# Sub-string/Pattern Matching in Sub-linear Time Using a Sparse Fourier Transform Approach
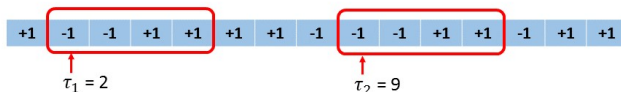
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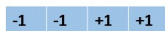Texas A&M University

## Problem Statement



Database (x)

| +1 | -1 | -1 | +1 | +1 | +1 | +1 | -1 | -1 | -1 | +1 | +1 | -1 | +1 | +1 |

$\tau_1 = 2$     $\tau_2 = 9$

Query (y)

| -1 | -1 | +1 | +1 |

- Database/String: $\underline{x} = [x[0], x[1], \cdots, x[N-1]]$ (length $N$)
- Query/Substring: $\underline{y} = [y[0], x[1], \cdots, x[M-1]]$ (length $M = N^\mu$)
- Determine all the $L$ locations $\underline{\tau} = [\tau_1, \tau_2, \cdots \tau_L]$ with high probability where
  1. Exact Matching: $y$ appears exactly in $\underline{x}$
     - $\underline{y} := \underline{x}[\tau : \tau + M - 1]$
  2. Approximate Matching: $y$ is a noisy substring of $\underline{x}$
     - $\underline{y} := \underline{x}[\tau : \tau + M - 1] \odot \underline{b}$
     - $\underline{b}$ is a noise sequence with $d_H(\underline{y}, \underline{x}[\tau : \tau + M - 1]) \leq K$

# Main Result

## Theorem 1

*Assume that a sketch of $\underline{x}$ of size $O(\frac{N}{M}\log N)$ can be precomputed and stored. Then for the exact pattern matching and approximate pattern matching (with $K = \eta M$, $0 \le \eta \le 1/6$) problems, with the number of matches $L$ scaling as $O(N^\lambda)$, our algorithm has*

- *a sketching function for $\underline{y}$ that computes $O(\frac{N}{M}\log N) = O(N^{1-\mu}\log N)$ samples*
- *a computational complexity of $O(\max(N^{1-\mu}\log^2 N, N^{\mu+\lambda}\log N))$*
- *a decoder that recovers all the $L$ matching positions with a failure probability that approaches zero asymptotically*

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**Note**

Particularly when $L = O(1)$ or $L < \frac{N}{M}$ (i.e. $\lambda < 1 - \mu$) our algorithm has a sub-linear time complexity.

# Some Prior Work

## Exact Matching

- [**?**]: First occurrence of the match (only $\tau_1$)
    - Average complexity - $O(N^{1-\mu} \log N)$ (sublinear)
    - Worst case complexity - $O(N \log N)$

## Approximate Matching

- [**?**]: Generalization of [**?**]
    - Average complexity - $O(NK/M \log N)$ (sub-linear only when $K \ll M$ )
- [**?**]: $O(N/M^{0.359})$ (sub-linear even when $K = O(M)$)
    - Combinatorial in nature

## Sparse Fourier Transform Approach

- [**?**]: Faster GPS receiver
    - Exploited sparsity in Correlation function $R_{XY}$
- [**?**]: Robust Sparse Fourier Transform
    - Sparse Graph code Approach
    - Computational complexity : $O(N \log N)$

# Motivation

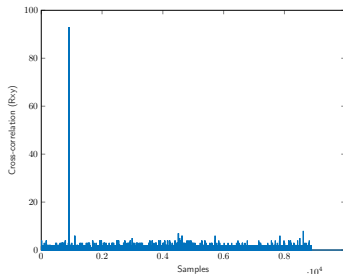- **Cross-correlation** ($\underline{r}$):

$$r[m] = (\underline{x} * \underline{y})[m] \triangleq \sum_{i=0}^{M-1} x[m+i]y[i], \quad 0 \leq m \leq N-1$$

- **Naive implementation**: $O(MN) = O(N^{1+\mu})$ (super-linear complexity)
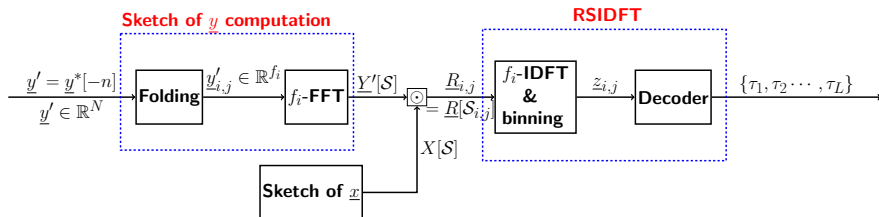- **Fourier Transform Approach**: $O(N \log N)$ complexity

