Applications of Spatially Coupled-LDPC codes & Sparse Graph Codes for Sparse Recovery

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Outline

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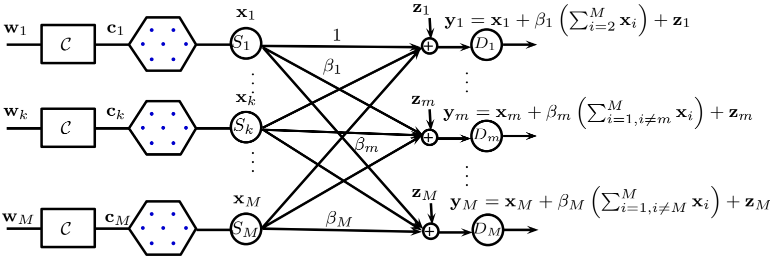
Lattices and Lattice Codes

- Efficient structures for packing, covering, channel coding & quantization
- Single user Gaussian channel - Erez and Zamir
- Coding with side information - Wyner-Ziv and Costa, Zamir, Erez and Shamai
- Secrecy - He and Yener
- Dirty multiple access channel - Philosof, Khisti, Erez and Zamir

"Lattices are everywhere" by Ram Zamir

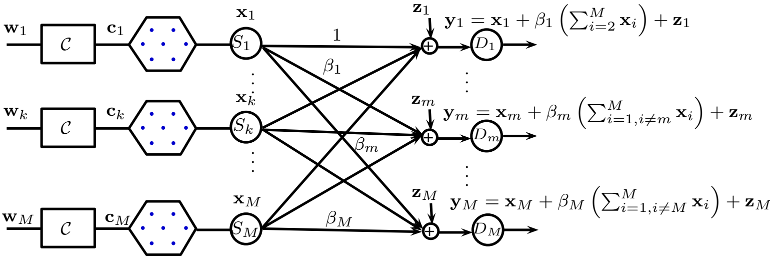
New perspectives for dealing with interference:

- Interference alignment - Sridharan, Jafarian, Vishwanath and Jafar



New perspectives for dealing with interference:

- Interference alignment - Sridharan, Jafarian, Vishwanath and Jafar
- Compute-and-forward - Nazer & Gastpar
- Physical layer network coding - Wilson et al, Nam et al



Lattices and Lattice Codes

- Above schemes are all based on good lattice codes.
- *Poltyrev*-good lattices are at the core of such lattice coding schemes

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Motivating questions

- These results are all based on Construction-A.
- Is this construction fundamental to good lattices?
- Can we work with just binary codes under practical decoding schemes?

Main Results in this Talk

Codes over $\mathbb{F}_2$ and BP decoding suffice

- Recall Forney et al's result - based on nested random binary linear codes
- Propose capacity-achieving nested SC LDPC ensemble
- Construct lattices using Construction-D, based on the above ensemble
- Show existence of sequence of lattices that are *Poltyrev*-good under BP

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Codes over $\mathbb{F}_2$ and BP decoding suffice

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Applications

- As an application, propose Symmetric Interference Channel
- Can be applied to other problems which adopt Construction A lattices

Construction D with L levels

- Barnes and Sloane '83, Forney, Chung and Trott '00, Yan, Ling, Wu ' 13
- Choose $G_1 \subseteq \ldots \subseteq G_L$ where G_l is a gen matrix of code $\mathcal{C}_l$ over $\mathbb{F}_2$.

$$\left[\begin{array}{ccc} \cdots & g_L & \cdots \\ \cdots & \vdots & \cdots \\ \hline \cdots & g_{k_2} & \cdots \\ \cdots & \vdots & \cdots \\ \hline \cdots & g_{k_1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & g_2 & \cdots \\ \cdots & g_1 & \cdots \end{array}\right]$$

G_1 $\quad \ldots \quad$ G_2 $\quad$ G_L

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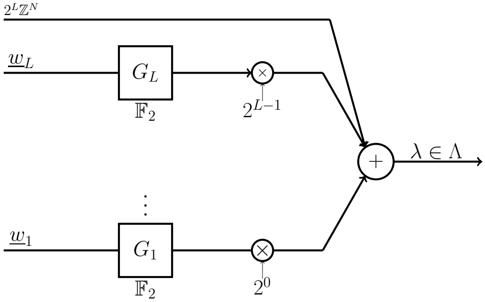
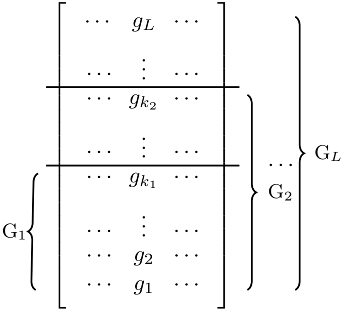
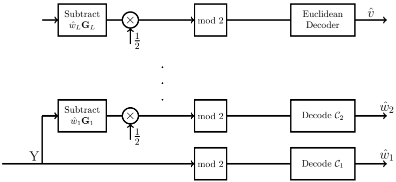
- $\underline{\lambda} = \underline{w}_1\mathbf{G}_1 + 2\underline{w}_2\mathbf{G}_2 \ldots + 2^{L-1}\underline{w}_{L-1}\mathbf{G}_{L-1} + 2^L\mathbb{Z}^N \in \Lambda$





Multi-Level Decoding(Successive Decoding)

- $\underline{y} = \boxed{\underline{w}_1\mathbf{G}_1 + 2\underline{w}_2\mathbf{G}_2 \ldots + 2^{L-1}\underline{w}_{L-1}\mathbf{G}_{L-1} + 2^L\mathbb{Z}^N} + \underline{n}$
- $\underline{y} \bmod 2 = [\underline{w}_1\mathbf{G}_1 + \underline{n}] \bmod 2 = \underline{w}_1 \odot \mathbf{G}_1 + \boxed{\underline{n} \bmod 2}$
- Decode $\underline{w}_1$, reconstruct $\underline{w}_1\mathbf{G}_1$ and subtract from $\underline{y}$



Theorem (Forney, Trott & Chung)

There exists a sequence of Construction D lattices based on $\mathcal{C}_1 \subseteq \mathcal{C}_2 \ldots \subseteq \mathcal{C}_L$ *such that the VNR* $\to 1$ *and the* $Pr(\lambda, \sigma^2) \to 0$.

- Take L large enough.
- It's sufficient that $\mathcal{C}_i$ at each level is capacity achieving for the mod-2 AWGN channel.

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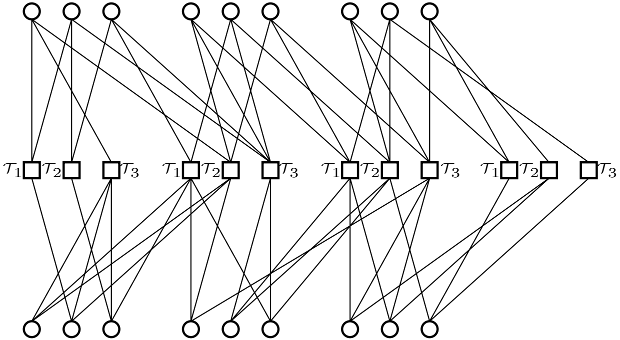
Objective:

- Capacity achieving nested code constructions, preferably under BP decoding.

