Compressed Sensing using Left and Right regular sparse graphs

Avinash Vem, Nagaraj Janakiraman, Krishna R. Narayanan

Department of Electrical and Computer Engineering
Texas A&M University

Outline

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Compressed Sensing

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}$$

- $\mathbf{x}$ -$N \times 1$ sparse signal
- $\mathrm{supp}(\mathbf{x}) := \{i : x_i \neq 0, i \in [N]\}$
- $K = |\mathrm{supp}(\mathbf{x})|$
- $\mathbf{A}$ -$M \times N$ measurement matrix
- $\mathbf{w}$ -additive noise
- $\mathbf{y}$ -$M \times 1$ measurement vector

Sparsity

$K \ll N$

Problem Statement

- Decoder: Given $\mathbf{y}$ reconstruct the vector $\mathbf{x}$ denoted by $\widehat{\mathbf{x}}$
- Prob. of failure of support recovery $\mathbb{P}_F := \Pr(\mathrm{supp}(\widehat{\mathbf{x}}) \neq \mathrm{supp}(\mathbf{x}))$
- Metrics of interest:
    - Sample complexity (M)
    - Decoding complexity
    - $\mathbb{P}_F$

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Objective

Devise a scheme with minimal num. of measurements M and minimal decoding complexity such that $\mathbb{P}_F \to 0$ as N(and K) $\to \infty$

Optimal order for Support Recovery [1]

- In the sub-linear sparsity regime, $K = o(N)$, necessary and sufficient conditions are shown to be:

$$C_1 K \log\left(\frac{N}{K}\right) < M < C_2 K \log\left(\frac{N}{K}\right)$$

- In the linear sparsity regime, $K = \alpha N$, it was shown that $M = \Theta(N)$ measurements are sufficient for asymptotically reliable recovery.

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- In [1], the minimum value of the signal space affects the bounds on M

$$x_i \in \mathcal{X} \triangleq \{Ae^{i\theta} : A \in \mathcal{A}, \theta \in \Omega\} \cup \{0\},$$
$$\mathcal{A} = \{A_{\min} + \rho l\}_{l=0}^{L_1}, \Omega = \{2\pi l / L_2\}_{l=0}^{L_2}$$

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Main result

Optimal Sample and Decoding Complexities

In the sub-linear sparsity regime, for a given SNR of $\frac{A_{\min}^2}{\sigma^2}$, our scheme has

- Sample complexity of $M = c_1 K \log(\frac{c_2 N}{K})$
- Decoding complexity of $O\left(K \log(\frac{N}{K})\right)$
- $\mathbb{P}_\mathsf{F} \to 0$ asymptotically in K

where the constants c_1 and c_2 are dependent on SNR, desired rate of decay of $\mathbb{P}_\mathsf{F}$ and left degree ℓ.

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Linear Sparsity Regime

In the linear sparsity regime our scheme has

- Sample complexity of $M = c_3 K \log(K)$
- Decoding complexity of $O\left(K \log(K)\right)$
- $\mathbb{P}_{\mathsf{F}} \to 0$ asymptotically in K

where the constant $c_3 > 1$ is a parameter dependent on left degree ℓ.

- Zhang and Pfister, "Verification Decoding of High-Rate LDPC Codes With Applications in Compressed Sensing", 2008
- Jafarpour, Xu, Hassibi and Calderbank, "Efficient and robust compressed sensing using optimized expander graphs", 2009
    - Sample complexity of $O(K)$ and measurement complexity of $O(N \log(\frac{N}{K}))$ for noiseless setting

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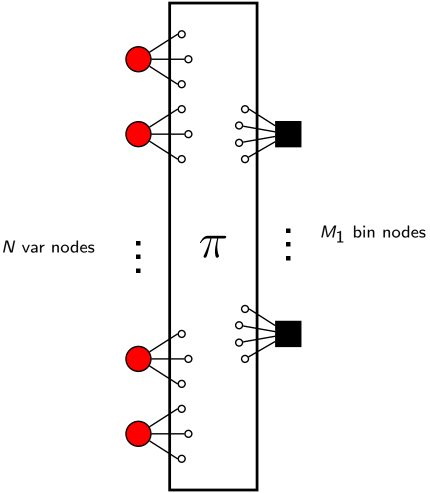
- Li, Pedarsani and Ramchandran, "Sub-linear compressed sensing for support recovery using sparse-graph codes", 2014
  - Introduced sparse-graph codes peeling decoder framework to CS
  - Sample and measurement complexities of $O(K \log N)$ for noisy setting
  - Sample and measurement complexities of $2K$ and $O(K)$ for noiseless setting

