector

   The approximation of internal virtual work is given by an ansatz polynomial which is applied to displacements and then the stiffness terms are integrated.Hence for the formulation of internal work,the strain vector components must be described in natural coordinates.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{bmatrix} u_1^{e1} \\ v_1^{e1} \\ u_2^{e2} \\ v_2^{e2} \\ u_3^{e3} \\ v_3^{e3} \\ u_4^{e4} \\ v_4^{e4} \end{bmatrix} \tag{3.13}$$

   Based on eq 3.13 ($\varepsilon = D_{\varepsilon\xi}\; u$),we can deduce the order of the matrix as follows,

$$\boxed{\varepsilon = D_{3X8}\; u_{8X1}}$$

   Hence,the strain vector is of the order (3X1).

3. Approximation of internal virtual work

   The internal virtual work is approximated with the help of eq 2.35 and eq 2.25 and modifying the surface element dA in eq 3.9 and integrating the over the natural coordinates [-1,1],we get,

$$\partial W_{int}^{e} = \int_{-1}^{1}\int_{-1}^{1} \delta\varepsilon(\xi) \cdot C\, \varepsilon(\xi)\, |J(\xi)|\, h\, d\xi d\eta \tag{3.14}$$

$$\partial \tilde{W}_{int}^{e} = \int_{-1}^{1}\int_{-1}^{1} \delta u^{e} \cdot B^{T}(\xi)\, C\, B(\xi)\, u^{e}\, |J(\xi)|\, h\, d\xi d\eta \tag{3.15}$$

$$\partial \tilde{W}_{int}^{e} = \delta u^{e} \cdot \int_{-1}^{1}\int_{-1}^{1} B^{T}(\xi)\, C\, B(\xi)\, |J(\xi)|\, h\, d\xi d\eta\, u^{e} = \delta u^{e} \cdot K^{e}\, u^{e} \tag{3.16}$$

Therefore,we have

$$K^{e} = \int_{-1}^{1}\int_{-1}^{1} B^{T}(\xi)\, C\, B(\xi)\, |J(\xi)|\, h\, d\xi d\eta \tag{3.17}$$

where,h=Element thickness(for plane strain state:h=constant)
& B = differential operator,given by,

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \eta} & \frac{\partial N_4}{\partial \xi} \end{bmatrix} \tag{3.18}$$

Based on the eq 3.17,we can deduce the order of the matrix as follows,

$$\boxed{K^{e} = (B^{T})_{3X8}\, C_{3X3}\, B_{3X8}}$$

Hence,the Stiffness matrix is a square matrix of the order (8X8).

4. Approximation of dynamic work

   The dynamic virtual work is approximated with the help of eq 2.35 and eq 2.25 and modifying the surface element dA in eq 3.9 and integrating the over the natural coordinates [-1,1],as explained previously,we get,

$$\delta W_{dyn}^{e} = \int_{-1}^{1}\int_{-1}^{1} \delta u(\xi) \cdot \ddot{u}(\xi)\, |J(\xi)|\, \rho\, h\, d\xi d\eta \tag{3.19}$$

   Then,by using the approximation obtained from eq 3.12 for continuous acceleration in eq 3.19,we get4,

$$\delta \tilde{W}_{dyn}^{e} = \delta u^{e} \cdot \int_{-1}^{1}\int_{-1}^{1} N^{T}(\xi)\, N(\xi)\ \, |J(\xi)|\, \rho\, h\, d\xi d\eta \ddot{u}^{e} = \delta u^{e} \cdot M^{e}\, \ddot{u}^{e} \tag{3.20}$$

Therefore,we have

$$M^e = \int_{-1}^{1}\int_{-1}^{1} N^T(\xi)\ N(\xi)\ \ |J(\xi)|\,\rho\,h\ d\xi d\eta \tag{3.21}$$

where,h=Element thickness(for plane strain state:h=constant)
Based on the eq 3.21,we can deduce the order of the matrix as follows,

$$\boxed{M^e = (N^T)_{3X8}\ N_{3X8}}$$

Hence,the Mass matrix is a square matrix of the order (8X8).

5. Approximation of virtual work of external loads

- Volume Loads

$$\delta\tilde{W}_{ext}^{\Omega e} = \delta u^e \cdot \int_{-1}^{1}\int_{-1}^{1} N^T(\xi)\ b(\xi)\,|J(\xi)|\ \ \rho\,h\ d\xi d\eta = \delta u^e \cdot r_p^e \tag{3.22}$$

where,
r_p^e=Element vector of volume forces

$$r_p^e = \int_{-1}^{1}\int_{-1}^{1} N^T(\xi)\ b(\xi)\,|J(\xi)|\ \ \rho\,h\ d\xi d\eta \tag{3.23}$$

- Boundary Forces

$$\delta\tilde{W}_{ext}^{\Omega e} = \delta u^e \cdot \int_{-1}^{1} N^T(\xi)\ t(\xi)\,|J(\xi)|\ \ \ h\ d\xi = \delta u^e \cdot r_n^e \tag{3.24}$$

r_n^e=Element vector of boundary forces

$$r_n^e = \int_{-1}^{1} N^T(\xi)\ t(\xi)\,|J(\xi)|\ \ \ h\ d\xi \tag{3.25}$$

Finally,

$$\delta\tilde{W}^e_{int} + \delta\tilde{W}^e_{dyn} = \delta\tilde{W}^e_{ext} \tag{3.26}$$

$$\delta u^e \cdot K^e\ u^e + \delta u^e \cdot M^e\ \ddot{u^e} = \delta u^e \cdot r_p^e + \delta u^e \cdot r_n^e \tag{3.27}$$

where,

$$f^e = \delta u^e \cdot r_p^e + \delta u^e \cdot r_n^e \tag{3.28}$$

This marks the end of linear elasticity,next we in-cooperate the principles of the linear elasticity into the dynamic case(generalized-α method).The linear elasticity is the foundation for the implementation of dynamic case.

3.2 Time integration scheme

Based on the previous explanation regarding the order of stiffness and mass matrix in section 3.1.1.3,now we use the Rayleigh's damping model to understand the formation of the damping matrix(C).

