Applications of Spatial Coupling

Avinash Vem
Advisor: Dr.Krishna Narayanan

Department of Electrical and Computer Engineering
Texas A&M University

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Outline

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An (ℓ, r) LDPC Code

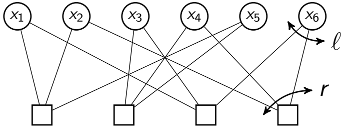
Parity-Check Matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$\ell = 2 \qquad r = 3$

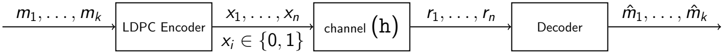
LDPC Code $\mathcal{C} = \{x : H \odot x = 0\}$

Tanner Graph



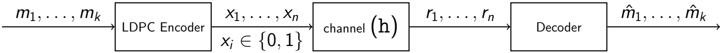
Compressed Representation

Belief Propagation(BP) Decoder



- For BEC(ε), $\mathtt{h} = \varepsilon$. For general BMS channel, $\mathrm{C}^{\mathrm{Sh}} = 1 - \mathtt{h}$

Belief Propagation(BP) Decoder



- For BEC(ε), $\mathtt{h} = \varepsilon$. For general BMS channel, $\mathrm{C}^{\mathrm{Sh}} = 1 - \mathtt{h}$

Belief Propagation (BP)

- Non-optimal
- Popular, low-complexity
- Threshold: $\mathtt{h}^{\mathrm{BP}}$

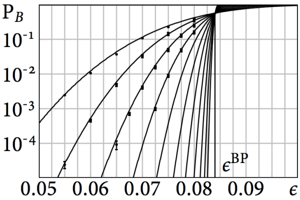
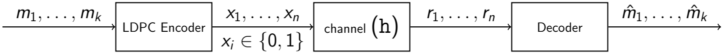


Figure: $(3, 6)$, P_{B} vs BSC(ε) for $n = 2^i$.

$\varepsilon^{\mathrm{BP}} = 0.084$, equivalent $\mathtt{h}^{\mathrm{BP}} = 0.4160$
whereas for rate $1/2$ code $\mathrm{h}^{\mathrm{Sh}} = 0.5$

BP vs MAP



- For BEC(ε), $\mathtt{h} = \varepsilon$. For general BMS channel, $\mathrm{C}^{\mathrm{Sh}} = 1 - \mathtt{h}$

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Maximum a Posteriori (MAP)

- Optimal decoder
- *Not realizable*
- Threshold: $\mathtt{h}^{\mathrm{MAP}}$

BP vs MAP

$$m_1,\dots,m_k \longrightarrow \boxed{\text{LDPC Encoder}} \xrightarrow[x_i\in\{0,1\}]{x_1,\dots,x_n} \boxed{\text{channel }(\mathtt{h})} \xrightarrow{r_1,\dots,r_n} \boxed{\text{Decoder}} \xrightarrow{\hat{m}_1,\dots,\hat{m}_k}$$

- For BEC(ε), $\mathtt{h} = \varepsilon$. For general BMS channel, $\mathrm{C}^{\mathrm{Sh}} = 1 - \mathtt{h}$

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- Threshold: $\mathtt{h}^{\mathrm{BP}}$

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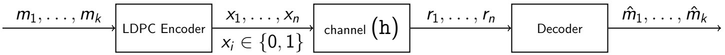
- Optimal decoder
- *Not realizable*
- Threshold: $\mathtt{h}^{\mathrm{MAP}}$

LDPC (ℓ, r)	Shannon $\mathtt{h}^{\mathrm{Sh}}$	AWGN		BSC	
		$\mathtt{h}^{\mathrm{BP}}$	$\mathtt{h}^{\mathrm{MAP}}$	$\mathtt{h}^{\mathrm{BP}}$	$\mathtt{h}^{\mathrm{MAP}}$
$(3,6)$	0.5000	0.4293	0.4794	0.4160	0.4681
$(4,6)$	0.6667	0.5211	0.6645	0.5203	0.6633

BP vs MAP



- For BEC(ε), $\mathtt{h} = \varepsilon$. For general BMS channel, $C^{\mathrm{Sh}} = 1 - \mathtt{h}$

Belief Propagation (BP)

- Non-optimal
- Popular, low-complexity
- Threshold: $\mathtt{h}^{\mathrm{BP}}$

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- Optimal decoder
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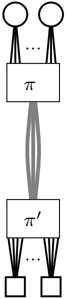
LDPC (ℓ, r)	Shannon $\mathtt{h}^{\mathrm{Sh}}$	AWGN $\mathtt{h}^{\mathrm{BP}}$	AWGN $\mathtt{h}^{\mathrm{MAP}}$	BSC $\mathtt{h}^{\mathrm{BP}}$	BSC $\mathtt{h}^{\mathrm{MAP}}$
(3, 6)	0.5000	0.4293	0.4794	0.4160	0.4681
(4, 6)	0.6667	0.5211	0.6645	0.5203	0.6633

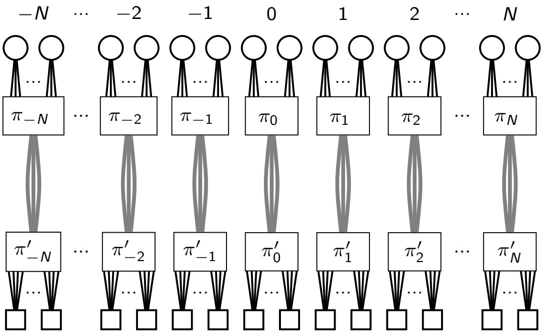
Spatial-Coupling aids in bridging this gap

(ℓ, r, N, w) Spatially-Coupled Ensemble

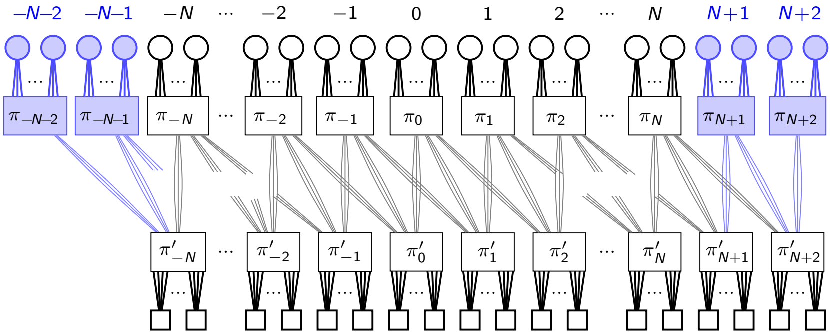


- An LDPC code of left-degree $\ell = 3$ and right-degree $r = 4$

(ℓ, r, N, w) Spatially-Coupled Ensemble

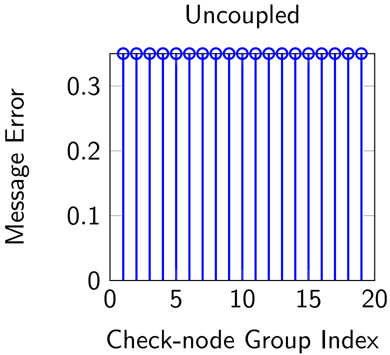


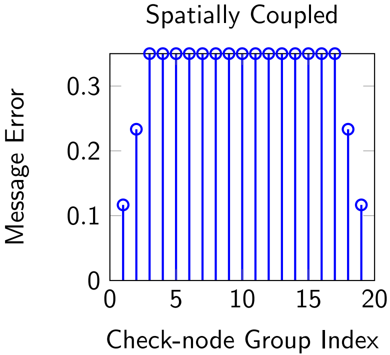
(ℓ, r, N, w) Spatially-Coupled Ensemble



- Shown for $\ell = 3$, $r = 4$, and $w = 3$
- Check-nodes at Section $\{i\}$ are connected to variable-nodes in Sections $\{i - (w-1), \ldots, i\}$
- Shown to have near optimal BP thresholds

Threshold Saturation via Spatial Coupling (1)





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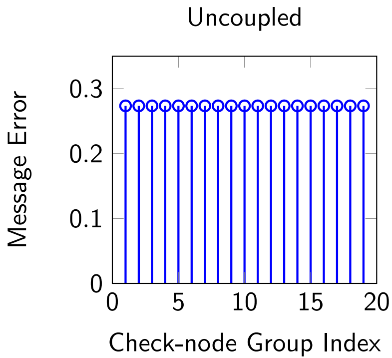

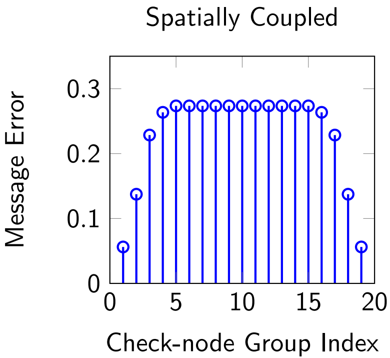

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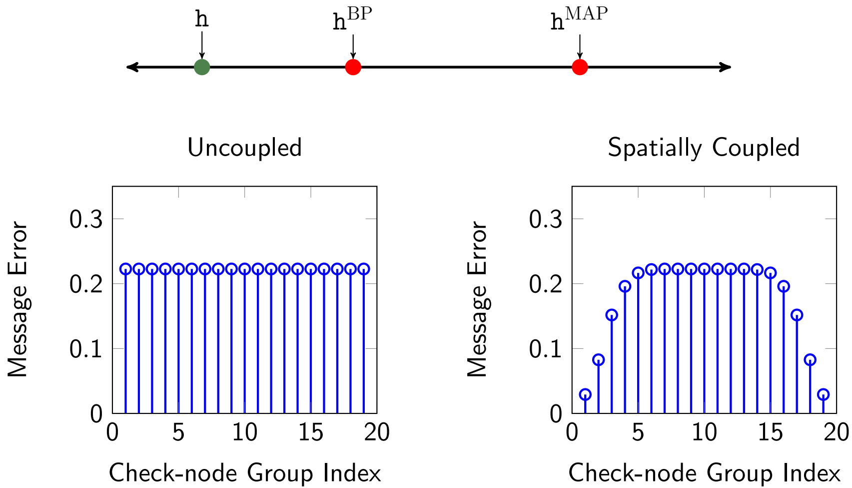

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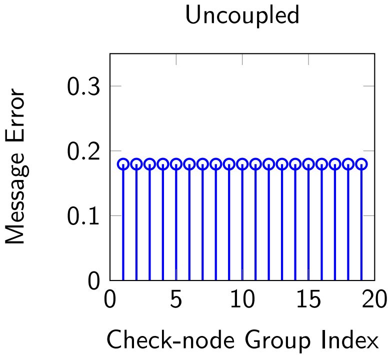

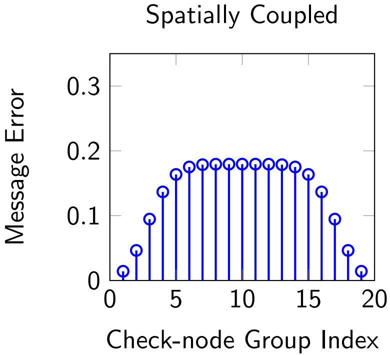

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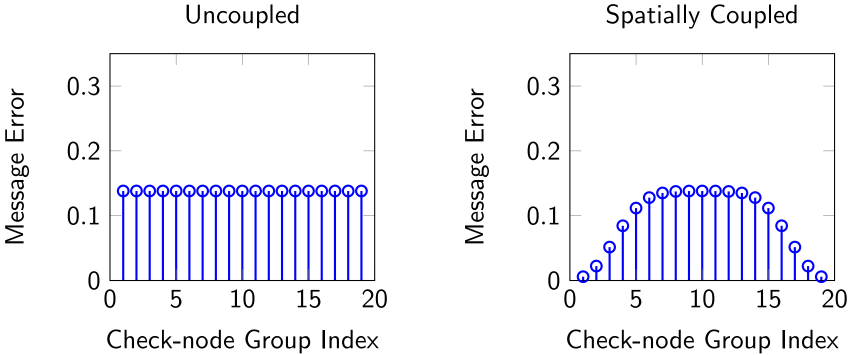

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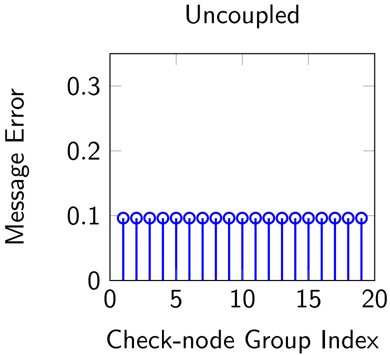

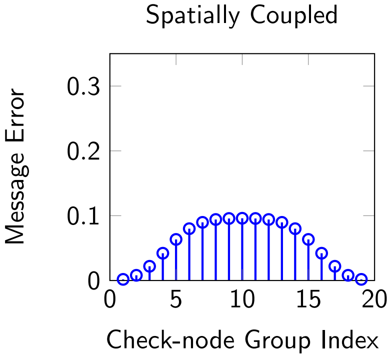

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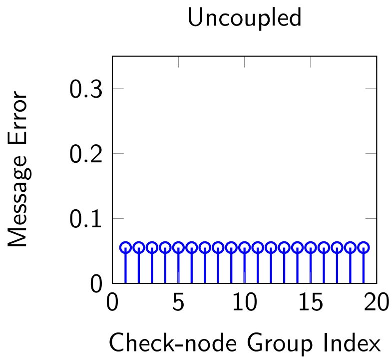

