# Computer Vision & Image Processing CSE 473 / 573

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Lecture 19

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Fitting / RANSAC

# Slide credits

- Silvio Savarese
- Richard Szeliski

# Computer vision

# Fitting methods

- Problem formulation
- Least square methods
- RANSAC
- Hough transforms
- Multi-model fitting
- Fitting helps matching!

# Fitting

**Goal:** Choose a parametric model to fit a certain quantity from data

- Lines
- Curves
- Homographic transformation
- Fundamental matrix
- Shape model

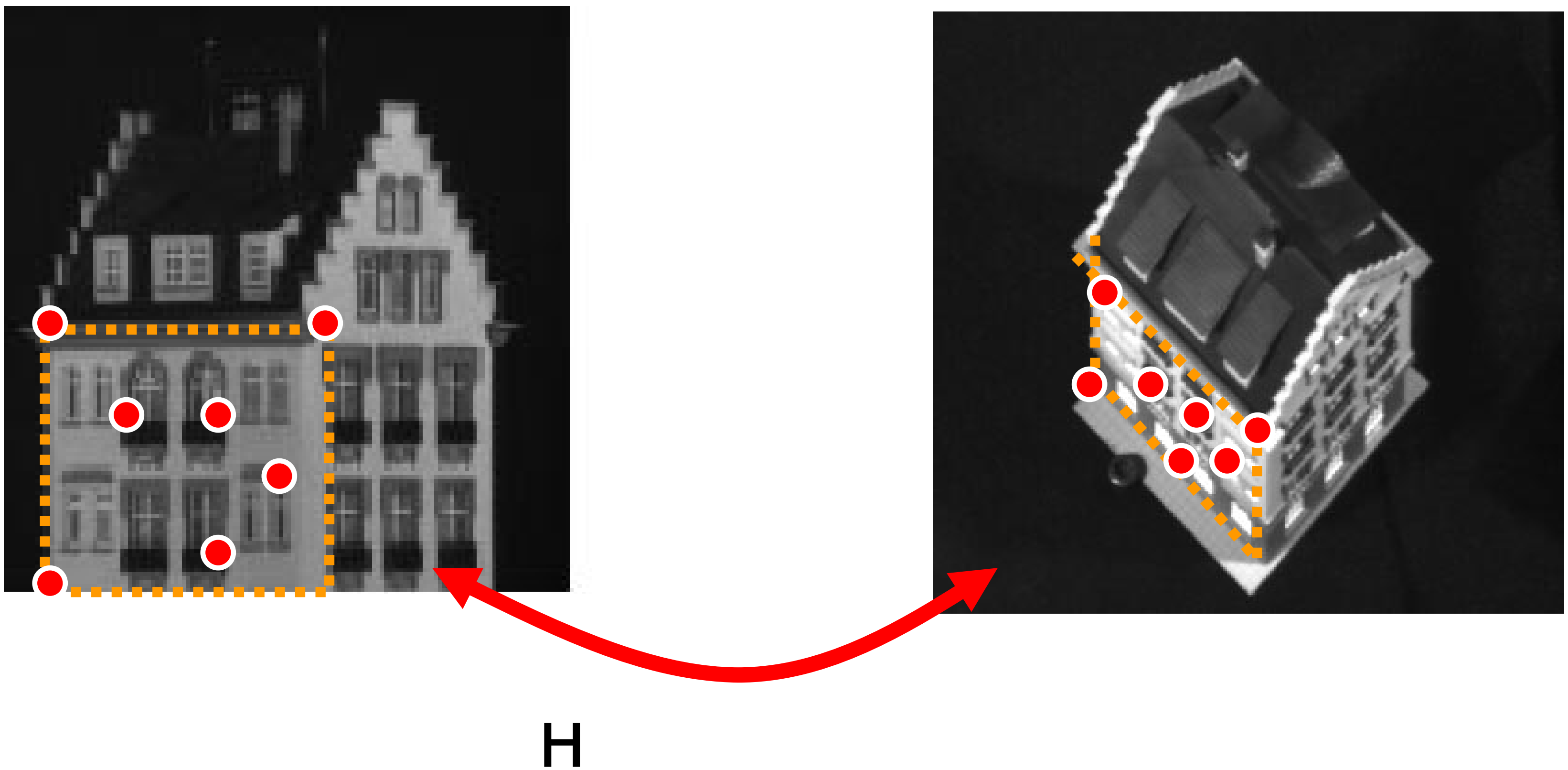
# Fitting

Goal: Choose a parametric model to fit a certain quantity from data

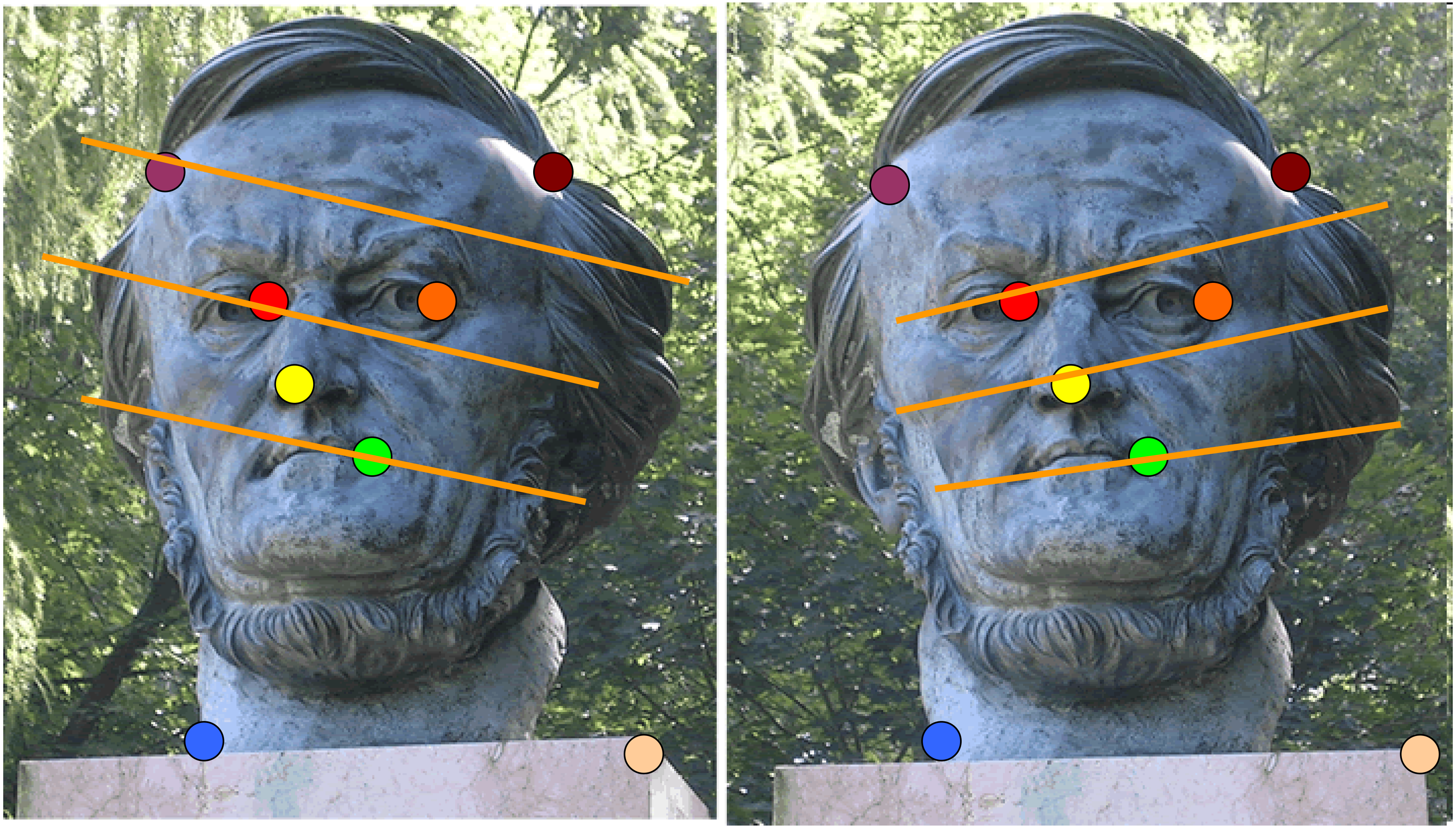
- Lines
- Curves
- *Homographic transformation*
- Fundamental matrix
- Shape model

# Example: Computing vanishing points

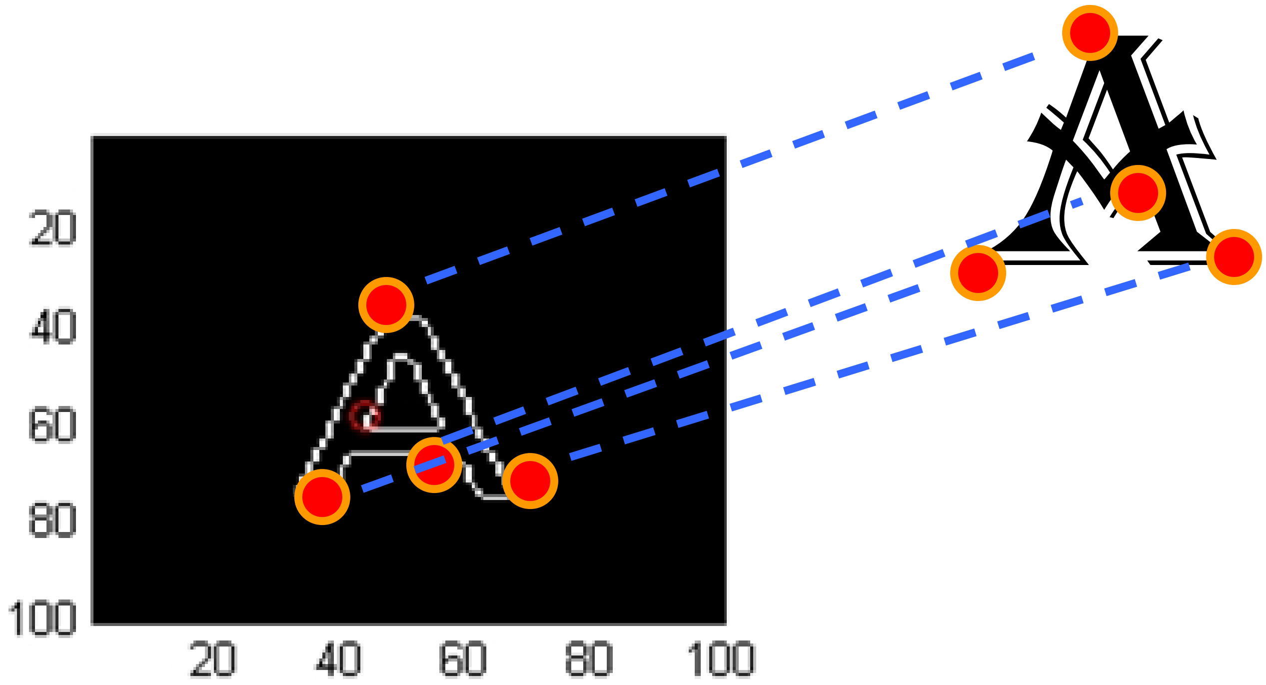
# Example: Estimating an homographic transformation