Outline

Graphical Representation

(N, ℓ, r, W) ensemble. $\ell N = rM_1$



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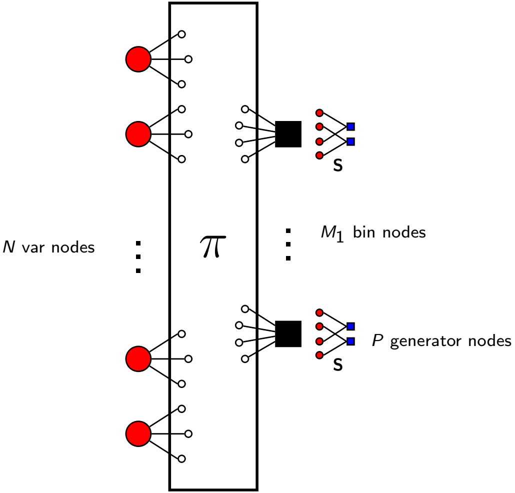

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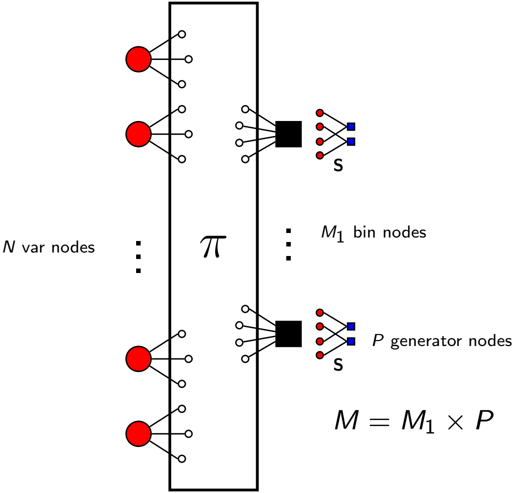


$M = M_1 \times P$ generator nodes

Matrix Representation

(N, ℓ, r, W) ensemble.

- $\mathbf{H}$ be the adjacency matrix (binning operation)- $M_1 \times N$
- $\mathbf{S}$ be the generator matrix at each bin - $P \times r$

$$\tilde{\mathbf{y}} = \mathbf{H}(\mathbf{x}) = \begin{bmatrix} \tilde{\mathbf{y}}_1 \\ \tilde{\mathbf{y}}_2 \\ \vdots \\ \tilde{\mathbf{y}}_{M_1} \end{bmatrix}, \dim(\tilde{\mathbf{y}}_i) = r \times 1,$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_{M_1} \end{bmatrix}, \text{where } \mathbf{y}_i = \mathbf{S}\tilde{\mathbf{y}}_i, \dim(\mathbf{y}_i) = P \times 1$$

- We define a tensor operation such that

$$\mathbf{y} = (\mathbf{S} \boxplus \mathbf{H})\mathbf{x}$$

Tensor Operation

- Sensing matrix $\mathbf{A}_{M_1 P \times N} = S_{P \times r} \boxplus H_{M_1 \times N}$ where

Tensor Operation

- Sensing matrix $\mathbf{A}_{M_1 P\times N} = S_{P\times r} \boxplus H_{M_1\times N}$ where
- $\forall i \in [1 : M_1]$, define a $P \times N$ matrix

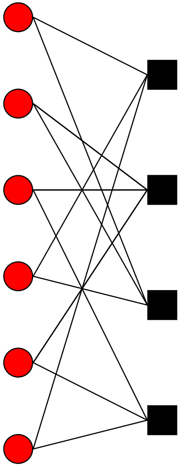
$$\mathbf{S}_i = \mathbf{h}_i \boxtimes \mathbf{S} \triangleq [\mathbf{0}, \ldots, \mathbf{0}, \mathbf{s}_1, \mathbf{0}, \ldots, \mathbf{s}_2, \ldots, \mathbf{0}, \mathbf{s}_r, \mathbf{0}]$$

  where the r columns are placed in the r non-zero indices of $\mathbf{h}_i$.

