

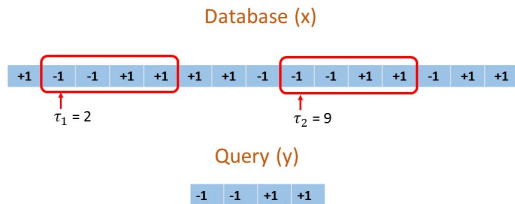
Sub-string/Pattern Matching in Sub-linear Time Using a Sparse Fourier Transform Approach

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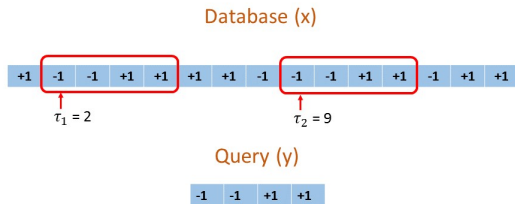


Problem Statement



- **Database/String:** $\underline{x} = [x[0], x[1], \dots, x[N-1]]$ (length N)
- **Query/Substring:** $\underline{y} = [y[0], y[1], \dots, y[M-1]]$ (length $M = N^\mu$)
- **Signal Model:** $x[i]$'s are i.i.d. r.v. from $\mathcal{A} = \{+1, -1\}$ (extensions possible)

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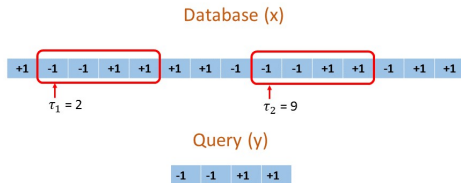


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Determine all the L locations $\underline{\tau} = [\tau_1, \tau_2, \dots, \tau_L]$ with high probability where

- 1 **Exact Matching:** \underline{y} appears exactly in \underline{x}
 - $\underline{y} := \underline{x}[\tau : \tau + M - 1]$
- 2 **Approximate Matching:** \underline{y} is a noisy substring of \underline{x}
 - $\underline{y} := \underline{x}[\tau : \tau + M - 1] \odot \underline{b}$
 - \underline{b} is a noise sequence with $d_H(\underline{y}, \underline{x}[\tau : \tau + M - 1]) \leq K$

Notations



| Symbol | Meaning |
|----------------------------|--|
| N | Size of the string or database in symbols |
| $M = N^\mu$ | Length of the query in symbols |
| $L = N^\lambda$ | Number of matches |
| K | $\max_{\tau} d_H(\underline{x}[\tau : \tau + M - 1], \underline{y})$ |
| η | $\frac{K}{M}$ |
| $G = N^\gamma$ | Number of blocks |
| $\tilde{N} = N^{1-\gamma}$ | Length of one block |
| $f_i = N^\alpha$ | Length of smaller point IDFT at each branch |
| $g_i = N/f_i$ | Sub-sampling parameter |
| B | Number of shifts also referred to as branches |
| d | Number of stages in the FFAST algorithm |

Main Result

Theorem 1

Assume that a sketch of \underline{x} of size $O(\frac{N}{M} \log N)$ can be precomputed and stored. Then for the exact pattern matching and approximate pattern matching (with $K = \eta M$, $0 \leq \eta \leq 1/6$) problems, with the number of matches L scaling as $O(N^\lambda)$, our algorithm has

- a sketching function for \underline{y} that computes $O(\frac{N}{M} \log N) = O(N^{1-\mu} \log N)$ *samples*
- a *computational complexity* of $O(\max(N^{1-\mu} \log^2 N, N^{\mu+\lambda} \log N))$
- a decoder that recovers all the L matching positions with a *failure probability that approaches zero asymptotically*

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Note

Particularly when $L = O(1)$ or $L < \frac{N}{M}$ (i.e. $\lambda < 1 - \mu$) our algorithm has a *sub-linear time* complexity.