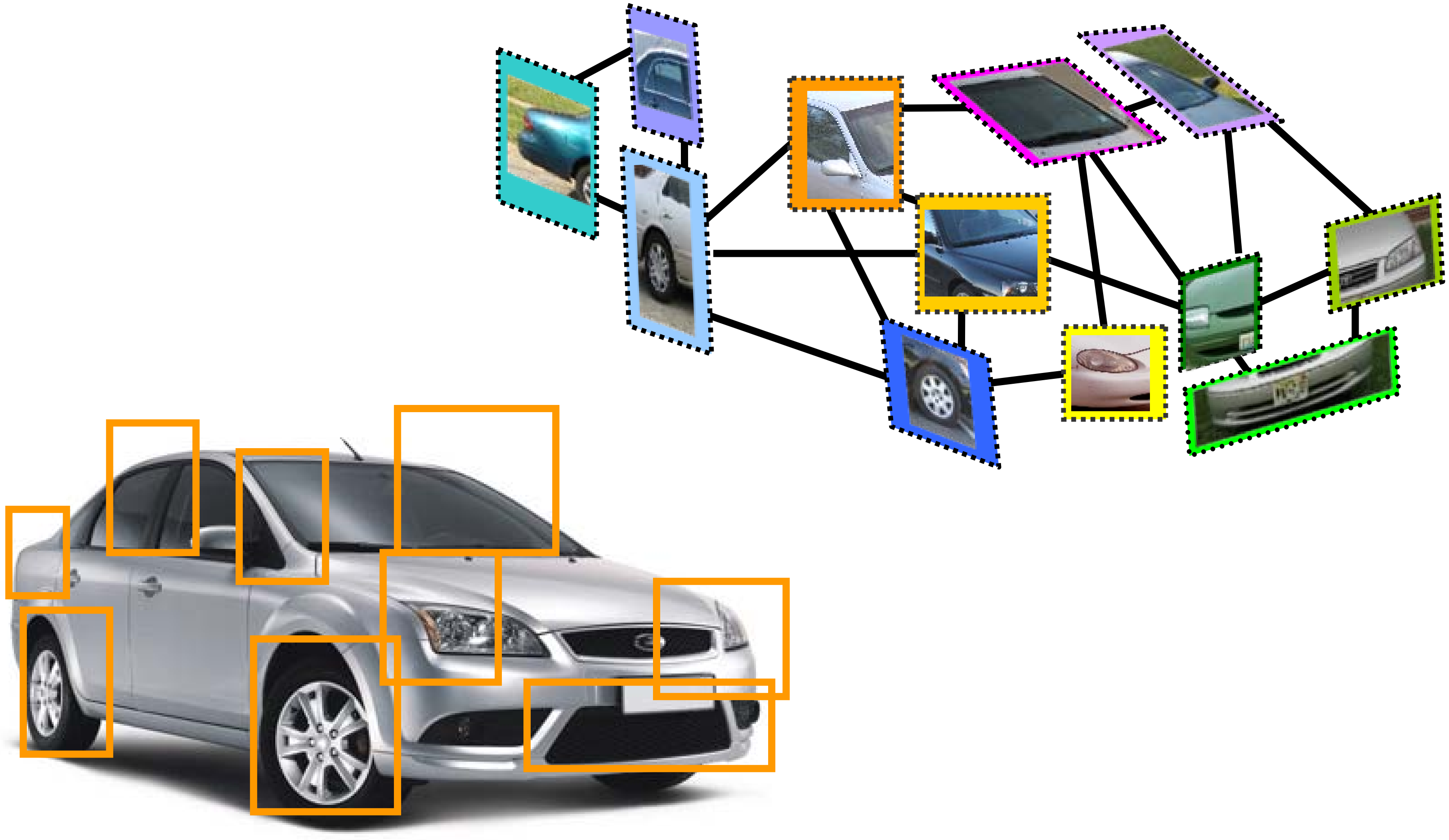
# Example: Estimating F



# Example: fitting an 2D shape template



# Example: fitting a 3D object model



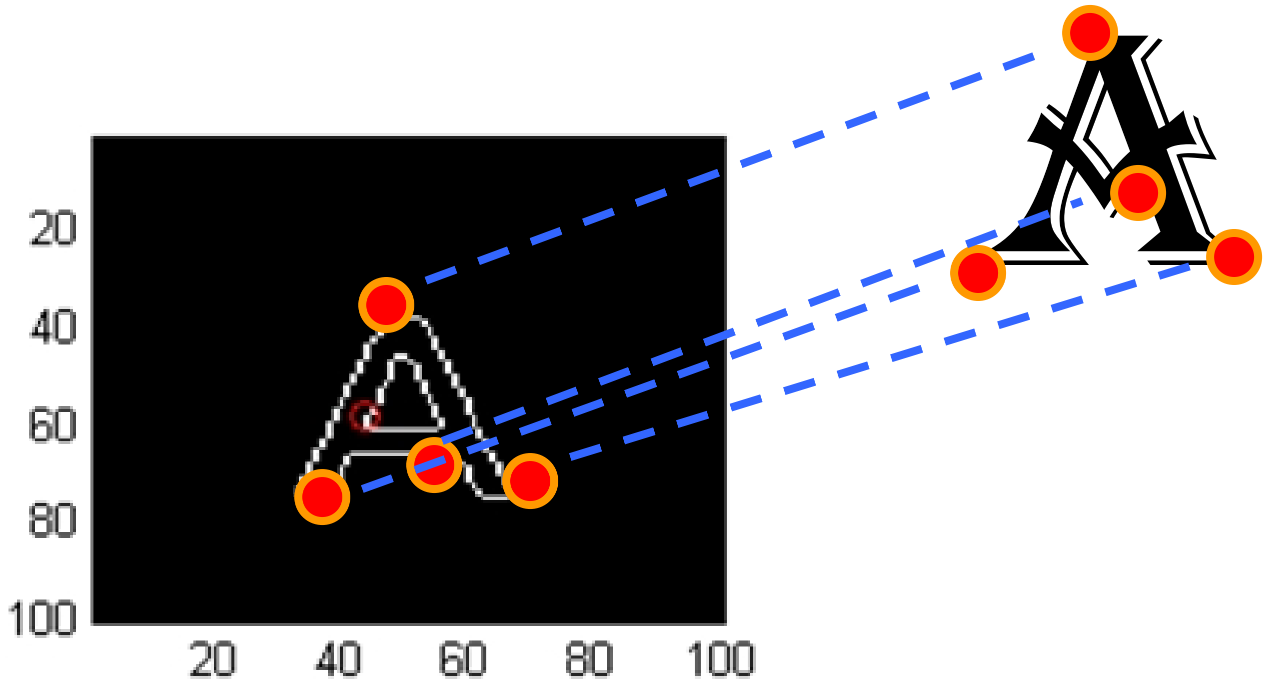
# Fitting

Goal: Choose a parametric model to fit a certain quantity from data

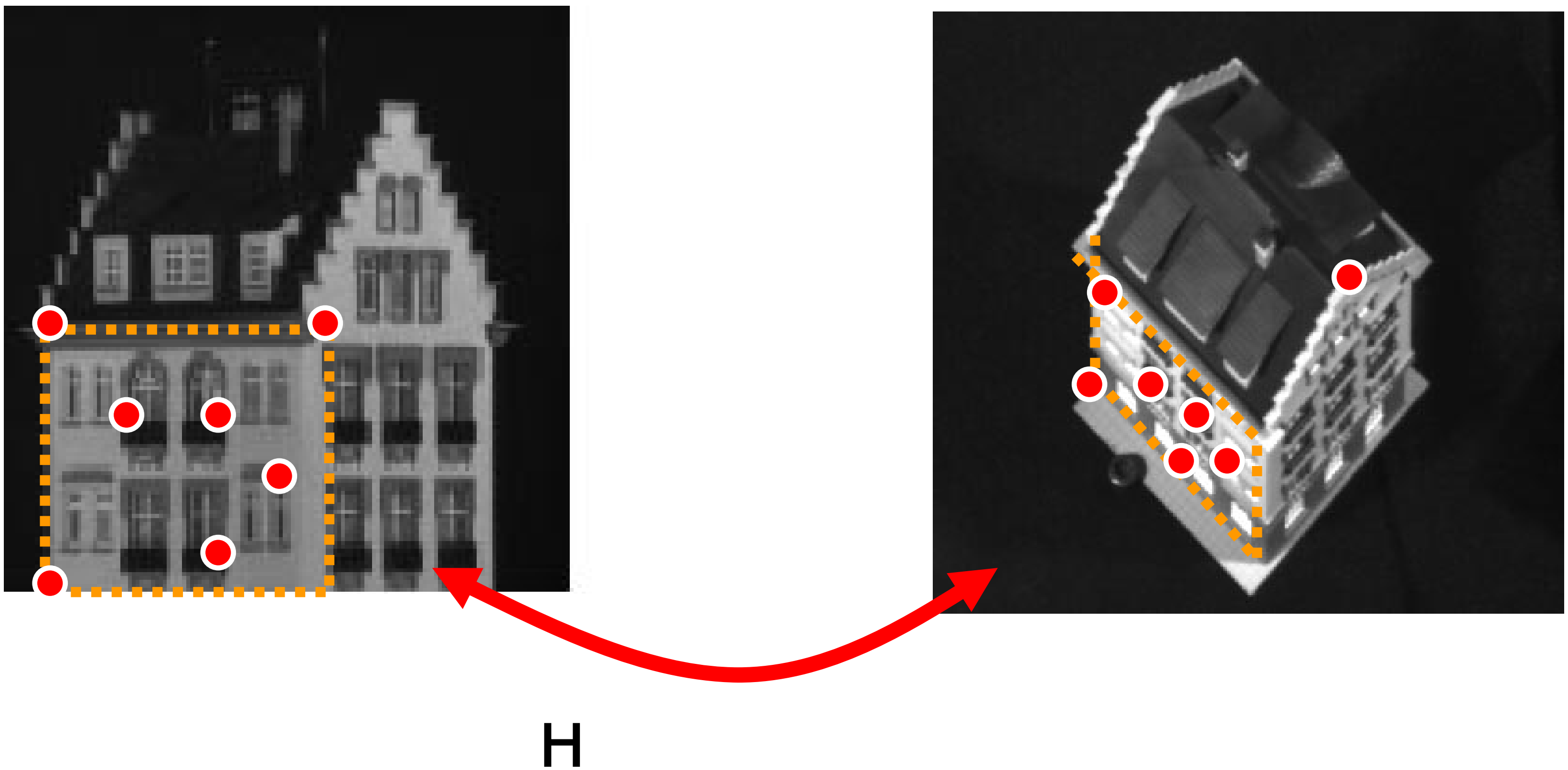
Critical issues:
- noisy data
- outliers
- missing data

# Critical issues: noisy data

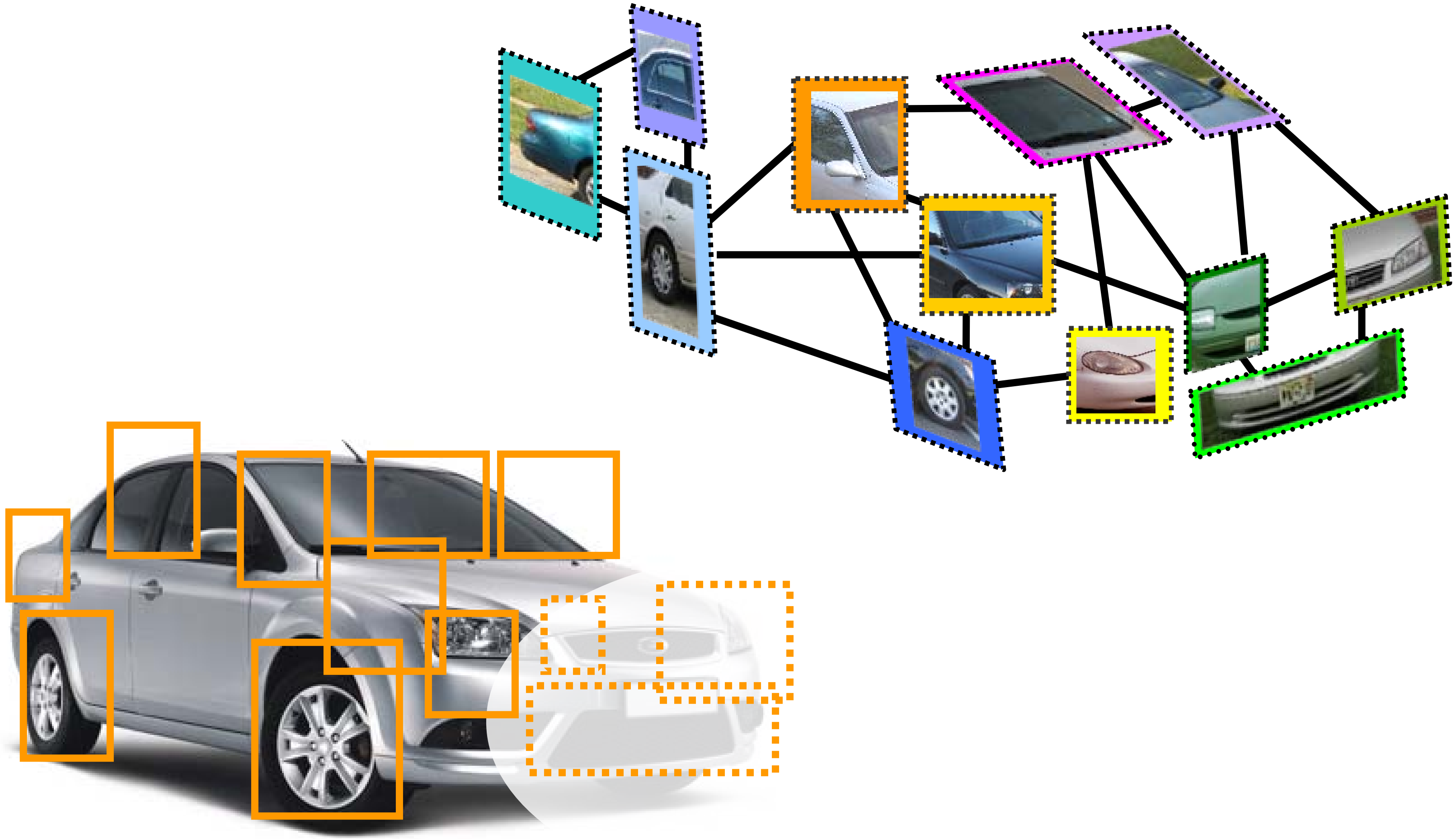
# Critical issues: noisy data (intra-class variability)



# Critical issues: outliers



# Critical issues: missing data (occlusions)



# Fitting

Goal: Choose a parametric model to fit a certain quantity from data

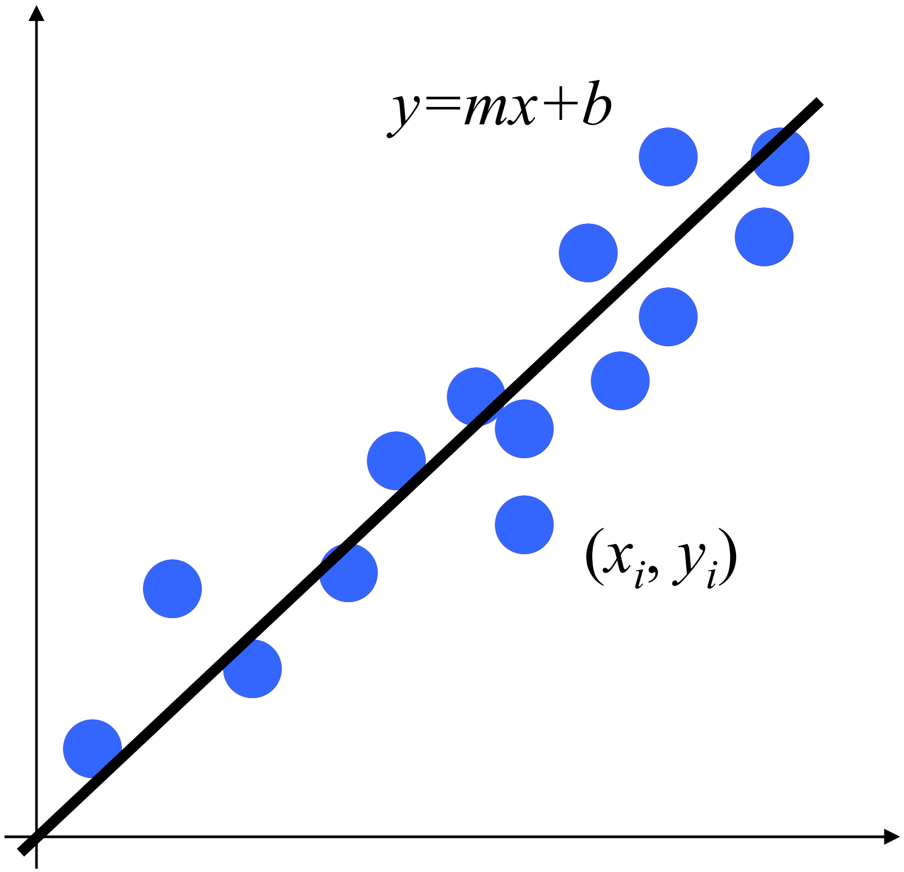
Techniques:

- Least square methods
- RANSAC
- Hough transform
- EM (Expectation Maximization)

# Least squares methods
## - fitting a line -

- Data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Line equation: $y_i = m\,x_i + b$
- Find $(m, b)$ to minimize

$$\boxed{E = \sum_{i=1}^{n} (y_i - m x_i - b)^2}$$



# Least squares methods
## - fitting a line -

$$\boxed{E=\sum_{i=1}^{n}(y_i-mx_i-b)^2}$$

$$\mathrm{E}=\sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\mathrm{y_i}-\begin{bmatrix}\mathrm{x_i} & 1\end{bmatrix}\begin{bmatrix}\mathrm{m}\\ \mathrm{b}\end{bmatrix}\right)^2=\left\|\begin{bmatrix}\mathrm{y_1}\\ \vdots\\ \mathrm{y_n}\end{bmatrix}-\begin{bmatrix}\mathrm{x_1} & 1\\ \vdots & \vdots\\ \mathrm{x_n} & 1\end{bmatrix}\begin{bmatrix}\mathrm{m}\\ \mathrm{b}\end{bmatrix}\right\|^2=\|\mathrm{Y}-\mathrm{XB}\|^2$$

$$=(\mathrm{Y}-\mathrm{XB})^{\mathrm{T}}(\mathrm{Y}-\mathrm{XB})=\mathrm{Y}^{\mathrm{T}}\mathrm{Y}-2(\mathrm{XB})^{\mathrm{T}}\mathrm{Y}+(\mathrm{XB})^{\mathrm{T}}(\mathrm{XB})$$

$$\frac{dE}{dB}=2X^TXB-2X^TY=0$$

$$\mathrm{X}^{\mathrm{T}}\mathrm{XB}=\mathrm{X}^{\mathrm{T}}\mathrm{Y}$$

*Normal equation*

$$\boxed{\mathrm{B}=\left(\mathrm{X}^{\mathrm{T}}\mathrm{X}\right)^{-1}\mathrm{X}^{\mathrm{T}}\mathrm{Y}}$$

# Least squares methods
## - fitting a line -

$$Ax = b$$

- More equations than unknowns

- Look for solution which minimizes $||Ax-b|| = (Ax-b)^T(Ax-b)$
- Solve $\frac{\partial (Ax-b)^T (Ax-b)}{\partial x_i} = 0$

- LS solution

$$x = (A^T A)^{-1} A^T b$$

# Least squares methods
## - fitting a line -

Solving $x = (A^t A)^{-1} A^t b$

$A^+ = (A^t A)^{-1} A^t$ = pseudo-inverse of A

$A = U \sum V^t$ = SVD decomposition of A

$A^{-1} = V \sum^{-1} U^t$

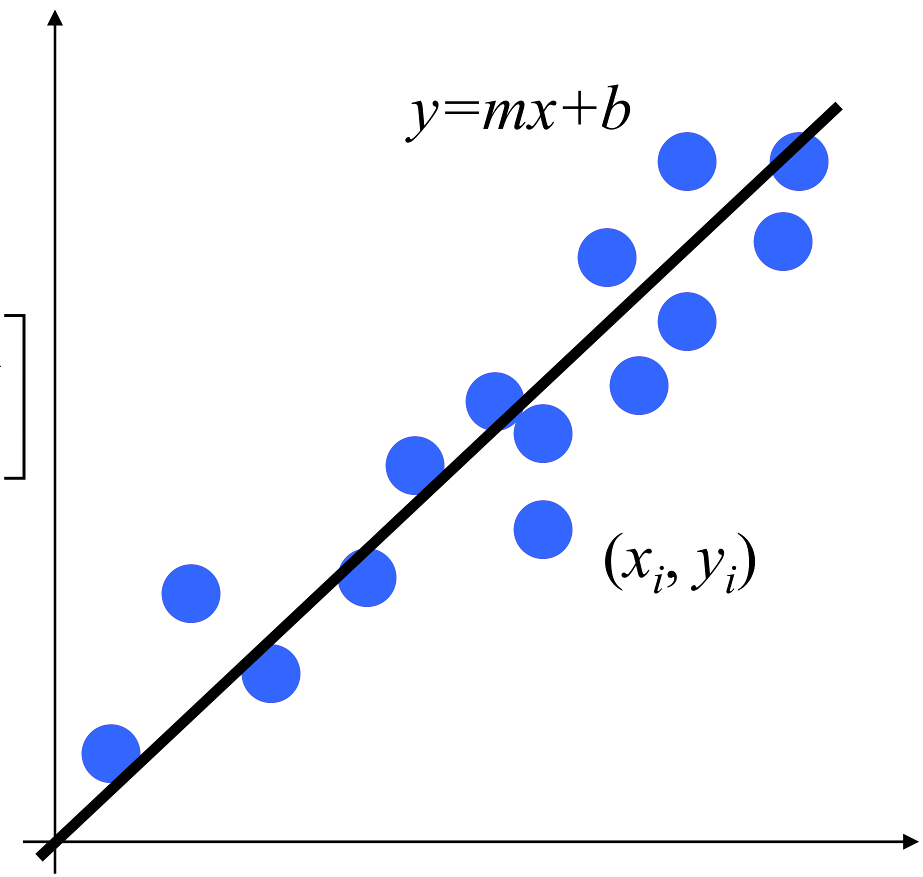
$A^+ = V \sum^+ U^t$

with $\sum^+$ equal to $\sum^{-1}$ for all nonzero singular values and zero otherwise

# Least squares methods
## - fitting a line -

$$E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

$$\boxed{B = \left(X^T X\right)^{-1} X^T Y} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$