Rayleigh's damping is defined as the damping obtained as a combination of mass and stiffness matrix,as shown below,

$$C = \alpha_1 M + \alpha_2 K \tag{3.29}$$

where,
α_1=mass-proportional damping coefficient
α_2=stiffness-proportional damping coefficient

Computation of $\alpha_1 \& \alpha_2$,the 2 natural frequencies (in rad/sec) needs to be known.The natural frequencies $(\omega_i \& \omega_j)$ are determined by extrapolating the values from the typical graph obtained by the following equation,

$\frac{\alpha_1}{2\omega_i} + \frac{\alpha_2 \omega_i}{2}$

The above equation is derived from the orthogonal transformation of damping matrix,as shown below,

$$\begin{gathered} M\ddot{u}_0 + C\dot{u}_0 + Ku = f_0 \\ \{\Phi^T\}[M]\Phi\{\ddot{u}\} + \{\Phi^T\}[C]\{\Phi\}\{\dot{u}\} + \{\Phi^T\}[K]\{\Phi\}\{u\} = \{\Phi^T\}\{f\} \end{gathered} \tag{3.30}$$

Eq 3.30 in matrix form,

$$\{\Phi^T\}[C]\{\Phi\}\{\dot{u}\} = \begin{bmatrix} \alpha_1+\alpha_2\omega_1^2 & 0 & \ldots & \ldots\ldots & 0 \\ 0 & \alpha_1+\alpha_2\omega_2^2 & \ldots & \ldots\ldots & 0 \\ . & . & \alpha_1+\alpha_2\omega_1^2 & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ 0 & . & . & \ldots\ldots & \alpha_1+\alpha_2\omega_n^2 \end{bmatrix} \tag{3.31}$$

As from eq 3.31,the orthogonal damping matrix is symmetric.Hence we arrive at the set of simultaneous equations,

$$\begin{aligned} 2\zeta_1\omega_1 &= \alpha_1+\alpha_2\omega_1^2 \\ 2\zeta_2\omega_2 &= \alpha_1+\alpha_2\omega_2^2 \end{aligned} \tag{3.32}$$

There generalized form of eq 3.32 is,

$$\begin{aligned} &2\zeta_i\omega_i = \alpha_1+\alpha_2\omega_i^2 \\ &\boxed{\zeta_i = \frac{\alpha_1}{2\omega_i}+\frac{\alpha_2\omega_i}{2}} \end{aligned} \tag{3.33}$$

Similarly,for the j-th mode,we get,

$$\begin{aligned} &2\zeta_j\omega_j = \alpha_1+\alpha_2\omega_j^2 \\ &\boxed{\zeta_j = \frac{\alpha_1}{2\omega_j}+\frac{\alpha_2\omega_j}{2}} \end{aligned} \tag{3.34}$$

General form of eq 3.33 & 3.34 becomes,

$$\Rightarrow \boxed{\zeta = \frac{\alpha_1}{2\omega}+\frac{\alpha_2\omega}{2}} \tag{3.35}$$

The typical curve of the eq 3.35 is shown in the Fig 3.2.

If the damping ratio ζ_i and ζ_j associated with the two specific frequencies (ω_i and ω_j) are known.Then two damping coefficients can be calculated by the following simultaneous equation,

Below shows the representation of eq 3.33 & 3.34 in matrix form,

$$\begin{bmatrix} \zeta_i \\ \zeta_j \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{\omega_i} & \omega_i \\ \frac{1}{\omega_j} & \omega_j \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{3.36}$$

Therefore,in order to determine α_1 and α_2 ,eq 3.36 is transformed into,

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \frac{2*\omega_i*\omega_j}{\omega_i^2-\omega_j^2}\begin{bmatrix}\omega_i & -\omega_j\\ -\frac{1}{\omega_i} & \frac{1}{\omega_j}\end{bmatrix}\begin{bmatrix}\zeta_i\\ \zeta_j\end{bmatrix} \tag{3.37}$$

Upon simplification of the eq 3.37 ,we get,

$$\begin{aligned}\alpha_1 &= \frac{2\omega_i\omega_j}{\omega_i^2-\omega_j^2}(\omega_i\zeta_i-\omega_j\zeta_j)\\ \alpha_2 &= \frac{2\omega_i\omega_j}{\omega_i^2-\omega_j^2}\left(-\frac{\zeta_i}{\omega_i}+\frac{\zeta_j}{\omega_j}\right)\end{aligned} \tag{3.38}$$

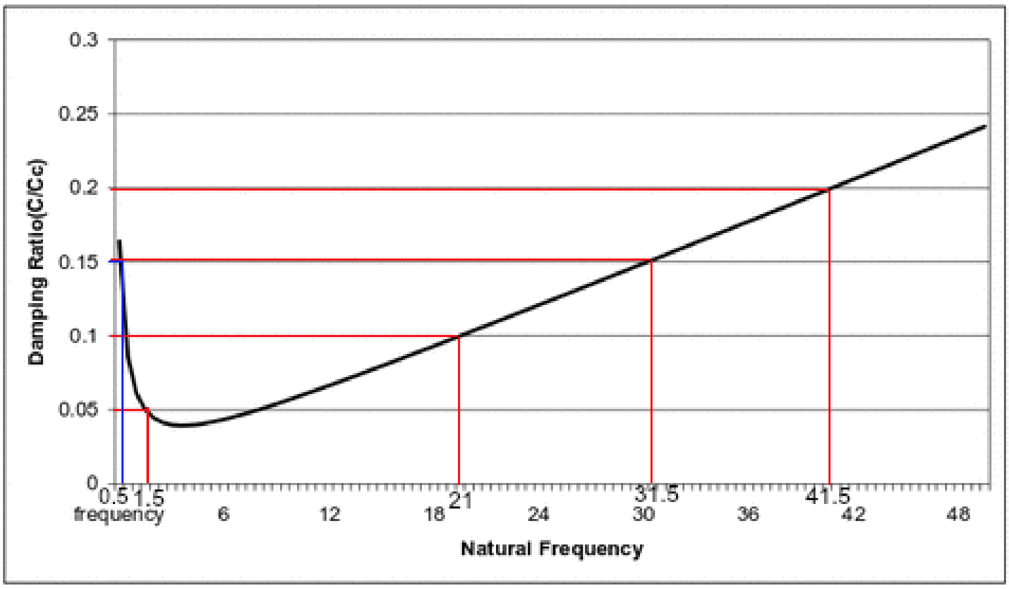


FIGURE 3.2: Natural frequency (in rad/sec) vs damping ratio.[6]

Note:For the various cases which are discussed in chapter 4.1.1,we choose the following damping factors and the natural frequency respectively as explained in theory according to [7],

