# Multilevel Lattices based on Spatially Coupled LDPC codes with Applications

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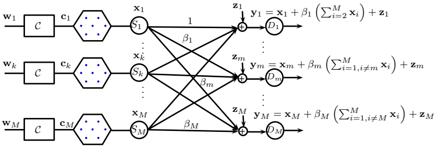
# Lattices and Lattice Codes

- Efficient structures for packing, covering, channel coding & quantization
- Single user Gaussian channel - Erez and Zamir
- Coding with side information - Wyner-Ziv and Costa, Zamir, Erez and Shamai
- Secrecy - He and Yener
- Dirty multiple access channel - Philosof, Khisti, Erez and Zamir

"Lattices are everywhere" by Ram Zamir

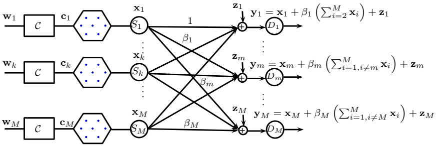
New perspectives for dealing with interference:

- Interference alignment - Sridharan, Jafarian, Vishwanath and Jafar



New perspectives for dealing with interference:

- Interference alignment - Sridharan, Jafarian, Vishwanath and Jafar
- Compute-and-forward - Nazer & Gastpar
- Physical layer network coding - Wilson et al, Nam et al



# Lattices and Lattice Codes

- Above schemes are all based on good lattice codes.
- *Poltyrev*-good lattices are at the core of such lattice coding schemes

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## Motivating questions

- These results are all based on Construction-A.
- Is this construction fundamental to good lattices?
- Can we work with just binary codes under practical decoding schemes?

# Main Results in this Talk

Codes over $\mathbb{F}_2$ and BP decoding suffice

- Recall Forney et al's result - based on nested random binary linear codes
- Propose capacity-achieving nested SC LDPC ensemble
- Construct lattices using Construction-D, based on the above ensemble
- Show existence of sequence of lattices that are *Poltyrev*-good under BP

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## Applications

- As an application, propose Symmetric Interference Channel
- Can be applied to other problems which adopt Construction A lattices

# Construction D with $L$ levels

- Barnes and Sloane '83, Forney, Chung and Trott '00, Yan, Ling, Wu ' 13
- Choose $G_1 \subseteq \ldots \subseteq G_L$ where $G_l$ is a gen matrix of code $\mathcal{C}_l$ over $\mathbb{F}_2$.

$$\begin{array}{c} \\ \\ \\ \\ \\ \\ \mathrm{G}_1 \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \end{array} \begin{bmatrix} \cdots & g_L & \cdots \\ \cdots & \vdots & \cdots \\ \hline \cdots & g_{k_2} & \cdots \\ \cdots & \vdots & \cdots \\ \hline \cdots & g_{k_1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_2 & \cdots \\ \cdots & g_1 & \cdots \end{bmatrix} \begin{array}{c} \\ \\ \left. \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \right\} \mathrm{G}_2 \end{array} \cdots \left. \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \mathrm{G}_L$$

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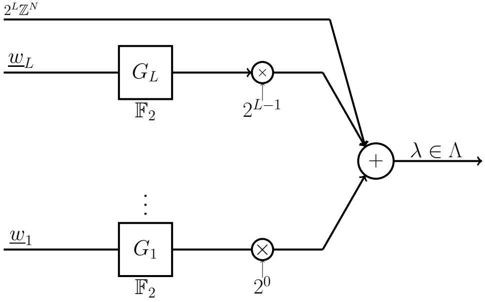
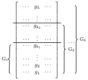
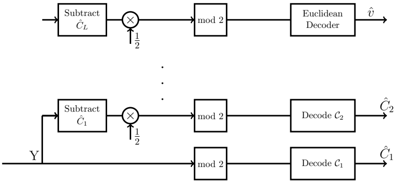
- $\underline{\lambda} = \underline{w}_1\mathbf{G}_1 + 2\underline{w}_2\mathbf{G}_2 \ldots + 2^{L-1}\underline{w}_{L-1}\mathbf{G}_{L-1} + 2^L\mathbb{Z}^N \in \Lambda$