Some Prior Work

Exact Matching

- **Boyer and Moore 1977**: First occurrence of the match (only τ_1)
 - Average complexity - $O(N^{1-\mu} \log N)$ (sublinear)
 - Worst case complexity - $O(N \log N)$

Approximate Matching

- **Chang and Marr 1994**: Generalization of Boyer and Moore 1977
 - Average complexity - $O(NK/M \log N)$ (sub-linear only when $K \ll M$)
- **Andoni et al. 2013**: $O(N/M^{0.359})$ (sub-linear even when $K = O(M)$)
 - Combinatorial in nature

Sparse Fourier Transform Approach

- **Hassanieh et al. 2012**: Faster GPS receiver
 - Exploited sparsity in Correlation function R_{XY}
- **Pawar and Ramchandran 2014**: Robust Sparse Fourier Transform
 - Sparse Graph code Approach
 - Computational complexity : $O(N \log N)$

Motivation

- **Cross-correlation** (\underline{r}):

$$r[m] = (\underline{x} * \underline{y})[m] \triangleq \sum_{i=0}^{M-1} x[m+i]y[i], \quad 0 \leq m \leq N-1$$

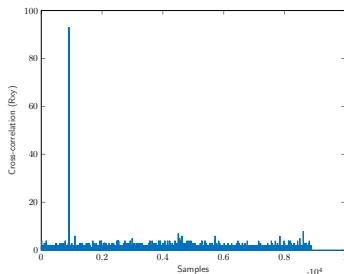
- **Naive implementation:** $O(MN) = O(N^{1+\mu})$ (**super-linear** complexity)
- **Fourier Transform Approach:** $O(N \log N)$ complexity

$$\underline{r} = \mathcal{F}_N^{-1} \{ \mathcal{F}_N \{ \underline{x} \} \odot \mathcal{F}_N \{ \underline{y}' \} \}, \quad \underline{y}' = \underline{y}^*[-n]$$

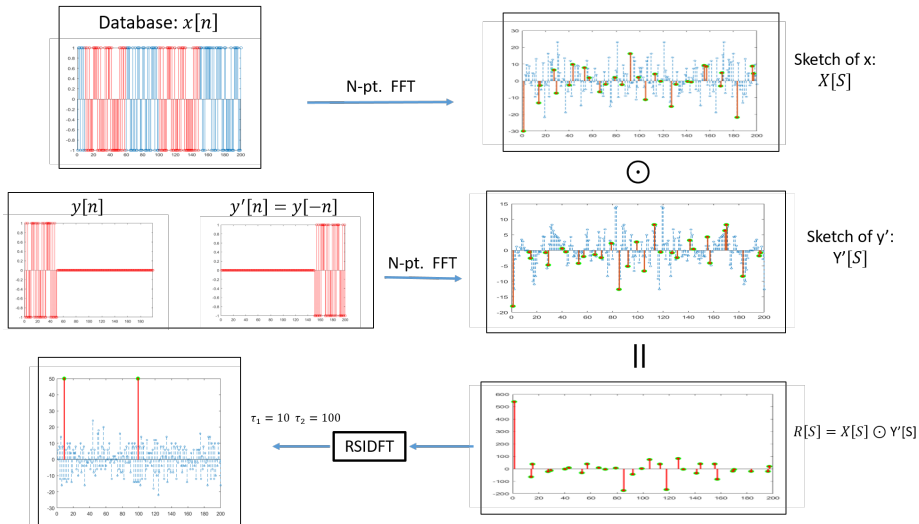
Key Observation

- \underline{r} is **Sparse** with some noise.

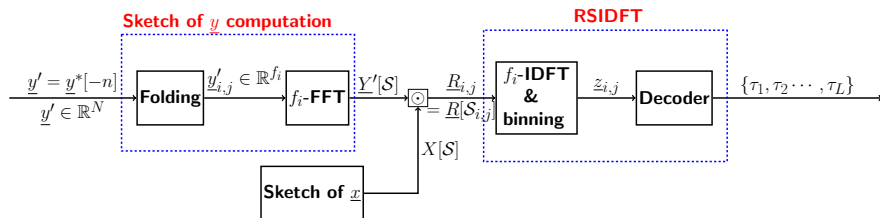
$$r[m] = \begin{cases} M, & \text{if } m \in \mathcal{T} \\ n_m, & m \in [N] - \mathcal{T} \end{cases}$$



Example



Sparse Fourier Transform Approach



$$\underline{r} = \underset{3}{\mathcal{F}_N^{-1}} \left\{ \underset{1}{\mathcal{F}_N\{\underline{x}\}} \odot \underset{2}{\mathcal{F}_N\{\underline{y}'\}} \right\}$$