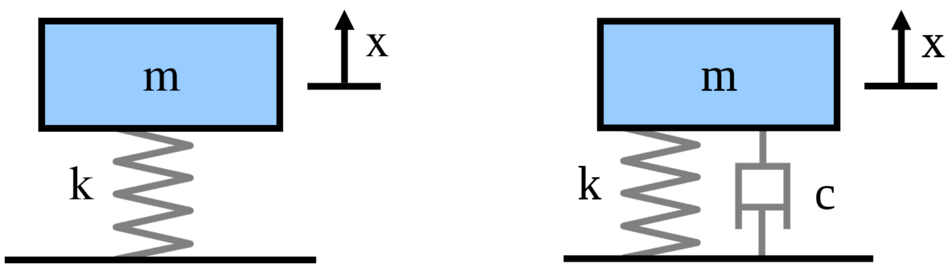
Sl.no	ζ_i in %	ζ_j in %	ω_i(rad/sec)	ω_j(rad/sec)
1	5	10	1.5	21.5
2	15	20	0.5	41.5
3	5	15	1.5	41.5
4	5	20	1.5	31.5

TABLE 3.1: Table showing the values of natural frequencies (in rad/sec) for respective damping factor (in %)

The values in the tabular column are obtained from the graph 3.2.

Importance of damping:

- Rayleigh's damping model helps to consider and understand the damping effects in non-linear dynamic structural analysis.
- The major advantage gained in converting the damping matrix into an equivalent Rayleigh's damping lies in the fact that by using orthogonal transformation a structure having n-degrees of freedom can be reduced to n-number of uncoupled equations.
- The damping ratio imparted to the overall structure consisting of the superstructure and the flexible isolators can be very high owing to the mass-proportional damping term.



(A) Free vibration without damping

(B) Free vibration damping

FIGURE 3.3: Mass-spring-damper model

3.2.1 Linear time integration

The generalized-α method Generalized-α method begins with linear change in acceleration in time interval $[t_n, t_{n+1}]$.

$$\ddot{u}_{n+1}(\tau) = \ddot{u}_n + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{\tau}{\Delta t}2\gamma \tag{3.39}$$

where, γ=Optimization parameter Next,integrating eq 3.39 in the time interval $[t_n, t_{n+1}]$ leads to velocity function,

$$\dot{u}_{n+1}(\tau) = \dot{u}_n + \ddot{u}_n\tau + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{2\gamma}{\Delta t}\frac{\tau^2}{2} \tag{3.40}$$

$$= \dot{u}_n + \ddot{u}_n\tau + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{\gamma}{\Delta t}\tau^2 \tag{3.41}$$

for $\tau = \Delta t$:

$$\dot{u}_{n+1} = \dot{u}_n + [(1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}]\Delta t \tag{3.42}$$

Next,integrating eq 3.42 in the time interval $[t_n, t_{n+1}]$ leads to displacement function,

$$u_{n+1}(\tau) = u_n + \dot{u}_n\tau + \ddot{u}_n\frac{\tau^2}{2} + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{2\gamma}{\Delta t}\frac{\tau^3}{6} \tag{3.43}$$

Displacement at the end of time interval $[t_n, t_{n+1}]$ is $\tau = \Delta t$.Hence,eq 3.43 becomes,

$$\begin{aligned} u_{n+1}(\tau) &= u_n + \dot{u}_n\Delta t + \ddot{u}_n\frac{\Delta t^2}{2} + (\ddot{u}_{n+1} - \ddot{u}_n)\frac{2\gamma}{\Delta t}\frac{\Delta t^3}{6} \\ &= u_n + \dot{u}_n\Delta t + \left[\left(\frac{1}{2} - \frac{\gamma}{3}\right)\ddot{u}_n + \frac{\gamma}{3}\ddot{u}_{n+1}\right]\Delta t \end{aligned} \tag{3.44}$$

$$u_{n+1} = u_n + \dot{u}_n\Delta t + [(1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}]\frac{\Delta t^2}{2} \tag{3.45}$$

where, $\beta = \frac{\gamma}{3}$ and, $\alpha\&\beta$ are parameters chosen to optimize stability and accuracy of the algorithm.

Next step is to represent $\dot{u}_{n+1}$ from eq 3.42 and $\ddot{u}_{n+1}$ from eq 3.45 as a function of u_{n+1},we get,

$$\dot{u}_{n+1}(u_{n+1}) = \frac{\gamma}{\beta \Delta t}(u_{n+1} - u_n) - \frac{\gamma - \beta}{\beta}\dot{u}_n - \frac{\gamma - 2\beta}{2\beta}\Delta t \ddot{u}_n \tag{3.46}$$

$$\ddot{u}_{n+1}(u_{n+1}) = \frac{1}{\beta \Delta t^2}(u_{n+1} - u_n) - \frac{1}{\beta \Delta t}\dot{u}_n - \frac{1 - 2\beta}{2\beta}\ddot{u}_n \tag{3.47}$$

Then substituting the eq 3.47 , eq 3.46 and eq 3.45 in the following equations respectively,

$$\begin{aligned} \ddot{u}_{n+1-\alpha_m} &= (1-\alpha_m)\ddot{u}_{n+1}(u_{n+1}) + \alpha_m \ddot{u}_n \\ \dot{u}_{n+1-\alpha_f} &= (1-\alpha_f)\dot{u}_{n+1}(u_{n+1}) + \alpha_f \dot{u}_n \\ u_{n+1-\alpha_f} &= (1-\alpha_f)u_{n+1} + \alpha_f u_n \\ f_{n+1-\alpha_f} &= (1-\alpha_f)f_{n+1} + \alpha_f f_n \end{aligned} \tag{3.48}$$

and then,finally,substituting the resulting equations in 3.48 into the equation of motion,i.e,

$$M\ddot{u}_{n+1-\alpha_m} + C\dot{u}_{n+1-\alpha_f} + Ku_{n+1-\alpha_f} = f_{n+1-\alpha_f} \tag{3.49}$$

and then rearranging we get,

$$\begin{aligned} K_{eff} &= M\frac{1-\alpha_m}{\beta \Delta t^2} + C\frac{\gamma(1-\alpha_f)}{\beta \Delta t} + K(1-\alpha_f) \\ f_{eff} &= f_{n+1-\alpha_f} - K\alpha_f u_n \\ &+ C\left[\frac{\gamma(1-\alpha_f)}{\beta \Delta t}u_n + \frac{\gamma - \gamma\alpha_f - \beta}{\beta}\dot{u}_n + \frac{(\gamma - 2\beta)(1-\alpha_f)}{2\beta}\Delta t \ddot{u}_n\right] \\ &+ M\left[\frac{1-\alpha_m}{\beta \Delta t^2}u_n + \frac{1-\alpha_m}{\beta \Delta t}\dot{u}_n + \frac{1-\alpha_m-2\beta}{2\beta}\ddot{u}_n\right] \end{aligned} \tag{3.50}$$

Note:
Mass matrix and Stiffness matrix are explained in section

This point marks the end of linear generalized-α method.Now,with the understanding of the linear part,lets move on to the non-linear generalized-α method (used in this studienarbeit).