- $S \boxplus H = \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \\ \vdots \\ \mathbf{S}_{M_1} \end{bmatrix}$

Example

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$



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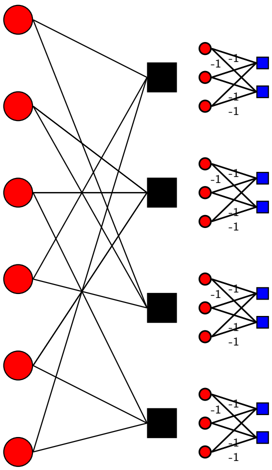
$$\mathbf{S} = \begin{bmatrix} +1 & -1 & -1 \\ -1 & +1 & -1 \end{bmatrix}.$$



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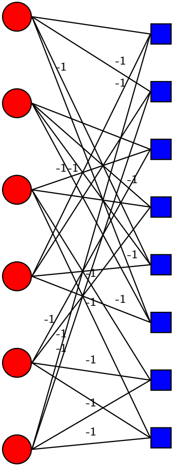
Sensing matrix $\mathbf{A}$ with $M = 8$:

$$\mathbf{A} = \mathbf{H} \boxplus \mathbf{S} = \begin{bmatrix} +1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & +1 & 0 & -1 \\ 0 & +1 & -1 & 0 & -1 & 0 \\ 0 & -1 & +1 & 0 & -1 & 0 \\ +1 & -1 & 0 & -1 & 0 & 0 \\ -1 & +1 & 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & +1 & -1 \end{bmatrix}$$



Bin Decoding

At each bin, input to the decoder is

$$\mathbf{y}_i = \sum_{j=1}^{r} x_{\mathbf{h}_i^j} \mathbf{s}_j + \mathbf{w}_i$$

- Zero-ton: Is it just noise?

$$\widehat{\mathcal{H}}_i = \mathcal{H}_Z, \quad \text{if } \frac{1}{P}\|\mathbf{y}_i\|^2 \leq (1+\gamma)\sigma^2$$

- Singleton: If a single variable is non-zero?

$$\begin{aligned} \alpha_k &= \frac{\mathbf{s}_k^\dagger \mathbf{y}_i}{\|\mathbf{s}_k\|^2} \\ \hat{k} &= \arg\min_k \ \ \|\mathbf{y}_i - \alpha_k s_k\| \\ \hat{x}[\hat{k}] &= \arg\min_{x \in \mathcal{X}} \|x - \alpha_{\hat{k}}\| \end{aligned}$$

- Multi-ton: More than one non-zero variable?

$$\widehat{\mathcal{H}}_i = \mathcal{H}_S(\hat{k}, \hat{x}[\hat{k}]), \quad \text{if } \frac{1}{P}\|\mathbf{y}_i - \hat{x}[\hat{k}]\mathbf{s}_{\hat{k}}\|^2 \leq (1+\gamma)\sigma^2$$

Peeling Decoding

while $\exists i \in [M_1] : \mathcal{H}_i = \mathcal{H}_Z$ or $\mathcal{H}_S$, **do**
if $\mathcal{H}_i = \mathcal{H}_Z$ **then**
Remove the bin i
Assign 0 to all the variables connected
else if $\mathcal{H}_i = \mathcal{H}_S(k, x[k])$ **then**
Assign $x[k]$ to k^{th} variable in bin i
Subtract $x[k]\mathbf{s}_k$ from connected $\mathbf{y}_i$
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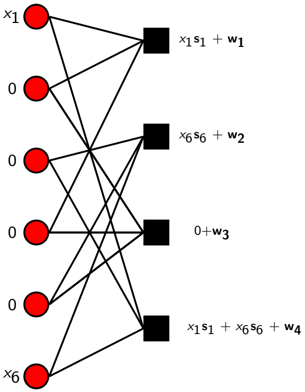

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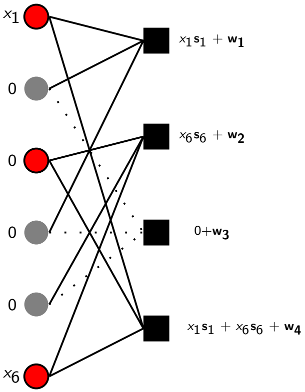

