Spatially-Coupled Codes for Write-Once Memories

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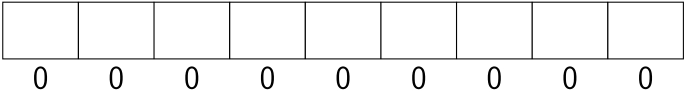
Allerton 2015

Write-Once Memories

Flash Memory

- In typical flash memory, changing from 0 to 1 is easy
- Resetting 1 to 0 requires rewriting whole block
- Write-once memories model such storage systems

Write-Once Memories



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Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed

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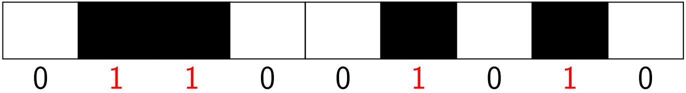

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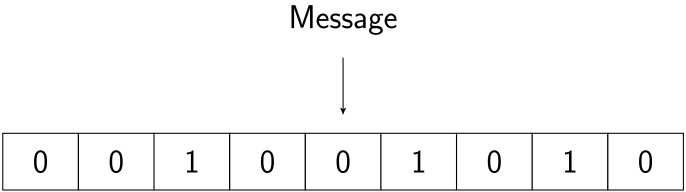
Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed
- $1 \longrightarrow 0$ is forbidden

Capacity Region (I) - Noiseless



Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- Only about $nt/\log(t)$ cells required to store n bits for t writes

Capacity Region (I) - Noiseless

Message

$\downarrow$

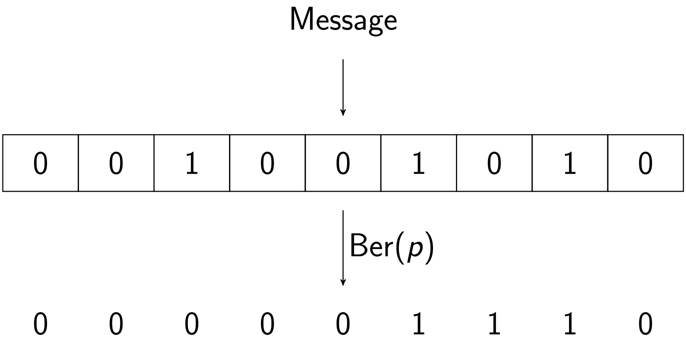
0	0	1	0	0	1	0	1	0

Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- Only about $nt/\log(t)$ cells required to store n bits for t writes
- In 1985, Heegard gave the capacity for t-write system
- For a 2-write system, it is

$$\left\{(R_1, R_2) \mid 0 \leq R_1 < h(\delta),\ 0 \leq R_2 < 1 - \delta\right\}$$

Capacity Region (II) - Read Errors



Write-Once Memory with Read Errors

- Different from write errors
- $Y = X \oplus \mathrm{Ber}(p)$, where $\mathrm{Ber}(p)$ denotes the Bernoulli noise
- Capacity region is unknown

Main Result

Objective

- Construct low-complexity coding schemes that achieve the capacity region of the WOM system
  - Low-complexity encoding and decoding

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  - For read errors, achieves

$$R_1 < h(\delta) - h(p), \qquad R_2 < 1 - \delta - h(p).$$

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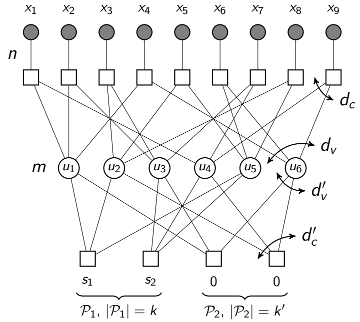
Idea

- Use compound LDGM/LDPC codes
- Encoding for second write is erasure quantization
- Use spatial coupling with message-passing

Compound LDGM/LDPC Codes



- Codebook $(n, m-k-k')$
- Message constraints

  $$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword $(x_1, \cdots, x_9)$:

  $$x_1 = u_1 \oplus u_4, \qquad x_2 = \cdots$$

- Parametrized by s^k: $\mathcal{C}(s^k)$

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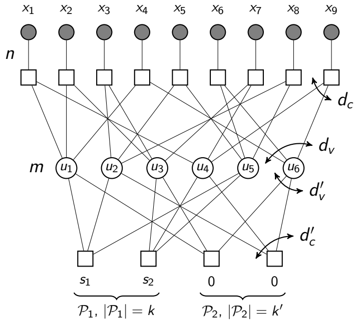

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Key Properties of Compound Codes

- a natural coset decomposition: $\mathcal{C} = \bigcup_{s^k \in \{0,1\}^k} \mathcal{C}(s^k)$
- achieves capacity over eras. chan. under MAP (when $m = n$)
- a good source code under optimal encoding
- a good channel code under optimal decoding

