CSE473/573:
Chapter 7:
Mathematical Morphology for Shape Analysis
(Textbook Chapter 13.1-13.5)

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Basic Morphological Concepts – 13.1-2

Mathematical Morphology – Origin

- Initially developed in the late 1960's; separating itself from other conventional image analysis developments
- Its principle of non-linear operations on object shapes is fundamentally different from those operations in linear system described mainly by convolution – however, many algorithms are some variations of convolutions – filtering, edge detection, and segmentation
- Mathematical morphology often performs the desired image processing and analysis task operations in a more efficient and faster fashion – Practically useful

Mathematical Morphology – Origin

- Original mathematical morphology was presented in a highly mathematical manner – not easy to follow by the practical engineers for implementation
- Recent efforts (past two decades) have resulted in the presentation of morphology easy to understand and implement by practical engineers
- Non-morphological approach to image processing and analysis is based on calculus and convolutions – all techniques we have covered are non-morphological
- Mathematical morphology uses non-linear algebra and operating on point sets, their connectivity and shape

Mathematical Morphology Applications

- Image Pre-processing – noisy filtering and shape simplification
- Object structure enhancement – skeletonizing, thinning, thickening, convex hull, object marking
- Object segmentation – foreground/background, multiple object segmentation
- Quantitative description of object – area, perimeter, projection, …
- Some difficult/impossible tasks in calculus-based image processing can be made possible and/or easier

Image as Point Sets of N-Dimension

- Planar shape description (for images) can naturally follow 2D Euclidean space and its subsets to design its operations
- Common set operations are assumed: inclusion ($\subset$ or $\supset$), intersection ($\cap$), union ($\cup$), empty set ($\emptyset$), and set complement (^{C})
- Set difference is defined as:

$$X \backslash Y = X \cap Y^C$$

- Use sets of integer pairs ($\in \mathcal{Z}^2$) for binary image morphology and sets of integer triples ($\in \mathcal{Z}^3$) for gray level morphology or binary 3D morphology

Binary Image Operations - Concepts

- **Binary images can be viewed as subsets of the 2D space of all integers – $\mathcal{Z}^2$**
- **A point in this case is represented by a pair of integers that denote the coordinates with respective to the two coordinate axes of the pixel raster**
- **Unit length of raster equals to the sampling period in each direction (x or y)**
- **For a well defined neighborhood relation between points, we consider a binary image as a discrete grid – rectangular grid is assumed**

Binary Image Operations - Concepts

- A binary image can be treated as a 2D point set – Points belonging to objects constitute set X with value "1" while points belonging to background constitute the complement set X^C with value "0"

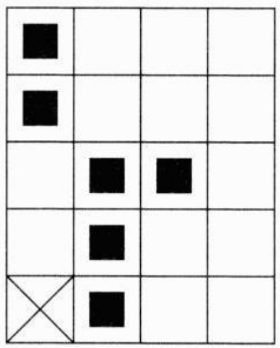


Figure 13.1: A point set example.

Binary Image Operations - Concepts

- **Figure 13.1 shows a point set example – the origin is marked as a diagonal cross and has coordinate** $(0, 0)$
- **The coordinates of any point follow the conventional** (x, y) **as we use in non-morphological operations**
- **The discrete image shown in Figure 13.1 can be represented by** $X = \{(1,0), (1,1), (1,2), (2,2), (0,3), (0,4)\}$
- **Each point** (x, y) **can be treated as a vector with respect to the origin** $(0, 0)$

Defining Morphological Transform

- A morphological transformation Ψ can be defined by the relation of the image (point set X) with another (usually) smaller point set B called **structuring element**
- The point set B is expressed with respect to a local origin $\mathcal{O}$ – called representative point
- Sometimes the origin $\mathcal{O}$ is not a member of structuring element

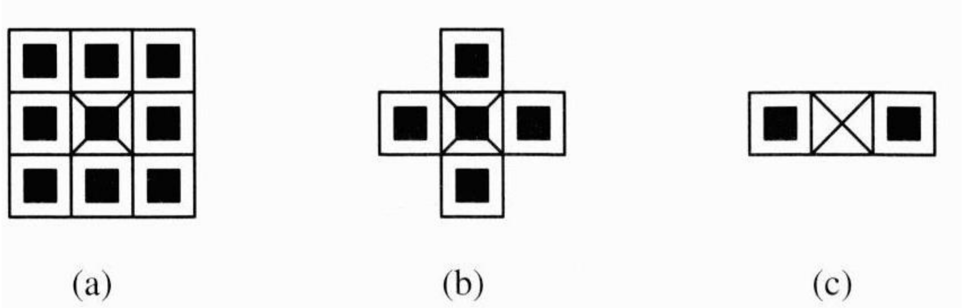


Figure 13.2: Typical structuring elements.

Applying Morphological Transform

- To apply the morphological transformation $\Psi(X)$ to a given image X, the structuring element B is moved **systematically** across the **entire image**
- When the structuring element B is positioned at a point (pixel) in the image, this pixel corresponds to the **representative point** or origin $\mathcal{O}$ of the structuring element and is called current pixel
- The transformation result (can be either zero or one) is stored in the output image at the current image pixel position

Morphological Operations

- Duality – deduced from the set complement – for each morphological transform $\Psi(X)$, there exists a dual transformation $\Psi^*(X)$

$$\psi(\mathrm{X}) = \left(\psi^*(\mathrm{X}^{\mathrm{C}})\right)^{\mathrm{C}}$$

- Translation by a vector h – X_h is defined by

$$X_h = \{p \in \mathcal{E}^2, p = x + h \text{ for some } x \in X\}$$

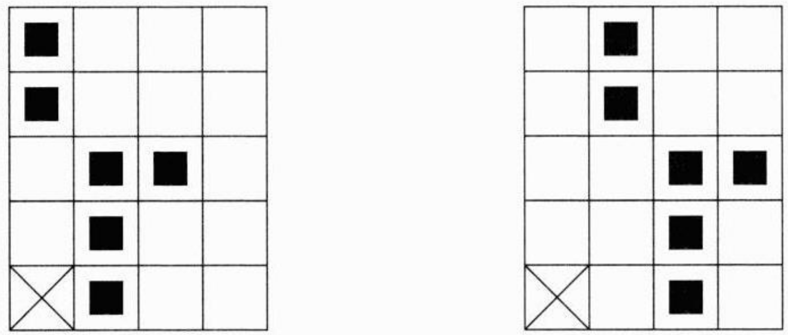


Figure 13.3: Translation by a vector.

Morphological Principles

- For image analysis tasks, some constraints are needed to restrict the set of possible morphological transformation
- Human usually have intuitive understanding of spatial structures – qualitative description of an object is relatively easy
- Computerized processing often requires more accurate quantitative description of an object
- Both qualitative and quantitative measures are required to restrict morphological transformation!

