

# Applications of Spatial Coupling & Sparse Graph Codes for Sparse Recovery

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## 1 Spatial Coupling(SC)

- Introduction
- Threshold Saturation Phenomenon

## 2 SC-LDPC Lattices

- Introduction
- Proposed Lattice Construction
- Poltyrev Goodness
- Application to Symmetric Interference Channel

## 3 Side-Information Problems

- Introduction
- Compound Codes
- Spatial Coupling

## 4 Write-Once Memory

# Outline

- 1 Spatial Coupling(SC)
  - Introduction
  - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
  - Introduction
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# An $(\ell, r)$ LDPC Code

## Parity-Check Matrix

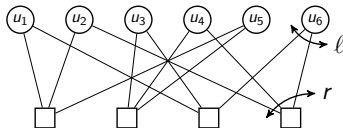
$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\ell = 2$$

$$r = 3$$

$$\text{LDPC Code } \mathcal{C} = \{x : H \odot x = 0\}$$

## Tanner Graph



## Compressed Representation



# Belief Propagation Decoder & Threshold

## Belief Propagation (BP)

- ▶ Popular choice
- ▶ Low-complexity
- ▶ Threshold:  $h^{\text{BP}}$

## Maximum a Posteriori (MAP)

- ▶ Optimal Decoder
- ▶ **Not Realizable**
- ▶ Threshold:  $h^{\text{MAP}}$

$$h^{\text{BP}} < h^{\text{MAP}}$$

# Threshold Comparison

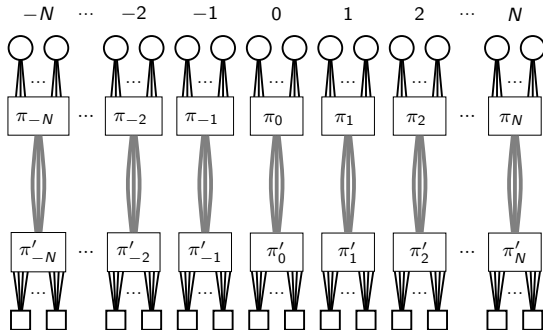
LDPC ( $\ell, r$ )	Shannon $h^{\text{Sh}}$	AWGN		BSC	
		$h^{\text{BP}}$	$h^{\text{MAP}}$	$h^{\text{BP}}$	$h^{\text{MAP}}$
(3,6)	0.5000	0.4293	0.4794	0.4160	0.4681
(4,6)	0.6667	0.5211	0.6645	0.5203	0.6633
(5,6)	0.8333	0.5731	0.8333	0.5773	0.8333

# $(\ell, r, N, w)$ Spatially-Coupled Ensemble



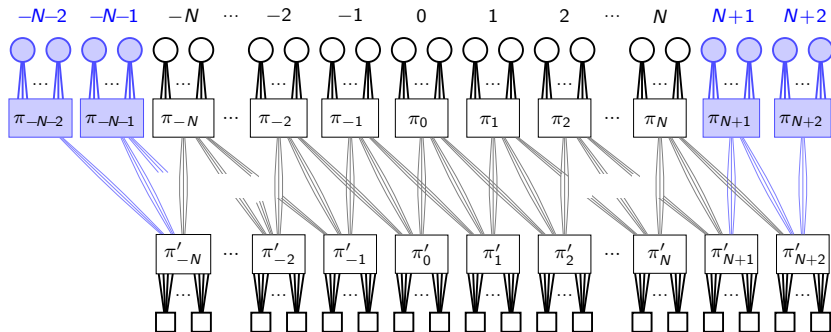
- An LDPC code of left-degree  $\ell = 3$  and right-degree  $r = 4$

# $(\ell, r, N, w)$ Spatially-Coupled Ensemble



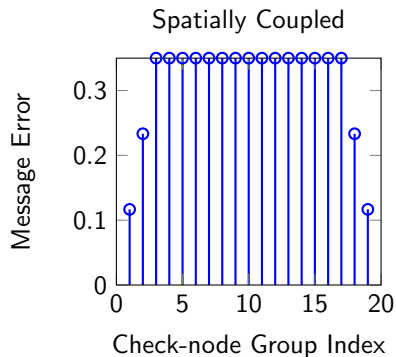
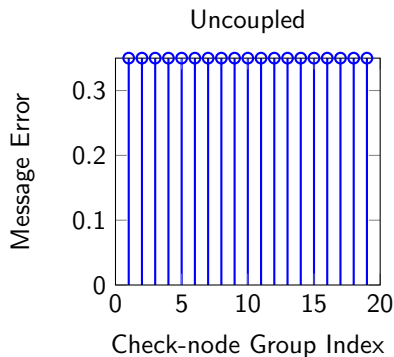
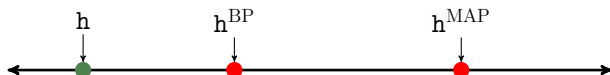


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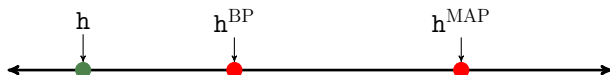


- Shown for  $\ell = 3$ ,  $r = 4$ , and  $w = 3$
- Check-nodes at Section  $\{i\}$  are connected to variable-nodes in Sections  $\{i - (w - 1), \dots, i\}$
- Shown to have **near optimal BP thresholds**

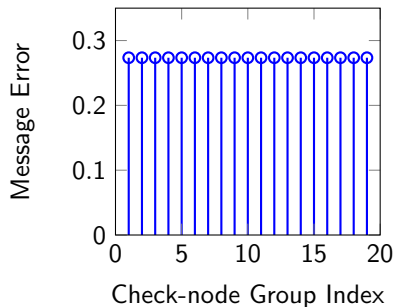
# Threshold Saturation via Spatial Coupling (1)



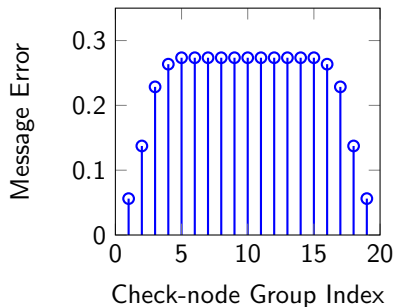
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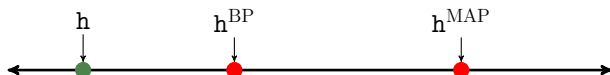
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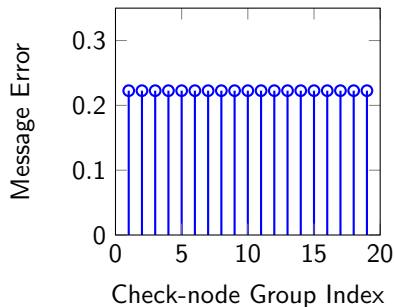
Spatially Coupled



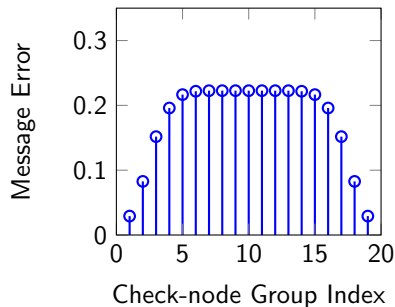
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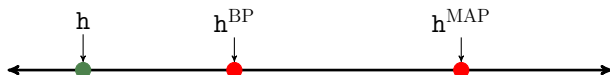
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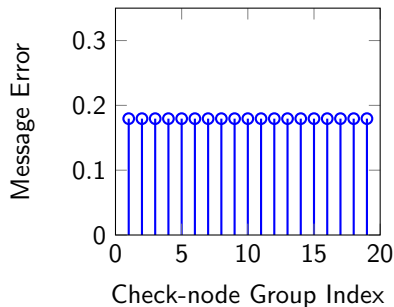
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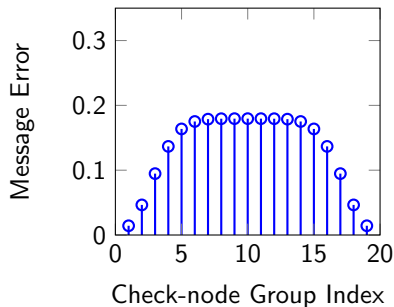
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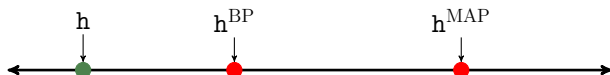
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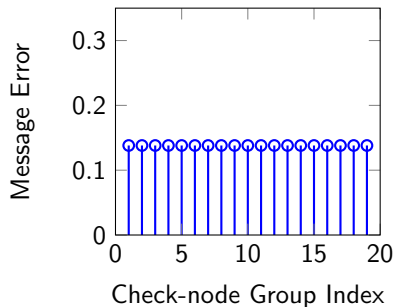
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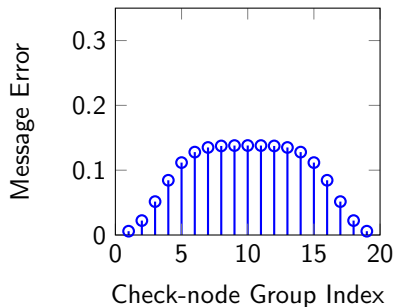
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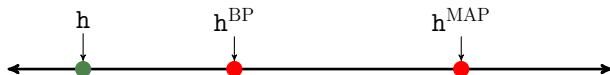
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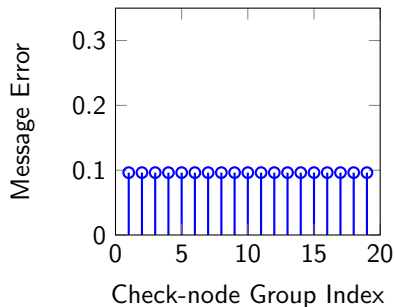
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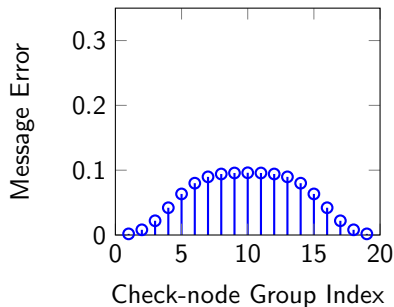
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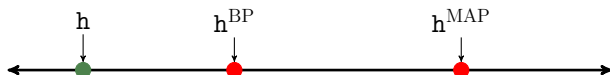
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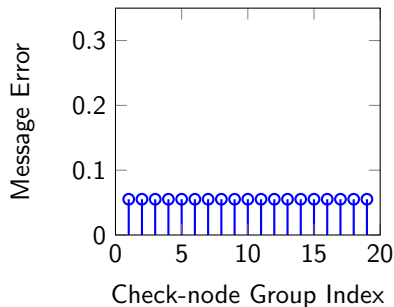
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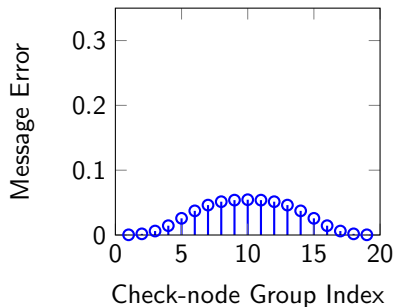
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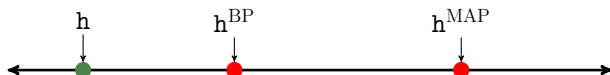