Good Code

"Good" source code

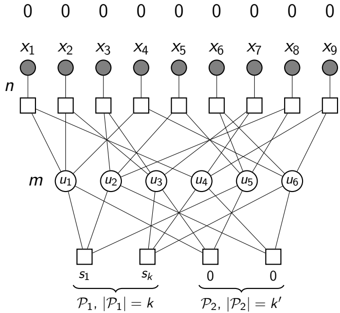
- Rate of the code is $R = 1 - h(\delta) + \varepsilon$
- When this code is used to optimally encode $\mathrm{Ber}(\frac{1}{2})$
- The average Hamming distortion is at most δ

"Good" channel code

- Rate of the code is $R = 1 - h(p) - \varepsilon$
- When this code is used for channel coding on $\mathrm{BSC}(p)$
- Message est. under optimal decoding with error at most ε

Coding Scheme for 2-write WOM: First Write

$$R_1 < h(\delta) - h(p)$$



$\frac{m-k-k'}{n} \approx 1 - h(\delta)$ $\qquad$ $\frac{m-k'}{n} \approx 1 - h(p)$

- With message s^k, encode 0^n to x^n (Distortion $\approx \delta$)
- Store x^n

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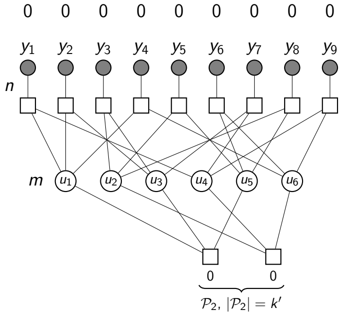

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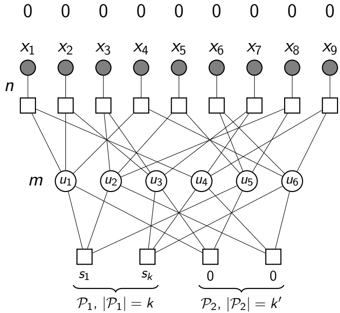

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- Dec. x^n and compute s^k
- $R_1 = \frac{k}{n} \approx h(\delta) - h(p)$

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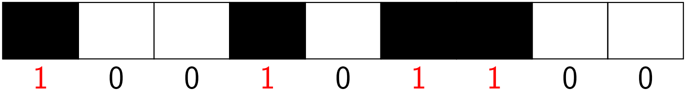

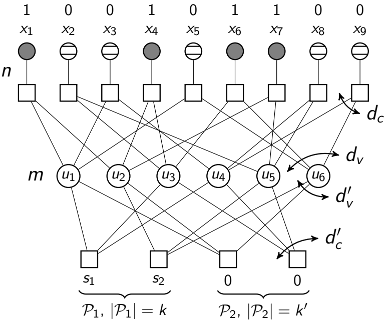

- Need to find a consistent codeword in $\mathcal{C}(s^k)$

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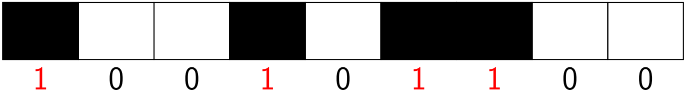
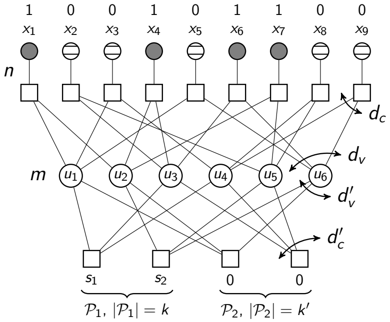




- Need to find a consistent codeword in $\mathcal{C}(s^k)$
- Closely related to Binary Erasure Quantization (BEQ)
- En Gad, Huang, Li and Bruck (ISIT 2015)

Binary Erasure Quantization

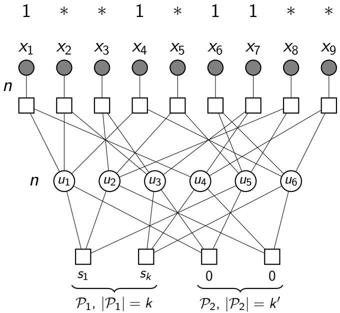
- Quantize a sequence in $\{0, 1, *\}^n$ to $x^n \in \mathcal{C} \subset \{0, 1\}^n$
  - 0's and 1's should match exactly
  - $*$'s can take either 0 or 1
- Can map the second write of 2-write WOM to BEQ
  - Map 0's to $*$'s and keep 1's
  - Quantize to codeword in $\mathcal{C}(s^k)$
- BEQ is the dual of decoding on binary erasure channel
  - Martinian and Yedidia (Allerton 2003)
  - Can quan. all seq. with erasure pattern $e^n \in \{0, 1\}^n$ to $\mathcal{C}$

    $$\Updownarrow$$

    Chan. dec. for $\mathcal{C}^{\perp}$ can correct all vectors with eras. $1^n \oplus e^n$
- Choose a good (dual) code $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