Morphological Principle #1

- **Principle #1 – Compatibility with translation**
- **Let the transformation ψ depend on the position of the origin $\mathcal{O}$ of the coordinate system, and denote this transformation by $\psi_{\mathcal{O}}$.**
- **If all points are translated by $-h$, expressed as ψ_{-h}, the compatibility with translation principle is given by**

$$\psi_{\mathcal{O}}(X_h) = \left(\psi_{-h}(X)\right)_h$$

- **If ψ does not depend on the origin $\mathcal{O}$, we have**

$$\psi(X_h) = \left(\psi(X)\right)_h$$

- **Then, the transformation is invariant under translation**

Morphological Principle #2

- Principle #2 – Compatibility with change of scale
- Let λX represent homothetic scaling of point set X. Let ψ_λ denote a transformation that depends on the positive parameter λ.
- The compatibility with change of scale is given by

$$\psi_\lambda(X) = \lambda\psi\left(\frac{1}{\lambda}X\right)$$

- If ψ does not depend on the scale λ, we have

$$\psi(\lambda X) = \lambda\psi(X)$$

- The transformation is invariant to change of scale

Morphological Principle #3

- Principle #3 – Local knowledge
- The local knowledge principle considers the situation in which only a part of a larger structure can be examined – due to restriction in grid size
- A transformation ψ satisfies the local knowledge principle if for any bounded point set Z' in the transformation $\psi(X)$, there exists a bounded set Z, knowledge of which is sufficient to provide ψ
- This can be written as:

$$\left(\psi(X \cap Z)\right) \cap Z' = \psi(X) \cap Z'$$

Morphological Principle #4

- Principle #4 – Upper semi-continuity
- The upper continuity principle simple states that – the morphological transformation does not exhibit any abrupt changes
- More details can be found from the following book:

Image Analysis and Mathematical Morphology (1982)

by Jean Serra

Binary Dilation and Erosion – 13.3

Binary Dilation and Erosion

- Binary images always consist of some sets of black and white pixels
- Sets of black pixels usually are considered objects while white pixels are considered as background
- Binary dilation and erosion are designed to process binary images
- Binary dilation and erosion are two most fundamental operations in mathematical morphology – almost all other complex morphological operations can be derived from binary dilation and erosion

Binary Dilation – Definition

- The morphological transformation binary dilation $\oplus$ combines two sets using vector addition – follows Minkowski set addition defined as

$$(a, b) + (c, d) = (a + c, b + d)$$

- The dilation $X \oplus B$ is defined as the point set of all possible vector additions of pairs of elements, one from each of the sets X and B

$$X \oplus B = \{p \in \mathcal{E}^2 : p = x + b, x \in X \text{ and } b \in B\}$$

- Dilation is not an invertible transformation

Binary Dilation – Simple Example

- $X = \{(1,0), (1,1), (1,2), (2,2), (0,3), (0,4)\}$
- $B = \{(0,0), (1,0)\}$
- $X \oplus B = \left\{ \begin{matrix} (1,0), (1,1), (1,2), (2,2), (0,3), (0,4), \\ (2,0), (2,1), (2,2), (3,2), (1,3), (1,4) \end{matrix} \right\}$

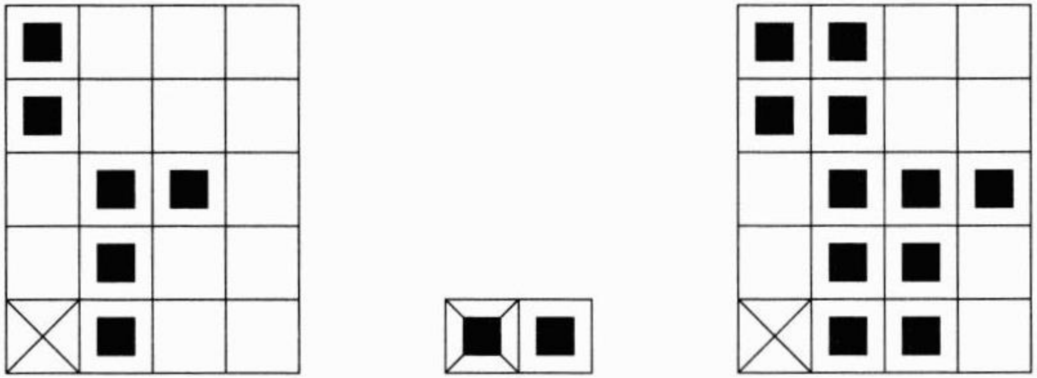


Figure 13.4: Dilation.

Binary Dilation – Graphics Example

- Figure 13.5 below shows the dilation operation on a graphics image with a 3×3 structuring element
- In this case, the dilation is an isotropic expansion – behave the same way in all directions – also called fill or grow – **all background pixels** neighboring **the object to become** new object pixels

Figure 13.5: Dilation as isotropic expansion.

Binary Dilation - Properties

- Dilation properties often facilitate easy hardware and software implementations
- Commutative

$$X \oplus B = B \oplus X$$

- Associative

$$X \oplus (B \oplus D) = (X \oplus B) \oplus D$$

- Represented as a union of shifted point sets

$$X \oplus B = \bigcup_{b \in B} X_b$$

- Dilation and intersection:

$$(X \cap Y) \oplus B = B \oplus (X \cap Y) \subseteq (X \oplus B) \cap (Y \oplus B)$$

Binary Dilation - Properties

- invariant to translation:

$$X_h \oplus B = (X \oplus B)_h$$

- Shift can speed up implementation – single instruction!
- Dilation is an increasing transformation:

$$\text{if } X \subseteq Y \text{ then } X \oplus B \subseteq Y \oplus B$$

- Dilation is used to fill small holes and narrow gulfs in objects – increase the object size
- If original size needs to be preserved – after filling the holes – dilation is followed by erosion – to be introduced next

Binary Dilation – A Special Case

- One special case of binary dilation is that the representative point is not a member of the structuring element B
- Such special dilation operation will generate a result substantially different from the original set – causing the loss of connectivity!

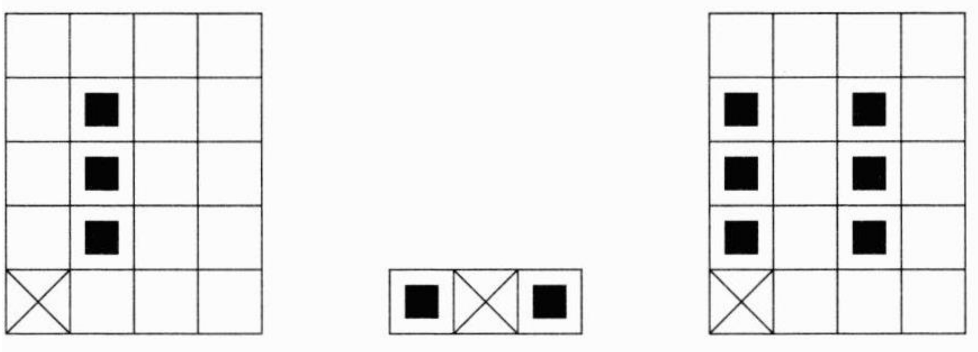


Figure 13.6: Dilation where the representative point is not a member of the structuring element.

Binary Erosion – Definition

- **Binary erosion $\ominus$ is defined by combining two sets using vector subtraction – a dual operator of dilation!**

$$X \ominus B = \{p \in \mathcal{E}^2 : \; p = x - b \in X \text{ for every } b \in B\}$$

- **The definition of vector subtraction also follow Minkowski set subtraction definition**
- **This definition states that for every point p from the image is tested, the result of erosion is given by those points p for which all possible $p + b$ are in X**
- **Erosion is again not an invertible transform**

Binary Erosion – Simple Example

- $X = \{(1,0), (1,1), (1,2), (0,3), (1,3), (2,3), (3,3), (1,4)\}$
- $B = \{(0,0), (1,0)\}$
- $X \ominus B = \{\ (0,3), (1,3), (2,3)\}$

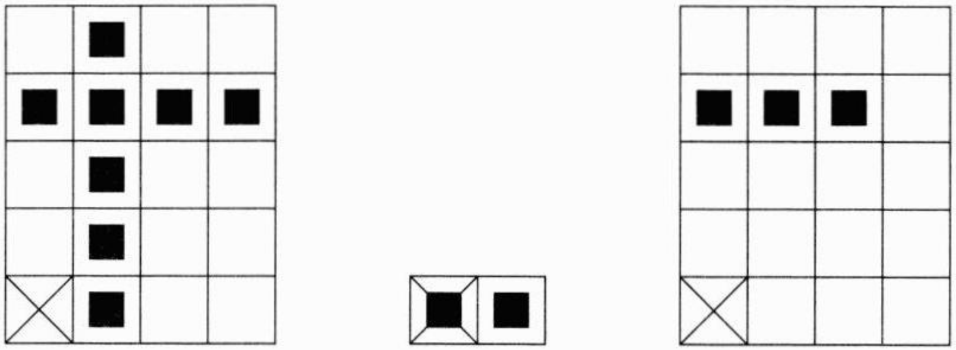


Figure 13.7: Erosion.

