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Impulse response analysis in nonlinear multivariate models

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Abstract

This paper presents a unified approach to impulse response analysis which can be used for both linear and nonlinear multivariate models. After discussing the advantages and disadvantages of traditional impulse response functions for nonlinear models, we introduce the concept of a generalized impulse response function which, we argue, is applicable to both linear and nonlinear models. We develop measures of shock persistence and asymmetric effects of shocks derived from the generalized impulse response function. We illustrate the use of these measures for a nonlinear bivariate model of US output and the unemployment rate.

Key words: Persistence; Impulse response functions; Threshold autoregressive models; Nonlinear vector autoregressions
JEL classification: C22; C32; E17; E37

1. Introduction

Many papers have attempted to measure the persistence effect of shocks on macroeconomic time series. Early work (e.g., Campbell and Mankiw, 1987) used univariate linear models and concluded that, at least at business cycle frequencies (e.g., eight to twelve quarters), shocks were persistent. The more recent work

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by Beaudry and Koop (1993) (BK hereafter), Potter (1995), and Pesaran and Potter (1994) (PP hereafter) has focused on nonlinear models. They make the argument that linear models are too restrictive. For example, they have a symmetry property which implies that shocks occurring in a recession are just as persistent as shocks occurring in an expansion. Hence, linear models cannot adequately capture asymmetries that may exist in business cycle fluctuations. The univariate nonlinear models cited above have found evidence that persistence varies over the business cycle. In particular they find that recessionary shocks are less persistent than are expansionary shocks. Other authors (e.g., Pesaran, Pierse and Lee, 1993; Lee and Pesaran, 1993; Blanchard and Quah, 1989) have extended the basic linear univariate literature to a consideration of linear multivariate models. They argue that a richer understanding of the persistence of shocks can be achieved by considering information from more than one macroeconomic time series (Blanchard and Quah, 1989) or from more than one sector of the economy (Lee and Pesaran, 1993).

In the present paper, we propose to combine these two strands of the literature and consider multivariate nonlinear models. Impulse response analysis of nonlinear multivariate models requires great care in the interpretation of results. As discussed in Potter (1994) nonlinear models produce impulse responses that are history- and shock-dependent. Further even in linear multivariate models there might be important interactions between shocks to different variables as examined in Lee and Pesaran (1993). We follow the approach of Potter and choose to consider the impulse response function as a random variable. We call this class the Generalized Impulse Response Function. In a recent paper Gallant, Rossi, and Tauchen (1993) (GRT hereafter) have examined similar issues although with a greater emphasis on providing measures of sampling uncertainty for impulse response functions produced from nonparametric estimates. We will provide some comparison of their approach with ours.

The remainder of the paper is organized as follows: Section 2 provides some background and develops the notation used in the rest of the paper. Section 3 discusses in some detail the pitfalls involved in the calculation of impulse responses in univariate and multivariate nonlinear models using traditional methods. Section 4 develops the properties of the generalized impulse response function. Section 5 describes the computational methods necessary to calculate the generalized impulse response function. Section 6 works through an application to a nonlinear model of U.S. GDP and unemployment.

2. Impulse response function analysis

2.1. Background

An impulse response function measures the time profile of the effect of a shock on the behaviour of a series. As such, we can think of an impulse response function as the outcome of a conceptual experiment. Such a basic idea can be implemented in a myriad of ways. We could, for example, conceptualize an experiment where we investigate the time profile of the effect of a positive *unit* shock hitting a series at time t, assuming no further shocks hit the system afterwards as compared with a baseline or benchmark profile suitably defined. At the same time, many other conceptual experiments corresponding to different assumptions about the size or timing of the shock are possible. Different ideas regarding the choice of the benchmark profile can also turn out to be important. The idea is very similar to Keynesian multiplier analysis, with the difference that the analysis is carried out with respect to shocks or 'innovations' of macroeconomic time series, rather than the series themselves (such as investment or government expenditure). The point of issue here is not whether a particular choice is right or wrong, but whether the choice is economically meaningful or interesting. In forthcoming sections, we discuss the pros and cons of several conceptual experiments which differ in their answers to the following:

1. What types of shocks (e.g., variable-specific or system-wide shocks) hit the system at time t?
2. What is the 'history' of the system at time $t-1$ (e.g., expansionary or recessionary) before the shock hits?
3. What future shocks are assumed to hit the system from $t+1$ to $t+n$?

2.2. Notation

To define various possible impulse response functions, we introduce some basic notations. For concreteness, we focus on nonlinear Markov multivariate models of order p, but our analysis readily extends to more general set-ups. In addition we assume that the random shock enters in an additive manner. The basic time series model we use is

$$Y_t = F(Y_{t-1}, \dots, Y_{t-p}) + H_t V_t, \tag{1}$$

where $F(\cdot)$ is a known function, Y_t is a $K \times 1$ random vector, V_t is $K \times 1$ vector of IID random disturbances, H_t is a $K \times K$ random matrix which is a function of $\{Y_{t-1}, \dots, Y_{t-p}\}$, and the shocks V_t have zero means and finite variances.

Many of our measures will be based on the conditional expectation of Y_{t+n}. Throughout the paper we will signify random variables with upper-case letters and realizations of these random variables by lower-case letters. Let Ω_{t-1} be the set containing information used to forecast Y_t. Because of the Markov assumption a sufficient set of information would be the realizations of Y_{t-j} for $j = 1, 2, \dots, p$. We use ω_{t-1} to denote a particular realization of Ω_{t-1}. The realization of the random variable at time $t+n$, y_{t+n}, will depend on ω_{t-1} and $\{v_t, \dots, v_{t+n}\}$. This is clear from iterating on (1). Throughout, we assume that the expectations of Y_{t+n} conditional on V_t and Ω_{t-1} (or any of their particular realizations, namely, v_t and ω_{t-1}) exist. The relevances of these conditional expectations to the impulse response analysis will be discussed in Section 4. But first we need to examine the impulse response analysis as it is traditionally carried out in the literature.

3. Traditional impulse response functions

3.1. Definition

We begin by considering the simplest type of impulse response function, which we refer to as the *traditional impulse response function.* This is the prevalent form of impulse response function used in the literature and is defined as the difference between two different realizations of y_{t+n} that are identical up to $t-1$. One realization assumes that between t and $t+n$ the system is hit only by a shock of size δ at period t (i.e., $V_t = \delta$), while the second realization, taken as the benchmark, assumes that the system is not hit by any shocks between t and $t+n$. Formally, we define a traditional impulse response function as

$$\begin{aligned} I_Y(n,\delta,\omega_{t-1}) = {} & \mathrm{E}[Y_{t+n} \mid V_t = \delta,\ V_{t+1} = 0,\ \dots,\ V_{t+n} = 0,\ \omega_{t-1}] \\ & - \mathrm{E}[Y_{t+n} \mid V_t = 0,\ V_{t+1} = 0,\ \dots,\ V_{t+n} = 0,\ \omega_{t-1}], \end{aligned}$$

for $n = 1, 2, 3, \dots$.

The traditional impulse response function is designed to provide an answer to the question: 'What is the effect of a shock of size δ hitting the system at time t on the state of the system at time $t+n$, given that no other shocks hit the system?' This definition of an impulse response function with all shocks except the current one 'turned off' accords well with the notion of a multiplier since it captures the properties of the model's propagation mechanism, and compares the value of y_{t+n} after the shock has occurred with its benchmark value where the economy has not been subject to any shocks. As the following examples demonstrate, the traditional impulse response function is more usefully applied to linear models than to nonlinear ones. In particular, it is worth noting that in the case of nonlinear models, the traditional impulse response function generally depends on ω_{t-1}, the particular history chosen as the basis for comparison of the two realizations. It also depends on the size and sign of the shock, δ, which must be selected by the researcher.

3.2. Univariate ARIMA models

Suppose that $\{Y_t\}$ is univariate and can be represented as an ARIMA (1,1,0):

$$\Delta Y_t = \phi \Delta Y_{t-1} + V_t,$$

where $|\phi| < 1$.

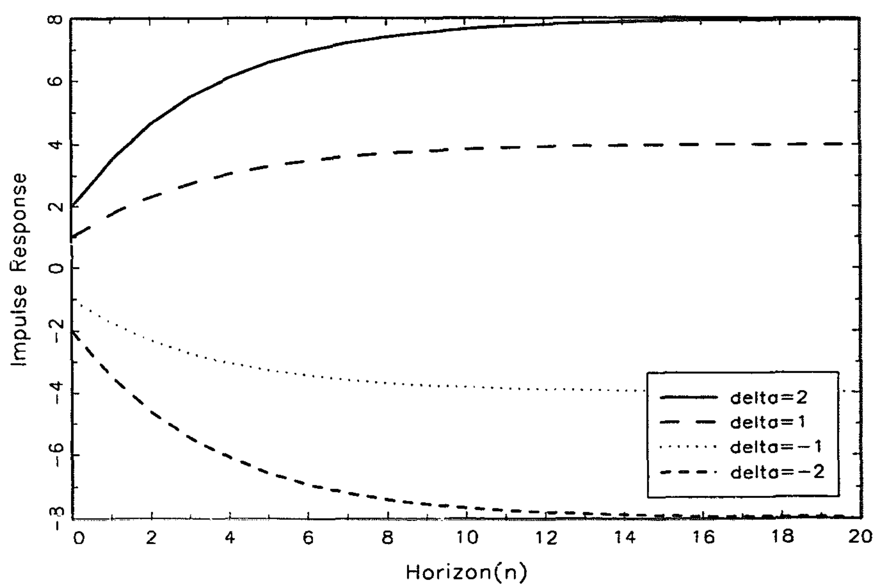

Fig. 1. ARIMA(1, 1, 0).