# Multi-Level Decoding(Successive Decoding)

- $\underline{y} = \boxed{\underline{w}_1\mathbf{G}_1 + 2\underline{w}_2\mathbf{G}_2 \ldots + 2^{L-1}\underline{w}_{L-1}\mathbf{G}_{L-1} + 2^L\mathbb{Z}^N} + \underline{n}$
- $\underline{y}$ mod 2 $= [\underline{w}_1\mathbf{G}_1 + \underline{n}]$ mod 2 $= \underline{w}_1 \odot \mathbf{G}_1 + \boxed{\underline{n} \mod 2}$
- Decode $\underline{w}_1$, reconstruct $\underline{w}_1\mathbf{G}_1$ and subtract from $\underline{y}$



**Theorem 1 (Forney, Trott & Chung)**

*There exists a sequence of Construction D lattices based on $\mathcal{C}_1 \subseteq \mathcal{C}_2 \ldots \subseteq \mathcal{C}_L$ such that the VNR $\to 1$ and the $Pr(\lambda, \sigma^2) \to 0$.*

- Take $L$ large enough.
- It's sufficient that $\mathcal{C}_i$ at each level is capacity achieving for the mod-2 AWGN channel.

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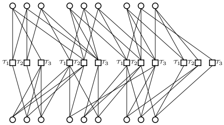
Objective:

- Capacity achieving nested code constructions, preferably under BP decoding.

# Proposed Nested Spatially-Coupled LDPC Ensemble

1. Begin with a $(d_v^1, d_c)$ SC LDPC code. For ex, $(d_v^1 = 3, d_c = 6, L = 3, w = 2)$.
2. Group check nodes into type $\mathcal{T}_k$, $k \in \{1, \ldots, d_v^1\}$



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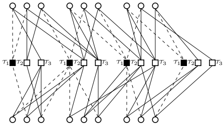

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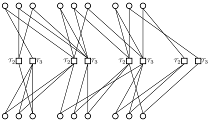
4. Results in a super-code that is a $(d_v^2, d_c)$ SC LDPC code.



# Lattice Design based on the proposed Nested SC LDPC ensemble

1. For a given $\sigma$, compute the capacity of the mod-2 AWGN channel at each level:

$$\underline{y_i} = \underline{w}_i \mathbf{G}_i + \frac{1}{2^{i-1}}\underline{n} \mod 2 = \underline{w}_i \odot \mathbf{G}_i + \boxed{\frac{1}{2^{i-1}}\underline{n} \mod 2}$$

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**Lemma 2**

*Given nested binary linear codes $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots \subseteq \mathcal{C}_r$ there exists nested generator matrices for these codes.*

# Proposed Ensemble is Capacity achieving

Theorem 3

*Each code ensemble in the proposed nested Spatially-Coupled LDPC ensemble is capacity achieving.*

Proof.

- Show that the mod 2 AWGN channel is BMS.
- Each derived protograph has the same spatially coupled structure.
- The proof follows from Kudekar & Urbanke, Kumar & Pfister's results.

□

# Proposed Lattices are Poltyrev-Good

**Theorem 4**

*There exists a sequence of SC LDPC lattices with VNR*$(\Lambda, \sigma^2) \to 1$ *for which, under multistage BP decoding,* $\mathbb{E}\left[P(\lambda, \sigma^2)\right] \to 0$ *as* $w, L, M \to \infty$.

**Proof.**

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**Proof.**

- The proposed nested ensemble achieve capacity.
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- Binary codes and more importantly practical BP decoding suffices.
- Practically we observe that two levels of coding gets you lattices very close to Poltyrev limit.

# Design Example of Poltyrev-Good Lattice

A target block error probability of $10^{-4}$ in the uncoded level gives $\sigma_L = 0.08$

1. Capacities for the mod 2 AWGN channel for respective levels:

| | Level L-1 | Level L-2 | Level L-3 |
|---|---|---|---|
| $\sigma_{\text{eff}}$ | 0.16 | 0.32 | 0.64 |
| Cap | 0.99 | 0.57 | 0.02 |
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2. Fix L=3 and use $(3,30)$, $(14,30)$ nested SC LDPC codes.

