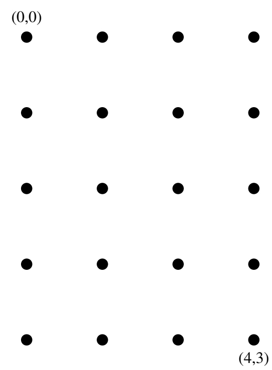
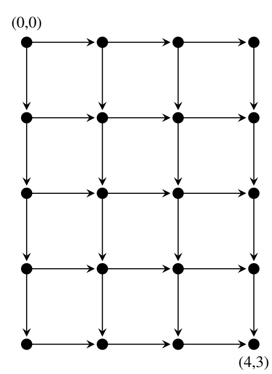
[redacted]

Least Cost Path in a Rectangular Grid

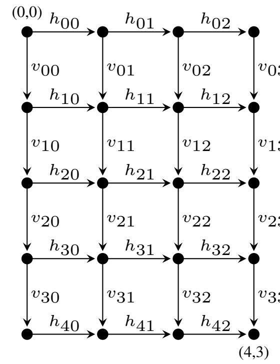
Consider a rectangular grid points/nodes at locations (i, j) in a matrix where i and j are integers with $0 \leq i \leq m$ and $0 \leq j \leq n$. Here, as usual for locations in a matrix, the index i refers to a row and index j refers to a column. For example, we can take $m = 4$ and $n = 3$ and our grid is shown in the following figure.



Assume that for each node we connect it to its neighbor on its right (if there is one) by a horizontal arrow, and to its neighbor below by a vertical arrow (if there is one).

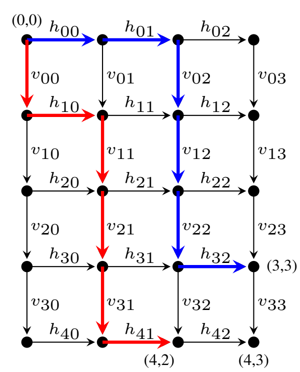


Assume each horizontal arrow from (i, j) to $(i, j + 1)$ is assigned a cost denoted by h_{ij}, and each vertical arrow from (i, j) to $(i + 1, j)$ is assigned a cost v_{ij}.



The problem is to find the least cost path from $(0,0)$ to (m,n) along a sequence of arrows, where the cost of a path is the sum of the costs at all of its individual arrows. This is an example of a problem that can be solved using dynamic programming. The basic idea is to work backwards. In order to get from $(0,0)$ to (m,n) we first have to first visit either the node $(m,n-1)$ or $(m-1,n)$.

For example, suppose we already know the best path from $(0,0)$ to $(4,2)$ shown in red, and the best path from $(0,0)$ to $(3,3)$ shown in blue, in the following figure.



Then we need only decide between the completions of two possible paths.

The best path from $(0,0)$ to $(4,3)$ would involve following the red path and then a final horizontal move or taking the blue path followed by a final vertical. Assuming the cost of the red path is $C_{4,2}$ the cost of it plus the final move is

$$C_{4,2} + h_{42}$$

and if the cost of the blue path is $C_{3,3}$ then the cost of that path plus the final move is

$$C_{3,3} + v_{33}.$$

Comparing these two costs leads to our decision as to which path to take. If there is a tie, we can take either path. We then conclude that

$$C_{4,3} = \min\{C_{4,2} + h_{42}, C_{3,3} + v_{33}\}.$$

We refer a node (i', j') as a *predecessor* of a node (i, j) if there is an arrow pointing from (i', j') to (i, j). Observer that a node can have 0, 1 or 2 predecessors.

- $(0, 0)$ has 0 predecessors,
- $(i, 0)$ has 1 predecessor if $i > 0$,
- $(0, j)$ has 1 predecessor if $j > 0$, and
- (i, j) has 2 predecessors if $i > 0$ and $j > 0$.

For any node (i, j) we denote the cost of an optimal path from $(0, 0)$ to (i, j) by $C_{i,j}$

We define a *direction* $d_{i,j}$ associated with any node with at least one predecessor. If there is a least cost path from $(0, 0)$ to (i, j) whose final step is from $(i - 1, j)$ to (i, j) we define $d_{ij} = V$. Otherwise, there must be a least cost path from $(0, 0)$ to (i, j) whose final step is from $(i, j - 1)$ to (i, j) and we define $d_{ij} = H$.

Key observation. Once we have determined $d_{i,j}$ for every node (i, j) we can find an optimal path from $(0, 0)$ to (m, n) working backwards from (m, n) to find out least cost path from $(0, 0)$ to (m, n). I.e. if $d_{m,n} = H$ then the optimal path visited $(m, n-1)$ and if $d_{m,n} = H$ then our optimal path visited $(m - 1, n)$ before reaching (m, n) in the next step. Whichever node we determine to have visited, we repeat this process to determine the previously visited node, until we get to $(0, 0)$.

Getting started

To get started with the algorithm, note that there is only one path along the right hand border of our grid from $(0, 0)$ to $(m, 0)$ so we can initialize $C_{0,0} = 0$ and $C_{i,0} = \sum_{p=0}^{m-1} v_{p0}$ for $i = 1, \ldots, m$, and similarly there is only one path across the top from $(0, 0)$ to $(0, n)$ we can immediately calculate the values of $C_{0,j} = \sum_{p=0}^{n-1} h_{0p}$ for $j = 1, \ldots, n$.