Proposed Nested Spatially-Coupled LDPC Ensemble

1. Begin with a (d_v^1, d_c) SC LDPC code. For ex, $(d_v^1 = 3, d_c = 6, L = 3, w = 2)$.
2. Group check nodes into type $\mathcal{T}_k$, $k \in \{1, \ldots, d_v^1\}$



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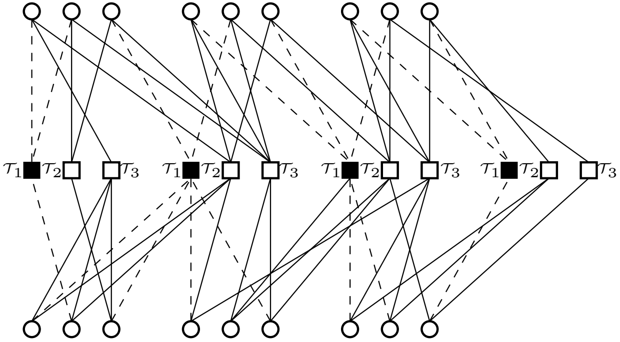

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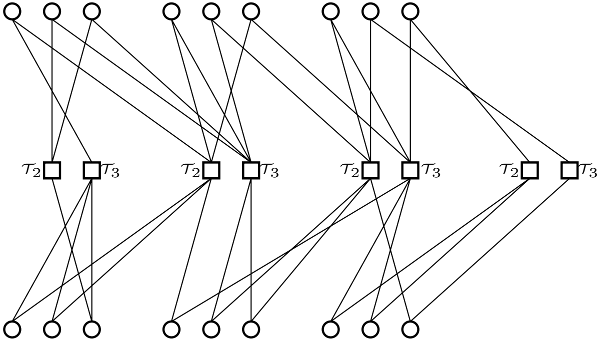
4. Results in a super-code that is a (d_v^2, d_c) SC LDPC code.



Lattice Design based on the proposed Nested SC LDPC ensemble

1. For a given σ, compute the capacity of the mod-2 AWGN channel at each level:

$$\underline{y}_i = \underline{w}_i \mathbf{G}_i + \frac{1}{2^{i-1}}\underline{n} \mod 2 = \underline{w}_i \odot \mathbf{G}_i + \boxed{\frac{1}{2^{i-1}}\underline{n} \mod 2}$$

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Lemma

Given nested binary linear codes $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots \subseteq \mathcal{C}_r$ there exists nested generator matrices for these codes.

Proposed Ensemble is Capacity achieving

Theorem

Each code ensemble in the proposed nested Spatially-Coupled LDPC ensemble is capacity achieving.

Proof.

- Show that the mod 2 AWGN channel is BMS.
- Each derived protograph has the same spatially coupled structure.
- The proof follows from Kudekar & Urbanke, Kumar & Pfister's results.

□

Proposed Lattices are Poltyrev-Good

Theorem

There exists a sequence of SC LDPC lattices with $VNR(\Lambda, \sigma^2) \to 1$ *for which, under multistage BP decoding,* $\mathbb{E}\left[P(\lambda, \sigma^2)\right] \to 0$ *as* $w, L, M \to \infty$.

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- The proposed nested ensemble achieve capacity.
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$\square$

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- The proposed nested ensemble achieve capacity.
- Follows from Forney's result.

$\square$

- Binary codes and more importantly practical BP decoding suffices.
- Practically we observe that two levels of coding gets you lattices very close to Poltyrev limit.

Design Example of Poltyrev-Good Lattice

A target block error probability of 10^{-4} in the uncoded level gives $\sigma_L = 0.08$

1. Capacities for the mod 2 AWGN channel for respective levels:

	Level L-1	Level L-2	Level L-3
σ_{eff}	0.16	0.32	0.64
Cap	0.99	0.57	0.02
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2. Fix L=3 and use $(3, 30)$, $(14, 30)$ nested SC LDPC codes.

(d_c, d_v^1, d_v^2)	(L,w)	$P(\mathbb{Z}_4, \sigma^2)$	$\sigma_{\max}$	VNR	$\text{VNR}_{\text{rate-loss}}$
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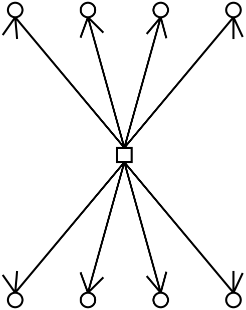
(60, 26, 3)	(72, 12)	5×10^{-10}	0.3200	0.482dB	0.927dB
(60, 27, 3)	(64, 9)	5×10^{-10}	0.3203	0.57dB	0.951dB

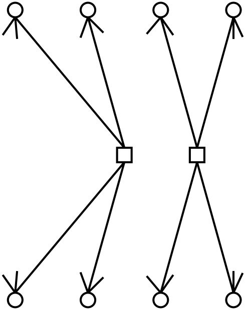
Alternate Nested SC LDPC ensemble

- Derive a lower rate code by "splitting the checks"
- Consider a $(3, 8)$ code

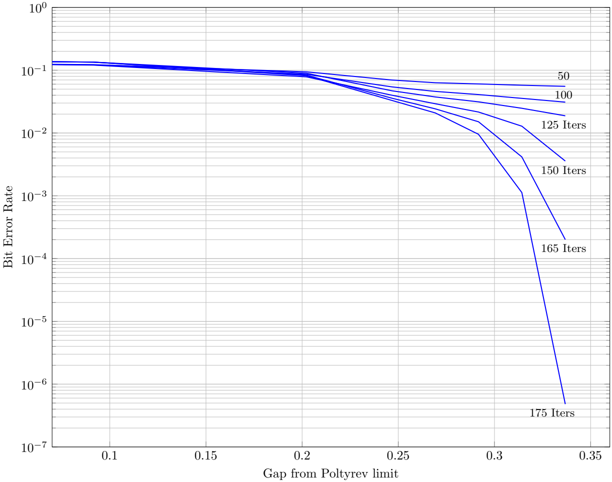


Alternate Nested SC LDPC ensemble

- Derive a lower rate code by "splitting the checks"
- Consider a $(3, 8)$ code
- Split each check into "two" checks to derive a $(3, 4)$ sub-code
- Easy to prove that resulting code is from the $(3, 4)$ SC LDPC ensemble