- **Note that single-pixel-wide line disappears after erosion – Can be used to eliminate thin or small structures**

Binary Erosion – Graphics Example

- Figure 13.8 below shows the erosion operation on a graphics image with a 3×3 structuring element
- In this case, the erosion is an isotropic operation –also called shrink or reduce – again, single-pixel-wide lines disappear after erosion

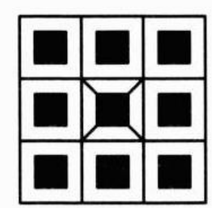


Figure 13.8: Erosion as isotropic shrink.

Binary Erosion – Applications

- Morphological transformation application – a quick and easy scheme to find contours of object
- Achieved by subtraction from the original object its eroded version – very nice single-pixel-wide contours!!!

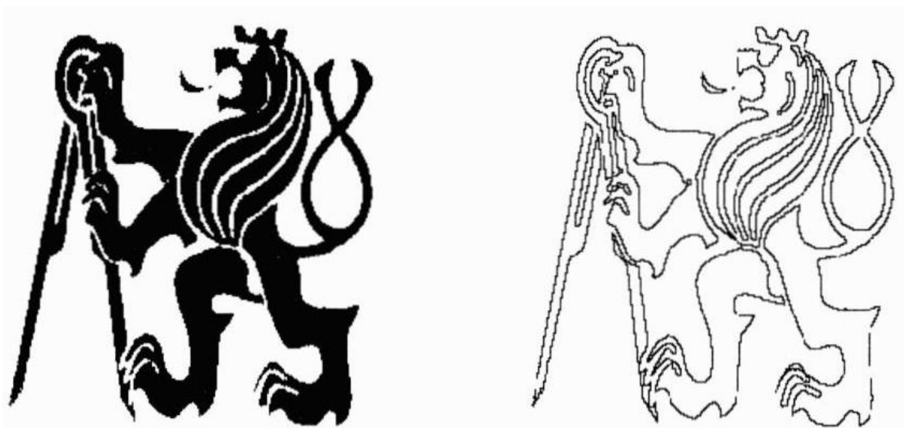


Figure 13.9: Contours obtained by subtraction of an eroded image from an original (left).

Binary Erosion – Alternative Definitions

- Define the translation of B by p as B_p, then, we have

$$X \ominus B = \{p \in \mathcal{E}^2 : B_p \in X\}$$

- This definition can be explained by sliding B across the image X and check whether or not a point belongs to the erosion result – only if B translated by p is still contained in the image X

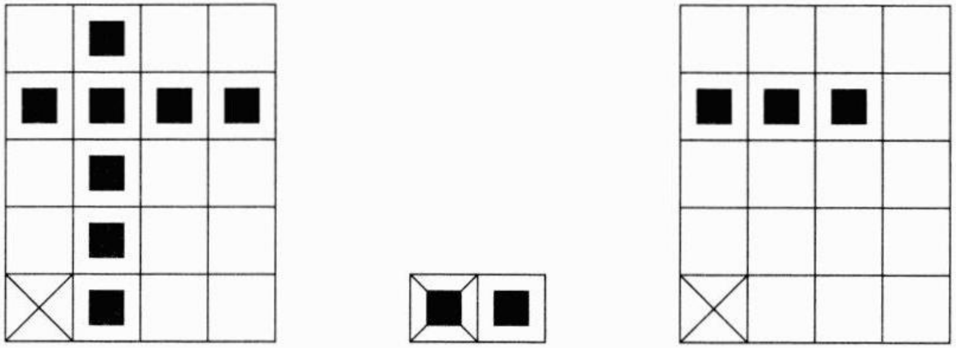


Figure 13.7: Erosion.

Binary Erosion – Alternative Definitions

- Another alternative definition for erosion for simple implementation purpose can be worked using the translation operations
- An image X eroded by the structuring element B can be expressed as an intersection of all translations of the image X by the vector $b \in B$

$$X \ominus B = \bigcap_{b \in B} X_{-b}$$

Binary Erosion - Properties

- **Anti-extensive – If the representative point is a member of the structuring element, then such erosion is anti-extensive – That is, if $(0,0) \in B$, then, $X \ominus B \subseteq X$**
- **Translation invariant:**

$$X_h \ominus B = (X \ominus B)_h$$

$$X \ominus B_h = (X \ominus B)_{-h}$$

- **Increasing transformation:**

$$\text{if } X \subseteq Y \text{ then } X \ominus B \subseteq Y \ominus B$$

- **It was mentioned before that the erosion can be combined with dilation as both are increasing transformations**

Binary Erosion - Properties

- **Ordering of erosion – If B, D are two structuring elements and D is contained in B, then the erosion by B is more aggressive than by D – If $D \subseteq B$, then $X \ominus B \subseteq X \ominus D$ – Enable the ordering of erosions with different sizes of structuring elements**
- **Non-commutative (different from dilation!)**

$$X \ominus B \neq B \ominus X$$

- **Erosion and intersection**

$$(X \cap Y) \ominus B = (X \ominus B) \cap (Y \ominus B)$$
$$B \ominus (X \cap Y) \supseteq (B \ominus X) \cup (B \ominus Y)$$

Binary Erosion - Properties

- Dual transformation with respect to dilation – Define symmetrical set to B as $\breve{B}$ – also called transpose or rational set

$$\breve{B} = \{-b : b \in B\}$$

- Example:

$$B = \{(1,2), (2,3)\}$$
$$\breve{B} = \{(-1,-2), (-2,-3)\}$$

- With $\breve{B}$, dual transformation is represented as:

$$(X \ominus Y)^C = X^C \oplus \breve{Y}$$

Binary Erosion - Properties

- **Erosion and set union – The order of erosion may be interchanged with set union – enable the structuring element to be decomposed a union of simple ones**

$$(X \cup Y) \ominus B \supseteq (X \ominus B) \cup (Y \ominus B)$$
$$B \ominus (X \cup Y) = (X \ominus B) \cap (Y \ominus B)$$

- **Successive erosion and dilation applications – image X is first applied erosion or dilation by B and then by D is equivalent to application of X by $B \oplus D$**

$$(X \oplus B) \oplus D = X \oplus (B \oplus D)$$
$$(X \ominus B) \ominus D = X \ominus (B \oplus D)$$

Hit-or-Miss Transformation

- A special morphological operator to find local patterns of pixels – variation of template matching
- Structuring element B usually consists of a pair of disjoint sets $B = (B_1, B_2)$, called composite structuring element
- This operator is denoted as $\otimes$ and defined as:

$$X \otimes B = \{x\colon\ B_1 \subset X \text{ and } B_2 \subset X^c\}$$

- The two conditions must be fulfilled simultaneously.
- May be expressed by erosions and dilations

$$X \otimes B = (X \ominus B_1) \cap (X^c \ominus B_2) = (X \ominus B_1) \backslash \left(X \oplus \breve{B}_2\right)$$

Hit-or-Miss Structuring Elements

- Hit-and-miss transform is a slight extension to the type that has been introduced for erosion and dilation
- The difference is that it can contain both foreground and background pixels, rather than just the foreground pixels

$$\begin{bmatrix} & 1 & \\ 0 & 1 & 1 \\ 0 & 0 & \end{bmatrix} \quad \begin{bmatrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} & 0 & 0 \\ 1 & 1 & 0 \\ & 1 & \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{bmatrix}$$

Opening and Closing Transformations

Opening and Closing Transformations

- Erosion and dilation are not inverse transformations – if an image is eroded and then dilated, original image cannot be retained – simplified and less detailed!
- Erosion followed by dilation defines an important morphological transformation called opening

$$X \circ B = (X \ominus B) \oplus B$$

- Dilation followed by erosion also defined an important morphological transformation called closing

$$X \cdot B = (X \oplus B) \ominus B$$

- If an image X is unchanged by opening – is called open with respect to B; Similarly, an image can be called closed with respect to B

Opening and Closing Transformations

- Opening and closing with an isotropic structuring element is used to eliminate specific image details smaller than the structuring element – global shape is retained
- Closing shall connect objects that are close to each other – filling up holes, smoothing object outline by filling up narrow gulfs
- All qualitative descriptions are relative to size and shape of structuring element

Opening and Closing Examples

Figure 13.10: Opening (original on the left).

