

Applications of Spatial Coupling

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Outline

- 1 Spatial Coupling(SC)
 - SC-LDPC Ensemble
 - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
 - Introduction to Lattices
 - Proposed Lattice Construction
 - Poltyrev Goodness
 - Symmetric Interference Channel
- 3 Side-Information Problems
 - Gelfand-Pinsker & Wyner-Ziv
 - Compound LDGM/LDPC Codes
 - Spatial Coupling of Compound Codes
- 4 Write-Once Memory
 - Problem Statement
 - Coding Scheme
- 5 Research Summary

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An (ℓ, r) LDPC Code

Parity-Check Matrix

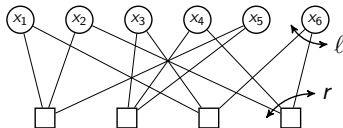
$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\ell = 2$$

$$r = 3$$

$$\text{LDPC Code } \mathcal{C} = \{x : H \odot x = 0\}$$

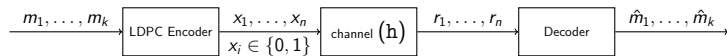
Tanner Graph



Compressed Representation

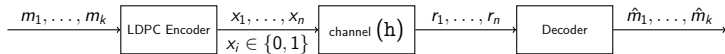


Belief Propagation(BP) Decoder



- For $\text{BEC}(\varepsilon)$, $h = \varepsilon$. For general BMS channel, $C^{\text{Sh}} = 1 - h$

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- Non-optimal
- Popular, low-complexity
- Threshold: h^{BP}

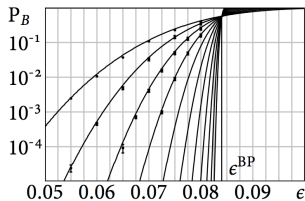
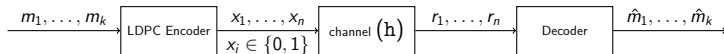


Figure: $(3, 6)$, P_B vs $\text{BSC}(\varepsilon)$ for $n = 2^i$.

$\varepsilon^{\text{BP}} = 0.084$, equivalent $h^{\text{BP}} = 0.4160$
 whereas for rate $1/2$ code $h^{\text{Sh}} = 0.5$

BP vs MAP



- For $\text{BEC}(\varepsilon)$, $h = \varepsilon$. For general BMS channel, $C^{\text{Sh}} = 1 - h$

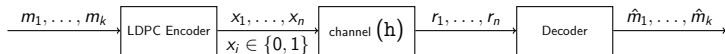
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Maximum a Posteriori (MAP)

- Optimal decoder
- *Not realizable*
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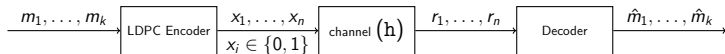
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LDPC (ℓ, r)	Shannon h^{Sh}	AWGN		BSC	
		h^{BP}	h^{MAP}	h^{BP}	h^{MAP}
(3, 6)	0.5000	0.4293	0.4794	0.4160	0.4681
(4, 6)	0.6667	0.5211	0.6645	0.5203	0.6633

BP vs MAP



- ▶ For BEC(ε), $h = \varepsilon$. For general BMS channel, $C^{\text{Sh}} = 1 - h$

Belief Propagation (BP)

- ▶ Non-optimal
- ▶ Popular, low-complexity
- ▶ Threshold: h^{BP}

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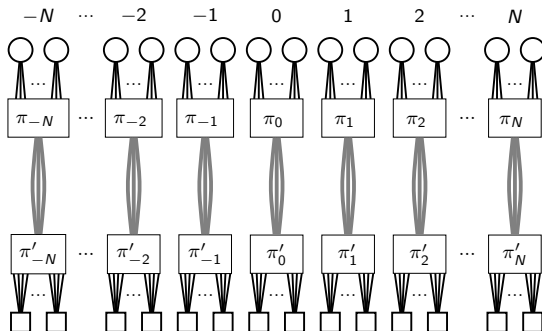
Spatial-Coupling aids in bridging this gap

(ℓ, r, N, w) Spatially-Coupled Ensemble

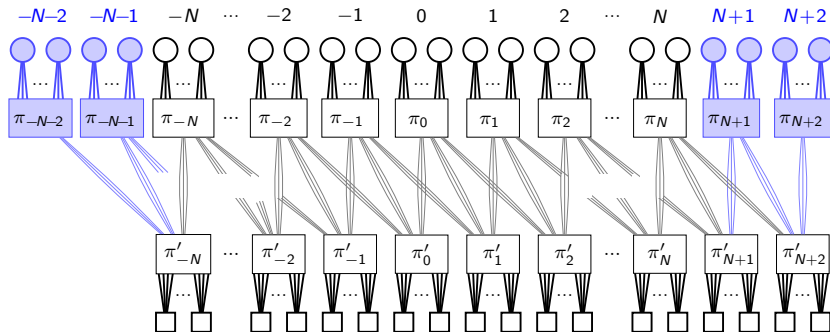


- An LDPC code of left-degree $\ell = 3$ and right-degree $r = 4$

(ℓ, r, N, w) Spatially-Coupled Ensemble

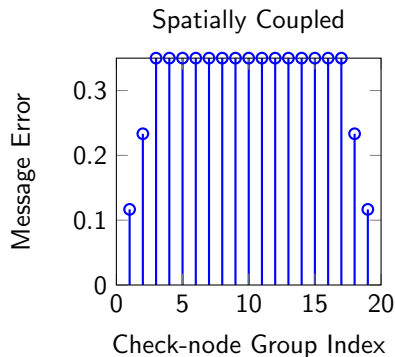
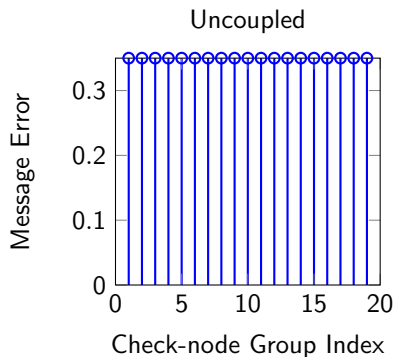
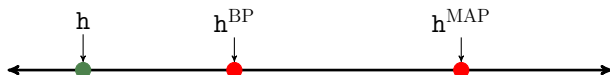


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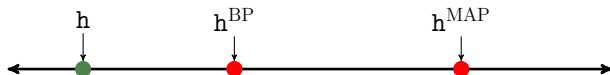


- Shown for $\ell = 3$, $r = 4$, and $w = 3$
- Check-nodes at Section $\{i\}$ are connected to variable-nodes in Sections $\{i - (w - 1), \dots, i\}$
- Shown to have **near optimal BP thresholds**

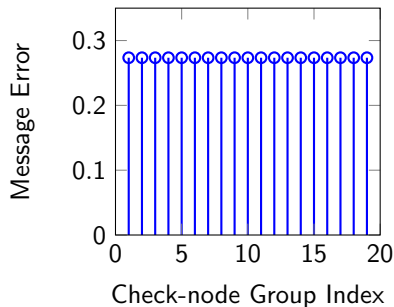
Threshold Saturation via Spatial Coupling (1)