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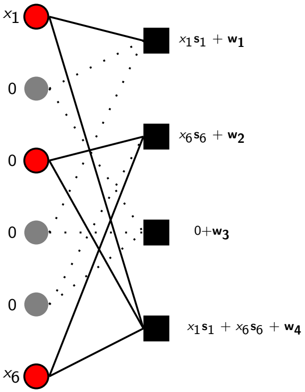

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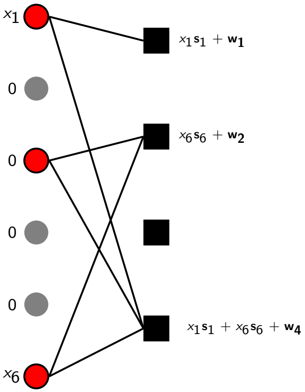

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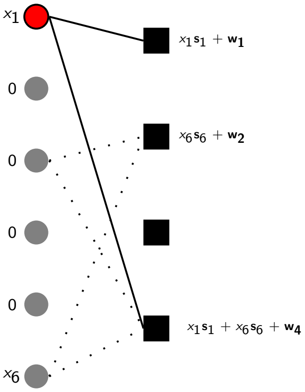

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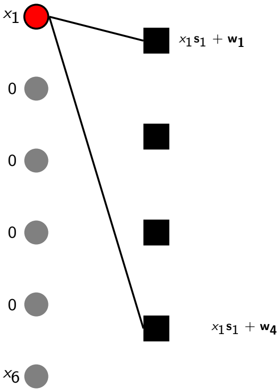

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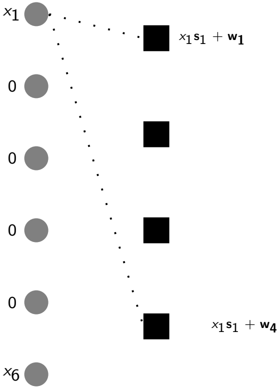

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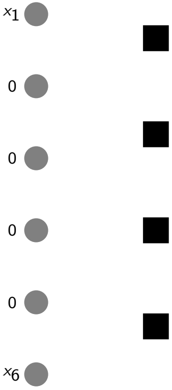

Outline

Oracle based Peeling Decoder

- Assume the hypothesis detection in each bin is correct
- Equivalence to peeling decoder on pruned graph- all zero variables are removed

Equivalence to (N, l, r) LDPC on BEC$(\epsilon = \frac{K}{N})$

If $\text{supp}(\mathbf{x}) = \{i : y_i = \mathcal{E}\}$, then $P^{(i)}_{\text{BEC}}(\mathbf{y}) = P^{(i)}_{\text{SR}}(\mathbf{z})$ for $\mathbf{z} = \mathbf{H}\mathbf{x}$.

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- Choose $M_1 = \eta K$ thus $r = \frac{\ell N}{\eta K}$

DE for Peeling decoder on LDPC -BEC channel

Fractional number of degree one checks remaining

$$\tilde{R}_1(y) = r\epsilon y^{l-1}[y - 1 + (1 - \epsilon y^{l-1})^{r-1}]$$

where $\epsilon = \frac{K}{N}$ and $r = \frac{\ell N}{\eta K}$

Peeling threshold

η^{Th} is defined to be the minimum value of η for which there is no non-zero solution for the equation:

$$\begin{aligned} y &= \lim_{\frac{N}{K}\to\infty} 1 - \left(1 - \frac{Ky^{\ell-1}}{N}\right)^{\frac{\ell N}{\eta K}} \\ &= 1 - e^{\frac{-\ell y^{\ell-1}}{\eta}} \end{aligned}$$

in the range $y \in [0,1]$.

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\end{aligned}
$$

in the range $y \in [0, 1]$.

Threshold behavior

For $M_1 > \eta^{\text{BP}} K$ bin nodes, the peeling decoder will be successful with probability $1 - O\left(\frac{1}{K^{\ell-2}}\right)$

Note that η^{Th} is a function of just the left degree ℓ.