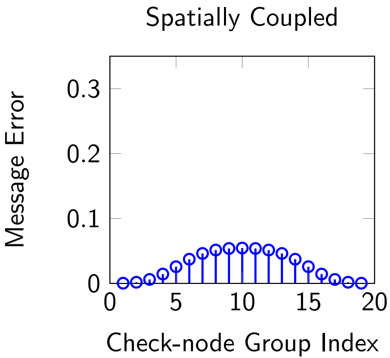

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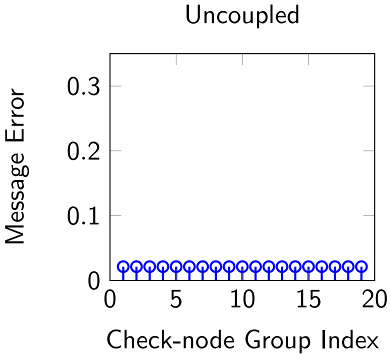

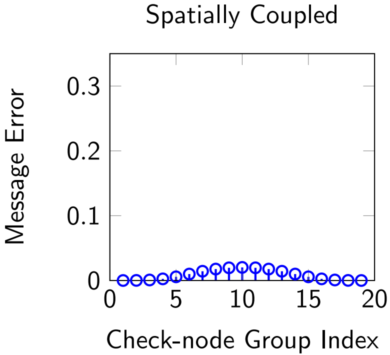

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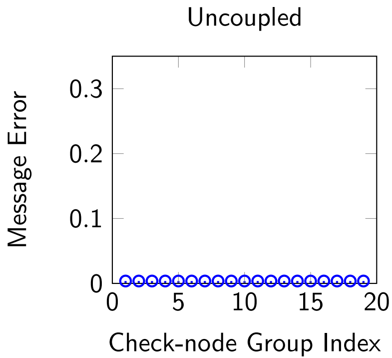
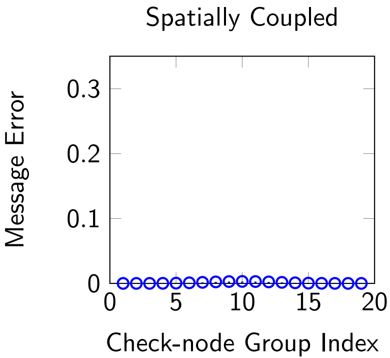
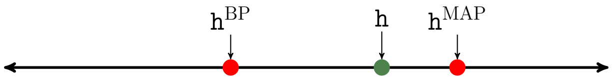
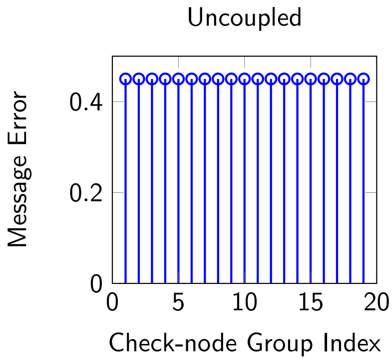
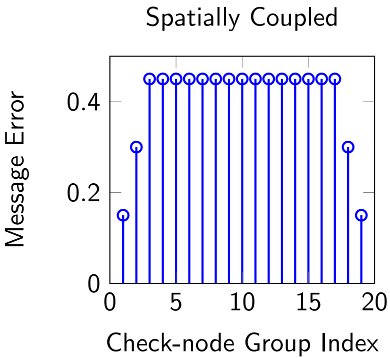




Threshold Saturation via Spatial Coupling (2)







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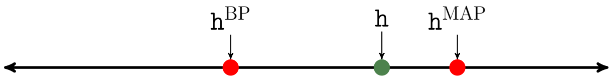

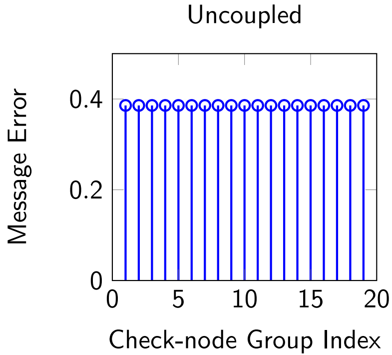

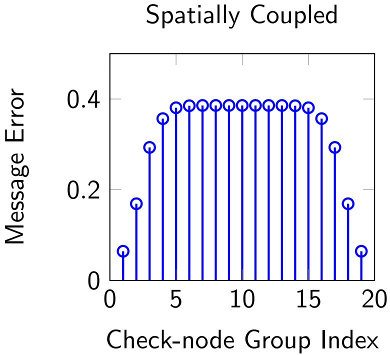

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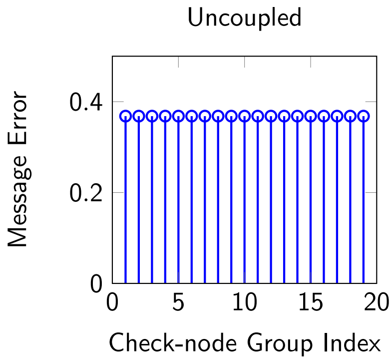

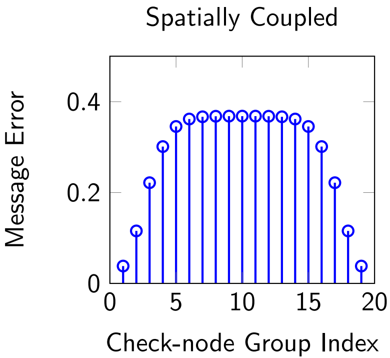

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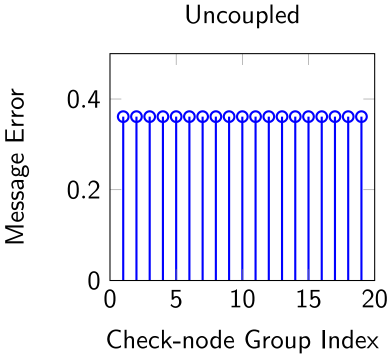

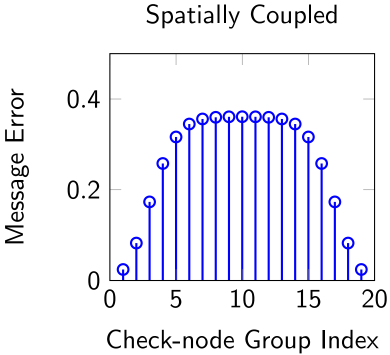

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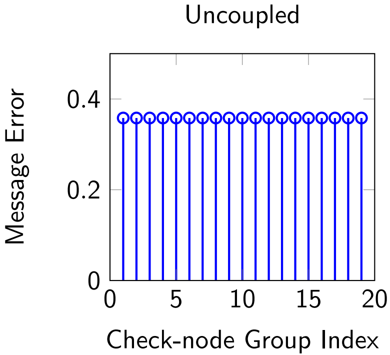

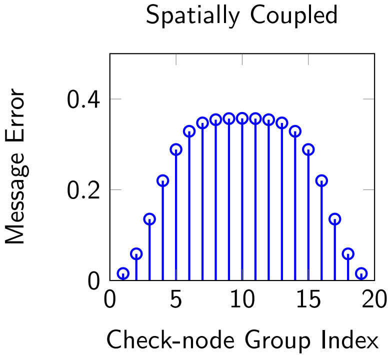

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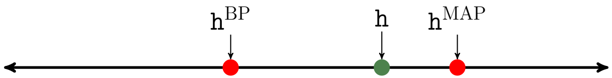
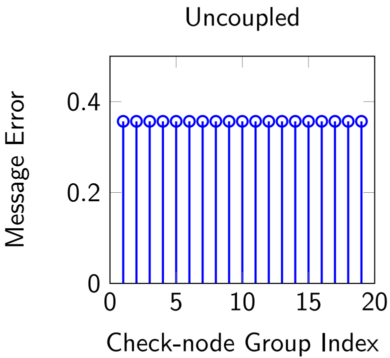
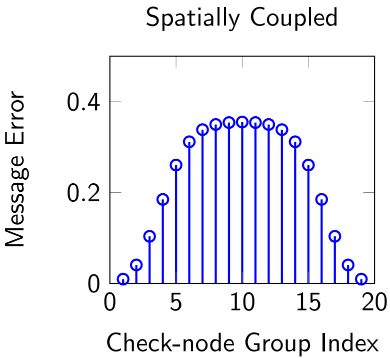
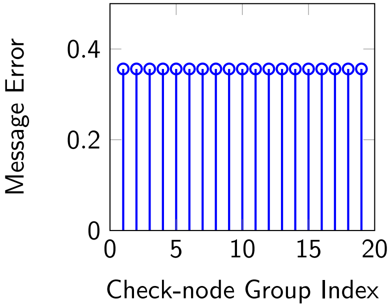




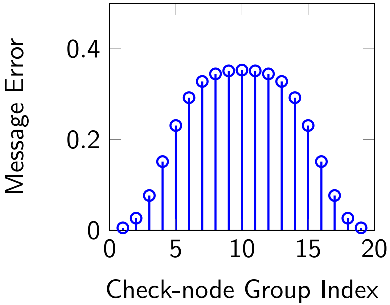


Threshold Saturation via Spatial Coupling (2)

Uncoupled



Spatially Coupled



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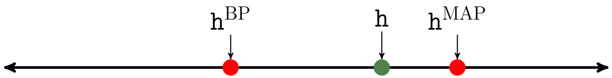

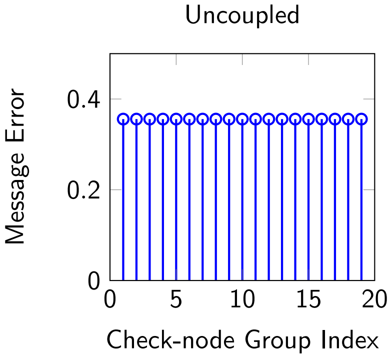

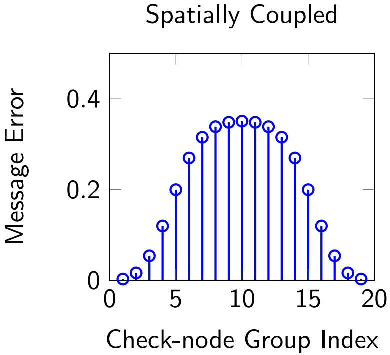

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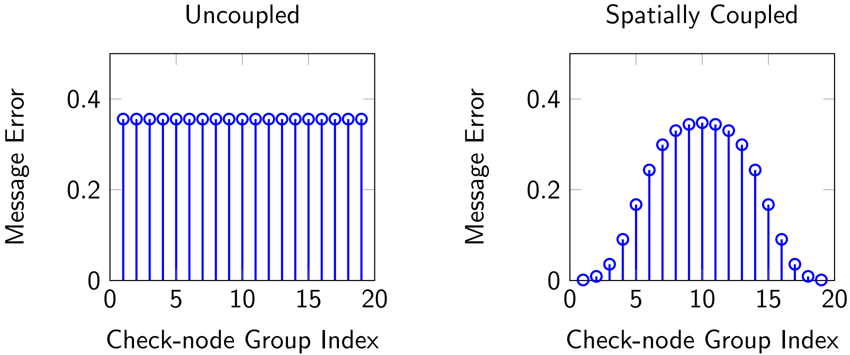

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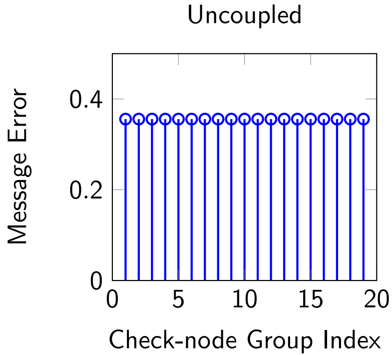

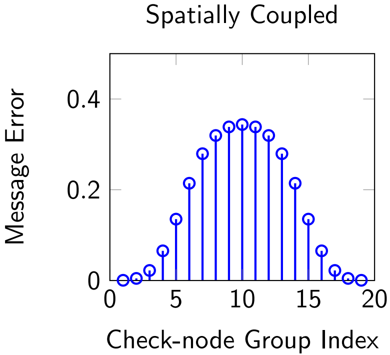

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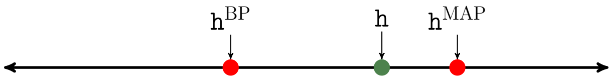

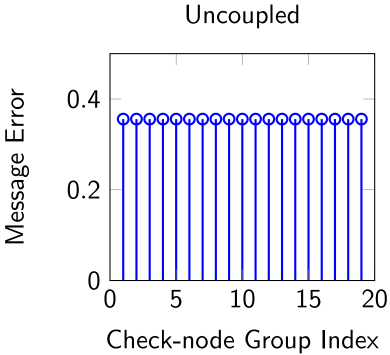

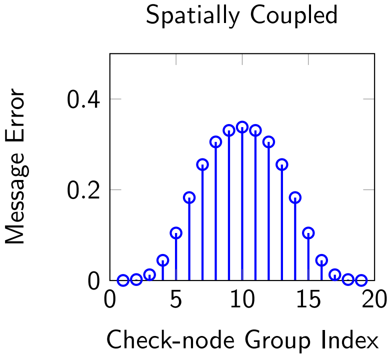

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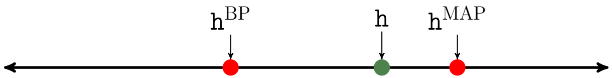

