PID:17 26th June, 2020

State-Space Model and Controllability

1 Aim

Finding out the Parametric State-Space Model. Substituting practical values and checking for controllability of the system.

2 Problem

- Consider the system of a Cyclebot. In the previous experiment we derived the state space model of the system. The input to this system was an input torque. Find the model in terms of an input voltage i.e. find the relation between voltage and torque and find the resultant model of the system.
- The hardware or the cyclebot that you have built. Find it's values, for example mass, moment of inertia, torque etc. Substituting these values in the A and B matrices, check whether the actual system is **controllabale** or not.

3 Solution

Let $[\theta, \dot{\theta}, \phi, \dot{\phi}]^T$ be the state variables of the system
where:
θ: deflection of the pendulum from the vertical
$\dot{\theta}$: angular velocity of the pendulum
ϕ: angle of the reaction wheel
$\dot{\phi}$: angular velocity of the reaction wheel

$\therefore$ State Space Representation of the system

$$\dot{x} = Ax + Bu \tag{1}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b}{a} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{a} \\ 0 \\ \frac{(a+I_2)}{aI_2} \end{bmatrix} T_r$$

where

$$a = (m_1 L_1^2 + m_2 L_2^2 + I_1) \tag{2}$$

$$b = (m_1 L_1 + m_2 L_2)g \tag{3}$$

To control the speed of the reaction wheel to the desired one, the following mathematical model is the physical behavior of the motor with a gear system:

$$V = L_m \frac{di}{dt} + R_m i + K_e \omega_m$$

$$T_m = K_t i$$

$$T_r = N_g T_m$$

where:

- V : motor supply voltage
- K_e : motor back emf constant
- ω_m : motor angular speed
- L_m : armature coil inductance
- R_m : armature coil resistance
- i : armature current
- T_m : motor generated torque
- K_t : motor torque constant
- N_g : gear ratio

The term of inductance can be neglected since, in general, the motor inductance value is much less than the motor resistance value (L_m«R_m). By using the relationship between the motor and the reaction wheel, i. e., $\dot{\phi} = \omega_r$, $\omega_m = N_g \omega_r$, where ω_r is reaction wheel angular speed, the motor supply voltage can be found in terms of the reaction wheel angle as follows:

$$T_r = N_g K_t \frac{V - K_e N_g \dot{\phi}}{R_m}$$

The Model:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 1 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ b_4 \end{bmatrix} V$$

where:

$a_{21} = \frac{b}{a}$, $a_{24} = \frac{K_t K_e {N_g}^2}{a R_m}$, $a_{41} = -\frac{b}{a}$, $a_{44} = -\frac{(a+I_2)}{a I_2} \frac{K_t K_e {N_g}^2}{R_m}$, $b_2 = -\frac{K_t N_g}{a R_m}$, $b_4 = \frac{(a+I_2)}{a I_2} \frac{K_t N_g}{R_m}$

Parameters of Our Cyclebot

Parameters	Values
Distance from pivot to COM of Body (L_1)	76.83mm
Distance from pivot to COM of Reaction Wheel (L_2)	148.53mm
Mass of Bicycle (M_1)	763g
Mass of Reaction Wheel (M_2)	357g
MOI of Body about COM (I_1)	2.1x$10^6 gmm^2$
MOI of Reaction wheel about COM (I_2)	8.2x$10^5 gmm^2$
Motor Resistance (R_m)	13.3 Ω
Motor Torque Constant (K_t)	2.77x$10^{-2} Nm/A$
Motor Back EMF Constant (K_e)	1.27x$10^{-2} Vs/rad$
Motor gear ratio (N_g)	1:30

The model obtained by substituting these values is:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 75.64 & 0 & 0 & 1.6482 \\ 0 & 0 & 0 & 1 \\ -75.64 & 0 & 0 & -2.0678e-05 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ -4.3151 \\ 0 \\ 80.512 \end{bmatrix} V$$

Using the above A and B matrices and the controllability matrix of the system

$$CO = [B, AB, A^2B, A^3B]$$

we check whether this system is controllable or not. Using the **ctrb(A,B)** function in octave, we find the controllability matrix of this system. Then using the **rank(CO)** function we get $rank = 4$, which proves that this system is controllable.