Simulation Results



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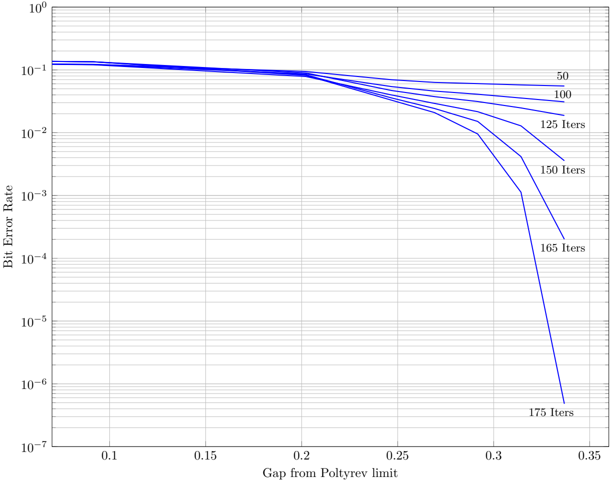
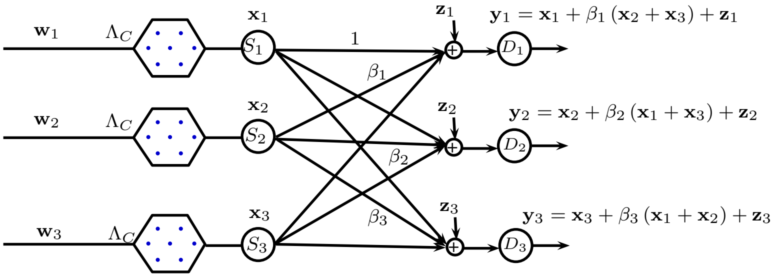


Note that the Block Error Probability is 10^{-4} at uncoded level.

3-User Symmetric Interference Channel



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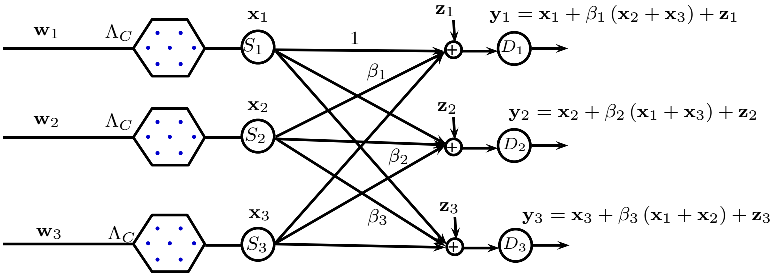


- $\mathbf{x}_i \in \Lambda_C \triangleq \Lambda \cap \mathbb{Z}_4^N$ is transmitted.

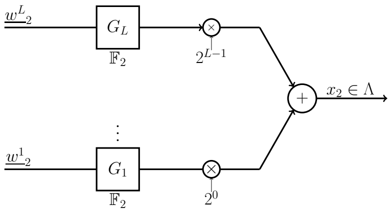
Symmetric Interference Channel - Decoding Sums

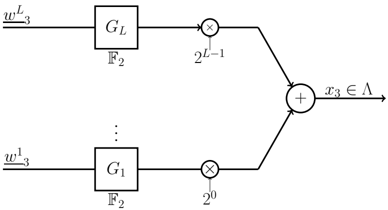
Interference at Destination 1:

$$\begin{aligned}\mathbf{x}_2 + \mathbf{x}_3 &= (\underline{w}_2^1 + \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{w}_2^2 + \underline{w}_3^2)\mathbf{G}_2 + 4\mathbf{k}_{23} \\ &= (\underline{w}_2^1 \oplus \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{c}_{23}^1 \oplus \underline{w}_2^2 \oplus \underline{w}_3^2)\mathbf{G}_2 + 4(\underline{c}_{23}^2 + \mathbf{k}_{23})\mathbf{Z}\end{aligned}$$

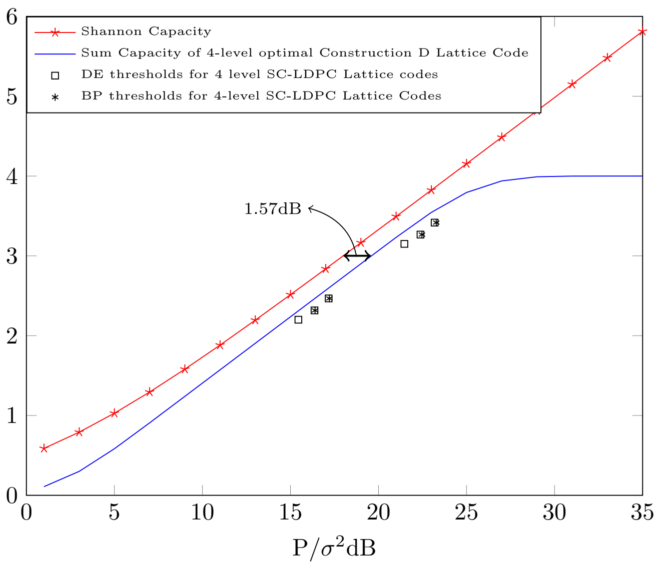
where the carry overs are

$$\begin{aligned}\underline{c}_{23}^1 &= 0.5\left(\underline{w}_1^1 + \underline{w}_1^2 - \underline{w}_1^1 \oplus \underline{w}_1^2\right), \\ \underline{c}_{23}^2 &= 0.5\left(\underline{c}_{23}^1 + \underline{w}_2^1 + \underline{w}_2^2 - \underline{c}_{23}^1 \oplus \underline{w}_2^1 \oplus \underline{w}_2^2\right)\end{aligned}$$





Achievable Information Rates



Concluding Remarks

- Multilevel constructions - efficient ways to decode integer combinations
- Need capacity achieving nested codes
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- Need capacity achieving nested codes
- Multilevel construction is provably good under message passing decoding
- Coding schemes based on Binary LDPC codes and iterative decoding suffice

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Lossy Source Coding Problem

$X^n = (X_1, \cdots, X_n)$, $X_i \sim \text{Bernoulli}(\frac{1}{2})$

Binary code $\mathcal{C} = (n, k)$, rate $R = k/n$

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Lossy Source Coding

- Compress X^n to $\hat{X}^n \in \mathcal{C}$
- Min. Hamming distortion

$$D = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i - \hat{X}_i|$$

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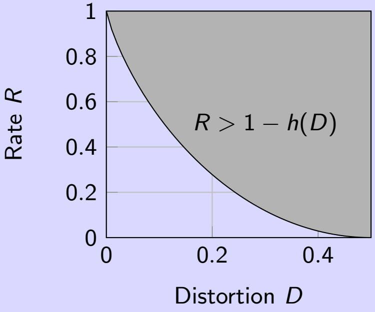
$$D = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}|X_i - \hat{X}_i|$$

- Rate-Distortion theory:

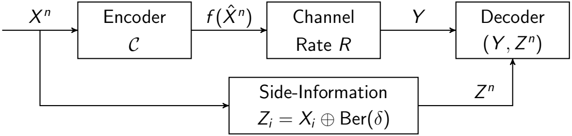
$$R > 1 - h(D)$$

- $h(\cdot)$ is binary entropy function

$$h(D) = -D\log_2 D - (1-D)\log_2(1-D)$$



Side-Information Problems: Wyner-Ziv



Wyner-Ziv Formulation

- Side-information Z^n about X^n
- Decoder additionally has Z^n
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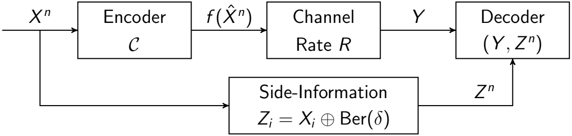

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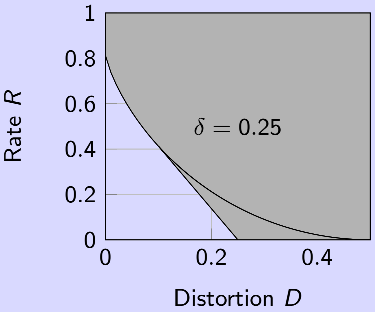
- Wyner-Ziv theory:

$$R > l.c.e\{h(D * \delta) - h(D), (\delta, 0)\}$$

- $D * \delta = D(1-\delta) + \delta(1-D)$



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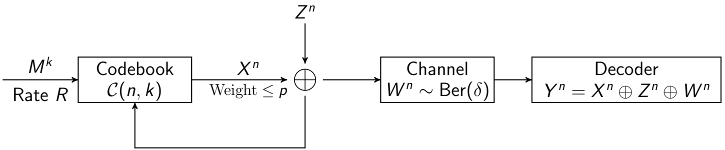

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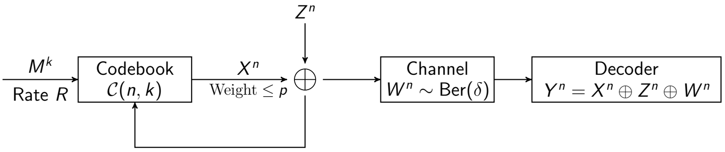

Gelfand-Pinsker Formulation

- Message M^k encoded to $X^n \in \mathcal{C}$ with $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] \leq p \leq \frac{1}{2}$
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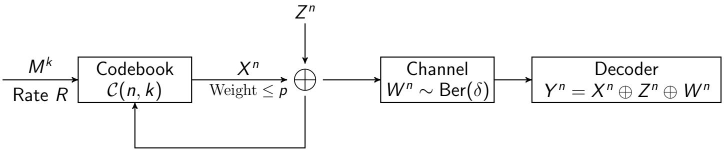

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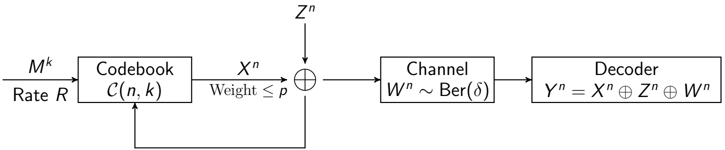

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- Capacity region by Gelfand-Pinsker:

$$R < h(p) - h(\delta)$$

Main Result

Objective

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  - Low-complexity encoding and decoding

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- Wainwright et al. used compound LDGM/LDPC codes with optimal encoding/decoding
- Message-passing algorithms have non-negligible gap

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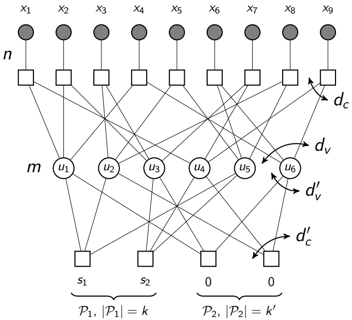
- Wainwright et al. used compound LDGM/LDPC codes with optimal encoding/decoding
- Message-passing algorithms have non-negligible gap
- Remedy via Spatial-Coupling
  - Channel coding in coupled compound codes (Kasai et al.)
  - Lossy source coding with spatially-coupled LDGM (Aref et al.)
  - Encoding with compound codes has additional challenges

Compound LDGM/LDPC Codes



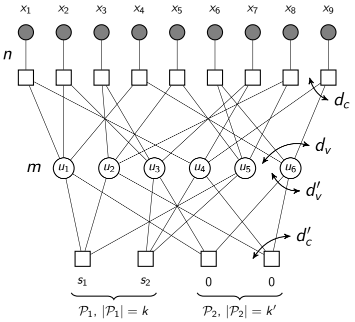
- Codebook $\mathcal{C}(n, m - k - k')$
- Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword $(x_1, \cdots, x_9)$:

$$x_1 = u_1 \oplus u_4, \qquad x_2 = \cdots$$

Compound LDGM/LDPC Codes



- Codebook $\mathcal{C}(n, m - k - k')$
- Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword $(x_1, \cdots, x_9)$:

$$x_1 = u_1 \oplus u_4, \qquad x_2 = \cdots$$

Key Properties

- Compound code is
  - a good source code under optimal encoding
  - a good channel code under optimal decoding
- LDGM code is
  - a good source code under optimal encoding
  - (side note) LDGM code is not a good channel code

Good Code

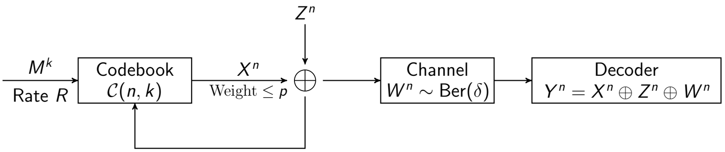
"Good" source code

- Rate of the code is $R = 1 - h(D) + \varepsilon$
- When this code is used to optimally encode $\mathrm{Ber}(\frac{1}{2})$
- The average Hamming distortion is at most D

"Good" channel code

- Rate of the code is $R = 1 - h(\delta) - \varepsilon$
- When this code is used for channel coding on $\mathrm{BSC}(\delta)$
- Message est. under optimal decoding with error at most ε

Coding Scheme: Gelfand-Pinsker



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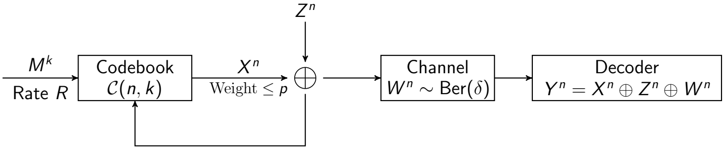

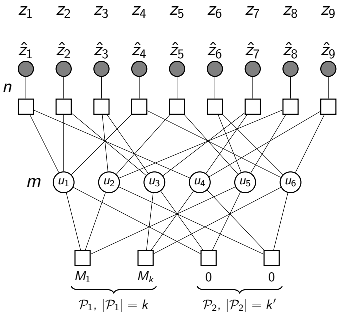

$\frac{m-k-k'}{n} \approx 1 - h(p) + \varepsilon \qquad \frac{m-k'}{n} \approx 1 - h(\delta) + \varepsilon$