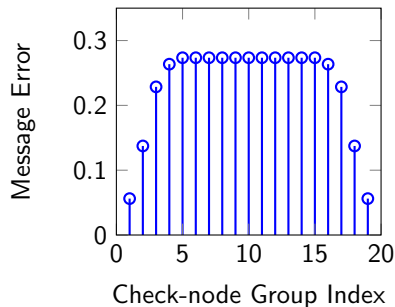
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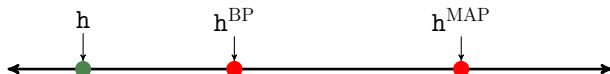
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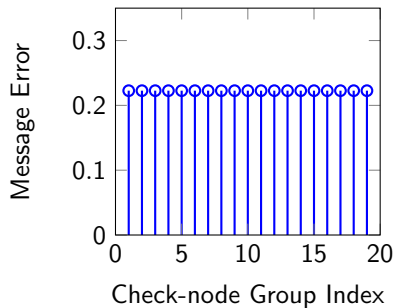
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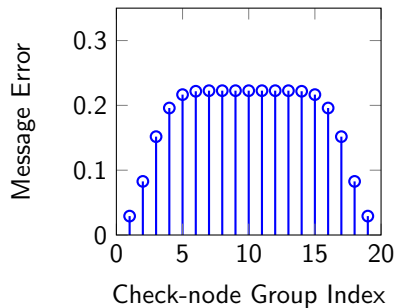
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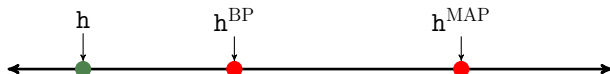
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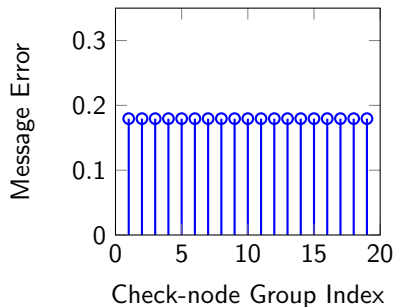
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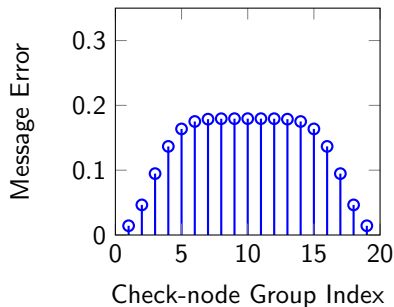
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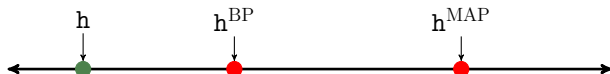
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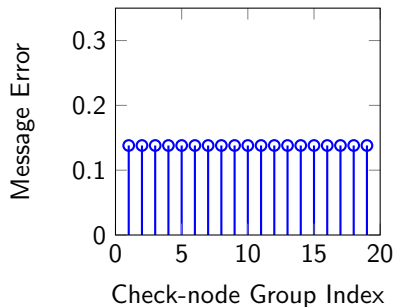
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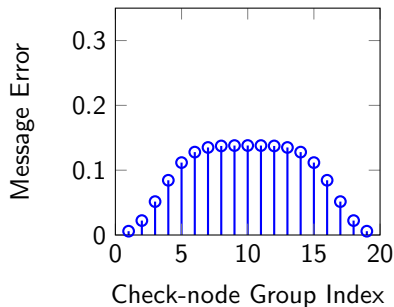
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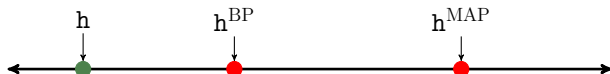
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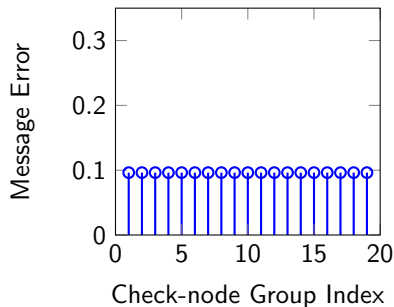
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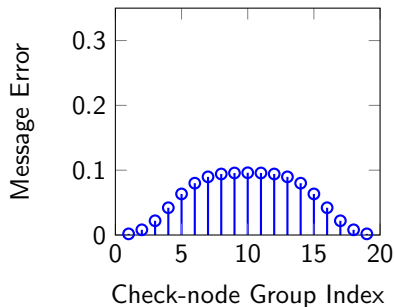
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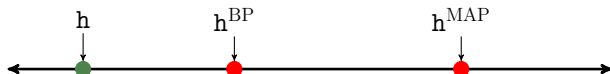
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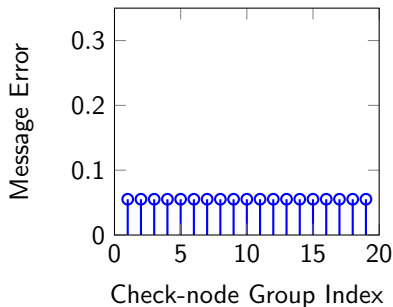
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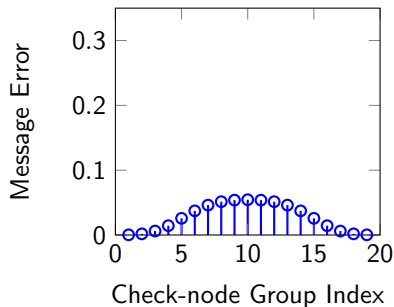
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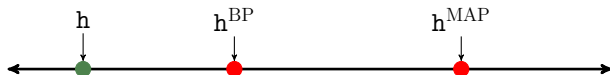
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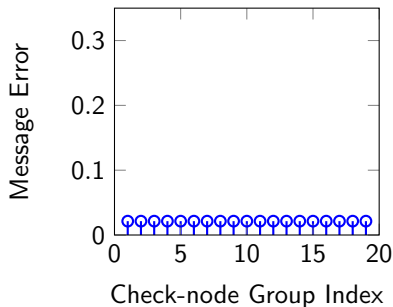
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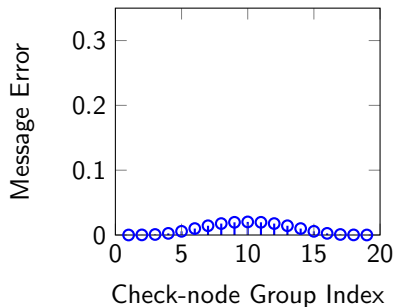
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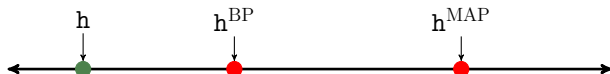
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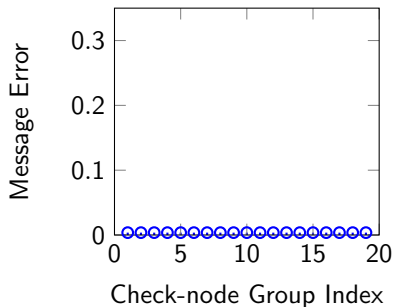
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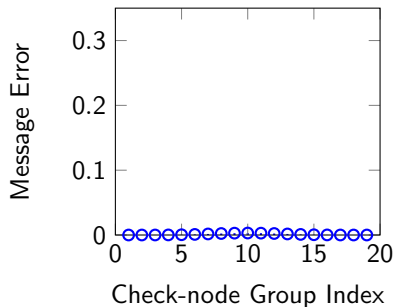
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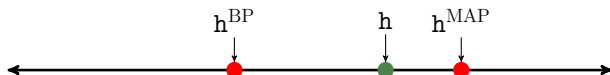
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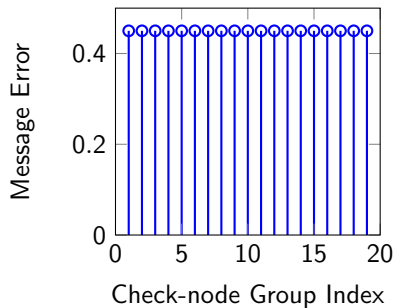
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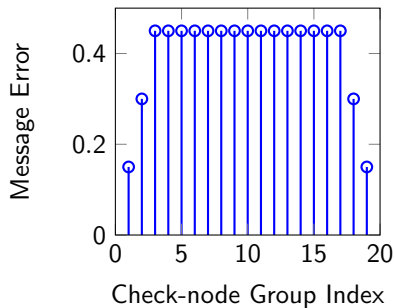
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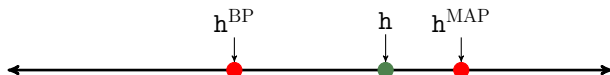
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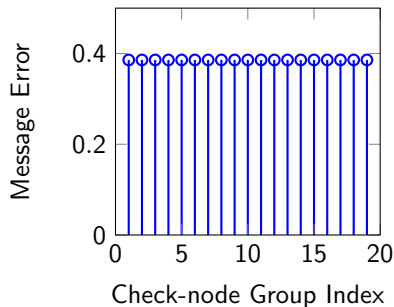
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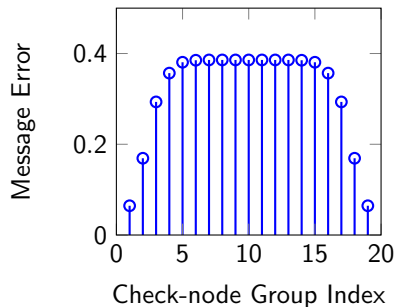
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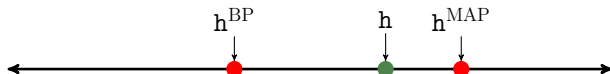
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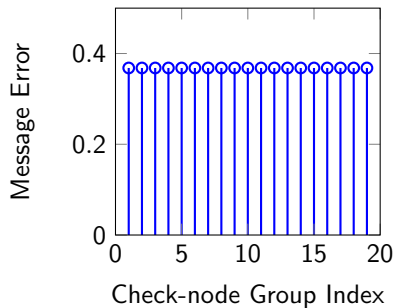
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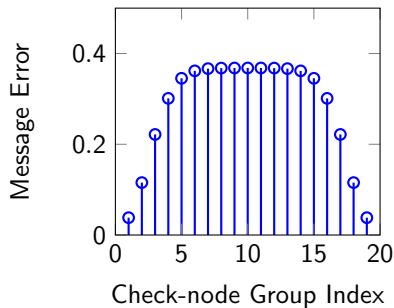
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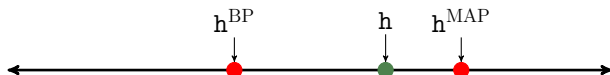
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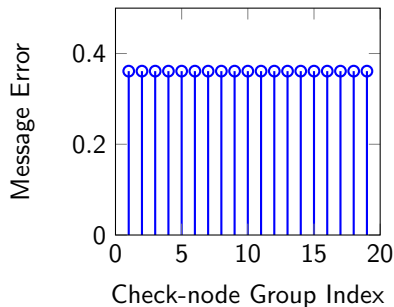
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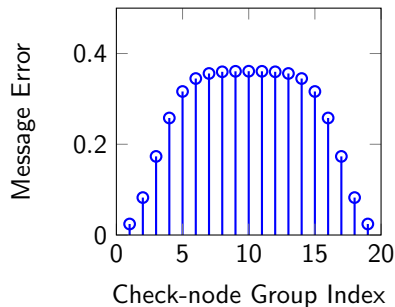
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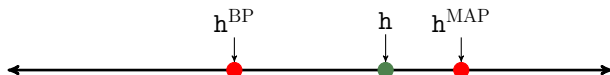
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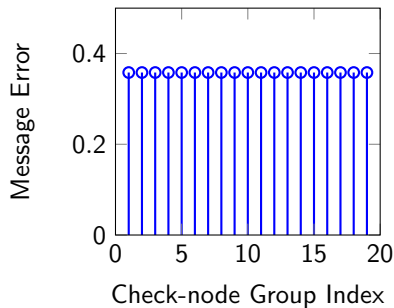
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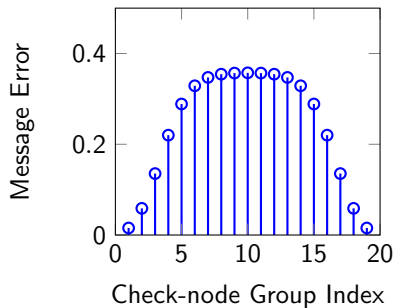
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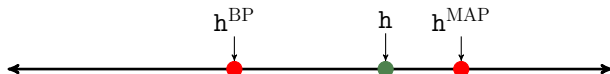
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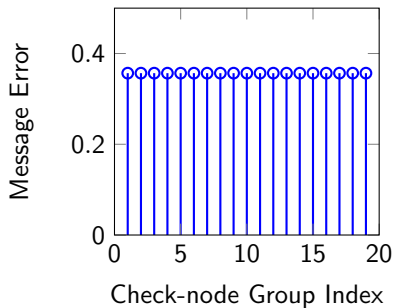
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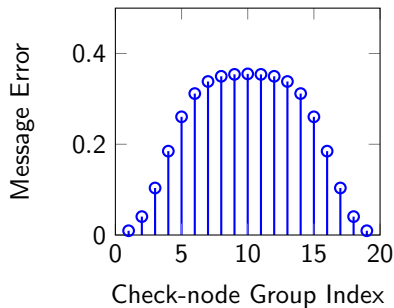
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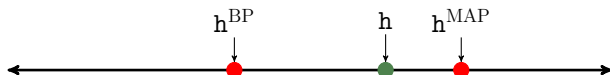
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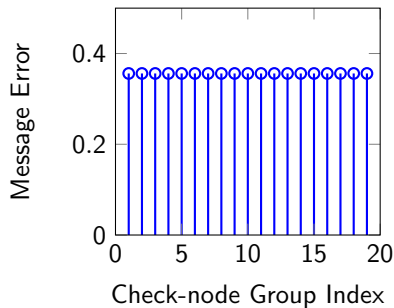
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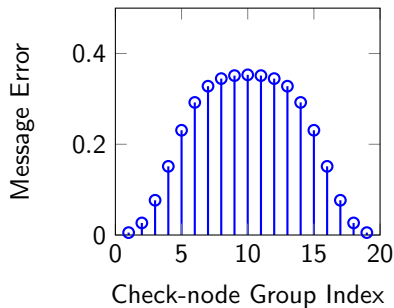
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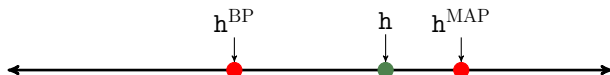
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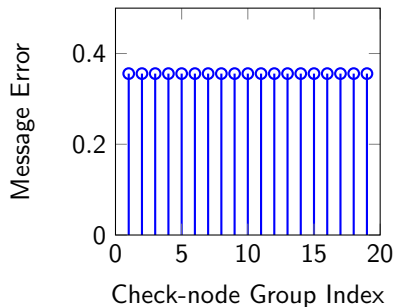
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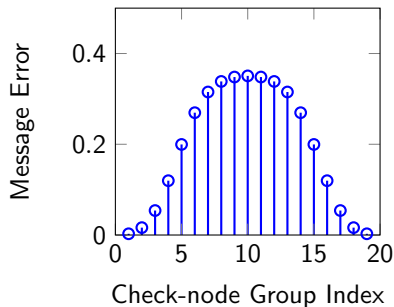
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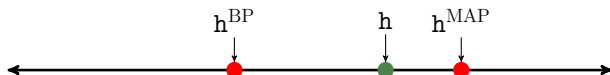
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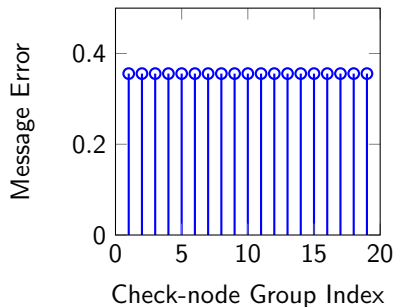
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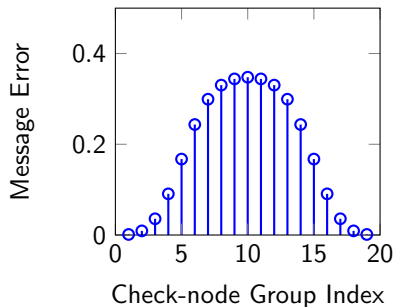
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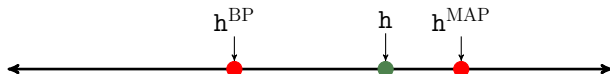
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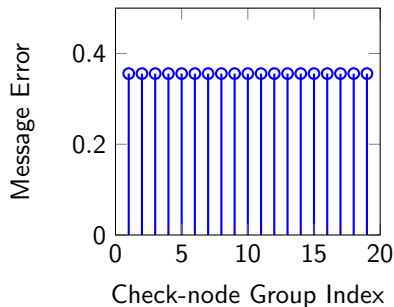
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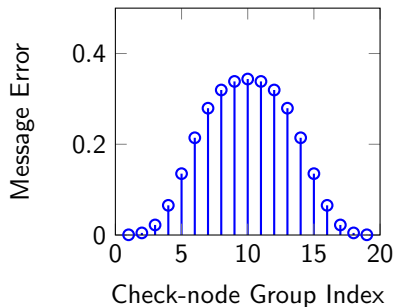
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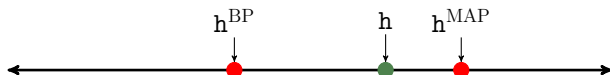
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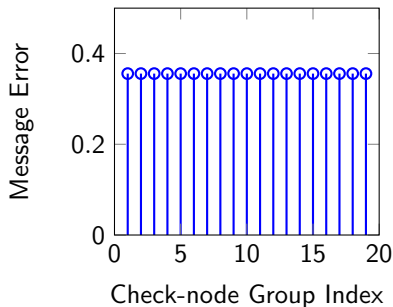
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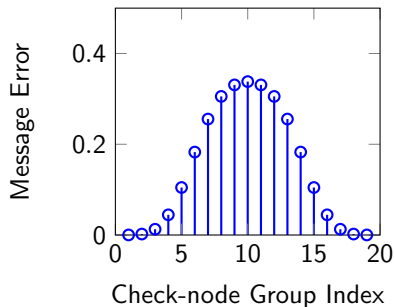
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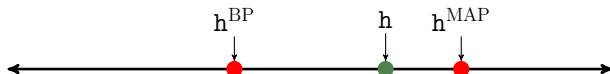
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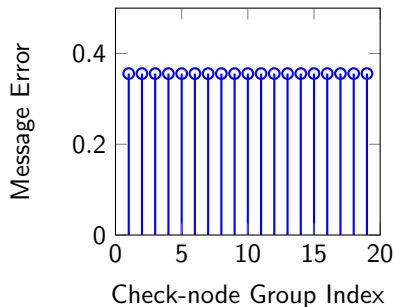
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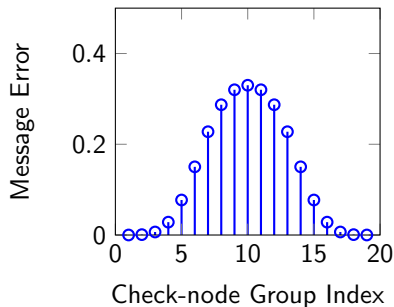
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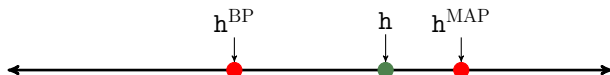
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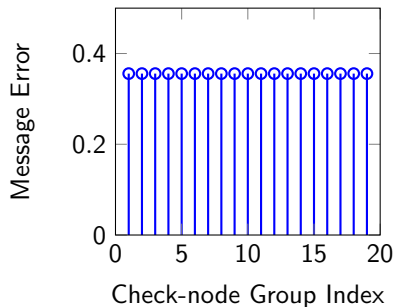
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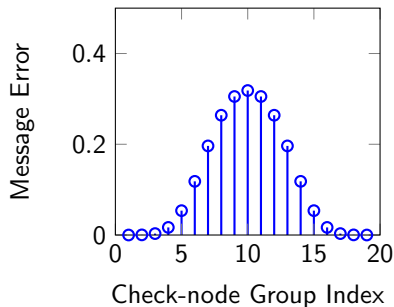
Threshold Saturation via Spatial Coupling (2)



