Mathematical Modelling of Reaction Wheel Balanced Inverted Pendulum

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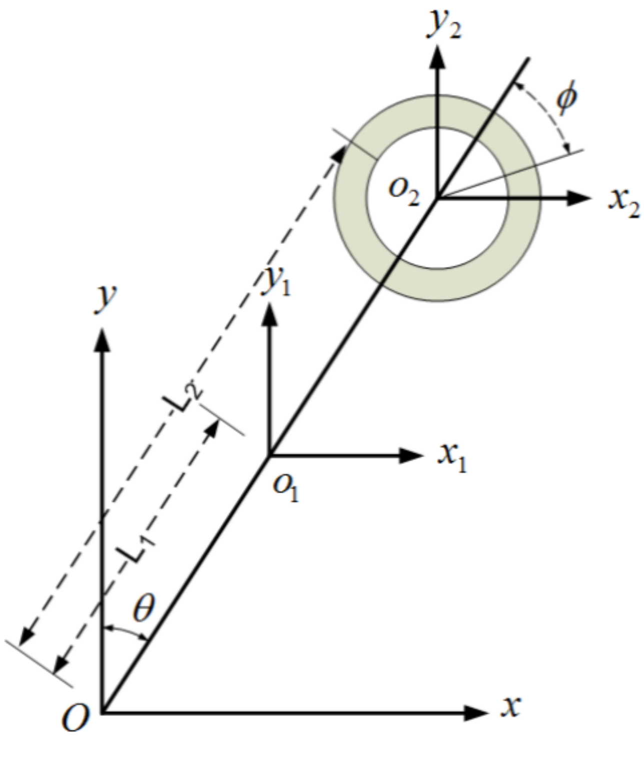
e-YSIP

The aim of this project is to build a cycle robot that can balance itself using a reaction wheel This bicycle robot has to be able to keep balancing at zero forward speed and to follow a given path without losing its balance at desired forward speed. The basic principle is that the reaction wheel attached to an electric motor can rotate clockwise or counterclockwise with desired speed in order to maintain balancing of the bicycle. Due to the fact that the motor applies a torque on the reaction wheel, which in turn applies an equal amount of torque back on the bicycle. With a proper control law, this action-reaction combination can balance the bicycle. Following is the mathematical modelling of a Reaction wheel balanced inverted pendulum.



A reaction wheel system based on the principle of an inverted pendulum system.

Let: For the inverted pendulum

m_1 : bicycle mass

o_1: center of mass

I_1: moment of inertia about center of mass

θ: angle between bicycle and vertical upright

L_1: distance from origin to center of mass of bicycle

For the reaction wheel
m_2 : reaction wheel mass
o_2: its center of mass
I_2: moment of inertia about center of mass of reaction wheel
θ: angle between bicycle and vertical upright
L_1: distance from origin to center of mass

Lagrange equation of system will be derived using

$$\frac{d}{dt}\frac{\partial u}{\partial \dot{q_i}} - \frac{\partial L}{\partial q_i} = \tau_i \tag{1}$$

τ_i is the external force corresponding to generalized coordinates q_i, and

$$L(q,\dot{q}) = KE(q,\dot{q}) - PE(q,\dot{q}) \tag{2}$$

where
L: Lagrangian Operator
KE: Kinetic energy of the system
PE: Potential energy of the system

Here we have θ and ϕ as generalized coordinates

Total KE:

$$KE = \frac{1}{2}(m_1L_1^2 + m_2L_2^2 + I_1)\dot{\theta^2} + \frac{1}{2}(I_2)(\dot{\theta} + \dot{\phi})^2 \tag{3}$$

$$= \frac{1}{2}(m_1L_1^2 + m_2L_2^2 + I_1 + I_2)\dot{\theta^2} + I_2\dot{\theta}\dot{\phi} + \frac{1}{2}I_2\dot{\phi^2} \tag{4}$$

Total PE:

$$PE = (m_1L_1 + m_2L_2)gCos\theta \tag{5}$$

$$\therefore L = KE - PE \tag{6}$$

$$= \frac{1}{2}(m_1L_1^2 + m_2L_2^2 + I_1 + I_2)\dot{\theta^2} + I_2\dot{\theta}\dot{\phi} + \frac{1}{2}I_2\dot{\phi^2} - (m_1L_1 + m_2L_2)gCos\theta \tag{7}$$

We know,

$$\frac{d}{dt}\frac{\partial u}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{8}$$

Using (7) in (8)

$$(m_1L_1^2 + m_2L_2^2 + I_1 + I_2)\ddot{\theta} + I_2\ddot{\phi} - (m_1L_1 + m_2L_2)gSin\theta = 0 \tag{9}$$

We also know,

$$\frac{d}{dt}\frac{\partial u}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = T_r \tag{10}$$

Using (7) in (10)

$$I_2\ddot{\theta} + I_2\ddot{\phi} = T_r \tag{11}$$

where T_r is the driving torque of reaction wheel.
When θ is very small, according to Jacobian linearization, we obtain:

$$(m_1L_1^2 + m_2L_2^2 + I_1 + I_2)\ddot{\theta} + I_2\ddot{\phi} - (m_1L_1 + m_2L_2)g\theta = 0 \tag{12}$$

$$I_2(\ddot{\theta} + \ddot{\phi}) = T_r \tag{13}$$

Let $[\theta, \dot{\theta}, \phi, \dot{\phi}]^T$ be the state variables of the system
where:
θ: deflection of the pendulum from the vertical
$\dot{\theta}$: angular velocity of the pendulum
ϕ: angle of the reaction wheel
$\dot{\phi}$: angular velocity of the reaction wheel

$\therefore$ State Space Representation of the system

$$\dot{x} = Ax + Bu \tag{14}$$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{b}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{b}{a} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{a} \\ 0 \\ \frac{(a+I_2)}{aI_2} \end{bmatrix} T_r$$

where

$$a = (m_1L_1^2 + m_2L_2^2 + I_1) \tag{15}$$

$$b = (m_1L_1 + m_2L_2)g \tag{16}$$

Using these A and B matrices we can find the optimal feedback matrix, K using LQR (Linear Quadratic Regulator) controller, which will then be used to control the system.