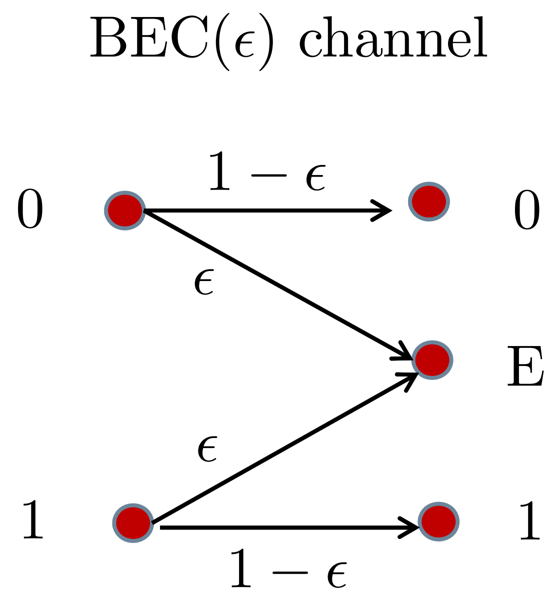
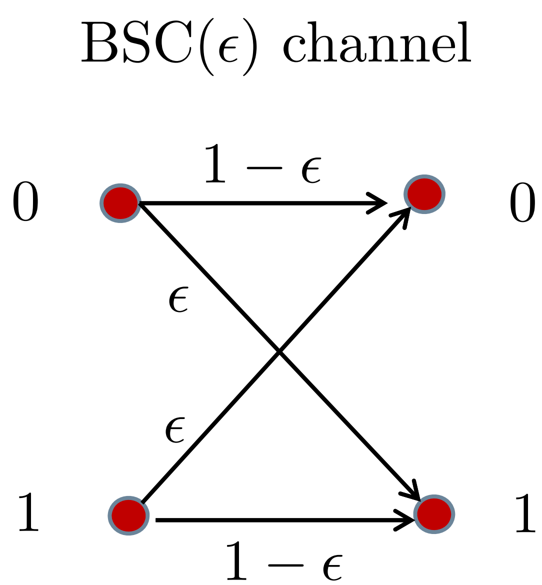
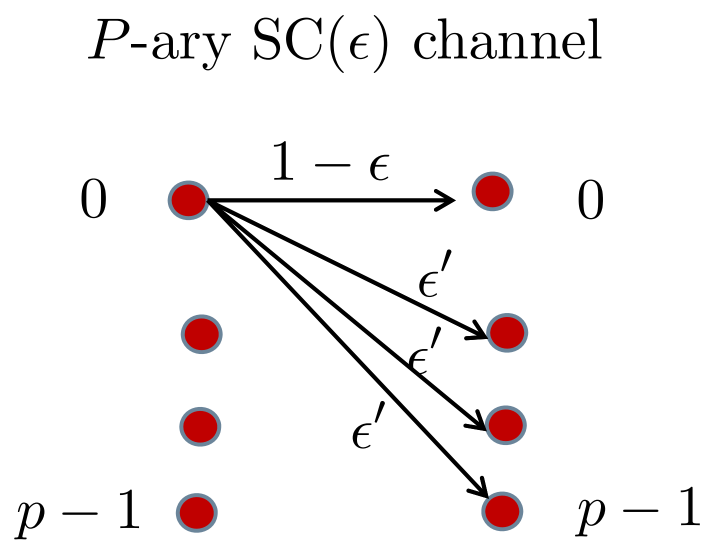
Transmission over a Binary Erasure Channel