## Limitations

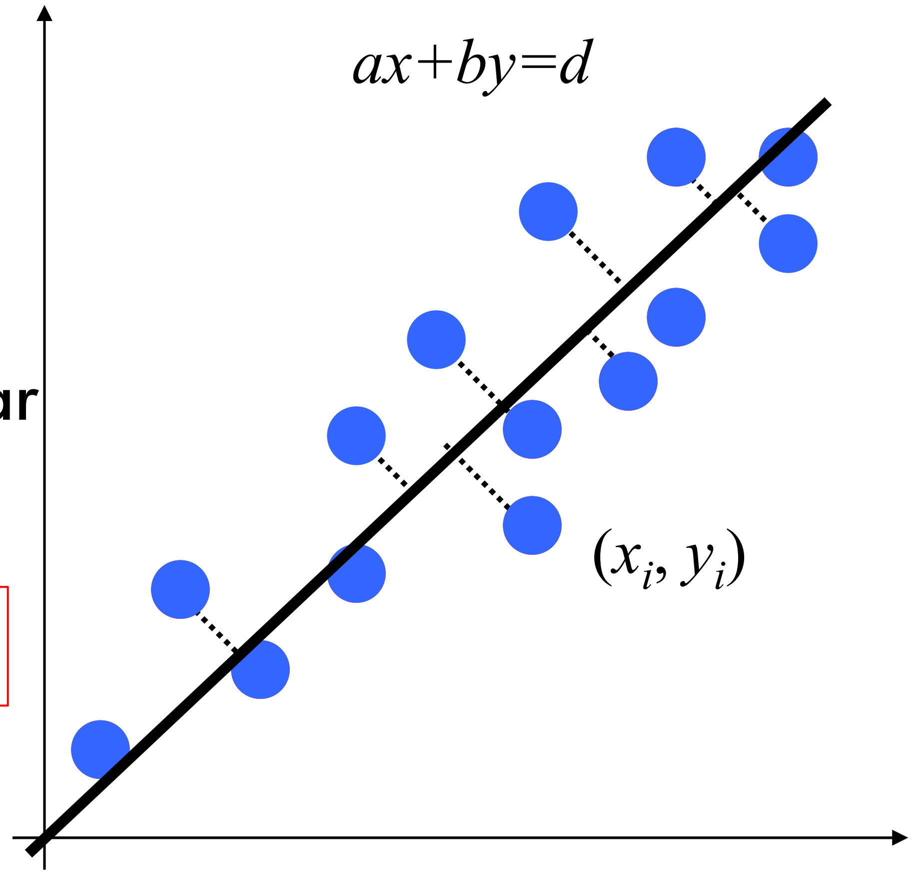
- Not rotation-invariant
- Fails completely for vertical lines

# Least squares methods
## - fitting a line -

- Distance between point $(x_n, y_n)$ and line $ax+by=d$

- Find $(a, b, d)$ to minimize the sum of squared perpendicular distances

$$E = \sum_{i=1}^{n} (a x_i + b y_i - d)^2$$



# Least squares methods

## - fitting a line -

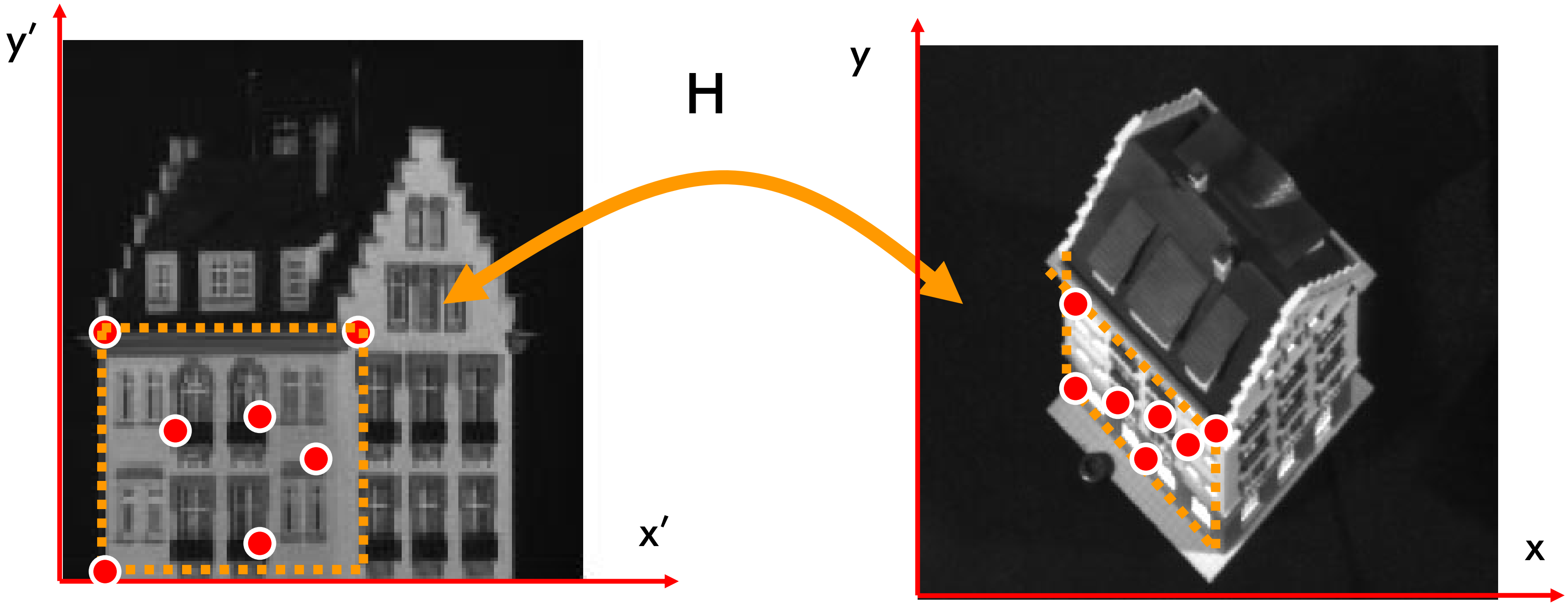
$$A\,h = 0$$

$$\text{Minimize} \; \| A\,h \| \quad \text{subject to} \quad \| h \| = 1$$

$$A = UDV^T$$

$$h = \text{last column of } V$$

# Least squares methods
## - fitting an homography -




$$\begin{bmatrix} wx' \\ wy' \\ w \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

# Least squares methods

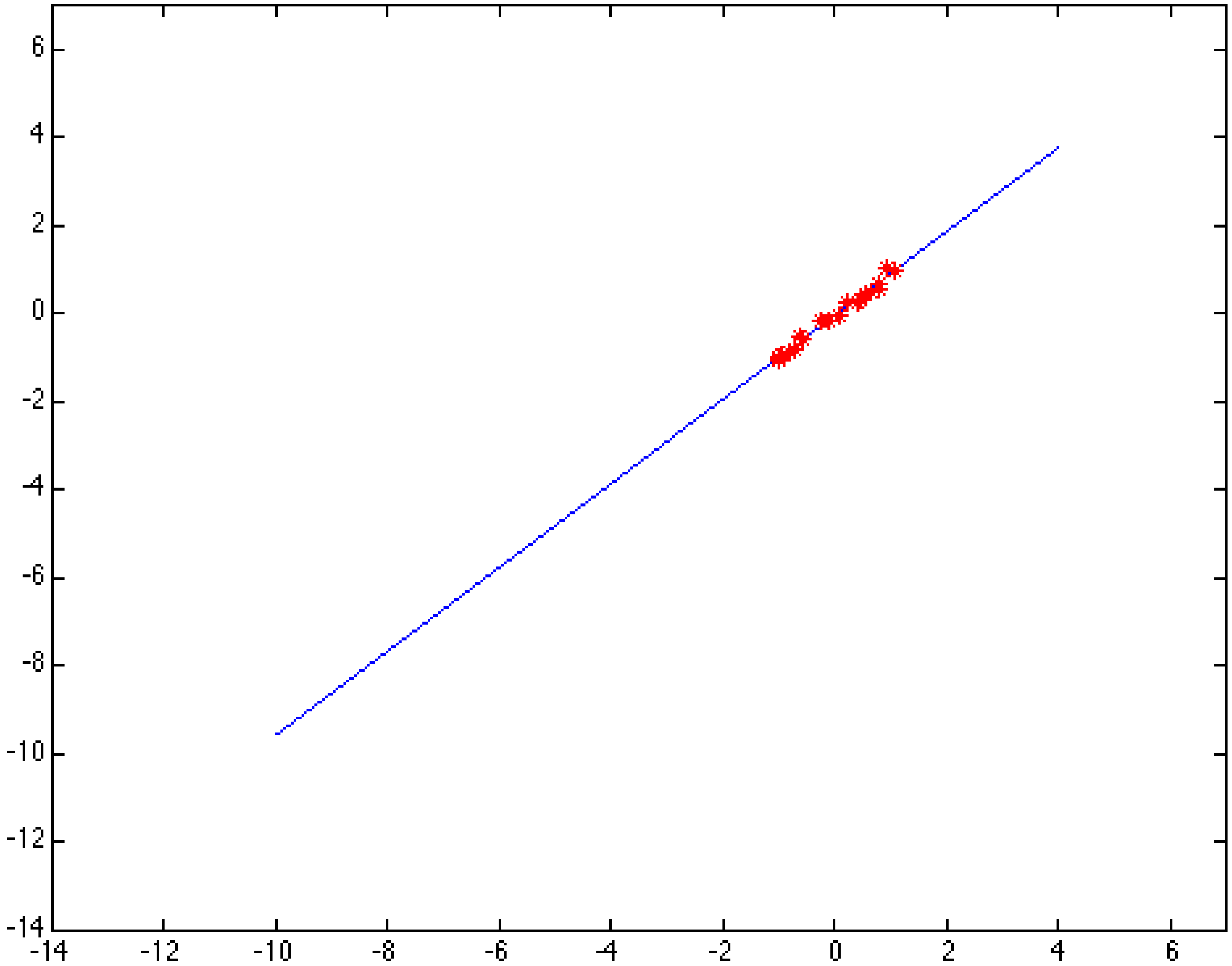
## - fitting an homography -

$$
\begin{aligned}
h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx' - x' &= 0 \\
h_{21}x + h_{22}y + h_{23} - h_{31}xy' - h_{32}yy' - y' &= 0
\end{aligned}
$$

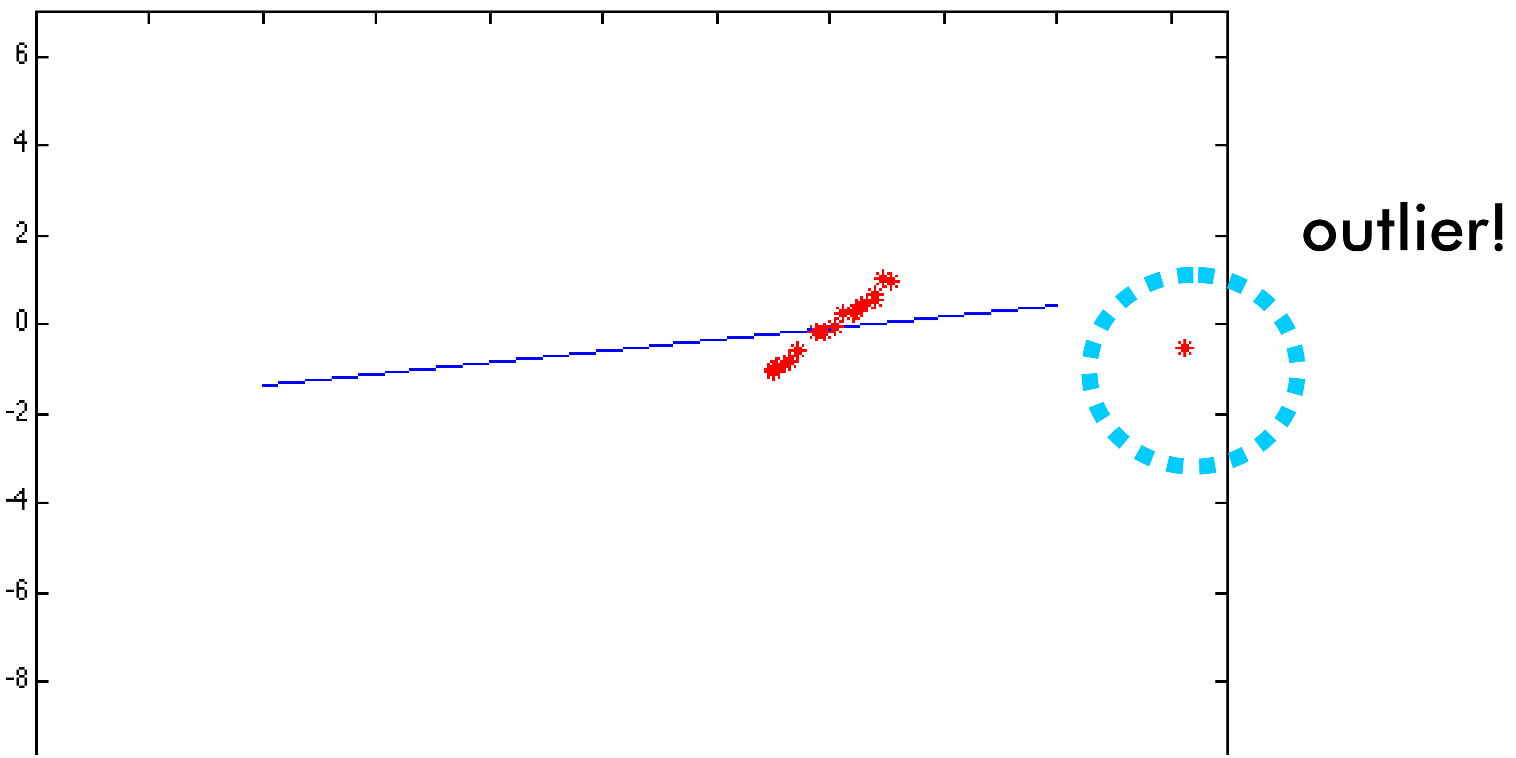
From n>=4 corresponding points: $\boxed{A\,h = 0}$

$$
\begin{pmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1x'_1 & -y_1x'_1 & -x'_1 \\
0 & 0 & 0 & x_1 & y_1 & 1 & -x_1y'_1 & -y_1y'_1 & -y'_1 \\
x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2x'_2 & -y_2x'_2 & -x'_2 \\
0 & 0 & 0 & x_2 & y_2 & 1 & -x_2y'_2 & -y_2y'_2 & -y'_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_n & y_n & 1 & 0 & 0 & 0 & -x_nx'_n & -y_nx'_n & -x'_n \\
0 & 0 & 0 & x_n & y_n & 1 & -x_ny'_n & -y_ny'_n & -y'_n
\end{pmatrix}
\begin{bmatrix} h_{1,1} \\ h_{1,2} \\ \vdots \\ h_{3,3} \end{bmatrix} = 0
$$