In the case of this model the traditional impulse response function for the level of Y_t takes the form[1]

$$I_Y(n, \delta, \omega_{t-1}) = \delta \frac{1 - \phi^{n+1}}{1 - \phi}.$$

Different values of δ only scale this measure, which is a restrictive property of all linear models in general. Thus, ARIMA models have impulse responses with a *symmetry property* (i.e., a shock of -1 has exactly the opposite effect of a shock of $+1$) and a *shock linearity property* (i.e., a shock of size 2 has exactly twice the effect of a shock of size 1). Further, the past does not effect the response in any way, a *history independence property.*

These properties are illustrated in Fig. 1, which plots $I_Y(n, \delta, \omega_{t-1})$ for an ARIMA(1,1,0) model with $\phi = 0.75$, for $\delta = -2, -1, +1, +2$. The above

[1] Note that the relationship between the impulse response for the first-differenced variable and the levels variable takes the form:

$$I_Y(n, \delta, \Omega_{t-1}) = I_Y(0, \delta, \Omega_{t-1}) + \sum_{j=1}^{n} I_{\Delta Y}(j, \delta, \Omega_{t-1}),$$

where $I_Y(\cdot)$ and $I_{\Delta Y}(\cdot)$ are the traditional impulse response functions for Y and ΔY, respectively. Since the error is additive in the present example, $I_Y(0, \delta, \Omega_{t-1}) = \delta$, so that the impulse response for the level variable is just a summation of the impulse response for the first-differenced variable.

result can be readily generalized to the ARIMA(p, 1, q) scheme

$$\phi(L)\Delta Y_t = \theta(L)V_t,$$

where $\phi(L)$ and $\theta(L)$ are polynomials in the lag operator.

Of particular interest is the long-run persistence effect of shocks on $\{Y_t\}$ a natural measure of which is an impulse response at an infinite horizon. The limit of the traditional impulse response function for the general ARIMA is

$$I_Y(\infty, \delta, \omega_{t-1}) = \frac{\theta(1)}{\phi(1)}\delta. \tag{2}$$

When δ is normalized to one we obtain the measure used in Campbell and Mankiw (1987). This simple example demonstrates that traditional impulse response functions in linear univariate models, when suitably scaled, are neither shock- nor histroy-dependent. These properties do not, however, carry over to nonlinear or multivariate models, as illustrated in the following examples. [For a discussion of these issues as they pertain to a particular nonlinear univariate model, see Beaudry and Koop (1993).]

3.3. A simple nonlinear univariate model

Consider the nonlinear AR(1) model

$$Y_t = \alpha Y_{t-1} - \beta Y_{t-1}^2 + V_t,$$

where Y_t is a scalar. The properties of the V_t sequence will be discussed shortly. If $\beta \neq 0$ and $-2 \leqslant \alpha < 2$, then this model can be rewritten as

$$X_t = \lambda X_{t-1}(1 - X_{t-1}) + U_t,$$

where $\lambda = 2 - \alpha$, $U_t = (\beta/(2 - \alpha))\, V_t$, and Y_t and X_t are related according to

$$Y_t = \frac{\alpha - 1}{\beta} + \frac{2 - \alpha}{\beta} X_t.$$

If the error term V_t is set to zero and X_t is restricted to the interval [0, 1], then we have the well-known logistic map (see, Frank and Stengos, 1988, for an introduction accessible to economists). If one uses the concept of the traditional impulse response function, then the benchmark and perturbed realizations correspond to two deterministic trajectories of the map produced by different initial conditions.

The behaviour of the logistic map depends crucially on the value of λ. If $\lambda > 4$, the system is explosive, so we restrict attention to $0 < \lambda \leqslant 4$. Even within this region the behaviour of the system can vary dramatically. For example, if we consider gradually increasing λ from 3 to 4, we move through a wide variety of dynamics: from a stable two-period cycle to an unstable one, from a stable four-period cycle to an unstable one, etc. For $\lambda > 3.57$ the periodicity of the cycle becomes infinite and the time path of the series depends crucially on starting values.

In order to introduce a stochastic error term that maintains the stability of the logistic map the distribution of the V_t must be bounded, and the bounds must vary over time. The exact form of the bounding is complicated but depends on λ and x_{t-1}; and since x_{t-1} depends on the previous realizations, v_{t-j}, $j = 1, \ldots, t$, the bounds are dependent.

Fig. 2 plots traditional impulse response functions for $\lambda = 3.3$, $x_{t-1} = 0.5$, and $\delta = +0.1, -0.1$. Clearly, the impulse responses are not symmetric. Furthermore, the two impulse response functions have different properties. Even though both exhibit periodic behaviour, $I_Y(n, \delta = -0.1, x_{t-1} = 0.5)$ goes to zero as n increases (i.e., the perturbed and benchmark realization become the same), whereas $I_Y(n, \delta = 0.1, x_{t-1} = 0.5)$ settles into a periodic pattern that does not dampen with n. This example illustrates a major difficulty with only considering one realization of the future shocks (in this case the zero future shocks): the constructed stochastic process is stationary but the traditional impulse response function does not converge to zero.

3.4. A self-exciting threshold autoregressive (SETAR) model

Consider the following simplified version of the model used in Potter (1995):

$$\Delta Y_t = \rho_1 \Delta Y_{t-1} + \rho_2 \Delta Y_{t-1}\, 1(\Delta Y_{t-1} \geqslant 0) + V_t,$$

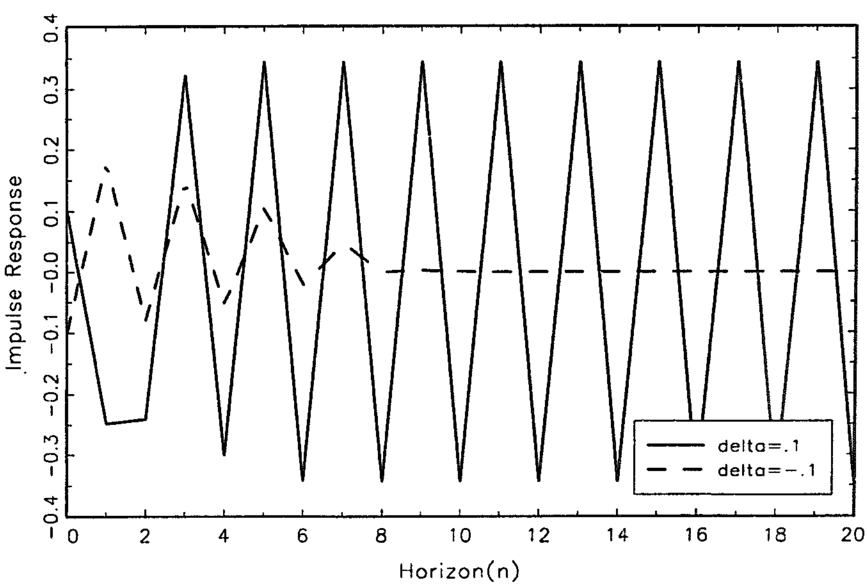

Fig. 2. Logistic map.

where $1(\cdot)$ is an indicator function with the property that $1(A) = 1$ if the event A occurs, and zero otherwise.

Similar models have been used for real US GNP (for example, Tiao and Tsay, 1994). Note that this model allows for different slopes depending on whether GNP growth in the previous period was positive or negative. Defining $\rho = \rho_1 + \rho_2$, assuming $0 < \rho_1 < 1$ and $0 < \rho < 1$. traditional impulse responses for the level of Y_t will depend on both δ and t. For this model there are four possible combinations of the past and present to consider:

1. $\delta \geqslant 0$ and $\Delta y_{t-1} \geqslant 0$.
2. $\delta < 0$ and $\Delta y_{t-1} \geqslant 0$.
3. $\delta \geqslant 0$ and $\Delta y_{t-1} < 0$.
4. $\delta < 0$ and $\Delta y_{t-1} < 0$.

These four cases can be collapsed into two possible outcomes for the particular history, $\Delta y_{t-1} = 0$. The traditional impulse response for the level of Y is then given by

$$I_Y(n, \delta, \Delta y_{t-1} = 0) = \begin{cases} \delta \dfrac{1 - \rho^{n+1}}{1 - \rho} & \text{if} \quad \delta \geqslant 0, \\ \delta \dfrac{1 - \rho^{n+1}}{1 - \rho_1} & \text{if} \quad \delta < 0. \end{cases}$$

Although this closed form solution is convenient, it illustrates the possible dangers of choosing a particular history and the bias introduced by setting all future shocks to zero. The latter bias means that the threshold effect is only active in the present and is inactive independent of the value of δ for the rest of the time even though the initial condition is exactly on the boundary between the regimes. Figs. 3 and 4 show the effect of $\delta = 2,\ +1,\ -1, 2$ shocks in the more general case where $\Delta y_{t-1} = -5, 2$ with $\rho_1 = 0.25$, $\rho = 0.75$. The figures show clearly the asymmetries across histories and across shocks in comparison to Fig. 1. However, one would expect that it understates the nonlinearity in cases where the perturbed value of ΔY is close to zero because of the use of the zero future realizations.