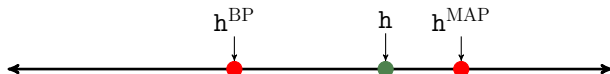
Uncoupled



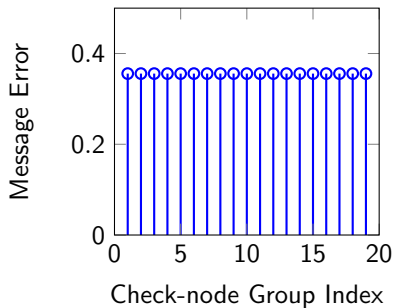
Spatially Coupled



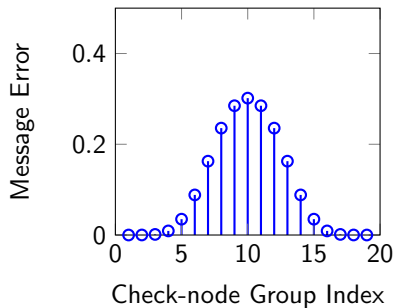
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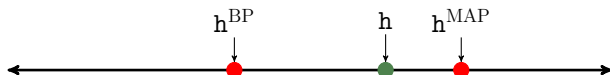
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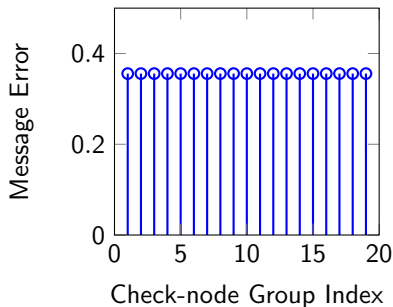
Spatially Coupled



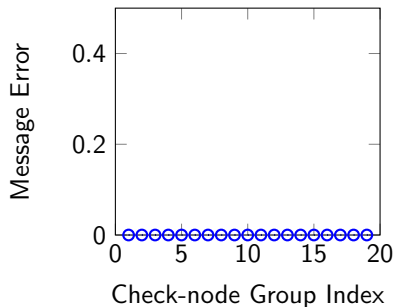
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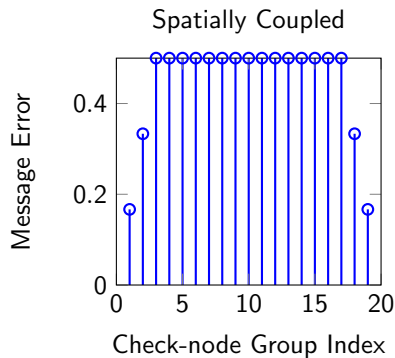
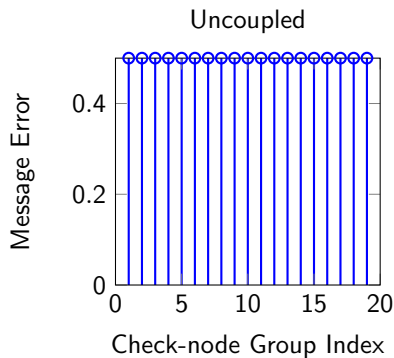
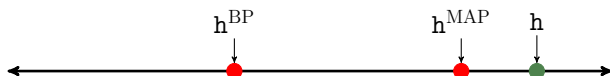
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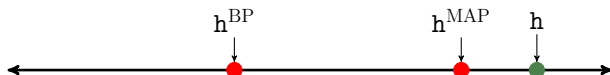
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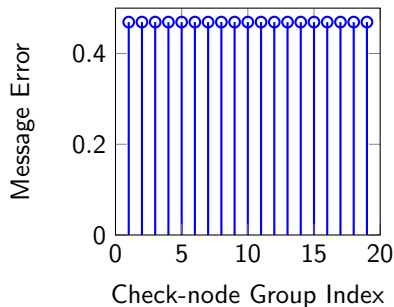
Threshold Saturation via Spatial Coupling (3)



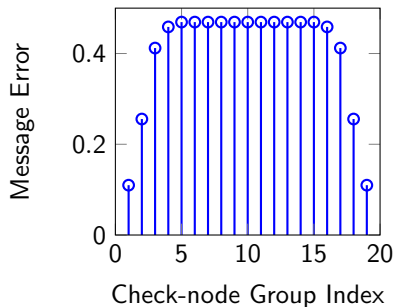
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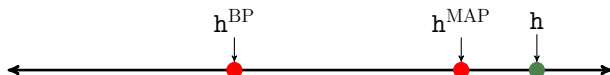
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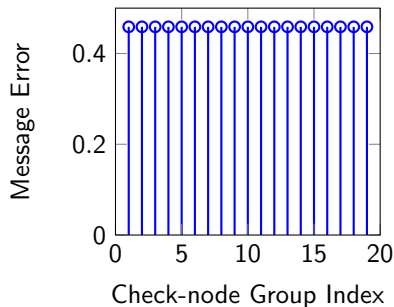
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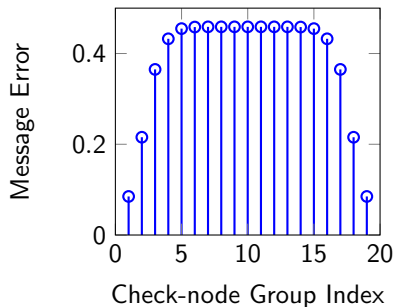
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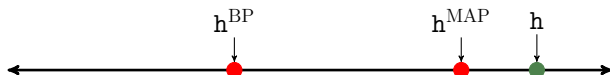
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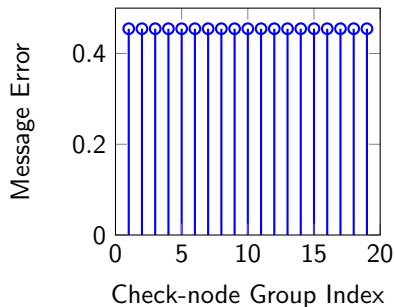
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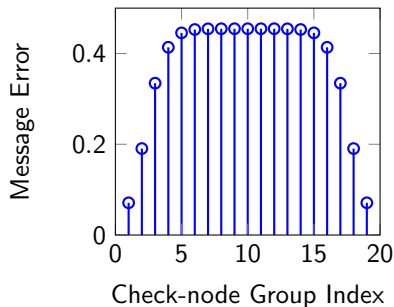
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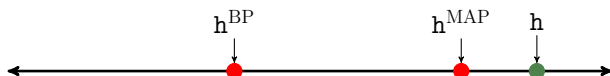
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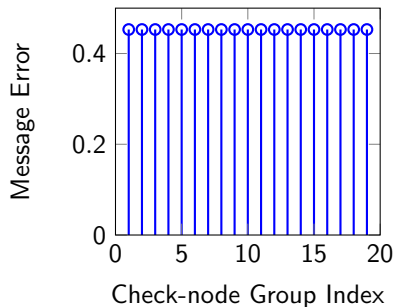
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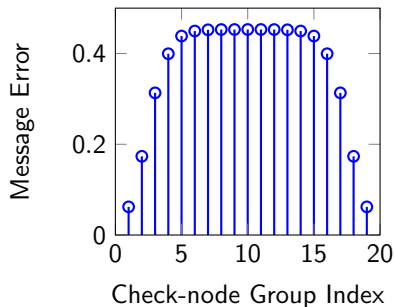
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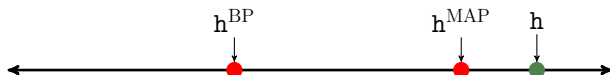
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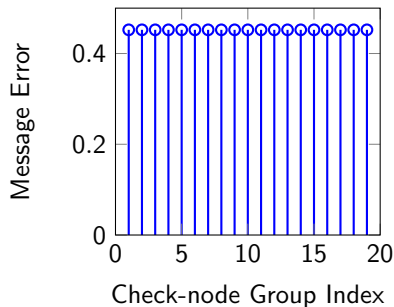
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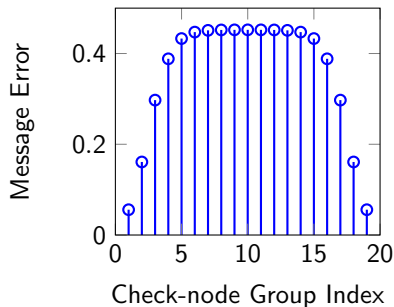
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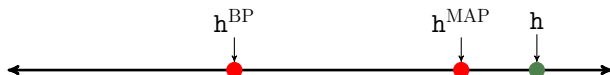
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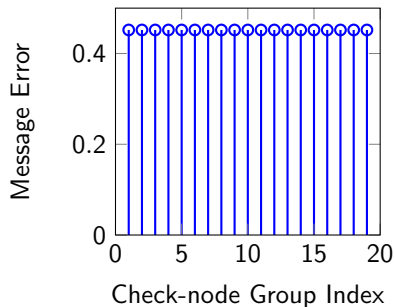
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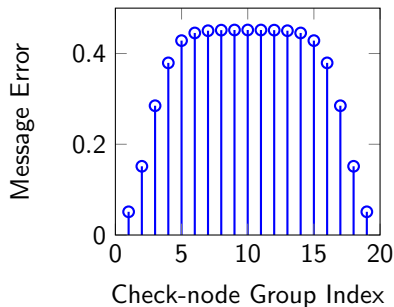
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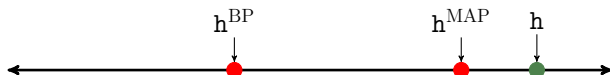
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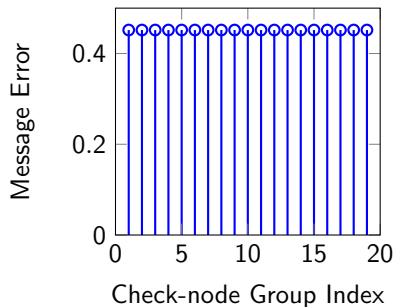
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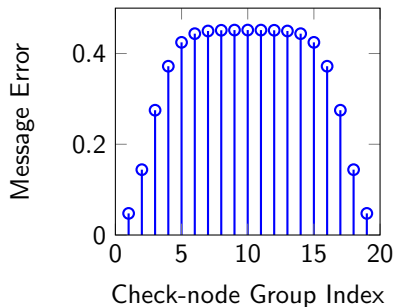
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Uncoupled



Spatially Coupled



Threshold Saturation Result

MAP Performance with a BP Decoder!

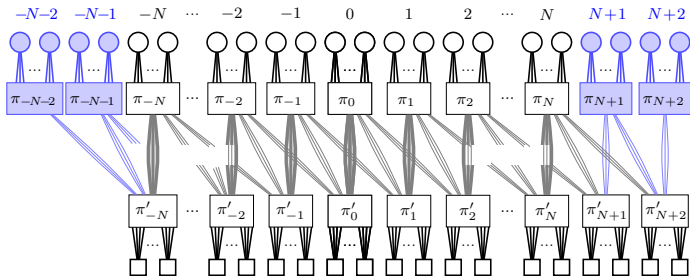
For large N , w $\mathbf{h}_c^{\text{BP}} = \mathbf{h}^{\text{MAP}}$

Threshold Saturation Result

MAP Performance with a BP Decoder!

For large N , w $\mathbf{h}_c^{\text{BP}} = \mathbf{h}^{\text{MAP}}$

SC-LDPC (ℓ, r)	Shannon \mathbf{h}^{Sh}	AWGN \mathbf{h}_c^{BP}	BSC \mathbf{h}_c^{BP}
(3, 6)	0.5000	0.4794	0.4681
(4, 6)	0.6667	0.6645	0.6633
(5, 6)	0.8333	0.8333	0.8333



The video link comes here

Rate loss for finite N and w

SC-LDPC (ℓ, r, N, w)	Shannon h^{Sh}	AWGN h_c^{BP}	BSC h_c^{BP}
(3, 6, 10, 3)	0.5434	0.4794	0.4681
(3, 6, 20, 3)	0.5222	0.4794	0.4681
(3, 6, 30, 3)	0.5149	0.4794	0.4681
(4, 6, 10, 3)	0.7245	0.6645	0.6633
(4, 6, 20, 3)	0.6963	0.6645	0.6633
(4, 6, 30, 3)	0.6866	0.6645	0.6633
(5, 6, 10, 3)	0.9056	0.8333	0.8333
(5, 6, 20, 3)	0.8704	0.8333	0.8333
(5, 6, 30, 3)	0.8582	0.8333	0.8333

Pros & Cons

Pros

- ▶ Significant improvement in thresholds
- ▶ Achieves capacity under [simple BP decoding](#) [KRU'11,KYMP'14]
- ▶ [Universality](#) - works for all channels models!