- With message M^k, encode Z^n to $\hat{Z}^n$ (Distortion $\approx p$)
- Transmit $X^n = Z^n \oplus \hat{Z}^n$

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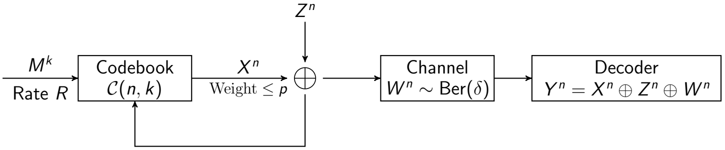

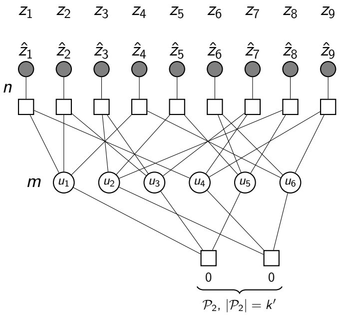

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$$\begin{aligned} Y^n &= X^n \oplus Z^n \oplus W^n \\ &= \hat{Z}^n \oplus W^n \end{aligned}$$
- Decode $\hat{Z}^n$ and compute M^k

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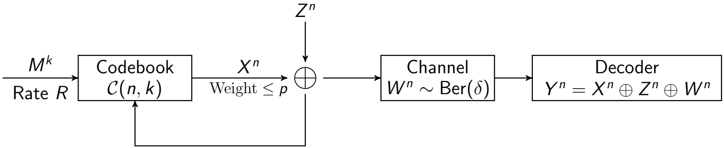

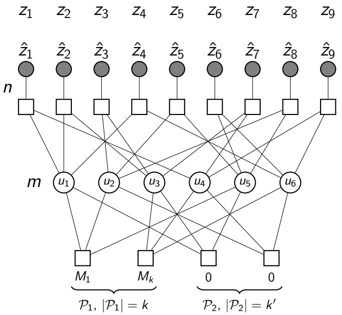

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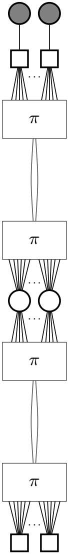
- $R = \frac{k}{n} \approx h(p) - h(\delta)$

Remarks

- Need codes that are simultaneously good for channel and source coding
- Use message-passing algorithms instead of optimal
- Use spatial-coupling for goodness of codes under message-passing

Spatially-Coupled Compound LDGM/LDPC Codes



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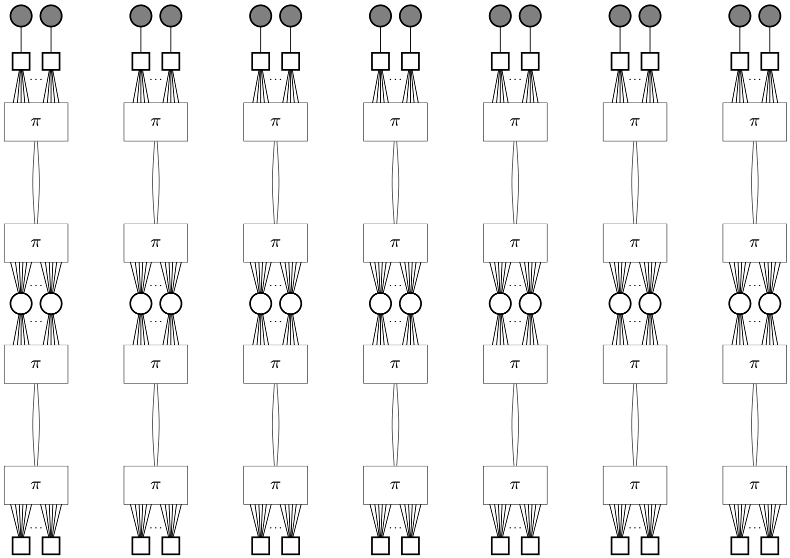

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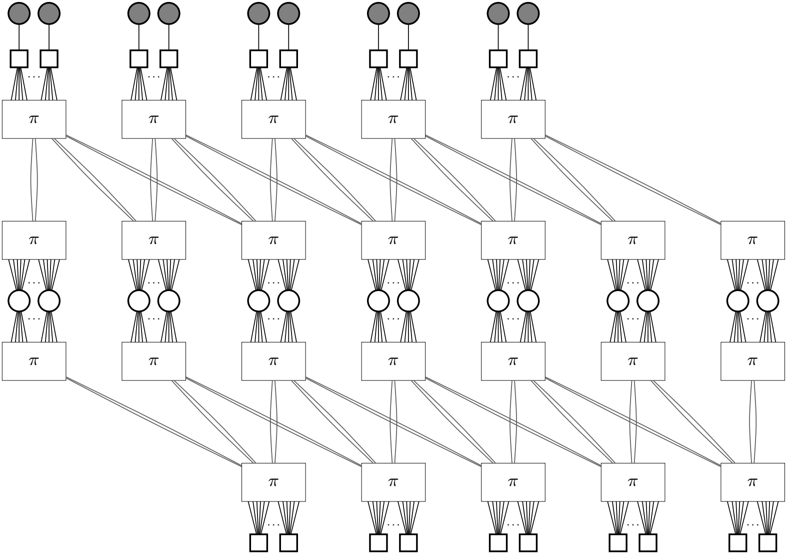

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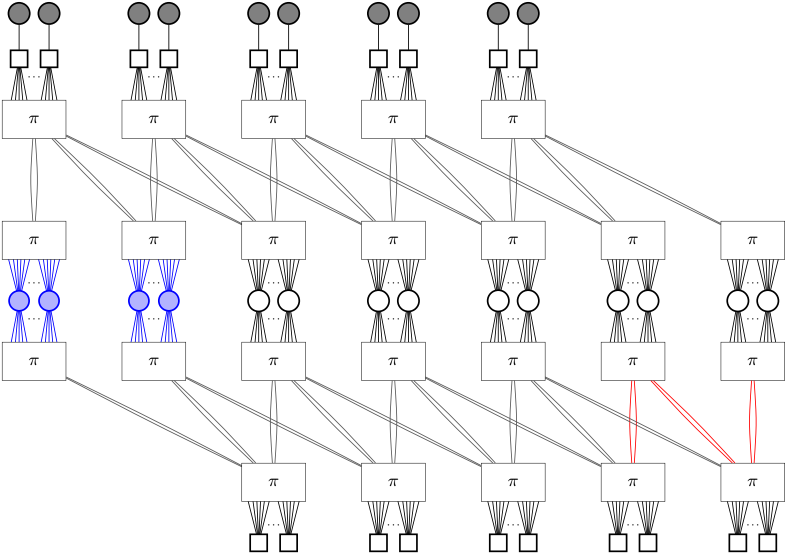
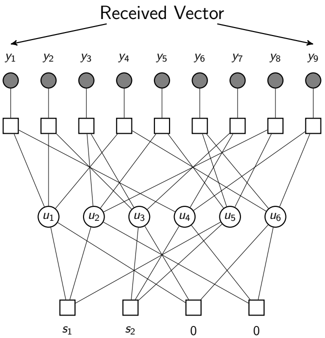
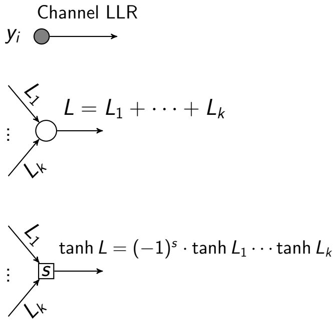