$$\epsilon' = \frac{\epsilon}{p-1}$$

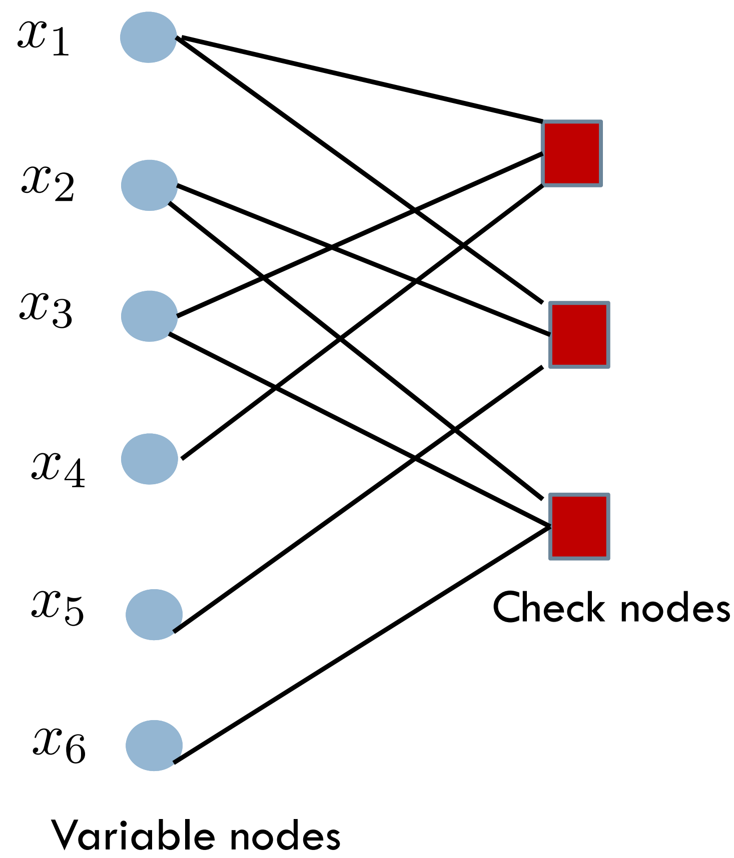
Tanner graph for the (6,3,3)

$$
\begin{array}{c}
\quad x_1, x_2, x_3, x_4, x_5, x_6 \\
H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

$$x_1 \oplus x_3 \oplus x_4 = y_1$$

$$x_1 \oplus x_2 \oplus x_5 = y_2$$

$$x_2 \oplus x_3 \oplus x_6 = y_3$$



- Parity check matrix and Tanner graph convey the same info
- A code can be specified by giving its Tanner graph

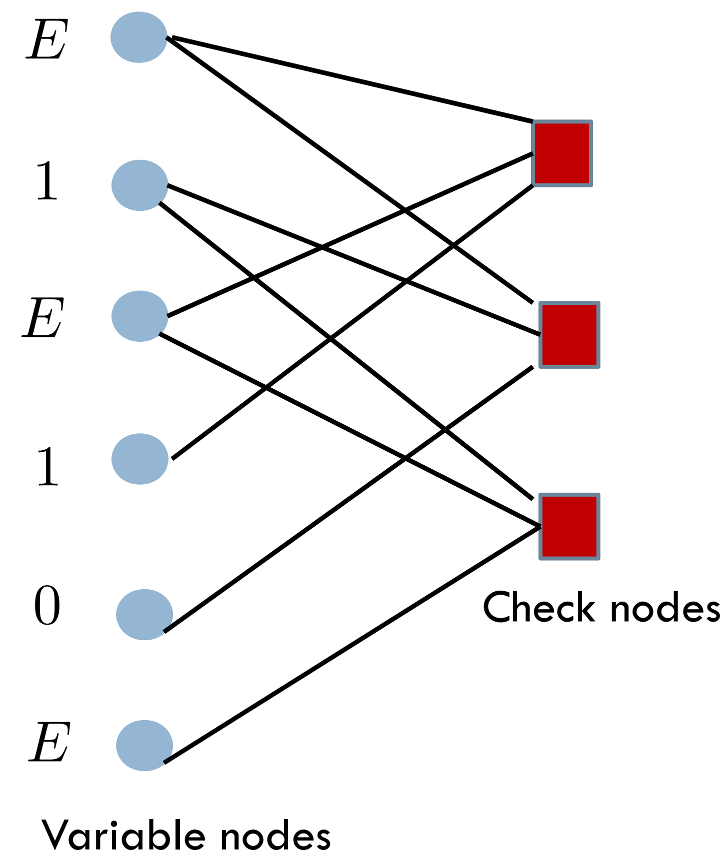
Decoding by peeling

$$H = \begin{array}{c} \begin{array}{cccccc} x_1, & x_2, & x_3, & x_4, & x_5, & x_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$x_1 \oplus x_3 \oplus x_4 = 0$$