Extensions of the above results to more general SETAR models are harder to obtain analytically. However, as we show in the empirical section, simulation methods can be used to calculate impulse responses and measures of persistence in a simple way for a wide variety of nonlinear models even when one considers integrating out the effect of future shocks.

The examples so far illustrate how the introduction of nonlinearity can cause the intrinsic linear properties of shock and history independence to be lost and, thus, how the traditional impulse response function can depend on t and δ in addition to n. The following example illustrates how switching to a multivariate framework introduces additional problems into the analysis.

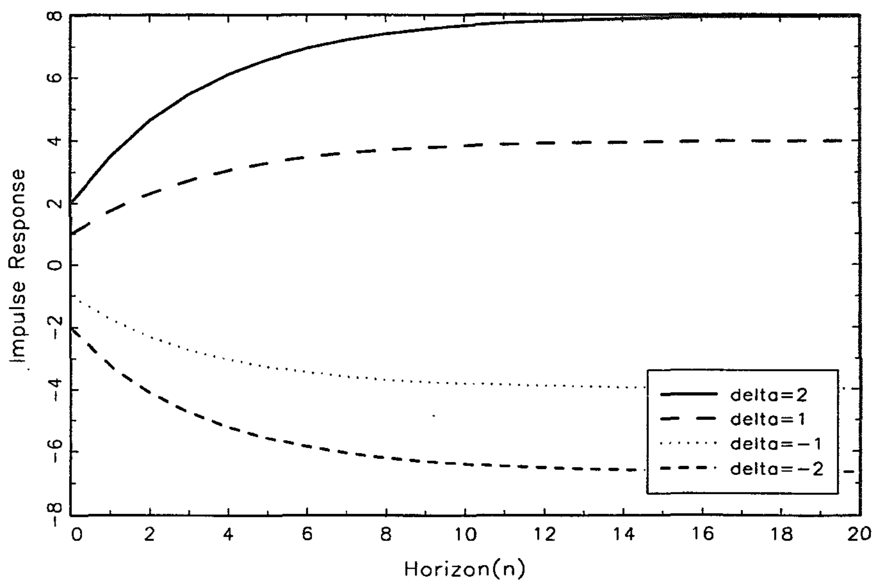

Fig. 3. SETAR positive history.

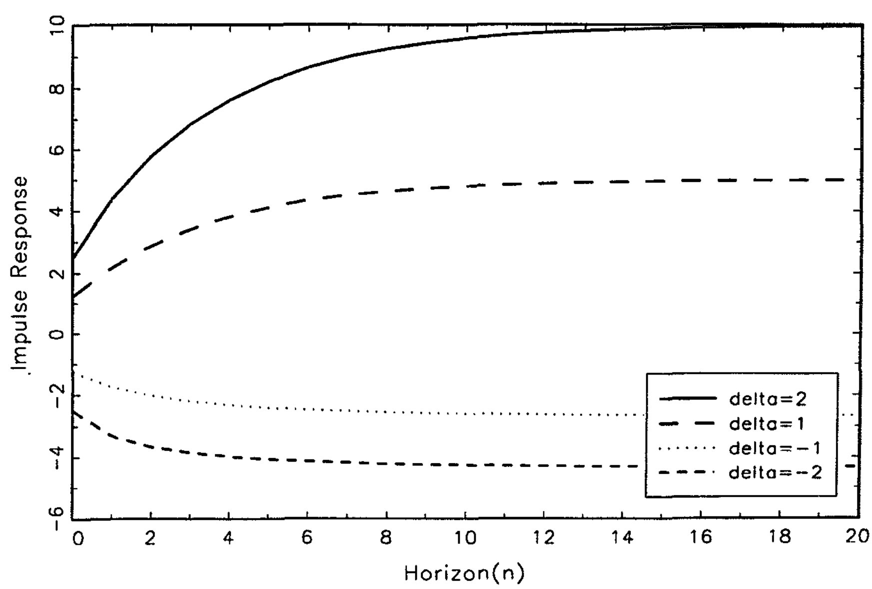

Fig. 4. SETAR negative history.

3.5. A bivariate linear AR

Consider the bivariate linear model

$$Y_t = AY_{t-1} + V_t,$$

where A is 2×2 matrix.

The focus of interest in multivariate models is the same as that in univariate models, namely the time profile of the effect of shocks on $Y_{t+1}, Y_{t+2}, \ldots$. However, in the multivariate case it is not clear what type of shock is of interest. For example, we will probably not be interested in the effect of a shock to V_{1t} since its interpretation in unclear. Given that V_{1t} and V_{2t} are generally correlated, the shock $(v_{1t}, v_{2t})' = \delta' = (1, 0)$, for example, is not likely to be economically relevant or meaningful; not the sort of shock that the Data Generation Process will often generate, in other words. Put another way, V_{1t} will not only have contemporaneous effects on Y_{1t} but on Y_{2t} as well. Hence, it is not appropriate to entertain perturbations in V_{1t} while keeping V_{2t} fixed. We refer to this property as the *composition effect*, since it highlights the fact that the impulse response function depends on the composition of shocks. One common solution to this problem is to transform the model so that the covariance matrix of the transformed shocks is a diagonal matrix. However, in general, such a transformation is not unique. In the absence of other a priori information (e.g., a logical causal ordering among the variables in the VAR), there is an arbitrariness implicit in such a transformation which can lead to difficulties in interpretation of the impulse response functions (see Section 4.3 below).

3.6. A simple multivariate SETAR model

Consider now a multivariate extension of the univariate SETAR model above. To provide some background, let $Y_t = (\Delta X_t, L_t)'$, where X_t is the logarithm of real GDP and L_t is the employment rate. The following simple model allows for different adjustment coefficients depending on whether one of the variables is above or below a threshold:

$$Y_t = AY_{t-1} + BY_{t-1}1(\Delta X_{t-1} \geqslant 0) + V_t,$$

where A and B are 2×2 matrices of adjustment coefficients with elements a_{ij} and b_{ij}. Assume furthermore that $a_{ij} \geqslant 0$ and $a_{ij} + b_{ij} \geqslant 0$. Formulae for traditional impulse response functions depend on the signs and sizes of both shocks, $\delta = (\delta_1, \delta_2)'$, as well as on $\Delta X_{t-1}, L_{t-1}$. Clearly, the composition effect discussed above continues to be present here. For example, if the shock, $\delta = (\delta_1, \delta_2)'$, and $\Delta X_{t-1}, L_{t-1}$ are all positive, then the 2×1 vector of traditional impulse responses is given by

$$I_Y(n, \delta, X_{t-1}) = (A + B)^n \delta,$$

which depends on both δ_1 and δ_2. Alternatively, the past and present value of output growth could be positive for both the baseline and perturbed paths but a decrease in employment could be associated with future negative output growth and trigger the threshold effect.

We now summarize the issues involved in the four types of model we have examined:

Model types	Conceptual problems
Univariate linear	*None*
All nonlinear models	*Treatment of the future*
Univariate nonlinear	*History and shock dependence*
Multivariate linear	*Composition dependence*
Multivariate nonlinear	*History, shock, composition dependence*

4. Generalized impulse response functions

4.1. Definition

The Generalized Impulse Response Function (*GI*) is designed to solve the problems categorized above. The problem of treatment of the future is dealt with by using the expectation operator conditioned on only the history and/or shock. That is, the future shocks are averaged out. Thus, the response constructed is an average of what might happen given the present and past. The natural baseline for the impulse response function is then defined as the conditional expectations, given only the history. This leaves the question of how to perturb the present to produce information on the dynamics of the system.[2] First consider the *GI* for the case of an arbitrary current shock, v_t, and history, ω_{t-1},

$$GI_Y(n, v_t, \omega_{t-1}) = \mathrm{E}[Y_{t+n} | v_t, \omega_{t-1}] - \mathrm{E}[Y_{t+n} | \omega_{t-1}] \quad \text{for} \quad n = 0, 1, \ldots. \tag{3}$$

The *GI* in this case is a function of v_t and ω_{t-1}. It is natural to treat v_t and ω_{t-1} as realizations from the same stochastic process that generates the realizations of $\{Y_t\}$. Thus, we can consider the *GI* defined above to be the realization of a random variable defined by[3]

$$GI_Y(n, V_t, \Omega_{t-1}) = \mathrm{E}[Y_{t+n} | V_t, \Omega_{t-1}] - \mathrm{E}[Y_{t+n} | \Omega_{t-1}]. \tag{4}$$

[2] The definitions below strictly only apply to stochastic time series where 'shocks' have a well-defined meaning. Deterministic chaotic time series are best treated by different methods.

[3] A similar idea can be applied to any function of $\{Y_t\}$ for which the conditional expectations, appropriately defined, exist.

