<u>E-YANTRA SUMMER INTERNSHIP</u>

Project- 17 / Shreya Rastogi 21/05/2020

<u>**CONTROLLABILITY** and **OBSERVABILITY**</u>

These are two important properties of state models which are to be studied prior to designing a controller. Let us start by asking certain questions and then in the process of answering them it will be explained about what is a controllable system and what is an observable system.

Consider an LTI (Linear Time Invariant) system:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Questions:

a) Can we move the state from one point in the state space to a desired location: $x_o \rightarrow x_d$, by choosing u properly?

b) Can we build a controller to stabilize the system?

c) Suppose that y contains all the quantities that we can measure. Can we evaluate the state x from y?

Let us first focus on the first two questions. The answer to both of them is related to the controllability of a system.

What is controllability?

Controllability measures the ability of a particular actuator configuration to control all the states of the system. A system is said to be completely state controllable if it is possible to transfer the system state from an initial state $X(t_o)$ to any other desired state $X(t_d)$ in specified time by a control vector U(t).

I) **Simple pendulum** system (Inherently non-linear)

- The system is linearized using state space modelling.
- The system equations are found out and consequently the A (state matrix) and B (input matrix) matrices are found out.
- Using Octave, we apply the test for controllability to this system.
- The **ctrb ()** function is applied to the A and B matrices.
- C = ctrb (A, B) returns the controllability matrix:

$$\boldsymbol{C} = [\boldsymbol{B}\ \boldsymbol{AB}\ \boldsymbol{A^2B} \ldots\ \boldsymbol{A^{n-1}B}]$$

where *A* is an *n*-by-*n* matrix, *B* is an *n*-by-*m* matrix, and *C* has *n* rows and *nm* columns.

C = ctrb (A, B) calculates the controllability matrix of the state-space LTI object system.

The system is controllable if C has full rank *n*.

Note: The rank of a matrix is the number of linearly independent rows or columns and equals the dimension of the row and column space.

Octave Output

```
>> m = 1;
>> g = 9.8;
>> L = 0.5;
>> A = [0 1; g/L 0];
>> B = [0;1/(m*L*L)];
>> C = ctrb(A,B)
C =

   0   4
   4   0

>> rank(C)
ans =  2
```

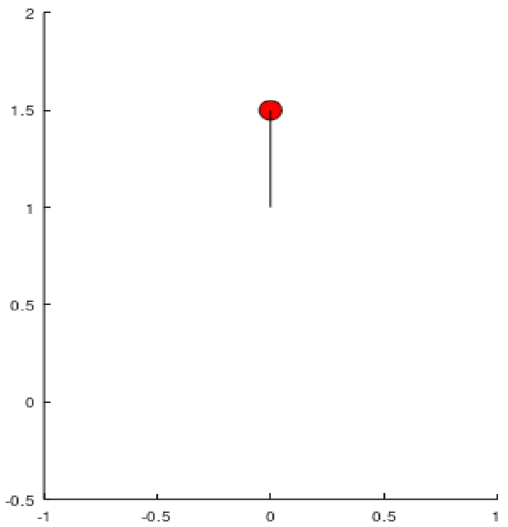
- Write all the relevant parameters in the command prompt of octave.
- Write the derived A and B matrices.
- Compute C matrix and it's rank.

Through this implementation we see that the rank of the simple pendulum system is equal to 2, which is equal to the number of state space variables for this system, hence the system is *controllable.*

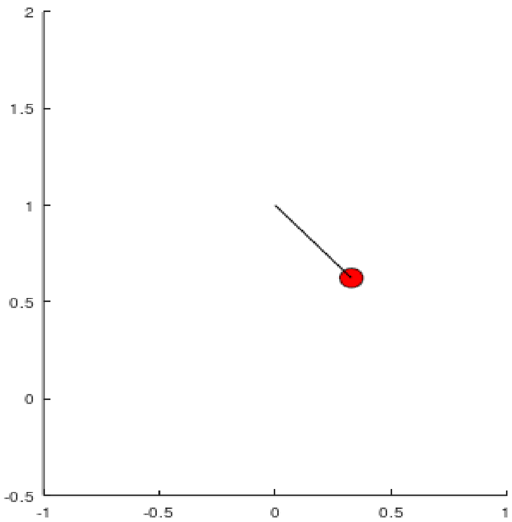
Simulation of Simple Pendulum in Octave

We can show the controllability of the system by stabilising it at inherently unstable points:

1. (π, 0)



2. (π/3, 0)



This shows that by changing the setpoint for the system, the system alters itself and responds to the external input. We were able to change it from an initial state to a desired state within a specific period of time.