Cons

- ▶ Need [large](#) blocklengths to leverage the gains

Outline

- 1 Spatial Coupling(SC)
 - SC-LDPC Ensemble
 - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
 - Introduction to Lattices
 - Proposed Lattice Construction
 - Poltyrev Goodness
 - Symmetric Interference Channel
- 3 Side-Information Problems
 - Gelfand-Pinsker & Wyner-Ziv
 - Compound LDGM/LDPC Codes
 - Spatial Coupling of Compound Codes
- 4 Write-Once Memory
 - Problem Statement
 - Coding Scheme
- 5 Research Summary

Lattice

Lattice

Let $\mathbf{G} \in \mathbb{R}^{n \times k}$. An n -dimensional real lattice Λ can be defined as

$$\Lambda = \{\mathbf{G}\mathbf{z}, \mathbf{z} \in \mathbb{Z}^k\}$$

Lattice

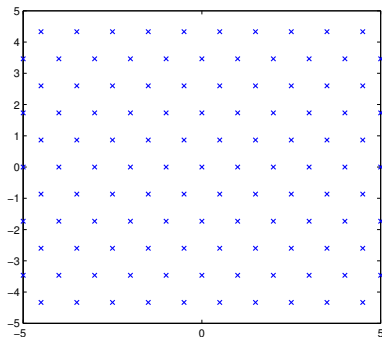
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Example:

$$\mathbf{G} = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$



Lattices and Lattice Codes

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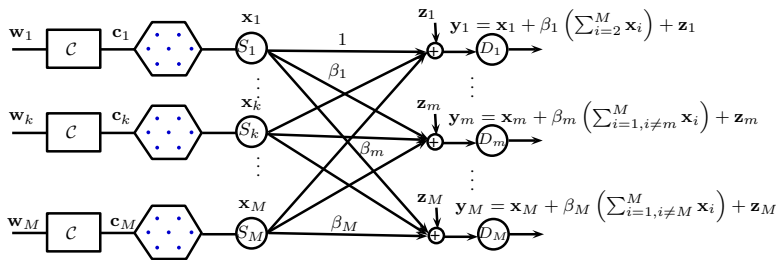
- ▶ Efficient structures for
 - **Mathematics**: sphere packing and sphere covering problems
 - **Information Theory**: channel coding & quantization
- ▶ Single user Gaussian channel - Erez and Zamir
- ▶ Coding with side information - Wyner-Ziv and Costa, Zamir, Erez and Shamai
- ▶ Secrecy - He and Yener
- ▶ Dirty multiple access channel - Philosof, Khisti, Erez and Zamir

“Lattices are everywhere” by Ram Zamir

Prior Work

New perspectives for dealing with interference:

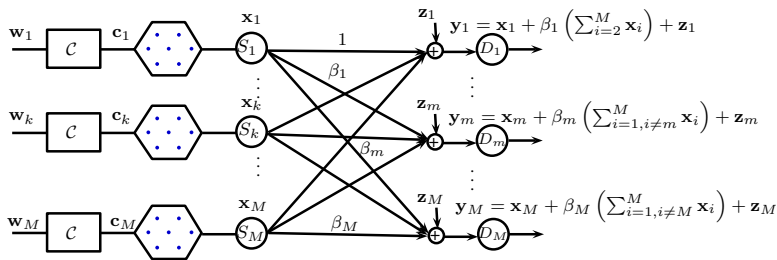
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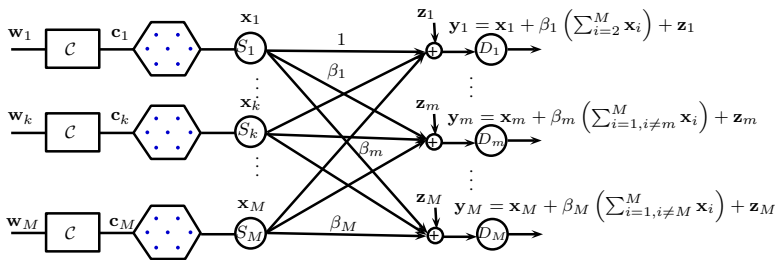
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- Above schemes are all based on lattices **good** for channel coding

Background on lattices

Voronoi region

The *fundamental Voronoi region* \mathcal{V} of a lattice, is the set of all points in \mathbb{R}^n that are closest to the zero vector.

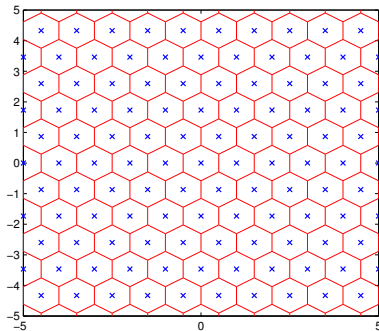
$$\mathcal{V} := \{\mathbf{x} : \|\mathbf{x} - \mathbf{0}\| \leq \|\mathbf{x} - \mathbf{c}\| \quad \forall \mathbf{c} \in \Lambda\}$$

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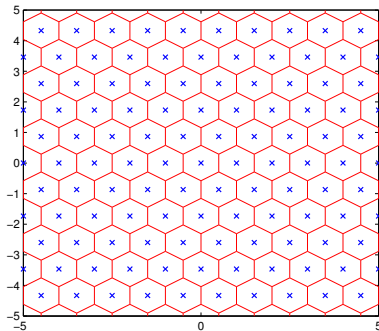


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► Fundamental volume of Λ , $V(\Lambda)$: $\text{Vol}(\mathcal{V})$

Goodness of Lattices for Channel Coding

- ▶ Let a lattice point $\lambda \in \Lambda$ is transmitted via AWGN channel of variance σ^2
- ▶ Volume-to-noise ratio(VNR) of Λ :

$$\text{VNR} := \frac{V(\Lambda)^{2/n}}{2\pi e\sigma^2}$$

- ▶ $P(\Lambda, \sigma^2) := \Pr(d(\lambda, \lambda + \mathbf{z}) \geq d(\lambda', \lambda' + \mathbf{z}))$ for some $\lambda' \in \Lambda$

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Poltyrev Goodness for Channel Coding

For any $\text{VNR} > 1 \exists \{\Lambda_n\}$ such that $P(\Lambda_n, \sigma^2) \rightarrow 0$ as $n \rightarrow \infty$.

- ▶ *Poltyrev-good* lattices are at the core of such lattice coding schemes

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Motivating questions

- ▶ All the existing results were based on Construction-A.
 - Linear codes over *increasing field sizes* and their ML decoding

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Main Result

- ▶ Codes over \mathbb{F}_2 and *BP decoding* suffice
- ▶ We show existence of sequence of lattices that are *Poltyrev-good* under BP decoding
- ▶ Apply proposed lattices to *Symmetric Interference Channel*
- ▶ Can be applied to other problems which adopt Construction A lattices

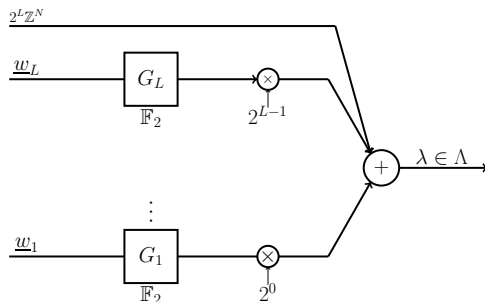
Construction D with L levels

- Barnes and Sloane '83, Forney, Chung and Trott '00, Yan, Ling, Wu ' 13
- Choose $G_1 \subseteq \dots \subseteq G_L$ where G_l is a gen matrix of code \mathcal{C}_l over \mathbb{F}_2 .

$$\begin{array}{c}
 \left[\begin{array}{ccc} \cdots & g_L & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_{k_2} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_{k_1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_2 & \cdots \\ \cdots & g_1 & \cdots \end{array} \right] \begin{array}{l} \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_L \\ \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} \dots \\ \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_2 \end{array} \end{array} \left. \vphantom{\begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array}} \right\} G_1$$

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- ▶ $\underline{\lambda} = \underline{w}_1 \mathbf{G}_1 + 2\underline{w}_2 \mathbf{G}_2 \dots + 2^{L-1} \underline{w}_{L-1} \mathbf{G}_{L-1} + 2^L \mathbb{Z}^N \in \Lambda$



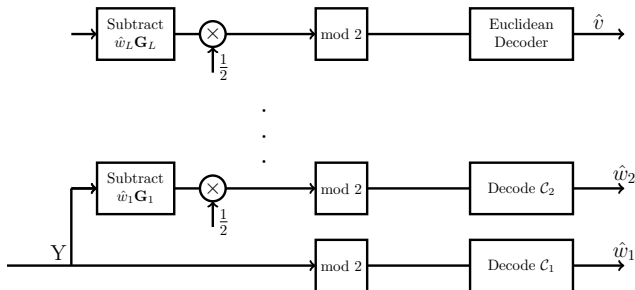
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Multi-Level Decoding(Successive Cancellation)

$$\blacktriangleright \underline{y} = \boxed{\underline{w}_1 \mathbf{G}_1 + 2\underline{w}_2 \mathbf{G}_2 \dots + 2^{L-1} \underline{w}_{L-1} \mathbf{G}_{L-1} + 2^L \mathbb{Z}^N} + \underline{n}$$

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- ▶ $\underline{y} \bmod 2 = [\underline{w}_1 \mathbf{G}_1 + \underline{n}] \bmod 2 = \underline{w}_1 \odot \mathbf{G}_1 + \boxed{\underline{n} \bmod 2}$
- ▶ Decode \underline{w}_1 , reconstruct $\underline{w}_1 \mathbf{G}_1$ and subtract from \underline{y}



Theorem (Forney, Trott & Chung)

There exists a sequence of Construction D lattices based on $\mathcal{C}_1 \subseteq \mathcal{C}_2 \dots \subseteq \mathcal{C}_L$ such that the $VNR \rightarrow 1$ and the $Pr(\lambda, \sigma^2) \rightarrow 0$.

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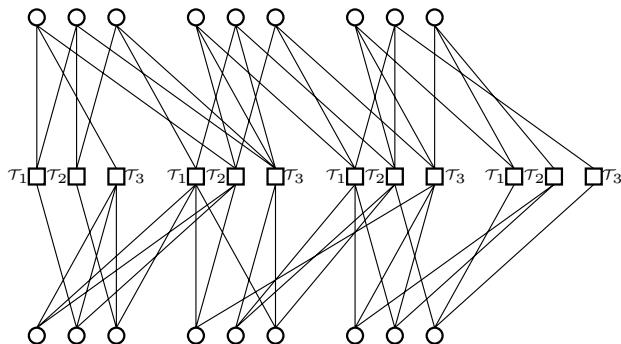
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Objective:

- ▶ Capacity achieving nested code constructions, preferably under BP decoding.

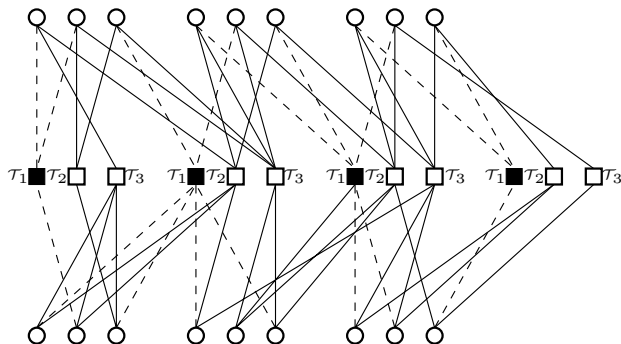
Proposed Nested Spatially-Coupled LDPC Ensemble

- 1 Begin with a (d_v^1, d_c) SC LDPC code. For ex, $(d_v^1 = 3, d_c = 6, L = 3, w = 2)$.
- 2 Group check nodes into type $\mathcal{T}_k, k \in \{1, \dots, d_v^1\}$



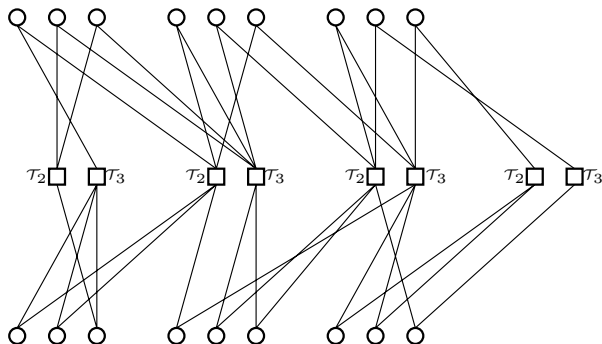
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- 4 Results in a super-code that is a (d_v^2, d_c) SC LDPC code.



Lattice Design based on the proposed Nested SC LDPC ensemble

- 1 For a given σ , compute the capacity of the mod-2 AWGN channel at each level:

$$\underline{y}_i = \underline{w}_i \mathbf{G}_i + \frac{1}{2^{i-1}} \underline{n} \mod 2 = \underline{w}_i \odot \mathbf{G}_i + \boxed{\frac{1}{2^{i-1}} \underline{n} \mod 2}$$

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Lemma

Given nested binary linear codes $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots \subseteq \mathcal{C}_r$ there exists nested generator matrices for these codes.

Proposed Ensemble is Capacity achieving

Theorem

Each code ensemble in the proposed nested Spatially-Coupled LDPC ensemble is capacity achieving.