Eq. (4) is the difference between two conditional expectations which are themselves random variables. Thus, $GI_Y(n, v_t, \omega_{t-1})$ represents a realization of this random variable.

Various conditional versions of the generalized impulse response function can be defined. For example, one could condition on a particular shock, and treat the variables generating the history, namely, $Y_{t-1}, Y_{t-2}, \dots, Y_{t-p}$ in the case of the Markov model (1), as random:

$$GI_Y(n, v_t, \Omega_{t-1}) = \mathrm{E}[Y_{t+n} \mid v_t, \Omega_{t-1}] - \mathrm{E}[Y_{t+n} \mid \Omega_{t-1}]. \tag{5}$$

Alternatively, one could condition on a particular history, ω_{t-1}, and treat the GI as a random variable in terms of V_t:

$$GI_Y(n, V_t, \omega_{t-1}) = \mathrm{E}[Y_{t+n} \mid V_t, \omega_{t-1}] - \mathrm{E}[Y_{t+n} \mid \omega_{t-1}]. \tag{6}$$

One could also consider conditioning on particular subsets of the history or the shock. For example, one might condition on all histories where the economy was in recession in the most recent period or on positive shocks:

$$GI_Y(n, A, B) \quad \text{conditional on} \quad v_t \in A, \quad \omega_{t-1} \in B.$$

The following properties are immediate:[4]

1) *Mean of unconditional GI*

$$\mathrm{E}[GI_Y(n, V_t, \Omega_{t-1})] = \mathbf{0}_K.$$

2) *Mean of GI conditional on history*

$$\mathrm{E}[GI_Y(n, V_t, \omega_{t-1})] = \mathbf{0}_K.$$

3) *Mean of GI conditional on shock*

$$\mathrm{E}[GI_Y(n, v_t, \Omega_{t-1})] = \mathrm{E}[Y_{t+n} \mid v_t] - \mathrm{E}[Y_{t+n}].$$

4.2. Comparison to Gallant, Rossi, and Tauchen approach

GRT concentrate on a measure derived from something similar to the mean of the GI conditional on particular shocks, except that they consider a different baseline forecast. They define the baseline forecast conditional on information up to time t as in the traditional impulse response function. They concentrate on the mean of these baseline forecasts in much of their work, which under stationarity is the unconditional mean. They then perturb a collection of histories by a shock of fixed size, δ, to generate differences in forecasts. The

[4] Below the expectations operator is used to integrate out the appropriate random variables. In 1) the random variables are the shock and history. In 2) the random variables are the shocks. In 3) the random variables are the histories up to time $t-1$.

average across these histories is then compared with the mean of the baseline forecasts. Since the effect of averaging across histories in our approach is to render the baseline equal to the unconditional mean, the differences in underlying baselines are averaged out in the mean of the *GI* conditional on a shock.

Although the average value can be of interest in many circumstances, the perspective taken in this paper is that it can hide a great deal of important information. For example, if shocks have asymmetric effects over the business cycle, then averaging across phases of the bussiness cycle will tend to weaken or hide the evidence of asymmetry. Further, the impulse response of interest is likely to be conditional, such as, to negative shocks in recessions. GRT, in addition to the mean (conditional on the shock), also present the 'profiles' of impulse response functions that produce the mean. In this paper we follow the more formal approach of directly considering the joint, conditional, and marginal densities of the GI. We do this by treating the *GI* as a random vector on the underlying probability space of the time series under consideration.

There is also an important difference between our approach and GRT with respect to the 'zero shock' (and, by continuity, small shocks) at time t, that is, when the baseline forecast exactly predicts the value of Y_t. GRT define the baseline for the impulse response function conditonal on information up to time t in constructing their profiles. Hence, when the initial perturbation is zero all future responses will be zero. In the case of the generalized impulse response function a zero initial response does not necessarily imply that all future responses are zero (see Potter, 1994, for a more thorough discussion). Thus, the properties of the conditional *GI* for a fixed shock could be quite different to those of the measure proposed by GRT. Further, in rational expectations models our definition of the generalized impulse response function is equivalent to the revision in forecast function. For example, the revision in forecast function determines the movement in stock prices in response to news about dividends or the movement in consumption in response to news about income.

In order to produce the appropriate distribution for the *GI* the shocks and histories must be drawn from the appropriate joint distributions. For the histories this will be from the $p \times K$-dimension marginal distribution of the time series. For the shocks this will be from the K-dimension distribution of the shocks.

The random variable perspective on the *GI* produces one important practical difference between the GRT approach and ours. In the GRT approach the investigator has to specify a 'typical' shock by consideration of scatterplots of the shocks. Instead we generate the shocks directly from either the empirical distribution of innovations in the time series or by estimating parameters of an assumed distributional form. One can then decide to focus on a particular aspect of marginal density of a conditional *GI*. Further, as it is discussed below, the random variable perspective leads to a natural and simple solution to the compositional problem.

In order to define measures of persistence and magnitude of response directly from the densities we concentrate on the dispersion of the generalized impulse response function. As shown in Potter (1994) the dispersion of the GI is related to a notion of long-run persistence of the time series. In particular if the time series is stationary, then as the horizon goes to infinity the GI random vector converges to a vector of zeros (with probability one). That is, under stationarity the baseline forecast and perturbed forecast converge to the same value for any history. Alternatively, if the time series under consideration are random walks, then the dispersion of the GI remains constant as the horizon is increased. These simple properties can be used to build up measures of magnitude of response based on dispersion at various horizons.

The unconditional GI can hide much interesting behaviour when an integrated series is examined in its first differences, but the responses are cumulated up to provide information on the response of the level of the series. Thus, conditional versions of the GI must be used to determine if, for example, 'positive shocks are more persistent than negative shocks'. One can assess the persistence of shocks for a particular history ω_{t-1}, the persistence of a particular shock v_t for all possible histories or for a particular subset of histories and shocks.

There is no unambigous measure of dispersion, but in cases where random variables have mean zero, second-order stochastic dominance has useful properties. In this paper we concentrate on presenting the density functions of the GI of individual time series at various horizons. One density has more dispersion than another in second-order stochastic dominance sense if it has 'fatter tails' and crosses the other density exactly two times. If a time series is not persistent, then its density should quickly narrow to a spike. Thus, the density of the GI as the horizon gets large is second-order stochastically dominated by the GI at the initial horizon. Time series that are persistent will have a GI whose densities have dispersion at all horizons. In particular if the propagation mechanism of the time series magnifies shocks, then the densities of the GI at future horizons will second-order stochastically dominate the density of the GI at horizon zero.

4.3. Application to univariate and multivariate linear models

All the measures defined in (3) to (6) produce the same expressions for the generalized impulse response functions when applied to linear univariate models. For the general linear model,

$$Y_t = \sum_{i=0}^{\infty} a_i V_{t-i}, \qquad V_t \sim \mathrm{IID}(0, 1), \tag{7}$$

where $\{a_i\}$ are absolutely summable. Using (4), we have

$$GI_Y(n, V_t, \Omega_{t-1}) = a_n V_t,$$

which is independent of Ω_{t-1} (and clearly any of its realizations, ω_{t-1}). The dependence of GI on V_t, the size of the current shock, is also proportional and can be readily dealt with by appropriate scaling of the GI function. Once the GI function is scaled by V_t, it reduces to a_n, the familiar traditional impulse response function.

Consider now the following multivariate version of (7)

$$Y_t = \sum_{i=0}^{\infty} A_i V_{t-i},$$

where the $K \times K$ matrices A_i are assumed to be square-summable.[5] For this case,

$$GI_Y(n, V_t, \Omega_{t-1}) = A_n V_t,$$

and, once again, does not depend on the history, Ω_{t-1}. However, since V_t is now a vector, its effect on the generalized impulse response function cannot be eliminated by an 'appropriate' scaling of the variables. Hence, the impulse response analysis is subject to the composition problem. In the linear case one can derive the distribution of the GI from the distribution of V_t. For example, in the case where $V_t \sim \mathrm{N}(0, \Sigma)$, then $GI_Y(n, V_t, \Omega_{t-1}) \sim \mathrm{N}(0, A_n \Sigma A_n')$, and the GI is fully characterised by the variance–covariance matrix $A_n \Sigma A_n'$, $n = 0, 1, \ldots$.[6]

The variances along the diagonal of $A_n \Sigma A_n'$ measure the effect of shocking the system at time t, by a random vector V_t, and then averaging the squares of the GI component by component against the joint distribution of these system-wide shocks. A pseudo-traditional impulse response function for linear models can be constructed by taking the square root of the diagonal of $A_n \Sigma A_n'$ for each horizon. This solution deals with the composition problem by ignoring it through focusing on the effect of system-wide shocks.

The GI also lends itself naturally to a solution of the compositional problem without the use of *a priori* theory in both the case of linear and nonlinear models. We can define the generalized impulse response function to be conditional not on all the shocks at time t but on just one of them. That is, we can consider fixing the ith shock from the vector of all shocks, V_t, and then integrating out the effects of the other shocks at time t given its value, v_{it},

$$GI_Y(n, v_{it}, \Omega_{t-1}) = \mathrm{E}[Y_{t+n} \mid V_{it} = v_{it}, \Omega_{t-1}] - \mathrm{E}[Y_{t+n} \mid \Omega_{t-1}], \tag{8}$$

where $\mathrm{E}[Y_{t+n} | V_{it} = v_{it}, \Omega_{t-1}]$ means that one is taking the expectations conditional on the information set Ω_{t-1} and for a fixed value of the ith shock at

[5] A necessary condition for A_i to be square-summable is given by $\sum_{i=0}^{\infty} \mathrm{tr}(A_i A_i') < \infty$.