Spatially Coupled

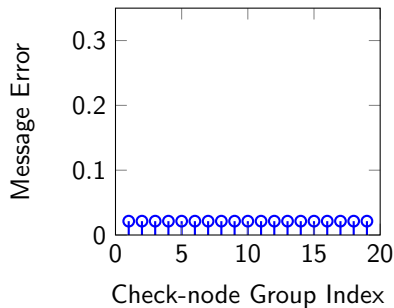




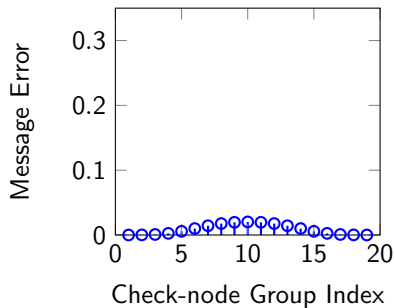
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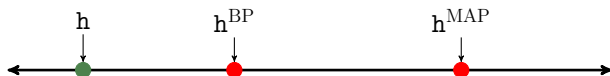
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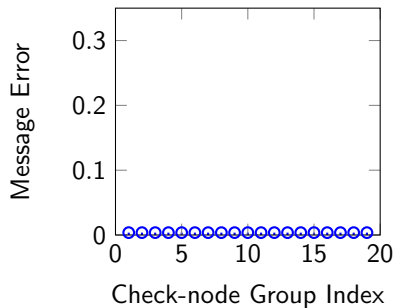
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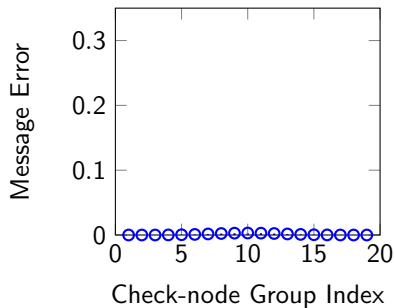
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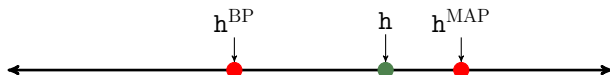
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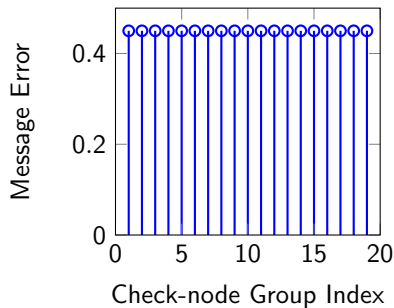
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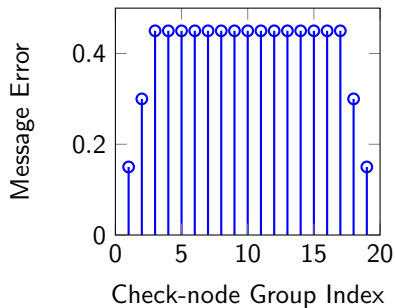
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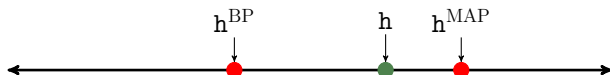
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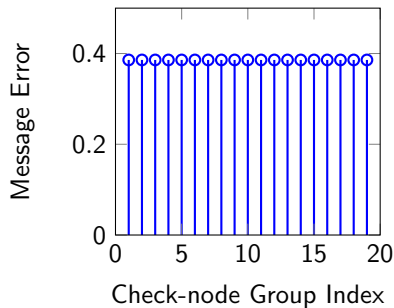
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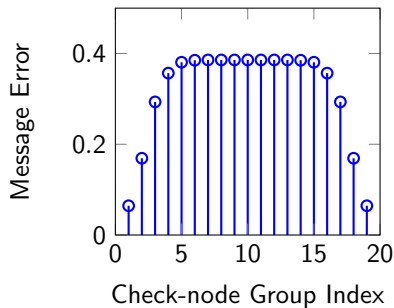
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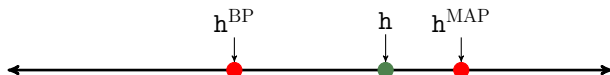
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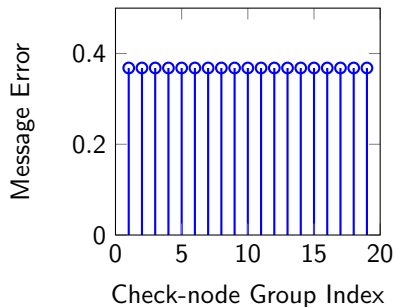
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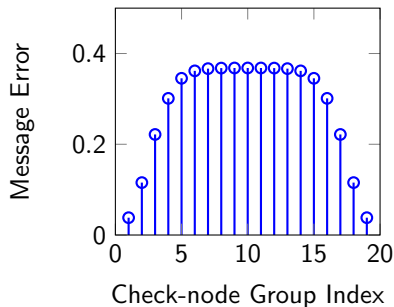
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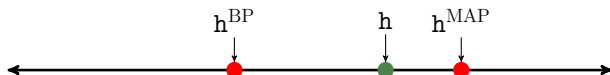
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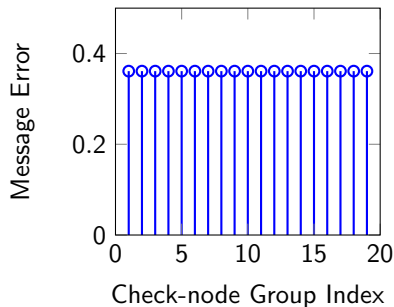
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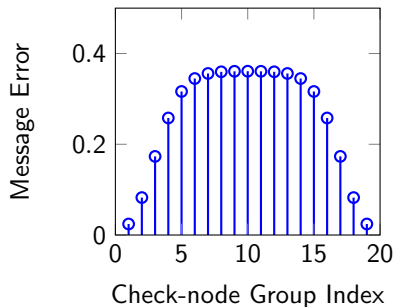
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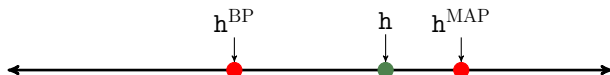
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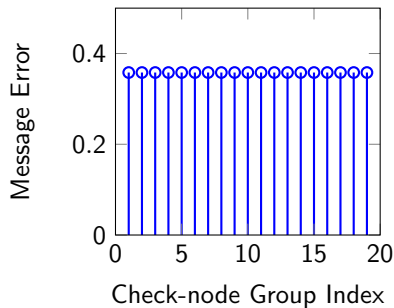
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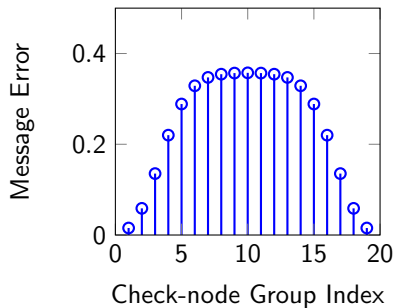
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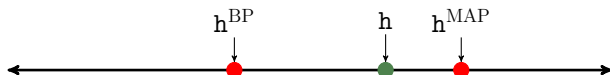
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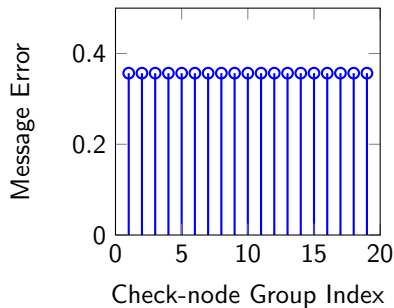
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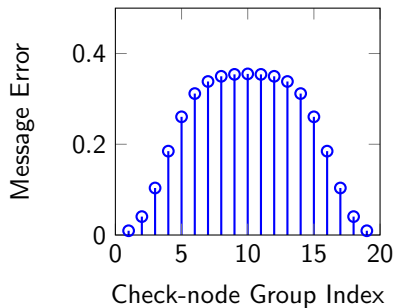
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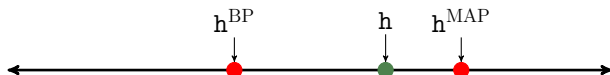


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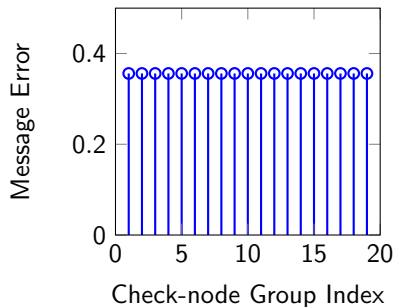




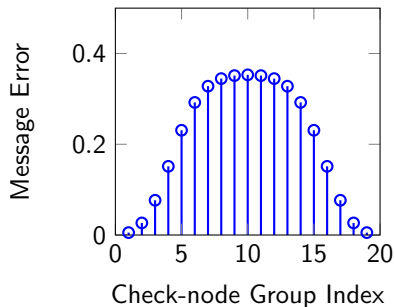
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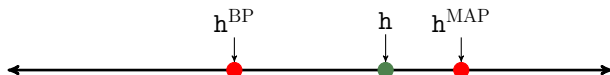
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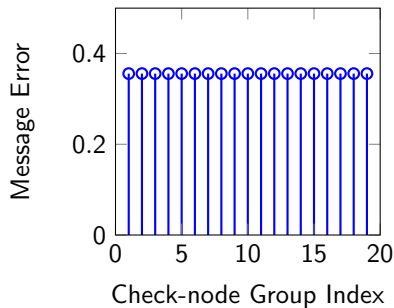
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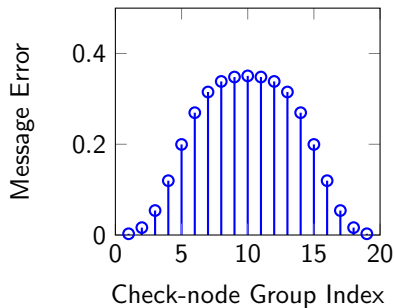
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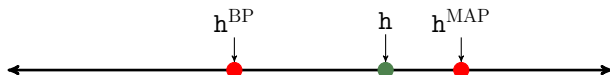
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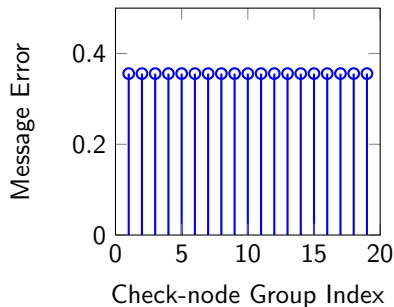
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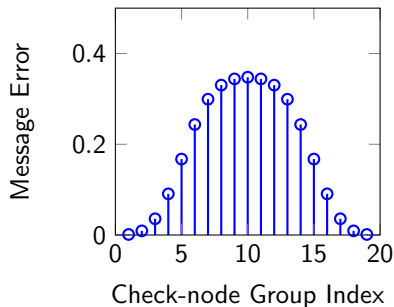
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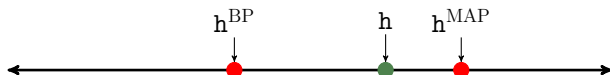
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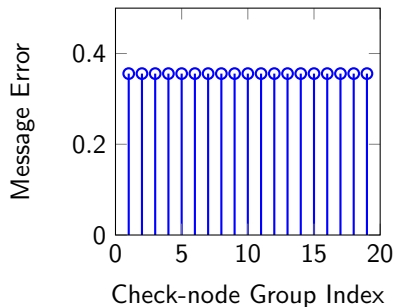
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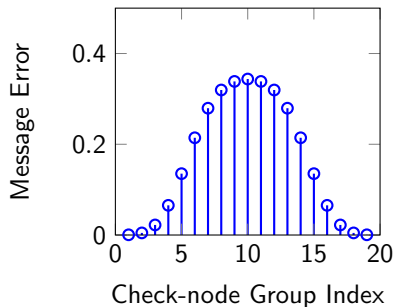
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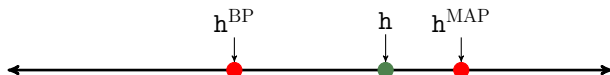
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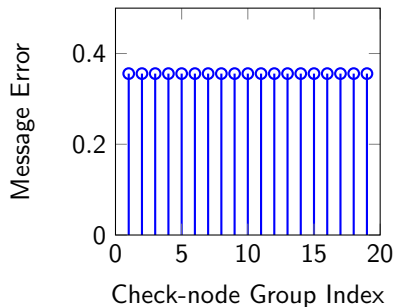
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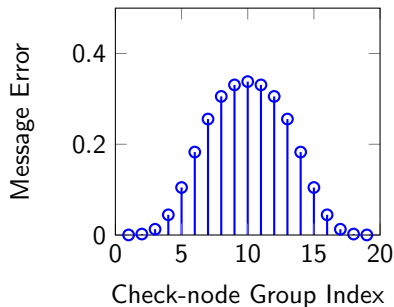
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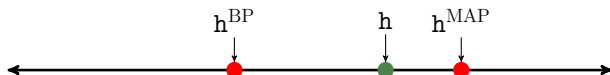
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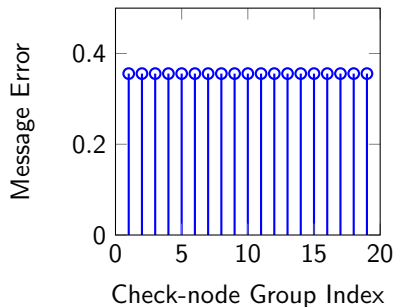
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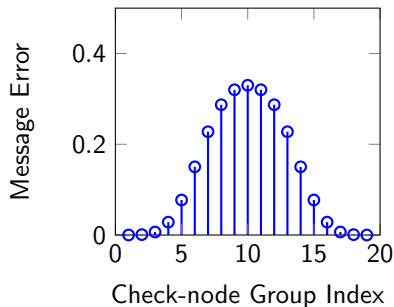
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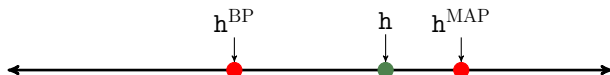
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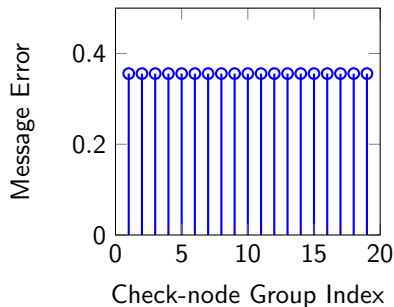
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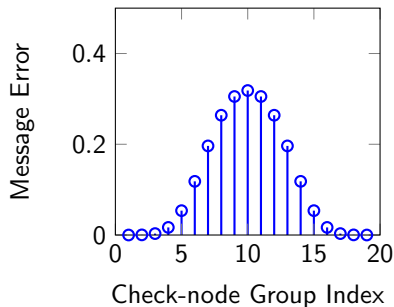
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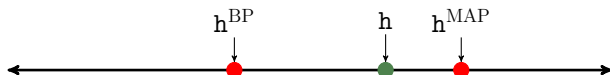
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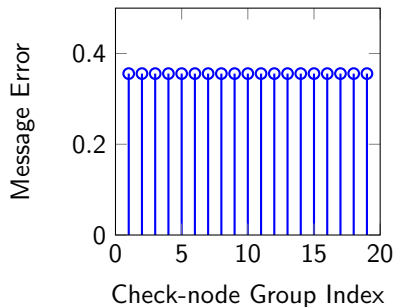
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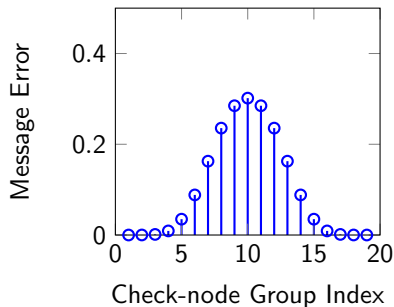
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Uncoupled