1. *Sketch of \underline{x}* : Assume $\underline{X}[l] = \mathcal{F}\{\underline{x}\}$ is precomputed at positions $l \in S$.
2. *Sketch of \underline{y}* : Compute $\underline{Y}'[l] = \mathcal{F}\{\underline{y}'\}$ for $l \in S$.
 - Only M non-zero values in \underline{y}' - Efficient computation (folding and adding)
3. *Sparse \mathcal{F}^{-1}* :
 - Robust Sparse Inverse Fourier Transform (RSIDFT)
 - Efficient Implementation- **sublinear** time and sampling complexity

Robust Sparse Inverse Fourier Transform(RSIDFT)

Main Idea

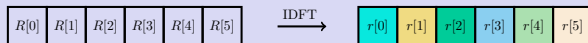
- **Sub-sampling** in frequency corresponds to **aliasing** in time
- Aliased coefficients \Leftrightarrow parity check constraints of **GLDPC codes**
- **CRT** guided sub-sampling induces a code good for **Peeling decoder**
- R-FFAST- proposed by Pawar and Ramchandran 2014

Key modifications

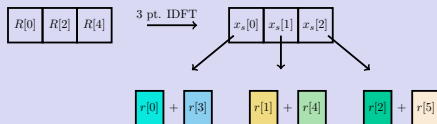
- Optimized for the induced noise model
- Correlation peak is always **positive**
- Take advantage in decoding algorithm - **sub-linear** time complexity

Aliasing and Sparse Graph Codes

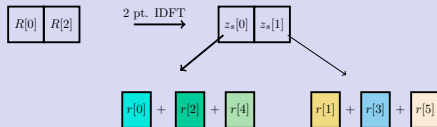
IDFT Computation ($N = 6$)



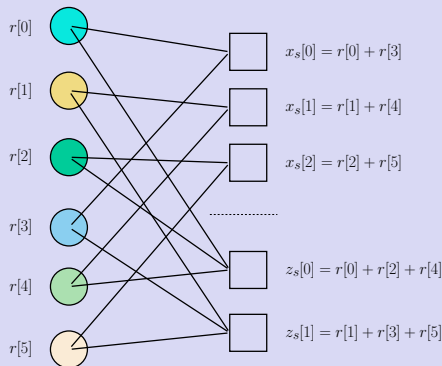
x_s : Sub-sampled by $P_1 = 2$



z_s : Sub-sampled by $P_2 = 3$

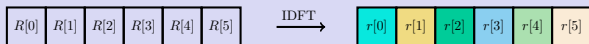


Factor graph

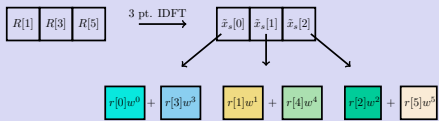


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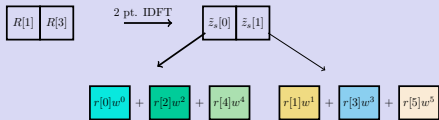
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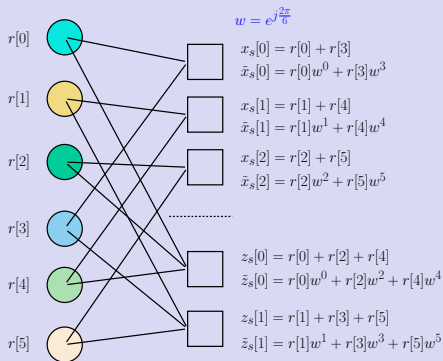
x_s : Sub-sampled by $P_1 = 2$ (shifted)



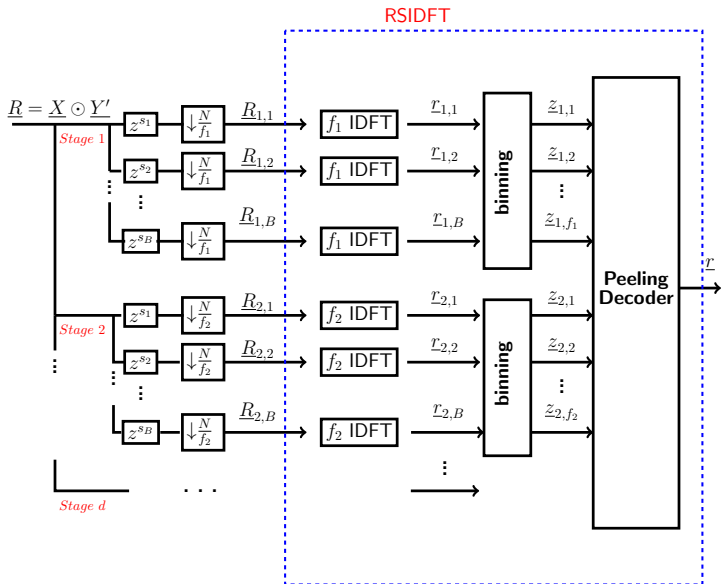
z_s : Sub-sampled by $P_2 = 3$ (shifted)