$$x_1 \oplus x_2 \oplus x_5 = 0$$

$$x_2 \oplus x_3 \oplus x_6 = 0$$



- If exactly 1 bit is unknown in every check, it can be recovered

Decoding by peeling

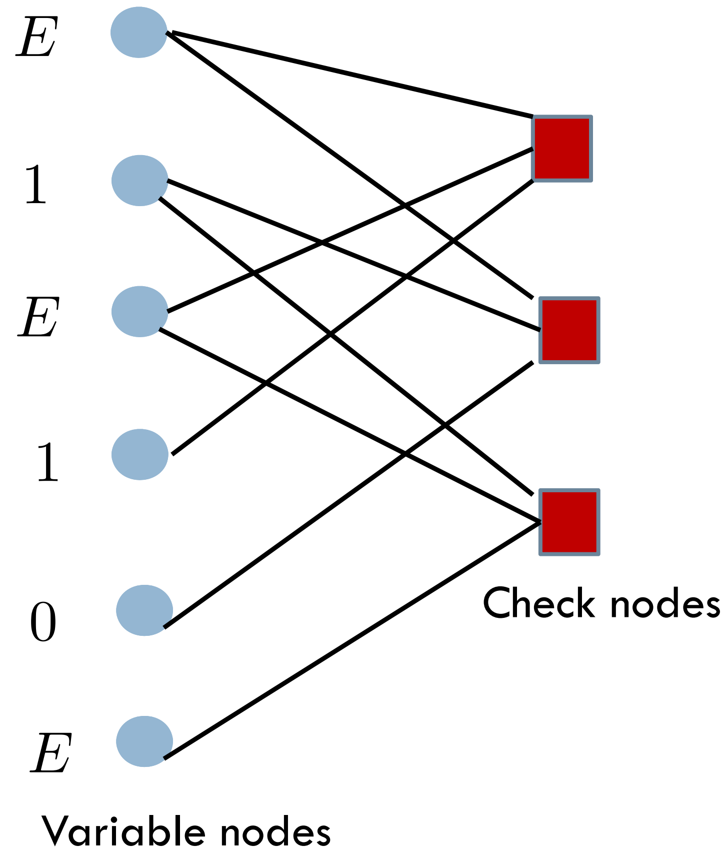
$$x_1, x_2, x_3, x_4, x_5, x_6$$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 \oplus x_3 \oplus x_4 = 0$$