| $(d_c, d_v^1, d_v^2)$ | (L,w) | $P(\mathbb{Z}_4, \sigma^2)$ | $\sigma_{\text{max}}$ | VNR | $\text{VNR}_{\text{rate-loss}}$ |
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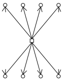
| (60, 26, 3) | (72, 12) | $5 \times 10^{-10}$ | 0.3200 | 0.482dB | 0.927dB |
| (60, 27, 3) | (64, 9) | $5 \times 10^{-10}$ | 0.3203 | 0.57dB | 0.951dB |

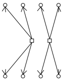
# Alternate Nested SC LDPC ensemble

- Derive a lower rate code by "splitting the checks"
- Consider a $(3, 8)$ code

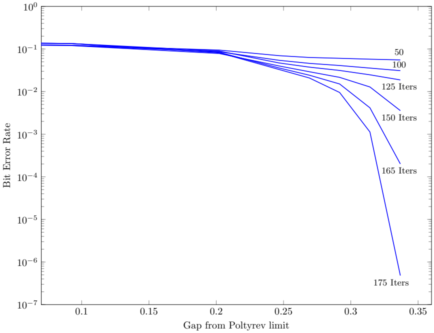


# Alternate Nested SC LDPC ensemble

- Derive a lower rate code by "splitting the checks"
- Consider a $(3, 8)$ code
- Split each check into "two" checks to derive a $(3, 4)$ sub-code
- Easy to prove that resulting code is from the $(3, 4)$ SC LDPC ensemble



# Simulation Results



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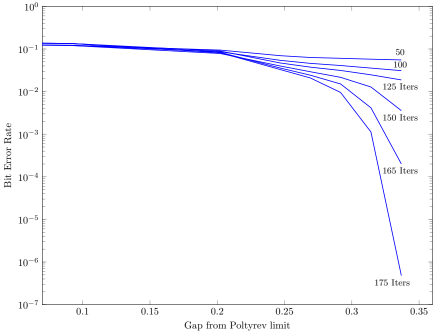
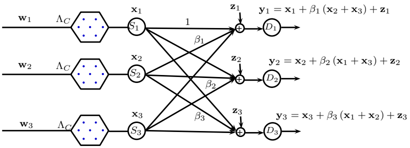


# 3-User Symmetric Interference Channel



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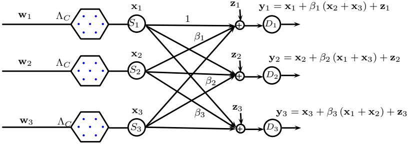


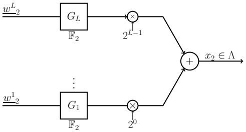
# Symmetric Interference Channel - Decoding Sums

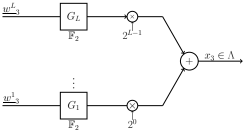
Interference at Destination 1:

$$
\begin{aligned}
\mathbf{x}_2 + \mathbf{x}_3 &= (\underline{w}_2^1 + \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{w}_2^2 + \underline{w}_3^2)\mathbf{G}_2 + 4\mathbf{k}_{23} \\
&= (\underline{w}_2^1 \oplus \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{c}_{23}^1 \oplus \underline{w}_2^2 \oplus \underline{w}_3^2)\mathbf{G}_2 + 4(\underline{c}_{23}^2 + \mathbf{k}_{23})\mathbf{Z}
\end{aligned}
$$

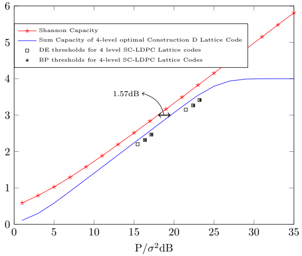
where the carry overs are

$$
\begin{aligned}
\underline{c}_{23}^1 &= 0.5\left(\underline{w}_1^1 + \underline{w}_1^2 - \underline{w}_1^1 \oplus \underline{w}_1^2\right), \\
\underline{c}_{23}^2 &= 0.5\left(\underline{c}_{23}^1 + \underline{w}_2^1 + \underline{w}_2^2 - \underline{c}_{23}^1 \oplus \underline{w}_2^1 \oplus \underline{w}_2^2\right)
\end{aligned}
$$



# Achievable Information Rates



# Concluding Remarks

- Multilevel constructions - efficient ways to decode integer combinations
- Need capacity achieving nested codes
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- Need capacity achieving nested codes
- Multilevel construction is provably good under message passing decoding
- Coding schemes based on Binary LDPC codes and iterative decoding suffice