Decoding in Spatially-Coupled Compound Codes

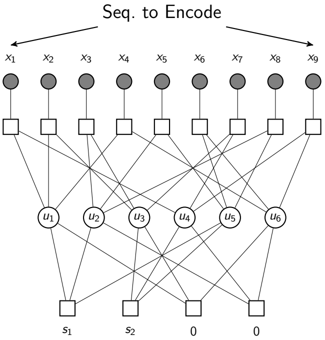


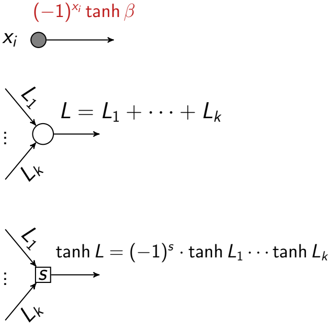


Remarks

- Standard message-passing algorithm
- Threshold saturation proven for SC compound codes on BEC
- Empirically observed for BMS channels

Encoding in Spatially-Coupled Compound Codes





Remarks

- Inverse temperature parameter β
- Message-passing rules are the same
- However, a crucial decimation step is needed

Encoding in SC Compound Codes: BPGD Algorithm

Encoding in SC Compound Codes: Remarks

- Randomization in setting u_{i^*} is crucial
- BPGD applied to uncoupled code always failed
- Spatially-coupled structure is crucial for successful encoding
  - In addition, distortion is close to optimal thresholds
  - Does not encode if decimated from both left and right
  - Does not encode if both left and right boundary is set to 0

Encoding in SC Compound Codes: Numerical Example

Block length (n)	4-cycles	Attempts $1/2/3/4/\geq 5$
9000	yes	5/3/5/2/35
9000	no	21/12/5/3/9
27000	no	35/15/0/0/0
45000	no	40/9/0/0/1
63000	no	44/6/0/0/0
81000	no	50/0/0/0/0

Remarks

- # Attempts to encode 50 seq. in $(6,3)$ LDGM / $(3,6)$ LDPC
- $L = 20,\ w = 4,\ \beta = 0.65,\ T = 10$
- Removing 4-cycles dramatically improves success
- How much do 6-cycles matter?

Numerical Results: Wyner-Ziv

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(D_*, δ_*)	(D, δ)
$(6, 3)$	(3,6)	(20,4)	(0.111,0.134)	(0.1174,0.122)
$(8, 4)$	(3,6)	(20,4)	(0.111,0.134)	(0.1149,0.120)
$(10, 5)$	(3,6)	(20,4)	(0.111,0.134)	(0.1139,0.122)

Remarks

- D_* and δ_* are calculated based on the rate of the respective code:

$$D_* = h^{-1}(1 - R1) \qquad \delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000$, $\beta = 1.04$, $T = 10$

Numerical Results: Gelfand-Pinsker

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(p_*, δ_*)	(p, δ)
$(6, 3)$	(3,6)	(20,4)	(0.215,0.157)	(0.2200,0.152)
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Concluding Remarks

Conclusion

- Spatially-coupled codes achieve the rate regions of Wyner-Ziv and Gelfand-Pinsker problems
- Coupling structure is also crucial
    - to achieve optimum thresholds
    - for encoding to succeed with decimation

Open Questions

- Effect of degree profiles, short-cycles on encoding success
- Precise trade-offs with polar codes

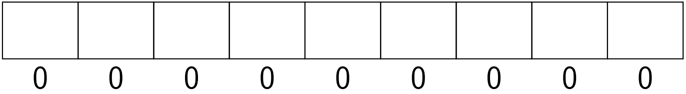
Outline

Write-Once Memories

Flash Memory

- In typical flash memory, changing from 0 to 1 is easy
- Resetting 1 to 0 requires rewriting whole block
- Write-once memories model such storage systems

Write-Once Memories



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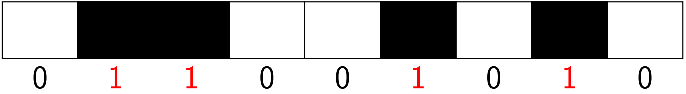
Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed

Write-Once Memories



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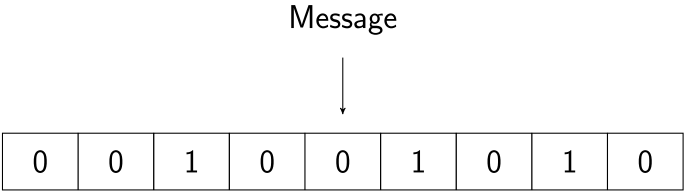
Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed
- $1 \longrightarrow 0$ is forbidden

Capacity Region (I) - Noiseless



Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- Only about $nt/\log(t)$ cells required to store n bits for t writes

Capacity Region (I) - Noiseless

Message

$\downarrow$

0	0	1	0	0	1	0	1	0

Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- Only about $nt/\log(t)$ cells required to store n bits for t writes
- In 1985, Heegard gave the capacity for t-write system
- For a 2-write system, it is

$$\left\{(R_1, R_2) \mid 0 \leq R_1 < h(\delta),\ 0 \leq R_2 < 1 - \delta\right\}$$

Capacity Region (II) - Read Errors

Message

0	0	1	0	0	1	0	1	0

Ber(p)

0 0 0 0 0 1 1 1 0

Write-Once Memory with Read Errors

- Different from write errors
- $Y = X \oplus \text{Ber}(p)$, where Ber($p$) denotes the Bernoulli noise
- Capacity region is unknown

Main Result

Objective

- Construct low-complexity coding schemes that achieve the capacity region of the WOM system
  - Low-complexity encoding and decoding

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  - For read errors, achieves

$$R_1 < h(\delta) - h(p), \qquad R_2 < 1 - \delta - h(p).$$

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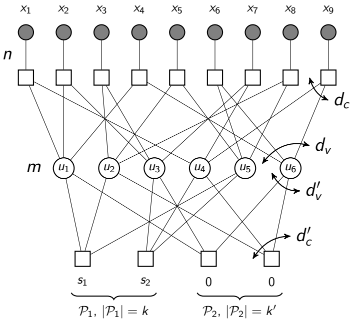
Idea

- Use compound LDGM/LDPC codes
- Encoding for second write is erasure quantization
- Use spatial coupling with message-passing

Compound LDGM/LDPC Codes



- Codebook $(n, m - k - k')$
- Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