$$x_1 \oplus x_2 \oplus x_5 = 0$$

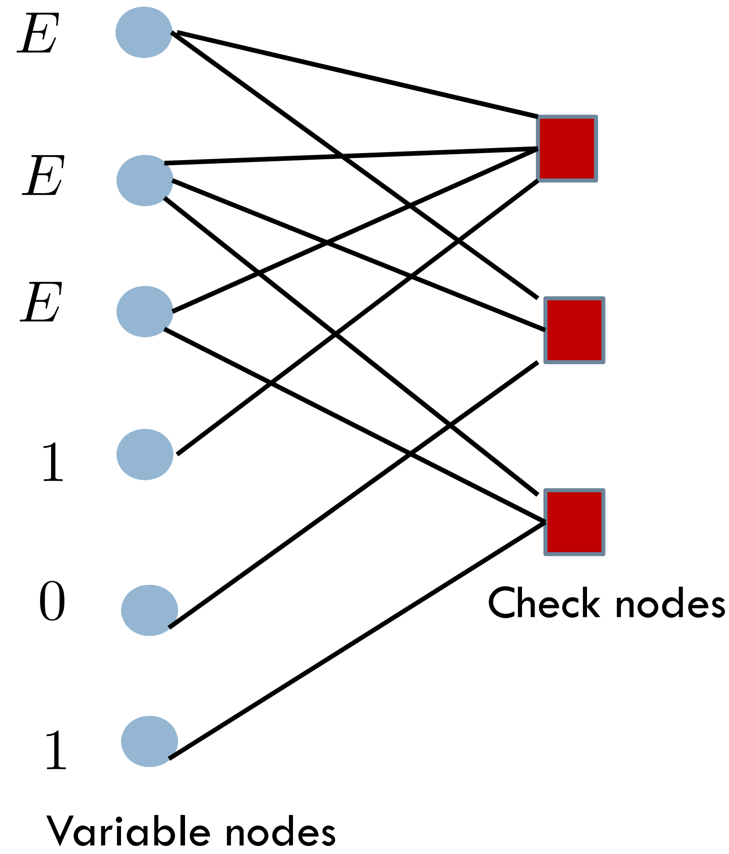
$$x_2 \oplus x_3 \oplus x_6 = 0$$



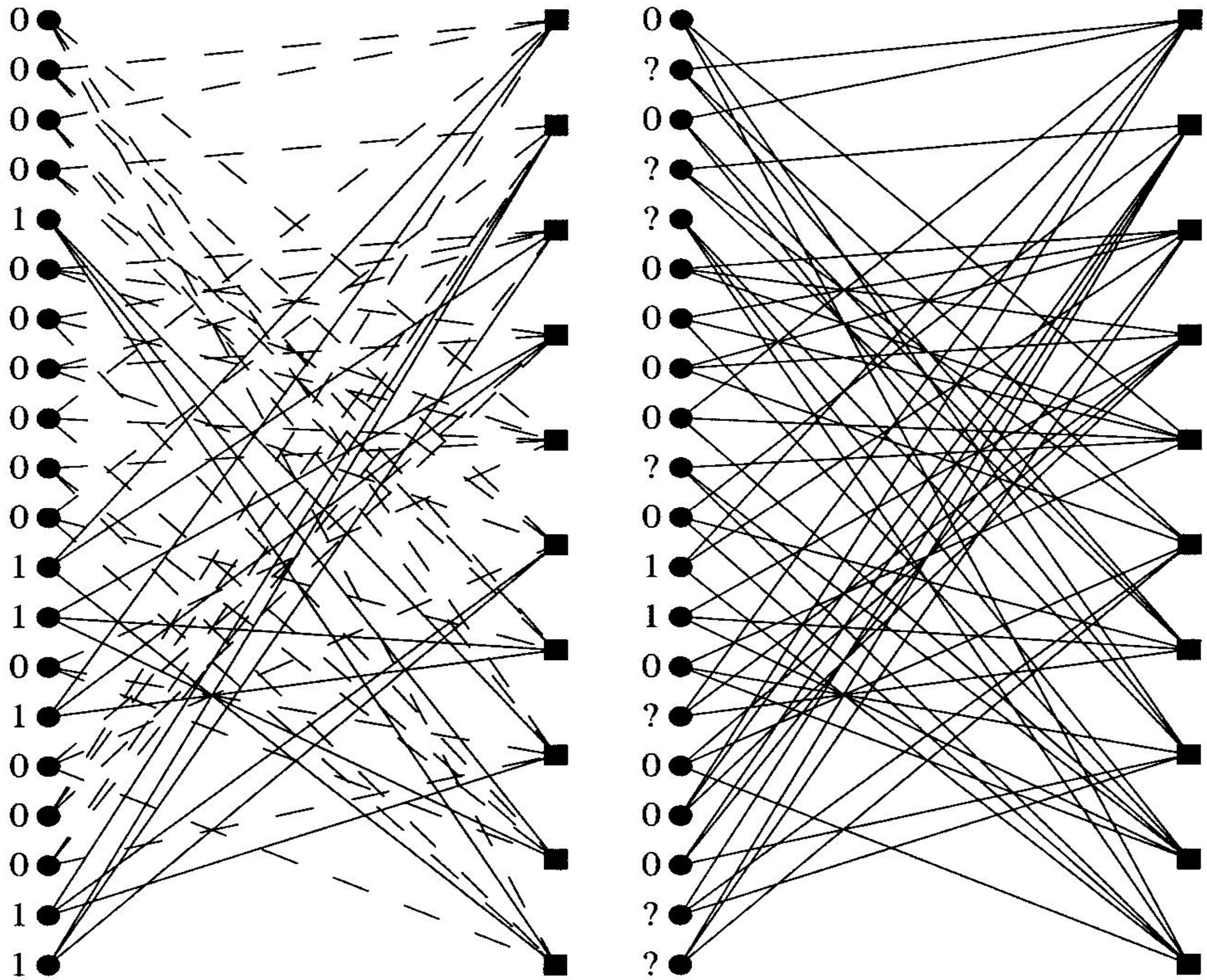
- If exactly 1 bit is unknown in every check, it can be recovered

Peeling decoder is suboptimal

$$
\begin{array}{cc}
 & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\
H = & \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$