# Least squares: Robustness to noise

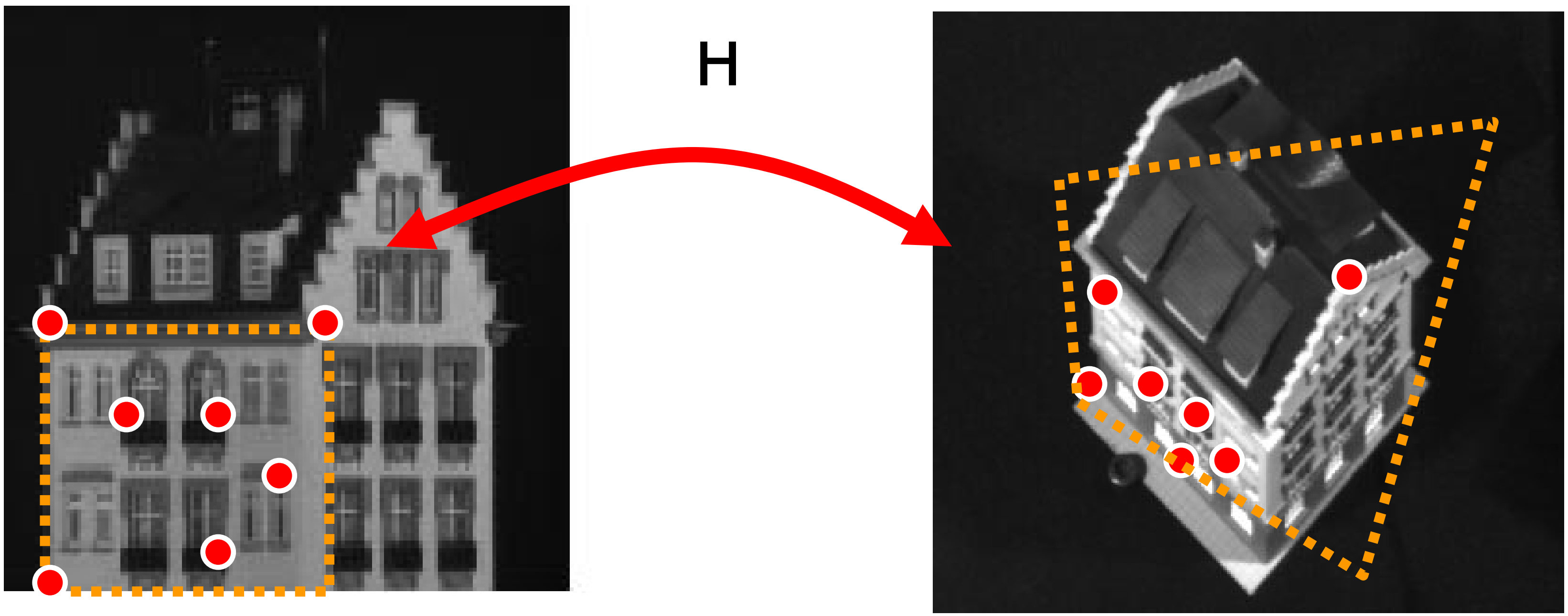


# Least squares: Robustness to noise




Problem: squared error heavily penalizes outliers

# Critical issues: outliers




CONCLUSION: Least square is not robust w.r.t. outliers

# Least squares: Robust estimators

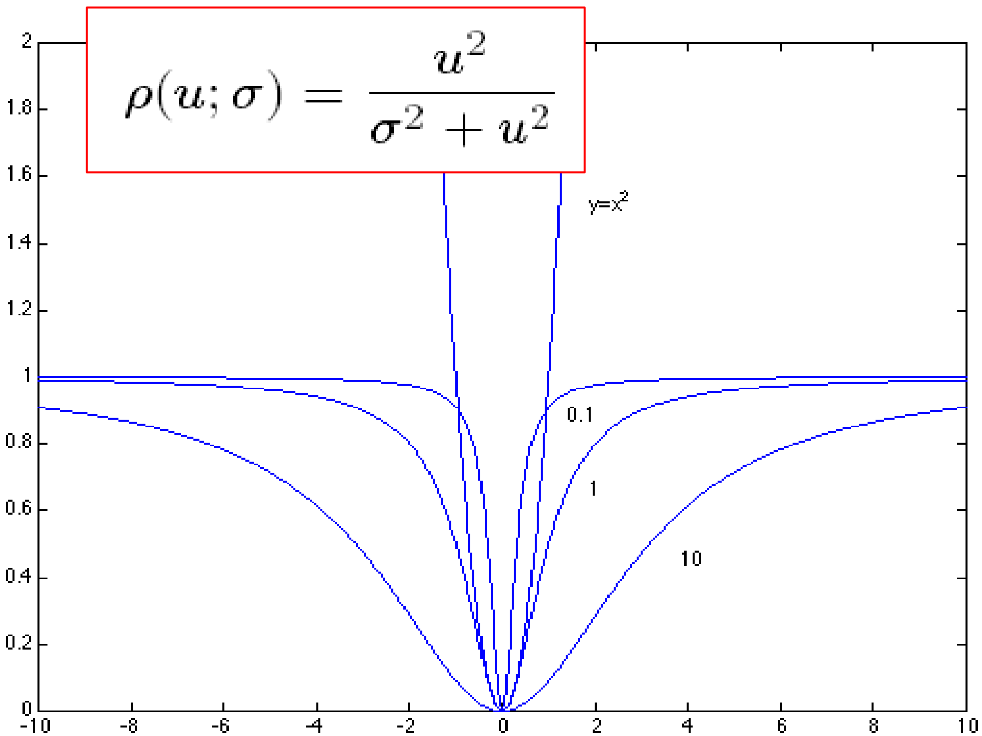
- General approach:

minimize $$\sum_i \rho\left(u_i\left(x_i, \theta\right); \sigma\right)$$

$$u = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$

- $u_i (x_i, \theta)$ – residual of i^th point w.r.t. model parameters $\theta$

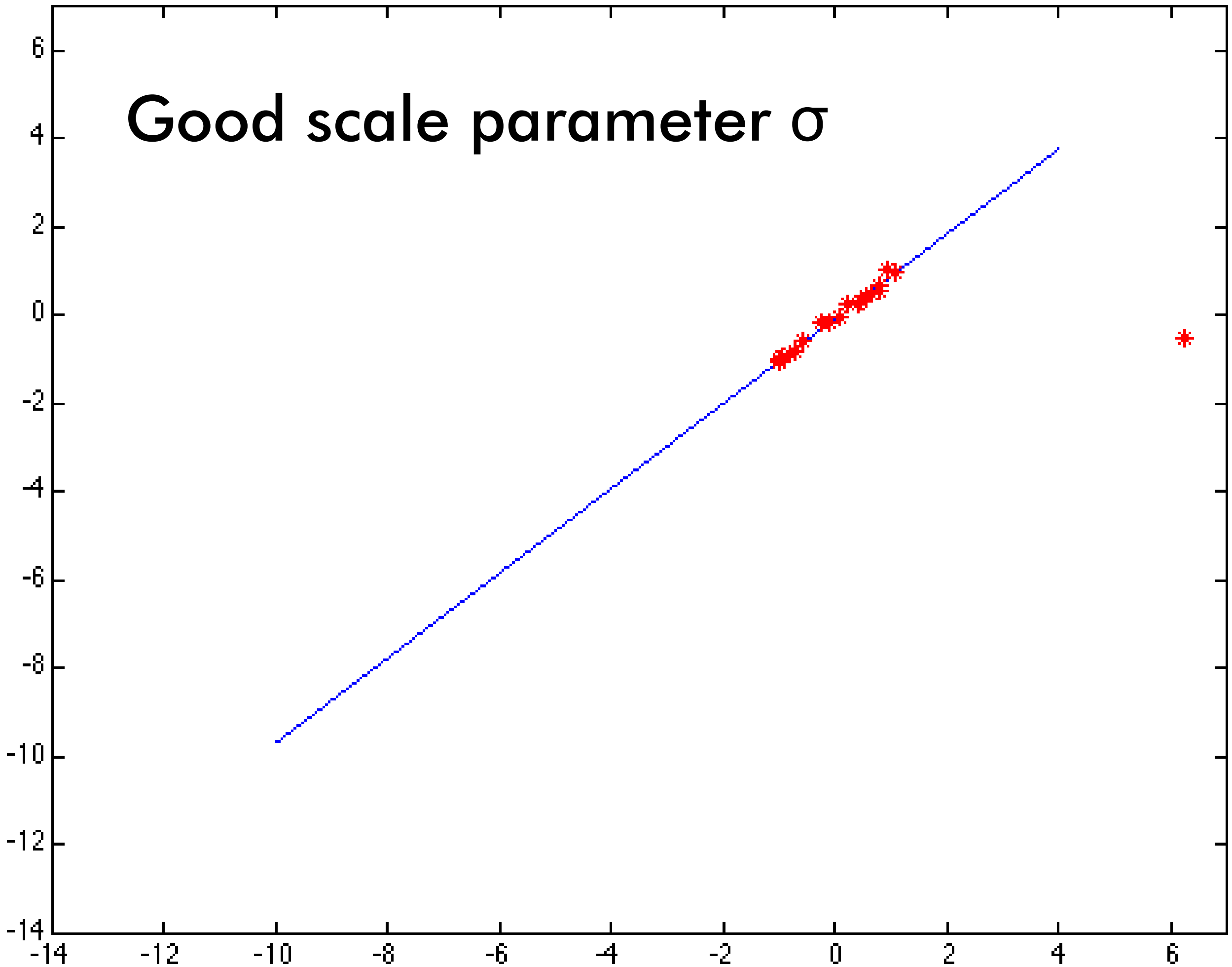
  $\rho$ – robust function with scale parameter $\sigma$




The robust function $\rho$

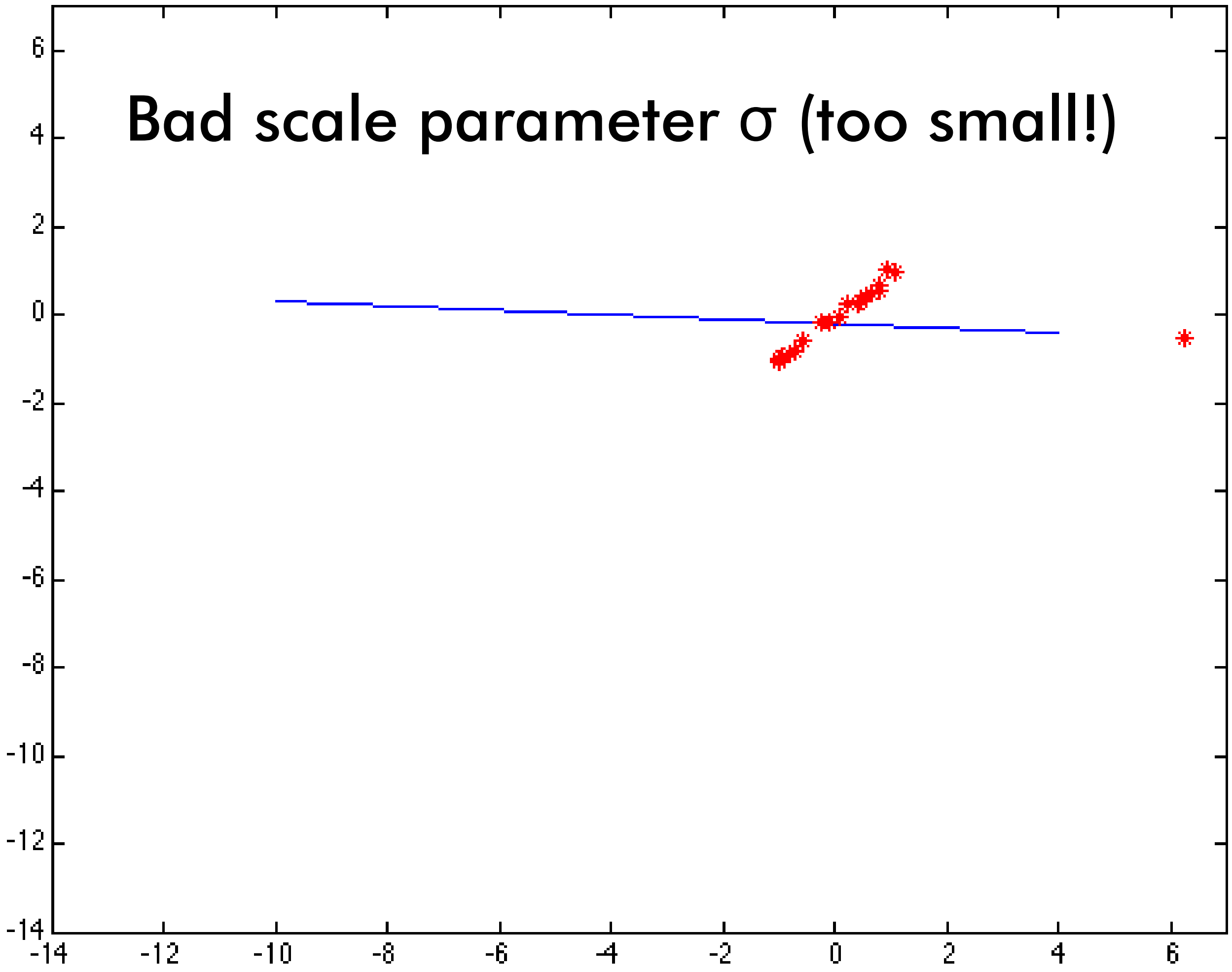
- favors a configuration with small residuals
- Penalizes large residuals

# Least squares: Robust estimators

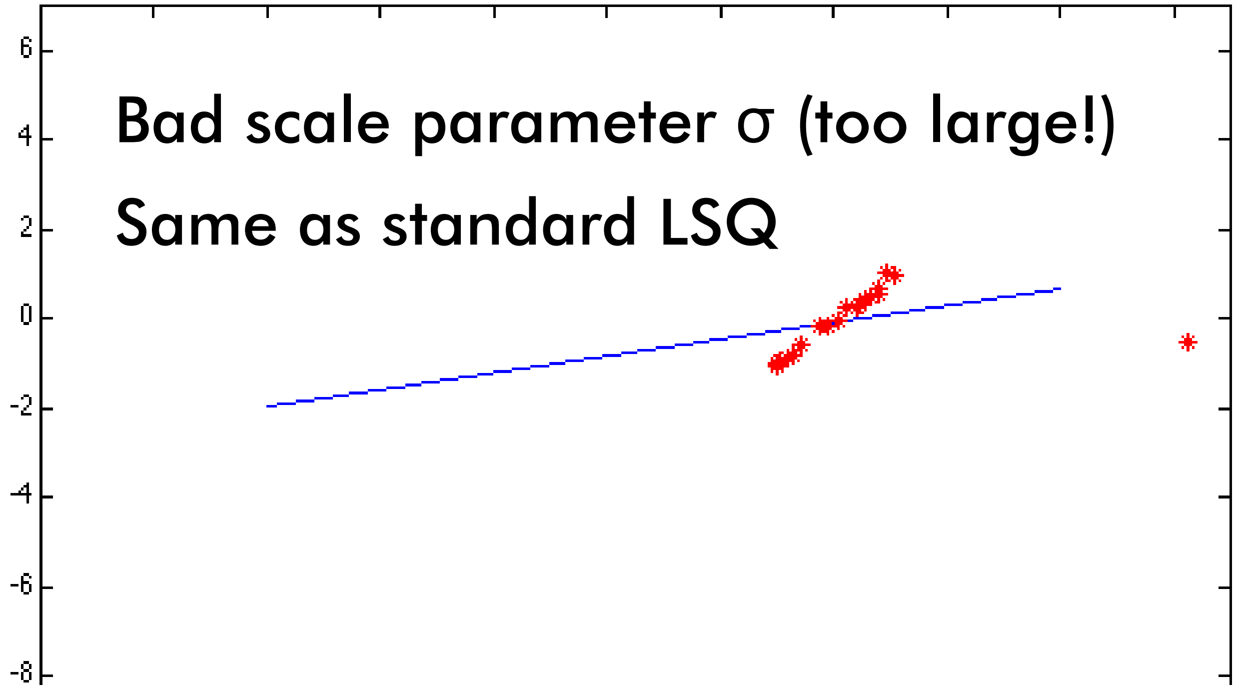



The effect of the outlier is eliminated

# Least squares: Robust estimators



# Least squares: Robust estimators




• CONCLUSION: Robust estimator useful if prior info about the distribution of points is known

- Robust fitting is a nonlinear optimization problem (iterative solution)
- Least squares solution provides good initial condition

# Fitting

Goal: Choose a parametric model to fit a certain quantity from data

Techniques:

- Least square methods
- RANSAC
- Hough transform

# Basic philosophy
## (voting scheme)

- Data elements are used to vote for one (or multiple) models

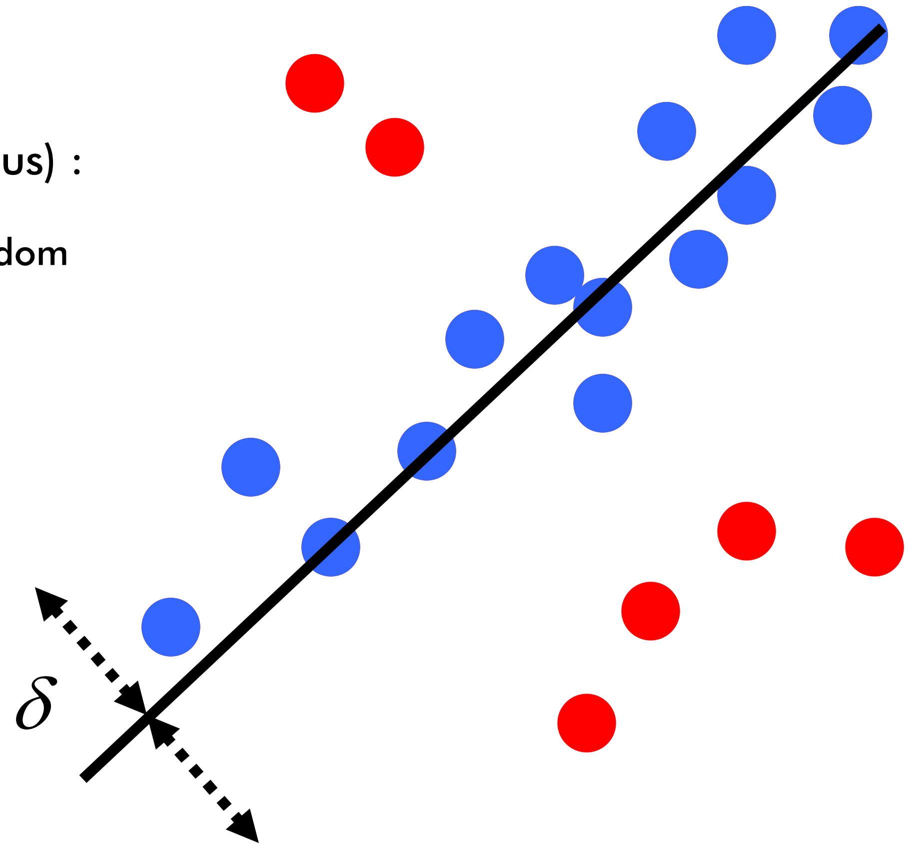
- Robust to outliers and missing data

- Assumption1: Noise features will not vote consistently for any single model ("few" outliers)
- Assumption2: there are enough features to agree on a good model ("few" missing data)

# RANSAC

(RANdom SAmple Consensus) :
Learning technique to estimate parameters of a model by random sampling of observed data

Fischler & Bolles in '81.