Figure 13.11: Closing (original on the left).

Opening and Closing Properties

- **Invariant to translation – both opening and closing are invariant to translation of the structuring element**
- **Increasing transformation – both opening and closing are increasing transformation – because both dilation and erosion are**
- **Extensive – closing is extensive ($X \subseteq X \cdot B$)**
- **Anti-extensive – opening is anti-extensive ($X \circ B \subseteq X$)**
- **Dual transformation between them:** $(X \cdot B)^C = X^C \circ \breve{B}$
- **Idempotent:** $X \circ B = (X \circ B) \circ B$ **and** $X \cdot B = (X \cdot B) \cdot B$ **– reapplication of transformations results no changes**

Gray-Scale Dilation and Erosion – 13.4

From Binary to Gray-Scale Morphology

- The gray-scale images can be considered from the topographic view – gray-scale value can be interpreted as the height at a given location – some hypothetical landscape view
- Light spot in the image – hills in the landscape; Dark spot in the image – hollows in the landscape
- Gray-scale morphological operations – to identify topographic features on the images – valley, mountain ridges (crests), and watersheds
- Definitions of gray-scale morphological operations are different from binary counterpart – higher dimensions

Spatial Domain and Functional Value

- For a n-dimensional Euclidean space, consider a point set A defined in this space, $A \subset \mathcal{E}^{n}$
- Assume that the first $(n-1)$ coordinates of the set constitute a spatial domain and the n^{th} coordinate corresponds to the value of a function at a point in the spatial domain $x \in \mathcal{E}^{n-1}$
- For gray-scale images, $n = 3$, the spatial domain is a 2D space (image domain)
- The third coordinate represents the height of the 2D spatial domain – matches the topographic view

Top Surface and Umbra Definition

- **Extension of binary morphological operations to gray-scale morphological operations needs several new definitions to characterize topological features**
- **Top Surface – The top surface of a set A is a function defined on the $(n-1)$-dimensional support. For each $(n-1)$-tuple, the top surface is the highest value of the n^{th} coordinate of A**
- **Let $A \subseteq \mathcal{E}^{\,n}$ and the support $F = \{x \in \mathcal{E}^{n-1} \text{ for some } y \in \mathcal{E}, (x, y) \in A\}$. The top surface of A, denoted as $T[A]$, is a mapping $F \rightarrow \mathcal{E}$ defined as:**

$$T[A](x) = \max\{y, (x, y) \in A\}$$

Top Surface Example in 2D

- In the case of 2D spatial domain, there are two spatial coordinates (x_1, x_2), the top surface will be $f(x_1, x_2)$

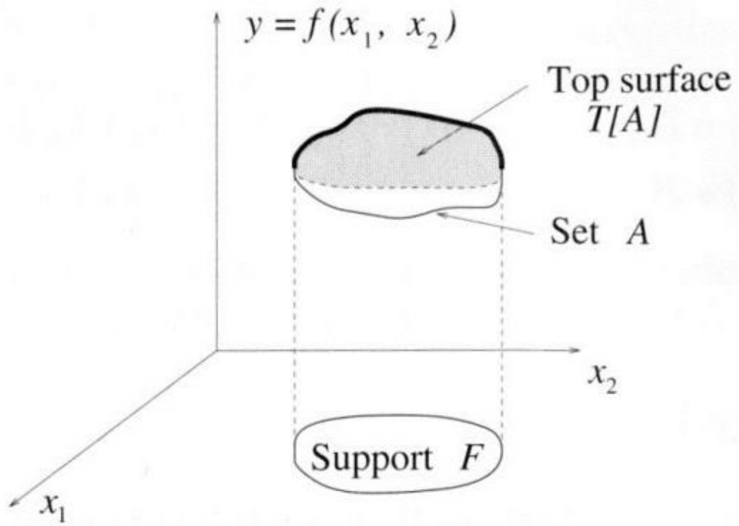


Figure 13.12: Top surface of the set A corresponds to maximal values of the function $f(x_1, x_2)$.

Top Surface and Umbra

- Umbra – The usual definition of umbra is a region of complete shadow resulting from obstructing the light by non-transparent object
- In morphology, the umbra of function f is a set that consists of the top surface of f and everything below this function
- Formal definition of umbra: Let $F \subseteq \mathcal{E}^{n-1}$ and $f: F \to \mathcal{E}$. The umbra of f, denoted as $U[f]$, in which $U[f] \subseteq F \times \mathcal{E}$, is defined as:

$$U[f] = \{(x, y) \in F \times \mathcal{E}, y \leq f(x)\}$$

- We see that the umbra of an umbra is also an umbra

Umbra Example in 2D

- This umbra example is for 2D spatial domain with two spatial coordinates (x_1, x_2), and top surface $f(x_1, x_2)$

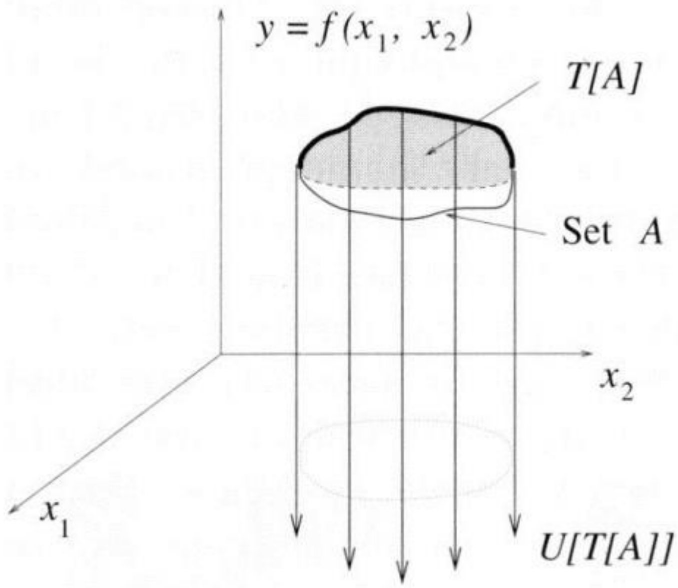


Figure 13.13: Umbra of the top surface of a set is the whole subspace below it.

1D Example – Function and Umbra

- Top surface and umbra in the case of a simple 1D gray-scale image

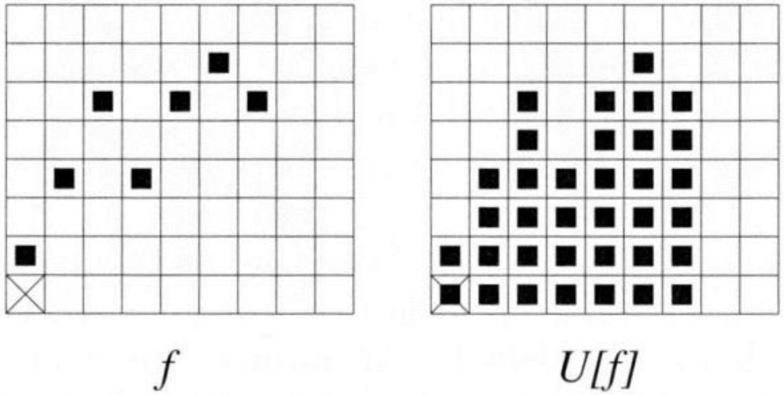


Figure 13.14: Example of a 1D function (left) and its umbra (right).