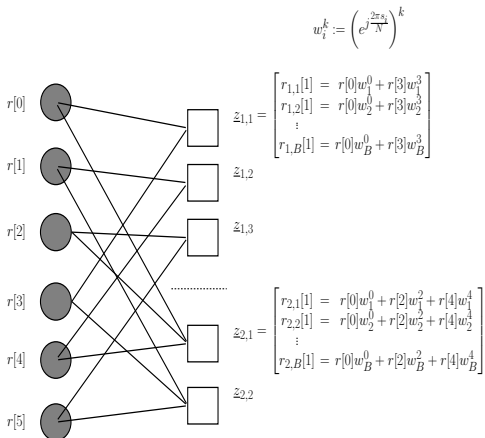
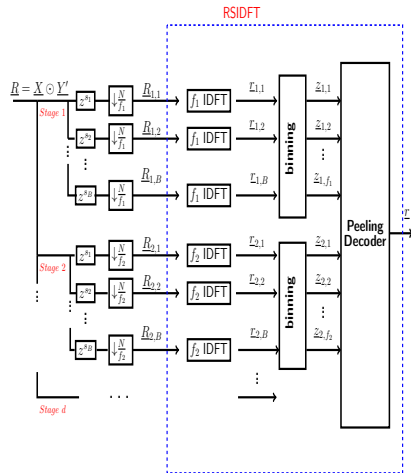
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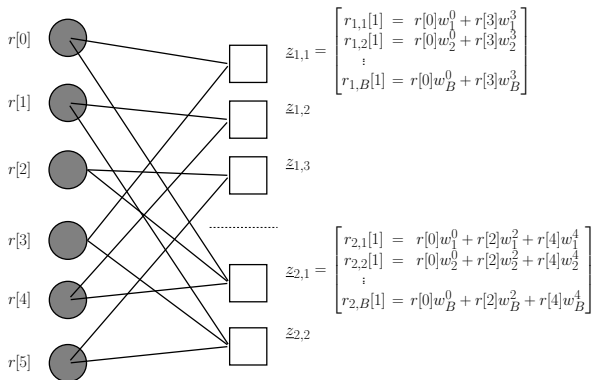
RSIDFT Framework



RSIDFT Framework



RSIDFT-Decoding (Peeling Decoder)



Observations:

$$\underline{z}_{i,k} = \begin{bmatrix} r_{i,1}[k] \\ r_{i,2}[k] \\ \vdots \\ r_{i,B}[k] \end{bmatrix}^T$$

Decoding- 3 steps

- 1 Bin Classification
- 2 Position Identification
- 3 Peeling Process

Decoder

Bin Classification

- Classify each check-node - Zero-ton / Single-ton / Multi-ton
- **Threshold constraints** on first observation $z_{i,k}[1] = z$
- Threshold varies with η
 - different for exact($\eta = 0$) and approximate matching

$$\hat{\mathcal{H}}_{i,j} = \begin{cases} \mathcal{H}_z & z/M < \gamma_1 \\ \mathcal{H}_s & \gamma_1 < z/M < \gamma_2 \\ \mathcal{H}_d & \gamma_2 < z/M < \gamma_3 \\ \mathcal{H}_m & z/M > \gamma_3 \end{cases}$$

where $(\gamma_1, \gamma_2, \gamma_3) = (\frac{1-2\eta}{2}, \frac{3-4\eta}{2}, \frac{5-6\eta}{2})$