3.2.2 Non-linear time integration scheme

Non-linear generalized-α method:
In this section,we are going to discuss about the time integration procedure and algorithm of semi-discrete non-linear equations.

Consider abstract,time dependent,non-linear problem for a vector-valued function $u(x,t)$.The variational form is as follows,

$$Weak\ \ Form: \qquad \eta(w; u, \dot{u}, \ddot{u}) = L(w) \tag{3.51}$$

Eq 3.51 represents a weak form of the non-linear equation of motion.
where,
u=displacement
$\dot{u}$=Velocity
$\ddot{u}$=Acceleration
Let control variable U (a vector) depend upon time,such that

$$u^h(\xi, t) = U_B(t) N_B(\xi) \tag{3.52}$$

where,
Displacement depends upon time
Shape functions depends upon the natural coordinates

Then,the residual vector is defined by the following equation,

$$R(U, \dot{U}, \ddot{U}) = \{R_P\} \tag{3.53}$$

where,

$$R_P = \eta(N_A e_i; u^h.\dot{u}^h, \ddot{u}^h) - L(N_A e_i) \tag{3.54}$$

&
$\dot{U}$=First time derivative of displacement (i.e Velocity)
$\ddot{U}$=Second time derivative of displacement (i.e Acceleration)

Then replacing $U, \dot{U}, \ddot{U}$ with $u, \dot{u}, \ddot{u}$ (for own understanding) ,the generalized-α time integration algorithms states:

Given $\rightarrow u_n, \dot{u}_n, \ddot{u}_n$
To find $\rightarrow u_{n+1}, \dot{u}_{n+1}, \ddot{u}_{n+1}, u_{n+\alpha_f}, \dot{u}_{n+\alpha_f}, \ddot{u}_{n+\alpha_m}$
Such that $\rightarrow R(u_{n+\alpha_f}, \dot{u}_{n+\alpha_f}, \ddot{u}_{n+\alpha_m}) = 0$

where,

$$u_{n+\alpha_f} = u_n + \alpha_f(u_{n+1} - u_n) \tag{3.55}$$
$$\dot{u}_{n+\alpha_f} = \dot{u}_n + \alpha_f(\dot{u}_{n+1} - \dot{u}_n) \tag{3.56}$$
$$\ddot{u}_{n+\alpha_m} = \ddot{u}_n + \alpha_m(\ddot{u}_{n+1} - \ddot{u}_n) \tag{3.57}$$
$$\dot{u}_{n+1} = \dot{u}_n + \Delta t((1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1}) \tag{3.58}$$
$$\ddot{u}_{n+1} = \ddot{u}_n + \Delta t\dot{u}_n + \frac{(\Delta t)^2}{2}((1-2\beta)\ddot{u}_n + 2\beta\ddot{u}_{n+1}) \tag{3.59}$$

In eq 3.55-3.59,α_f,α_m,γ and β are real valued parameters.These parameters are selected to ensure second order accuracy and unconditional stability of the algorithm.
Therefore,

$$\alpha_m = \frac{2 - p_\infty}{1 + p_\infty} \tag{3.60}$$

$$\alpha_f = \frac{1}{1 + p_\infty} \tag{3.61}$$

where,
p_∞=Spectral radius of amplification matrix

Unconditional stability condition:

$$\alpha_m \geq \alpha_f \geq \frac{1}{2} \tag{3.62}$$

Second order accuracy condition:

$$\gamma = \frac{1}{2} - \alpha_f + \alpha_m \tag{3.63}$$

$$\beta = \frac{1}{4}(1 - \alpha_f + \alpha_m)^2 \tag{3.64}$$

In order to solve the non-linear system of eq 3.55-3.59,we employ the Newton-Raphson method. Here,the Newton-Raphson method consists of 2-phase predictor-corrector algorithm.

- Predictor phase:
  In this phase,we select a zero acceleration predictor and it should be consistent with the eq 3.55-3.59.
  Now,set

$$\ddot{u}^i_{n+1} = 0 \tag{3.65}$$

$$\dot{u}^i_{n+1} = \dot{u}_n + \Delta t(1 - \gamma)\ddot{u}_n \tag{3.66}$$

$$u_{n+1} = u_n + \Delta t \dot{u}_n + \frac{(\Delta t)^2}{2}(1 - 2\beta)\ddot{u}_n \tag{3.67}$$

- Multi-corrector phase:
  During this phase,we apply the Newton-Raphson method in order to achieve convergence of the non-linear system of equation.
  At first,we iterate the values at intermediate time levels as,

$$u^i_{n+\alpha_f} = u_n + \alpha_f(u^i_{n+1} - u_n) \tag{3.68}$$
$$\dot{u}^i_{n+\alpha_f} = \dot{u}_n + \alpha_f(\dot{u}^i_{n+1} - \dot{u}_n) \tag{3.69}$$
$$\ddot{u}^i_{n+\alpha_m} = \ddot{u}_n + \alpha_m(\ddot{u}^i_{n+1} - \ddot{u}_n) \tag{3.70}$$

Then,with the help of the intermediate values from eq 3.68-3.70,we assemble the residuals of continuity and momentum equations as,

$$\frac{\mathrm{d}R^i}{\mathrm{d}\ddot{u}_{n+1}}\Delta a = -R^i_{n+1} \tag{3.71}$$
$$\tag{3.72}$$

where,

$$R^i_{n+1} = R(u_{n+\alpha_f}, \dot{u}_{n+\alpha_f}, \ddot{u}_{n+\alpha_m}) \tag{3.73}$$

and

$$\frac{\mathrm{d}R^i}{\mathrm{d}\ddot{u}_{n+1}} = \frac{\mathrm{d}R}{\mathrm{d}\ddot{u}_{n+1}}(u^i_{n+\alpha_f}, \dot{u}^i_{n+\alpha_f}, \ddot{u}^i_{n+\alpha_m}) \tag{3.74}$$

where,

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}\ddot{u}_{n+1}} =& \frac{\partial R}{\partial \ddot{u}_{n+\alpha_m}} \cdot \frac{\partial \ddot{u}_{n+\alpha_m}}{\partial \ddot{u}_{n+1}} \\ &+ \frac{\partial R}{\partial \dot{u}_{n+\alpha f}} \cdot \frac{\partial \dot{u}_{n+\alpha f}}{\partial \dot{u}_{n+1}} \cdot \frac{\partial \dot{u}_{n+1}}{\partial \ddot{u}_{n+1}} \\ &+ \frac{\partial R}{\partial u_{n+\alpha_f}} \cdot \frac{\partial u_{n+\alpha_f}}{\partial u_{n+1}} \cdot \frac{\partial u_{n+1}}{\partial \ddot{u}_{n+1}}\end{aligned} \tag{3.75}$$