Gray-Scale Dilation Definition

- **Gray-scale dilation of two functions (an image and a structuring element) can be defined via top surface of the dilation of their umbras**
- **Let** $F, K \subseteq \mathcal{E}^{n-1}$ **and the mappings** $f: F \to \mathcal{E}$ **and** $k: K \to \mathcal{E}$**. The dilation** $\oplus$ **of f by k,** $f \oplus k: F \oplus K \to \mathcal{E}$ **is defined by:**

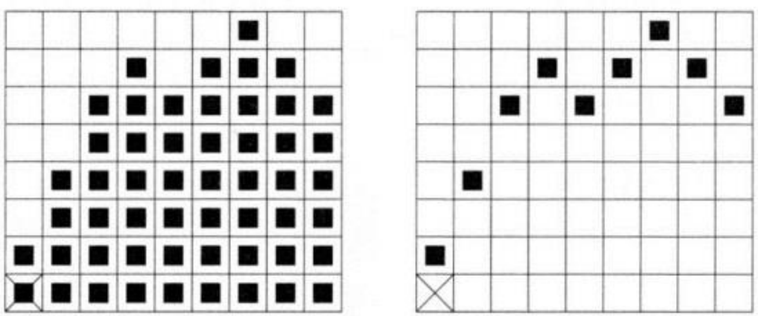
$$(f \oplus k) = T\{U[f] \oplus U[k]\}$$

- **The operator in the left-hand side is gray-scale dilation while the operator in the right-hand side is binary dilation – no new notation is needed as no confusion is expected**

Illustration of Gray-Scale Dilation

k $U[k]$

Figure 13.15: A structuring element: 1D function (left) and its umbra (right).



$U[f] \oplus U[k]$ $T[U[f] \oplus U[k]] = f \oplus k$

Figure 13.16: 1D example of gray-scale dilation. The umbras of the 1D function f and structuring element k are dilated first, $U[f] \oplus U[k]$. The top surface of this dilated set gives the result, $f \oplus k = T\left[U[f] \oplus U[k]\right]$.

Implementing Gray-Scale Dilation

- The definition and illustration for gray-scale dilation are easy to understand – not necessarily easy to implement
- Computationally well-defined algorithm – gray-scale dilation can be obtained by taking the maximum of a set of sums:

$$(f \oplus k)(x) = \max\{f(x-z) + k(z), z \in K, x - z \in F\}$$

- The complexity of this max of sum operation is the same as convolution in linear filtering – sum of product is performed

Gray-Scale Erosion Definition

- **Gray-scale erosion can be defined similarly as dilation**
- **Gray-scale erosion of two functions is defined in three steps: (1). Take their umbras, (2). Erode them using binary erosion, (3). Take the top surface as the results**
- **Let** $F, K \subseteq \mathcal{E}^{n-1}$ **and** $f: F \to \mathcal{E}$ **and** $k: K \to \mathcal{E}$**. The erosion** $\ominus$ **of** f **by** k**,** $f \ominus k: F \ominus K \to \mathcal{E}$ **is defined by:**

$$(f \ominus k) = T\{U[f] \ominus U[k]\}$$

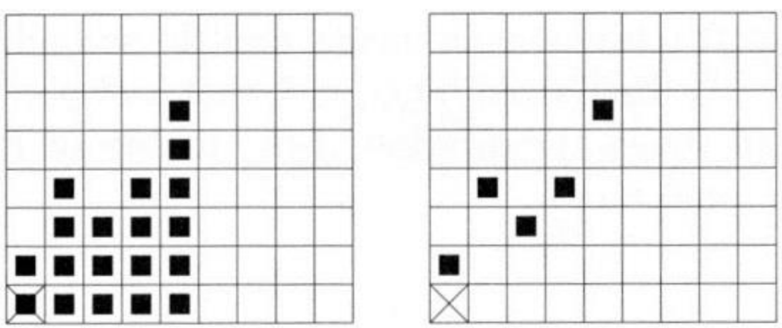
- **Again, gray-scale erosion can be obtained by taking the minimum of a set of differences:**

$$(f \ominus k)(x) = \min_{z \in K}\{f(x+z) - k(z)\}$$

Illustration of Gray-Scale Erosion

k $U[k]$

Figure 13.15: A structuring element: 1D function (left) and its umbra (right).



$U[f] \ominus U[k]$ $T[U[f] \ominus U[k]] = f \ominus k$

Figure 13.17: 1D example of gray-scale erosion. The umbras of 1D function f and structuring element k are eroded first, $U[f] \ominus U[k]$. The top surface of this eroded set gives the result, $f \ominus k = T\big[U[f] \ominus U[k]\big]$.

Microscopic Image Processing

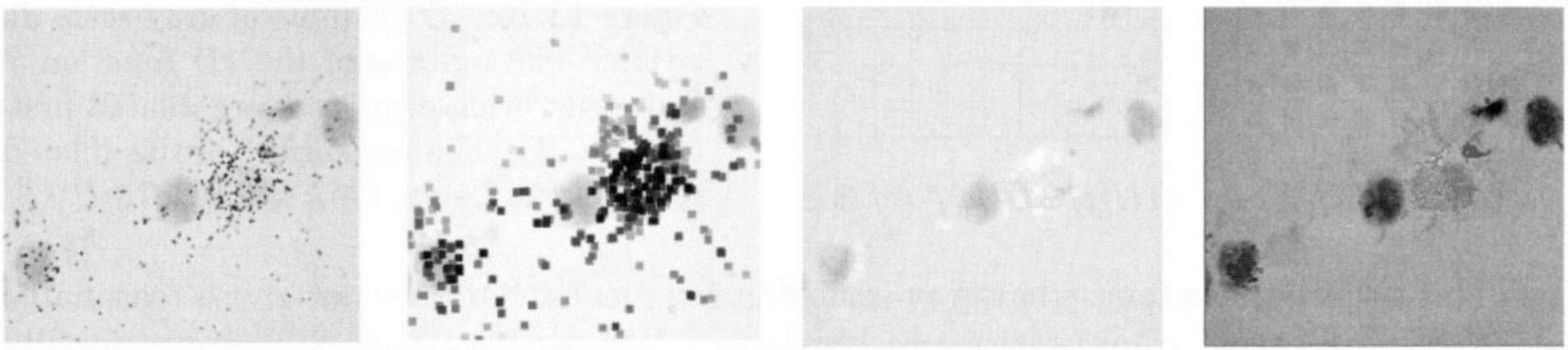


Figure 13.18: Morphological pre-processing: (a) cells in a microscopic image corrupted by noise; (b) eroded image; (c) dilation of (b), the noise has disappeared; (d) reconstructed cells. *Courtesy of P. Kodl, Rockwell Automation Research Center, Prague, Czech Republic.*

Gray-Scale Morphology Applications

Umbra Homeomorphism – Definition

- **Top surface always inverts the umbra operation – actually is a left inverse of the umbra:**

$$T\big[U[f]\big] = f$$

- **Umbra is NOT an inverse of the top surface – Umbra is a special operation that links gray-scale and binary morphology**
- **Umbra homeomorphism theorem – Let $F, K \subseteq \mathcal{E}^{n-1}$ and $f: F \to \mathcal{E}$ and $k: K \to \mathcal{E}$. Then,**

$$(a) \qquad U[f \oplus k] = U[f] \oplus U[k]$$

$$(b) \qquad U[f \ominus k] = U[f] \ominus U[k]$$

Umbra Homeomorphism – Applications

- The umbra homeomorphism is used for deriving properties of gray-scale morphological operations
- The operation is expressed in terms of umbra and top surface – these can be transformed to binary sets using the umbra homeomorphism property – finally transformed back using the definition of gray-scale dilation and erosion
- Properties known from binary morphology can be derived and used in gray-scale morphology
  - Commutativity of dilation, chain rules of structural element decomposition, duality between dilation and erosion, …