Proof.

- ▶ Show that the mod 2 AWGN channel is BMS.
- ▶ Each derived protograph has the same spatially coupled structure.
- ▶ The proof follows from Kudekar & Urbanke, Kumar & Pfister's results.



Proposed Lattices are Poltyrev-Good

Theorem

There exists a sequence of SC LDPC lattices with $VNR(\Lambda, \sigma^2) \rightarrow 1$ for which, under multistage BP decoding, $\mathbb{E} [P(\lambda, \sigma^2)] \rightarrow 0$ as $w, L, M \rightarrow \infty$.

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- ▶ The proposed nested ensemble achieve capacity.
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- ▶ **Binary codes** and more importantly **practical BP decoding** suffices.
- ▶ Practically we observe that **two levels** of coding gets you lattices very close to Poltyrev limit.

Design Example of Poltyrev-Good Lattice

Target error probability $P(2^L \mathbb{Z}^n, \sigma_L^2) = 10^{-4}$ in the uncoded level $\implies \sigma_L = 0.08$

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- Fix $L=3$ and use $(3, 30)$, $(14, 30)$ nested SC-LDPC codes.

- Note $P(4\mathbb{Z}^n, \sigma^2) \approx nP(4\mathbb{Z}, \sigma^2)$
- We fix $n = 2 \times 10^5$

(d_c, d_v^1, d_v^2)	(L, w)	$P(4\mathbb{Z}, \sigma^2)$	σ_{max}	VNR	VNR _{rate-loss}
$(30, 14, 3)$	$(32, 4)$	5×10^{-10}	0.3184	1.02dB	1.347dB

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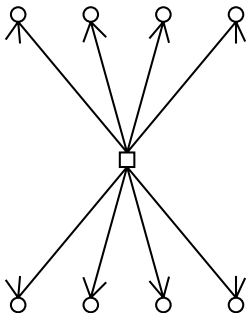
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(60, 26, 3)	(72, 12)	5×10^{-10}	0.3200	0.482dB	0.927dB
(60, 27, 3)	(64, 9)	5×10^{-10}	0.3203	0.57dB	0.951dB

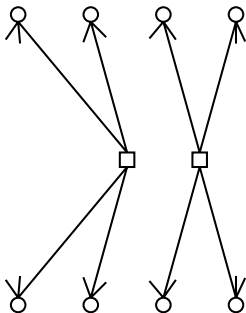
Alternate Nested SC LDPC ensemble

- Derive a lower rate code by “splitting the checks”
- Consider a $(3, 8)$ code

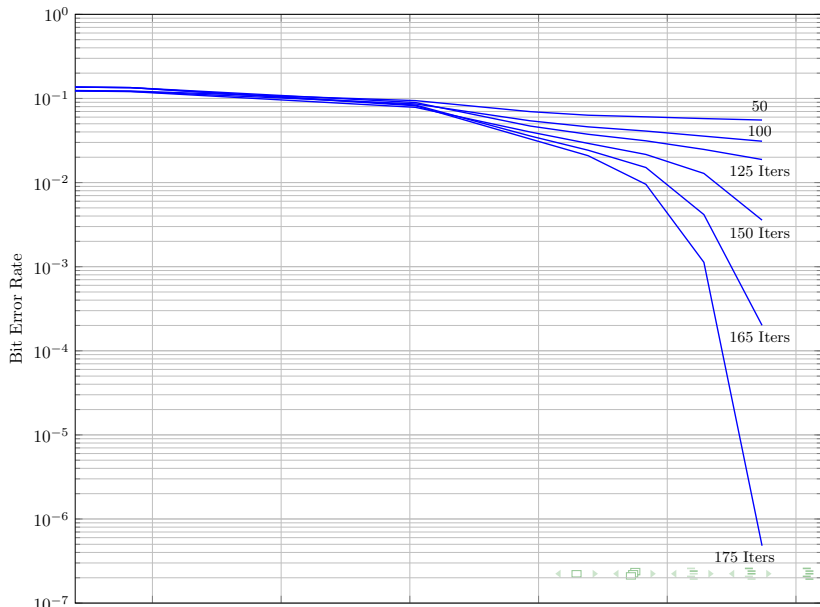


Alternate Nested SC LDPC ensemble

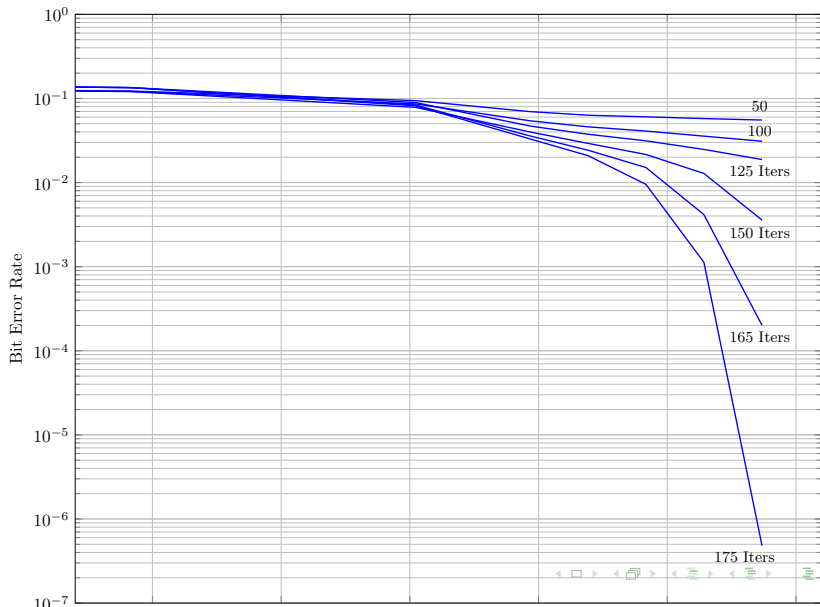
- ▶ Derive a lower rate code by “splitting the checks”
- ▶ Consider a $(3, 8)$ code
- ▶ Split each check into “two” checks to derive a $(3, 4)$ sub-code
- ▶ Easy to prove that resulting code is from the $(3, 4)$ SC LDPC ensemble



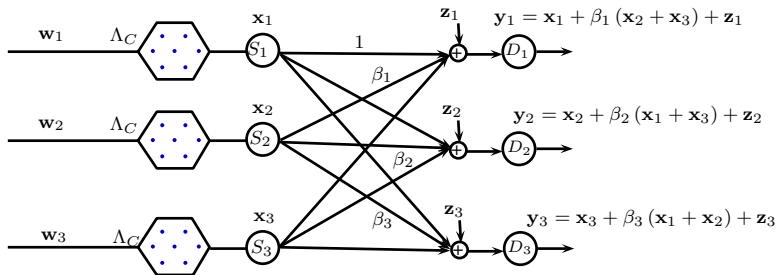
Simulation Results



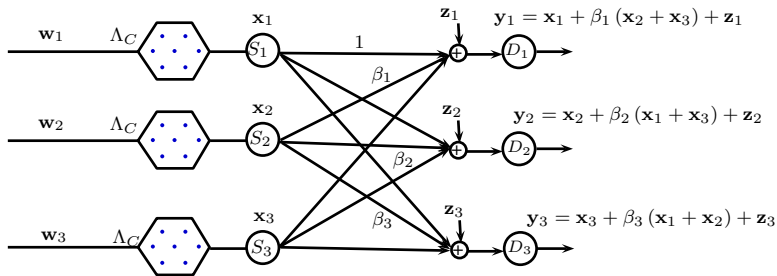
Simulation Results



3-User Symmetric Interference Channel



3-User Symmetric Interference Channel



► $\mathbf{x}_i \in \Lambda_C \triangleq \Lambda \cap \mathbb{Z}_4^N$ is transmitted.

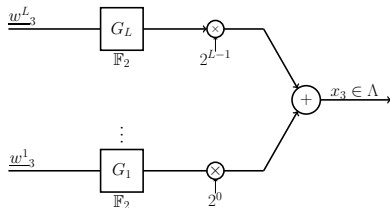
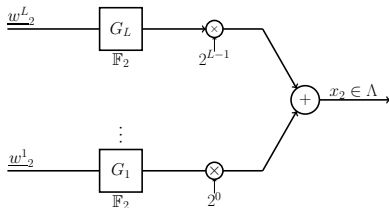
Symmetric Interference Channel - Decoding Sums

Interference at Destination 1:

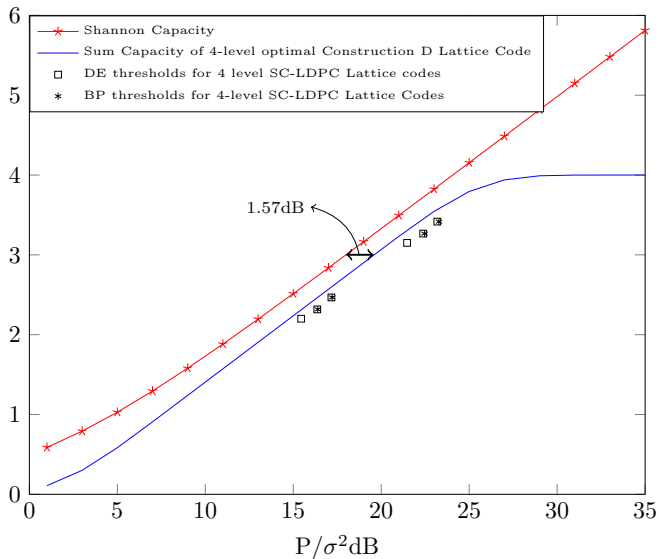
$$\begin{aligned}\mathbf{x}_2 + \mathbf{x}_3 &= (\underline{w}_2^1 + \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{w}_2^2 + \underline{w}_3^2)\mathbf{G}_2 + 4\mathbf{k}_{23} \\ &= (\underline{w}_2^1 \oplus \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{c}_{23}^1 \oplus \underline{w}_2^2 \oplus \underline{w}_3^2)\mathbf{G}_2 + 4(\underline{c}_{23}^2 + \mathbf{k}_{23})\mathbf{Z}\end{aligned}$$

where the carry overs are

$$\begin{aligned}\underline{c}_{23}^1 &= 0.5 (\underline{w}_1^1 + \underline{w}_1^2 - \underline{w}_1^1 \oplus \underline{w}_1^2), \\ \underline{c}_{23}^2 &= 0.5 (\underline{c}_{23}^1 + \underline{w}_2^1 + \underline{w}_2^2 - \underline{c}_{23}^1 \oplus \underline{w}_2^1 \oplus \underline{w}_2^2)\end{aligned}$$



Achievable Information Rates



Concluding Remarks

- ▶ Multilevel constructions - efficient ways to decode integer combinations
- ▶ Need capacity achieving nested codes
- ▶ Multilevel construction is provably good under message passing decoding

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- ▶ Need capacity achieving nested codes
- ▶ Multilevel construction is provably good under message passing decoding
- ▶ Coding schemes based on *binary codes and iterative decoding* suffice

Outline

- 1 Spatial Coupling(SC)
 - SC-LDPC Ensemble
 - Threshold Saturation Phenomenon
- 2 SC-LDPC Lattices
 - Introduction to Lattices
 - Proposed Lattice Construction
 - Poltyrev Goodness
 - Symmetric Interference Channel
- 3 Side-Information Problems
 - Gelfand-Pinsker & Wyner-Ziv
 - Compound LDGM/LDPC Codes
 - Spatial Coupling of Compound Codes
- 4 Write-Once Memory
 - Problem Statement
 - Coding Scheme
- 5 Research Summary

Lossy Source Coding Problem

$$X^n = (X_1, \dots, X_n), X_i \sim \text{Bernoulli}(\frac{1}{2})$$

Binary code $\mathcal{C} = (n, k)$, rate $R = k/n$

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- ▶ Compress X^n to $\hat{X}^n \in \mathcal{C}$
- ▶ Min. Hamming distortion

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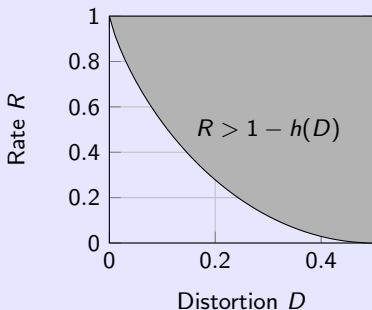
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- ▶ Rate-Distortion theory:

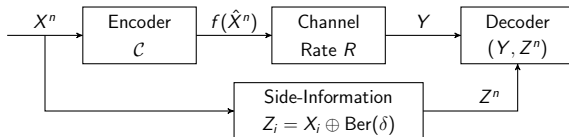
$$R > 1 - h(D)$$

- ▶ $h(\cdot)$ is binary entropy function

$$h(D) = -D \log_2 D - (1-D) \log_2 (1-D)$$



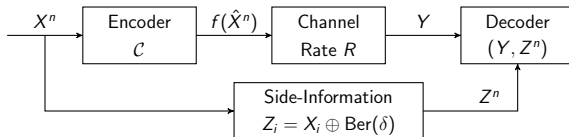
Side-Information Problems: Wyner-Ziv



Wyner-Ziv Formulation

- ▶ **Side-information** Z^n about X^n
- ▶ Decoder **additionally** has Z^n
- ▶ Say $Z_i = X_i \oplus \text{Ber}(\delta)$