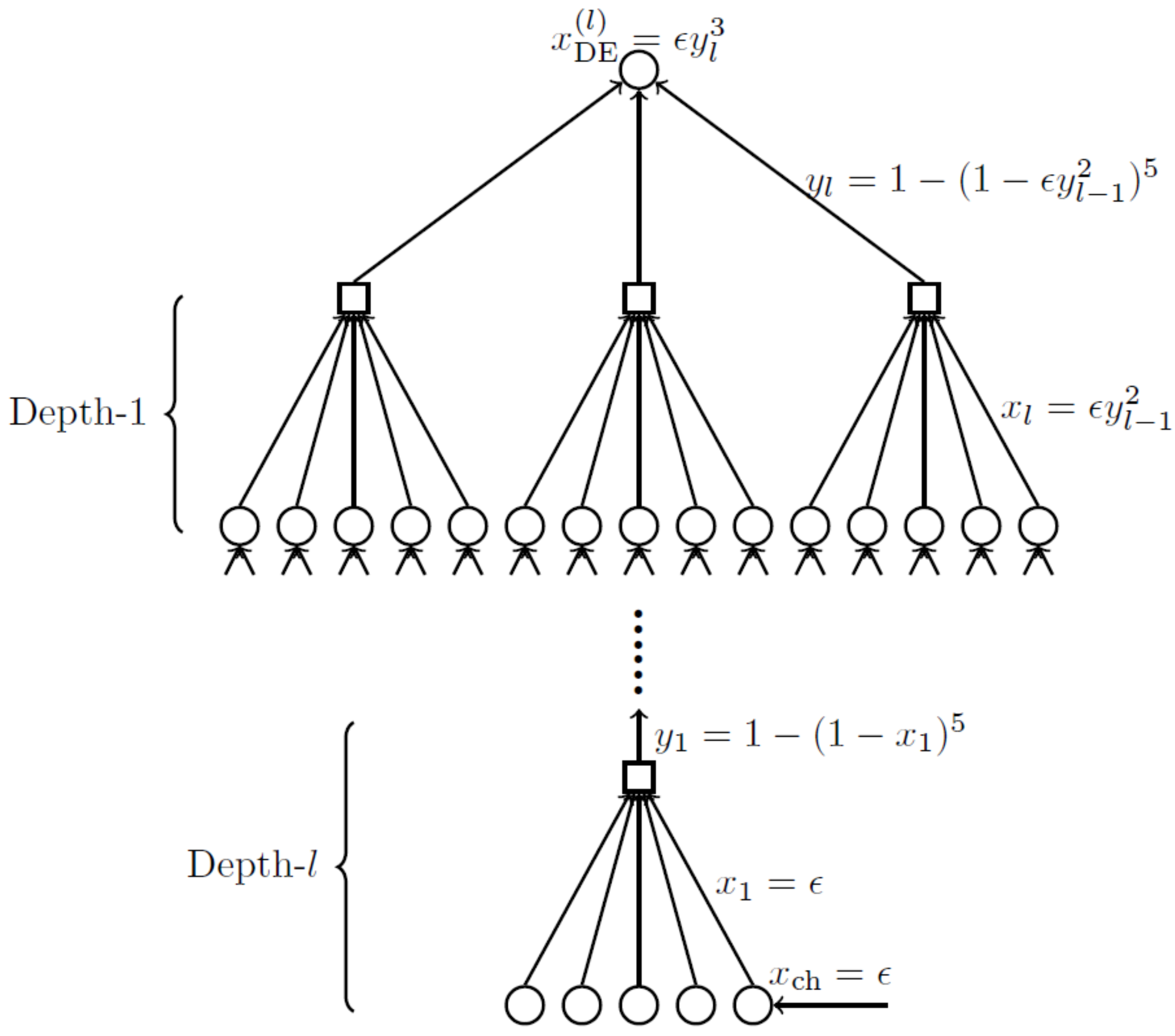
LDPC Code from a (3,6) regular ensemble



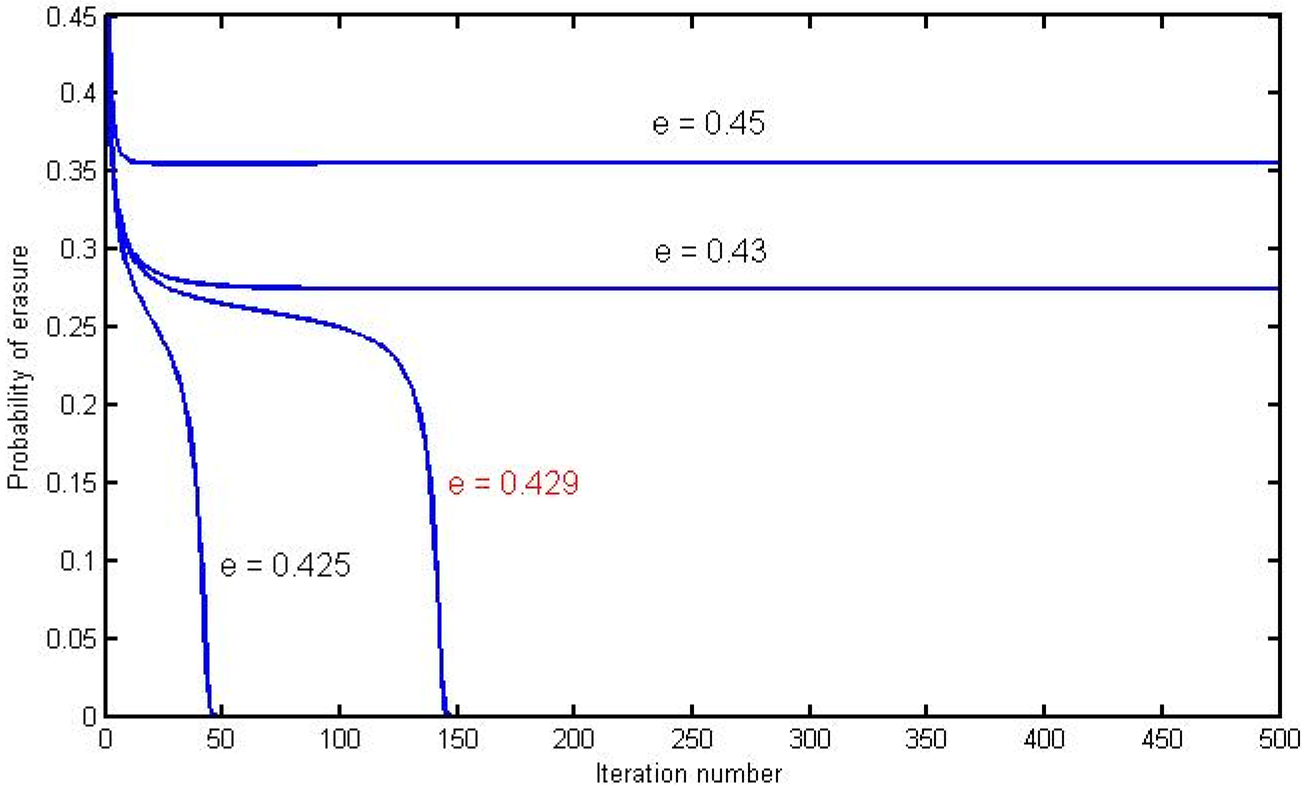
How good are LDPC codes ? Density Evolution

- Assume $n \longrightarrow \infty$
- So no short cycles, we can assume graph cycle free
- What is the highest fraction e for which we can correct all errors?