### Key Observation

- $\underline{r}$ is Sparse with some noise.

$$r[m] = \begin{cases} M, & \text{if } m \in \mathcal{T} \\ n_m, & m \in [N] - \mathcal{T} \end{cases}$$
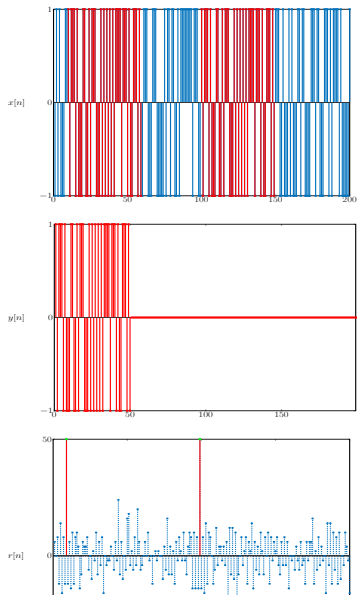
# Sparse Fourier Transform Approach



$$\underline{r} = \underset{3}{\mathcal{F}_N^{-1}} \left\{ \underset{1}{\mathcal{F}_N\{\underline{x}\}} \odot \underset{2}{\mathcal{F}_N\{\underline{y}'\}} \right\}$$

1. *Sketch of $\underline{x}$ :* Assume $\underline{X}[l] = \mathcal{F}\{\underline{x}\}$ is precomputed at positions $l \in \mathcal{S}$.

2. *Sketch of $\underline{y}$:*
   - Compute $\underline{Y}'[l] = \mathcal{F}\{\underline{y}'\}$ for $l \in \mathcal{S}$.
   - Only $M$ non-zero values in $\underline{y}'$ - Efficient computation (folding and adding)

3. *Sparse $\mathcal{F}^{-1}$:*
   - Robust Sparse Inverse Fourier Transform (RSIDFT)
   - Efficient Implementation- sublinear time and sampling complexity
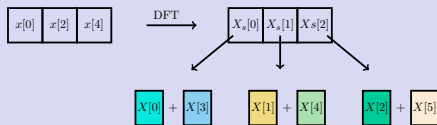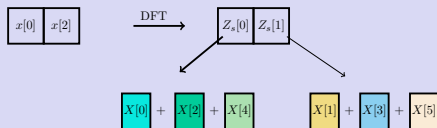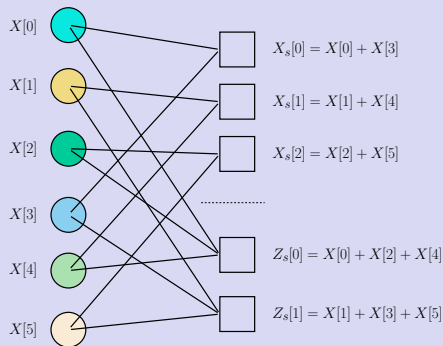
# Example

# Aliasing and Sparse Graph Codes



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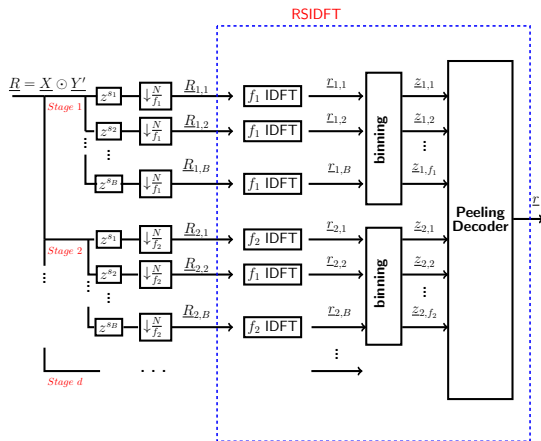
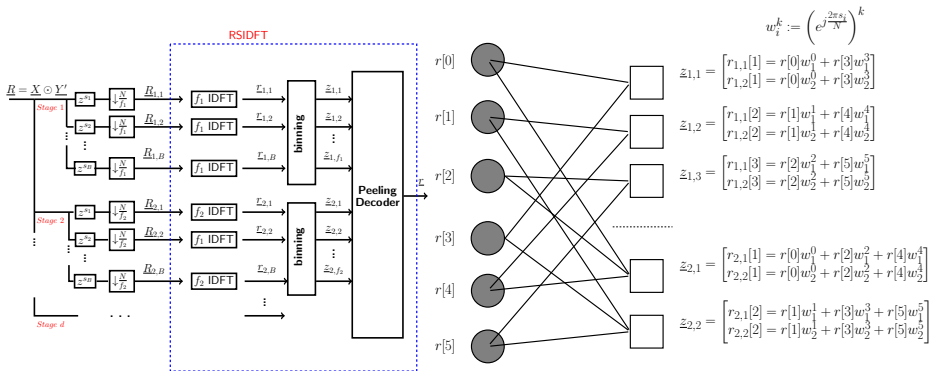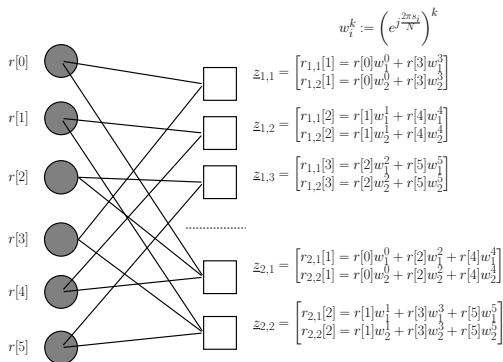# RSIDFT Framework