Gray-Scale Opening and Closing

- **Gray-scale opening is defined as**

$$f \circ k = (f \ominus k) \oplus k$$

- **Gray-scale closing is defined as**

$$f \cdot k = (f \oplus k) \ominus k$$

- **Duality between opening and closing can be expressed similarly with symmetric set definition**

$$-(f \circ k)(x) = \left((-f) \cdot \breve{k}\right)(x)$$

- **The opening by structuring element k on landscape f can be interpreted as the position of all highest points reached by some part of k when sliding k on f**

Application – Top Hat Transformation

- Top hat transformation is a simple morphological tool for finding objects in the gray-scale images which have significant brightness difference than the background – even when the background is changing
- Definition – top hat transformation is the residue of opening as compared to the original image – $X \setminus (X \circ K)$
- Top hat transformation is good for extracting light (or dark) objects from dark (or light) but slowly changing background
- Only those parts of the image (objects) cannot fit into structuring element K can be extracted

Top Hat Transformation Illustration

- The following illustration shows the process and the origin of the transformation name

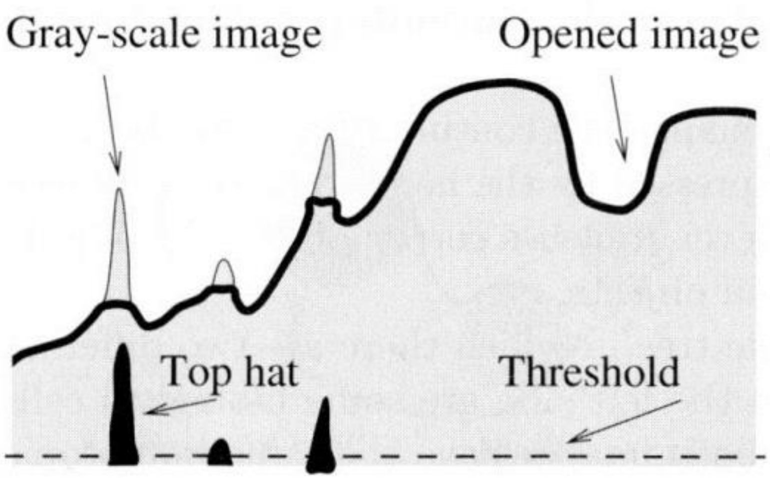


Figure 13.19: The top hat transform permits the extraction of light objects from an uneven background.

Top Hat Transformation Example

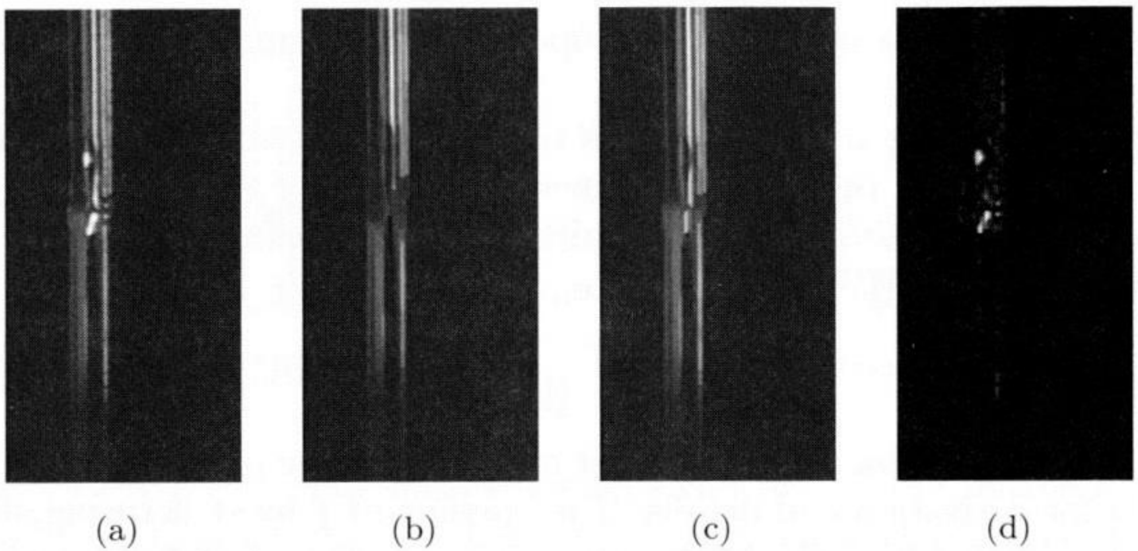


Figure 13.20: An industrial example of gray-scale opening and top hat segmentation, i.e., image-based control of glass tube narrowing by gas flame. (a) Original image of the glass tube, 512×256 pixels. (b) Erosion by a one-pixel-wide vertical structuring element 20 pixels long. (c) Opening with the same element. (d) Final specular reflection segmentation by the top hat transformation. *Courtesy of V. Smutný, R. Šára, CTU Prague, P. Kodl, Rockwell Automation Research Center, Prague, Czech Republic.*

Skeletons – Basic Concepts

Homotopic Transformations

- One of the important topological properties of an object (a region in an image) is its continuity
- It is crucial to maintain such continuity when applying morphological operations to the region (object)
- A morphological transformation is called homotopic if it does not change the continuity relation between regions and holes within an image
- The continuity relation can be expressed by the homotopic tree – its root corresponds to background; first level branches corresponds to objects (regions); second level branches match the holes within regions

Homotopic Tree Examples

- Two different images may have same topological relation – correspond to the same homotopic tree
- Examples for both biological cells and house with a spruce tree – same homotopic tree!

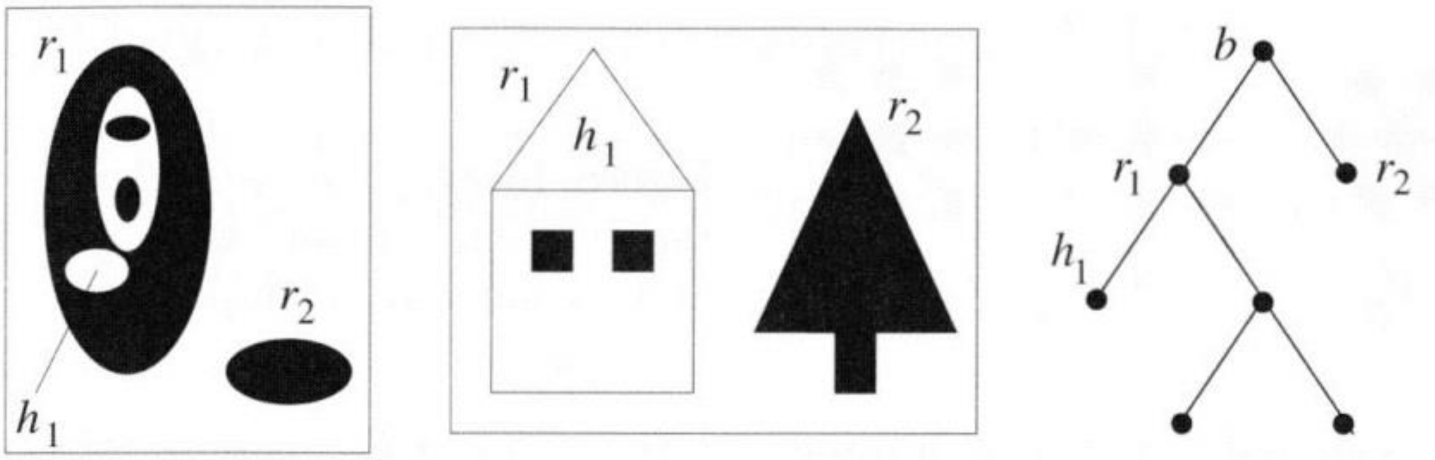


Figure 13.21: The same homotopic tree for two different images.

