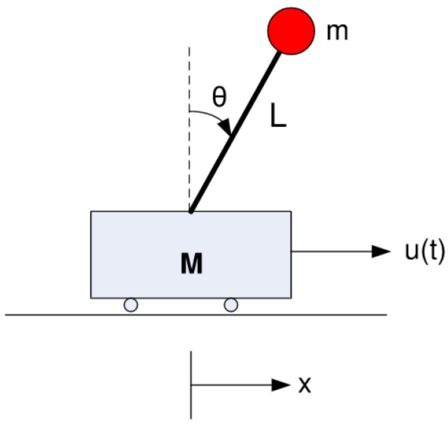
Eigenvalues and Stability Criteria

Solution to Experiment 1

Aim:

To comment on stability of system by using given model.

Problem Statement:



Consider a cart pendulum system. State variables include angle of the pendulum θ, angular velocity $\dot{\theta}$, horizontal displacement x, and velocity $\dot{x}$. Mathematical Model of this system in state space form:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L}(\frac{m}{M}+1) & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u$$

Find eigenvalues of matrix A and comment on behaviour of this system when input u=0, and

$\theta = 0$.

1 Solution

Here,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{L}(\frac{m}{M}+1) & 0 \end{bmatrix}$$

To find eigenvalues of this matrix,

$$\begin{bmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & \frac{mg}{M} & 0 \\ 0 & 0 & 0-\lambda & 1 \\ 0 & 0 & \frac{g}{L}(\frac{m}{M}+1) & 0-\lambda \end{bmatrix} = 0$$

$$\therefore \lambda^2(\lambda^2 - \frac{g}{L}(\frac{m}{M}+1)) = 0$$
$$\therefore \lambda^2 = 0 \qquad\qquad or\ (\lambda^2 - \frac{g}{L}(\frac{m}{M}+1)) = 0$$

Two solutions are $\lambda = 0$. If $\lambda \neq 0$,

$$\lambda^2 = \frac{g}{L}(\frac{m}{M}+1)$$
$$\therefore \lambda = \pm\sqrt{\frac{g}{L}(\frac{m}{M}+1)}$$

Here, $\lambda = 0, 0, \sqrt{\frac{g}{L}(\frac{m}{M}+1)}, -\sqrt{\frac{g}{L}(\frac{m}{M}+1)}$.

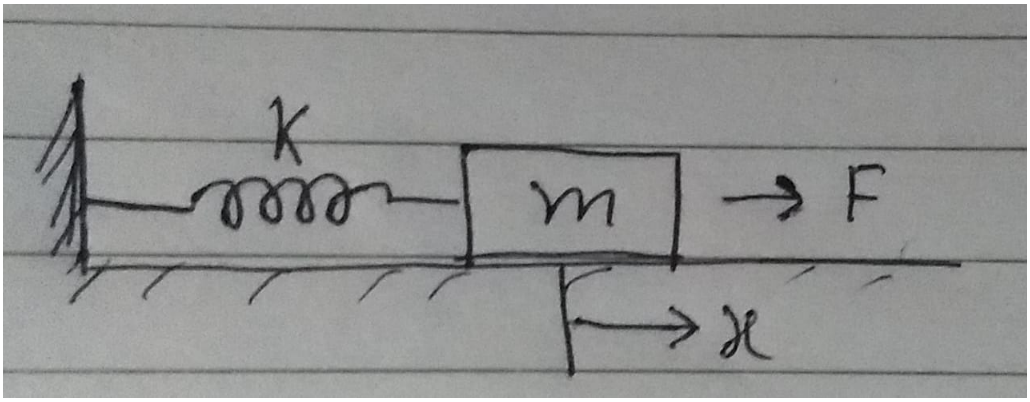
One of the eigenvalues is positive. So, the system is unstable at $\theta = 0$.

Solution to Experiment 2

Aim:

To find range of K matrix for which system is stable

Problem Statement:



Consider a spring mass system. A variable force F is applied on the mass. Model of this system can be given as

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

Let a matrix $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$. Let $F = -Kx$.

As $\dot{x} = Ax + Bu$, $u = F$, and $F = -Kx$,

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) \\ &= (A - BK)x \end{aligned}$$

Now this new coefficient of x, $A - BK$, can be considered the new A matrix of this system. To find stability of the system with this input, apply the stability criteria on the new matrix, $A' = A - BK$.

Find the range of K_1 and K_2 for which the system is stable.

Solution

According to the theory given in problem statement, eigenvalues of $A - BK$ should be negative.
Here, $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$, $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$.
So, $BK = \begin{bmatrix} 0 & 0 \\ \frac{K_1}{m} & \frac{K_2}{m} \end{bmatrix}$

$$A - BK = A' = \begin{bmatrix} 0 & 1 \\ \frac{-k-K_1}{m} & -\frac{K_2}{m} \end{bmatrix}$$

To find eigenvalues of A',

$$\begin{bmatrix} 0-\lambda & 1 \\ \frac{-k-K_1}{m} & -\frac{K_2}{m}-\lambda \end{bmatrix} = 0$$

$$\therefore \lambda^2 + \frac{K_2}{m}\lambda + \frac{k+K_1}{m} = 0$$

Let λ_1 and λ_2 be the solutions of this equation. For both of these to be negative,

$$\begin{aligned} \lambda_1 + \lambda_2 &< 0 \\ \lambda_1\lambda_2 &> 0 \end{aligned}$$

So,

$$\begin{aligned} \frac{-K_2}{m} &< 0 && \Rightarrow K_2 > 0 \\ \frac{k+K_1}{m} &> 0 && \Rightarrow K_1 > -k \end{aligned}$$