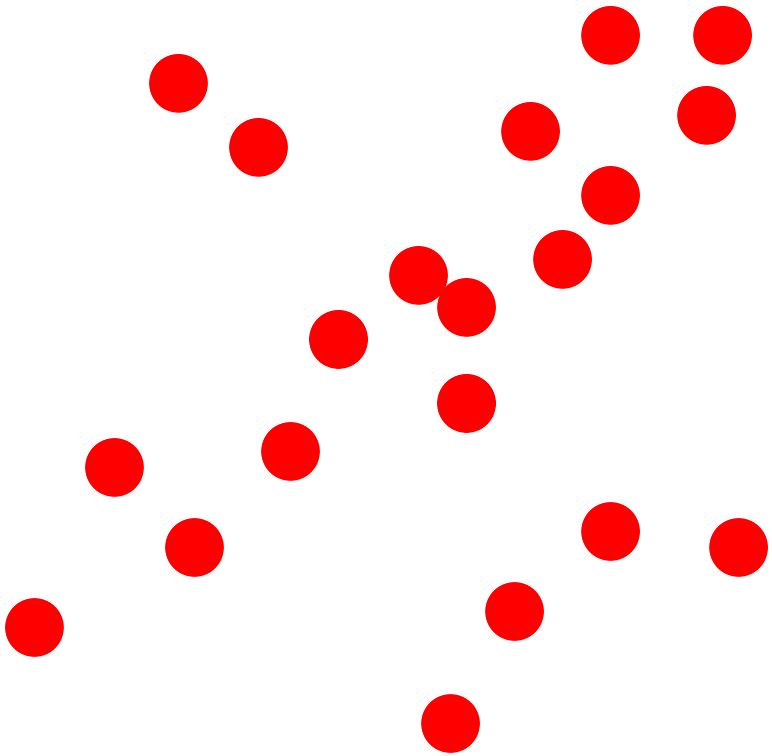

$$\pi : \boldsymbol{I} \rightarrow \{\boldsymbol{P}, \boldsymbol{O}\} \qquad \min_{\pi} |\boldsymbol{O}|$$

such that:

Model parameters

$$\boldsymbol{f}(\boldsymbol{P}, \beta) < \delta \qquad \boldsymbol{f}(\boldsymbol{P}, \beta) = \left\| \beta - \left(\boldsymbol{P}^{T}\boldsymbol{P}\right)^{-1}\boldsymbol{P}^{T} \right\|$$

# RANSAC



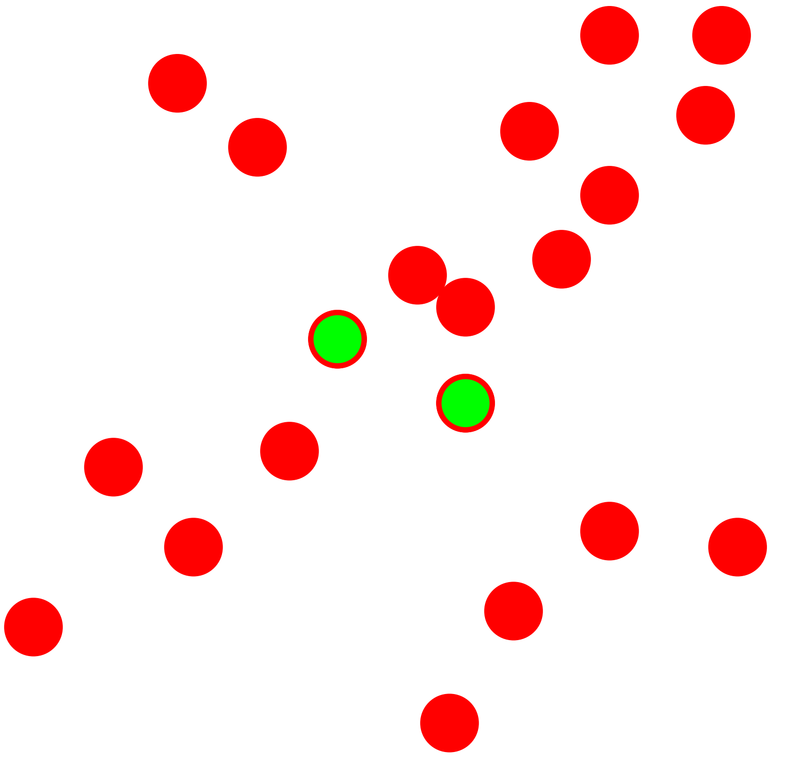
Sample set = set of points in 2D

Algorithm:

1. Select random sample of minimum required size to fit model
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC



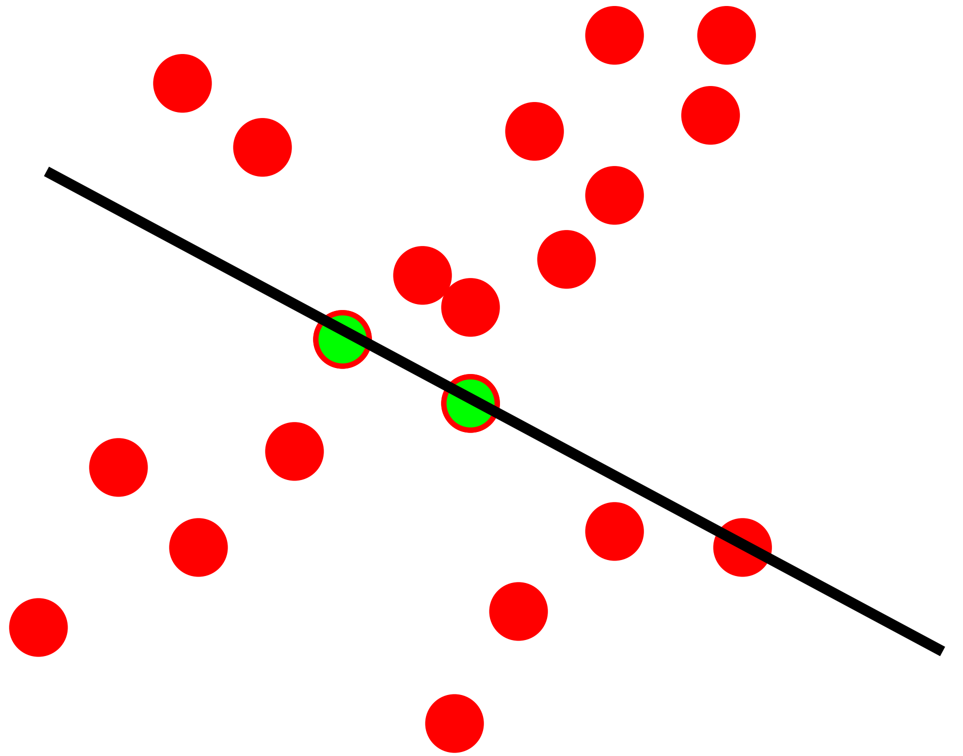
Sample set = set of points in 2D

Algorithm:

1. Select random sample of minimum required size to fit model [?] =[2]
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC



Sample set = set of points in 2D

## Algorithm:

1. Select random sample of minimum required size to fit model [?] =[2]
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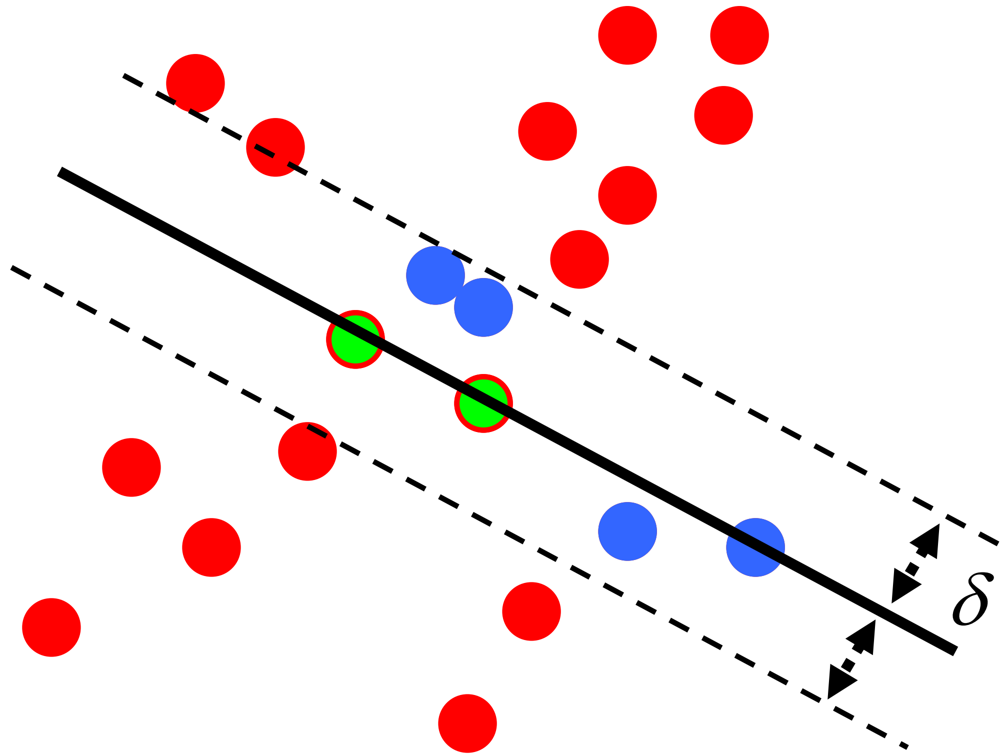
Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC




Sample set = set of points in 2D

$$\left|O\right| = 14$$

## Algorithm:

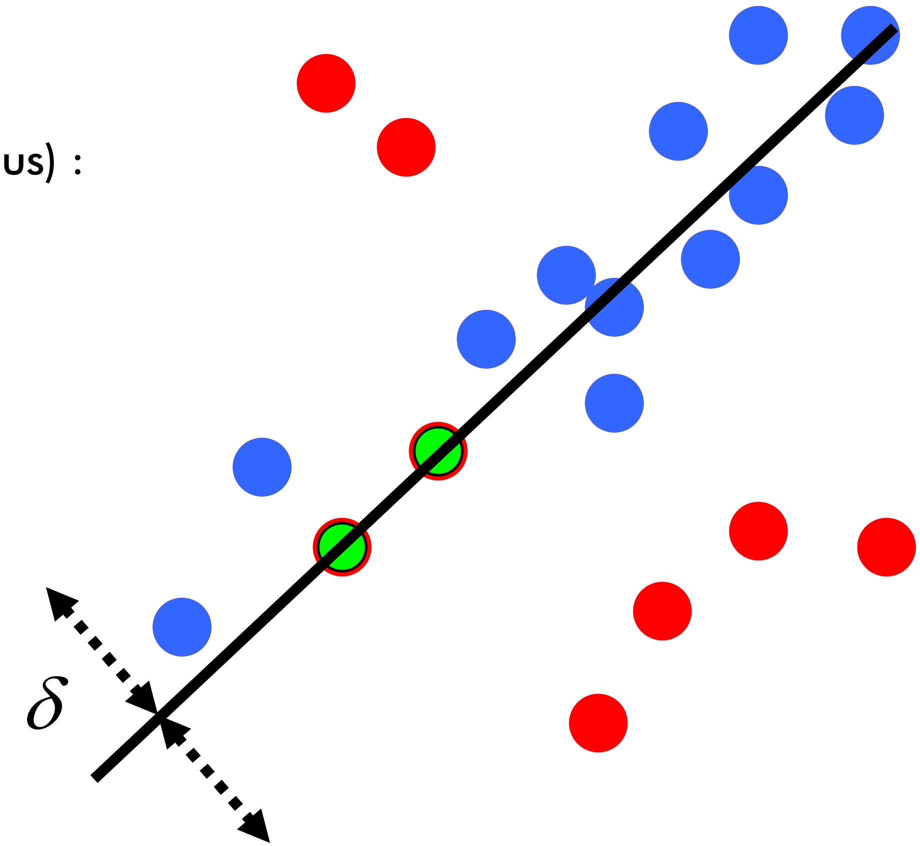
1. Select random sample of minimum required size to fit model [?] = [2]
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found

# RANSAC

(RANdom SAmple Consensus) :

Fischler & Bolles in '81.




$|O| = 6$

## Algorithm:

1. Select random sample of minimum required size to fit model [?]
2. Compute a putative model from sample set
3. Compute the set of inliers to this model from whole data set

Repeat 1-3 until model with the most inliers over all samples is found

# How many samples?

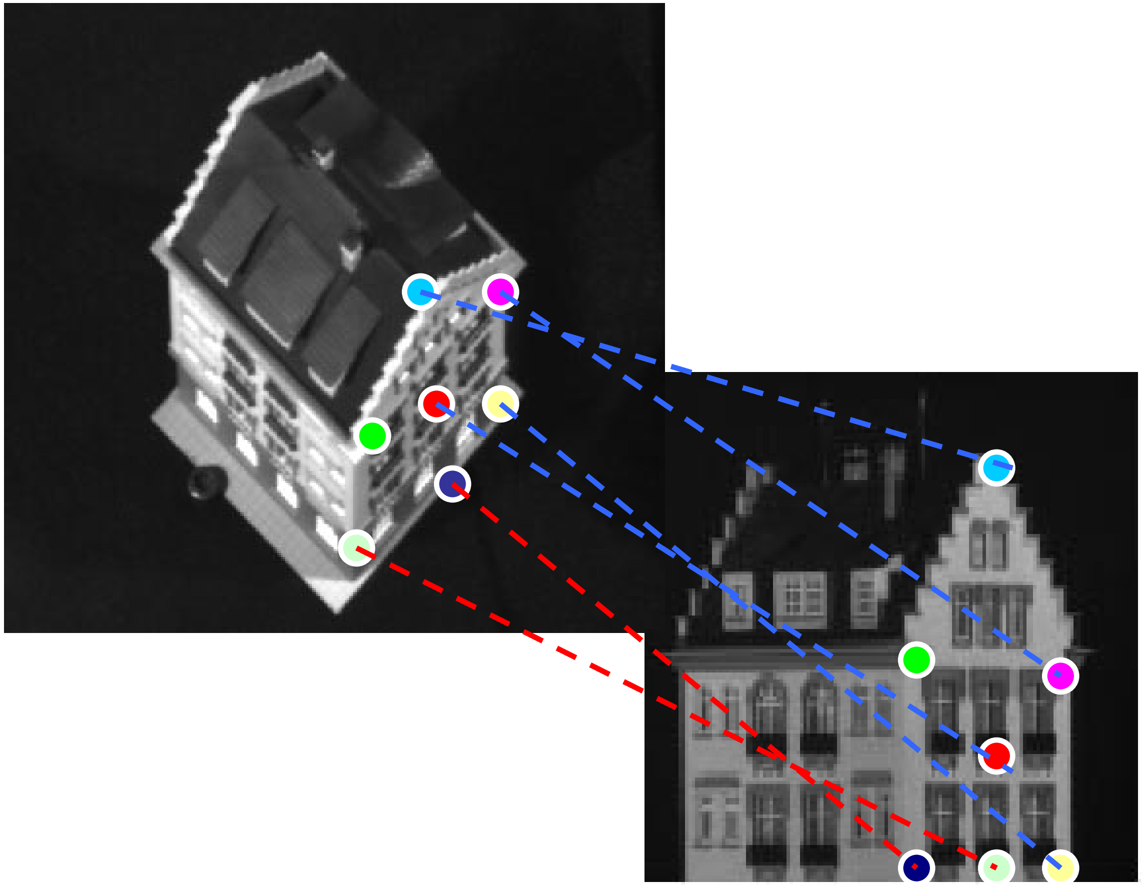
- Number of samples *N*
  - Choose *N* so that, with probability *p*, at least one random sample is free from outliers (e.g. *p*=0.99) (outlier ratio: *e* )
- Initial number of points *s*
  - Typically minimum number needed to fit the model
- Distance threshold $\delta$
  - Choose $\delta$ so probability for inlier is *p* (e.g. 0.95)
  - Zero-mean Gaussian noise with std. dev. σ: $t^2=3.84\sigma^2$

$$\boxed{N = \log(1-p)/\log\left(1-(1-e)^s\right)}$$

| | proportion of outliers $e$ | | | | | | |
|---|---|---|---|---|---|---|---|
| s | 5% | 10% | 20% | 25% | 30% | 40% | 50% |
| 2 | 2 | 3 | 5 | 6 | 7 | 11 | 17 |
| 3 | 3 | 4 | 7 | 9 | 11 | 19 | 35 |
| 4 | 3 | 5 | 9 | 13 | 17 | 34 | 72 |
| 5 | 4 | 6 | 12 | 17 | 26 | 57 | 146 |
| 6 | 4 | 7 | 16 | 24 | 37 | 97 | 293 |
| 7 | 4 | 8 | 20 | 33 | 54 | 163 | 588 |
| 8 | 5 | 9 | 26 | 44 | 78 | 272 | 1177 |