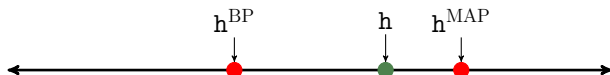


Spatially Coupled

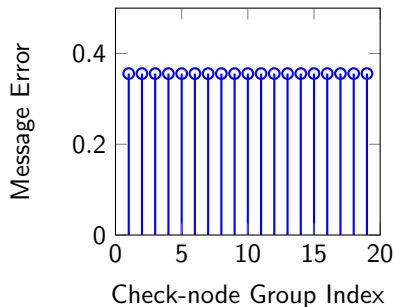




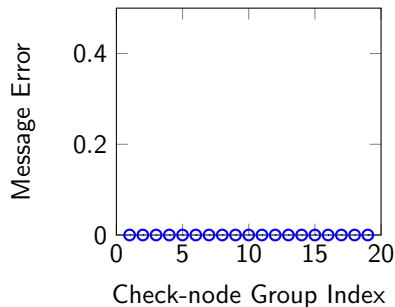
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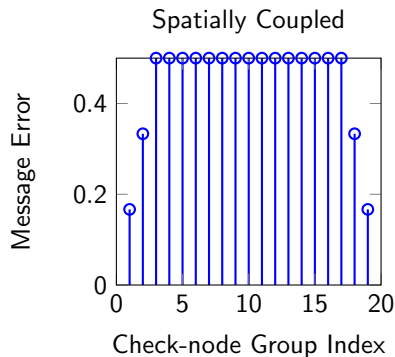
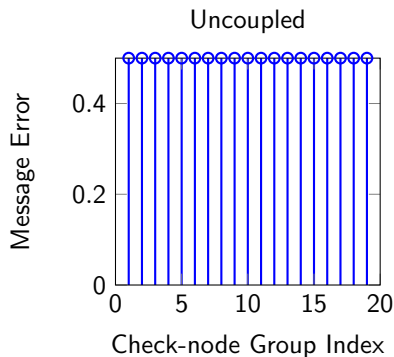
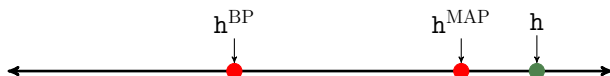
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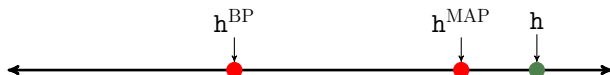
Spatially Coupled



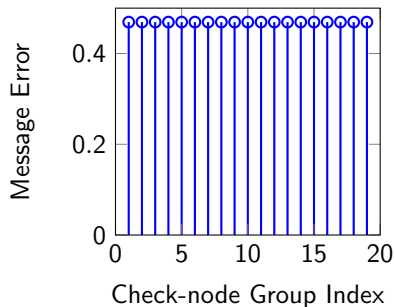
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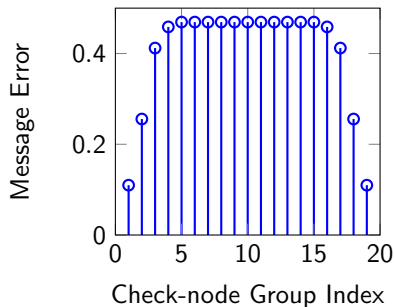
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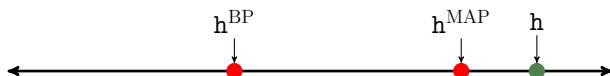
Uncoupled



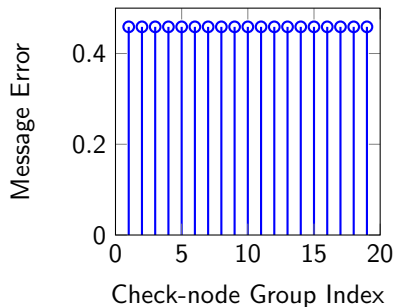
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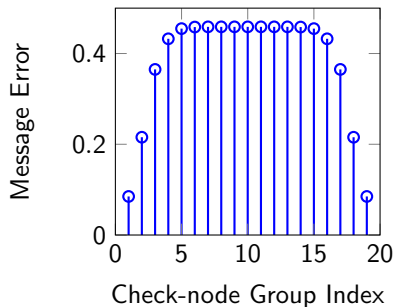
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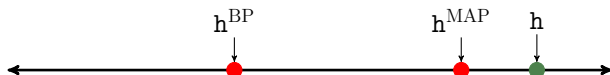
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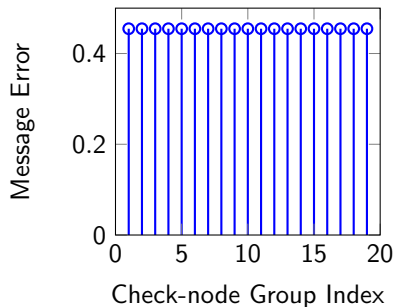
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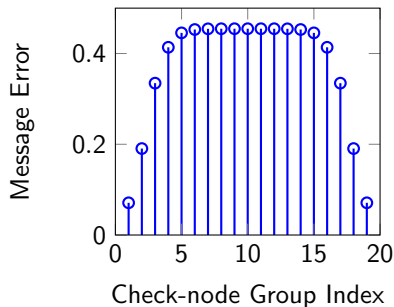
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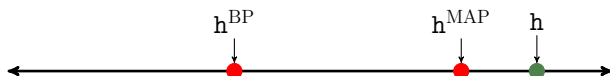
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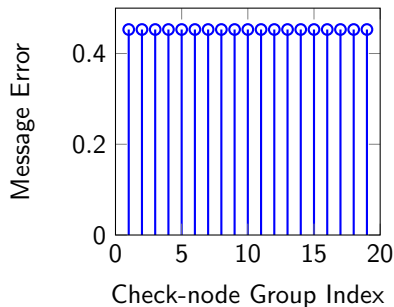
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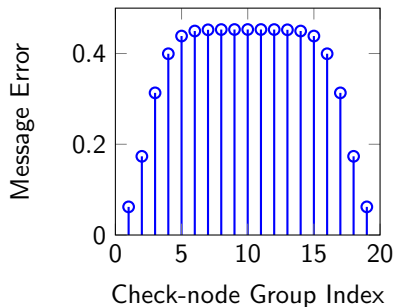
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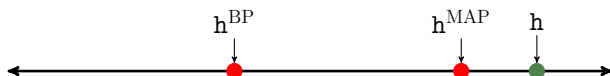
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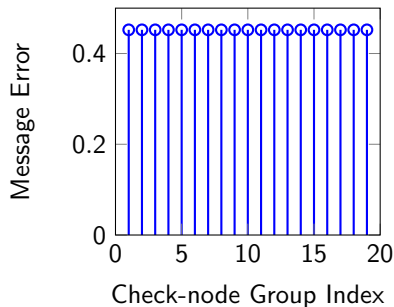
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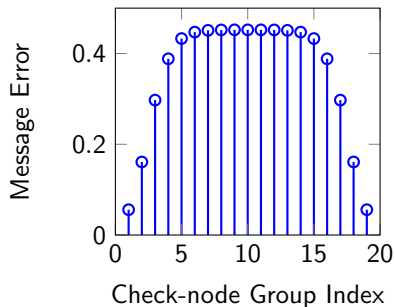
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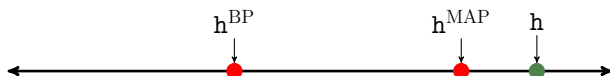
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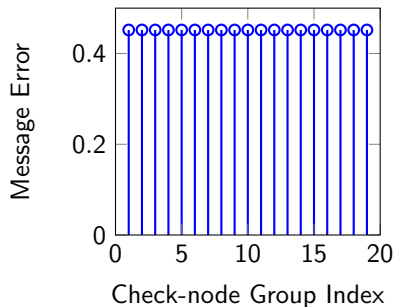
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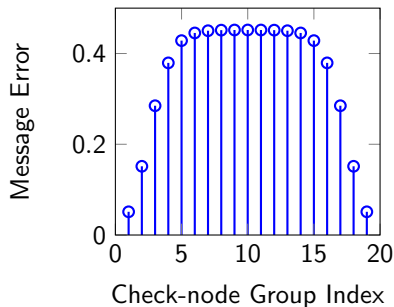
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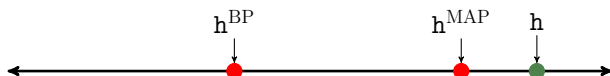


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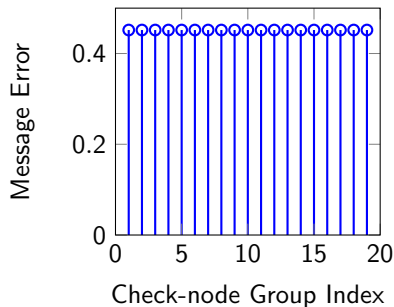




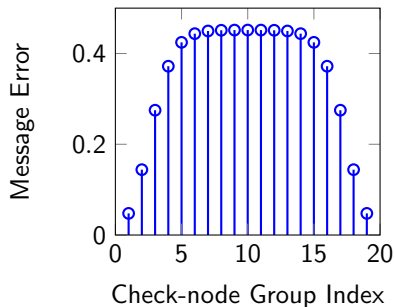
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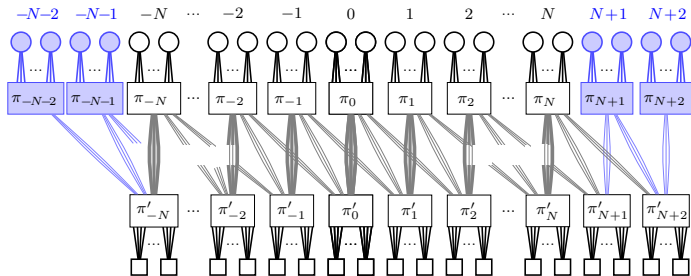


# Threshold Saturation Result

MAP Performance with a BP Decoder!

For large  $N$ , w  $\mathbf{h}_c^{\text{BP}} = \mathbf{h}^{\text{MAP}}$

SC-LDPC ( $\ell, r$ )	Shannon $\mathbf{h}^{\text{Sh}}$	AWGN $\mathbf{h}_c^{\text{BP}}$	BSC $\mathbf{h}_c^{\text{BP}}$
(3, 6)	0.5000	0.4794	0.4681
(4, 6)	0.6667	0.6645	0.6633
(5, 6)	0.8333	0.8333	0.8333



# Rate loss for finite $N$ and $w$

SC-LDPC ( $\ell, r, N, w$ )	Shannon $h^{\text{Sh}}$	AWGN $h_c^{\text{BP}}$	BSC $h_c^{\text{BP}}$
(3, 6, 10, 3)	0.5434	0.4794	0.4681
(3, 6, 20, 3)	0.5222	0.4794	0.4681
(3, 6, 30, 3)	0.5149	0.4794	0.4681
(4, 6, 10, 3)	0.7245	0.6645	0.6633
(4, 6, 20, 3)	0.6963	0.6645	0.6633
(4, 6, 30, 3)	0.6866	0.6645	0.6633
(5, 6, 10, 3)	0.9056	0.8333	0.8333
(5, 6, 20, 3)	0.8704	0.8333	0.8333
(5, 6, 30, 3)	0.8582	0.8333	0.8333

# Pros & Cons

## Pros

- ▶ Significant improvement in **thresholds**
- ▶ Achieves capacity under **simple BP decoding**
- ▶ **Universality** - works for all channels models!