Skeleton Concept and Definition

- **Another topological property of regions (objects) is the representation of regions by their skeletons**
- **In 1967, Blum first introduced the concept of skeleton under the name of medial axis transform – understood with "grassfire" scenario**
- **Assume a region $X \subset \mathcal{R}^2$: A grassfire is lit on the entire region boundary at the same time. The fire propagates towards the region interior with constant speed – The skeleton $S(X)$ is set of points where two or more fire fronts meet**

Skeleton Concept and Definition

- Conceptually clear illustration of “grassfire” operation to obtain the skeleton of an arbitrary region
- This illustration is not a rigorous definition for the concept of skeleton

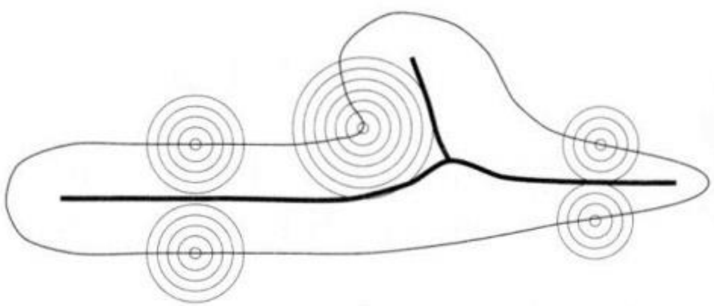


Figure 13.22: Skeleton as points where two or more firefronts of grassfire meet.

Concept of Maximal Ball

- Maximal ball is key to rigorous definition of skeleton
- A ball $B(p, r)$ with center p and radius r, $r \geq 0$ is the set of points with distance d from the center less than or equal to r
- The ball B included in a set X is said to be maximal if and only if there is no larger ball included in X that contains B – each ball B', $B \subseteq B' \subseteq X \Longrightarrow B' = B$.
- The distance metric d used in the process of generating skeleton depends on the grid and connectivity definitions – discrete plane $\mathcal{Z}^2$

Illustration of Ball and Maximal Ball

- The following figure shows the difference between balls and maximal balls
- These conceptual explanations are based on continuous domain objects and regions – care must be taken when discrete grid is considered

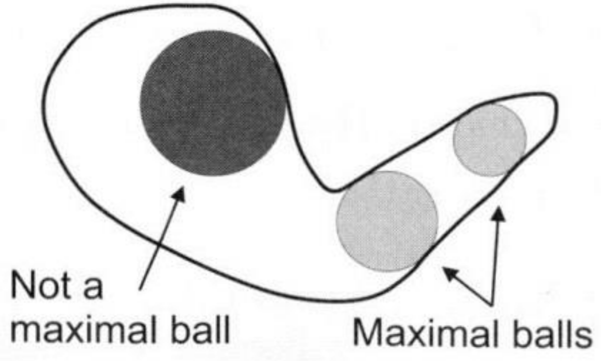


Figure 13.23: Ball and two maximal balls in a Euclidean plane.

Balls in Euclidean and Discrete Planes

- Previous examples are given with $\mathcal{R}^2$ and are based on Euclidean distances
- Three distances in discrete planes and corresponding unit balls can be defined - B_H, B_4, and B_8

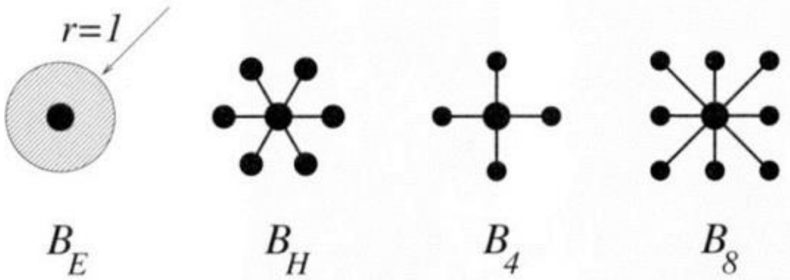


Figure 13.24: Unit-size disk for different distances, from left side: Euclidean distance, 6-, 4-, and 8-connectivity, respectively.

Skeleton by Maximal Balls

- Skeleton obtained via maximal balls can be defined as the set of centers p of maximal balls applied to set X
- Mathematically, we have

$$S(X) = \{p \in X : \exists r \geq 0, B(p,r) \text{ is a maximal ball of } X\}$$

- This definition is intuitive in the Euclidean plane – the skeleton of a disk (circular object) is reduced to its center while skeleton of a stripe with rounded ending is reduced to unit thickness line at its center
- Two representation shortcomings: (1) such skeleton may not preserve homotopy (connectivity) (2) some lines of the skeleton may be more than one pixel wide

Skeleton Examples

- Skeleton of rectangle is not reduced to single line!
- Skeleton of two touching circles is reduced to three distinct points – not a straight line!
- Skeleton of a ring is indeed a single pixel wide ring

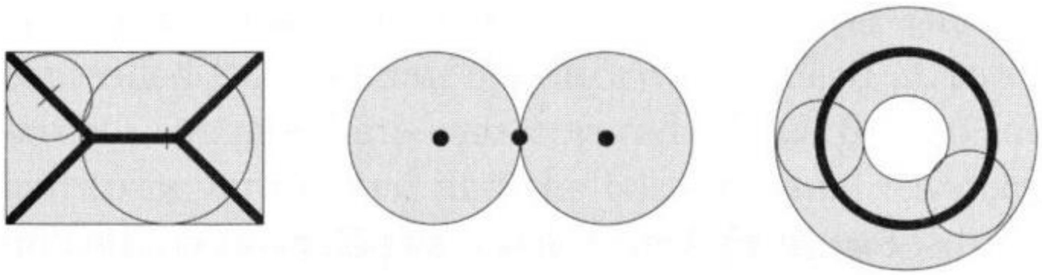


Figure 13.25: Skeletons of rectangle, two touching balls, and a ring.

Simple Skeleton by Discrete Balls

- For discrete planes (binary images), the unit size disk B can be dilated n times to obtain a ball of radius n
- Let nB be the ball of radius n, we have

$$nB = \underbrace{B \oplus B \oplus \cdots \oplus B}_{n}$$

- Skeleton of an object can be extracted via maximal balls as the union of the residues of opening the set X at all scales:

$$S(X) = \bigcup_{n=0}^{\infty} \left((X \ominus nB) \backslash (X \ominus nB) \circ B \right)$$

Thinning, Thickening, Homotopic Skeleton

Thinning and Thickening

- **Skeleton generation can be considered as a process of "thinning" an object's by hit-or-miss transformation**
- **Definition of thinning – For an image X and a composite structuring element $B = (B_1, B_2)$ (B here is not a ball!), thinning can be defined as:**

$$X \oslash B = X \setminus (X \otimes B)$$

- **Definition of thickening – Similar definition can be expressed as:**

$$X \odot B = X \cup (X \otimes B)$$

- **Thinning and thickening are dual transformations**

$$(X \odot B)^C = X^C \oslash B, \qquad B = (B_2, B_1)$$

Thinning and Thickening

- Intuitively, the thinning operation will subtract part of object boundary from the original object; thickening will add part of background boundary to object
- Both thinning and thickening are very often applied sequentially with a sequence of composite structuring elements $\{B_{(1)}, B_{(2)}, B_{(3)}, \dots, B_{(n)}\}$, $B_{(i)} = (B_{i1}, B_{i2})$
- Sequential thinning:

$$X \oslash \{B_{(i)}\} = \left(\left(\left(X \oslash B_{(1)}\right) \oslash B_{(2)}\right) \dots \oslash B_{(n)}\right)$$