- Compute the probability that the message along a randomly chosen edge is an erasure and see how this changes as the iterations proceed

Analysis – computation tree, density evolution



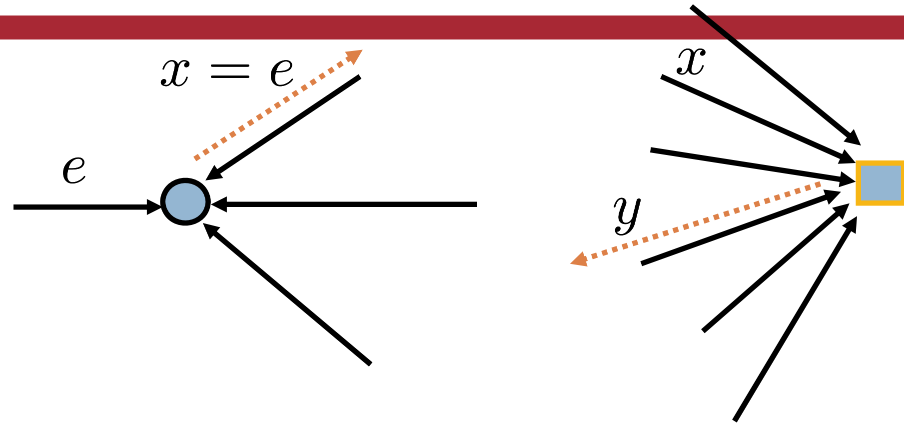
Threshold for the (3,6) regular LDPC code