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- Parametrized by s^k: $\mathcal{C}(s^k)$

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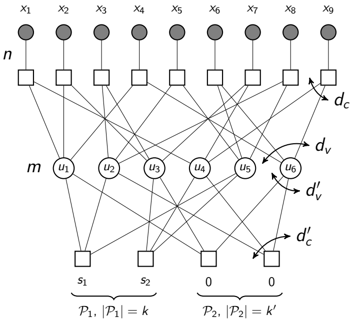

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Key Properties of Compound Codes

- a natural coset decomposition: $\mathcal{C} = \bigcup_{s^k \in \{0,1\}^k} \mathcal{C}(s^k)$
- achieves capacity over eras. chan. under MAP (when $m = n$)
- a good source code under optimal encoding
- a good channel code under optimal decoding

Good Code

"Good" source code

- Rate of the code is $R = 1 - h(\delta) + \varepsilon$
- When this code is used to optimally encode $\mathrm{Ber}(\frac{1}{2})$
- The average Hamming distortion is at most δ

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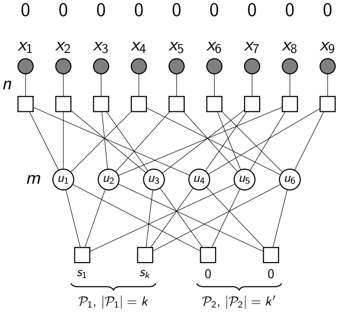
Coding Scheme for 2-write WOM: First Write

$$R_1 < h(\delta) - h(p)$$



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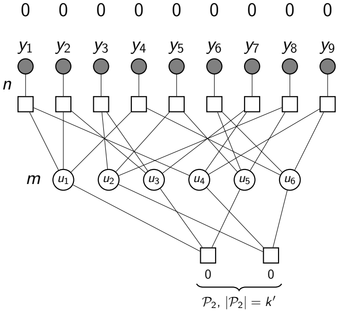

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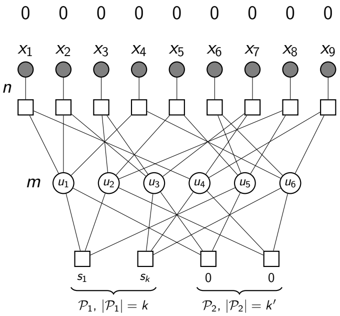

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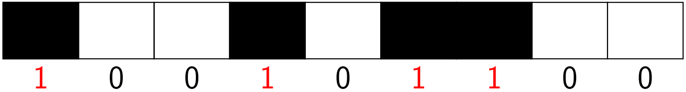

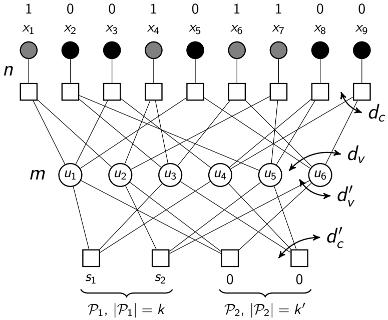

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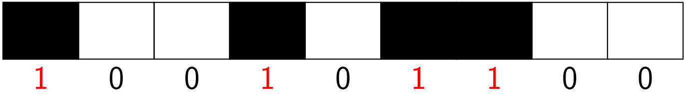
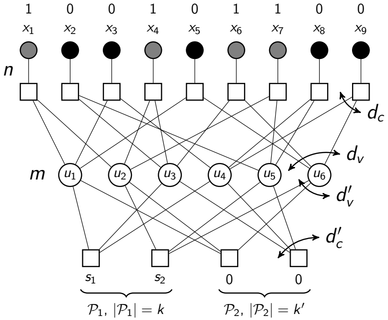




- Need to find a consistent codeword in $\mathcal{C}(s^k)$
- Closely related to Binary Erasure Quantization (BEQ)
- En Gad, Huang, Li and Bruck (ISIT 2015)

Binary Erasure Quantization

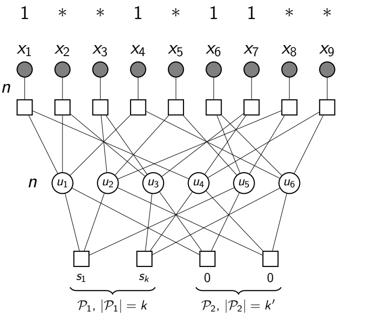
- Quantize a sequence in $\{0,1,*\}^n$ to $x^n \in \mathcal{C} \subset \{0,1\}^n$
  - 0's and 1's should match exactly
  - $*$'s can take either 0 or 1
- Can map the second write of 2-write WOM to BEQ
  - Map 0's to $*$'s and keep 1's
  - Quantize to codeword in $\mathcal{C}(s^k)$
- BEQ is the dual of decoding on binary erasure channel
  - Martinian and Yedidia (Allerton 2003)
  - Can quan. all seq. with erasure pattern $e^n \in \{0,1\}^n$ to $\mathcal{C}$

    $$\Updownarrow$$

    Chan. dec. for $\mathcal{C}^\perp$ can correct all vectors with eras. $1^n \oplus e^n$
- Choose a good (dual) code $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



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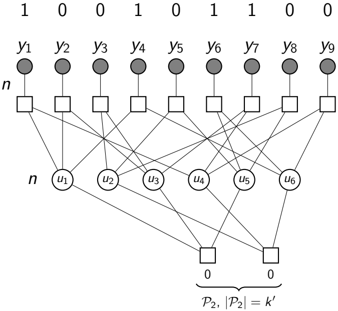

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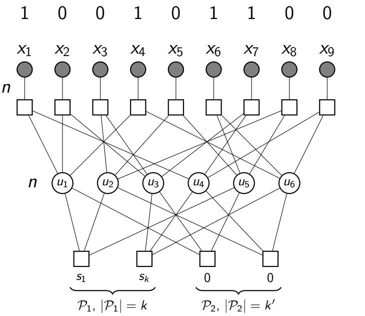

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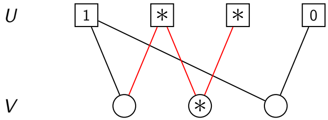
- Dec. x^n and compute s^k
- $R_2 = \frac{k}{n} \approx 1 - \delta - h(p)$

Iterative Erasure Quantization Algorithm



- Peeling type encoder

Remarks

- Need codes that are simultaneously good for channel/source coding and erasure quantization
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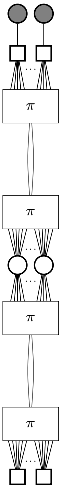