$\frac{n-k-k'}{n} \approx \delta$ $\frac{n-k'}{n} \approx 1 - h(p)$

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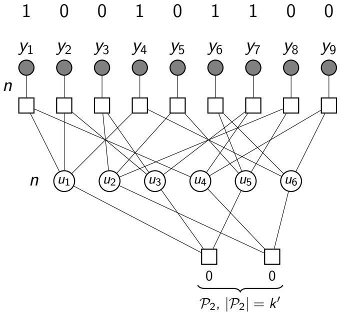

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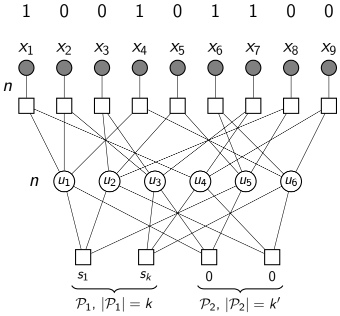

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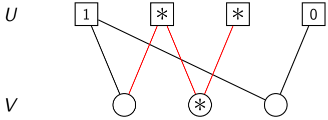
- $R_2 = \frac{k}{n} \approx 1 - \delta - h(p)$

Iterative Erasure Quantization Algorithm



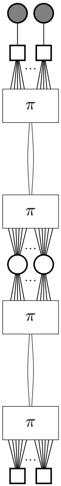
- Peeling type encoder

while $\exists$ non-erasures in V **do**
  if $\exists$ non-erased $u \in U$ such that only one of its neighbors $v \in V$ is not erased **then**
    Pair (u, v).
    Erase u and v.
  else
    FAIL.
    break.
  end if
end while

Remarks

- Need codes that are simultaneously good for channel/source coding and erasure quantization
- Use message-passing algorithms instead of optimal
- Use spatial-coupling for goodness of codes under message-passing

Spatially-Coupled Compound LDGM/LDPC Codes



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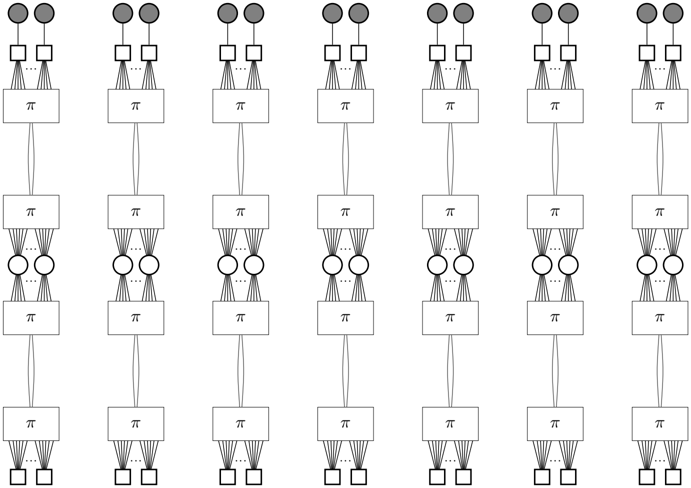

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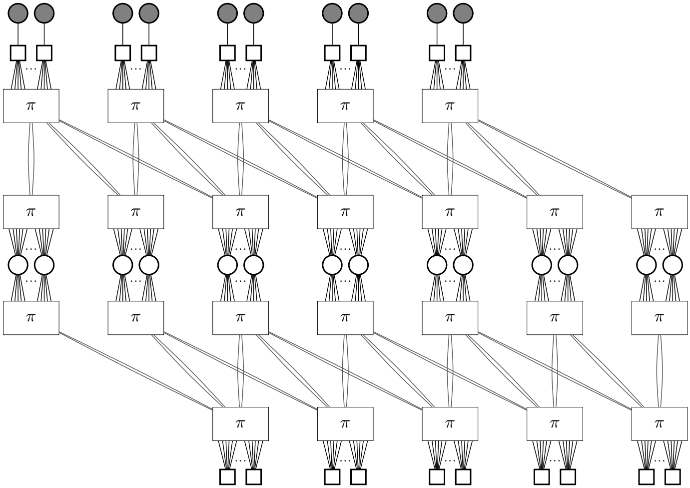