By substituting eq 3.57 ,3.56 ,3.55 into eq 3.75 and simplifying,we get,

$$\frac{\partial R}{\partial \ddot{u}_{n+1}} = M^* = M\alpha_m + (\Delta t)C\alpha_f + (\Delta t)^2\beta K\alpha_f \tag{3.76}$$

where,

$$\begin{aligned}M &= \frac{\partial R}{\partial \ddot{u}_{n+\alpha_m}} \\ C &= \frac{\partial R}{\partial \dot{u}_{n+\alpha_f}} \\ K &= \frac{\partial R}{\partial u_{n+\alpha_f}}\end{aligned} \tag{3.77}$$

After determining M^*,calculate the correction factor (Δa) from the following equation,

$$M^* \Delta a = -R^i_{n+1} \tag{3.78}$$

Finally,update the iterates using the correction factor $(\Delta \ddot{u})$ from eq 3.78 as shown below,

$$\begin{aligned} \ddot{u}^{i+1}_{n+1} &= \ddot{u}^i_{n+1} + \Delta \ddot{u} \\ \dot{u}^{i+1}_{n+1} &= \dot{u}^i_{n+1} + \gamma \Delta t \Delta \ddot{u} \\ u^{i+1}_{n+1} &= u^i_{n+1} + \beta (\Delta t)^2 \Delta \ddot{u} \end{aligned} \tag{3.79}$$

Therefore,from the steps involved in the 2-phase Newton-Raphson method,we can then calculate the the values of displacement $(\ddot{u})$,velocity $(\dot{u})$ and acceleration (u) for a each individual time step.The process continues until the user defined time-step is reached.

This procedure of the 2-phase Newton-Raphson method is summarized in the figure 3.4 as shown below.

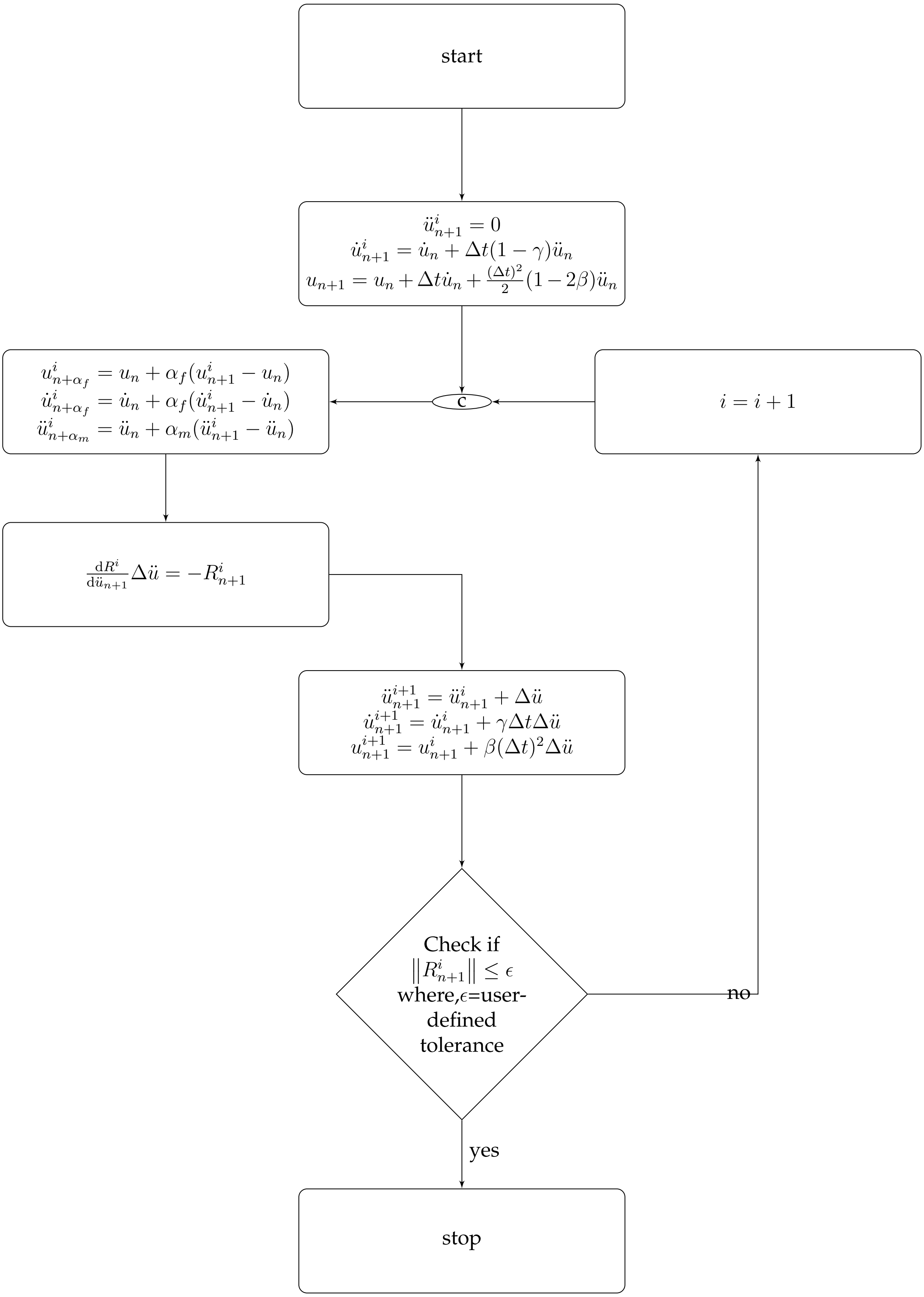


FIGURE 3.4: Flowchart showing the detailed algorithm involved in generalize-α method for each time step.[8]

Chapter 4

Results and conclusion

4.1 Results

This chapter deals with the numerical implementation of the finite element discretization equations which were obtained from the previous chapters using MATLAB.