## Cons

- ▶ Need **large** blocklengths to leverage the gains

# Outline

- 1 Spatial Coupling(SC)
  - Introduction
  - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
  - Introduction
  - Proposed Lattice Construction
  - Poltyrev Goodness
  - Application to Symmetric Interference Channel
- 3 Side-Information Problems
  - Introduction
  - Compound Codes
  - Spatial Coupling
- 4 Write-Once Memory

# Lattices and Lattice Codes

## Lattice

A lattice of dimension  $n$  is a discrete subgroup of  $\mathbb{R}^n$  isomorphic to  $\mathbb{Z}^n$

$$\Lambda = \{\mathbf{G}\mathbf{z}, \mathbf{z} \in \mathbb{Z}^n\}$$

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- ▶ Efficient structures for
  - **Mathematics**: sphere packing and sphere covering problems
  - **Information Theory**: channel coding & quantization
- ▶ Single user Gaussian channel - Erez and Zamir
- ▶ Coding with side information - Wyner-Ziv and Costa, Zamir, Erez and Shamai
- ▶ Secrecy - He and Yener
- ▶ Dirty multiple access channel - Philosof, Khisti, Erez and Zamir

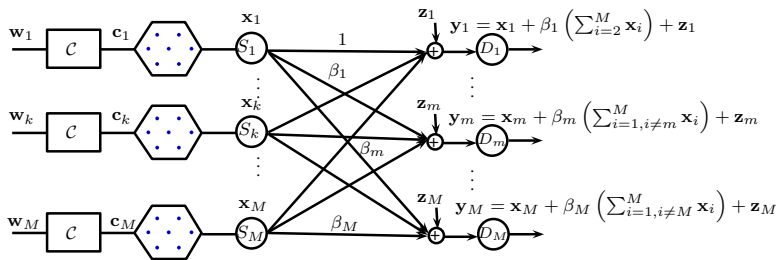
“Lattices are everywhere” by Ram Zamir



# Prior Work

New perspectives for dealing with interference:

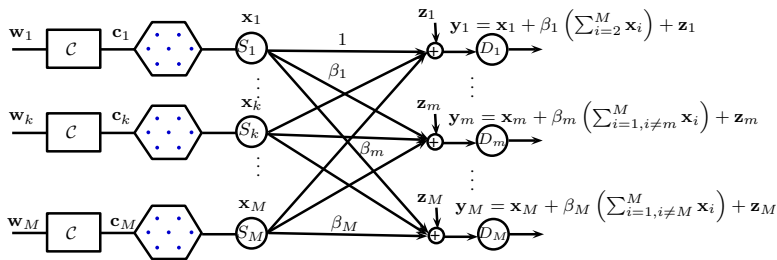
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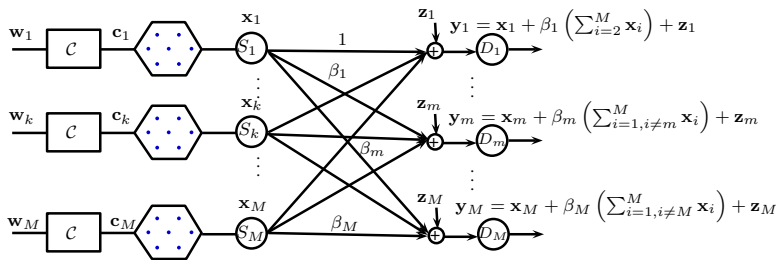
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- Above schemes are all based on lattices **good** for channel coding

# Lattices good for Channel Coding

- ▶ Voronoi region  $\mathcal{V}$  of a lattice  $\Lambda$ ,  $\mathcal{V} := \{\mathbf{x} : \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{c}\| \quad \forall \mathbf{c} \in \Lambda\}$
- ▶ Fundamental volume of  $\Lambda$ ,  $V(\Lambda)$ :  $\text{Vol}(\mathcal{V})$
- ▶ Let a lattice point  $\lambda \in \Lambda$  is transmitted via AWGN channel of variance  $\sigma^2$
- ▶ Volume-to-noise ratio(VNR) of  $\Lambda$ :

$$\text{VNR} = \frac{V(\Lambda)^{2/n}}{2\pi e \sigma^2}$$

- ▶  $P(\Lambda, \sigma^2) := \Pr(d(\lambda, \lambda + \mathbf{z}) \geq d(\lambda', \lambda' + \mathbf{z}))$  for some  $\lambda' \in \Lambda$

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## Poltyrev Goodness for Channel Coding

For any  $\text{VNR} > 1 \exists \{\Lambda_n\}$  such that  $P(\Lambda_n, \sigma^2) \rightarrow 0$  as  $n \rightarrow \infty$ .

- ▶ *Poltyrev-good* lattices are at the core of such lattice coding schemes

# Objective

## Motivating questions

- ▶ These results are all based on Construction-A.
  - Linear codes over increasing field sizes and their ML decoding

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## Motivating questions

- ▶ These results are all based on Construction-A.
  - Linear codes over increasing field sizes and their ML decoding
- ▶ Is this construction fundamental to good lattices?
- ▶ Can we work with just binary codes under practical decoding schemes?

# Main Results

## Codes over $\mathbb{F}_2$ and BP decoding suffice

- ▶ Recall Forney et al's result - based on nested random binary linear codes
- ▶ Propose capacity-achieving nested SC LDPC ensemble
- ▶ Construct lattices using Construction-D, based on the above ensemble
- ▶ Show existence of sequence of lattices that are *Poltyrev*-good under BP



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## Applications

- ▶ As an application, propose [Symmetric Interference Channel](#)
- ▶ Can be applied to other problems which adopt Construction A lattices

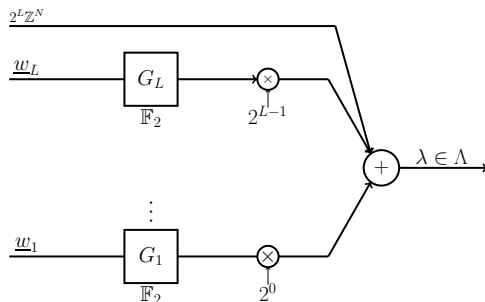
# Construction D with $L$ levels

- Barnes and Sloane '83, Forney, Chung and Trott '00, Yan, Ling, Wu ' 13
- Choose  $G_1 \subseteq \dots \subseteq G_L$  where  $G_l$  is a gen matrix of code  $\mathcal{C}_l$  over  $\mathbb{F}_2$ .

$$\begin{array}{c}
 \left[ \begin{array}{ccc} \cdots & g_L & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_{k_2} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_{k_1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_2 & \cdots \\ \cdots & g_1 & \cdots \end{array} \right] \begin{array}{l} \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_L \\ \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} \dots \\ \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_2 \\ \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_1 \end{array}
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- ▶  $\underline{\lambda} = \underline{w}_1 \mathbf{G}_1 + 2\underline{w}_2 \mathbf{G}_2 \dots + 2^{L-1} \underline{w}_{L-1} \mathbf{G}_{L-1} + 2^L \mathbb{Z}^N \in \Lambda$



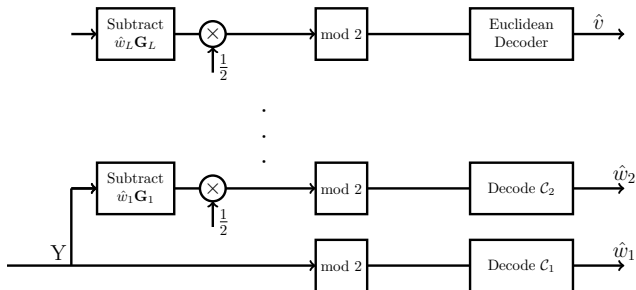
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# Multi-Level Decoding(Successive Cancellation)

$$\blacktriangleright \underline{y} = \boxed{\underline{w}_1 \mathbf{G}_1 + 2\underline{w}_2 \mathbf{G}_2 \dots + 2^{L-1} \underline{w}_{L-1} \mathbf{G}_{L-1} + 2^L \mathbb{Z}^N} + \underline{n}$$

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- ▶  $\underline{y} \bmod 2 = [\underline{w}_1 \mathbf{G}_1 + \underline{n}] \bmod 2 = \underline{w}_1 \odot \mathbf{G}_1 + \boxed{\underline{n} \bmod 2}$
- ▶ Decode  $\underline{w}_1$ , reconstruct  $\underline{w}_1 \mathbf{G}_1$  and subtract from  $\underline{y}$



### Theorem (Forney, Trott & Chung)

*There exists a sequence of Construction D lattices based on  $\mathcal{C}_1 \subseteq \mathcal{C}_2 \dots \subseteq \mathcal{C}_L$  such that the  $VNR \rightarrow 1$  and the  $Pr(\lambda, \sigma^2) \rightarrow 0$ .*

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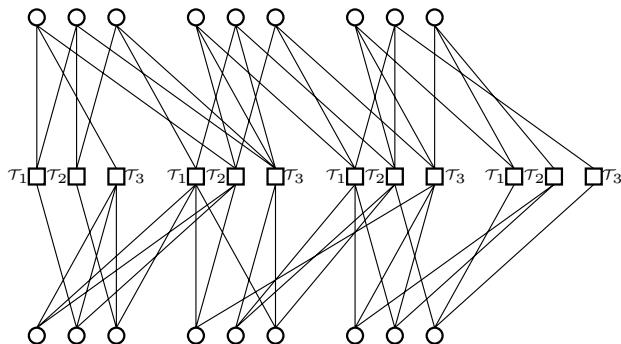
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- ▶ It's sufficient that  $\mathcal{C}_i$  at each level is capacity achieving for the mod-2 AWGN channel.

Objective:

- ▶ Capacity achieving nested code constructions, preferably under BP decoding.

# Proposed Nested Spatially-Coupled LDPC Ensemble

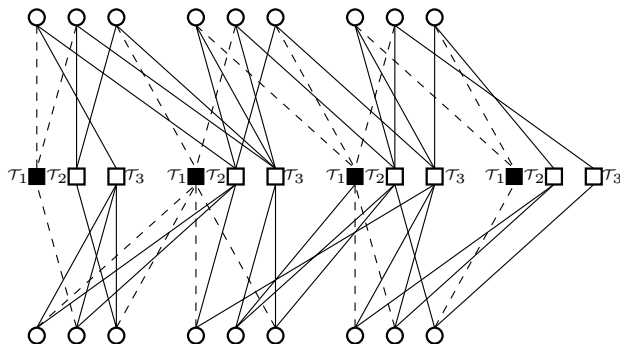
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- 2 Group check nodes into type  $\mathcal{T}_k, k \in \{1, \dots, d_v^1\}$





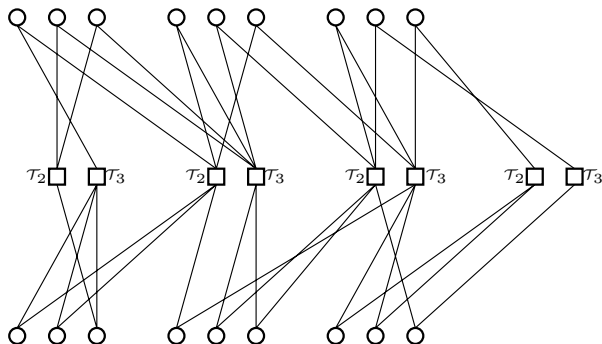
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- 3 Remove all check nodes of type  $\mathcal{T}_1, \dots, \mathcal{T}_{d_v^1 - d_v^2}$ . Ex:  $(d_v^2 = 2, 6)$  sup-code.
- 4 Results in a super-code that is a  $(d_v^2, d_c)$  SC LDPC code.