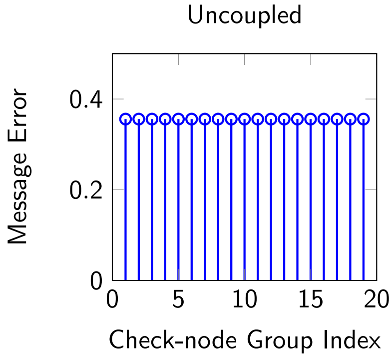

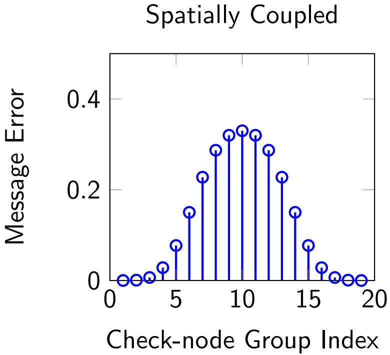

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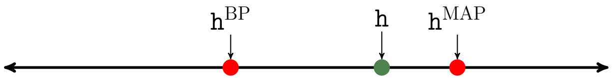

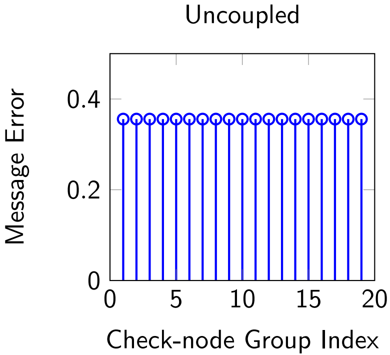

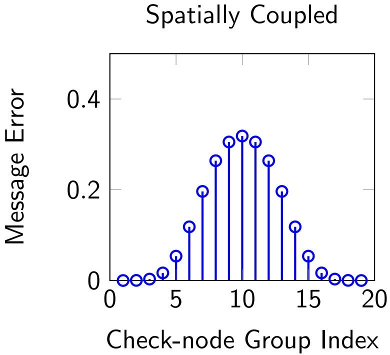

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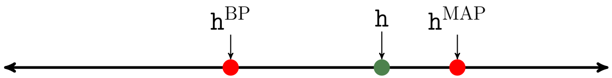

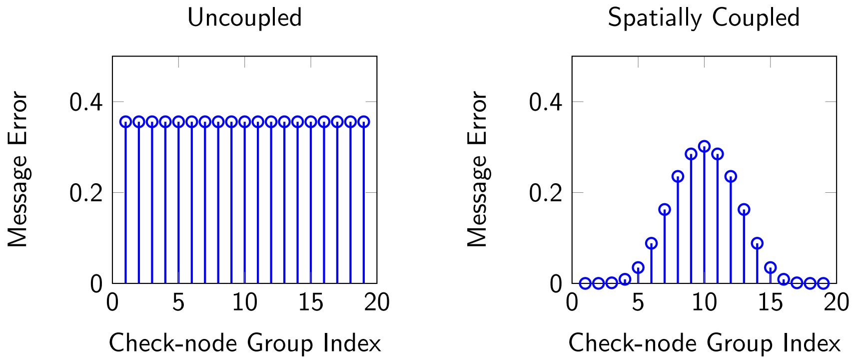

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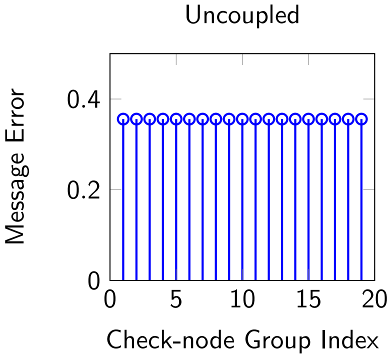
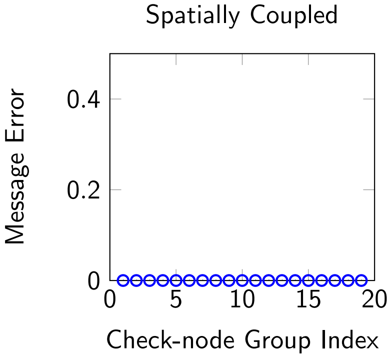
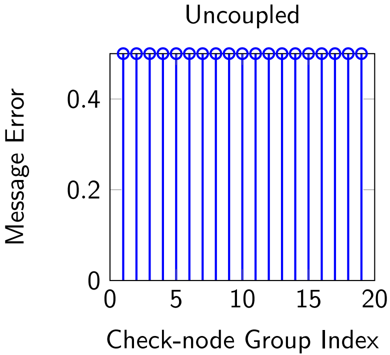
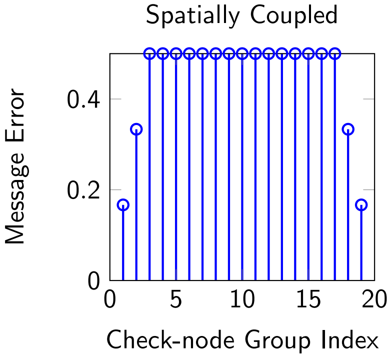




Threshold Saturation via Spatial Coupling (3)





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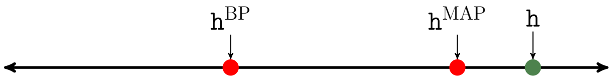

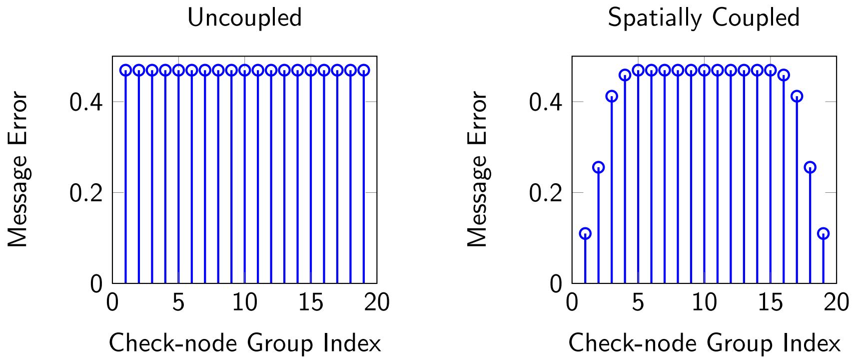

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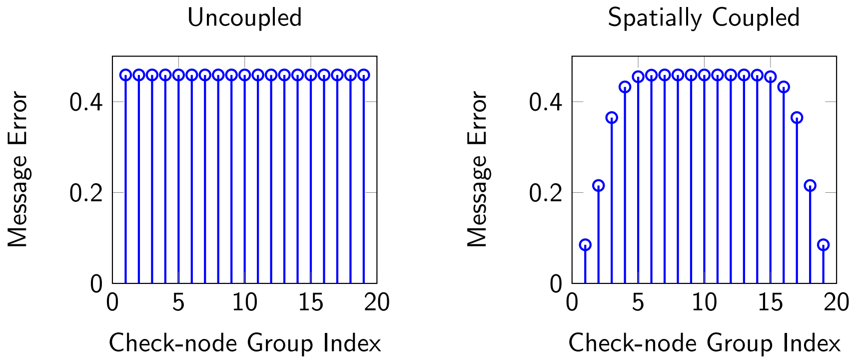

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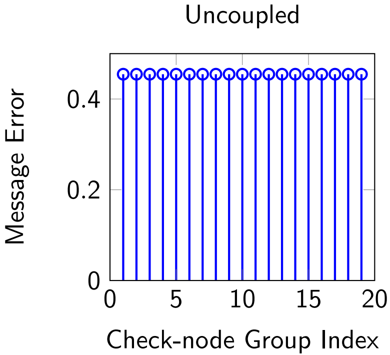

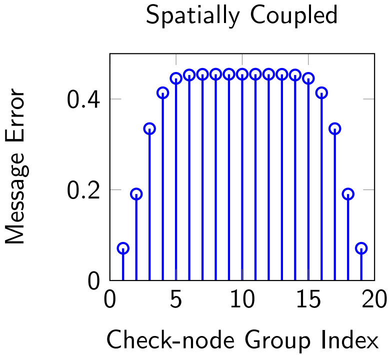

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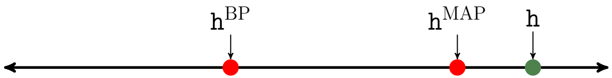

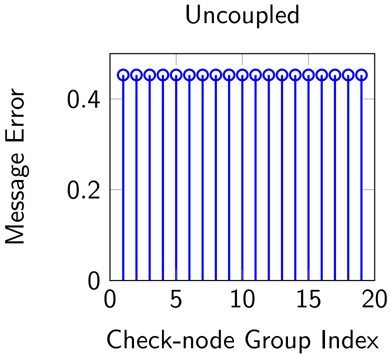

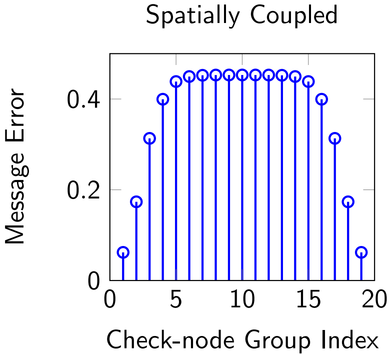

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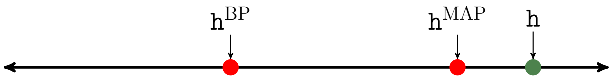

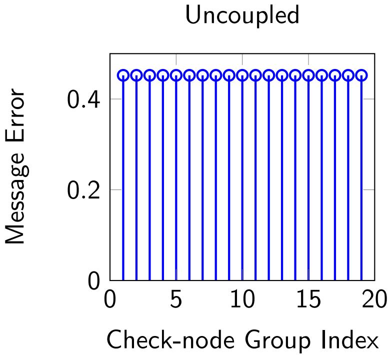

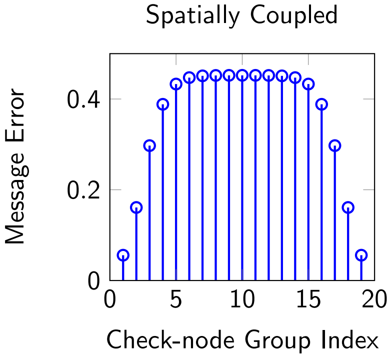

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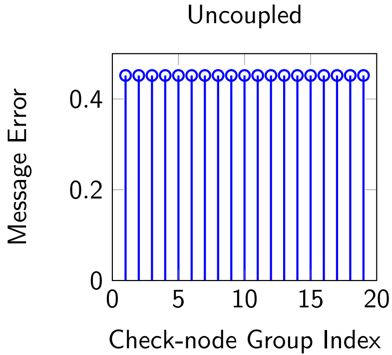

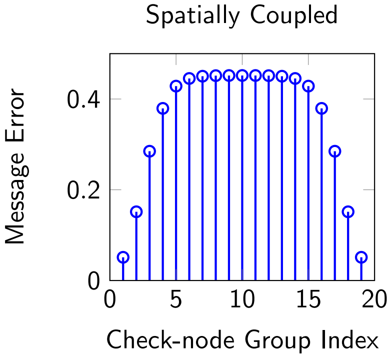

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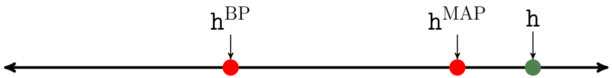

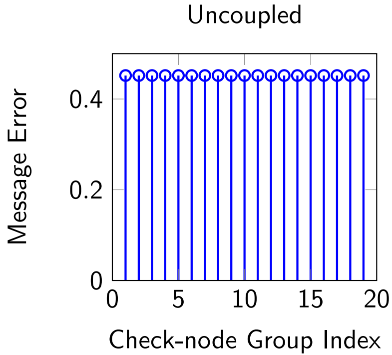

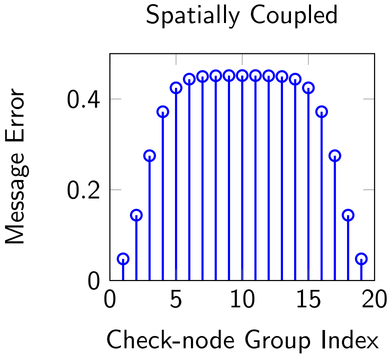

Threshold Saturation Result

MAP Performance with a BP Decoder!