Source: M. Pollefeys

# Estimating H by RANSAC



- H $\rightarrow$ 8 DOF
- Need 4 correspondences

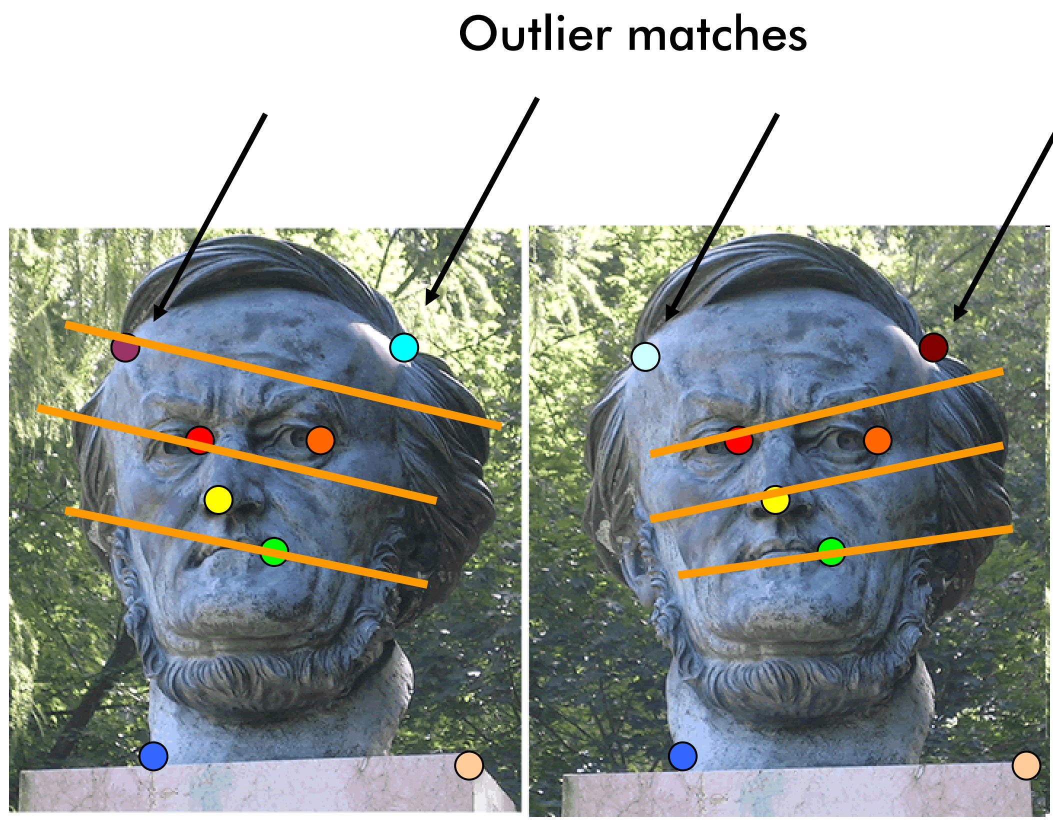
Sample set = set of matches between 2 images

## Algorithm:

1. Select a random sample of minimum required size [?]
2. Compute a putative model from these
3. Compute the set of inliers to this model from whole sample space

Repeat 1-3 until model with the most inliers over all samples is found

# Estimating F by RANSAC




- F $\rightarrow$ 7 DOF
- Need 7 (8) correspondences

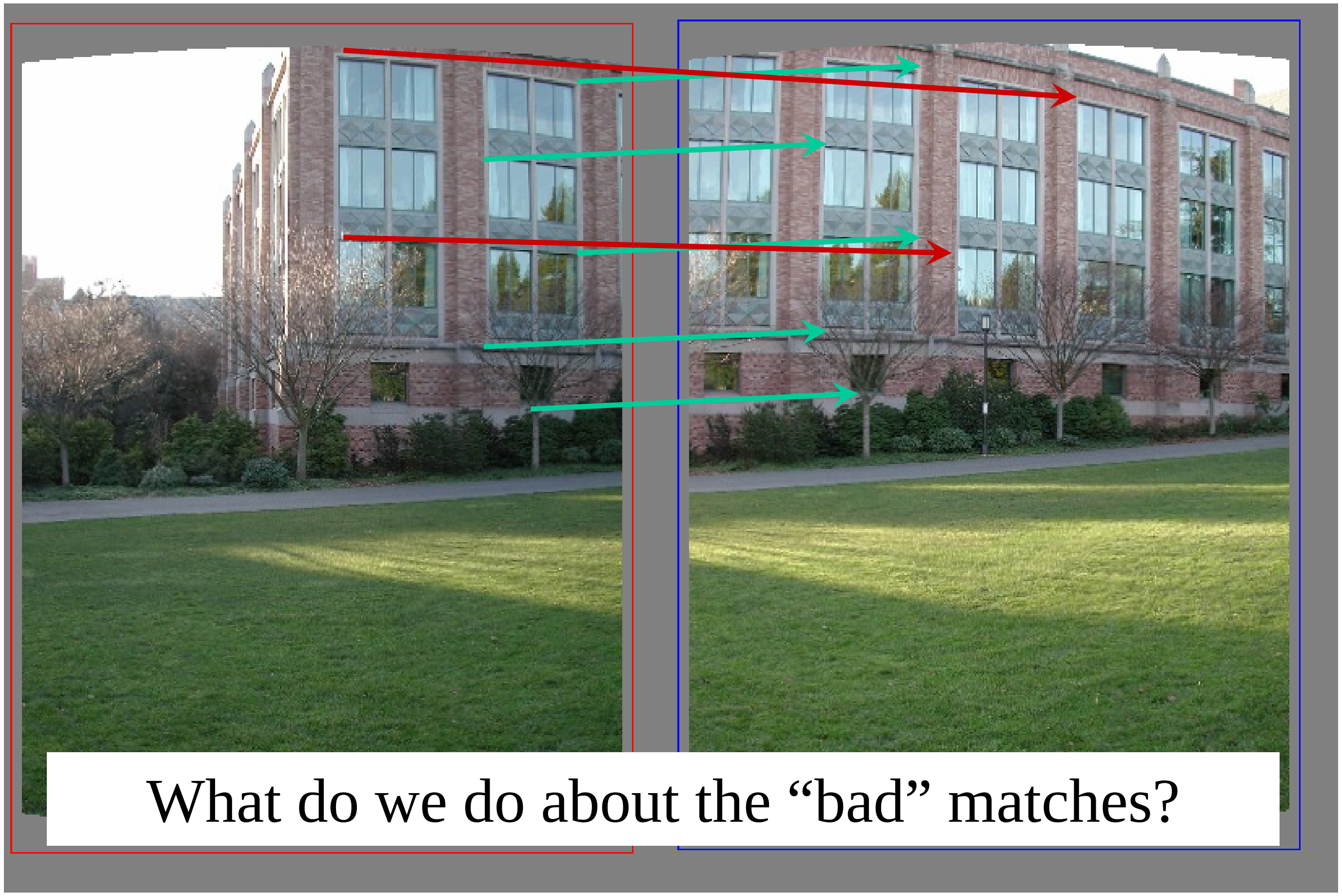
Sample set = set of matches between 2 images

## Algorithm:

1. Select a random sample of minimum required size [?]
2. Compute a putative model from these
3. Compute the set of inliers to this model from whole sample space

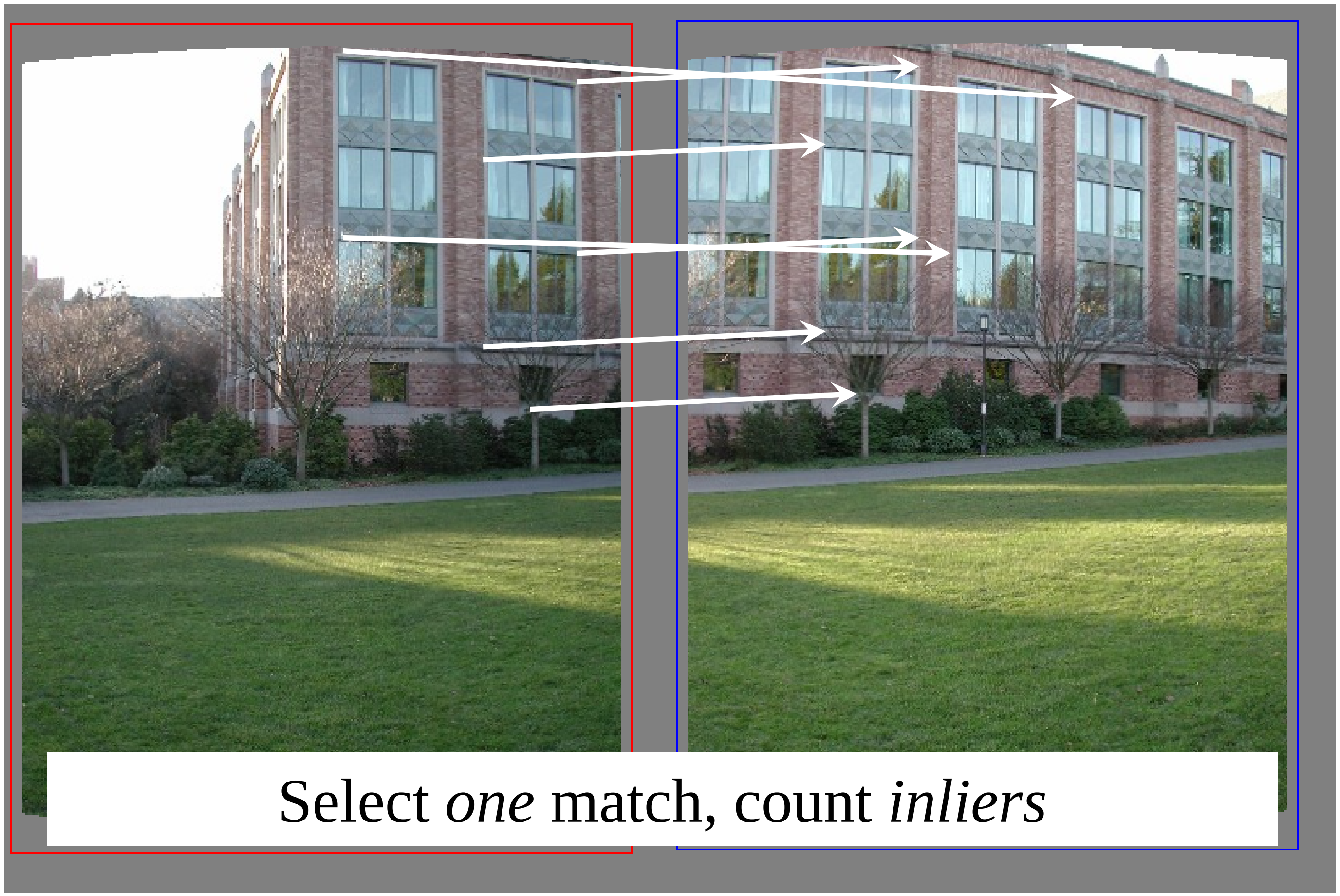
Repeat 1-3 until model with the most inliers over all samples is found

# Matching features



What do we do about the “bad” matches?

# RAndom SAmple Consensus



Select *one* match, count *inliers*

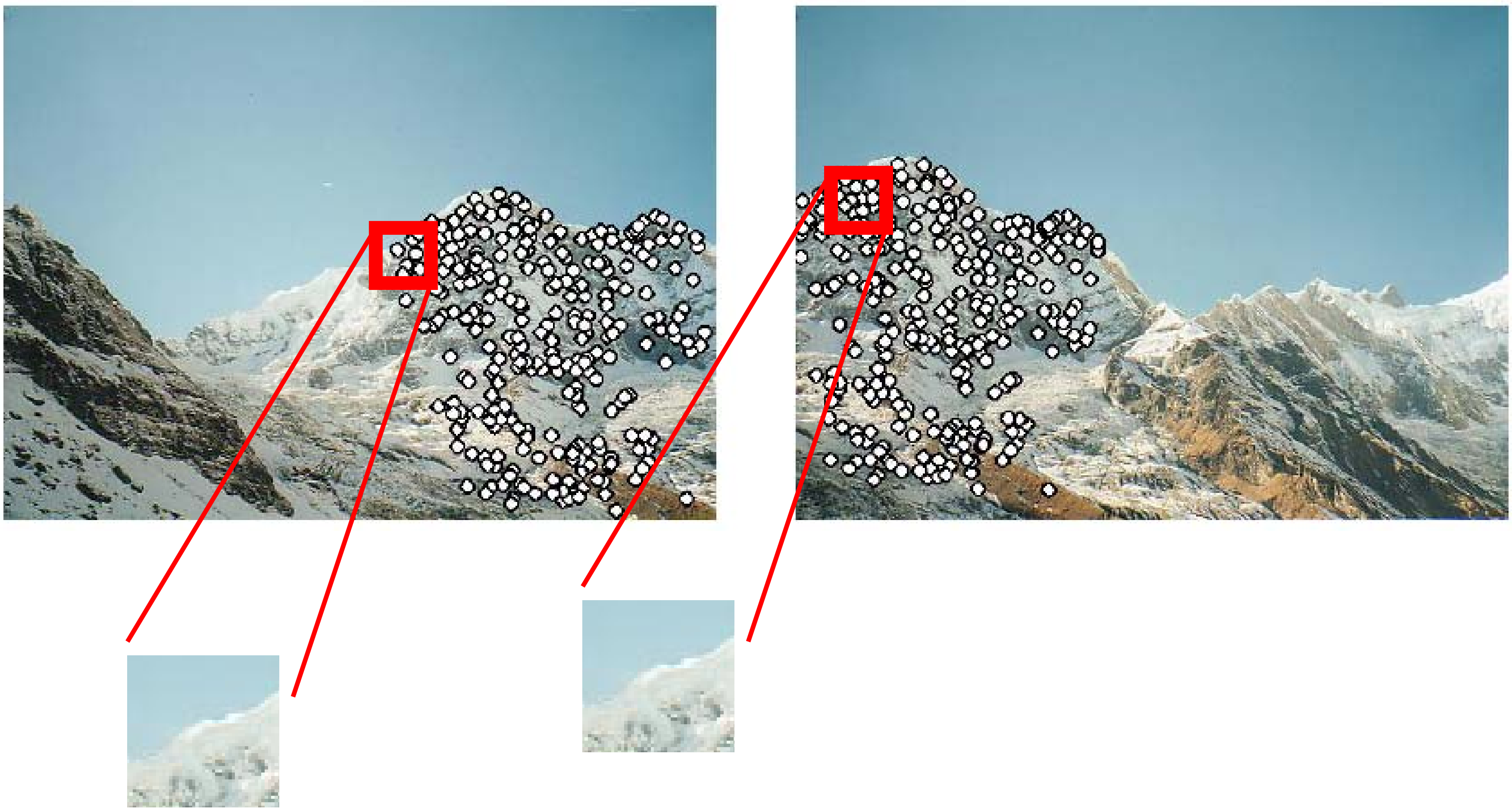
# Fitting helps matching!

Suppose we need to align these 2 images

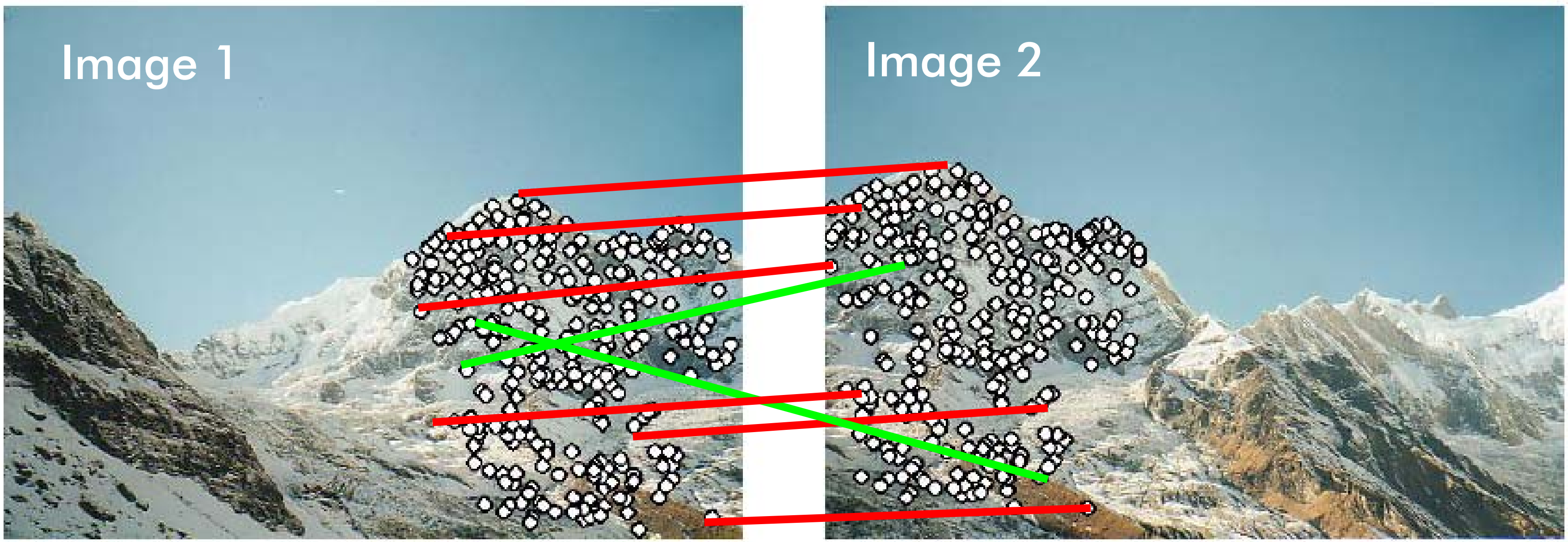
Images courtesy of M. Brown

# Fitting helps matching!



Feature are matched (for instance, based on correlation)

# Fitting helps matching!




Matches bases on appearance only

Red: good matches

Green: bad matches

Idea:

- Fitting an homography H (by RANSAC) mapping features from images 1 to 2
- Bad matches will be labeled as outliers (hence rejected)!