- Similar to FFAST
- Key modifications
  - Correlation peak is always positive
  - Take advantage in decoding algorithm - sub-linear time complexity

# RSIDFT Framework

# RSIDFT-Decoding (Peeling Decoder)



$$w_i^k := \left( e^{\frac{j 2\pi s_i}{N}} \right)^k$$

$r[0]$

$r[1]$

$r[2]$

$r[3]$

$r[4]$

$r[5]$

$\underline{z}_{1,1} = \begin{bmatrix} r_{1,1}[1] = r[0]w_1^0 + r[3]w_1^3 \\ r_{1,2}[1] = r[0]w_2^0 + r[3]w_2^3 \end{bmatrix}$

$\underline{z}_{1,2} = \begin{bmatrix} r_{1,1}[2] = r[1]w_1^1 + r[4]w_1^1 \\ r_{1,2}[2] = r[1]w_2^1 + r[4]w_2^1 \end{bmatrix}$

$\underline{z}_{1,3} = \begin{bmatrix} r_{1,1}[3] = r[2]w_1^2 + r[5]w_1^5 \\ r_{1,2}[3] = r[2]w_2^2 + r[5]w_2^5 \end{bmatrix}$

$\underline{z}_{2,1} = \begin{bmatrix} r_{2,1}[1] = r[0]w_1^0 + r[2]w_1^2 + r[4]w_1^4 \\ r_{2,2}[1] = r[0]w_2^0 + r[2]w_2^2 + r[4]w_2^4 \end{bmatrix}$

$\underline{z}_{2,2} = \begin{bmatrix} r_{2,1}[2] = r[1]w_1^1 + r[3]w_1^3 + r[5]w_1^5 \\ r_{2,2}[2] = r[1]w_2^1 + r[3]w_2^3 + r[5]w_2^5 \end{bmatrix}$

- Observations: $\underline{z}_{i,k} = \begin{bmatrix} r_{i,1}[k], r_{i,2}[k], \cdots, r_{i,B}[k] \end{bmatrix}^T$
- Decoding- 3 steps
  1. Bin Classification
  2. Position Identification
  3. Peeling Process

# Decoder

## Bin Classification

- Classify each check-node - Zero-ton / Single-ton / Multi-ton
- Threshold constraints on first observation $z_{i,k}[1] = z$
- Threshold varies with $\eta$
    - different for exact($\eta = 0$) and approximate matching

$$\widehat{\mathcal{H}}_{i,j} = \begin{cases} \mathcal{H}_z & z/M < \gamma_1 \\ \mathcal{H}_s & \gamma_1 < z/M < \gamma_2 \\ \mathcal{H}_d & \gamma_2 < z/M < \gamma_3 \\ \mathcal{H}_m & z/M > \gamma_3 \end{cases}$$

where $(\gamma_1, \gamma_2, \gamma_3) = (\frac{1-2\eta}{2}, \frac{3-4\eta}{2}, \frac{5-6\eta}{2})$

# Decoder

## Position Identification

- Observation:

$$\underline{z}_{i,k} = \mathbb{W}_{i,k} \times \begin{bmatrix} r[k + (0)f_i] \\ r[k + (1)f_i] \\ \vdots \\ r[k + (g_i - 1)f_i] \end{bmatrix}$$