Decoder

Position Identification

- Observation:

$$\underline{z}_{i,k} = \begin{bmatrix} e^{\frac{j2\pi k s_1}{N}} & e^{\frac{j2\pi (k+f_i) s_1}{N}} & \dots & e^{\frac{j2\pi (k+(g_i-1)f_i) s_1}{N}} \\ e^{\frac{j2\pi k s_2}{N}} & e^{\frac{j2\pi (k+f_i) s_2}{N}} & \dots & e^{\frac{j2\pi (k+(g_i-1)f_i) s_2}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{j2\pi k s_B}{N}} & e^{\frac{j2\pi (k+f_i) s_B}{N}} & \dots & e^{\frac{j2\pi (k+(g_i-1)f_i) s_B}{N}} \end{bmatrix} \times \begin{bmatrix} r[k + (0)f_i] \\ r[k + (1)f_i] \\ \vdots \\ r[k + (g_i - 1)f_i] \end{bmatrix}$$

- Column that gives **maximum correlation** with the observation

$$\hat{k} = \arg \max_{k \in \{j+lg_i\}} \underline{z}_{i,j}^\dagger \underline{w}^k$$

Decoder

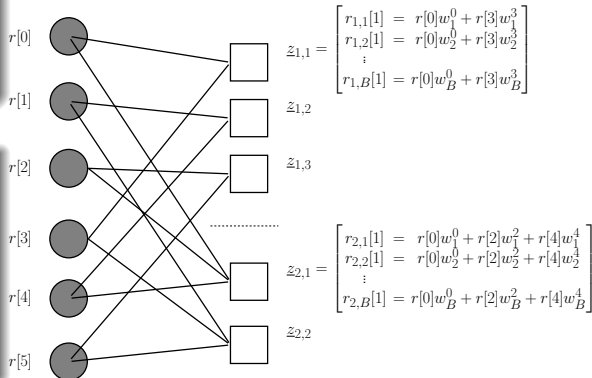
Peeling Process:

Exact Matching

- Remove a decoded variable node's contribution from **all participating bin nodes**

Approximate Matching

- Remove a decoded variable node's contribution only from neighboring **single-tons and double-tons**
- Avoid error propagation



Simulation Results

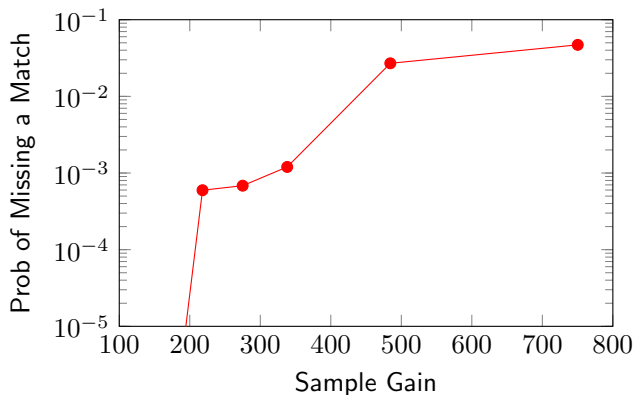


Figure: Plot of Probability of Missing a Match vs. Sample Gain for Exact Matching of a substring of length $M = 10^5$ from a equiprobable binary $\{+1, -1\}$ sequence of length $N = 10^{12}$, divided into $G = 10^5$ blocks each of length $\tilde{N} = 10^7$. The substring was simulated to repeat in $L = 10^6$ locations uniformly at random.

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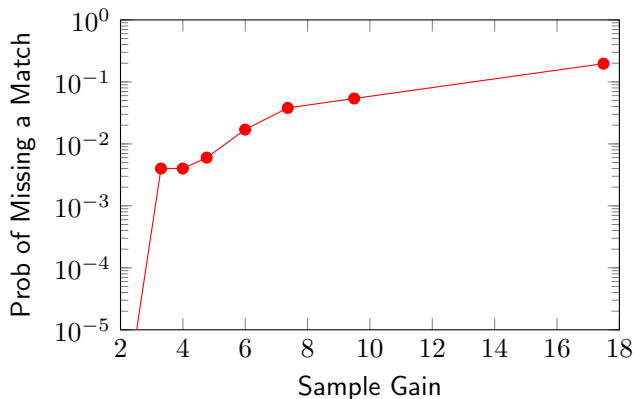


Figure: Plot of Probability of Missing a Match vs. Sample Gain for Exact Matching of a substring of length $M = 10^3$ from a equiprobable binary $\{+1, -1\}$ sequence of length $N = 10^{12}$, divided into $G = 10^6$ blocks each of length $\tilde{N} = 10^6$. The substring was simulated to repeat in $L = 10^6$ locations uniformly at random.

Questions?



Thank you!