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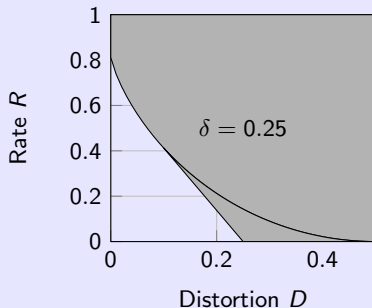


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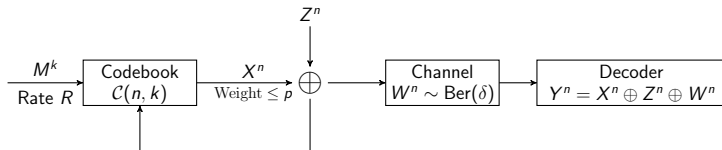
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- ▶ Say $Z_i = X_i \oplus \text{Ber}(\delta)$
- ▶ Wyner-Ziv theory:

$$R > \text{l.c.e}\{h(D * \delta) - h(D), (\delta, 0)\}$$

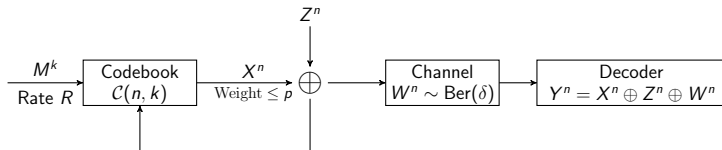
- ▶ $D * \delta = D(1 - \delta) + \delta(1 - D)$



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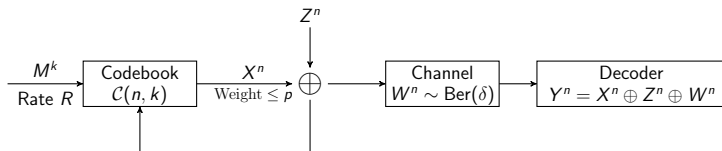


Gelfand-Pinsker Formulation

- ▶ Message M^k encoded to $X^n \in \mathcal{C}$ with $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] \leq p \leq \frac{1}{2}$
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- ▶ The output at the decoder is

$$Y^n = X^n \oplus Z^n \oplus W^n, \quad \{W_i\} \sim \text{Ber}(\delta)$$

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- ▶ Capacity region by Gelfand-Pinsker:

$$R < h(p) - h(\delta)$$

Main Result

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- ▶ Message-passing algorithms have *non-negligible gap*
- ▶ Remedy via **Spatial-Coupling**
 - Channel coding in coupled compound codes (Kasai et al.)
 - Lossy source coding with spatially-coupled LDGM (Aref et al.)
 - Encoding with compound codes has additional challenges

An (ℓ, r) LDGM Code

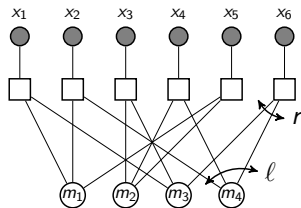
Generator Matrix

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\ell = 3$$

$$r = 2$$

Tanner Graph



$$\text{LDGM Code } \mathcal{C} = \{x : x = m \odot G\}$$

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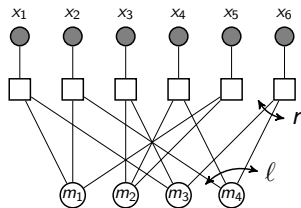
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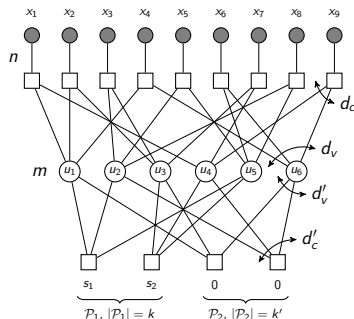
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$$\text{LDGM Code } \mathcal{C} = \{x : x = m \odot G\}$$

$$x_1 = m_1 \oplus m_3 \iff x_1 \oplus m_1 \oplus m_3 = 0$$

Compound LDGM/LDPC Codes



► Codebook $\mathcal{C}(n, m - k - k')$

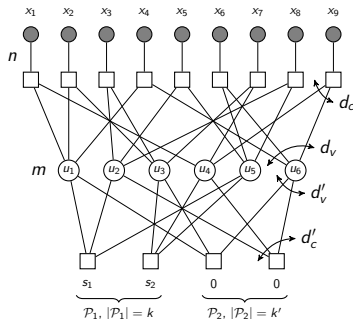
► Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

► Codeword (x_1, \dots, x_9) :

$$x_1 = u_1 \oplus u_4, \quad x_2 = \dots$$

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Key Properties

► Compound code is

- a **good source code** under optimal encoding
- a **good channel code** under optimal decoding

► LDGM code is

- a **good source code** under optimal encoding
- (side note) LDGM code is **not** a good channel code

Good Code

“Good” source code

- ▶ Rate of the code is $R = 1 - h(D) + \varepsilon$
- ▶ When this code is used to *optimally encode* $\text{Ber}(\frac{1}{2})$ source
- ▶ The average Hamming *distortion is at most* D

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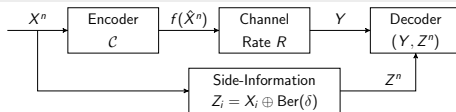
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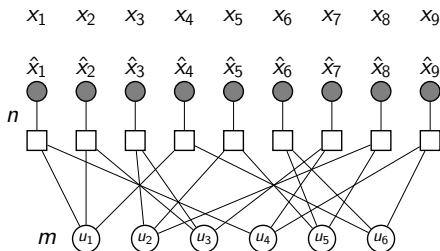
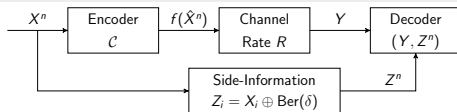
“Good” channel code

- ▶ Rate of the code is $R = 1 - h(\delta) - \varepsilon$
- ▶ When this code is used for channel coding on $\text{BSC}(\delta)$
- ▶ Message est. under *optimal decoding* with *error at most* ε

Coding Scheme: Wyner-Ziv



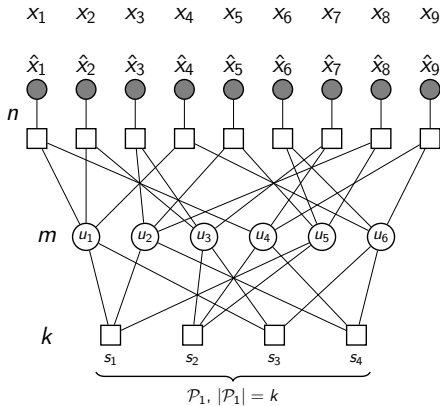
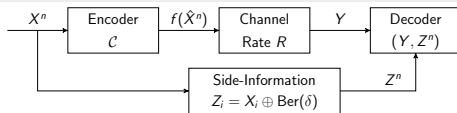
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► Encode X^n to \hat{X}^n using **LDGM** w/
Distortion $\approx D$

$$\frac{m}{n} \approx 1 - h(D) + \varepsilon$$

Coding Scheme: Wyner-Ziv



► Encode X^n to \hat{X}^n using **LDGM** w/
Distortion $\approx D$

► Compute & **transmit** s_i 's
 $R = \frac{k}{n} \approx h(D * \delta) - h(D)$

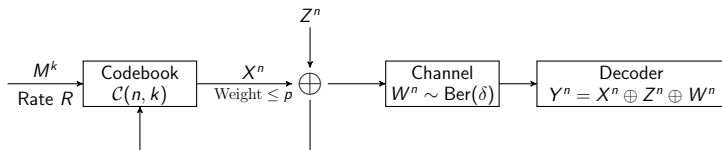
► Decoder has Z^n :

$$\begin{aligned} Z_i &= X_i \oplus \text{Ber}(\delta) \\ &\approx \hat{X}_i \oplus \text{Ber}(D) \oplus \text{Ber}(\delta) \\ &= \hat{X}_i \oplus \text{Ber}(D * \delta) \end{aligned}$$

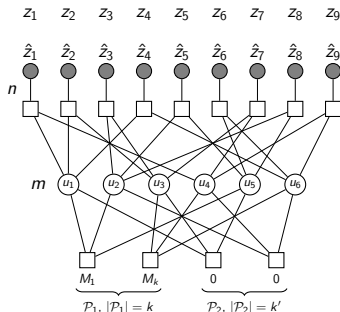
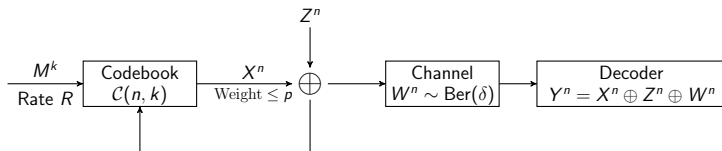
► Decode \hat{X}^n from Z^n & s_i

$$\frac{m}{n} \approx 1 - h(D) + \varepsilon \quad \frac{m-k}{n} \approx 1 - h(D * \delta) + \varepsilon$$

Coding Scheme: Gelfand-Pinsker



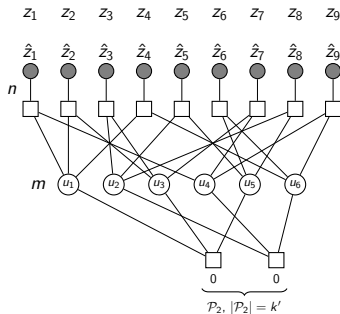
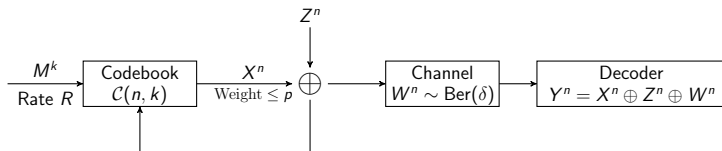
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- ▶ With message M^k , encode Z^n to \hat{Z}^n (Distortion $\approx p$)
- ▶ Transmit $X^n = Z^n \oplus \hat{Z}^n$

$$\frac{m-k-k'}{n} \approx 1 - h(p) + \varepsilon \quad \frac{m-k'}{n} \approx 1 - h(\delta) + \varepsilon$$

Coding Scheme: Gelfand-Pinsker



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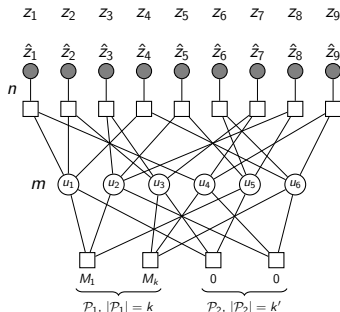
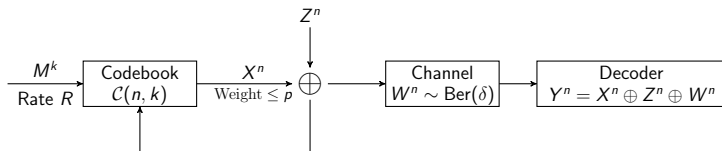
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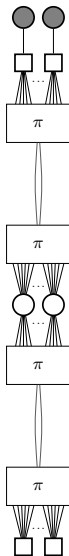
► Decode \hat{Z}^n and compute M^k

► $R = \frac{k}{n} \approx h(p) - h(\delta)$

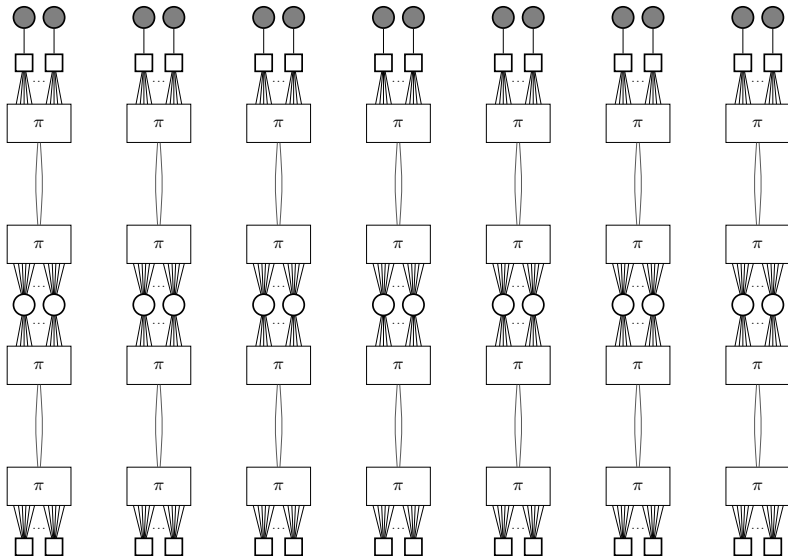
Remarks

- ▶ Need codes that are *simultaneously good* for channel and source coding
- ▶ Use *message-passing algorithms* instead of *optimal*
- ▶ Use spatial-coupling for *goodness* of codes under message-passing