Now whenever we know the values of $C_{i',j'}$ for every predecessor of (i', j') of (i, j) so we can calculate $C_{i,j}$ and $d_{i,j}$, so we proceed to compute these quantities for the nodes in the following order

$$\begin{array}{ccccccccccc}
 & (1,1) & \to & (1,2) & \to & (1,3) & \cdots & \to & (1,n) & \to \\
\to & (2,1) & \to & (2,2) & \to & (2,3) & \cdots & \to & (2,n) & \to \\
\to & (3,1) & \to & (3,2) & \to & (3,3) & \cdots & \to & (3,n) & \to \\
\to & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \to \\
\to & (m-1,1) & \to & (m-1,2) & \to & (m-1,3) & \cdots & \to & (m-1,n-1) & \to \\
\to & (m,1) & \to & (m,2) & \to & (m,3) & \cdots & \to & (m,n) &
\end{array}$$

Observe that if we use this ordering of the nodes, whenever we reach node (i, j) we will have already found $C_{i',j'}$ and $d_{i',j'}$ for its two predecessors.

Pseudo-Code

We can now write some pseudo-code for finding a solution to our problem in the general case. The inputs to our problem are:

- m, n
- an $m \times (n+1)$ matrix $V = (v_{ij})_{i=0,\ldots,m-1}^{j=0,\ldots,n}$ with v_{ij} giving the cost of a vertical (down) move from (i, j) to $(i+1, j)$
- an $(m+1) \times n$ matrix $H = (h_{ij})_{i=0,\ldots,m}^{j=0,\ldots,n-1}$ with h_{ij} giving the cost of a horizontal (right) move from (i, j) to $(i, j+1)$.

We'll use an $(m+1) \times (n+1)$ matrix $C = (c_{ij})_{i=0,\ldots,m}^{j=0,\ldots,n}$ to store the *cost* of a shortest path from $(0, 0)$ to (i, j) for $i = 0, \ldots, m$ and $j = 0, \ldots, n$.

We'll use an $(m+1) \times (n+1)$ matrix of strings $D = (d_{ij})_{i=0,\ldots,m}^{j=0,\ldots,n}$ to store, at each node (i, j) the value "H" if there is a best path from $(0, 0)$ to (i, j) whose last step is a move from predecessor $(i, j-1)$ to (i, j) otherwise the value is "V" since there must be a best path from $(0, 0)$ to (i, j) whose last step is a move from the predecessor $(i-1, j)$ to (i, j).

Now we proceed as follows to calculate the terms in the matrices C and D.

Initialize $c_{0,0} = 0$
Initialize $c_{i,0} = \sum_{p=0}^{i-1} v_{p0}$ and $d_{i,0} =$"V" for $i = 1, \ldots, m$.
Initialize $c_{0j} = \sum_{p=0}^{j-1} h_{0p}$ and $d_{0,j} =$"H' for $j = 1, \ldots, n$.
For $i = 1, \ldots, m$:
    For $j = 1, \ldots, n$:
        If $C_{i-1,j} + v_{i-1,j} \leq C_{i,j-1} + h_{i,j-1}$
            Take $C_{i,j} = C_{i-1,j} + v_{i-1,j}$ and $d_{i,j} =$"V"
        Else

Take $C_{i,j} = C_{i,j-1} + h_{i,j-1}$ and $d_{i,j}$ =”H”

Observe that once we run this algorithm, for every node (i, j) except for $(0, 0)$ it is the case that

- either $(i - 1, j)$ is a predecessor of (i, j) and $d_{i,j}$ =”V”, or
- $(i, j - 1)$ is a predecessor of (i, j) and $d_{i,j}$ =”H”.

This gives us the cost $C_{i,j}$ of the optimal path from $(0, 0)$ to (i, j) with the cost of $(0, 0)$ to (m, n) as a special case. Next we carry out the following steps to find the actual optimal path - the list of nodes visited by the optimal path.

Initialize list of OptimalPathNodes list as $[(m, n)]$
While length of Optimal PathNodes list is less than $m + n + 1$
  Let (i, j) be the last element in OptimalPathNodes.
  If $(i - 1, j)$ is a predecessor of (i, j) and $d_{(i,j)}$ =”V”
    Append $(i - 1, j)$ to OptimalPathNodes
  Else
    Append $(i, j - 1)$ to OptimalPathNodes
Reverse the OptimalPathNodes list
Return OptimalPathNodes list

Solving by brute-force

For small values of m and n it is possible to enumerate every possible path from $(0, 0)$ to (m, n). Since every such path must contain m horizontal moves and n vertical moves the paths are in one-to-one correspondence with the permutations of the list consisting of m values of “H” and n values of “V”. The number of such permutations is

$$\binom{m+n}{n}.$$

There is a function in the *sympy* package that can be used to generate all permutations of a list that contains repeated elements. The following code

```
import numpy as np
m=3
n=2
L=["V" for i in range(m)]+["H" for i in range(n)]
allpaths=multiset_permutations(L)
```

```
for a in allpaths:
   print(a)
```

yields the output

```
['H', 'H', 'V', 'V', 'V']
['H', 'V', 'H', 'V', 'V']
['H', 'V', 'V', 'H', 'V']
['H', 'V', 'V', 'V', 'H']
['V', 'H', 'H', 'V', 'V']
['V', 'H', 'V', 'H', 'V']
['V', 'H', 'V', 'V', 'H']
['V', 'V', 'H', 'H', 'V']
['V', 'V', 'H', 'V', 'H']
['V', 'V', 'V', 'H', 'H']
```

To get the optimal path, we simply compute the cost of each path and take a path that gives the lowest cost.