- Sequential thickening

$$X \odot \{B_{(i)}\} = \left(\left(\left(X \odot B_{(1)}\right) \odot B_{(2)}\right) \dots \odot B_{(n)}\right)$$

Golay Alphabet – Useful Sequences

- Golay alphabet represents a sequence of structuring elements defined in a given discrete planes
- For 8-connectivity distance measure, a sequence of 3×3 matrices can be defined from all permissible rotations of a structuring element
- One commonly used structuring element sequence L:

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ * & 1 & * \\ 1 & 1 & 1 \end{bmatrix}, L_2 = \begin{bmatrix} * & 0 & 0 \\ 1 & 1 & 0 \\ * & 1 & * \end{bmatrix}, \quad \ldots \ldots$$

- A total of eight elements; the other six elements are given by rotation

Sequential Thinning – Process

- One important property of sequential thinning is its ability to preserve connectivity of a given image – This is NOT true for maximal ball based skeleton
- Sequential thinning and thickening transformation shall converge to some image – the number of iteration depends on the shape of the image and the structuring elements used
- When two successive results of thinning (thickening) are identical, the process will stop
- The final thinned image will consist only of lines with one pixel width and isolated points

Sequential Thinning (L) Example

- The following images shows the results of five iterations (intermediate results) of sequential thinning with structuring element L

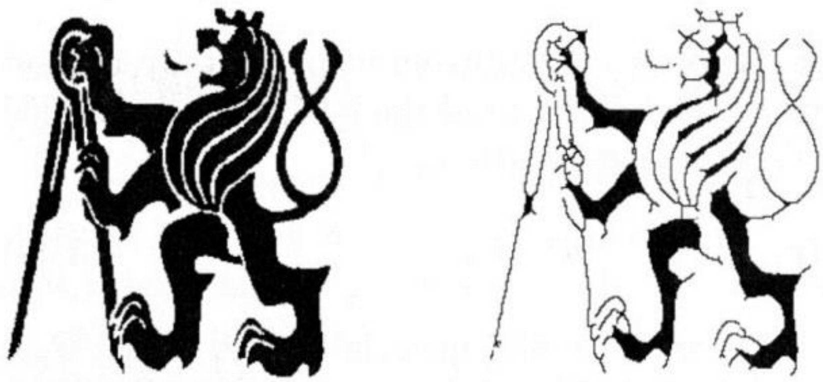


Figure 13.26: Sequential thinning using element L after five iterations.

Sequential Thinning (L) Example

- The following images shows the final results of sequential thinning with structuring element L – This is homotopic skeleton when idempotent state is reached

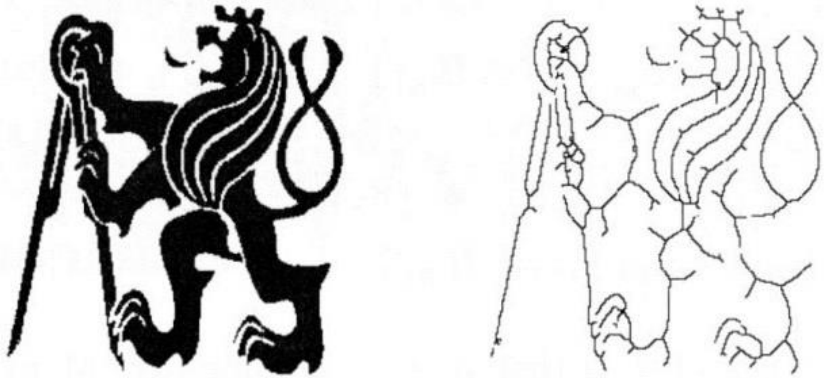


Figure 13.27: Homotopic substitute of the skeleton (element L).

Sequential Thinning with Element E

- Structural element L often generate a skeleton with jagged thin lines – due to sharp points on the boundary of the image (object)
- Applying sequential thinning with structural element E can smooth the jagged skeleton
- Structuring element sequence E:

$$E_1 = \begin{bmatrix} * & 1 & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \ldots\ldots$$

- A total of eight elements; the other six elements are obtained by rotation

Sequential Thinning (E) Example

- The following example shows the results of only five iterations – idempotent state may be reached via this sequential thinning – only closed contours will remain!

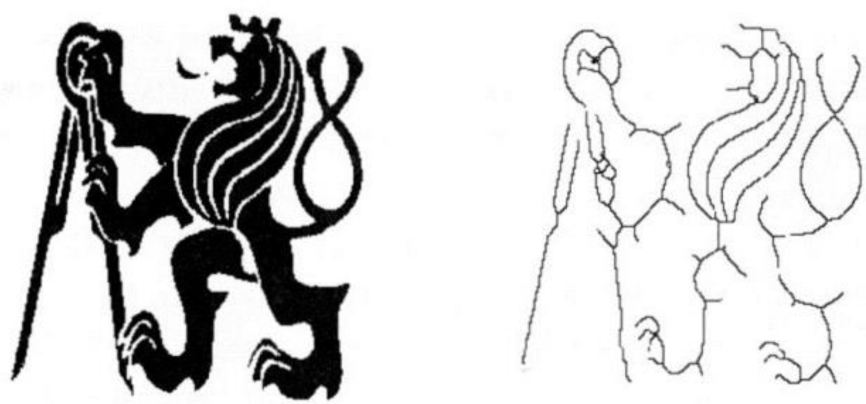
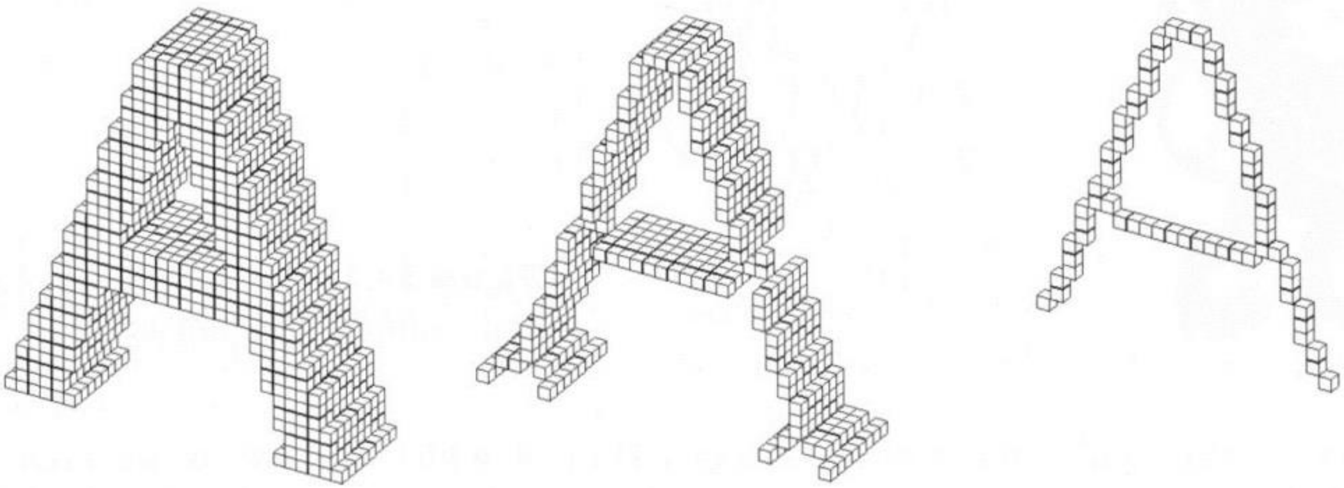


Figure 13.28: Five iterations of sequential thinning by element E.

Morphological Thinning in 3D



(a) (b) (c)

Figure 13.30: Morphological thinning in 3D. (a) Original 3D data set, a character A. (b) Thinning performed in one direction. (c) One voxel thick skeleton obtained by thinning image (b) in second direction. *Courtesy of K. Palágyi, University of Szeged, Hungary.*