Finding Equilibrium points and Linearization

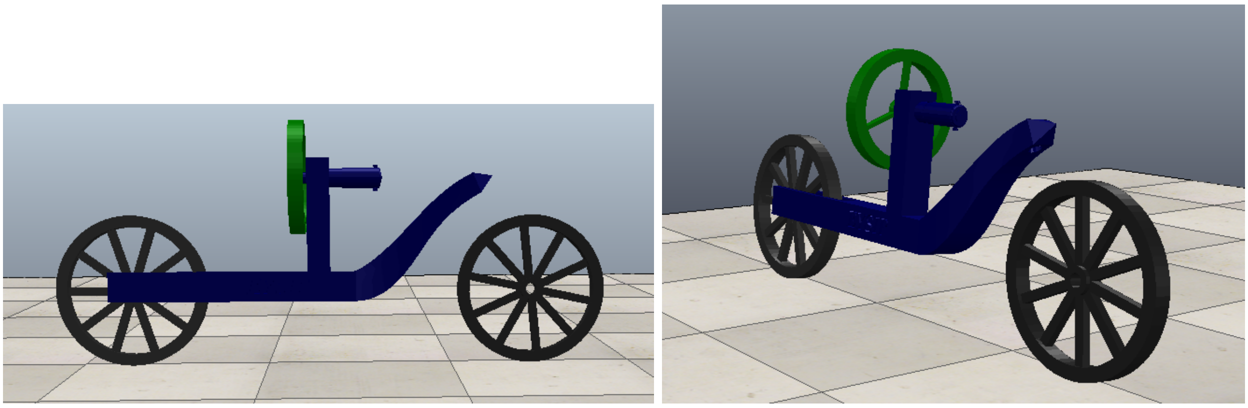
Experiment

1 Aim:

To find equilibrium points in a system, and linearize the system around those points.

2 Problem Statement:

Model of a system, an inverted pendulum with a reaction wheel, is given.



(a) (b)

Figure 1: Inverted Pendulum System with Reaction Wheel

$$\ddot{\theta} = \frac{(m_1L_1 + m_2L_2)g}{m_1L_1^2 + m_2L_2^2 + I_1} sin\theta - \frac{1}{m_1L_1^2 + m_2L_2^2 + I_1} T \tag{1}$$

$$\ddot{\phi} = -\frac{(m_1L_1 + m_2L_2)g}{m_1L_1^2 + m_2L_2^2 + I_1} sin\theta + \frac{m_1L_1^2 + m_2L_2^2 + I_1 + I_2}{I_2(m_1L_1^2 + m_2L_2^2 + I_1)} T \tag{2}$$

Where:

- m_1 : mass of pendulum
- L_1 : Distance between pivot and COM of pendulum
- I_1 : Moment of inertia of pendulum about it's COM
- m_2 : mass of reaction wheel
- L_2 : Distance between pivot and COM of reaction wheel
- I_2 : Moment of inertia of reaction wheel about it's COM
- $\theta, \dot{\theta}$: Tilt angle and angular velocity of tilt
- $\phi, \dot{\phi}$: Angular position and velocity of reaction wheel
- T : Torque provided to the reaction wheel

Find the equilibrium point(s) of this system, and linearize the system around these points using Jacobian technique. Also find the type of equilibrium.

3 Solution:

At equilibrium point(s), rate of change of the system is zero.

$$\therefore \ddot{\theta} = 0$$

and

$$\ddot{\phi} = 0$$

According to equation 1 and 2,

$$\frac{(m_1L_1 + m_2L_2)g}{m_1L_1^2 + m_2L_2^2 + I_1} sin\theta = 0$$
$$\therefore sin\theta = 0$$
$$\therefore \theta = 0, \pi$$

The system will be at equilibrium when $\theta = 0, \pi$.
We have to represent this in state space form, where 2 matrices are required. First, taking Jacobian with respect to θ and ϕ,

$$J_A = \begin{bmatrix} \frac{\partial\ddot{\theta}}{\partial\theta} & \frac{\partial\ddot{\theta}}{\partial\phi} \\ \frac{\partial\ddot{\phi}}{\partial\theta} & \frac{\partial\ddot{\phi}}{\partial\phi} \end{bmatrix}$$

And with respect to T,

$$J_B = \begin{bmatrix} \frac{\partial\ddot{\theta}}{\partial T} \\ \frac{\partial\ddot{\phi}}{\partial T} \end{bmatrix}$$

Solving these we get,

$$J_A = \begin{bmatrix} \frac{(m_1L_1+m_2L_2)g}{m_1L_1^2+m_2L_2^2+I_1} cos\theta & 0 \\ -\frac{(m_1L_1+m_2L_2)g}{m_1L_1^2+m_2L_2^2+I_1} cos\theta & 0 \end{bmatrix}$$
$$and$$
$$J_B = \begin{bmatrix} \frac{1}{m_1L_1^2+m_2L_2^2+I_1} \\ \frac{m_1L_1^2+m_2L_2^2+I_1+I_2}{I_2(m_1L_1^2+m_2L_2^2+I_1)} \end{bmatrix}$$

So, equations linearized about point $\theta = \theta$ are

$$\ddot{\theta} = \frac{(m_1L_1 + m_2L_2)gcos\theta}{m_1L_1^2 + m_2L_2^2 + I_1}\theta - \frac{1}{m_1L_1^2 + m_2L_2^2 + I_1}T \tag{3}$$

$$\ddot{\phi} = -\frac{(m_1L_1 + m_2L_2)gcos\theta}{m_1L_1^2 + m_2L_2^2 + I_1}\theta + \frac{m_1L_1^2 + m_2L_2^2 + I_1 + I_2}{I_2(m_1L_1^2 + m_2L_2^2 + I_1)}T \tag{4}$$

To represent the system in state space representation, state variables need to be chosen. Let's consider $\theta, \dot{\theta}, \phi$, and $\dot{\phi}$ as state variables, and T as input to the system. Considering this,

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b}{a} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{a} \\ 0 \\ \frac{(a+I_2)}{aI_2} \end{bmatrix} T$$

where

$$a = m_1 L_1^2 + m_2 L_2^2 + I_1$$
$$b = g(m_1 L_1 + m_2 L_2) cos\theta$$

To determine type of equilibrium at points, eigenvalues of A are needed. To calculate those,

$$\begin{bmatrix} 0-\lambda & 1 & 0 & 0 \\ \frac{b}{a} & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 1 \\ -\frac{b}{a} & 0 & 0 & 0-\lambda \end{bmatrix} = 0$$

$$\therefore (\lambda^2)(\lambda^2 - \frac{b}{a}) = 0$$

2 of the solutions are $\lambda = 0$. If $\lambda \neq 0$,

$$\lambda = \pm\sqrt{\frac{b}{a}} = \pm\sqrt{\frac{g(m_1 L_1 + m_2 L_2) cos\theta}{m_1 L_1^2 + m_2 L_2^2 + I_1}}$$

At $\theta = 0$, $cos\theta = 1$. So, one of the eigenvalues is positive.
Hence, the system is unstable at $\theta = 0$.

At $\theta = \pi$, $cos\theta = -1$. So, both nonzero eigenvalues are purely imaginary, i.e. real part is 0.
Hence the system is marginally stable at $\theta = \pi$.