# Lattice Design based on the proposed Nested SC LDPC ensemble

- 1 For a given  $\sigma$ , compute the capacity of the mod-2 AWGN channel at each level:

$$\underline{y}_i = \underline{w}_i \mathbf{G}_i + \frac{1}{2^{i-1}} \underline{n} \mod 2 = \underline{w}_i \odot \mathbf{G}_i + \boxed{\frac{1}{2^{i-1}} \underline{n} \mod 2}$$

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## Lemma

*Given nested binary linear codes  $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{C}_r$  there exists nested generator matrices for these codes.*

# Proposed Ensemble is Capacity achieving

## Theorem

*Each code ensemble in the proposed nested Spatially-Coupled LDPC ensemble is capacity achieving.*

## Proof.

- ▶ Show that the mod 2 AWGN channel is BMS.
- ▶ Each derived protograph has the same spatially coupled structure.
- ▶ The proof follows from Kudekar & Urbanke, Kumar & Pfister's results.



# Proposed Lattices are Poltyrev-Good

## Theorem

*There exists a sequence of SC LDPC lattices with  $VNR(\Lambda, \sigma^2) \rightarrow 1$  for which, under multistage BP decoding,  $\mathbb{E} [P(\lambda, \sigma^2)] \rightarrow 0$  as  $w, L, M \rightarrow \infty$ .*

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- ▶ Binary codes and more importantly practical BP decoding suffices.
- ▶ Practically we observe that two levels of coding gets you lattices very close to Poltyrev limit.

# Design Example of Poltyrev-Good Lattice

Target error probability  $P(2^L \mathbb{Z}^n, \sigma_L^2) = 10^{-4}$  in the uncoded level  $\implies \sigma_L = 0.08$



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	Level L-1	Level L-2	Level L-3
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- ② Fix  $L=3$  and use  $(3, 30), (14, 30)$  nested SC LDPC codes.

- Note  $P(4\mathbb{Z}^n, \sigma^2) \approx nP(4\mathbb{Z}, \sigma^2)$
- We fix  $n = 2 \times 10^5$

$(d_c, d_v^1, d_v^2)$	$(L, w)$	$P(\mathbb{Z}_4, \sigma^2)$	$\sigma_{\text{max}}$	VNR	VNR <sub>rate-loss</sub>
$(30, 14, 3)$	$(32, 4)$	$5 \times 10^{-10}$	0.3184	1.02dB	1.347dB

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Cap	0.99	0.57	0.02
(14,30) (3,30)	0.9	0.533	0

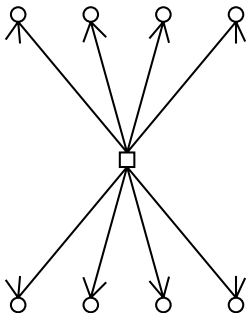
- ② Fix  $L=3$  and use (3, 30), (14, 30) nested SC LDPC codes.

- Note  $P(4\mathbb{Z}^n, \sigma^2) \approx nP(4\mathbb{Z}, \sigma^2)$
- We fix  $n = 2 \times 10^5$

$(d_c, d_v^1, d_v^2)$	(L,w)	$P(\mathbb{Z}_4, \sigma^2)$	$\sigma_{\text{max}}$	VNR	VNR <sub>rate-loss</sub>
(30,14,3)	(32,4)	$5 \times 10^{-10}$	0.3184	1.02dB	1.347dB
(60, 26, 3)	(72, 12)	$5 \times 10^{-10}$	0.3200	0.482dB	0.927dB
(60, 27, 3)	(64, 9)	$5 \times 10^{-10}$	0.3203	0.57dB	0.951dB

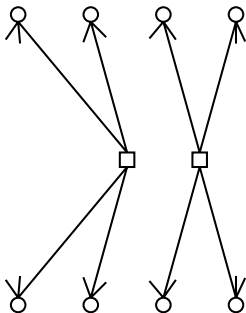
# Alternate Nested SC LDPC ensemble

- Derive a lower rate code by “splitting the checks”
- Consider a  $(3, 8)$  code

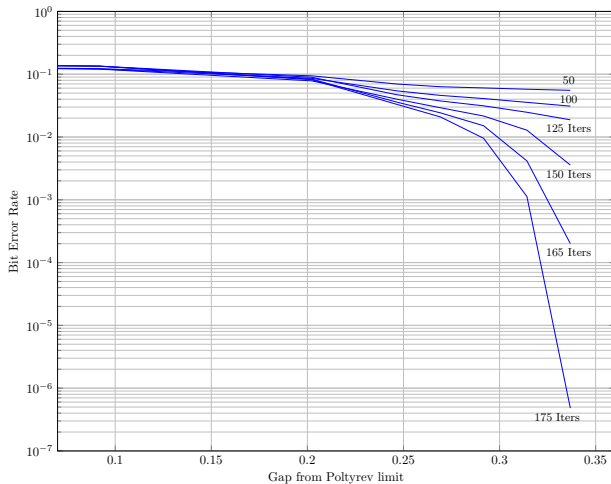


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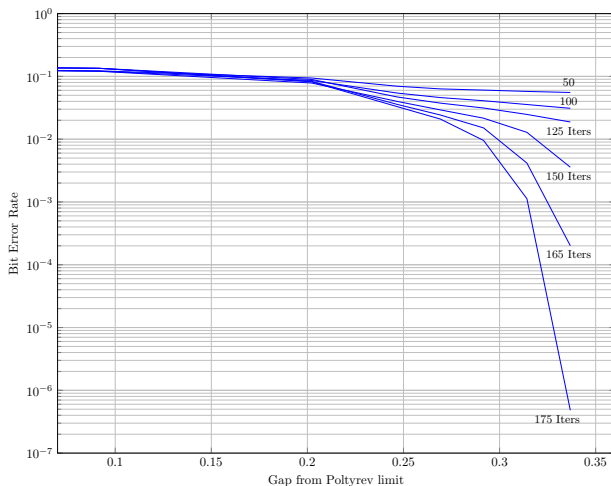
- ▶ Derive a lower rate code by “splitting the checks”
- ▶ Consider a  $(3, 8)$  code
- ▶ Split each check into “two” checks to derive a  $(3, 4)$  sub-code
- ▶ Easy to prove that resulting code is from the  $(3, 4)$  SC LDPC ensemble



# Simulation Results

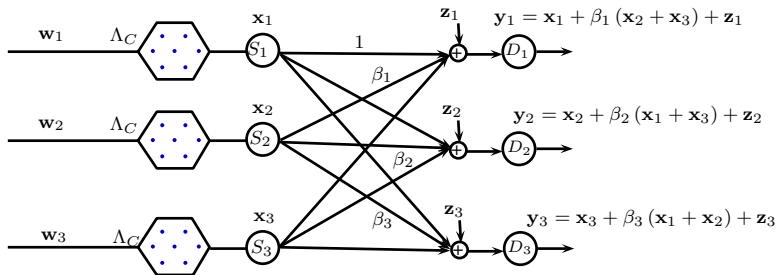


# Simulation Results



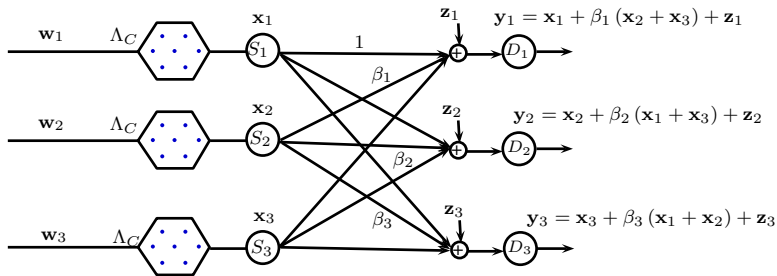
Note that the Block Error Probability is  $10^{-4}$  at uncoded level.

# 3-User Symmetric Interference Channel





# 3-User Symmetric Interference Channel



►  $\mathbf{x}_i \in \Lambda_C \triangleq \Lambda \cap \mathbb{Z}_4^N$  is transmitted.

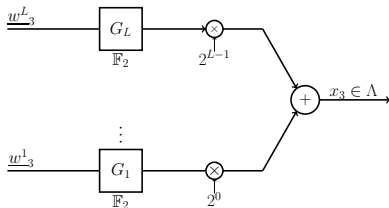
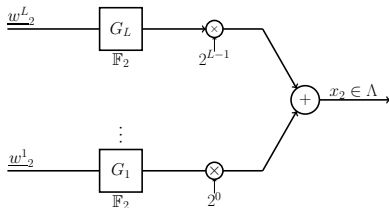
# Symmetric Interference Channel - Decoding Sums

Interference at Destination 1:

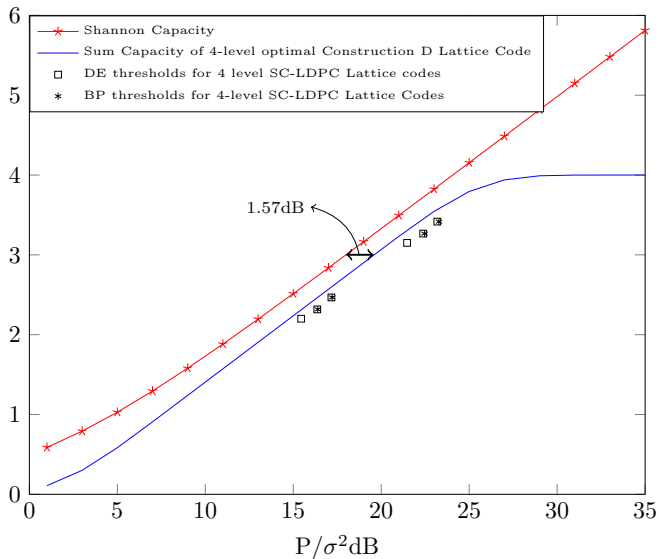
$$\begin{aligned}\mathbf{x}_2 + \mathbf{x}_3 &= (\underline{w}_2^1 + \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{w}_2^2 + \underline{w}_3^2)\mathbf{G}_2 + 4\mathbf{k}_{23} \\ &= (\underline{w}_2^1 \oplus \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{c}_{23}^1 \oplus \underline{w}_2^2 \oplus \underline{w}_3^2)\mathbf{G}_2 + 4(\underline{c}_{23}^2 + \mathbf{k}_{23})\mathbf{Z}\end{aligned}$$

where the carry overs are

$$\begin{aligned}\underline{c}_{23}^1 &= 0.5 (\underline{w}_1^1 + \underline{w}_1^2 - \underline{w}_1^1 \oplus \underline{w}_1^2), \\ \underline{c}_{23}^2 &= 0.5 (\underline{c}_{23}^1 + \underline{w}_2^1 + \underline{w}_2^2 - \underline{c}_{23}^1 \oplus \underline{w}_2^1 \oplus \underline{w}_2^2)\end{aligned}$$



# Achievable Information Rates



# Concluding Remarks

- ▶ Multilevel constructions - efficient ways to decode integer combinations
- ▶ Need capacity achieving nested codes
- ▶ Multilevel construction is provably good under message passing decoding

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- ▶ Multilevel constructions - efficient ways to decode integer combinations
- ▶ Need capacity achieving nested codes
- ▶ Multilevel construction is provably good under message passing decoding
- ▶ Coding schemes based on Binary LDPC codes and iterative decoding suffice

# Outline

- 1 Spatial Coupling(SC)
  - Introduction
  - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
  - Introduction
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  - Application to Symmetric Interference Channel
- 3 Side-Information Problems
  - Introduction
  - Compound Codes
  - Spatial Coupling
- 4 Write-Once Memory

# Lossy Source Coding Problem

$$X^n = (X_1, \dots, X_n), \text{ } X_i \sim \text{Bernoulli}(\frac{1}{2})$$

Binary code  $\mathcal{C} = (n, k)$ , rate  $R = k/n$

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## Lossy Source Coding

- ▶ Compress  $X^n$  to  $\hat{X}^n \in \mathcal{C}$
- ▶ Min. Hamming distortion

$$D = \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i - \hat{X}_i|$$



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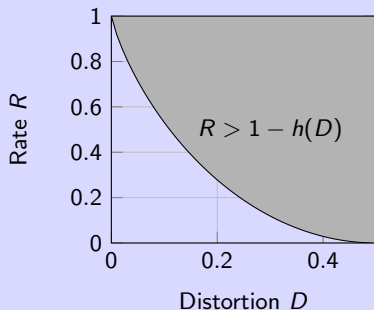
$$D = \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i - \hat{X}_i|$$

- Rate-Distortion theory:

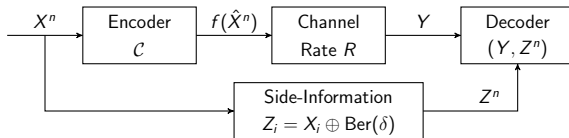
$$R > 1 - h(D)$$

- $h(\cdot)$  is binary entropy function

$$h(D) = -D \log_2 D - (1-D) \log_2 (1-D)$$



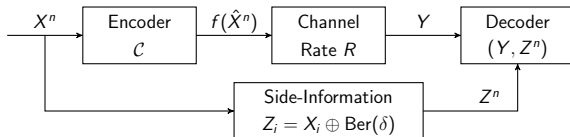
# Side-Information Problems: Wyner-Ziv



## Wyner-Ziv Formulation

- ▶ Side-information  $Z^n$  about  $X^n$
- ▶ Decoder additionally has  $Z^n$
- ▶ Say  $Z_i = X_i \oplus \text{Ber}(\delta)$