# Fitting helps matching!

# Fitting helps matching!

Images courtesy of Brandon Lloyd

# RANSAC - conclusions

## Good:

- Simple and easily implementable
- Successful in different contexts

## Bad:

- Many parameters to tune
- Trade-off accuracy-vs-time
- Cannot be use if ratio inliers/outliers is too small

# Fitting

Goal: Choose a parametric model to fit a certain quantity from data

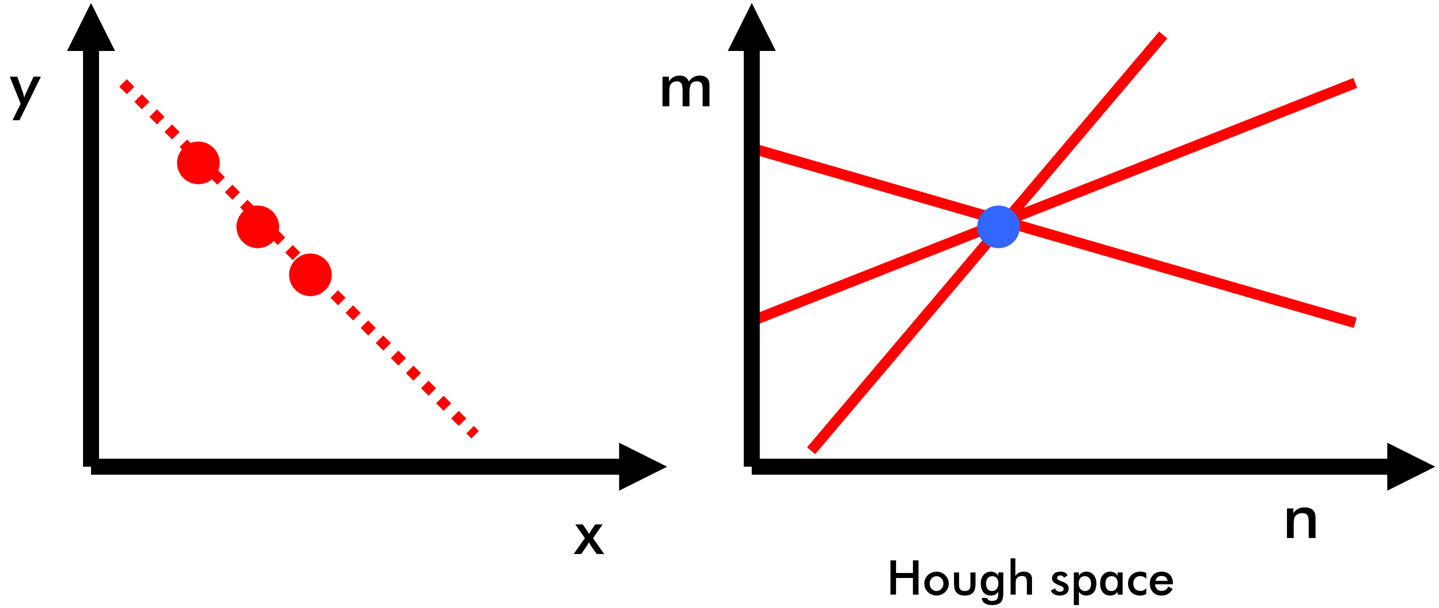
Techniques:

- Least square methods
- RANSAC
- Hough transform

# Hough transform

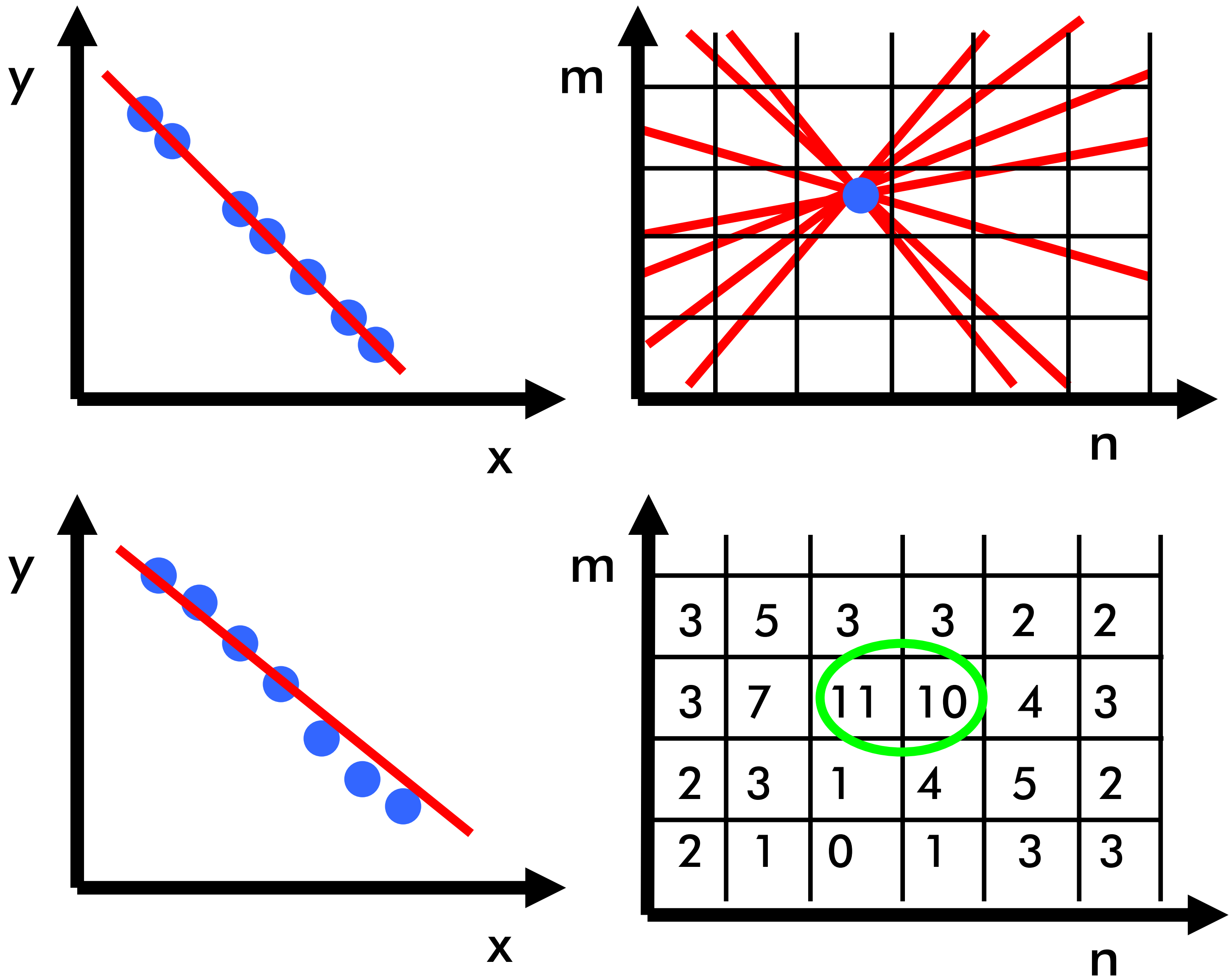
P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures,* Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

Given a set of , find the curve or line that explains the data points best




$$y = m\,x + n$$

# Hough transform

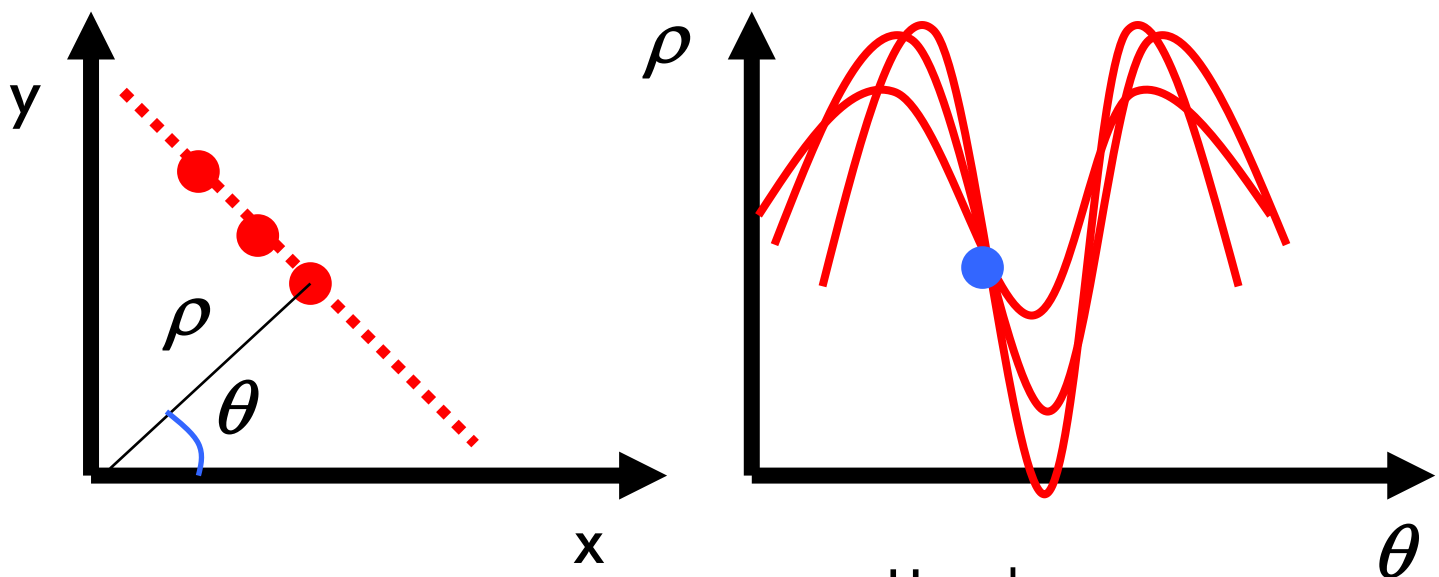


# Hough transform

P.V.C. Hough, *Machine Analysis of Bubble Chamber Pictures,* Proc. Int. Conf. High Energy Accelerators and Instrumentation, 1959

Issue : parameter space [m,n] is unbounded…

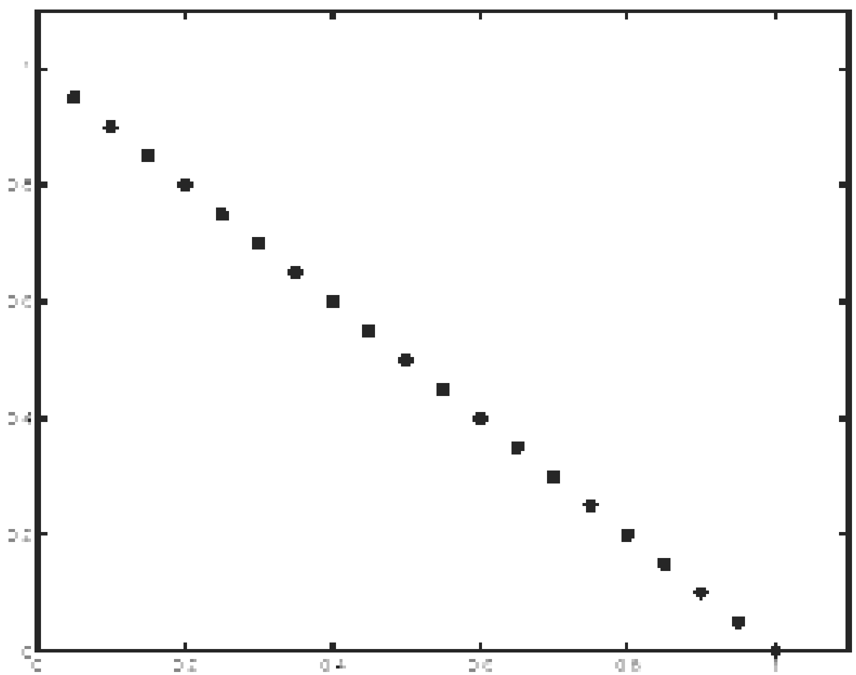
- Use a polar representation for the parameter space