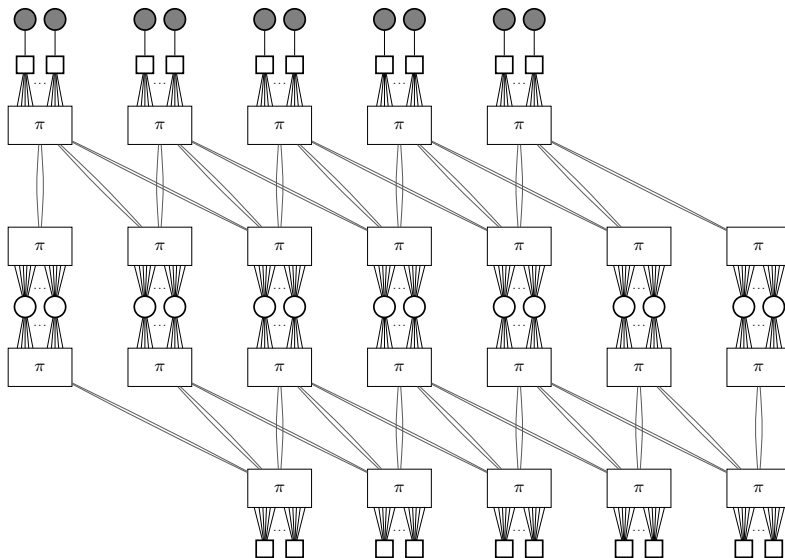
Spatially-Coupled Compound LDGM/LDPC Codes



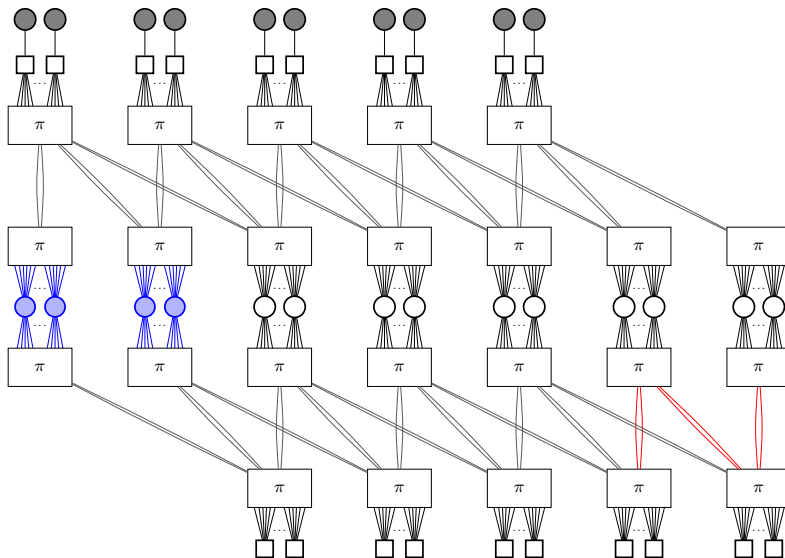
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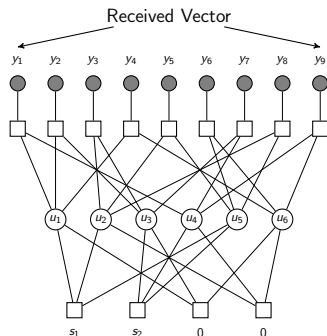
Spatially-Coupled Compound LDGM/LDPC Codes



Spatially-Coupled Compound LDGM/LDPC Codes



Decoding in Spatially-Coupled Compound Codes



Channel LLR

y_i

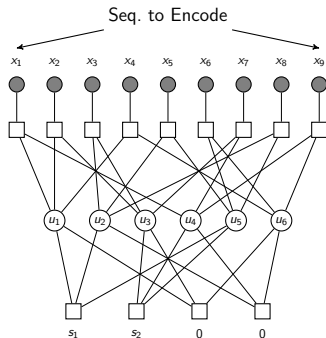
$$L = L_1 + \dots + L_k$$

$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

Remarks

- ▶ Standard message-passing algorithm
- ▶ Threshold saturation proven for SC compound codes on BEC
- ▶ Empirically observed for BMS channels

Encoding in Spatially-Coupled Compound Codes



$$x_i \xrightarrow{(-1)^x \tanh \beta}$$

$$L = L_1 + \dots + L_k$$

$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

Remarks

- ▶ Inverse temperature parameter β
- ▶ Message-passing rules are the same
- ▶ However, a *crucial decimation step is needed*

Encoding in SC Compound Codes: BPGD Algorithm

```

while There are active LDPC bit-nodes do
  for  $t = 1$  to  $T$  do
    Run the BP equations
  end for
  Evaluate LLRs  $m_i$  for each LDPC bit-node
  Choose max. of  $|m_i|$  in left-most  $w$  active sections
  if  $|m_{i^*}| = 0$  then
    Set  $u_{i^*}$  to 0 or 1 uniformly randomly
  else
    Set  $u_{i^*}$  to 0 or 1 with prob.  $\frac{1+\tanh m_{i^*}}{2}$  or  $\frac{1-\tanh m_{i^*}}{2}$ 
  end if
  Decimate (remove) LDPC bit-node  $i^*$  and update parities
end while
If  $\{u_i\}$  fail to satisfy LDPC checks, then re-encode

```

Encoding in SC Compound Codes: Remarks

- ▶ Randomization in setting u_{j^*} is crucial
- ▶ BPGD applied to uncoupled code *always failed*
- ▶ Spatially-coupled structure is crucial for successful encoding
 - In addition, distortion is close to optimal thresholds
 - *Does not encode* if decimated from both left and right
 - *Does not encode* if both left and right boundary is set to 0

Encoding in SC Compound Codes: Numerical Example

Block length (n)	4-cycles	Attempts $1/2/3/4/ \geq 5$
9000	yes	5/3/5/2/35
9000	no	21/12/5/3/9
27000	no	35/15/0/0/0
45000	no	40/9/0/0/1
63000	no	44/6/0/0/0
81000	no	50/0/0/0/0

Remarks

- ▶ # Attempts to encode 50 seq. in $(6, 3)$ LDGM / $(3, 6)$ LDPC
- ▶ $L = 20$, $w = 4$, $\beta = 0.65$, $T = 10$
- ▶ Removing 4-cycles dramatically improves success
- ▶ How much do 6-cycles matter?

Numerical Results: Wyner-Ziv

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(D_*, δ_*)	(D, δ)
(6, 3)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1174, 0.122)
(8, 4)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1149, 0.120)
(10, 5)	(3, 6)	(20, 4)	(0.111, 0.134)	(0.1139, 0.122)

Remarks

- D_* and δ_* are calculated based on the rate of the respective code:

$$D_* = h^{-1}(1 - R1)$$

$$\delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000$, $\beta = 1.04$, $T = 10$

Numerical Results: Gelfand-Pinsker

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(p_*, δ_*)	(p, δ)
(6, 3)	(3, 6)	(20, 4)	(0.215, 0.157)	(0.2200, 0.152)
(8, 4)	(3, 6)	(20, 4)	(0.215, 0.157)	(0.2230, 0.151)
(10, 5)	(3, 6)	(20, 4)	(0.215, 0.157)	(0.2200, 0.151)

Remarks

- p_* and δ_* are calculated based on the rate of the respective code:

$$p_* = h^{-1}(1 - R1)$$

$$\delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000$, $\beta = 0.65$, $T = 10$

Concluding Remarks

Conclusion

- ▶ Spatially-coupled codes achieve the rate regions of Wyner-Ziv and Gelfand-Pinsker problems
- ▶ **Coupling structure** is also crucial
 - to achieve optimum thresholds
 - for encoding to succeed with decimation

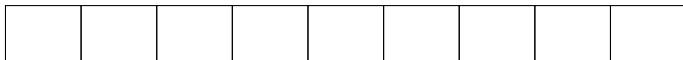
Open Questions

- ▶ Effect of degree profiles, short-cycles on encoding success
- ▶ Precise trade-offs with **polar codes**

Outline

- 1 Spatial Coupling(SC)
 - SC-LDPC Ensemble
 - Threshold Saturation Phenomenon
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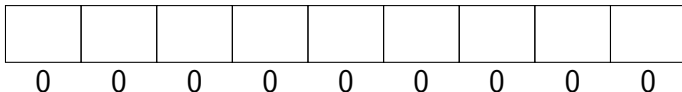
Write-Once Memories



Flash Memory

- ▶ In typical flash memory, changing from 0 to 1 is easy
- ▶ Resetting 1 to 0 requires **rewriting whole block**
- ▶ Write-once memories model such storage systems

Write-Once Memories



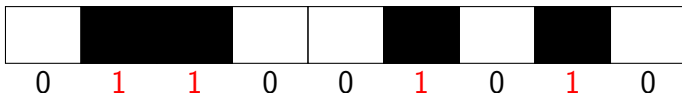
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Binary Write-Once Memories

- ▶ **0 \rightarrow 1** is allowed

Write-Once Memories



Flash Memory

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- ▶ Resetting 1 to 0 requires **rewriting whole block**
- ▶ Write-once memories model such storage systems

Binary Write-Once Memories

- ▶ 0 \rightarrow 1 is allowed
- ▶ 1 \rightarrow 0 *is forbidden*

Capacity Region (I) - Noiseless

Message



0	0	1	0	0	1	0	1	0
---	---	---	---	---	---	---	---	---

Write-Once Memory without Noise

- ▶ In 1982, Rivest and Shamir gave first WOM codes
 - 2 bits in 2 writes with only 3 cells

2 + 2 bits in 2-write WOM

x	$r(x)$	$r'(x)$
00		
01		
10		
11		

2 + 2 bits in 2-write WOM

x	$r(x)$	$r'(x)$
00	000	
01	001	
10	010	
11	100	

2 + 2 bits in 2-write WOM

x	$r(x)$	$r'(x)$
00	000	111
01	001	110
10	010	101
11	100	011

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Write-Once Memory without Noise

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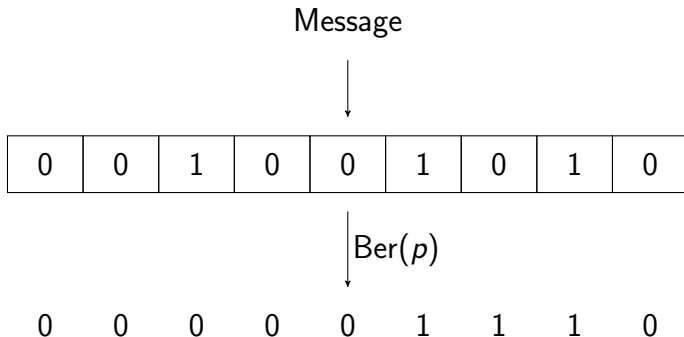
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Write-Once Memory without Noise

- ▶ In 1982, Rivest and Shamir gave first WOM codes
 - 2 bits in 2 writes with only 3 cells
- ▶ Only about $nt / \log(t)$ cells required to store n bits for t writes
- ▶ In 1985, Heegard gave the *capacity* for t -write system
- ▶ For a 2-write system, it is

$$\left\{ (R_1, R_2) \mid 0 \leq R_1 < h(\delta), 0 \leq R_2 < 1 - \delta \right\}$$

Capacity Region (II) - Read Errors



Write-Once Memory with Read Errors

- ▶ Different from write errors
- ▶ $Y = X \oplus \text{Ber}(p)$, where $\text{Ber}(p)$ denotes the Bernoulli noise
- ▶ Capacity region is *unknown*

Main Result

Objective

- ▶ Construct *low-complexity* coding schemes that achieve the *capacity region* of the WOM system
 - Low-complexity encoding and decoding

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- ▶ Focus on the 2-write WOM system
 - Achieves the capacity region of the noiseless system
 - For read errors, achieves

$$R_1 < h(\delta) - h(p),$$

$$R_2 < 1 - \delta - h(p).$$

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- ▶ Extension to multi-write systems *seems possible with BPGD*

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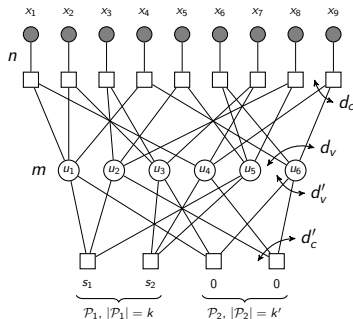
$$R_2 < 1 - \delta - h(p).$$

- ▶ Extension to multi-write systems *seems possible with BPGD*

Idea

- ▶ Use compound LDGM/LDPC codes
- ▶ Encoding for second write is *erasure quantization*
- ▶ Use *spatial coupling with message-passing*

Compound LDGM/LDPC Codes



► Codebook $(n, m - k - k')$

► Message constraints

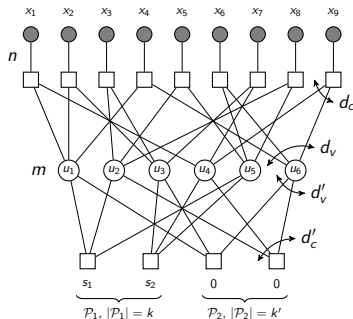
$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

► Codeword (x_1, \dots, x_9) :

$$x_1 = u_1 \oplus u_4, \quad x_2 = \dots$$

► Parametrized by s^k : $\mathcal{C}(s^k)$

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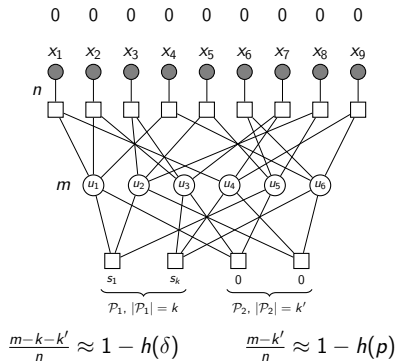
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Key Properties of Compound Codes