For large N, w $\quad \mathtt{h}_{\mathrm{c}}^{\mathrm{BP}} = \mathtt{h}^{\mathrm{MAP}}$

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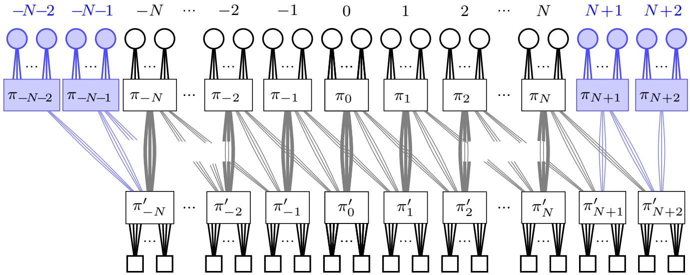
SC-LDPC (ℓ, r)	Shannon $\mathtt{h}^{\mathrm{Sh}}$	AWGN $\mathtt{h}_c^{\mathrm{BP}}$	BSC $\mathtt{h}_c^{\mathrm{BP}}$
$(3, 6)$	0.5000	0.4794	0.4681
$(4, 6)$	0.6667	0.6645	0.6633
$(5, 6)$	0.8333	0.8333	0.8333



The video link comes here

Rate loss for finite N and w

SC-LDPC (ℓ, r, N, w)	Shannon $\mathtt{h}^{\mathrm{Sh}}$	AWGN $\mathtt{h}_{\mathrm{c}}^{\mathrm{BP}}$	BSC $\mathtt{h}_{\mathrm{c}}^{\mathrm{BP}}$
$(3, 6, 10, 3)$	0.5434	0.4794	0.4681
$(3, 6, 20, 3)$	0.5222	0.4794	0.4681
$(3, 6, 30, 3)$	0.5149	0.4794	0.4681
$(4, 6, 10, 3)$	0.7245	0.6645	0.6633
$(4, 6, 20, 3)$	0.6963	0.6645	0.6633
$(4, 6, 30, 3)$	0.6866	0.6645	0.6633
$(5, 6, 10, 3)$	0.9056	0.8333	0.8333
$(5, 6, 20, 3)$	0.8704	0.8333	0.8333
$(5, 6, 30, 3)$	0.8582	0.8333	0.8333

Pros & Cons

Pros

- Significant improvment in thresholds
- Achieves capacity under simple BP decoding [KRU'11,KYMP'14]
- Universality - works for all channels models!

Cons

- Need large blocklengths to leverage the gains

Outline

Lattice

Lattice

Let $\mathbf{G} \in \mathbb{R}^{n \times k}$. An n-dimensional real lattice Λ can be defined as

$$\Lambda = \{\mathbf{G}\mathbf{z}, \mathbf{z} \in \mathbb{Z}^k\}$$

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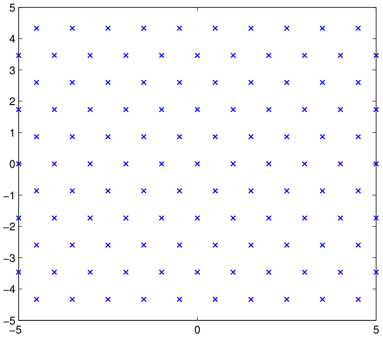
Example:

$$\mathbf{G} = \left[\begin{array}{cc} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{array} \right]$$



Lattices and Lattice Codes

Lattice

Let $\mathbf{G} \in \mathbb{R}^{n\times k}$. An n-dimensional real lattice Λ can be defined as

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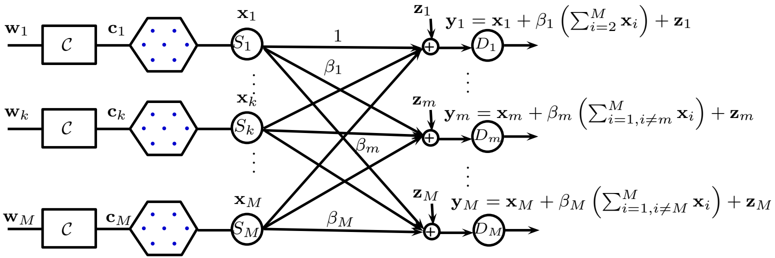
- Efficient structures for
  - Mathematics: sphere packing and sphere covering problems
  - Information Theory: channel coding & quantization
- Single user Gaussian channel - Erez and Zamir
- Coding with side information - Wyner-Ziv and Costa, Zamir, Erez and Shamai
- Secrecy - He and Yener
- Dirty multiple access channel - Philosof, Khisti, Erez and Zamir

"Lattices are everywhere" by Ram Zamir

Prior Work

New perspectives for dealing with interference:

- Interference alignment - Sridharan, Jafarian, Vishwanath and Jafar



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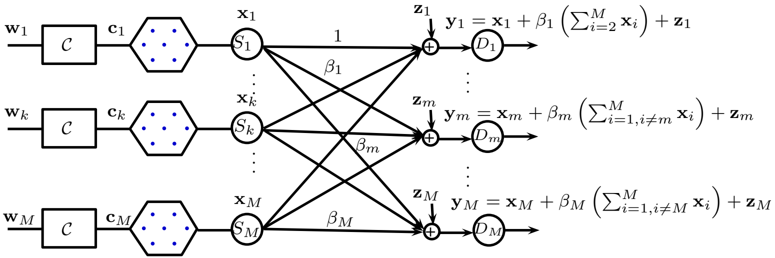

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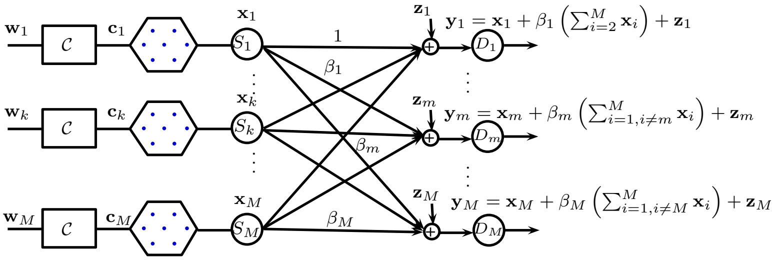

- Above schemes are all based on lattices good for channel coding

Background on lattices

Voronoi region

The *fundamental Voronoi region* $\mathcal{V}$ of a lattice, is the set of all points in $\mathbb{R}^n$ that are closest to the zero vector.

$$\mathcal{V} := \{\mathbf{x} : \|\mathbf{x} - \mathbf{0}\| \leq \|\mathbf{x} - \mathbf{c}\| \quad \forall \mathbf{c} \in \Lambda\}$$

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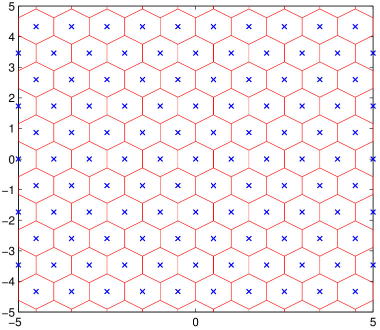

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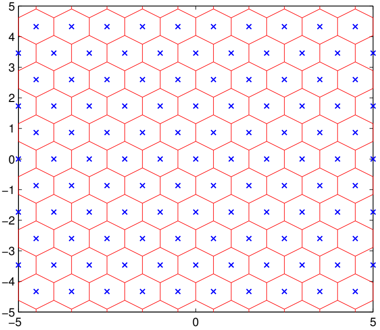

- Fundamental volume of Λ, $V(\Lambda)$: $\mathrm{Vol}(\mathcal{V})$

Goodness of Lattices for Channel Coding

- Let a lattice point $\lambda \in \Lambda$ is transmitted via AWGN channel of variance σ^2
- Volume-to-noise ratio(VNR) of Λ:

$$\mathrm{VNR} := \frac{V(\Lambda)^{2/n}}{2\pi e \sigma^2}$$

- $P(\Lambda, \sigma^2) := \Pr(d(\lambda, \lambda + \mathbf{z}) \geq d(\lambda', \lambda' + \mathbf{z}))$ for some $\lambda' \in \Lambda$

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Poltyrev Goodness for Channel Coding

For any VNR> 1 $\exists\{\Lambda_n\}$ such that $P(\Lambda_n, \sigma^2) \to 0$ as $n \to \infty$.

- *Poltyrev*-good lattices are at the core of such lattice coding schemes

Objective

Motivating questions

- All the existing results were based on Construction-A.
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Main Result

- Codes over $\mathbb{F}_2$ *and BP decoding* suffice
- We show existence of sequence of lattices that are *Poltyrev*-good under BP
- Apply proposed lattices to Symmetric Interference Channel
- Can be applied to other problems which adopt Construction-A lattices

Construction D with L levels

- Barnes and Sloane '83, Forney, Chung and Trott '00, Yan, Ling, Wu ' 13
- Choose $G_1 \subseteq \ldots \subseteq G_L$ where G_l is a gen matrix of code $\mathcal{C}_l$ over $\mathbb{F}_2$.

$$
\left[\begin{array}{ccc}
\cdots & g_L & \cdots \\
\cdots & \vdots & \cdots \\ \hline
\cdots & g_{k_2} & \cdots \\
\cdots & \vdots & \cdots \\ \hline
\cdots & g_{k_1} & \cdots \\
\cdots & \vdots & \cdots \\
\cdots & g_2 & \cdots \\
\cdots & g_1 & \cdots
\end{array}\right]
$$

G_1 G_2 $\ldots$ G_L

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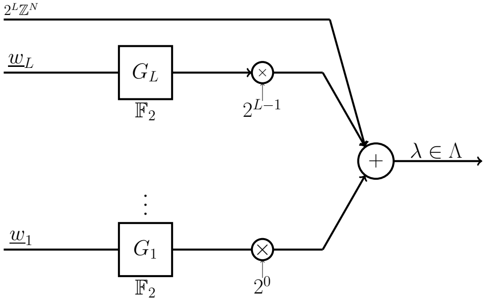
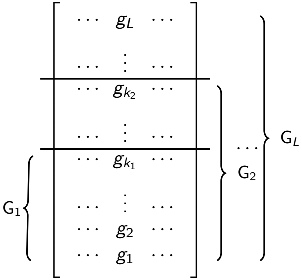
- $\underline{\lambda} = \underline{w}_1 \mathbf{G}_1 + 2\underline{w}_2 \mathbf{G}_2 \ldots + 2^{L-1} \underline{w}_{L-1} \mathbf{G}_{L-1} + 2^L \mathbb{Z}^N \in \Lambda$





Multi-Level Decoding(Successive Cancellation)

- $\underline{y} = \boxed{\underline{w}_1\mathbf{G}_1 + 2\underline{w}_2\mathbf{G}_2 \ldots + 2^{L-1}\underline{w}_{L-1}\mathbf{G}_{L-1} + 2^L\mathbb{Z}^N} + \underline{n}$

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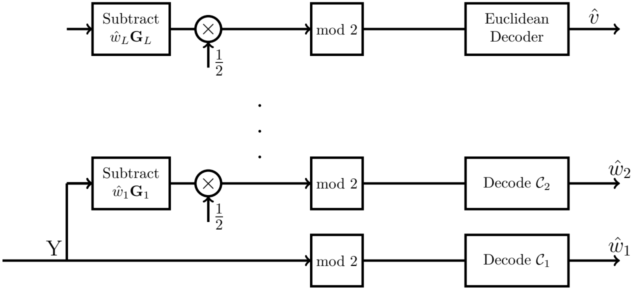
- $\underline{y} \bmod 2 = [\underline{w}_1\mathbf{G}_1 + \underline{n}] \bmod 2 = \underline{w}_1 \odot \mathbf{G}_1 + \boxed{\underline{n} \bmod 2}$
- Decode $\underline{w}_1$, reconstruct $\underline{w}_1\mathbf{G}_1$ and subtract from $\underline{y}$



Theorem (Forney, Trott & Chung)

There exists a sequence of Construction D lattices based on $\mathcal{C}_1 \subseteq \mathcal{C}_2 \ldots \subseteq \mathcal{C}_L$ such that the VNR $\to 1$ and the $Pr(\lambda, \sigma^2) \to 0$.

- Take L large enough.
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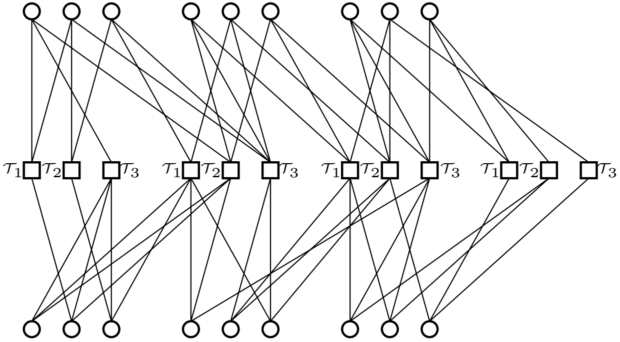
Objective:

- Capacity achieving nested code constructions, preferably under BP decoding.