Bin detection matrix

- Singleton detection is the crucial part of bin decoding:

$$\mathbf{y}_i = x_k \mathbf{s}_k + \mathbf{w}_i$$

- Error correction coding: $\mathbf{S}$ be the codebook, where each $\mathbf{s}_i$ is a codeword.
- Block length $= P$. # codewords $\geq \frac{N\ell}{\eta K}$
- Choose a code with rate $R(\beta)$ s.t. fractional minimum distance $\beta > \mathbb{P}_e := e^{-\frac{A_{\min}^2}{2\sigma^2}}$
- Thus $P = \frac{\lceil \log_2(\frac{N\ell}{\eta K}) \rceil}{R(\beta)}$.

Sample Complexity

$$\begin{aligned} M &= M_1 \times P \\ &\geq \left[\frac{\eta^{\mathrm{Th}}}{R(\mathbb{P}_e)}\right] K \log\left(\frac{\ell N}{\eta^{\mathrm{Th}} K}\right) \end{aligned}$$

Analysis of Bin Decoding

- Let $\mathrm{E}_{\mathrm{bin}}$ be the event an error was made in overall bin decoding
- Union bounding: $\mathrm{E}_{\mathrm{bin}} \leq (\eta K + \ell K)\mathrm{Pr}(\mathrm{E})$

Error Probability of a bin - Ramchandran *et al*, 2014

$$\mathrm{Pr}(\mathrm{E}) \leq 3e^{-\frac{P}{4}\frac{\gamma^2}{1+4\gamma}} + 2e^{-\frac{P}{4}(\sqrt{1+2\gamma}-1)^2} + 4e^{-c_6 P\left(1-\frac{2\gamma\sigma^2}{A_{\mathrm{min}}^2}\right)} + 2e^{-P\frac{(\beta-\mathbb{P}_e)^2}{2\mathbb{P}_e(1-\mathbb{P}_e)}}$$

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Sub-Linear sparsity

- Order optimal sample complexity with precise constants given
- $\mathbb{P}_F \to 0$ as N(and K) $\to \infty$
- Trade-off between the constants in M, rate of decay of $\mathbb{P}_F$ and SNR
- Optimal decoding complexity of $O\left(K \log\left(\frac{N}{K}\right)\right)$

Implications

Error Probability of a bin - Ramchandran *et al*, 2014

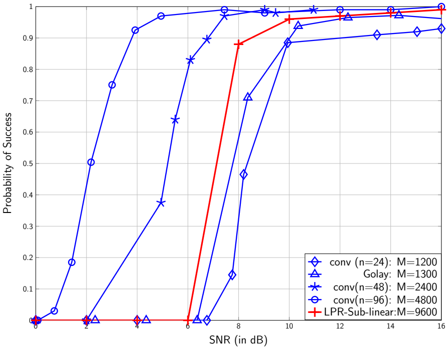
$$\Pr(\mathrm{E}) \leq 3e^{-\frac{P}{4}\frac{\gamma^2}{1+4\gamma}} + 2e^{-\frac{P}{4}(\sqrt{1+2\gamma}-1)^2} + 4e^{-c_6 P\left(1-\frac{2\gamma\sigma^2}{A^2_{\text{min}}}\right)} + 2e^{-P\frac{(\beta-\mathbb{P}_e)^2}{2\mathbb{P}_e(1-\mathbb{P}_e)}}$$

Linear sparsity: $K = \alpha N$

- Choice of $P = c_1 \log\left(c_2 \frac{N}{K}\right)$ doesn't work
- We choose $P = \log(K)$ and rate $R(\beta)$ as earlier
- A sub-code of size $\frac{\ell}{\alpha\eta}$ of the codebook is chosen as **S**
- Sample complexity of $\eta^{\text{Th}} K \log K$
- Can we do $\Theta(K)$ with practical decoding?

Outline

- $K = 50, N = 10^5$. $\mathcal{X} = \{+1, -1\}$
- $\ell = 4, \eta = 2(M_1 = 2K)$. Note that $\eta^{\mathrm{Th}}(\ell = 4) = 1.295$
- $r = \frac{N\ell}{\eta K} = 4000$. $\log_2(r) = 12$
- $(12, 24)$ Golay code and $(12, n)$ convolutional codes for $n = 24, 48, 96$ with QAM form $\mathbf{S}$.



Questions