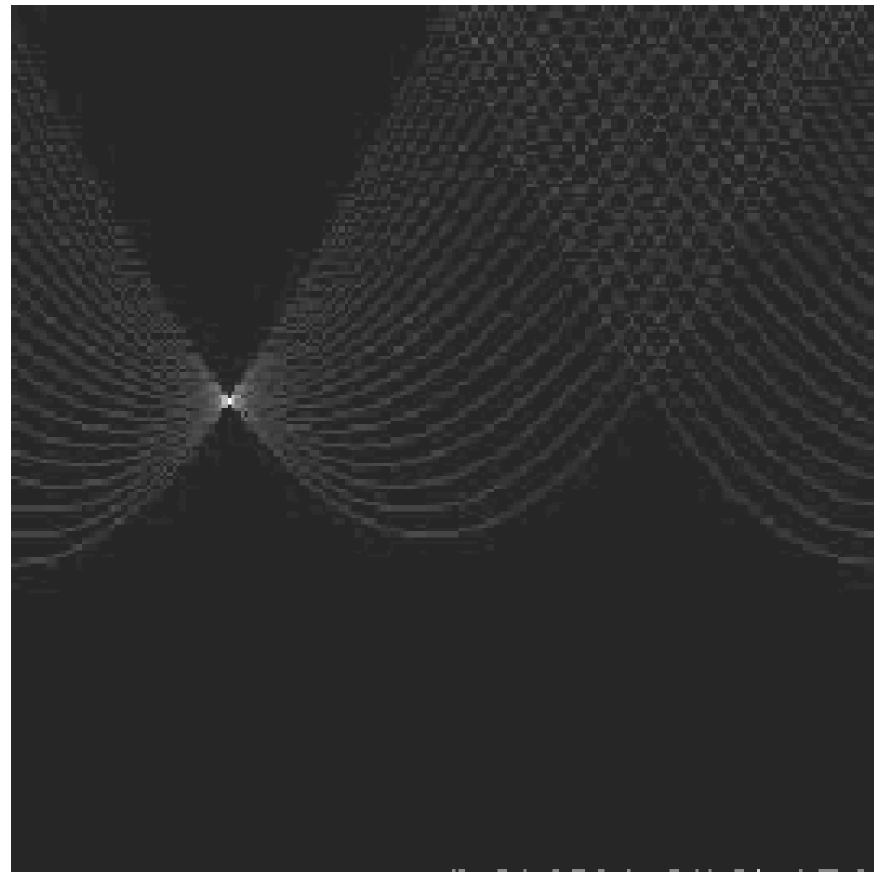

Hough space

$$x \cos\theta + y \sin\theta = \rho$$

# Hough transform - experiments



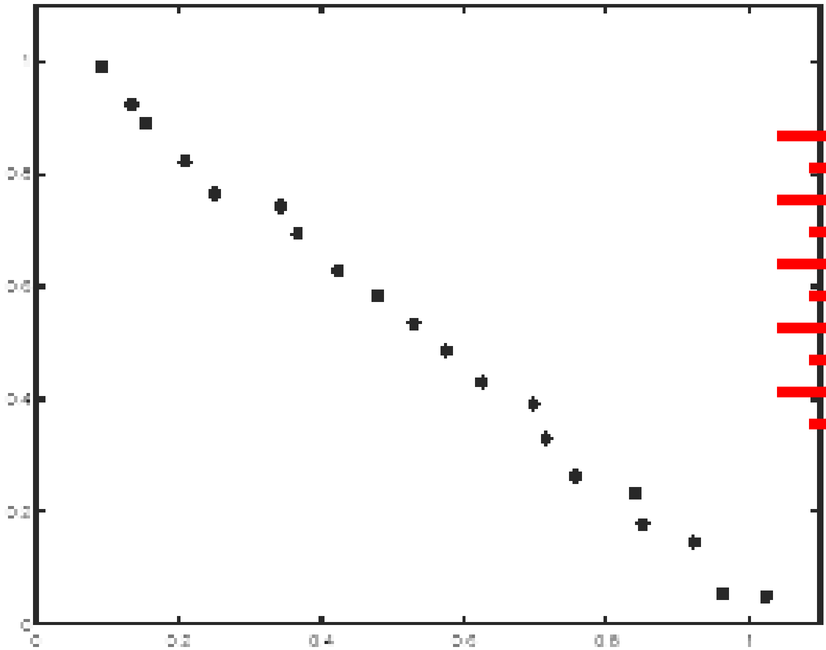
features

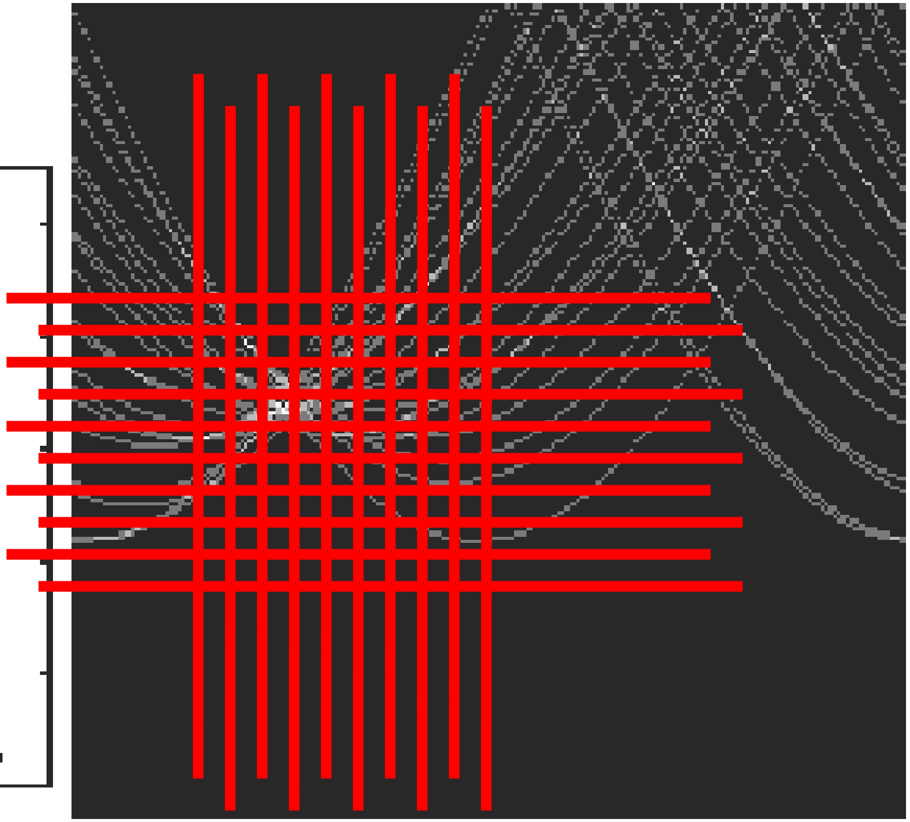


votes

# Hough transform - experiments

Noisy data



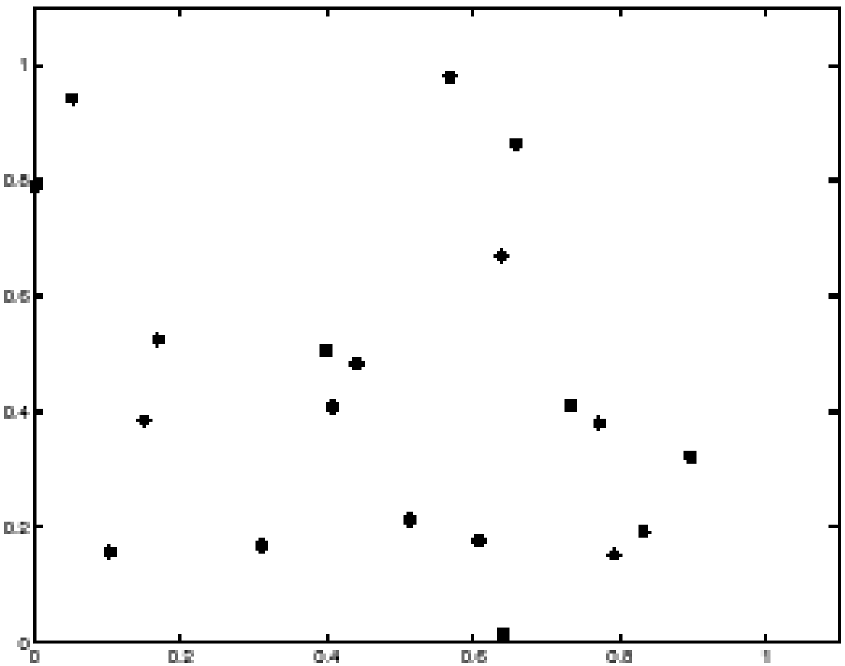


features

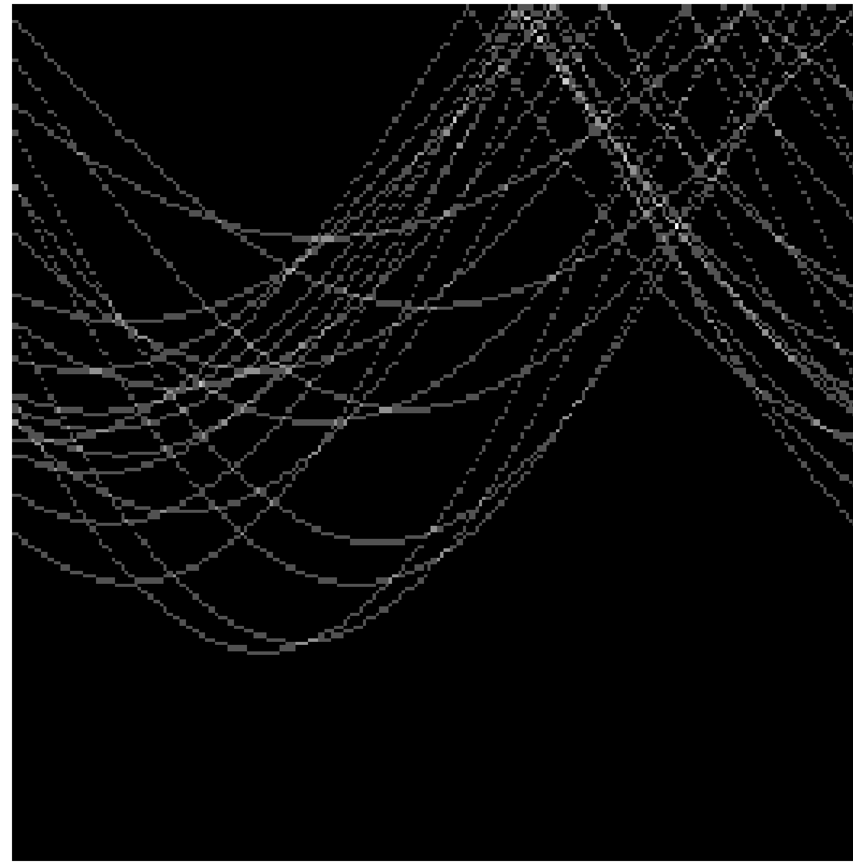
votes

Issue: Grid size needs to be adjusted…

# Hough transform - experiments




features



votes

Issue: spurious peaks due to uniform noise

# Hough transform - conclusions

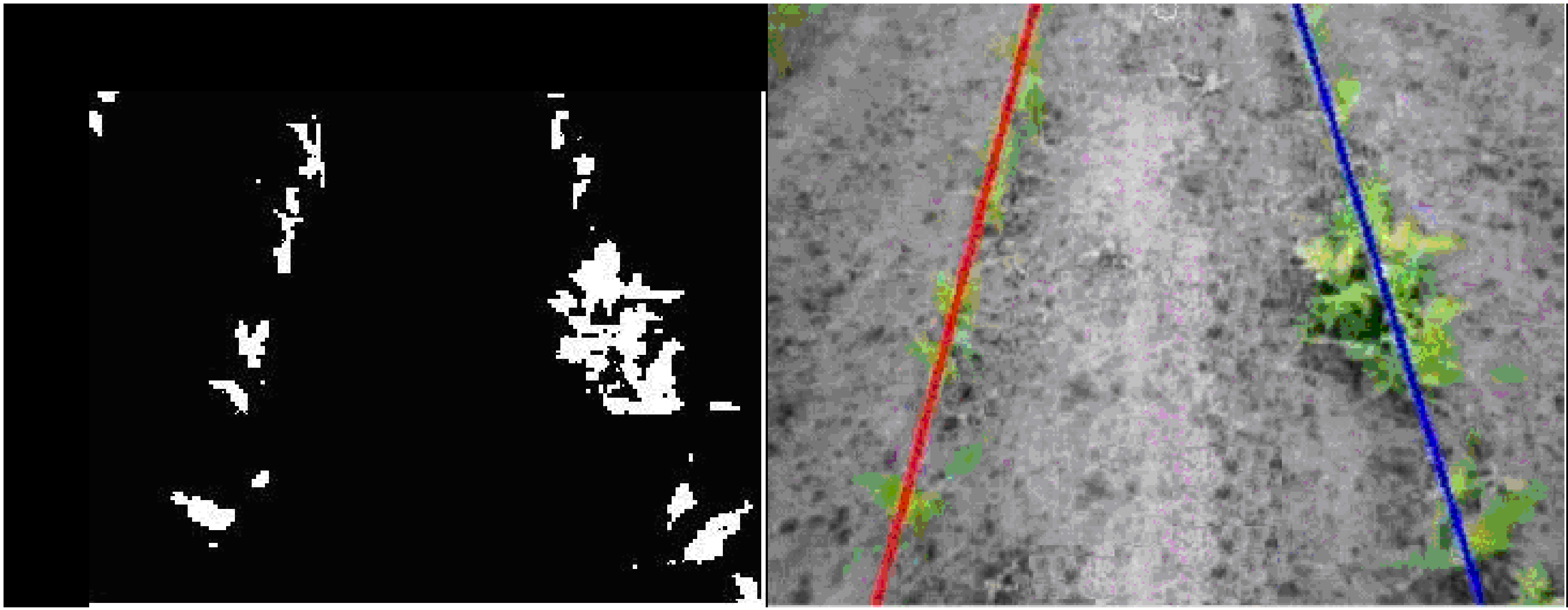
## Good:

- All points are processed independently, so can cope with occlusion/outliers
- Some robustness to noise: noise points unlikely to contribute consistently to any single bin

## Bad:

- Spurious peaks due to uniform noise
- Trade-off noise-grid size (hard to find sweet point)

# Hough transform - experiments

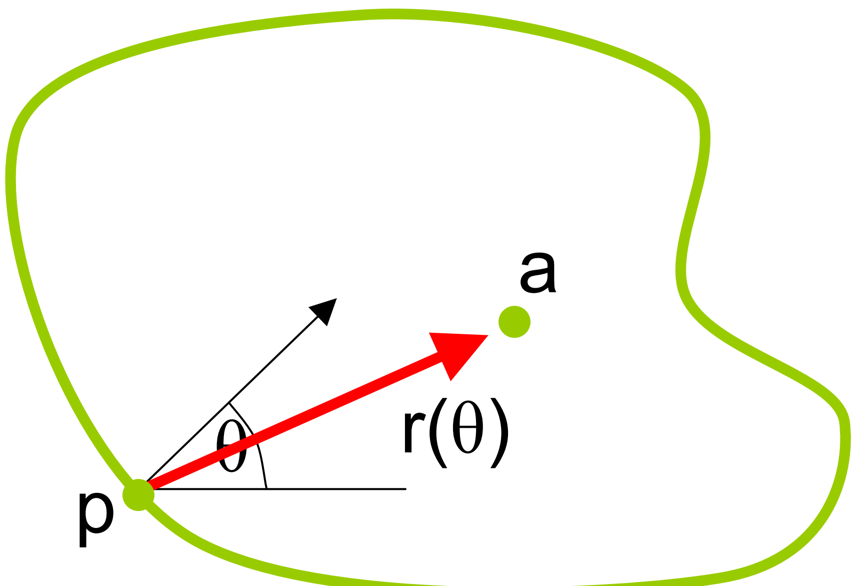


Courtesy of TKK Automation Technology Laboratory

# Generalized Hough transform

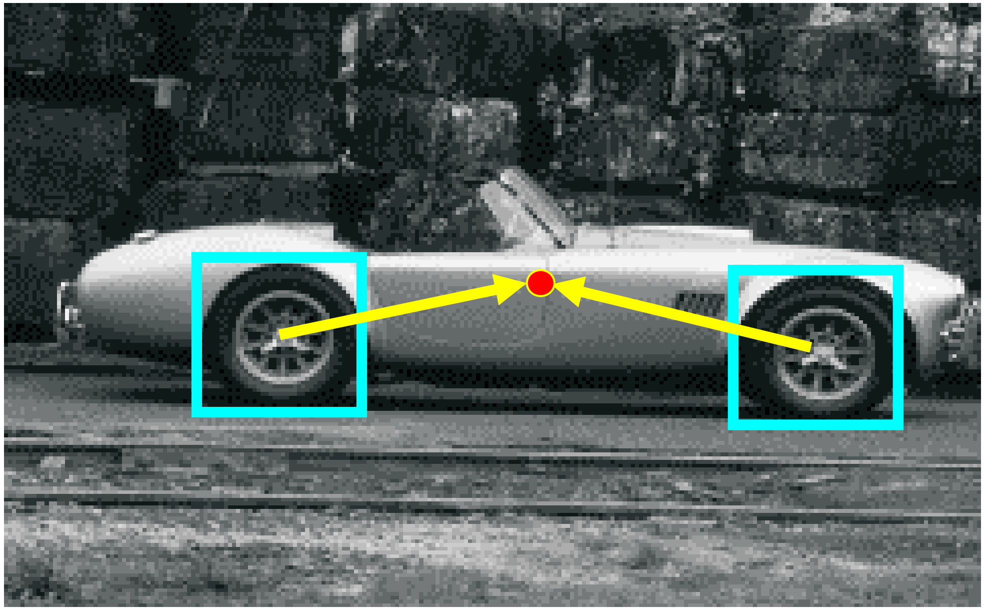
D. Ballard, Generalizing the Hough Transform to Detect Arbitrary Shapes, Pattern Recognition 13(2), 1981

- Identify a shape model by measuring the location of its parts and shape centroid
- Measurements: orientation theta, location of p
- Each measurement casts a vote in the Hough space: $p + r(\theta)$



# Generalized Hough transform

B. Leibe, A. Leonardis, and B. Schiele, [Combined Object Categorization and Segmentation with an Implicit Shape Model](), ECCV Workshop on Statistical Learning in Computer Vision 2004



# Hough transform

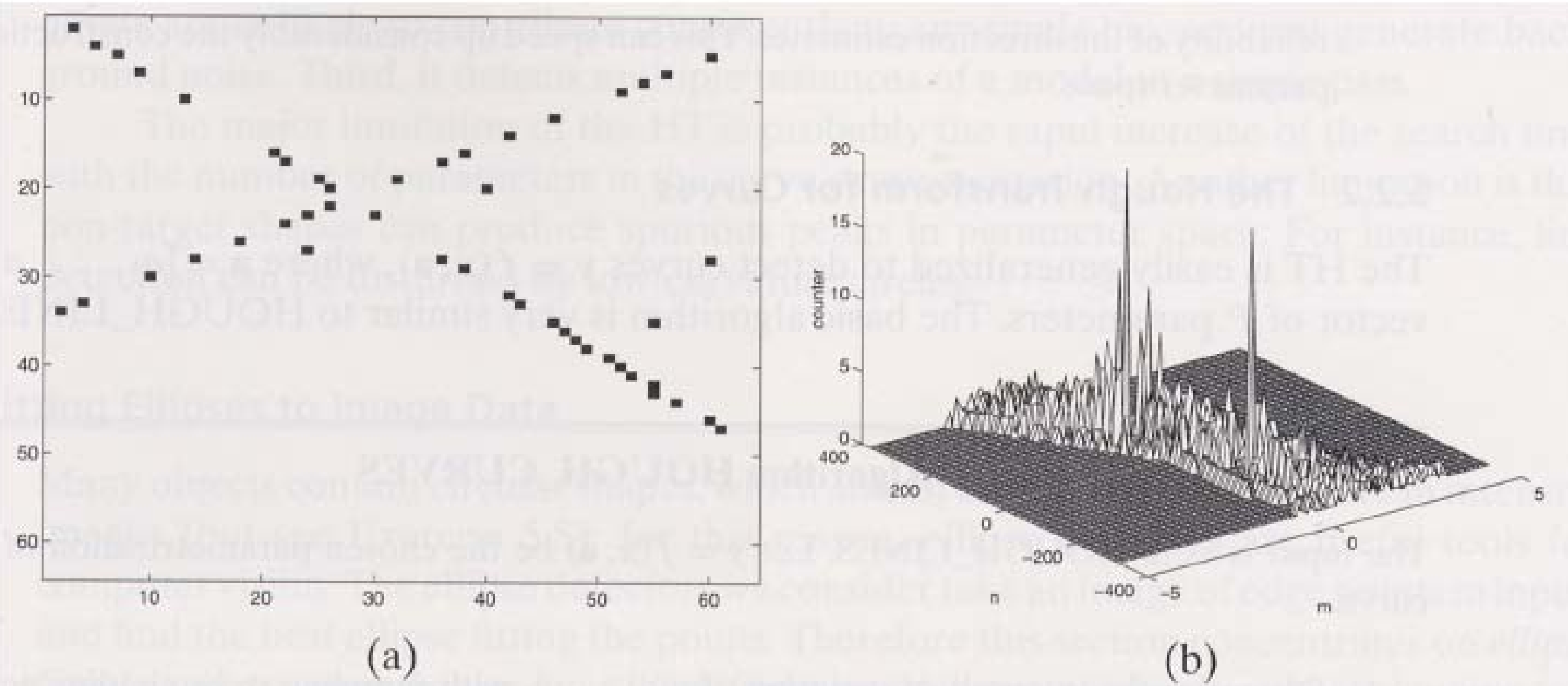



Figure 5.2 (a) An image containing two lines, sampled irregularly, and several random points. (b) Plot of the counters in the corresponding parameter space (how many points contribute to each cell $(m, n)$). Notice that the main peaks are obvious, but there are many secondary peaks.

Courtesy of unknown

Same cons and pros as before…

# QUESTIONS?