Proposed Nested Spatially-Coupled LDPC Ensemble

1. Begin with a (d_v^1, d_c) SC LDPC code. For ex, $(d_v^1 = 3, d_c = 6, L = 3, w = 2)$.
2. Group check nodes into type $\mathcal{T}_k$, $k \in \{1, \ldots, d_v^1\}$



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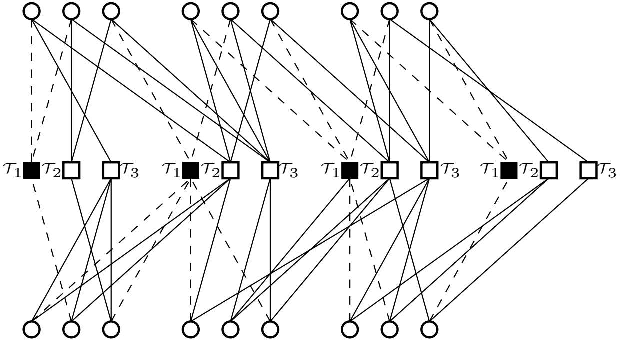

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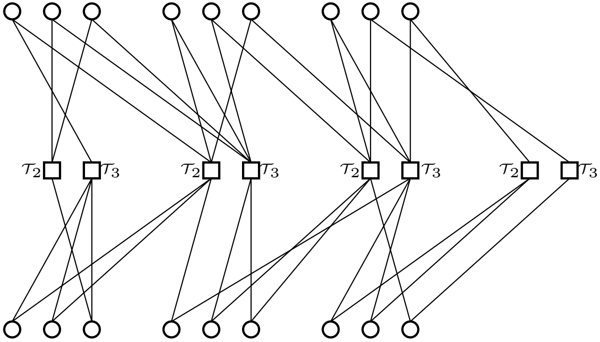
4. Results in a super-code that is a (d_v^2, d_c) SC LDPC code.



Lattice Design based on the proposed Nested SC LDPC ensemble

1. For a given σ, compute the capacity of the mod-2 AWGN channel at each level:

$$\underline{y}_i = \underline{w}_i \mathbf{G}_i + \frac{1}{2^{i-1}} \underline{n} \mod 2 = \underline{w}_i \odot \mathbf{G}_i + \boxed{\frac{1}{2^{i-1}} \underline{n} \mod 2}$$

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Lemma

Given nested binary linear codes $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots \subseteq \mathcal{C}_r$ there exists nested generator matrices for these codes.

Proposed Ensemble is Capacity achieving

Theorem

Each code ensemble in the proposed nested Spatially-Coupled LDPC ensemble is capacity achieving.

Proof.

- Each derived protograph has the same spatially coupled structure.
- Show that the mod 2 AWGN channel is BMS.
- The proof follows from [KRU'12] & [KYMP'13]'s results.

□

Proposed Lattices are Poltyrev-Good

Theorem

There exists a sequence of SC LDPC lattices with $VNR(\Lambda, \sigma^2) \to 1$ *for which, under multistage BP decoding,* $\mathbb{E}\left[P(\lambda, \sigma^2)\right] \to 0$ *as* $w, L, M \to \infty$.

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- The proposed nested ensemble achieve capacity.
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□

- Binary codes and more importantly practical BP decoding suffices.
- Practically we observe that two levels of coding gets you lattices very close to Poltyrev limit.

Design Example of Poltyrev-Good Lattice

Target error probability $P(2^L\mathbb{Z}^n, \sigma_L^2) = 10^{-4}$ in the uncoded level $\implies \sigma_L = 0.08$

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σ_{eff}	0.16	0.32	0.64
Cap	0.99	0.57	0.02
(14,30) (3,30)	0.9	0.533	0

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2. Fix L=3 and use $(3, 30)$, $(14, 30)$ nested SC-LDPC codes.
    - Note $P(4\mathbb{Z}^n, \sigma^2) \approx nP(4\mathbb{Z}, \sigma^2)$
    - We fix $n = 2 \times 10^5$

(d_c, d_v^1, d_v^2)	(L,w)	$P(4\mathbb{Z}, \sigma^2)$	$\sigma_{\max}$	VNR	$\text{VNR}_{\text{rate-loss}}$
(30,14,3)	(32,4)	5×10^{-10}	0.3184	1.02dB	1.347dB

Design Example of Poltyrev-Good Lattice

Target error probability $P(2^L\mathbb{Z}^n, \sigma_L^2) = 10^{-4}$ in the uncoded level $\implies \sigma_L = 0.08$

1. Capacities for the mod 2 AWGN channel for respective levels:

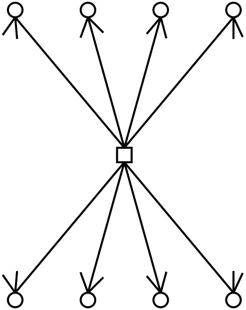
	Level L-1	Level L-2	Level L-3
σ_{eff}	0.16	0.32	0.64
Cap	0.99	0.57	0.02
(14,30) (3,30)	0.9	0.533	0

2. Fix L=3 and use $(3, 30)$, $(14, 30)$ nested SC-LDPC codes.
   - Note $P(4\mathbb{Z}^n, \sigma^2) \approx nP(4\mathbb{Z}, \sigma^2)$
   - We fix $n = 2 \times 10^5$

(d_c, d_v^1, d_v^2)	(L,w)	$P(4\mathbb{Z}, \sigma^2)$	$\sigma_{\max}$	VNR	$\text{VNR}_{\text{rate-loss}}$
(30,14,3)	(32,4)	5×10^{-10}	0.3184	1.02dB	1.347dB
(60, 26, 3)	(72, 12)	5×10^{-10}	0.3200	0.482dB	0.927dB
(60, 27, 3)	(64, 9)	5×10^{-10}	0.3203	0.57dB	0.951dB

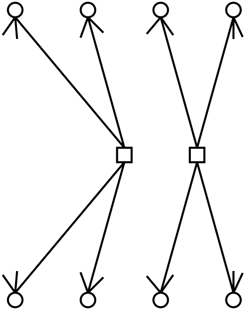
Alternate Nested SC LDPC ensemble

- Derive a lower rate code by "splitting the checks"
- Consider a $(3, 8)$ code

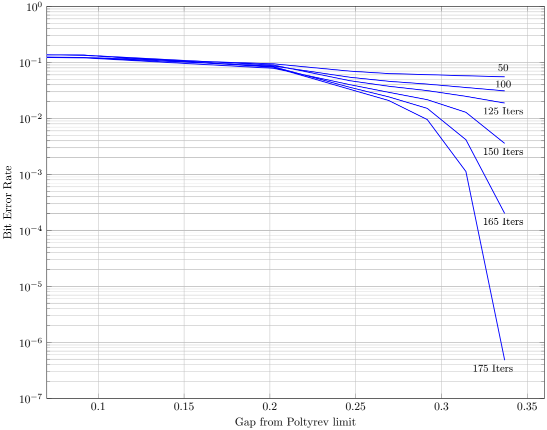


Alternate Nested SC LDPC ensemble

- Derive a lower rate code by “splitting the checks”
- Consider a $(3, 8)$ code
- Split each check into “two” checks to derive a $(3, 4)$ sub-code
- Easy to prove that resulting code is from the $(3, 4)$ SC LDPC ensemble



Simulation Results



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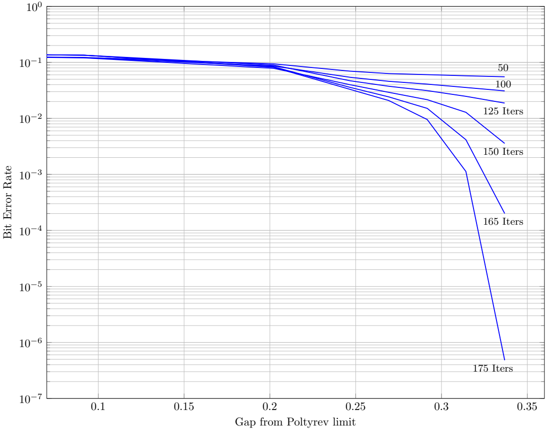
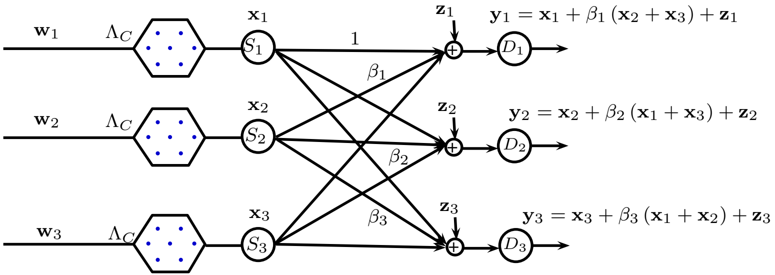


Note that the Block Error Probability is 10^{-4} at uncoded level.

3-User Symmetric Interference Channel



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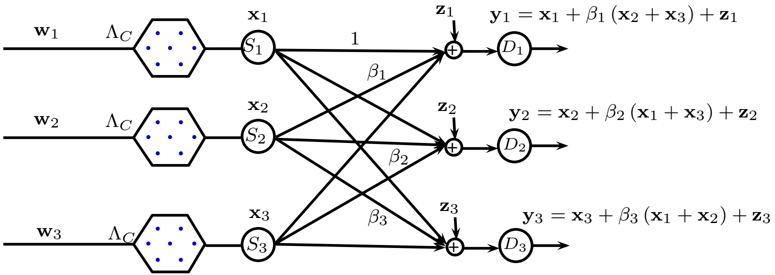


- $\mathbf{x}_i \in \Lambda_C \triangleq \Lambda \cap \mathbb{Z}_4^N$ is transmitted.

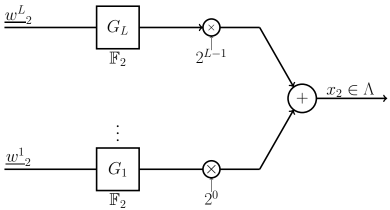
Symmetric Interference Channel - Decoding Sums

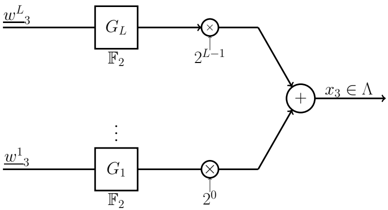
Interference at Destination 1:

$$\begin{aligned}\mathbf{x}_2 + \mathbf{x}_3 &= (\underline{w}_2^1 + \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{w}_2^2 + \underline{w}_3^2)\mathbf{G}_2 + 4\mathbf{k}_{23} \\ &= (\underline{w}_2^1 \oplus \underline{w}_3^1)\mathbf{G}_1 + 2(\underline{c}_{23}^1 \oplus \underline{w}_2^2 \oplus \underline{w}_3^2)\mathbf{G}_2 + 4(\underline{c}_{23}^2 + \mathbf{k}_{23})\mathbf{Z}\end{aligned}$$

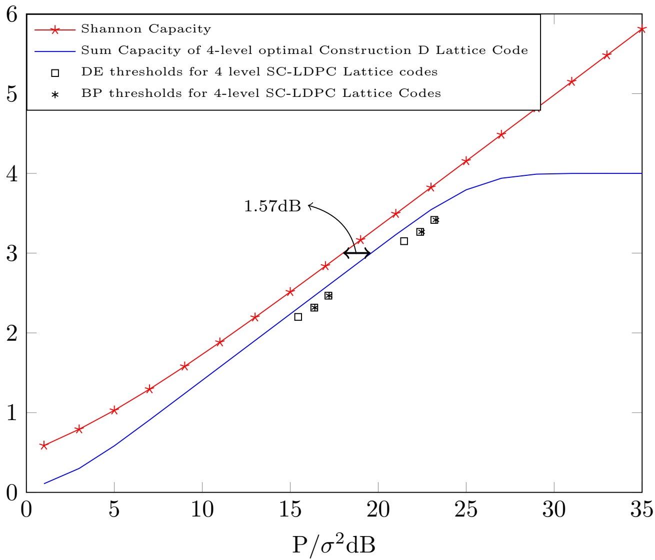
where the carry overs are

$$\begin{aligned}\underline{c}_{23}^1 &= 0.5\left(\underline{w}_1^1 + \underline{w}_1^2 - \underline{w}_1^1 \oplus \underline{w}_1^2\right), \\ \underline{c}_{23}^2 &= 0.5\left(\underline{c}_{23}^1 + \underline{w}_2^1 + \underline{w}_2^2 - \underline{c}_{23}^1 \oplus \underline{w}_2^1 \oplus \underline{w}_2^2\right)\end{aligned}$$





Achievable Information Rates



Concluding Remarks

- Multilevel constructions - efficient ways to decode integer combinations
- Need capacity achieving nested codes
- Multilevel construction is provably good under message passing decoding

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- Need capacity achieving nested codes
- Multilevel construction is provably good under message passing decoding
- Coding schemes based on *binary codes and iterative decoding* suffice

Outline

Lossy Source Coding Problem

$X^n = (X_1, \cdots, X_n)$, $X_i \sim \text{Bernoulli}(\frac{1}{2})$

Binary code $\mathcal{C} = (n, k)$, rate $R = k/n$

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Lossy Source Coding

- Compress X^n to $\hat{X}^n \in \mathcal{C}$
- Min. Hamming distortion

$$D = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}|X_i - \hat{X}_i|$$

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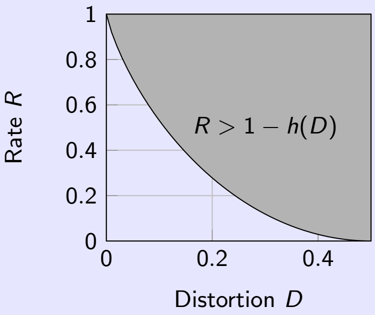
- Rate-Distortion theory:

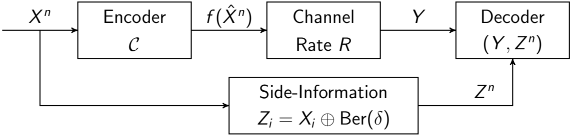
$$R > 1 - h(D)$$

- $h(\cdot)$ is binary entropy function

$$h(D) = -D\log_2 D - (1-D)\log_2(1-D)$$



Side-Information Problems: Wyner-Ziv



Wyner-Ziv Formulation

- Side-information Z^n about X^n
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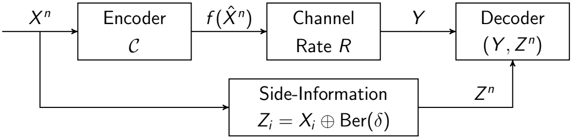

Wyner-Ziv Formulation

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- Say $Z_i = X_i \oplus \mathrm{Ber}(\delta)$
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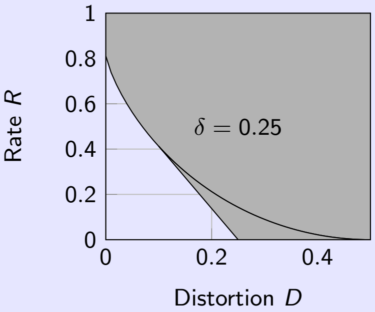
$$R > l.c.e\{h(D * \delta) - h(D), (\delta, 0)\}$$

- $D * \delta = D(1-\delta) + \delta(1-D)$



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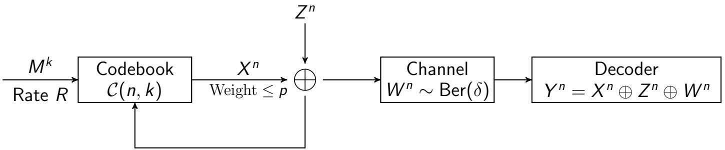

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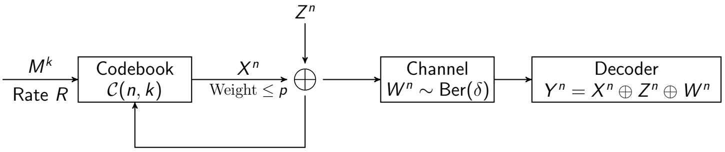

Gelfand-Pinsker Formulation

- Message M^k encoded to $X^n \in \mathcal{C}$ with $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] \leq p \leq \frac{1}{2}$
- Side-information Z^n is available *only at the encoder*
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$$Y^n = X^n \oplus Z^n \oplus W^n, \quad \{W_i\} \sim \text{Ber}(\delta)$$

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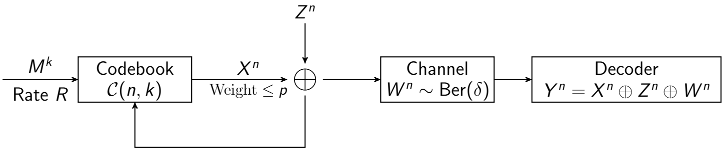

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- Capacity region by Gelfand-Pinsker:

$$R < h(p) - h(\delta)$$

Main Result

Objective

- Construct low-complexity coding schemes that achieve the complete rate regions of Wyner-Ziv and Gelfand-Pinsker
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Idea

- Wainwright et al. used compound LDGM/LDPC codes with *optimal encoding/decoding*
- Message-passing algorithms have *non-negligible gap*
- Remedy via Spatial-Coupling
  - Channel coding in coupled compound codes (Kasai et al.)
  - Lossy source coding with spatially-coupled LDGM (Aref et al.)
  - Encoding with compound codes has additional challenges

An (ℓ, r) LDGM Code

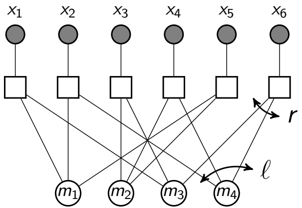
Generator Matrix

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$\ell = 3$ $\qquad$ $r = 2$

LDGM Code $\mathcal{C} = \{x : x = m \odot G\}$

Tanner Graph



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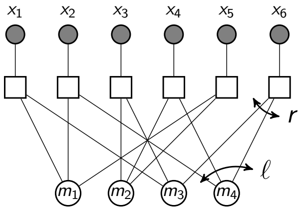
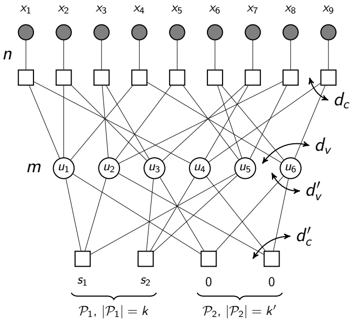


$$x_1 = m_1 \oplus m_3 \iff x_1 \oplus m_1 \oplus m_3 = 0$$

Compound LDGM/LDPC Codes



- Codebook $\mathcal{C}(n, m-k-k')$
- Message constraints

$$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword $(x_1, \cdots, x_9)$:

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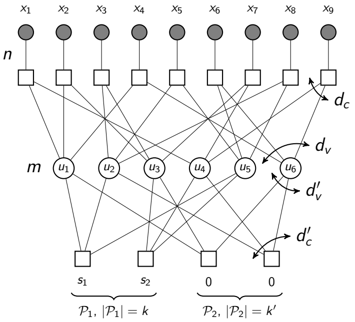

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Key Properties

- Compound code is
  - a good source code under optimal encoding
  - a good channel code under optimal decoding
- LDGM code is
  - a good source code under optimal encoding
  - (side note) LDGM code is *not* a good channel code

Good Code

“Good” source code

- Rate of the code is $R = 1 - h(D) + \varepsilon$
- When this code is used to *optimally encode* $\mathrm{Ber}(\frac{1}{2})$ source
- The average Hamming distortion is at most D

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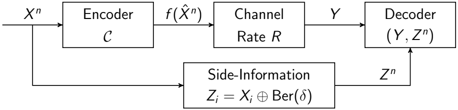
"Good" channel code

- Rate of the code is $R = 1 - h(\delta) - \varepsilon$
- When this code is used for channel coding on $\mathrm{BSC}(\delta)$
- Message est. under *optimal decoding* with error at most ε

Coding Scheme: Wyner-Ziv



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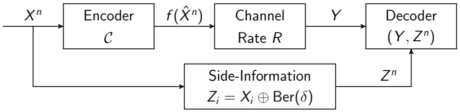

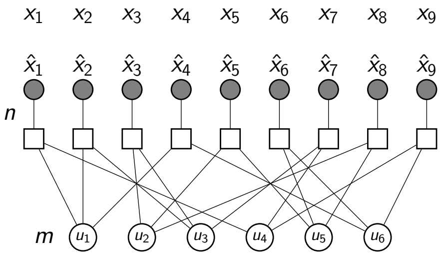

- Encode X^n to $\hat{X}^n$ using *LDGM* w/ Distortion $\approx D$

$$\frac{m}{n} \approx 1 - h(D) + \varepsilon$$

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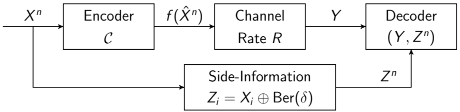
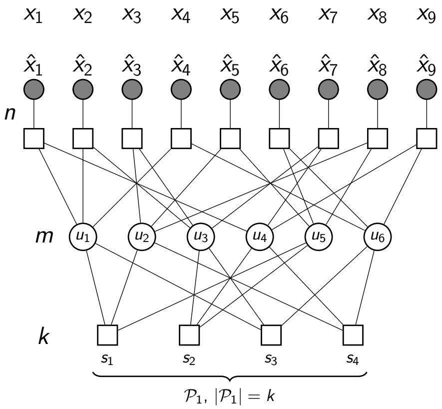




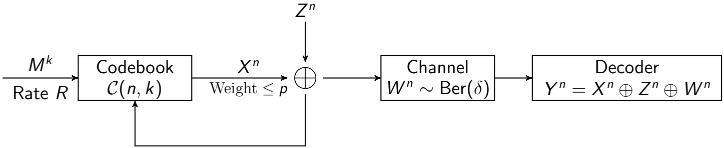
- Encode X^n to $\hat{X}^n$ using *LDGM* w/ Distortion $\approx D$
- Compute & transmit s_i's
  $R = \frac{k}{n} \approx h(D * \delta) - h(D)$
- Decoder has Z^n:

$$\begin{aligned} Z_i &= X_i \oplus \mathrm{Ber}(\delta) \\ &\approx \hat{X}_i \oplus \mathrm{Ber}(D) \oplus \mathrm{Ber}(\delta) \\ &= \hat{X}_i \oplus \mathrm{Ber}(D * \delta) \end{aligned}$$

- Decode $\hat{X}^n$ from Z^n & s_i

$$\frac{m}{n} \approx 1 - h(D) + \varepsilon \qquad \frac{m-k}{n} \approx 1 - h(D * \delta) + \varepsilon$$

Coding Scheme: Gelfand-Pinsker



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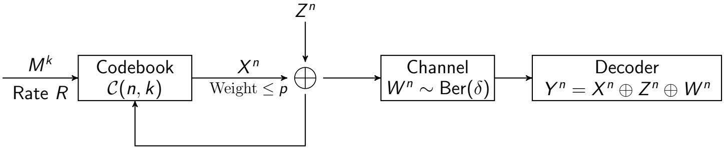

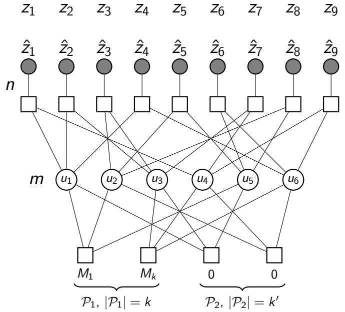

$\frac{m-k-k'}{n} \approx 1 - h(p) + \varepsilon$ $\quad$ $\frac{m-k'}{n} \approx 1 - h(\delta) + \varepsilon$

- With message M^k, encode Z^n to $\hat{Z}^n$ (Distortion $\approx p$)
- Transmit $X^n = Z^n \oplus \hat{Z}^n$

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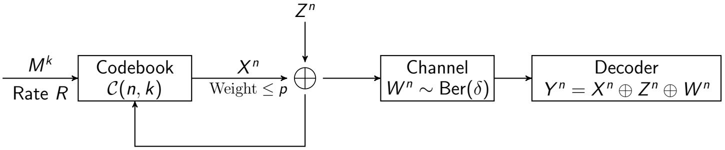

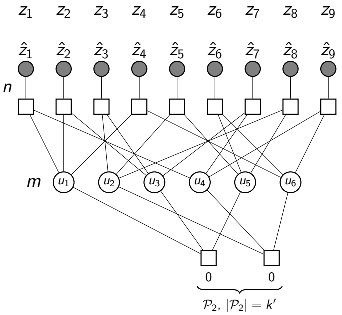

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- Decoder has

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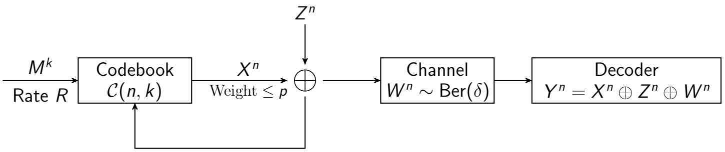

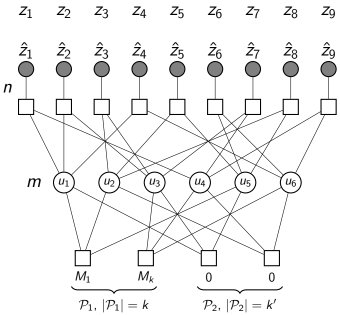

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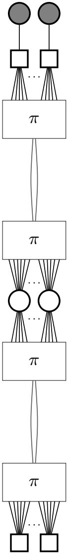
- Decode $\hat{Z}^n$ and compute M^k
- $R = \frac{k}{n} \approx h(p) - h(\delta)$

Remarks

- Need codes that are *simultaneously good* for channel and source coding
- Use message-passing algorithms instead of *optimal*
- Use spatial-coupling for *goodness* of codes under message-passing

Spatially-Coupled Compound LDGM/LDPC Codes



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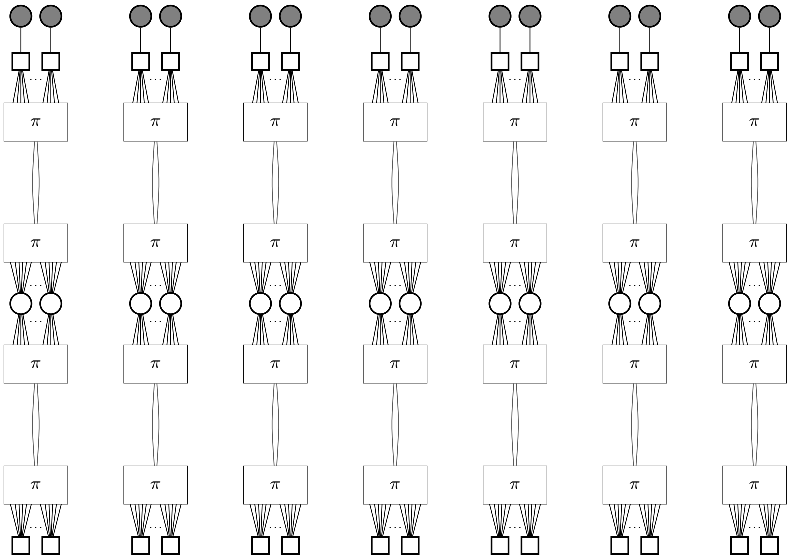