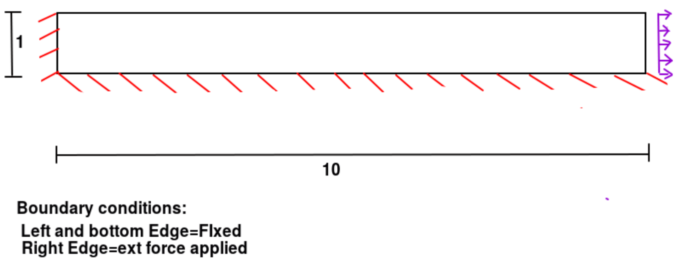
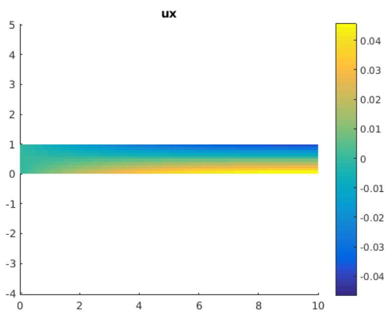
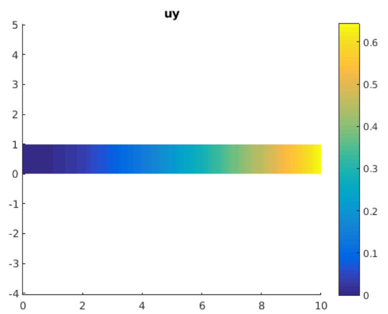


FIGURE 4.1: Rectangular domain with Boundary conditions

Below are the displacement,velocity and acceleration distribution for t=0.005 and damping is zero.

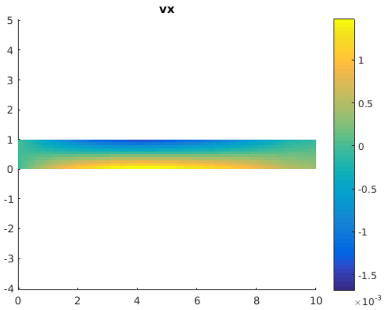


(A) Displacement in x-direction

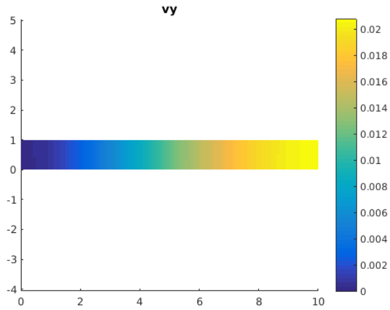


(B) Displacement in y-direction

FIGURE 4.2: Displacement distribution

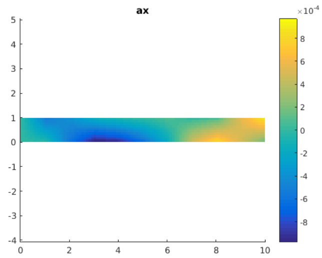


(A) Velocity in x-direction

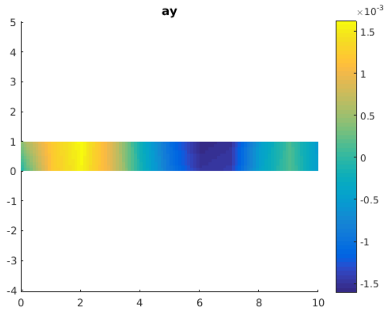


(B) Velocity in y-direction

FIGURE 4.3: Velocity distribution



(A) Acceleration in x-direction



(B) Acceleration in y-direction

FIGURE 4.4: Acceleration distribution

4.1.1 Generalized-α method and graphs

In this case,we consider the effects of Rayleigh's damping matrix and with help of different cases,let us learn how damping matrix affects the equation of motion.
As explained in section 3.2 Rayleigh's damping is given by,

$$C = \alpha_1 M + \alpha_2 K \tag{4.1}$$

There are three different subclauses which come into picture when Rayleigh's damping model is considered.They are:

- When $\alpha_1 = 0$
  Eq 4.1 becomes,
  $C = \alpha_2 K \Rightarrow$ Stiffness proportional damping
- When $\alpha_2 = 0$
  Eq 4.1 becomes,
  $C = \alpha_1 M \Rightarrow$ Mass proportional damping

- When $\alpha_1 \neq 0 \& \alpha_2 \neq 0$
  Eq 4.1 becomes,
  $C = \alpha_1 M + \alpha_2 K \Rightarrow$ Combination of both mass and stiffness proportional damping

We are going to study the effects of external load(t=0.005,0.01,0.0025) on the rectangular domain for the all of the aforementioned subclauses of Rayleigh's damping model.

4.1.1.1 Case 1:$C = \alpha_2 K$

1. Discussion regarding displacements

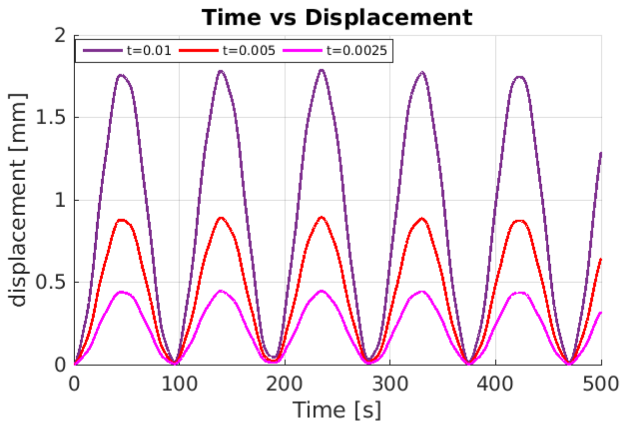


FIGURE 4.5: Time vs displacement graph for various external loads and $C = \alpha_2 K$

2. Discussion regarding velocity

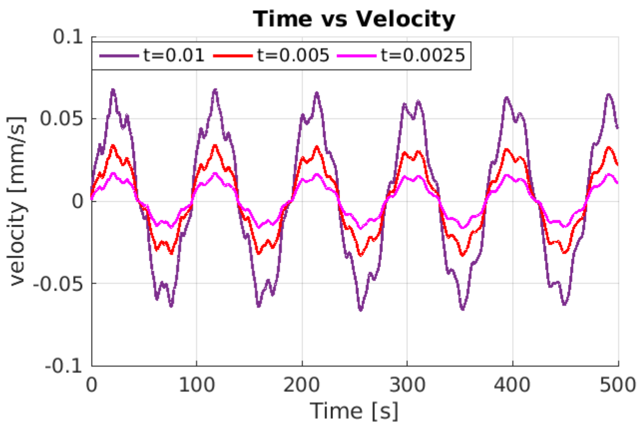


FIGURE 4.6: Time vs velocity graph for various external loads and $C = \alpha_2 K$