- For (3,6) regular codes threshold is 0.429
- Although it is not 0.5, it is still quite good
- How can we close the gap? – Irregular LDPC Codes

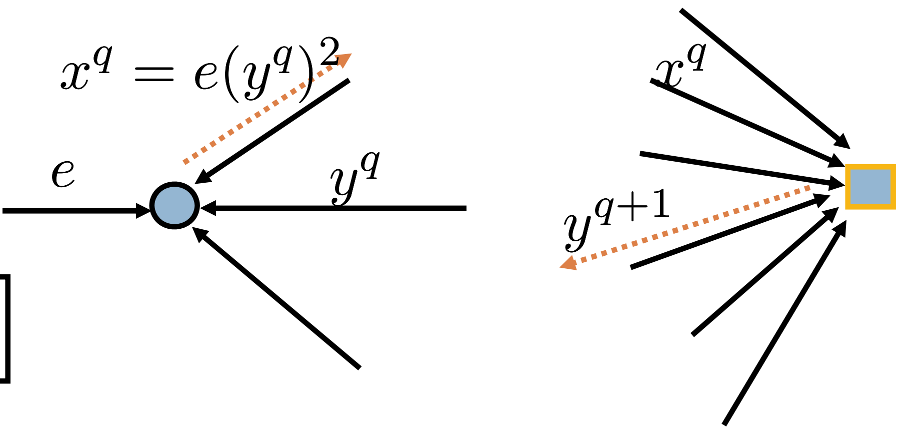
(3,6) LDPC Code

- First iteration: $x = e$



- $y = 1 - (1 - x)^5$

- During the q^{th} iteration
  - $x^q = e(y^q)^2$
  - $y^{q+1} = 1 - (1 - x^q)^5$
  - $\boxed{x^{q+1} = e(1 - (1 - x^q)^5)^2}$