[6] A scaled version of the variance measure, called 'persistence profiles', is also proposed in Lee and Pesaran (1993) and applied by Pesaran and Shin (1995) to analyse the speed of convergence of cointegrating relations to equilibrium.

time t, while integrating out all other contemporaneous and future shocks. Note that in many cases the conditional expectations function, $\mathrm{E}[V_t | V_{it}]$, could be nonlinear.

In order to illustrate this use of the generalized impulse response function, consider tne case where the vector of random shocks, V_t, is jointly normally distributed. Under this assumption the conditional expectation of the shocks is a linear function of v_{it}:

$$\mathrm{E}[V_t \mid V_{it} = v_{it}] = \eta_i \sigma_{ii}^{-1} v_{it},$$

where $\sigma_{ii} = \mathrm{E}[V_{it}^2]$ and $\eta_i = \mathrm{E}[V_t V_{it}]$. The generalised impulse response of the effect of a shock to the ith disturbance term at time t on Y_{t+n} for the multivariate linear model is then given by

$$GI(n, v_{it}, \Omega_{t-1}) = \left(A_n \eta_i / \sqrt{\sigma_{ii}}\right)\left(v_{it} / \sqrt{\sigma_{ii}}\right).$$

Scaling the GI by $v_{it}/\sqrt{\sigma_{ii}}$, we obtain the effect of a 'unit' shock (i.e., a shock of one standard deviation in size) to the ith disturbance term on Y_{kt+n}, namely

$$A_n \eta_i e_k / \sqrt{\sigma_{ii}},$$

where e_k is a selection vector with its kth element equal to unity and zeros elsewhere.[7]

This conditional GI appropriately takes account of the dependence between the different (Gaussian) shocks, and reduces to the traditional impulse responses produced by a Cholesky factorizataion only in the case of a diagonal covariance matrix. Also, unlike the orthogonalised impulse responses obtained using a Cholesky factorization, the GI responses $A_n \eta_i e_k / \sqrt{\sigma_{ii}}$ are unique and are not affected by reordering of the variables in Y_t.

The above ideas can be readily extended to nonlinear multivariate models, but their implementation requires Monte Carlo integration (see Section 5 below). It is easy to extend this analysis to a subset of shocks. However, there is still a compositional problem to deal with. An obvious solution is again to use the random variable interpretation and consider the distributional characteristics of the random variable produced by the subsystem shock,

$$\mathrm{E}[V_t \mid \{v_{it}\}],$$

where $\{v_{it}\}$ is a subset of the K shocks. This allows an investigator with some prior information about a subset of the shocks to conduct impulse response analysis without having to impose restrictions on the relation of this subset to the rest of the shocks or amongst the remaining shocks.

[7] By the 'unit' shock, we mean a shock with size equal to one standard deviation.

5. Monte Carlo techniques for computation of *GI*

In what follows we consider the computation of the generalized impulse response function assuming that the nonlinear model is known, that is, we ignore sampling variability. The basic object we require is a conditional expectation. Granger and Teräsvirta (1993, Ch. 8) give an excellent description of various methods of computing the conditional expectation. In this section we describe the steps involved in computing the conditional expectations in the *GI* function by means of Monte Carlo integration.

1. Pick a history and shock, ω_{t-1}, v_t, by some combination of:

a) Using the observed values of the time series for ω_{t-1} but randomly drawing v_t from the joint density of V_t. There are two main ways one can do this:
   i) Under the assumption that V_t is Gaussian or some other parametric form, use standard methods to draw from its joint density.
   ii) When Gaussianity or some other complete parametric description is not assumed, bootstrap methods can be used.
      A. If the all contemporaneous dependence across shocks is captured in a constant covariance matrix for the innovations, estimate the covariance matrix. Then one can transform the observed V_t to contemporaneous independence by using the inverse of a Cholesky factorization of the estimated covariance matrix. This creates a sample of size $K \times T$ to bootstrap from. The individual independent draws are then grouped into K vectors and the estimated Cholesky factor used to return the dependence.[8]
      B. If one does not want to make any assumptions about the form of dependence but has some knowledge of conditional heteroscedasticity, then one can make draw weighted shocks from the *joint* empirical distribution. That is, given T observations from the joint distribution of V_t use the conditional weights

$$\left\{\psi_t(\omega_{t-1}) \colon 0 \leqslant \psi_t(\omega_{t-1}) \leqslant 1,\ \sum_{t=1}^{T} \psi_t(\omega_{t-1}) = 1,\ \forall \omega_{t-1}\right\}$$

to define the resampling scheme. In the case of innovations that are independent from the history of the time series, the weights would be constant across different histories and set equal to $1/T$. In practice this method is most useful for the case of threshold models, where the regimes split the innovations into separate groups and one can bootstrap the shocks within each regime.

[8] For more details see the Appendix to Pesaran and Shin (1995).

(b) Use the time series model (functional form and shocks generated using one of the methods above) to produce simulated realizations that can be used to form draws from the joint distribution of $Y_{t-1}, \ldots, Y_{t-p}$.

2. For a given horizon N, randomly sample $(N+1)\times R$ values of the (K-dimensional) innovation.

3. Use the first N random shocks (obtained under step 2) to compute the realization, $y^0_{t+n}(v_t, \omega_{t-1})$ for $n = 1, 2, \ldots, N$, iterating on the nonlinear time series model under consideration from the given initial conditions ω_{t-1}, v_t.

4. Use the same draw of N random shocks plus one additional draw of the random shock to produce a realization, $y^0_{t+n}(\omega_{t-1})$, of the time series for $n = 0, 1, \ldots, N$ based on the initial condition ω_{t-1}.

5. Repeat steps 3 and 4 R times and form the averages for each individual component:

$$\bar{y}_{R,t+n}(v_t,\omega_{t-1}) = \frac{1}{R}\sum_{i=0}^{R-1} y^i_{t+n}(v_t,\omega_{t-1}), \qquad n = 1, 2, \ldots, N,$$

$$\bar{y}_{R,t+n}(\omega_{t-1}) = \frac{1}{R}\sum_{i=0}^{R-1} y^i_{t+n}(\omega_{t-1}), \qquad n = 0, 1, \ldots, N.$$

As $R \to \infty$, by the Law of Large Numbers these averages across individual Monte Carlo replications will converge to the conditional expectations, namely $\mathrm{E}[Y_{t+n} \mid v_t, \omega_{t-1}]$ and $\mathrm{E}[Y_{t+n} \mid \omega_{t-1}]$, required in the definition of the generalized impulse response function, (3).

6. Take the difference between the two averages to form a Monte Carlo estimate of the GI, where $\bar{y}_t(v_t, \omega_{t-1})$ merely involves one realization. Under certain monotonicity conditions on the function $F(\cdot)$ the use of 'common random variables' in steps 3 and 4 is efficient for minimizing the sampling variance of the difference (see Ripley, 1987). Even when these conditions are not satisfied, the approach has the intuitive appeal of presenting the average over realizations which differ only at time t. That is, a version of traditional impulse response function where the future is switched on.

7. Repeat steps 1 to 6 a sufficient number of times to allow accurate estimation of aspects of interest of the GI random vector. Again, as the number of repetitions increases pointwise convergence will be guaranteed by the Law of Large Numbers. In practice, nonparametric kernel methods can be used to accurately estimate the densities from a smaller number of replications.

In practice there is uncertainty about the choice of the nonlinear model and the joint density of the innovations. A strength of the approach of GRT is the direct way in which their methods address these issues. It would be possible to allow for parameter uncertainty within the class of nonlinear models under consideration by constructing generalized impulse response functions for different draws from a distribution reflecting the parameter uncertainty (see Koop, 1994, for a Bayesian approach).

6. Empirical application

6.1. A bivariate model of U.S. output and the unemployment rate

An enormous variety of nonlinear forms exists in the literature. However, much economic theory implies that economic behaviour should vary over the business cycle. BK, for instance, argue that, if positive innovations in real output reflect technological progress and technological regress is unlikely, then positive shocks should have a permanent effect while recessionary shocks should have only a temporary effect. Empirical evidence from a univariate model for real U.S. GNP is presented by them which supports this contention. However, BK use only traditional impulse response functions. PP extend the analysis of BK to allow for nonlinear effects in expansions as well as recessions. PP find effects similar to BK in recessions, but also additional nonlinearities during high-growth periods. Both analyses are univariate in nature, and it is possible that the introduction of an additional conditioning variable could weaken or alter the nonlinearity in important ways. In particular, both papers find evidence that persistence in the flow variable output varies over the business cycle. How robust is this conclusion to the introduction of a stock variable such as unemployment?