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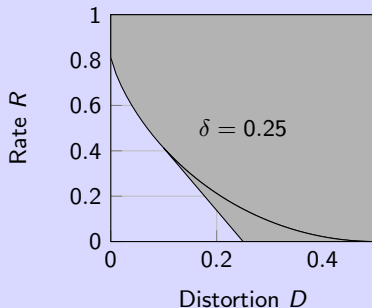


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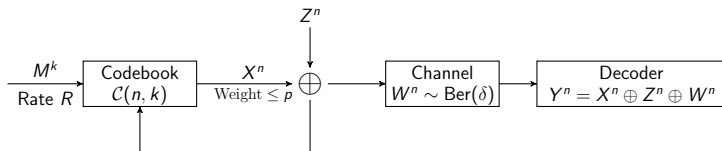
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- ▶ Wyner-Ziv theory:

$$R > \text{l.c.e}\{h(D * \delta) - h(D), (\delta, 0)\}$$

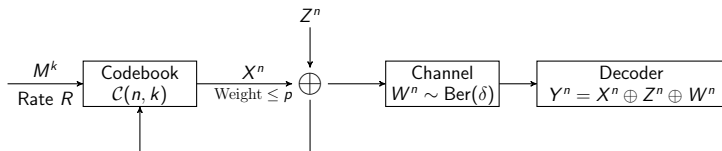
- ▶  $D * \delta = D(1 - \delta) + \delta(1 - D)$



# Side-Information Problems: Gelfand-Pinsker



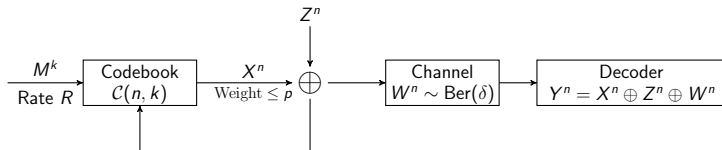
# Side-Information Problems: Gelfand-Pinsker



## Gelfand-Pinsker Formulation

- Message  $M^k$  encoded to  $X^n \in \mathcal{C}$  with  $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] \leq p \leq \frac{1}{2}$
- Side-information  $Z^n$  is available **only at the encoder**

# Side-Information Problems: Gelfand-Pinsker

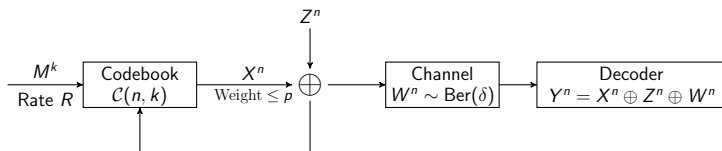


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$$Y^n = X^n \oplus Z^n \oplus W^n, \quad \{W_i\} \sim \text{Ber}(\delta)$$

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- ▶ Capacity region by Gelfand-Pinsker:

$$R < h(p) - h(\delta)$$

# Main Result

## Objective

- ▶ Construct **low-complexity** coding schemes that achieve the **complete rate regions** of Wyner-Ziv and Gelfand-Pinsker
  - Low-complexity encoding and decoding



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## Idea

- ▶ Wainwright et al. used compound LDGM/LDPC codes with **optimal encoding/decoding**
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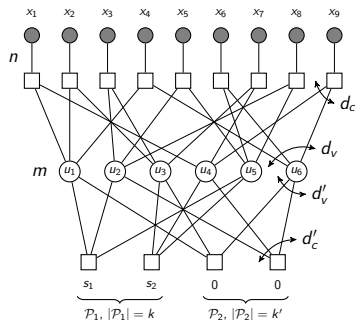
## Objective

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## Idea

- ▶ Wainwright et al. used compound LDGM/LDPC codes with **optimal encoding/decoding**
- ▶ Message-passing algorithms have **non-negligible gap**
- ▶ Remedy via **Spatial-Coupling**
  - Channel coding in coupled compound codes (Kasai et al.)
  - Lossy source coding with spatially-coupled LDGM (Aref et al.)
  - Encoding with **compound codes has additional challenges**

# Compound LDGM/LDPC Codes



► Codebook  $\mathcal{C}(n, m - k - k')$

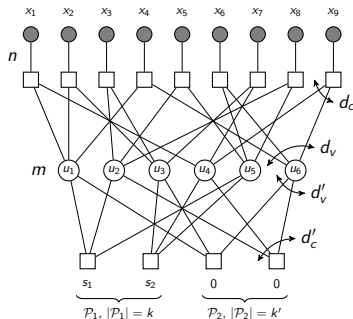
► Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

► Codeword  $(x_1, \dots, x_9)$ :

$$x_1 = u_1 \oplus u_4, \quad x_2 = \dots$$

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## Key Properties

- Compound code is
  - a **good source code** under optimal encoding
  - a **good channel code** under optimal decoding
- LDGM code is
  - a **good source code** under optimal encoding
  - (side note) LDGM code is **not** a good channel code

# Good Code

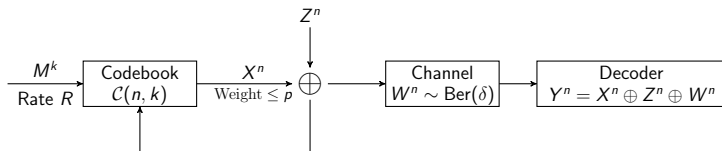
## “Good” source code

- ▶ Rate of the code is  $R = 1 - h(D) + \varepsilon$
- ▶ When this code is used to **optimally encode**  $\text{Ber}(\frac{1}{2})$
- ▶ The average Hamming **distortion is at most**  $D$

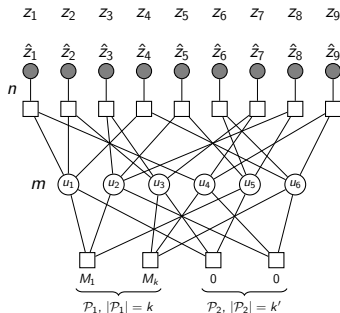
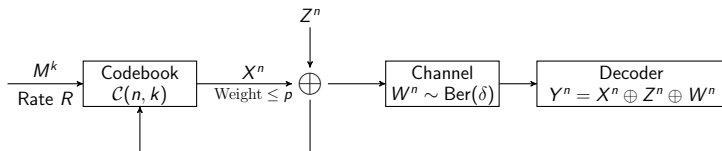
## “Good” channel code

- ▶ Rate of the code is  $R = 1 - h(\delta) - \varepsilon$
- ▶ When this code is used for channel coding on  $\text{BSC}(\delta)$
- ▶ Message est. under **optimal decoding** with **error at most**  $\varepsilon$

# Coding Scheme: Gelfand-Pinsker



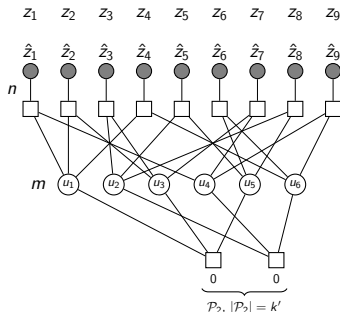
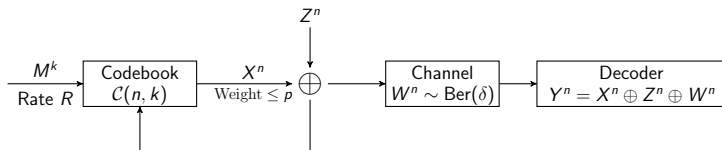
# Coding Scheme: Gelfand-Pinsker



- With message  $M^k$ , encode  $Z^n$  to  $\hat{Z}^n$  (Distortion  $\approx p$ )
- Transmit  $X^n = Z^n \oplus \hat{Z}^n$

$$\frac{m-k-k'}{n} \approx 1 - h(p) + \varepsilon \quad \frac{m-k'}{n} \approx 1 - h(\delta) + \varepsilon$$

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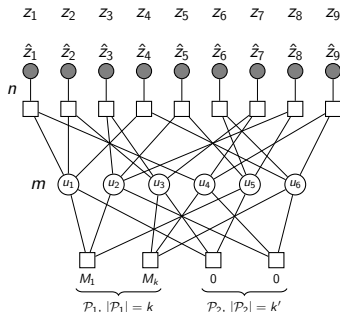
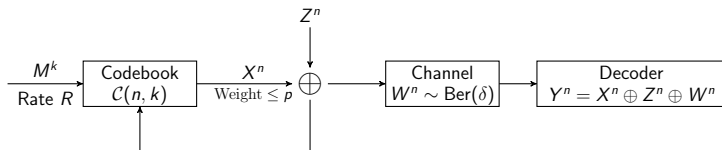
► Decoder has

$$\begin{aligned} Y^n &= X^n \oplus Z^n \oplus W^n \\ &= \hat{Z}^n \oplus W^n \end{aligned}$$

► Decode  $\hat{Z}^n$  and compute  $M^k$



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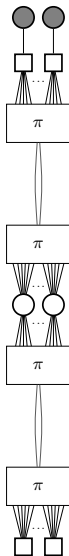
► Decode  $\hat{Z}^n$  and compute  $M^k$

►  $R = \frac{k}{n} \approx h(p) - h(\delta)$

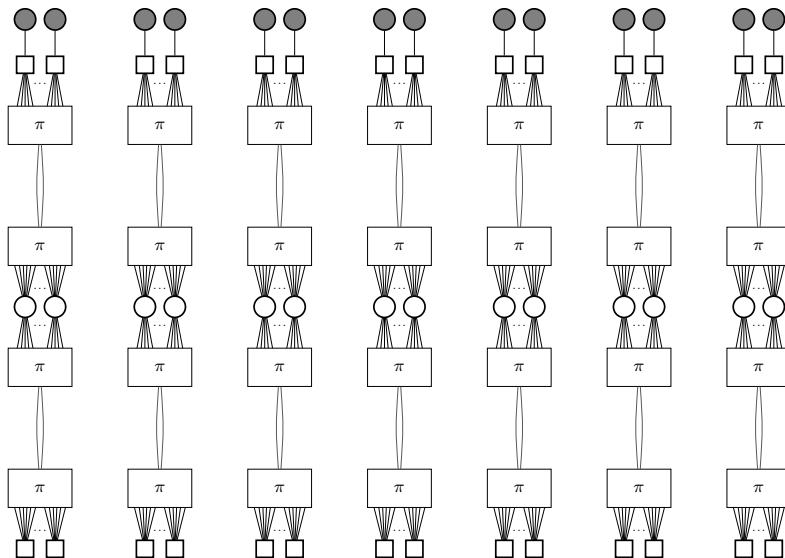
# Remarks

- ▶ Need codes that are **simultaneously good** for channel and source coding
- ▶ Use **message-passing algorithms** instead of **optimal**
- ▶ Use spatial-coupling for **goodness** of codes under message-passing

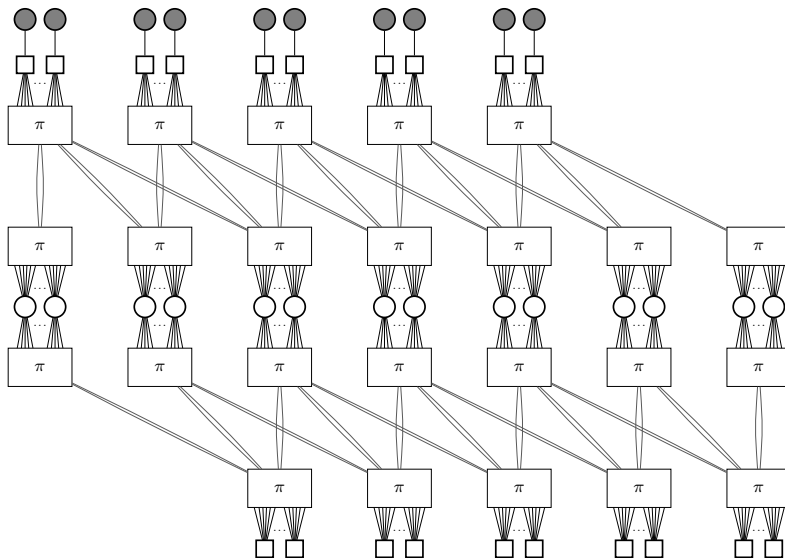
# Spatially-Coupled Compound LDGM/LDPC Codes