- a natural **coset decomposition**: $\mathcal{C} = \bigcup_{s^k \in \{0,1\}^k} \mathcal{C}(s^k)$
- achieves capacity over eras. chan. under MAP (when $m = n$)
- a **good source code** under optimal encoding
- a **good channel code** under optimal decoding

Coding Scheme for 2-write WOM: First Write

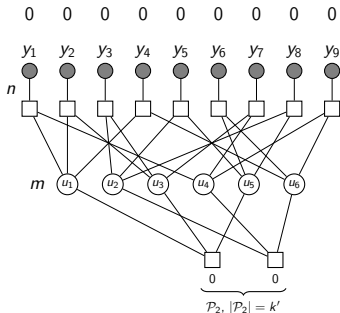
$$R_1 < h(\delta) - h(p)$$



- With message s^k , encode 0^n to x^n (Distortion $\approx \delta$)
- Store x^n

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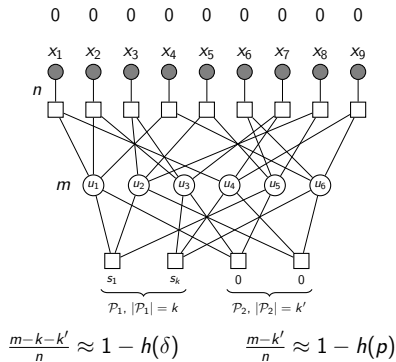
$$\frac{m-k'}{n} \approx 1 - h(p)$$

- ▶ With message s^k , encode 0^n to x^n (Distortion $\approx \delta$)
- ▶ Store x^n
- ▶ Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

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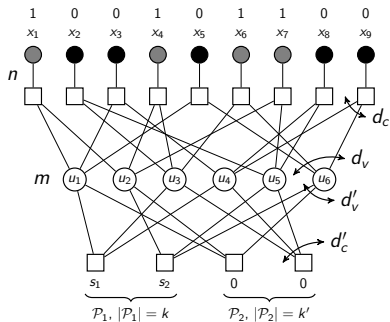


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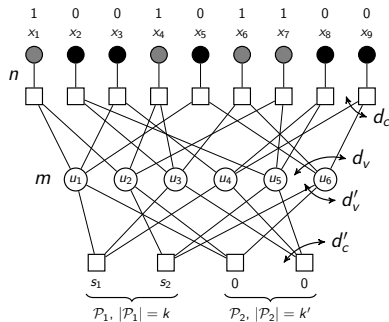
- ▶ Dec. x^n and compute s^k
- ▶ $R_1 = \frac{k}{n} \approx h(\delta) - h(p)$

Coding Scheme for 2-write WOM: Second Write



- Need to find a *consistent* codeword in $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write



- Need to find a *consistent* codeword in $\mathcal{C}(s^k)$
- Closely related to **Binary Erasure Quantization (BEQ)**
- En Gad, Huang, Li and Bruck (ISIT 2015)

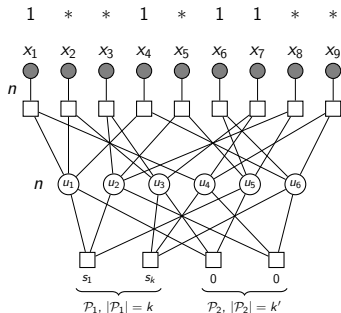
Binary Erasure Quantization

- ▶ Quantize a sequence in $\{0, 1, *\}^n$ to $x^n \in \mathcal{C} \subset \{0, 1\}^n$
 - 0's and 1's should *match exactly*
 - *'s can take *either 0 or 1*
- ▶ Can map the second write of 2-write WOM to BEQ
 - Map 0's to *'s and keep 1's
 - Quantize to codeword in $\mathcal{C}(s^k)$
- ▶ BEQ is the dual of decoding on binary erasure channel
 - Martinian and Yedidia (Allerton 2003)
 - Can quan. all seq. with erasure pattern $e^n \in \{0, 1\}^n$ to \mathcal{C}

\Updownarrow

Chan. dec. for \mathcal{C}^\perp can correct all vectors with eras. $1^n \oplus e^n$
- ▶ Choose a good (dual) code $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write

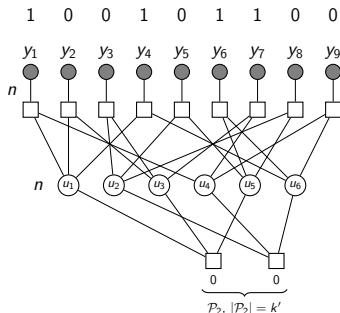


$$\frac{n-k-k'}{n} \approx \delta$$

$$\frac{n-k'}{n} \approx 1 - h(p)$$

- *Change 0's to *'s*
- With message s^k , encode seq. to $\mathcal{C}(s^k)$

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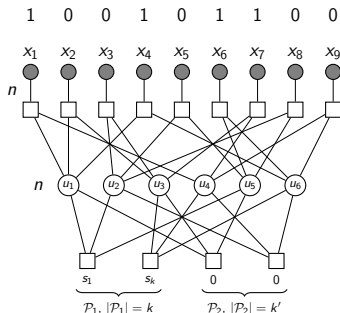
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$$y_i = x_i \oplus \text{Ber}(p)$$

Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



$$\frac{n-k-k'}{n} \approx \delta$$

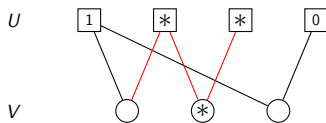
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- *Change 0's to *'s*
- With message s^k , encode seq. to $\mathcal{C}(s^k)$
- Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

- Dec. x^n and compute s^k
- $R_2 = \frac{k}{n} \approx 1 - \delta - h(p)$

Iterative Erasure Quantization Algorithm

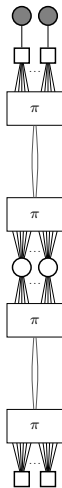


► *Peeling type encoder*

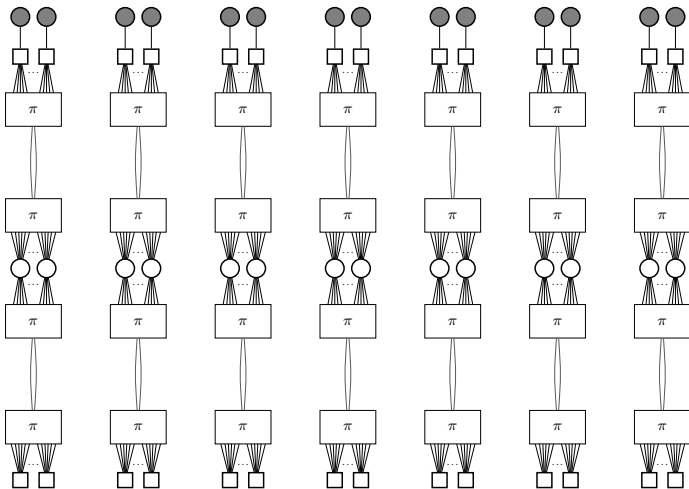
Remarks

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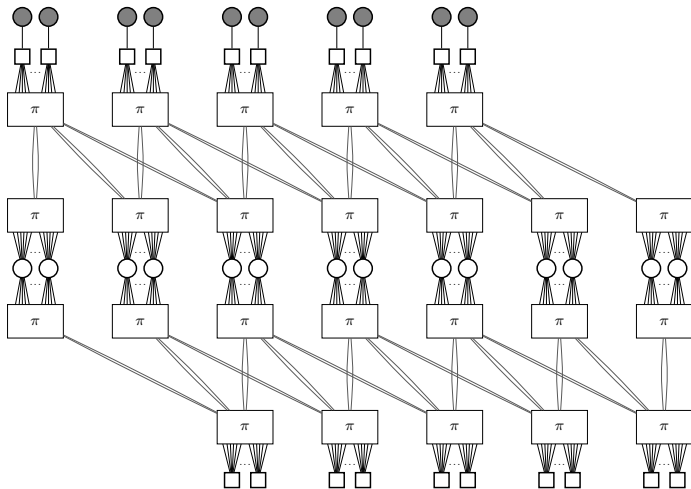
Spatially-Coupled Compound LDGM/LDPC Codes



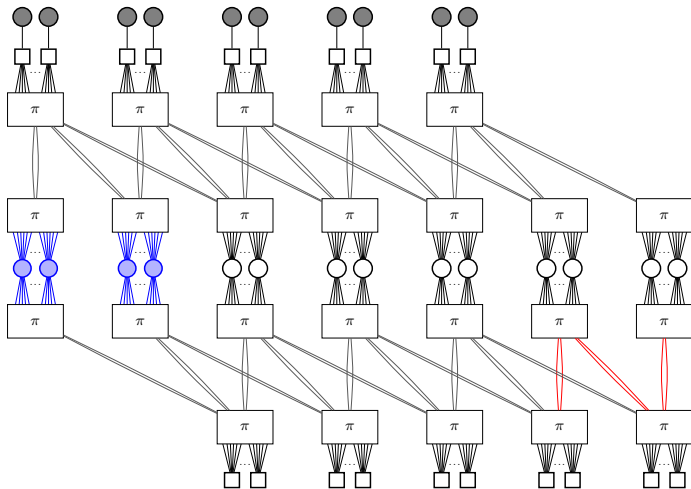
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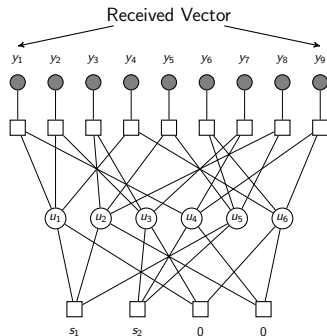
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


Decoding in Spatially-Coupled Compound Codes




Channel LLR

y_i  \rightarrow



$$L = L_1 + \dots + L_k$$



$$\tanh L = (-1)^s \cdot \tanh L_1 \cdots \tanh L_k$$

Remarks

- ▶ Standard message-passing algorithm
- ▶ Threshold saturation proven for SC compound codes on BEC
- ▶ Empirically observed for BMS channels

Numerical Results: Noiseless WOM

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	δ^*	δ $w = 2$	δ $w = 3$	δ $w = 4$
(3, 3, 3, 6)	0.500	0.477	0.492	0.494
(3, 3, 4, 6)	0.333	0.294	0.324	0.326
(3, 3, 5, 6)	0.167	0.095	0.156	0.158
(4, 4, 3, 6)	0.500	0.461	0.491	0.492
(4, 4, 4, 6)	0.333	0.278	0.323	0.325
(4, 4, 5, 6)	0.167	0.086	0.155	0.159
(5, 5, 3, 6)	0.500	0.436	0.488	0.491
(5, 5, 4, 6)	0.333	0.260	0.320	0.324
(5, 5, 5, 6)	0.167	0.079	0.154	0.159

Remarks

- ▶ δ^* is the Shannon threshold
- ▶ $L = 30$, Single system length ≈ 24000

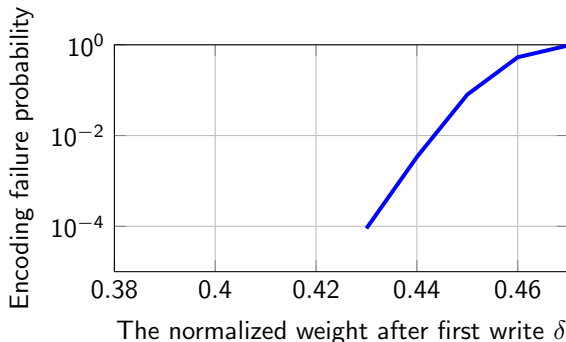
Numerical Results: WOM with Read Errors

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	w	(δ^*, p^*)	(δ, p)
(3, 3, 4, 6)	3	(0.333, 0.0615)	(0.321, 0.0585)
(3, 3, 4, 8)	3	(0.500, 0.0417)	(0.490, 0.0387)
(3, 3, 6, 8)	4	(0.250, 0.0724)	(0.239, 0.0684)
(4, 4, 4, 6)	4	(0.333, 0.0615)	(0.324, 0.0585)
(4, 4, 4, 8)	4	(0.500, 0.0417)	(0.492, 0.0387)
(4, 4, 6, 8)	4	(0.250, 0.0724)	(0.241, 0.0694)

Remarks

- δ^* and p^* are the Shannon thresholds
- $L = 30$, Single system length ≈ 30000

Numerical Results: Small Blocklength



Remarks

- ▶ $(L, w) = (30, 3)$, Single system length 1200, Shannon threshold of 0.5
- ▶ A total of 10^5 were attempted to encode
- ▶ No failures for $\delta < 0.43$

Concluding Remarks

Conclusion

- ▶ Spatially-coupled compound codes achieve the capacity of 2-write systems
- ▶ **Coupling structure** is also crucial
 - to achieve optimum thresholds
 - for encoding to succeed

Multi-Write Systems

- ▶ Will BPGD work for multi-write systems?

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- ▶ SC-LDPC Lattices [C1]
- ▶ SC-Compound Codes
 - Side-Information Problems [C2]
 - Coding for WOM [C3]
- ▶ Sparse graph coding tools for solving sparse recovery problems
 - Regular bipartite sparse graphs for compressed sensing [C5]
 - Group testing*
 - Pattern matching*
- ▶ Uncoordinated multiple access
 - Universal schemes for massive uncoordinated multiple access [C4]
 - Optimal distributions for finite user multiple access*
- ▶ Coding for low latency requirements*

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*-To be submitted