We work with a bivariate nonlinear model of 100 times the change in the log of real GDP growth (ΔX_t) and the unemployment rate (U_t). The nonlinearity of the model considered is due to the inclusion of a variables which measure the current depth of the recession (CDR_t) and the extent of overheating (OH_t) in the economy. In BK, the current depth of recession variable was defined as: $CDR_t = \max_{j \geqslant 0}\{X_{t-j}\} - X_t$, where X_t is the log level of output. As long as the economy is growing this variable will be zero, but it measures deviations from past highs during recessions.[9] We use the extension of BK's work to floors and ceilings developed by PP. They define three regimes, a corridor regime when the growth rate is 'normal', a ceiling regime when the growth rate is fast (the idea is that the economy is overheating, as measured by the variable OH_t), and a floor

[9] A plot of this variable is given in Fig. 1 of BK.

regime when output growth has been low. The regimes and variables measuring depth of recession and overheating are constructed as follows:

$$F_t = \begin{cases} 1(\Delta X_t < r_F) & \text{if} \quad F_{t-1} = 0, \\ 1(CDR_{t-1} + \Delta X_t < 0) & \text{if} \quad F_{t-1} = 1, \end{cases}$$

$$CDR_t = \begin{cases} (\Delta X_t - r_F) F_t & \text{if} \quad F_{t-1} = 0, \\ (CDR_{t-1} + Y_t) F_t & \text{if} \quad F_{t-1} = 1, \end{cases}$$

$$C_t = 1(F_t = 0) 1(\Delta X_t > rc) 1(\Delta X_{t-1} > rc),$$

$$OH_t = C_t\{OH_{t-1} + \Delta X_t - rc\},$$

$$COR_t = F_t + C_t,$$

where $1(A)$ is an indicator function taking the value unity when A holds and zero otherwise, r_F, r_C are thresholds defining the regimes, and where $F_{t-1} = 1$ means that the floor regime is active, $C_{t-1} = 1$ that the ceiling regime is active, and $COR_t = 0$ that the corridor regime is active.

In order to introduce the nonlinear effects we start with a linear model. If Y_t denotes $(\Delta X_t, U_t)'$ and $A(L)$ is a matrix polynomial in the lag operator $A_0 + A_1 L + A_2 L^2 + \cdots + A_p L^p$, then the linear model is given by

$$Y_t = a + A(L) Y_{t-1} + V_t, \tag{9}$$

where a is a 2×1 vector of intercepts.

Now consider including the variables CDR_{t-1}, OH_{t-1} in the linear model. These will introduce nonlinear effects in a particular simply manner given time series on F_{t-1}, C_{t-1}. This is the route we follow in this paper to illustrate our techniques.

Defining $Z_t = (CDR_t, OH_t)'$. The nonlinear model is given by

$$Y_t = a + A(L) Y_{t-1} + B Z_{t-1} + H_t V_t, \tag{10}$$

where $H_t = \{H_0 1(COR_{t-1} = 0) + H_1 F_{t-1} + H_2 C_{t-1}\}$, B, H_i, $i = 0,\ 1,\ 2$, are 2×2 matrices of constant coefficients, and $V_t \sim \text{IID}\ (0, I_2)$.

The nonlinear model allows for the effect of shocks to vary across the business cycle both through changes in the conditional mean and through changes in the contemporaneous relationships between the variables captured in the H_i matrices. For fixed choices of r_F, r_C the nonlinear model is very easy to estimate since it is linear in parameters, the difficulty comes in searching for the estimates of r_F, r_C. The focus of this paper is on impulse response functions, hence we do not attempt a full estimation of the best nonlinear model. Instead we take the observations on Z_t directly from PP and then estimate the parameters of (10) by least squares over the sample period 1952q1 to 1993q1.[10]

[10] The data on output and unemployment was taken from Citibase and are seasonally adjusted.

6.2. Empirical results

Our focus is on how the responses might vary across the different regimes of the nonlinear model (10) described above.

1. The impulse responses reported are obtained using least squares estimates of the unknown parameters (a, $A(L)$, B), the estimates of r_F, r_C from PP, and bootstrap method (B) to simulate from the empirical distribution of the estimated residuals.
2. We fix the maximum autoregressive lag at 3.
3. We use 164 ω_{t-1}'s in the observed sample rather than generating more values from the estimated nonlinear time series model and innovations.
4. We draw 100 times from the joint (conditional on the regime) distribution of the innovations at each history to produce 100 realizations of the shock for each ω_t. Thus, given 164 histories we generate 16,400 realizations of the generalized impulse response function.
5. The maximum horizon, N, is set to 20 and we average over 1000 futures (R). We only examine the 'system-wide' shock.
6. We use a normal kernel to nonparametrically estimate the marginal densities of the GI for output and unemployment at horizons 0, 8, 20.

Our focus will be on describing how the changing shapes of various marginal densities of the GI can be related to the traditional impulse response function analysis. In order to do this we use the GI for the linear model (9) as our first example. The standard deviation of the GI for output and employment can be interpreted as a traditional impulse response function for a particular type of shock as noted above in Section 4.3. We present the standard deviation of the GI alongside densities of the GI from the linear model in Figs. 5 to 8.

The results using the pseudo-traditional impulse response functions are:

1. The response of output to the shock exhibits some magnification between horizons 0 and 8 and the response at horizon 20 is smaller than the initial shock (this is known as the typical hump shape) but bounded well above zero.

2. The response of unemployment at horizon 20 is close to zero. However, the initial response to the shock shows substantial magnification, again producing the hump shape typical of so many economic time series.

As a guide to understanding Figs. 6 and 8 note that the actual response is on the x axis and the density associated with this response on the y axis. Only horizons 0, 8, and 20 are considered. For output the marginal densities at different horizons are all similar, but for unemployment at horizon 20 the density shows much less dispersion. This indicates the strong mean reversion implied by the linear model. The weaker mean reversion implied by output can be seen in the relationship of the marginal density for horizon 20 with the others.

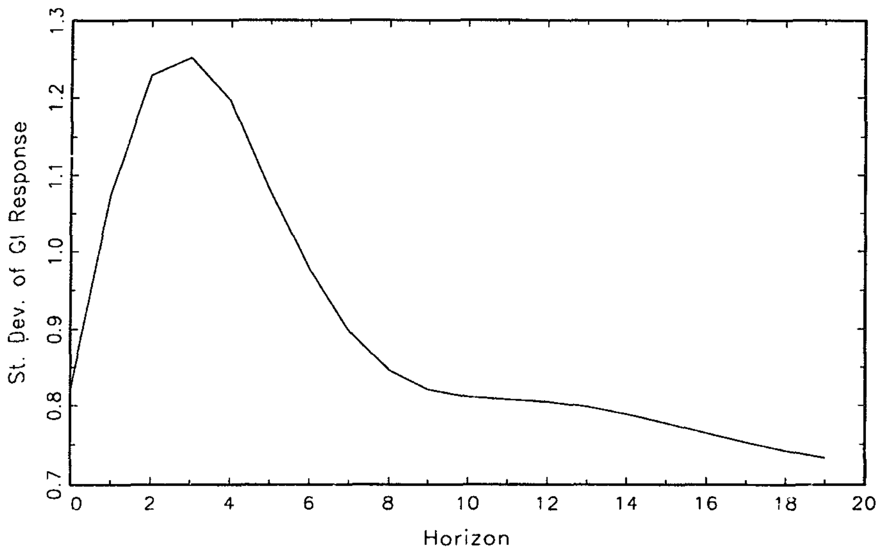

Fig. 5. Traditional impulse response, linear model, GNP.

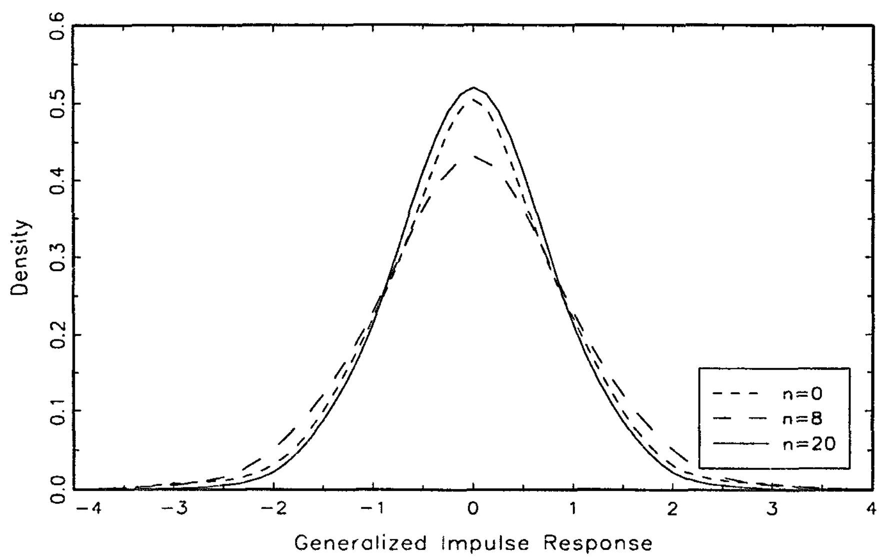

Fig. 6. Generalized impulse response, linear model, GNP.