- Sensing Matrix:

$$\mathbb{W}_{i,k} = \left[ \underline{w}^k, \underline{w}^{k+f_i}, \ldots, \underline{w}^{k+(g_i-1)f_i} \right] \text{ where } \underline{w}^k = \begin{bmatrix} e^{\frac{j2\pi ks_1}{N}} \\ e^{\frac{j2\pi ks_2}{N}} \\ \vdots \\ e^{\frac{j2\pi ks_B}{N}} \end{bmatrix}, \quad g_i = \frac{N}{f_i}$$

- Column that gives maximum correlation with the observation

$$\hat{k} = \underset{k \in \{j + lg_i\}}{\arg\max} \; \underline{z}_{i,j}^{\dagger} \underline{w}^k$$
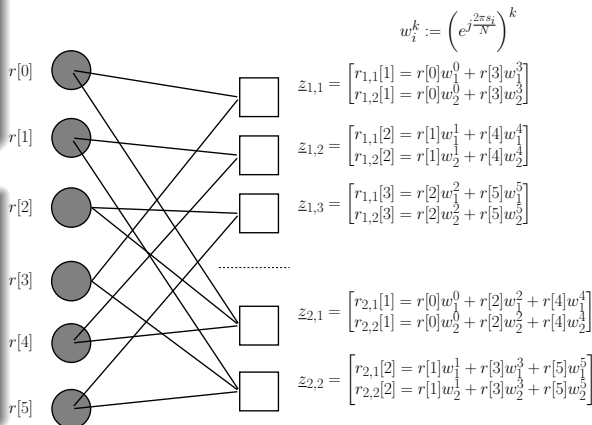
# Decoder

## Peeling Process:



**Exact Matching**
- Remove a decoded variable node's contribution from all participating bin nodes

**Approximate Matching**
- Remove a decoded variable node's contribution only from neighboring single-tons and double-tons
- Avoid error propagation

$$w_i^k := \left( e^{j\frac{2\pi s_i}{N}} \right)^k$$

$$\underline{z}_{1,1} = \begin{bmatrix} r_{1,1}[1] = r[0]w_1^0 + r[3]w_1^3 \\ r_{1,2}[1] = r[0]w_2^0 + r[3]w_2^3 \end{bmatrix}$$

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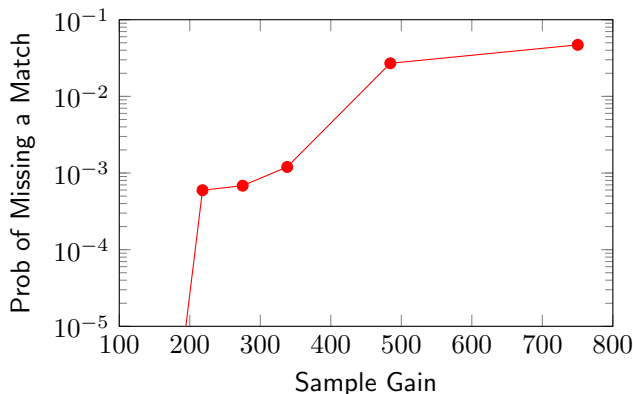
# Simulation Results



Figure: Plot of Probability of Missing a Match vs. Sample Gain for Exact Matching of a substring of length $M = 10^5$ from a equiprobable binary $\{+1,-1\}$ sequence of length $N = 10^{12}$, divided into $G = 10^5$ blocks each of length $\tilde{N} = 10^7$. The substring was simulated to repeat in $L = 10^6$ locations uniformly at random.
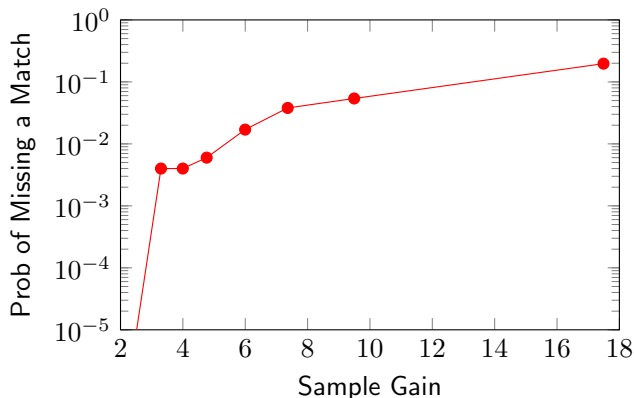
# Simulation Results



Figure: Plot of Probability of Missing a Match vs. Sample Gain for Exact Matching of a substring of length $M = 10^3$ from a equiprobable binary $\{+1,-1\}$ sequence of length $N = 10^{12}$, divided into $G = 10^6$ blocks each of length $\tilde{N} = 10^6$. The substring was simulated to repeat in $L = 10^6$ locations uniformly at random.

Thank you!