II) **Cart Pendulum** system (Inherently non-linear)

- We repeat the same steps as in the simple pendulum example above.

- The **ctrb ()** function is applied to the A and B matrices.

Octave Output

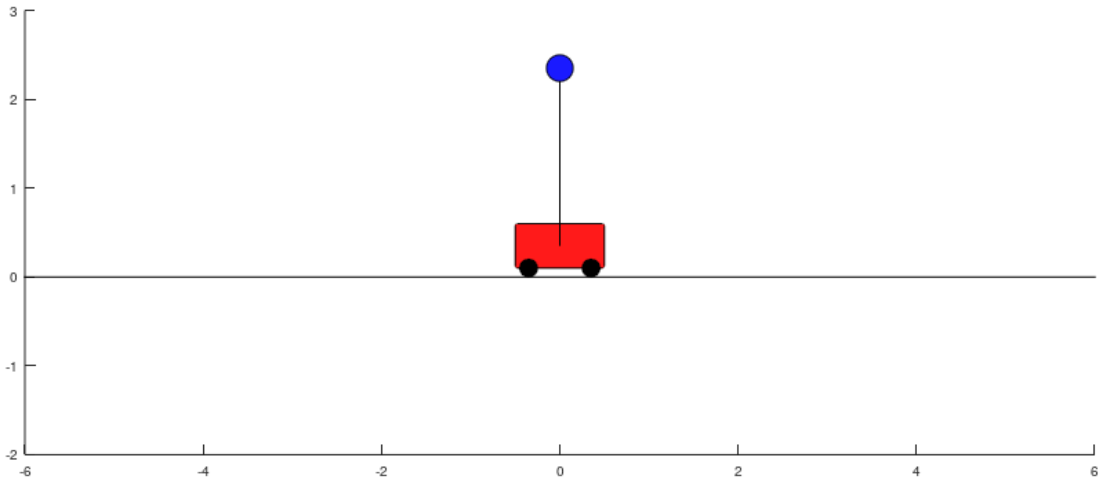
```
>>  m = 1;
>>   M = 5;
>>   L = 2;
>>   g = 9.8;
>>  A = [0 1 0 0;
       0 0 (-1*m*g)/(M) 0;
       0 0 0 1;
       0 0 ((M+m)*g)/(M*L) 0];
>>   B = [0; 1/(M); 0; -1/(M*L)];
>> C = ctrb(A,B)
C =

   0.00000   0.20000   0.00000   0.19600
   0.20000   0.00000   0.19600   0.00000
   0.00000  -0.10000   0.00000  -0.58800
  -0.10000   0.00000  -0.58800   0.00000

>> rank(C)
ans =  4
>> length(A) - rank(C)
ans = 0
```

Through this implementation we see that the rank of the cart pendulum system is equal to 4, which is equal to the number of state space variables for this system, hence the system is *controllable.*

Simulation of Cart Pendulum in Octave

We can show the controllability of the system by stabilising it at an inherently unstable point:

1. (0,0,0,0)



Since in the octave implementation above we proved that the system is controllable, we made use of the LQR controller to stabilise the system at an inherently unstable point (0,0,0,0).

So, by looking at these examples you must have understood the meaning of a controllable system.

- Thus far we have seen controllable systems. Let us now look at an example of an **uncontrollable** system:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{\mathbf{x}}
\end{aligned}
$$

The octave output for this system:

```
>> A = [-2, 0; 0, -1];
>> B = [2; 0];
>> CO = ctrb(A,B)
CO =

   2   -4
   0    0

>> rank(CO)
ans =   1
```

Since the rank of the controllability matrix is 1 which is less than the number of states for the system, that is 2, we conclude that this system is *not controllable*.

Another concept related to **Controllability** is **Stabilizability**.

STABILIZABILITY

- A system is said to be *stabilizable* if any initial state can be asymptotically steered to the origin by choosing the input *u* appropriately.
- Stabilizability is weaker than controllability.
- *A system is stabilizable if uncontrollable subspace is naturally stable.*
  What this means is that even if all the state space variables are not controllable, the system can still be partially controlled or stabilized. If the uncontrollable state space variables are already stable or can be made to achieve stability, then in that case we say that a system is **Stabilizable.**

- **Coming to the third question. The answer to it is related to the observability of a system.**

What is observability?