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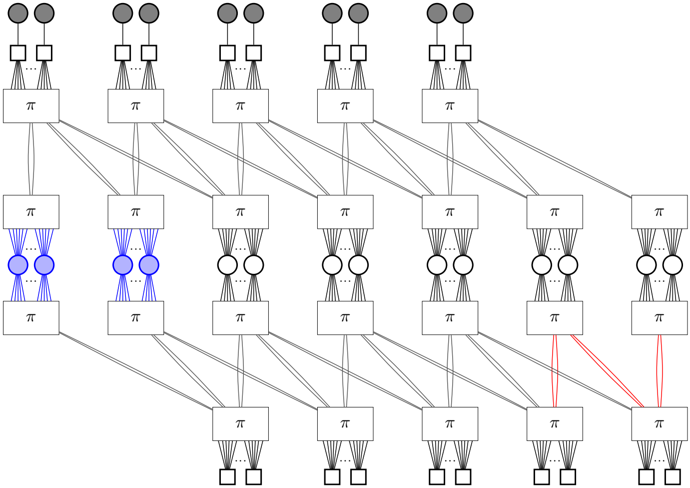
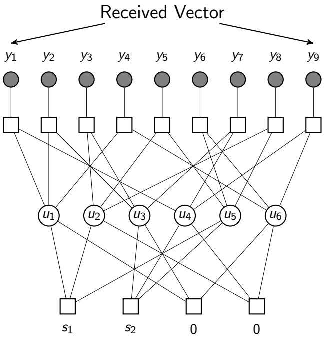
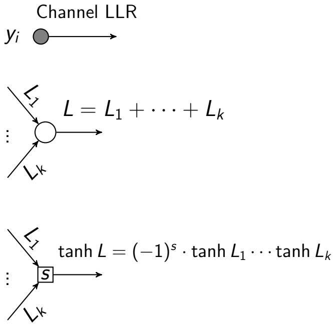


Decoding in Spatially-Coupled Compound Codes





Remarks

- Standard message-passing algorithm
- Threshold saturation proven for SC compound codes on BEC
- Empirically observed for BMS channels

Numerical Results: Noiseless WOM

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	δ^*	δ $w = 2$	δ $w = 3$	δ $w = 4$
$(3, 3, 3, 6)$	0.500	0.477	0.492	0.494
$(3, 3, 4, 6)$	0.333	0.294	0.324	0.326
$(3, 3, 5, 6)$	0.167	0.095	0.156	0.158
$(4, 4, 3, 6)$	0.500	0.461	0.491	0.492
$(4, 4, 4, 6)$	0.333	0.278	0.323	0.325
$(4, 4, 5, 6)$	0.167	0.086	0.155	0.159
$(5, 5, 3, 6)$	0.500	0.436	0.488	0.491
$(5, 5, 4, 6)$	0.333	0.260	0.320	0.324
$(5, 5, 5, 6)$	0.167	0.079	0.154	0.159

Remarks

- δ^* is the Shannon threshold
- $L = 30$, Single system length ≈ 24000

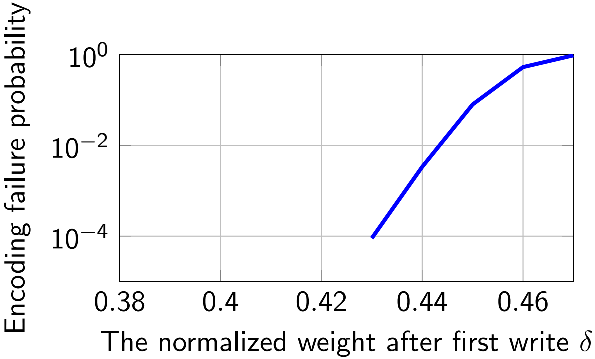
Numerical Results: WOM with Read Errors

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	w	(δ^*, p^*)	(δ, p)
$(3, 3, 4, 6)$	3	$(0.333, 0.0615)$	$(0.321, 0.0585)$
$(3, 3, 4, 8)$	3	$(0.500, 0.0417)$	$(0.490, 0.0387)$
$(3, 3, 6, 8)$	4	$(0.250, 0.0724)$	$(0.239, 0.0684)$
$(4, 4, 4, 6)$	4	$(0.333, 0.0615)$	$(0.324, 0.0585)$
$(4, 4, 4, 8)$	4	$(0.500, 0.0417)$	$(0.492, 0.0387)$
$(4, 4, 6, 8)$	4	$(0.250, 0.0724)$	$(0.241, 0.0694)$

Remarks

- δ^* and p^* are the Shannon thresholds
- $L = 30$, Single system length ≈ 30000

Numerical Results: Small Blocklength



Remarks

- $(L, w) = (30, 3)$, Single system length 1200, Shannon threshold of 0.5
- A total of 10^5 were attempted to encode
- No failures for $\delta < 0.43$

Concluding Remarks

Conclusion

- Spatially-coupled compound codes achieve the capacity of 2-write systems
- Coupling structure is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed

Multi-Write Systems

- Will BPGD work for multi-write systems?