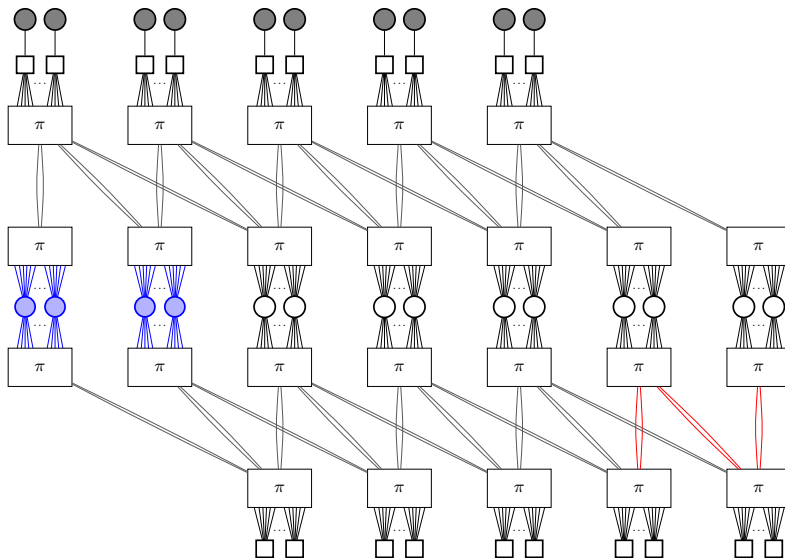
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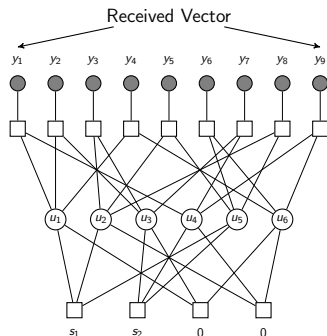
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# Decoding in Spatially-Coupled Compound Codes



Channel LLR

$y_i$

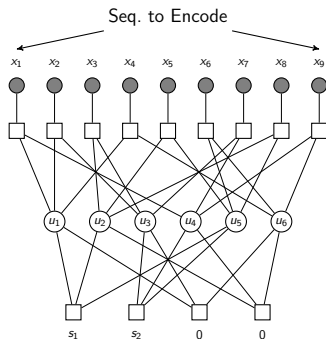
$$L = L_1 + \dots + L_k$$

$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

## Remarks

- ▶ Standard message-passing algorithm
- ▶ Threshold saturation proven for SC compound codes on BEC
- ▶ Empirically observed for BMS channels

# Encoding in Spatially-Coupled Compound Codes



$$(-1)^x \tanh \beta$$

$x_i$

$$L = L_1 + \dots + L_k$$

$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

## Remarks

- ▶ Inverse temperature parameter  $\beta$
- ▶ Message-passing rules are the same
- ▶ However, a **crucial decimation step is needed**



# Encoding in SC Compound Codes: BPGD Algorithm

# Encoding in SC Compound Codes: Remarks

- ▶ Randomization in setting  $u_{j^*}$  is crucial
- ▶ BPGD applied to uncoupled code always failed
- ▶ Spatially-coupled structure is crucial for successful encoding
  - In addition, distortion is close to optimal thresholds
  - Does not encode if decimated from both left and right
  - Does not encode if both left and right boundary is set to 0

# Encoding in SC Compound Codes: Numerical Example

Block length ( $n$ )	4-cycles	Attempts 1/2/3/4/ $\geq 5$
9000	yes	5/3/5/2/35
9000	no	21/12/5/3/9
27000	no	35/15/0/0/0
45000	no	40/9/0/0/1
63000	no	44/6/0/0/0
81000	no	50/0/0/0/0

## Remarks

- ▶ # Attempts to encode 50 seq. in  $(6, 3)$  LDGM /  $(3, 6)$  LDPC
- ▶  $L = 20$ ,  $w = 4$ ,  $\beta = 0.65$ ,  $T = 10$
- ▶ Removing 4-cycles dramatically improves success
- ▶ How much do 6-cycles matter?

# Numerical Results: Wyner-Ziv

LDGM ( $d_v, d_c$ )	LDPC ( $d'_v, d'_c$ )	( $L, w$ )	( $D_*, \delta_*$ )	( $D, \delta$ )
(6, 3)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1174, 0.122)
(8, 4)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1149, 0.120)
(10, 5)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1139, 0.122)

## Remarks

- $D_*$  and  $\delta_*$  are calculated based on the rate of the respective code:

$$D_* = h^{-1}(1 - R1)$$

$$\delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000$ ,  $\beta = 1.04$ ,  $T = 10$

# Numerical Results: Gelfand-Pinsker

LDGM ( $d_v, d_c$ )	LDPC ( $d'_v, d'_c$ )	( $L, w$ )	( $p_*, \delta_*$ )	( $p, \delta$ )
(6, 3)	(3,6)	(20,4)	(0.215,0.157)	(0.2200,0.152)
(8, 4)	(3,6)	(20,4)	(0.215,0.157)	(0.2230,0.151)
(10, 5)	(3,6)	(20,4)	(0.215,0.157)	(0.2200,0.151)

## Remarks

- $p_*$  and  $\delta_*$  are calculated based on the rate of the respective code:

$$p_* = h^{-1}(1 - R_1)$$

$$\delta_* = h^{-1}(1 - R_2)$$

- $n \approx 140000$ ,  $\beta = 0.65$ ,  $T = 10$

# Concluding Remarks

## Conclusion

- ▶ Spatially-coupled codes achieve the rate regions of Wyner-Ziv and Gelfand-Pinsker problems
- ▶ **Coupling structure** is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed with decimation

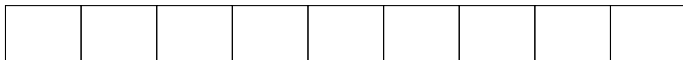
## Open Questions

- ▶ Effect of degree profiles, short-cycles on encoding success
- ▶ Precise trade-offs with **polar codes**

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# Write-Once Memories

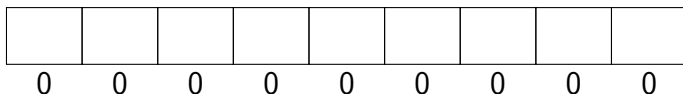


## Flash Memory

- ▶ In typical flash memory, changing from 0 to 1 is easy
- ▶ Resetting 1 to 0 requires **rewriting whole block**
- ▶ Write-once memories model such storage systems



# Write-Once Memories



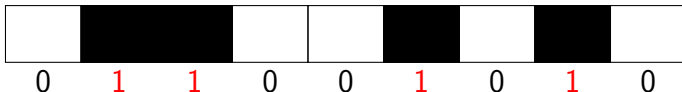
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## Binary Write-Once Memories

- ▶ **0 → 1 is allowed**

# Write-Once Memories



## Flash Memory

- ▶ In typical flash memory, changing from 0 to 1 is easy
- ▶ Resetting 1 to 0 requires **rewriting whole block**
- ▶ Write-once memories model such storage systems

## Binary Write-Once Memories

- ▶ 0  $\rightarrow$  1 is allowed
- ▶ 1  $\rightarrow$  0 is forbidden

## Capacity Region (I) - Noiseless

Message



0	0	1	0	0	1	0	1	0
---	---	---	---	---	---	---	---	---

### Write-Once Memory without Noise

- ▶ In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- ▶ Only about  $nt/\log(t)$  cells required to store  $n$  bits for  $t$  writes

# Capacity Region (I) - Noiseless

Message



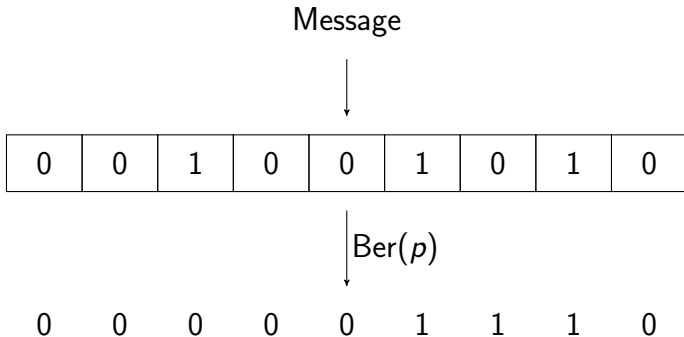
0	0	1	0	0	1	0	1	0
---	---	---	---	---	---	---	---	---

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  - 2 bits in 2 writes with only 3 cells
- ▶ Only about  $nt / \log(t)$  cells required to store  $n$  bits for  $t$  writes
- ▶ In 1985, Heegard gave the **capacity** for  $t$ -write system
- ▶ For a 2-write system, it is

$$\left\{ (R_1, R_2) \mid 0 \leq R_1 < h(\delta), 0 \leq R_2 < 1 - \delta \right\}$$

## Capacity Region (II) - Read Errors



### Write-Once Memory with Read Errors

- ▶ Different from write errors
- ▶  $Y = X \oplus \text{Ber}(p)$ , where  $\text{Ber}(p)$  denotes the Bernoulli noise
- ▶ Capacity region is **unknown**

# Main Result

## Objective

- ▶ Construct **low-complexity** coding schemes that achieve the **capacity region** of the WOM system
  - Low-complexity encoding and decoding

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$$R_1 < h(\delta) - h(p),$$

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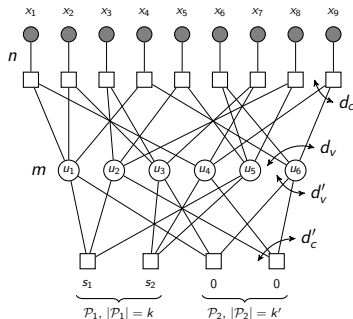
$$R_2 < 1 - \delta - h(p).$$

- ▶ Extension to multi-write systems **seems possible with BPGD**

## Idea

- ▶ Use compound LDGM/LDPC codes
- ▶ Encoding for second write is **erasure quantization**
- ▶ Use **spatial coupling with message-passing**

# Compound LDGM/LDPC Codes



- Codebook  $(n, m - k - k')$

- Message constraints

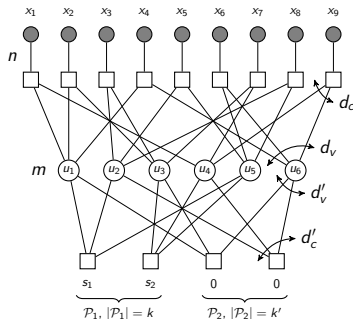
$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword  $(x_1, \dots, x_9)$ :

$$x_1 = u_1 \oplus u_4, \quad x_2 = \dots$$

- Parametrized by  $s^k$ :  $\mathcal{C}(s^k)$

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## Key Properties of Compound Codes

- a natural **coset decomposition**:  $\mathcal{C} = \bigcup_{s^k \in \{0,1\}^k} \mathcal{C}(s^k)$
- achieves capacity over eras. chan. under MAP (when  $m = n$ )
- a **good source code** under optimal encoding
- a **good channel code** under optimal decoding

# Good Code

## “Good” source code

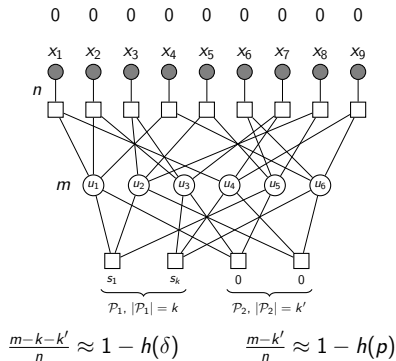
- ▶ Rate of the code is  $R = 1 - h(\delta) + \varepsilon$
- ▶ When this code is used to **optimally encode**  $\text{Ber}(\frac{1}{2})$
- ▶ The average Hamming **distortion is at most**  $\delta$

## “Good” channel code

- ▶ Rate of the code is  $R = 1 - h(p) - \varepsilon$
- ▶ When this code is used for channel coding on  $\text{BSC}(p)$
- ▶ Message est. under **optimal decoding** with **error at most**  $\varepsilon$

# Coding Scheme for 2-write WOM: First Write

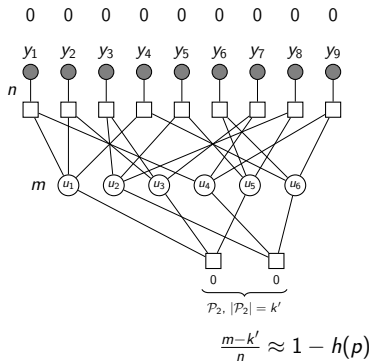
$$R_1 < h(\delta) - h(p)$$



- With message  $s^k$ , encode  $0^n$  to  $x^n$  (Distortion  $\approx \delta$ )
- Store  $x^n$

