AM5600: Computational Methods in Mechanics (July-Nov. 2018)

Assignment #6

Due: At the beginning of class on Oct. 25, 2018

1. Are the following functions Lipschitz in y? If so, find L
   a) $f(t,y) = -ty, 0 \leq t \leq 5, -\infty < y < \infty$
   b) $f(t,y) = -(1+t^2)y^2, 1 \leq t \leq 3, -\infty < y < \infty$
   c) $f(t,y) = t\sqrt{y-3}, t \geq 4, y(4) = 3$
2. Derive the general third-order Runge-Kutta method:
$$y_{n+1} = y_n + C_1 k_1 + C_2 k_2 + C_3 k_3$$
   Show that one such method is given by:
$$y_{n+1} = y_n + \frac{(k_1 + 3k_2 + k_3)}{4},\ k_1 = hf(t_n, y_n)$$
$$k_2 = hf\left(t_n + \frac{h}{3}, y_n + \frac{k_1}{3}\right),\ k_3 = hf\left(t_n + \frac{2h}{3}, y_n + \frac{2k_2}{3}\right)$$
3. Suppose that the local truncation error (single step) for a numerical scheme for ODE satisfies:
$$|\varepsilon_{n+1}(h)| \leq Ch^p$$
   and y satisfies Lipschitz condition for all $t \in [a,b]$ and $h > 0$. Now, we encounter round errors, such that $\varepsilon_{n+1}(h) = Ch^p + e_{n+1}h^{-1}$, where, e_{n+1} is the roundoff errors on the $n+1$ step. Derive the generalized expression for global error.
4. Use Euler's method to approximate the solution $y(t) = t^{3/2}$ of the IVP
$$y' = 1.5y^{1/3},\ y(0) = 0,\ t \geq 0$$
   Explain your results with the help of Lipschitz theorem. Furthermore, can you utilize Taylor's method to approximate the solution. Give appropriate justification.
5. Solve the differential equation below
$$y' = -1000\big(y - (\mathrm{t}+2)\big) + 1,\ y(0) = 1,\ t \geq 0$$
   The exact solution is $y(t) = -e^{-1000t} + \mathrm{t} + 2,\ t \geq 0$
   a) using Euler's explicit method with a step size $h = [5 \times 10^{-4}, 20 \times 10^{-4}, 25 \times 10^{-4}]$ till $t = 0.01$
   b) using Euler's implicit method with different step sizes (h) till $t = 0.01$.

   Compare your findings from a) and b) with the exact solutions and discuss which method performs better with appropriate justification.

AM5801/AM5810: Computational Lab (optional for students crediting AM5600)

Due: At the end of lab on Oct. 24, 2018

I. Solve the system of equations:

$$\frac{dx}{dt} = 10(y - x)$$
$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = xy - \frac{8}{3}z$$

Use the initial conditions as $x(0) = 3$, $y(0) = 3$, $z(0) = 20$. Develop a code for RK4 to solve this system of equations. Compare your results with in-built functions such as *ode45* (RK45) and *ode113* by using *tic-toc* to find which is the fastest method for computing the solution for $t \in [0,100]$. Set absolute and relative error tolerance at 10^{-6}. Use *plot3* to plot your solution in 3D. The trajectory of your solution should have a butterfly shape (known as Lorenz attractor, https://en.wikipedia.org/wiki/Lorenz_system). How accurate are your solution at $t = 100$ (select a tighter tolerance for comparison)?

II. Consider the differential equation

$$y' = \sin(t)(\, y^2 - \cos^2(t) - 1),\ y(0) = 1,\ t \geq 0$$

The exact solution to this non-linear ODE is $y(t) = \cos(t)$.

a) Using RK4 method solve the equation for $t \in [0,50]$. Plot the error vs. h for several values of h (use loglog). Does the error decay as $O(h^4)$?

b) Compare your previous results with explicit Euler's method. Solve the problem for several values of h. On a loglog plot, show the error vs. the number of function evaluations used by each method.

III. Compute the solution to the systems of equation for the given initial conditions using RK4. Plot your results in the phase space $\big(y(t)\ vs.\, x(t)\big)$

Using initial conditions with $x(0)^2 + y(0)^2$ both smaller and larger than 1 (inside and outside the unit circle), solve:

$$x' = -4y + x(1 - x^2 - y^2),\ \ y' = 4x + y(1 - x^2 - y^2)$$

for $t \in [0,10]$. What is the final state of the system? Justify your solution by plotting the trajectories in the XY plane for different conditions. Note: supply initial conditions separately.