The system is completely observable if the state x can be determined from the knowledge of u(input) and y(output) over a finite time segment. A linear time invariant system of order *n* is said to be *observable* if the state at any instant can be determined by observing the output *y* over a finite interval of time.

- **obsv** () computes the observability matrix for state-space systems. For an *n*-by-*n* matrix A and a *p*-by-*n* matrix C, obsv (A, C) returns the observability matrix

$$\mathbf{Ob} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

with *n* columns and *np* rows.

- Here the C matrix is the output matrix of the system. It is a constant matrix having the dimensions of p x n for a MIMO (multi-input, multi-output) system, where p is the number of outputs and n is the number of states.

I) Simple Pendulum

We will now see whether a simple pendulum system is observable or not.

Octave Output

```
>> m1 = 7.5;
>> m2 = 7.51;
>> g  = 9.8;
>> r  = 0.2;
>> A = [0 1; 0 0];
>> B = [0;1/(r*(m1+m2))];
>> C = [1,0; 0, 1];
>> Ob = obsv(A,C);
>> rank(Ob)
ans =  2
```

As we can see that rank = 2 for the observability matrix which is equal to the number of states for the simple pendulum system.

Therefore, this system is observable.

II) Cart Pendulum

We will now see whether a cart pendulum system is observable or not.

Octave Output

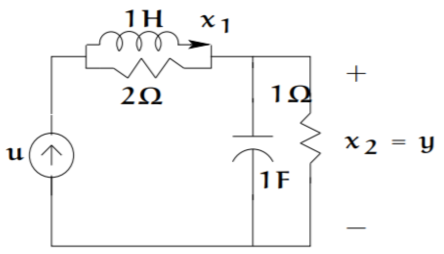
```
>> m = 1;
>> M = 5;
>> L = 2;
>> g = 9.8;
>> A = [0 1 0 0;
     0 0 (-1*m*g)/(M) 0;
     0 0 0 1;
     0 0 ((M+m)*g)/(M*L) 0];
>> B = [0; 1/(M); 0; -1/(M*L)];
>> C = [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]
>> Ob = obsv(A,C)
Ob =

    1.00000    0.00000    0.00000    0.00000
    0.00000    1.00000    0.00000    0.00000
    0.00000    0.00000    1.00000    0.00000
    0.00000    0.00000    0.00000    1.00000
    0.00000    1.00000    0.00000    0.00000
    0.00000    0.00000   -1.96000    0.00000
    0.00000    0.00000    0.00000    1.00000
    0.00000    0.00000    5.88000    0.00000
    0.00000    0.00000   -1.96000    0.00000
    0.00000    0.00000    0.00000   -1.96000
    0.00000    0.00000    5.88000    0.00000
    0.00000    0.00000    0.00000    5.88000
    0.00000    0.00000    0.00000   -1.96000
    0.00000    0.00000  -11.52480    0.00000
    0.00000    0.00000    0.00000    5.88000
    0.00000    0.00000   34.57440    0.00000

>> rank(Ob)
ans =   4
```

As we can see that rank = 4 for the observability matrix which is equal to the number of states for the cart pendulum system.

Therefore, this system is observable.

Thus far we have learnt about observable systems.

- Here is an example of an **Unobservable** system:



The network shown in the figure below has two state variables: the current x_1 through the inductor and the voltage x_2 across the capacitor. The input u is a current source.

If u = 0, $x_2(0) = 0$ and $x_1(0) = a \neq 0$, then the output is identically zero. Any x (0) = $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and $u(t) \equiv 0$ yields the same output $y(t) \equiv 0$. Thus, there is no way to uniquely determine the initial state $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and the system is unobservable.

Another concept related to **Observability** is **Detectability**.

DETECTABILITY

Detectability means that all unobservable states are stable. It means that all the unobservable states are convergent.

- The system is detectable if all the unstable eigenvalues are observable, or equivalently, if the unobservable subspace is stable.
- Detectability is weaker than observability.
- A system in which all the state variables are not observable, if the unobservable subspace is stable or in other words can be made to approach zero or can be made to converge, then we call that system as detectable.

DUALITY

- The concept of controllability and observability are often referred to as **duals.**
- The pair (A, B) is controllable (stabilizable) if and only if (A^T, B^T) is observable (detectable). The system (A, C) is observable (detectable) if and only if (A^T, C^T) is controllable (stabilizable).