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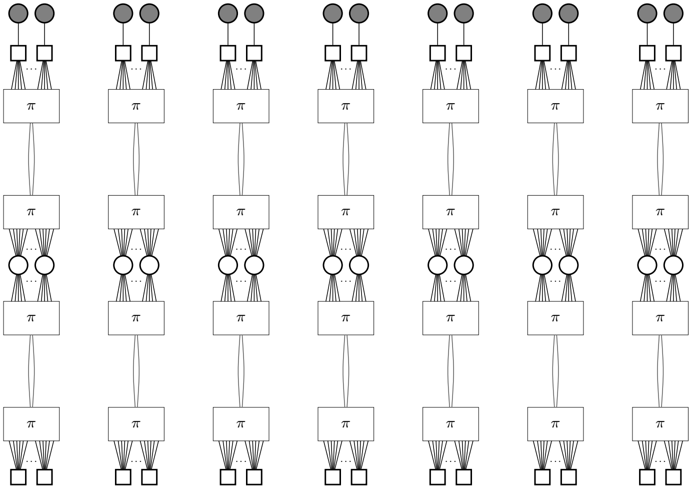
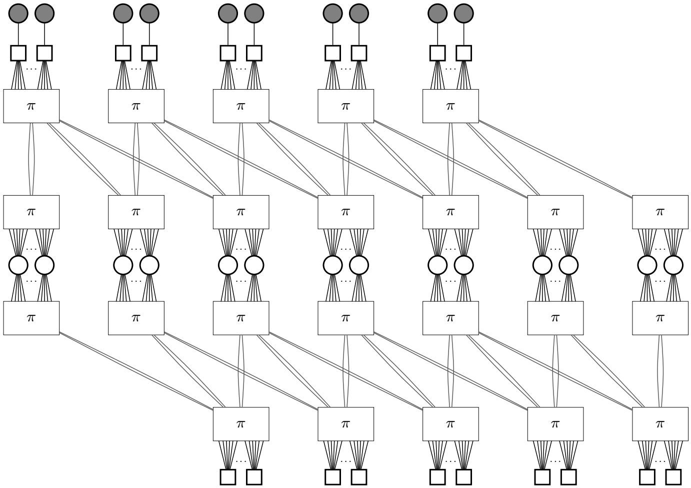
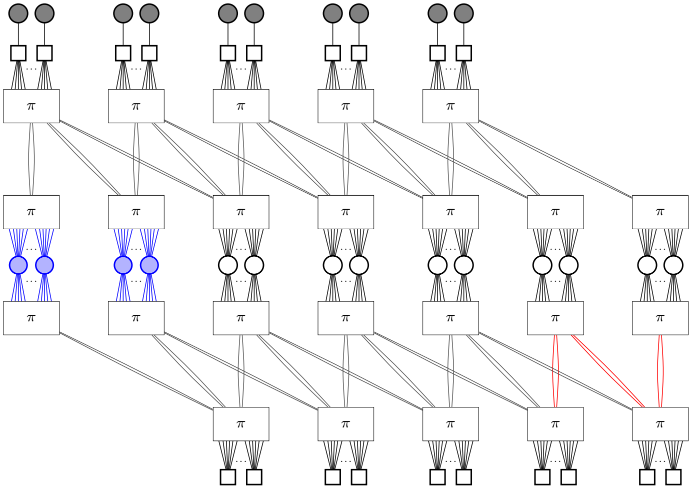


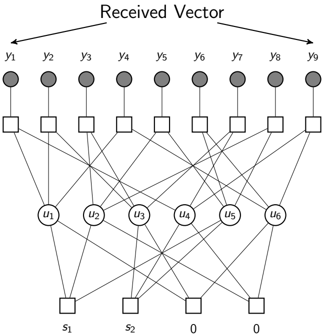
Spatially-Coupled Compound LDGM/LDPC Codes

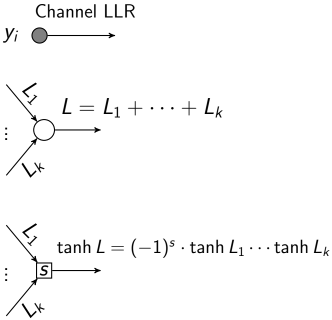


Spatially-Coupled Compound LDGM/LDPC Codes



Decoding in Spatially-Coupled Compound Codes





Remarks

- Standard message-passing algorithm
- Threshold saturation proven for SC compound codes on BEC
- Empirically observed for BMS channels

Numerical Results: Noiseless WOM

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	δ^*	δ $w = 2$	δ $w = 3$	δ $w = 4$
$(3, 3, 3, 6)$	0.500	0.477	0.492	0.494
$(3, 3, 4, 6)$	0.333	0.294	0.324	0.326
$(3, 3, 5, 6)$	0.167	0.095	0.156	0.158
$(4, 4, 3, 6)$	0.500	0.461	0.491	0.492
$(4, 4, 4, 6)$	0.333	0.278	0.323	0.325
$(4, 4, 5, 6)$	0.167	0.086	0.155	0.159
$(5, 5, 3, 6)$	0.500	0.436	0.488	0.491
$(5, 5, 4, 6)$	0.333	0.260	0.320	0.324
$(5, 5, 5, 6)$	0.167	0.079	0.154	0.159

Remarks

- δ^* is the Shannon threshold
- $L = 30$, Single system length ≈ 24000

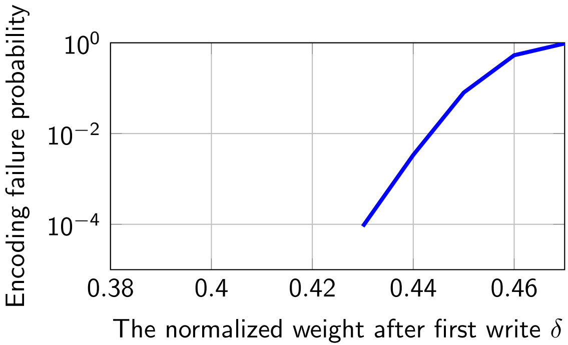
Numerical Results: WOM with Read Errors

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	w	(δ^*, p^*)	(δ, p)
$(3, 3, 4, 6)$	3	$(0.333, 0.0615)$	$(0.321, 0.0585)$
$(3, 3, 4, 8)$	3	$(0.500, 0.0417)$	$(0.490, 0.0387)$
$(3, 3, 6, 8)$	4	$(0.250, 0.0724)$	$(0.239, 0.0684)$
$(4, 4, 4, 6)$	4	$(0.333, 0.0615)$	$(0.324, 0.0585)$
$(4, 4, 4, 8)$	4	$(0.500, 0.0417)$	$(0.492, 0.0387)$
$(4, 4, 6, 8)$	4	$(0.250, 0.0724)$	$(0.241, 0.0694)$

Remarks

- δ^* and p^* are the Shannon thresholds
- $L = 30$, Single system length ≈ 30000

Numerical Results: Small Blocklength



Remarks

- $(L, w) = (30, 3)$, Single system length 1200, Shannon threshold of 0.5
- A total of 10^5 were attempted to encode
- No failures for $\delta < 0.43$

Concluding Remarks

Conclusion

- Spatially-coupled compound codes achieve the capacity of 2-write systems
- Coupling structure is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed

Multi-Write Systems

- Will BPGD work for multi-write systems?