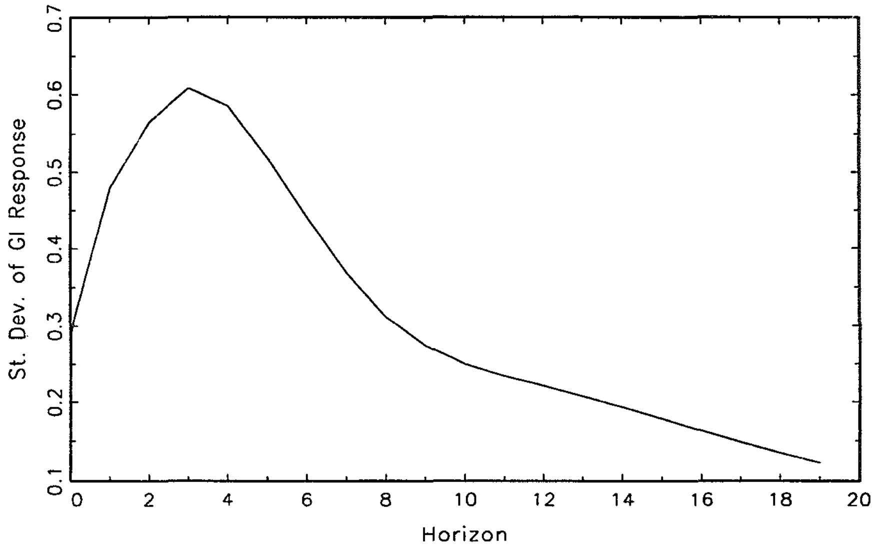

Fig. 7. Traditional impulse response, linear model, unemployment.

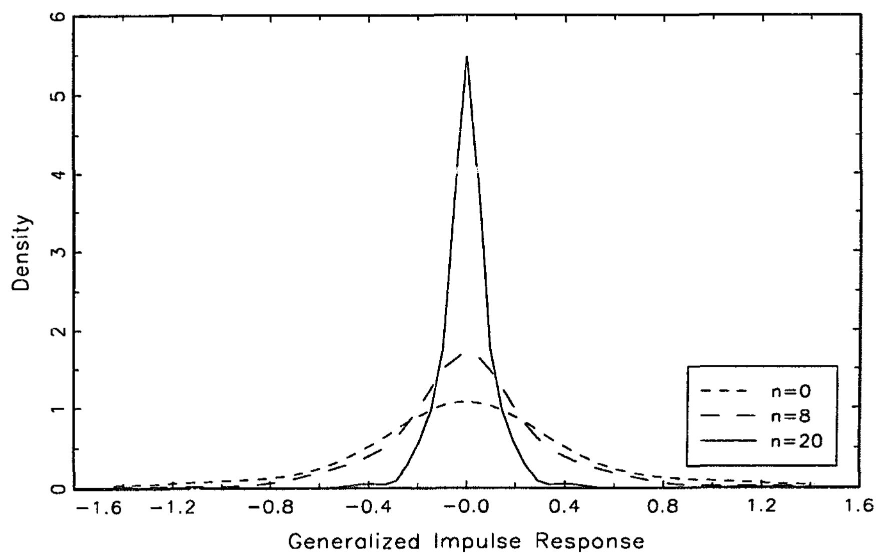

Fig. 8. Generalized impulse response, linear model, unemployment.

On the other hand, the mean reversion is unambiguous because the densities at 0 and 8 horizons second-order stochastically dominate the density at horizon 20.

In Figs. 9 to 16 information on the *GI* is presented for output and unemployment from the nonlinear model (10). For the conditioning event we use the state of the time series at time $t - 1$: floor, ceiling, or corridor. If we consider the result without any conditioning, both variables show more persistence at horizon 20 using the nonlinear model compared to the linear model. This statement is relative to the dispersion of the initial shock which is of course smaller in the nonlinear case than in the linear case because of the extra explanatory power of the nonlinear model. For output there is left-skewness in the actual response in the floor regime indicating that the median response is positive. This is consistent with the univariate output results of BK and PP. Shocks are more persistent in the floor and corridor regimes with the latter being the most persistent. The ceiling regime is close to exhibiting mean reversion and contains the least persistence of shocks.

For unemployment there is little evidence of asymmetry in the marginal densities but there is evidence of nonlinearity in the differences between the densities of the unconditional and the conditional *GI*. Observe how the dispersion of the initial shocks is higher in the floor regime, and dispersion is increased at horizon 8 but then shows substantial reversion at horizon 20. This is capturing two features: the lagged relationship unemployment has with output growth,

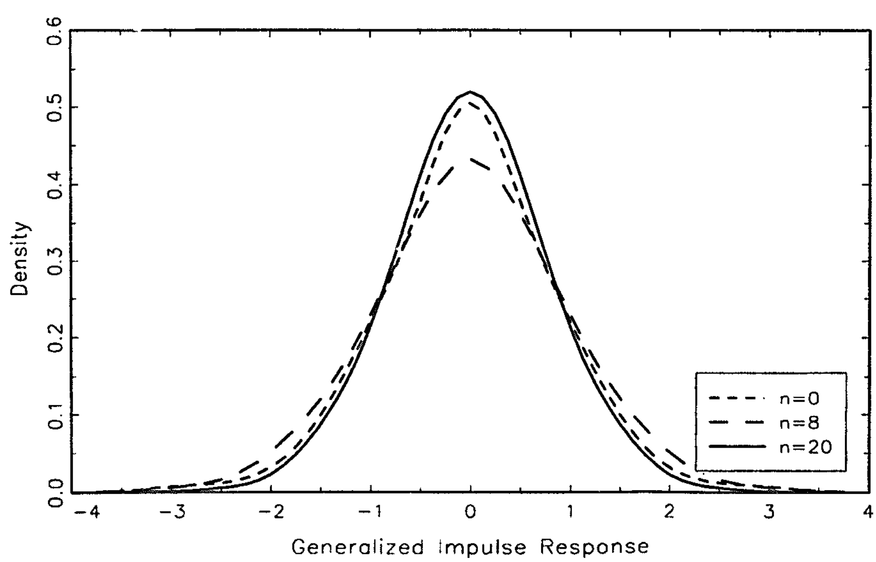

Fig. 9. Nonlinear model, output, unconditional.

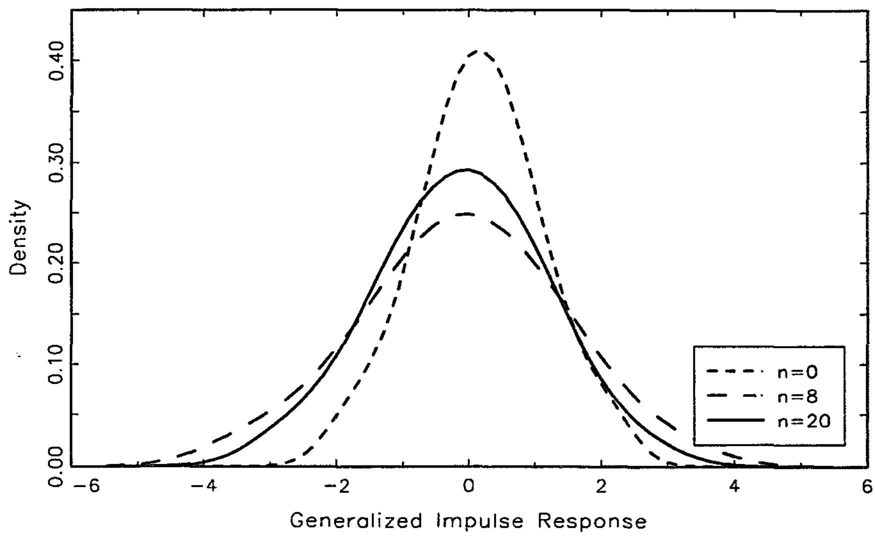

Fig. 10. Nonlinear model, output, floor regime.

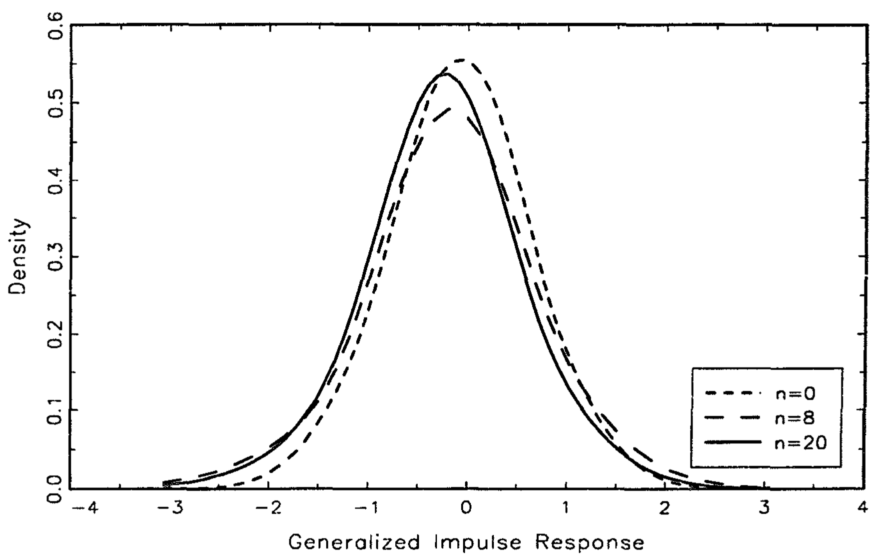

Fig. 11. Nonlinear model, output, ceiling regime.