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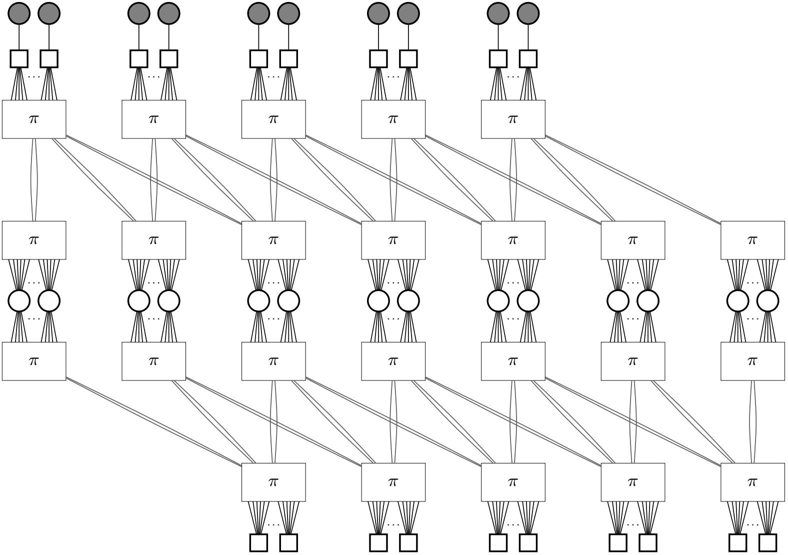

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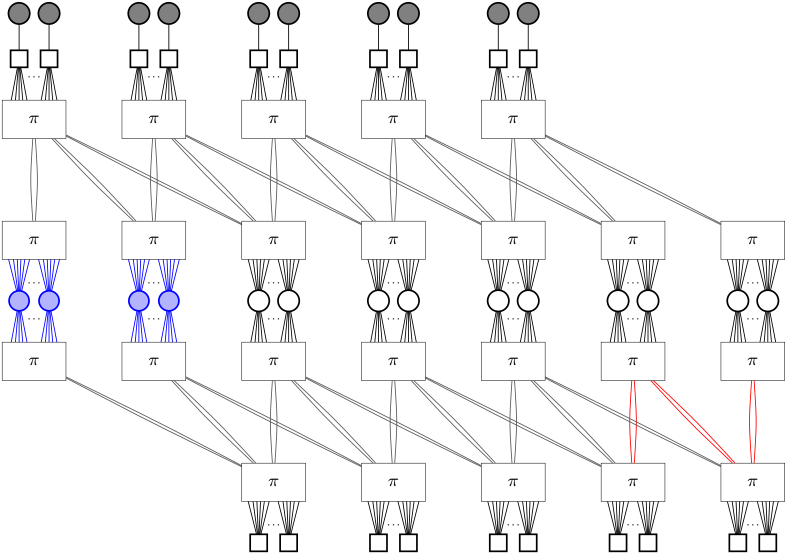
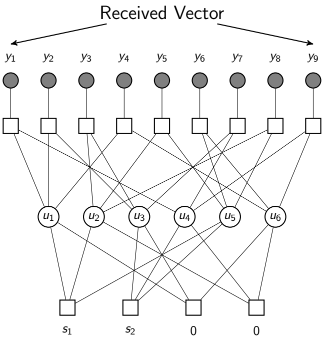
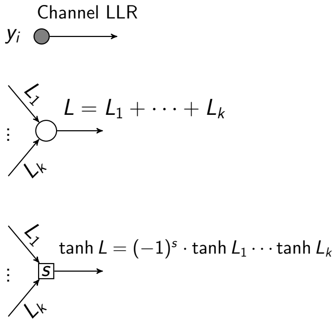


Decoding in Spatially-Coupled Compound Codes

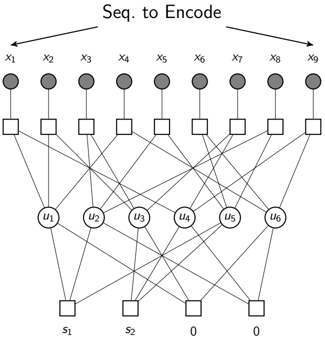


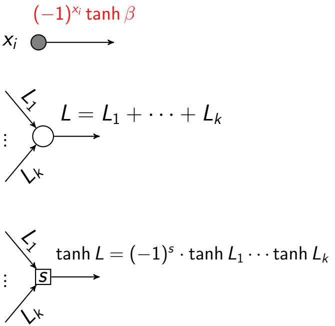


Remarks

- Standard message-passing algorithm
- Threshold saturation proven for SC compound codes on BEC
- Empirically observed for BMS channels

Encoding in Spatially-Coupled Compound Codes





Remarks

- Inverse temperature parameter β
- Message-passing rules are the same
- However, a *crucial decimation step is needed*

Encoding in SC Compound Codes: BPGD Algorithm

while There are active LDPC bit-nodes **do**
  for $t = 1$ to T **do**
    Run the BP equations
  end for
  Evaluate LLRs m_i for each LDPC bit-node
  Choose max. of $|m_i|$ in *left-most w active sections*
  if $|m_{i^*}| = 0$ **then**
    Set u_{i^*} to 0 or 1 uniformly randomly
  else
    Set u_{i^*} to 0 or 1 with *prob.* $\frac{1+\tanh m_{i^*}}{2}$ *or* $\frac{1-\tanh m_{i^*}}{2}$
  end if
  Decimate (remove) LDPC bit-node i^* and update parities
end while
If $\{u_i\}$ fail to satisfy LDPC checks, then **re-encode**

Encoding in SC Compound Codes: Remarks

- Randomization in setting u_{i^*} is crucial
- BPGD applied to uncoupled code *always failed*
- Spatially-coupled structure is crucial for successful encoding
  - In addition, distortion is close to optimal thresholds
  - *Does not encode* if decimated from both left and right
  - *Does not encode* if both left and right boundary is set to 0

Encoding in SC Compound Codes: Numerical Example

Block length (n)	4-cycles	Attempts $1/2/3/4/\geq 5$
9000	yes	5/3/5/2/35
9000	no	21/12/5/3/9
27000	no	35/15/0/0/0
45000	no	40/9/0/0/1
63000	no	44/6/0/0/0
81000	*no*	50/0/0/0/0

Remarks

- \# Attempts to encode 50 seq. in $(6,3)$ LDGM / $(3,6)$ LDPC
- $L = 20$, $w = 4$, $\beta = 0.65$, $T = 10$
- Removing *4-cycles* dramatically improves success
- How much do 6-cycles matter?

Numerical Results: Wyner-Ziv

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(D_*, δ_*)	(D, δ)
$(6, 3)$	(3,6)	(20,4)	(0.111,0.134)	(0.1174,0.122)
$(8, 4)$	(3,6)	(20,4)	(0.111,0.134)	(0.1149,0.120)
$(10, 5)$	(3,6)	(20,4)	(0.111,0.134)	(0.1139,0.122)

Remarks

- D_* and δ_* are calculated based on the rate of the respective code:

$$D_* = h^{-1}(1 - R1) \qquad \delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000,\ \beta = 1.04,\ T = 10$

Numerical Results: Gelfand-Pinsker

LDGM (d_v, d_c)	LDPC (d'_v, d'_c)	(L, w)	(p_*, δ_*)	(p, δ)
$(6, 3)$	(3,6)	(20,4)	(0.215,0.157)	(0.2200,0.152)
$(8, 4)$	(3,6)	(20,4)	(0.215,0.157)	(0.2230,0.151)
$(10, 5)$	(3,6)	(20,4)	(0.215,0.157)	(0.2200,0.151)

Remarks

- p_* and δ_* are calculated based on the rate of the respective code:

$$p_* = h^{-1}(1 - R1) \qquad\qquad \delta_* = h^{-1}(1 - R2)$$

- $n \approx 140000,\ \beta = 0.65,\ T = 10$

Concluding Remarks

Conclusion

- Spatially-coupled codes achieve the rate regions of Wyner-Ziv and Gelfand-Pinsker problems
- Coupling structure is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed with decimation

Open Questions

- Effect of degree profiles, short-cycles on encoding success
- Precise trade-offs with polar codes

Outline

Write-Once Memories

Flash Memory

- In typical flash memory, changing from 0 to 1 is easy
- Resetting 1 to 0 requires rewriting whole block
- Write-once memories model such storage systems

Write-Once Memories

0	0	0	0	0	0	0	0	0

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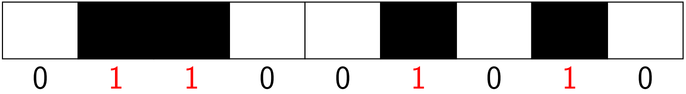
Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed

Write-Once Memories



Flash Memory

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Binary Write-Once Memories

- $0 \longrightarrow 1$ is allowed
- $1 \longrightarrow 0$ *is forbidden*

Capacity Region (I) - Noiseless

Message

0	0	1	0	0	1	0	1	0

Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
    - 2 bits in 2 writes with only 3 cells

$2 + 2$ bits in 2-write WOM

x	$r(x)$	$r'(x)$
00		
01		
10		
11		

$2 + 2$ bits in 2-write WOM

x	$r(x)$	$r'(x)$
00	000	
01	001	
10	010	
11	100	

$2 + 2$ bits in 2-write WOM

x	$r(x)$	$r'(x)$
00	000	111
01	001	110
10	010	101
11	100	011

Capacity Region (I) - Noiseless

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Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
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- Only about $nt/\log(t)$ cells required to store n bits for t writes

Capacity Region (I) - Noiseless

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$\downarrow$

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Write-Once Memory without Noise

- In 1982, Rivest and Shamir gave first WOM codes
  - 2 bits in 2 writes with only 3 cells
- Only about $nt/\log(t)$ cells required to store n bits for t writes
- In 1985, Heegard gave the *capacity* for t-write system
- For a 2-write system, it is

$$\left\{(R_1, R_2) \mid 0 \leq R_1 < h(\delta),\ 0 \leq R_2 < 1 - \delta\right\}$$

Capacity Region (II) - Read Errors

Message

$\downarrow$

0	0	1	0	0	1	0	1	0

$\downarrow$ Ber(p)

0 0 0 0 0 1 1 1 0

Write-Once Memory with Read Errors

- Different from write errors
- $Y = X \oplus \text{Ber}(p)$, where Ber($p$) denotes the Bernoulli noise
- Capacity region is *unknown*

Main Result

Objective

- Construct *low-complexity* coding schemes that achieve the capacity region of the WOM system
  - Low-complexity encoding and decoding

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    - For read errors, achieves

$$R_1 < h(\delta) - h(p), \qquad R_2 < 1 - \delta - h(p).$$

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- Extension to multi-write systems *seems possible with BPGD*

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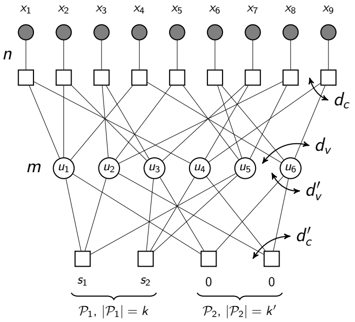
- Extension to multi-write systems *seems possible with BPGD*

Idea

- Use compound LDGM/LDPC codes
- Encoding for second write is *erasure quantization*
- Use spatial coupling with message-passing

Compound LDGM/LDPC Codes



- Codebook $(n, m - k - k')$
- Message constraints

  $$u_1 \oplus u_2 \oplus u_5 = s_1, \quad u_1 \oplus u_3 \oplus u_6 = 0$$

- Codeword $(x_1, \cdots, x_9)$:

  $$x_1 = u_1 \oplus u_4, \qquad x_2 = \cdots$$

- Parametrized by s^k: $\mathcal{C}(s^k)$

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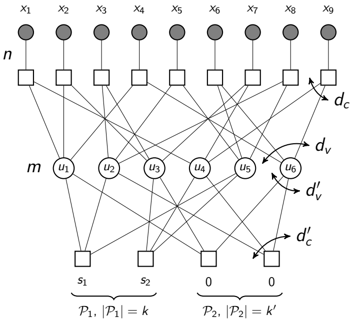

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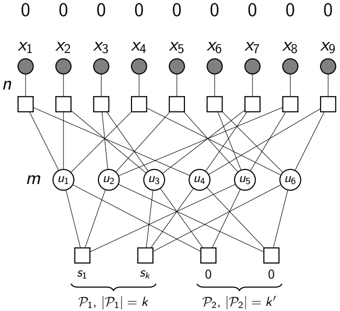
Key Properties of Compound Codes

- a natural coset decomposition: $\mathcal{C} = \bigcup_{s^k \in \{0,1\}^k} \mathcal{C}(s^k)$
- achieves capacity over eras. chan. under MAP (when $m = n$)
- a good source code under optimal encoding
- a good channel code under optimal decoding