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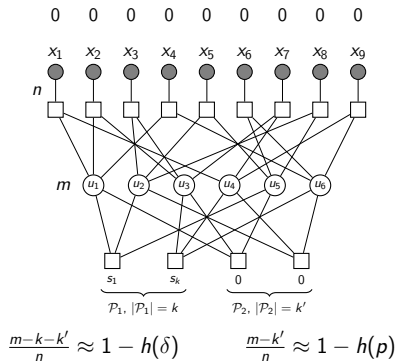


- ▶ With message  $s^k$ , encode  $0^n$  to  $x^n$  (Distortion  $\approx \delta$ )
- ▶ **Store  $x^n$**
- ▶ Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

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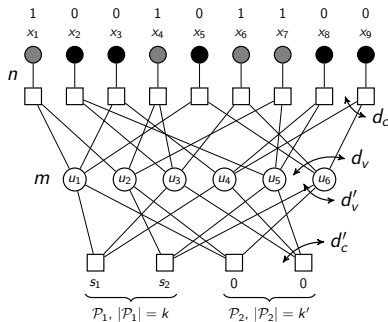


- ▶ With message  $s^k$ , encode  $0^n$  to  $x^n$  (Distortion  $\approx \delta$ )
- ▶ Store  $x^n$
- ▶ Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

- ▶ Dec.  $x^n$  and compute  $s^k$
- ▶  $R_1 = \frac{k}{n} \approx h(\delta) - h(p)$

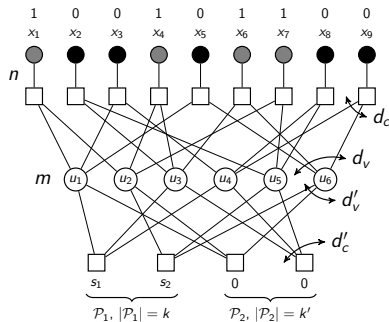
# Coding Scheme for 2-write WOM: Second Write



- Need to find a **consistent** codeword in  $\mathcal{C}(s^k)$



# Coding Scheme for 2-write WOM: Second Write



- Need to find a **consistent** codeword in  $\mathcal{C}(s^k)$
- Closely related to **Binary Erasure Quantization (BEQ)**
- En Gad, Huang, Li and Bruck (ISIT 2015)

# Binary Erasure Quantization

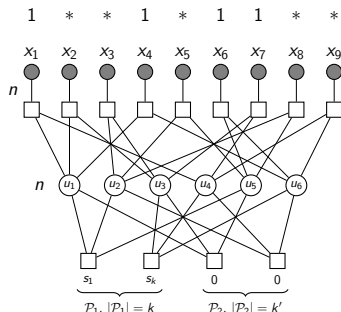
- ▶ Quantize a sequence in  $\{0, 1, *\}^n$  to  $x^n \in \mathcal{C} \subset \{0, 1\}^n$ 
  - 0's and 1's should **match exactly**
  - \*'s can take **either 0 or 1**
- ▶ Can map the second write of 2-write WOM to BEQ
  - Map 0's to \*'s and keep 1's
  - Quantize to codeword in  $\mathcal{C}(s^k)$
- ▶ BEQ is the dual of decoding on binary erasure channel
  - Martinian and Yedidia (Allerton 2003)
  - Can quan. all seq. with erasure pattern  $e^n \in \{0, 1\}^n$  to  $\mathcal{C}$

$\Updownarrow$

Chan. dec. for  $\mathcal{C}^\perp$  can correct all vectors with eras.  $1^n \oplus e^n$
- ▶ Choose a good (dual) code  $\mathcal{C}(s^k)$

# Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



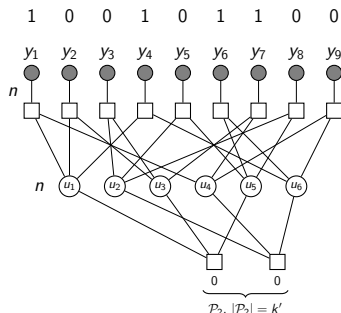
$$\frac{n-k-k'}{n} \approx \delta$$

$$\frac{n-k'}{n} \approx 1 - h(p)$$

- Change 0's to \*'s
- With message  $s^k$ , encode seq. to  $\mathcal{C}(s^k)$

# Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



$$\mathcal{P}_2, |\mathcal{P}_2| = k'$$

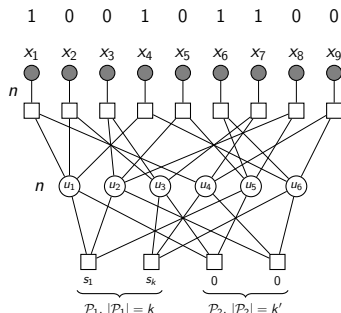
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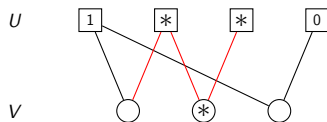
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- With message  $s^k$ , encode seq. to  $\mathcal{C}(s^k)$
- Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

- Dec.  $x^n$  and compute  $s^k$
- $R_2 = \frac{k}{n} \approx 1 - \delta - h(p)$

# Iterative Erasure Quantization Algorithm

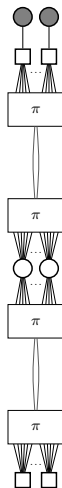


- Peeling type encoder

# Remarks

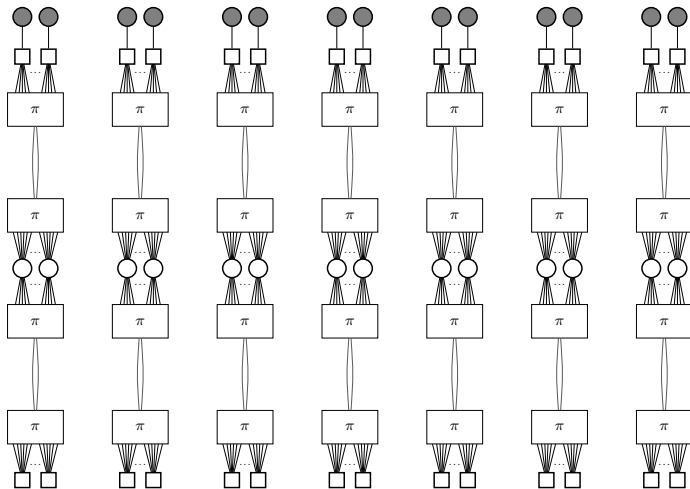
- ▶ Need codes that are **simultaneously good** for channel/source coding and erasure quantization
- ▶ Use **message-passing algorithms** instead of **optimal**
- ▶ Use spatial-coupling for **goodness** of codes under message-passing

# Spatially-Coupled Compound LDGM/LDPC Codes

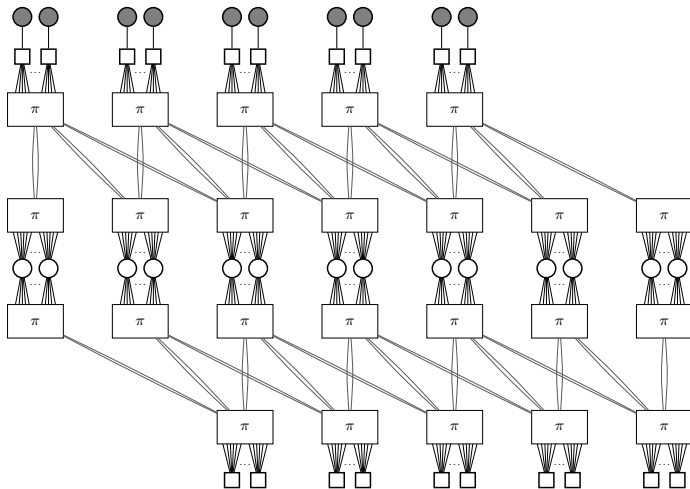




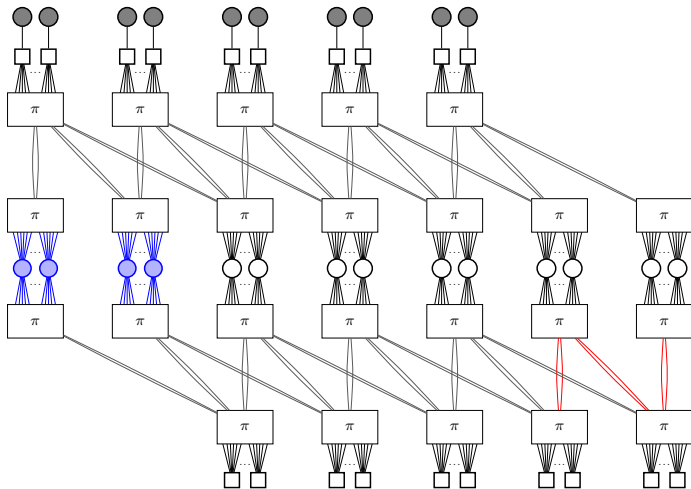
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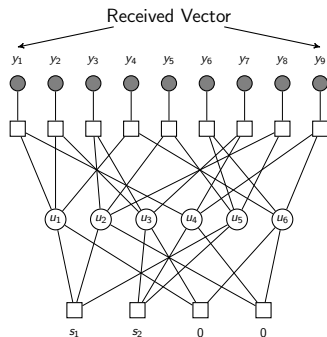
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


# Decoding in Spatially-Coupled Compound Codes




Channel LLR

$y_i$    $\rightarrow$



$$L = L_1 + \dots + L_k$$



$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

## Remarks

- ▶ Standard message-passing algorithm
- ▶ Threshold saturation proven for SC compound codes on BEC
- ▶ Empirically observed for BMS channels

# Numerical Results: Noiseless WOM

LDGM/LDPC ( $d_v, d_c, d'_v, d'_c$ )	$\delta^*$	$\delta$ $w = 2$	$\delta$ $w = 3$	$\delta$ $w = 4$
(3, 3, 3, 6)	0.500	0.477	0.492	0.494
(3, 3, 4, 6)	0.333	0.294	0.324	0.326
(3, 3, 5, 6)	0.167	0.095	0.156	0.158
(4, 4, 3, 6)	0.500	0.461	0.491	0.492
(4, 4, 4, 6)	0.333	0.278	0.323	0.325
(4, 4, 5, 6)	0.167	0.086	0.155	0.159
(5, 5, 3, 6)	0.500	0.436	0.488	0.491
(5, 5, 4, 6)	0.333	0.260	0.320	0.324
(5, 5, 5, 6)	0.167	0.079	0.154	0.159

## Remarks

- ▶  $\delta^*$  is the Shannon threshold
- ▶  $L = 30$ , Single system length  $\approx 24000$

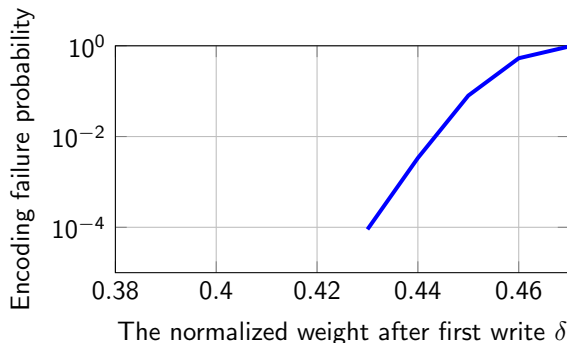
# Numerical Results: WOM with Read Errors

LDGM/LDPC ( $d_v, d_c, d'_v, d'_c$ )	$w$	$(\delta^*, p^*)$	$(\delta, p)$
(3, 3, 4, 6)	3	(0.333, 0.0615)	(0.321, 0.0585)
(3, 3, 4, 8)	3	(0.500, 0.0417)	(0.490, 0.0387)
(3, 3, 6, 8)	4	(0.250, 0.0724)	(0.239, 0.0684)
(4, 4, 4, 6)	4	(0.333, 0.0615)	(0.324, 0.0585)
(4, 4, 4, 8)	4	(0.500, 0.0417)	(0.492, 0.0387)
(4, 4, 6, 8)	4	(0.250, 0.0724)	(0.241, 0.0694)

## Remarks

- $\delta^*$  and  $p^*$  are the Shannon thresholds
- $L = 30$ , Single system length  $\approx 30000$

# Numerical Results: Small Blocklength



## Remarks

- ▶  $(L, w) = (30, 3)$ , Single system length 1200, Shannon threshold of 0.5
- ▶ A total of  $10^5$  were attempted to encode
- ▶ No failures for  $\delta < 0.43$

# Concluding Remarks

## Conclusion

- ▶ Spatially-coupled compound codes achieve the capacity of 2-write systems
- ▶ **Coupling structure** is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed

## Multi-Write Systems

- ▶ Will BPGD work for multi-write systems?