- Threshold is the maximum value of e for which

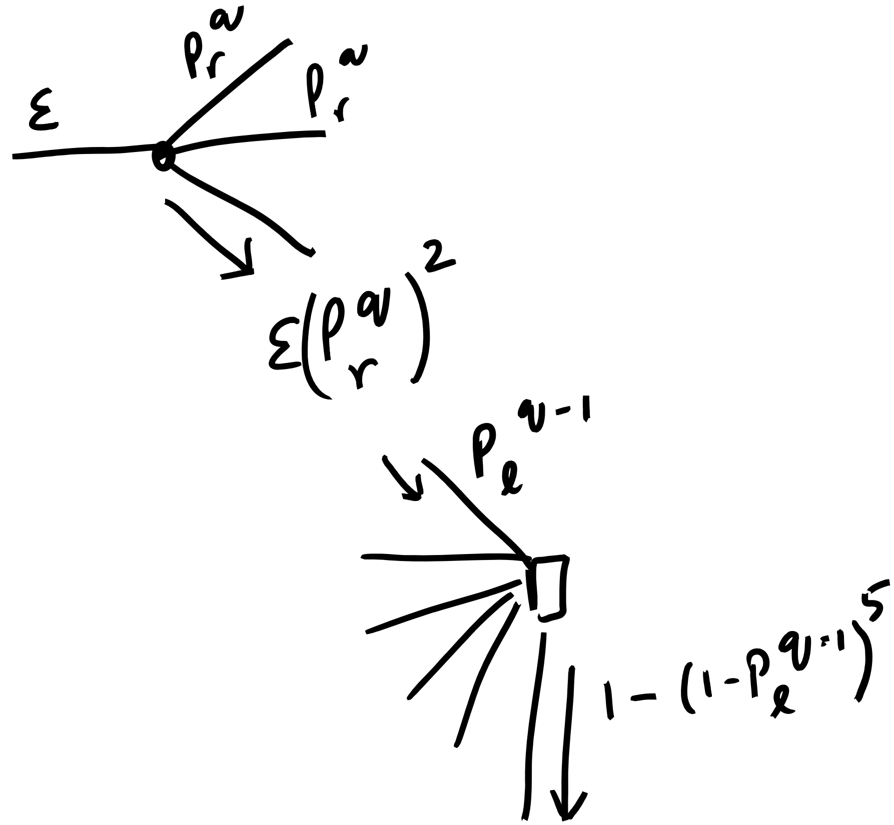
$$x^q \to 0 \text{ as } q \to \infty$$

(3,6) LDPC Code

- First iteration: $p_l = e$

- $p_r = 1-(1-p_l)^5$

- During the q^{th} iteration
  - $p_l^q = e\,(p_r^q)^2$
  - $p_r^q = 1-(1-p_l^{q-1})^5$
  - $\boxed{p_l^q = e\,(1-(1-p_l^{q-1})^5)^2}$



- <u>Threshold</u> is the maximum value of e for which

$$p_r^q \longrightarrow 0 \text{ as } q \longrightarrow \infty$$

2 erasures can be corrected

$$[x_1\ x_2\ x_3\ x_4\ x_5\ x_6]$$

$$\underline{x} = [u_1, u_2, u_3] \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= u_2 \\ x_3 &= u_3 \\ x_4 &= u_1 \oplus u_2 \\ x_5 &= u_2 \oplus u_3 \\ x_6 &= u_1 \oplus u_3 \end{aligned}$$

- Solve the set of simultaneous equations
- Mathematical tools are needed to design good matrices

Erasures to Non-binary Errors – Tensoring construction

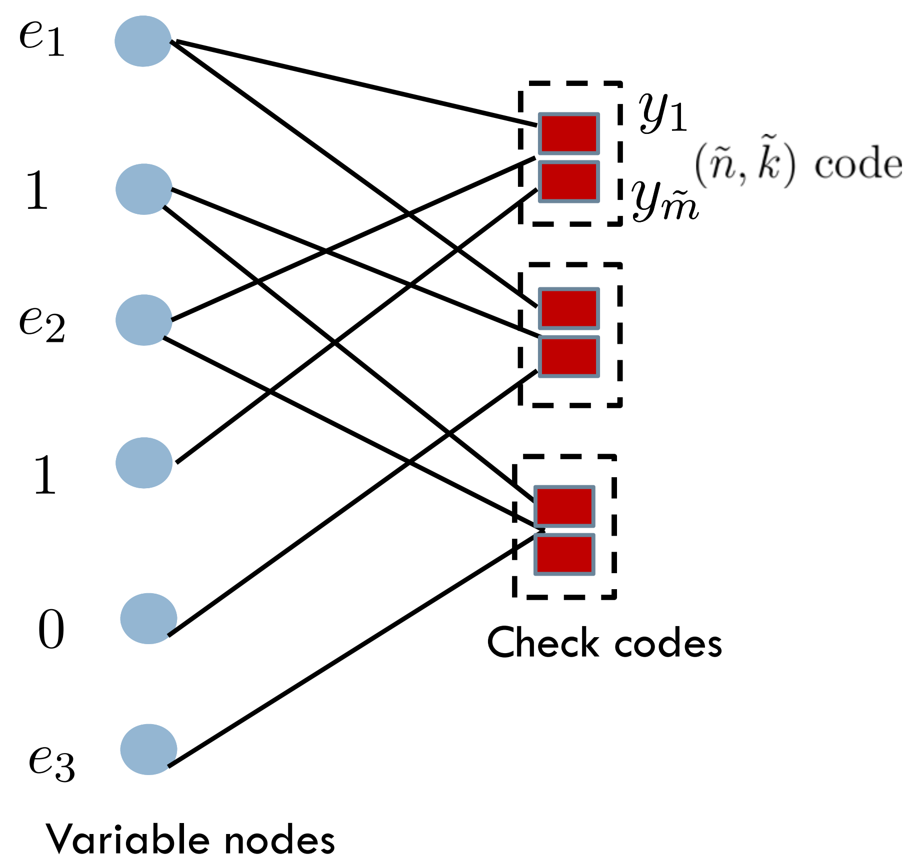
$$x_1, x_2, x_3, x_4, x_5, x_6$$

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\otimes$$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & W^2 & W^3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & W & 0 & 0 & W^4 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & W & W^2 & 0 & 0 & W^5 \end{bmatrix}$$



- If exactly 1 error in every check, it can be recovered

Syndrome source coding is the same as decoding