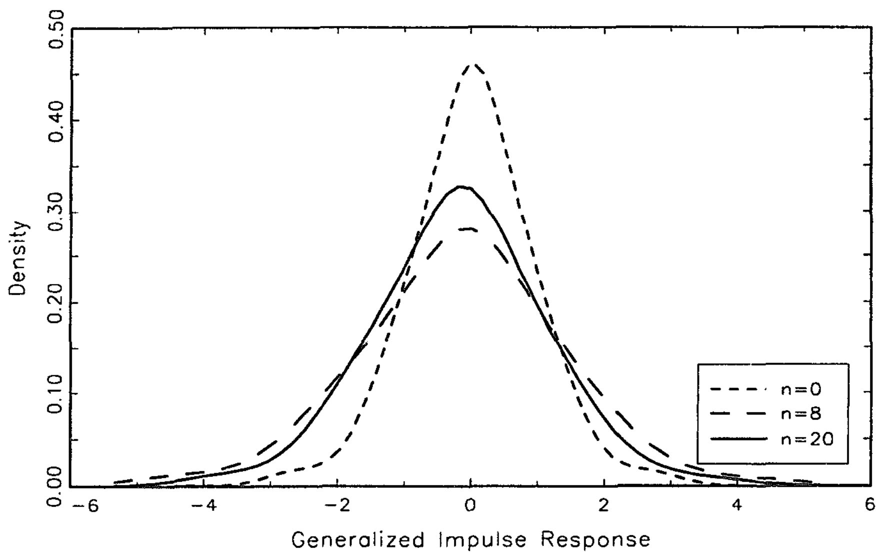

Fig. 12. Nonlinear model, output, corridor regime.

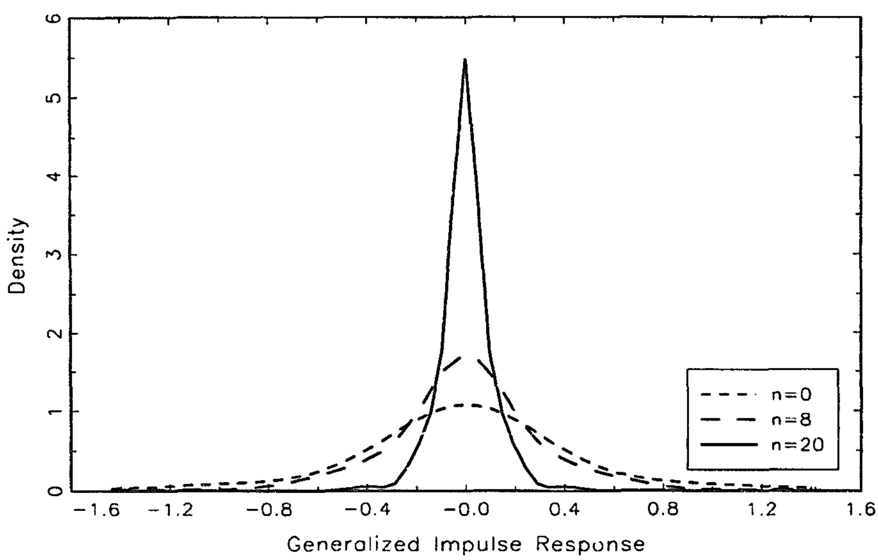

Fig. 13. Nonlinear model, unemployment, unconditional.

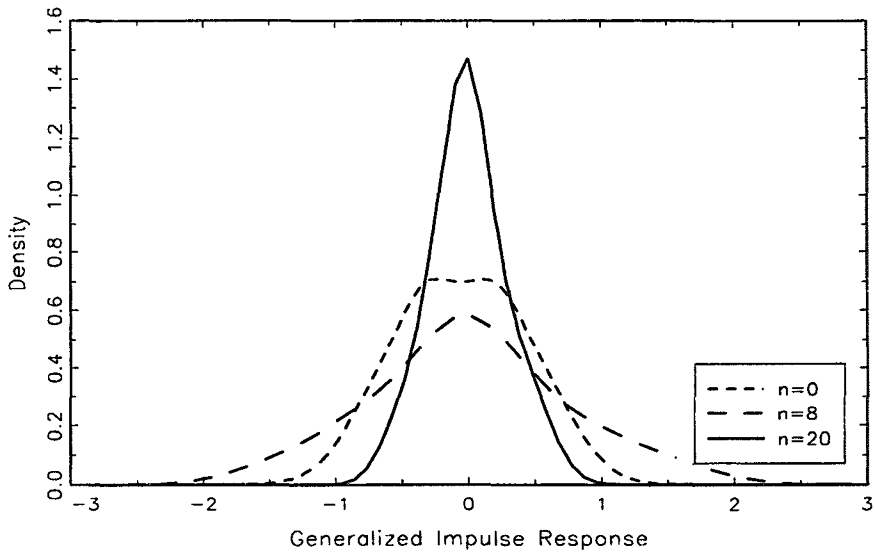

Fig. 14. Nonlinear model, unemployment, floor regime.

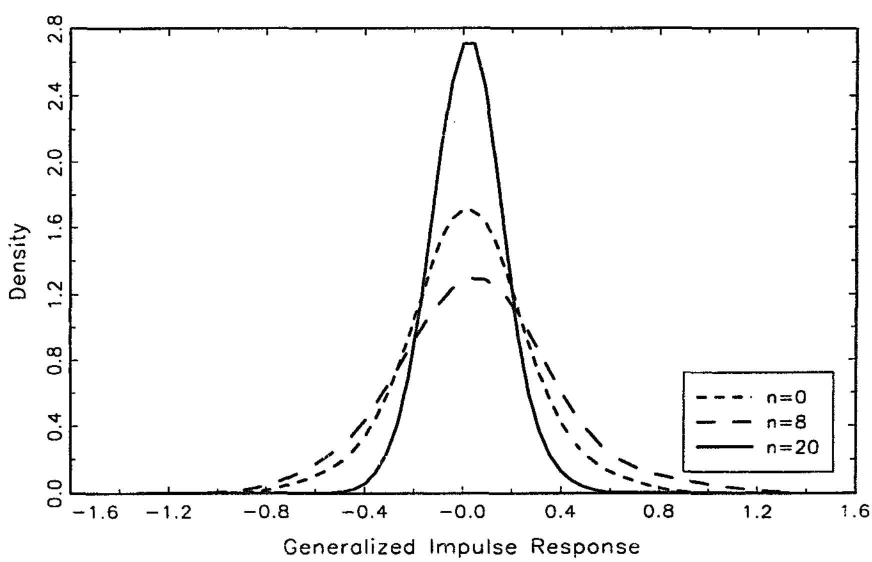

Fig. 15. Nonlinear model, unemployment, ceiling regime.

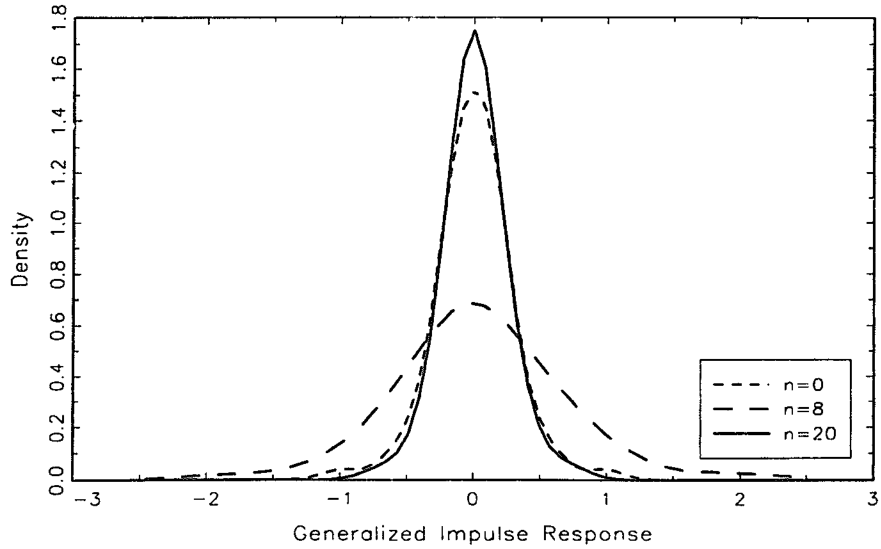

Fig. 16. Nonlinear model, unemployment, corridor regime.

and between horizons 0 and 8 the additional slow response of unemployment in recessions first discussed by Neftçi (1984). However, it also captures a response faster than average after 8 quarters. Indeed the *GI* of unemployment is most persistent relative to the dispersion of the initial shock in the corridor regime.

7. Conclusions

In this paper impulse response analysis has been generalized to the nonlinear multiple time series case. Much of what macroeconomists understand about the dynamics of the economy is based on traditional impulse response functions applied to linear multiple time series models. It has been shown that the generalized impulse response function can be applied to both linear and nonlinear models and provides a natural method of dealing with the problems of history, shock, and compositional dependence of impulse responses for multiple linear and nonlinear time series.

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