3. Discussion regarding acceleration

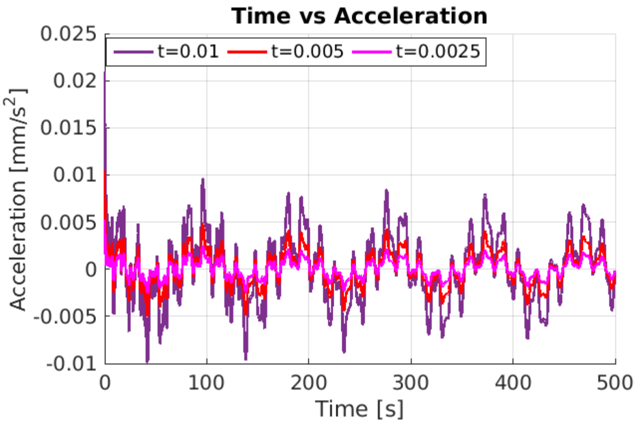


FIGURE 4.7: Time vs acceleration graph for various external loads and $C = \alpha_2 K$

The maximum values of the displacement,velocity and acceleration vectors are tabulated below,

max Values	t=0.01	t=0.005	t=0.0025
Displacement(u)	1.289515466842	0.644757733421	0.322378866710
Velocity(v)	0.043893750613	0.021946875306	0.010973437653
Acceleration(a)	0.000422283661274	0.0002111418306	0.00010557091531

TABLE 4.1: The maximum values of u, v & a for stiffness-proportional damping

As observed from the graphs 4.5,4.6 and 4.7,the following inferences are drawn,

(a) The stiffness proportional damping corresponds physically to linear viscous dampers that interconnect the degrees of freedom of a structure.Hence,the rectangular domain oscillates over a chosen period of time [0sec,500sec].As the damping effect now is directly proportional to natural frequency.

(b) A state of static equilibrium is not achieved.

4.1.1.2 Case 2:$C = \alpha_1 M$

1. Discussion regarding displacements

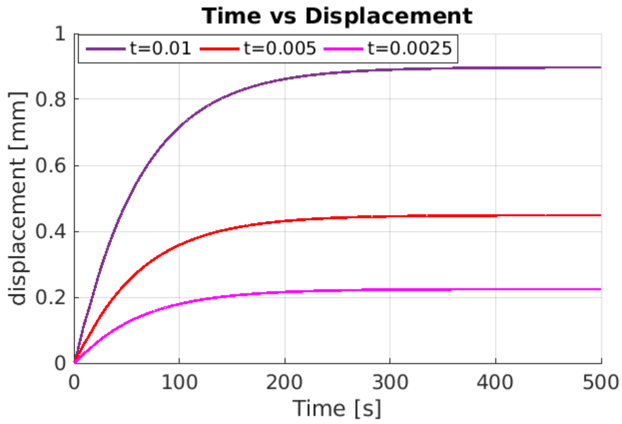


FIGURE 4.8: Time vs displacement graph for various external loads and $C = \alpha_1 M$

2. Discussion regarding velocity

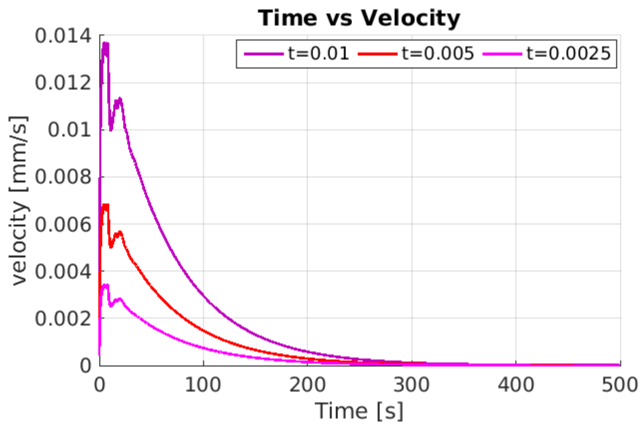


FIGURE 4.9: Time vs velocity graph for various external loads and $C = \alpha_1 M$

3. Discussion regarding acceleration

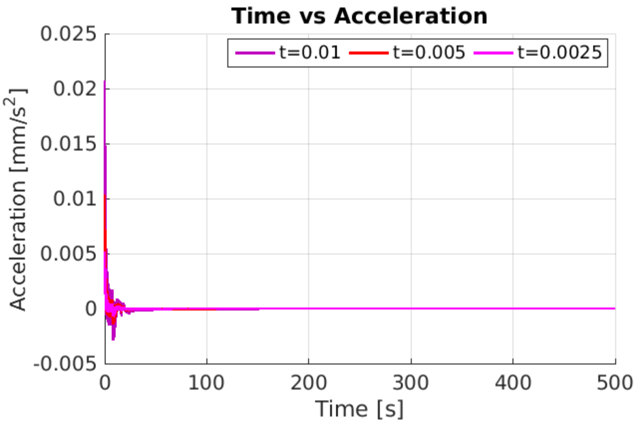


FIGURE 4.10: Time vs acceleration graph for various external loads and $C = \alpha_1 M$

max Values	t=0.01	t=0.005	t=0.0025
Displacement(u)	0.896642097948	0.448321048974	0.224160524487
Velocity(v)	0.00000433993798	0.00000216996899	0.00000108498449
Acceleration(a)	0.0000000048517	0.00000000242585	0.00000000121292

TABLE 4.2: The maximum values of u, v & a for mass-proportional damping

As observed from the graphs 4.8,4.9 and 4.10,the following inferences are drawn,

(a) The rectangular domain oscillates until till it reaches the maximum displacement(in mm) and then a state of equilibrium is achieved.

(b) For all external loads under consideration(t=0.01,t=0.005,t=0.0025),the displacement and acceleration quantities become constant(parallel to x-axis),the velocity quantities drops to zero.This point in time marks the state of equilibrium.

