Stability Criteria of a System

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1 State Space Representation

In state space representation, all different state variables are packed together to form a state vector. Rate of change of this vector is dependant on the state itself, and the inputs applied to the system. All the inputs to the system are grouped into input vector. The standard state space equation is:

$$\dot{x} = Ax + Bu$$

Where

- x represents current state. Also called as state vector.
- A represents how current state affects rate of change of state. All inherent properties if the system are described by this matrix.
- u represents the input matrix.
- B represents effect of input on rate of change of state.

2 Stability Criteria

A system is said to be stable if all the parameters of state vector remain bounded as time approaches infinity. If there exists even a single state variable that approaches infinity with time, system becomes unstable.

2.1 Continuous Time

To predict the stability system, only matrix A is considered. Eigenvalues of matrix A are poles of this system. So, solution to the differential equation

$$\dot{x} = Ax \Rightarrow x_i(t) = \sum_{k=0}^{n} x_i(0)e^{\lambda_k t}$$

where $x_i(t)$ represents behaviour of i^{th} state variable, and λ_k represents eigenvalues of matrix A. By looking at this, it can be said that if:

- $Re(\lambda_k) < 0$: System will approach zero, asymptotically stable
- $Re(\lambda_k) = 0$: System will stay at initial conditions, or will oscillate with amplitude equal to initial conditions depending on $Im(\lambda_k)$.This is known as critically stable system.
- $Re(\lambda_k) < 0$: System will keep on rising exponentially, becoming unstable.

2.2 Discrete Time

In discrete time, state space equation modifies to

$$x_{n+1} = A_d x_n + B_d u_n$$

This means that $(n+1)^{th}$ state will depend on n^{th} state and n^{th} input. Just like in continuous time, eigenvalues of A_d represent poles of system in discrete time. Solution to the *difference* equation

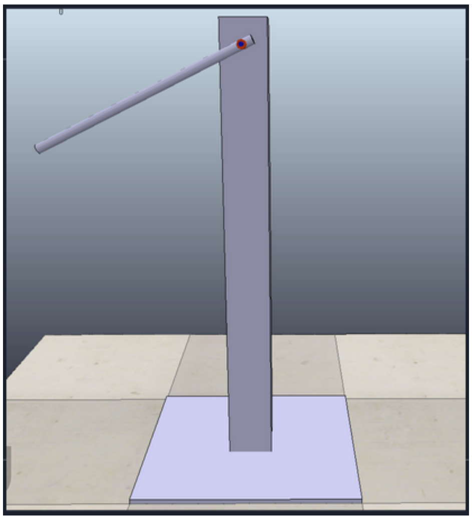
$$x_{n+1} = A_d x_n \Rightarrow x_i(n) = \sum_{k=0}^{n} x_i(0)\lambda_k^n$$

This means that if:

- $|\lambda_k| < 1$: system will approach zero, asymptotically stable.
- $|\lambda_k| = 1$: System will stay at initial conditions, or will oscillate with amplitude equal to initial conditions depending on $\angle\lambda_k$. This is known as critically stable system.
- $|\lambda_k| > 1$: System will keep on rising exponentially, becoming unstable.

3 Examples

3.1 Simple Pendulum System



State variables are chosen as the angle from vertical, θ, and angular velocity ω. Mathematical model of this system:

$$\dot{\omega} = -\frac{g}{l} sin\theta$$

Let $sin\theta \approx \theta$. Model in state space form:

$$\begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

Here, $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$

Finding eigenvalues of this matrix,

$$\begin{bmatrix} 0-\lambda & 1 \\ -\frac{g}{l} & 0-\lambda \end{bmatrix} = 0$$

$$\therefore \lambda = \pm\sqrt{\frac{g}{l}}j$$

Here, both the eigenvalues are purely imaginary. As real part is zero, system is critically stable. As imaginary part is nonzero, the system oscillates. We know this is right, as a pendulum will keep on oscillating forever if friction is not considered.