4.1.1.3 Case 3:$C = \alpha_1 M + \alpha_2 K$

1. Discussion regarding displacements

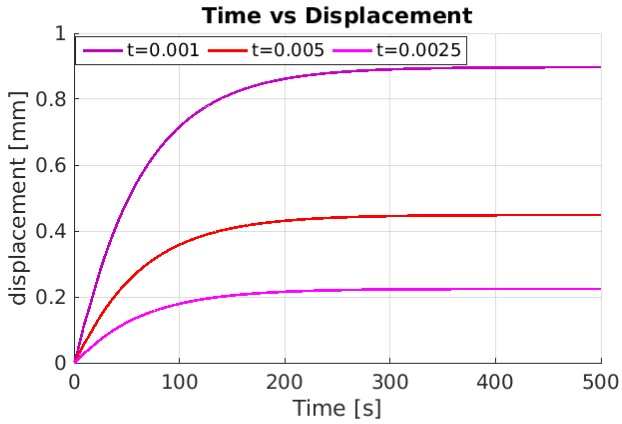


FIGURE 4.11: Time vs displacement graph for various external loads and $C = \alpha_1 M + \alpha_2 K$

2. Discussion regarding velocity

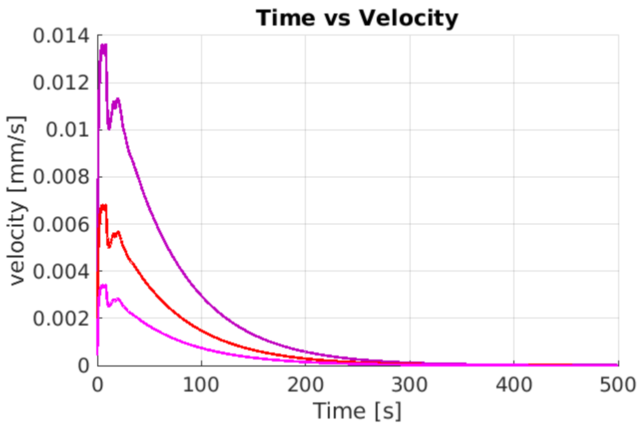


FIGURE 4.12: Time vs velocity graph for various external loads and $C = \alpha_1 M + \alpha_2 K$

3. Discussion regarding acceleration

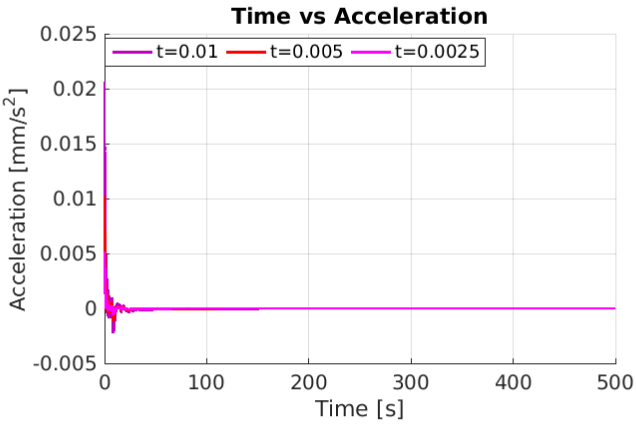


FIGURE 4.13: Time vs acceleration graph for various external loads and $C = \alpha_1 M + \alpha_2 K$

max Values	t=0.01	t=0.005	t=0.0025
Displacement(u)	0.896641948939	0.448320974469	0.224160487234
Velocity(v)	0.00000434206551	0.00000217103275	0.00000108551637
Acceleration(a)	0.00000000485376	0.00000000242688	0.00000000121344

TABLE 4.3: The maximum values of u, v & a for both mass and stiffness-proportional damping

As observed from the graphs 4.11,4.12 and 4.13,the following inferences are drawn,

(a) By comparing the values of a and b as determined in section 3.2,value of b is really small as compared to a.When damping related to both mass and stiffness is chosen,the characteristics are similar to that of the mass proportional damping because 'a' dominates the model.

(b) *NOTE*:When C=0,the Rayleigh's model functions as a stiffness-proportional damping,where the rectangular domain oscillates upto the maximum displacement and for a prescribed time range [0sec,500sec].Therefore,no state of equilibrium is gained.

Now for the same case in hand i.e,$C = \alpha_1 M + \alpha_2 K$ below shows the time-vs-displacement graph for different damping factor,

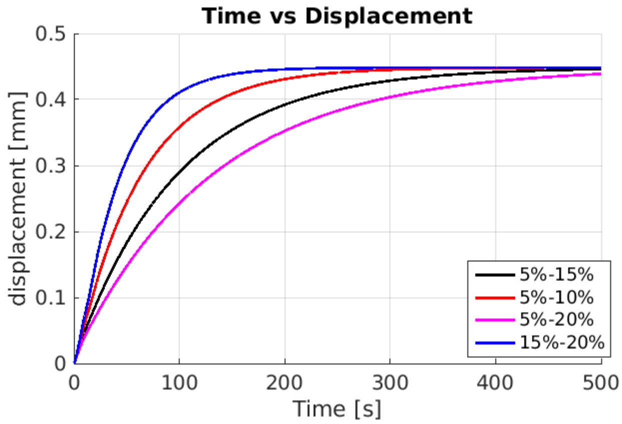


FIGURE 4.14: Time-vs-Displacement graph for different damping factors

As observed from the graph 4.14,the following inferences are drawn,

4.2 Conclusion

- The stiffness-proportional damping part of the Rayleigh's damping model corresponds physically to linear viscous dampers that interconnects the degree of freedom of the structure.
  If only stiffness proportional damping i.e, $c = \alpha_2 K$ is used then any high frequency response will be over-damped.

  Also,the stiffness-proportional damping models provide a better correlation with experimental results.

- When we use $C = \alpha_1 M$,this approach will help in the attainment of static equilibrium but the displacements responses will exhibit significant higher frequency responses.

Bibliography

[1] [Online]. Available: https://en.wikipedia.org/wiki/Stress_%28mechanics%29.

[2] R. D. Kriz, Tech. Rep., 2006. [Online]. Available: http://www.sv.vt.edu/classes/ESM4714/methods/EEG.html.

[3] K. . Meschke, “Finite element methods in linear structural mechanics”, in. 2005.

[4] [Online]. Available: https://en.wikipedia.org/wiki/Mixed_boundary_condition.

[5] D. Systemes, 2012. [Online]. Available: http://solidworks.com/.

[6] I. Chowdhury and S. P. Dasgupta, “Computation of rayleigh da mping coefficients for large systems”, pp. 2–4,

[7] M. Mukhopadhyay, “Vibrations, dynamics and structural systems”, in. 2000, vol. 2nd edition.

[8] Y. J.A.Cottrell T.J.R Hughes, in.