Coding Scheme for 2-write WOM: First Write

$$R_1 < h(\delta) - h(p)$$



$\frac{m-k-k'}{n} \approx 1 - h(\delta)$ $\frac{m-k'}{n} \approx 1 - h(p)$

- With message s^k, encode 0^n to x^n (Distortion $\approx \delta$)
- Store x^n

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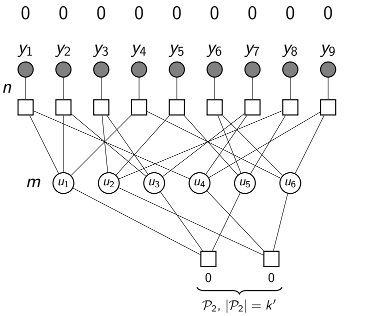

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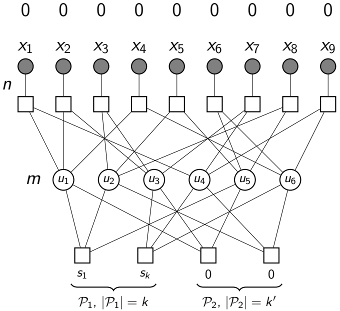


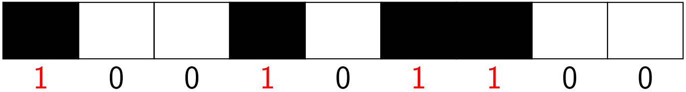
$\frac{m-k-k'}{n} \approx 1 - h(\delta)$ $\qquad$ $\frac{m-k'}{n} \approx 1 - h(p)$

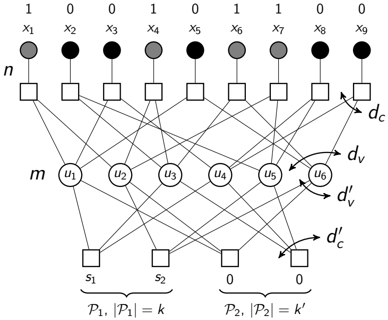
- With message s^k, encode 0^n to x^n (Distortion $\approx \delta$)
- Store x^n
- Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

- Dec. x^n and compute s^k
- $R_1 = \frac{k}{n} \approx h(\delta) - h(p)$

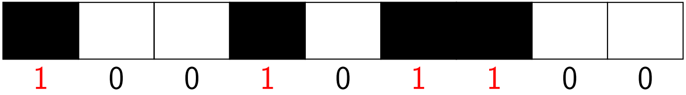
Coding Scheme for 2-write WOM: Second Write

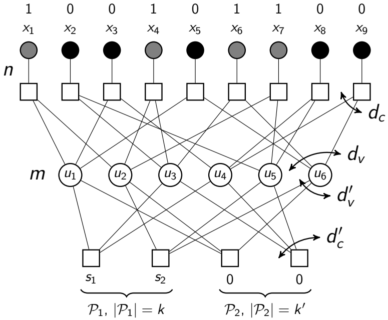




- Need to find a *consistent* codeword in $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write





- Need to find a *consistent* codeword in $\mathcal{C}(s^k)$
- Closely related to Binary Erasure Quantization (BEQ)
- En Gad, Huang, Li and Bruck (ISIT 2015)

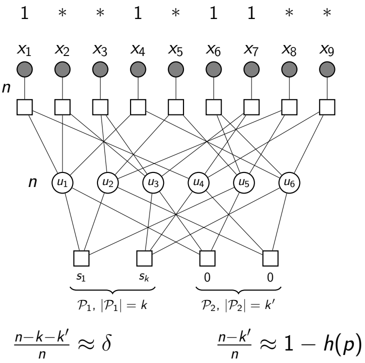
Binary Erasure Quantization

- Quantize a sequence in $\{0, 1, *\}^n$ to $x^n \in \mathcal{C} \subset \{0, 1\}^n$
  - 0's and 1's should *match exactly*
  - $*$'s can take either 0 or 1
- Can map the second write of 2-write WOM to BEQ
  - Map 0's to $*$'s and keep 1's
  - Quantize to codeword in $\mathcal{C}(s^k)$
- BEQ is the dual of decoding on binary erasure channel
  - Martinian and Yedidia (Allerton 2003)
  - Can quan. all seq. with erasure pattern $e^n \in \{0, 1\}^n$ to $\mathcal{C}$

    $\Updownarrow$

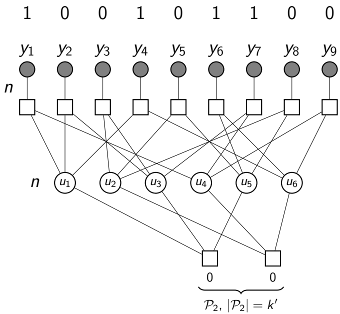
    Chan. dec. for $\mathcal{C}^\perp$ can correct all vectors with eras. $1^n \oplus e^n$
- Choose a good (dual) code $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write



- *Change 0's to $*$'s*
- With message s^k, encode seq. to $\mathcal{C}(s^k)$

Coding Scheme for 2-write WOM: Second Write



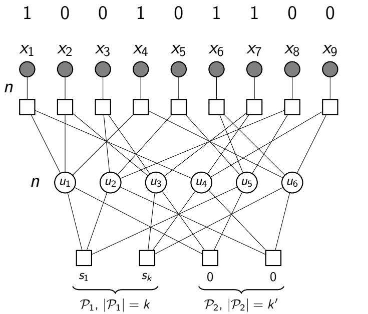
$$\frac{n-k'}{n} \approx 1 - h(p)$$

- *Change 0's to $*$'s*
- With message s^k, encode seq. to $\mathcal{C}(s^k)$
- Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

Coding Scheme for 2-write WOM: Second Write

$$R_2 < 1 - \delta - h(p)$$



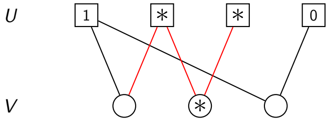
$\frac{n-k-k'}{n} \approx \delta$ $\frac{n-k'}{n} \approx 1 - h(p)$

- *Change 0's to $*$'s*
- With message s^k, encode seq. to $\mathcal{C}(s^k)$
- Decoder has

$$y_i = x_i \oplus \text{Ber}(p)$$

- Dec. x^n and compute s^k
- $R_2 = \frac{k}{n} \approx 1 - \delta - h(p)$

Iterative Erasure Quantization Algorithm

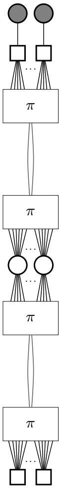


- *Peeling type encoder*

Remarks

- Need codes that are *simultaneously good* for channel/source coding and erasure quantization
- Use message-passing algorithms instead of *optimal*
- Use spatial-coupling for *goodness* of codes under message-passing

Spatially-Coupled Compound LDGM/LDPC Codes



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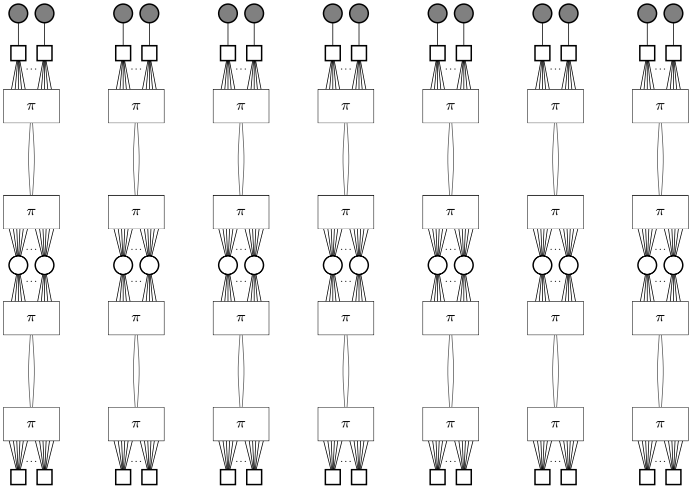

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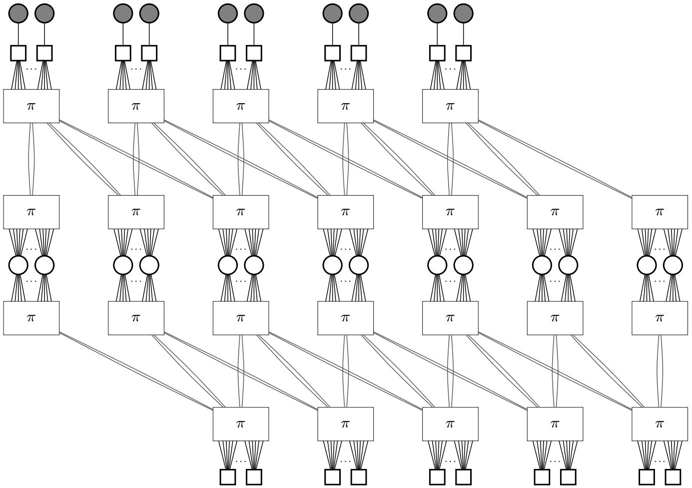

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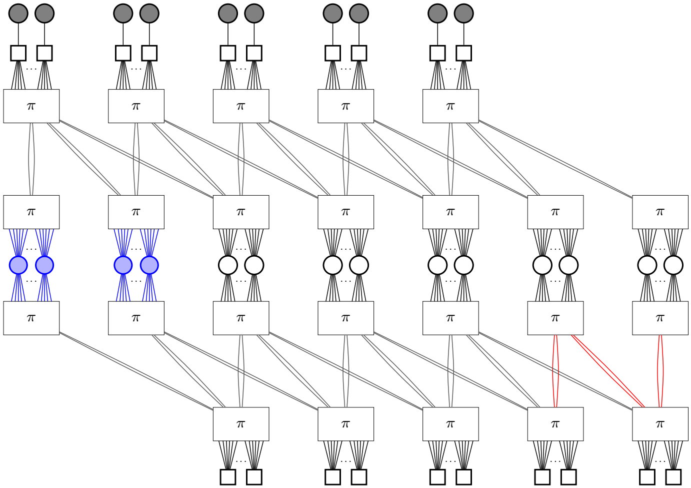
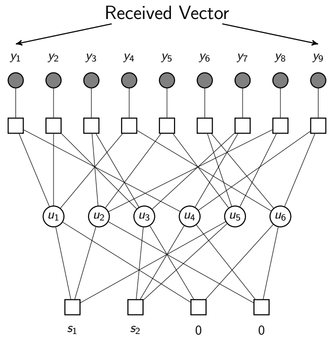
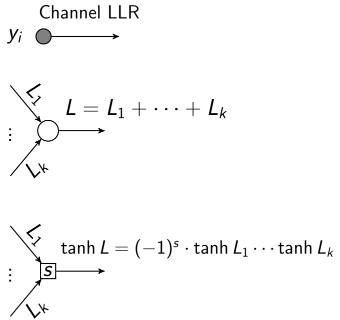


Decoding in Spatially-Coupled Compound Codes





Remarks

- Standard message-passing algorithm
- Threshold saturation proven for SC compound codes on BEC
- Empirically observed for BMS channels

Numerical Results: Noiseless WOM

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	δ^*	δ $w = 2$	δ $w = 3$	δ $w = 4$
$(3, 3, 3, 6)$	0.500	0.477	0.492	0.494
$(3, 3, 4, 6)$	0.333	0.294	0.324	0.326
$(3, 3, 5, 6)$	0.167	0.095	0.156	0.158
$(4, 4, 3, 6)$	0.500	0.461	0.491	0.492
$(4, 4, 4, 6)$	0.333	0.278	0.323	0.325
$(4, 4, 5, 6)$	0.167	0.086	0.155	0.159
$(5, 5, 3, 6)$	0.500	0.436	0.488	0.491
$(5, 5, 4, 6)$	0.333	0.260	0.320	0.324
$(5, 5, 5, 6)$	0.167	0.079	0.154	0.159

Remarks

- δ^* is the Shannon threshold
- $L = 30$, Single system length ≈ 24000

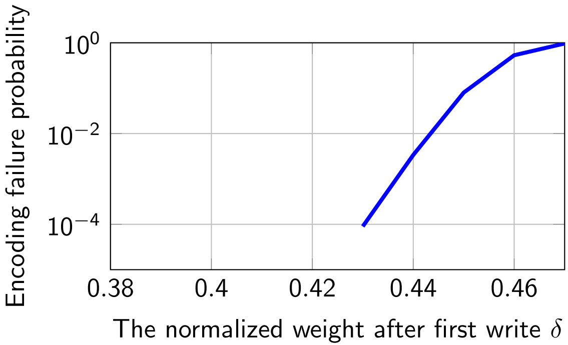
Numerical Results: WOM with Read Errors

LDGM/LDPC (d_v, d_c, d'_v, d'_c)	w	(δ^*, p^*)	(δ, p)
$(3, 3, 4, 6)$	3	$(0.333, 0.0615)$	$(0.321, 0.0585)$
$(3, 3, 4, 8)$	3	$(0.500, 0.0417)$	$(0.490, 0.0387)$
$(3, 3, 6, 8)$	4	$(0.250, 0.0724)$	$(0.239, 0.0684)$
$(4, 4, 4, 6)$	4	$(0.333, 0.0615)$	$(0.324, 0.0585)$
$(4, 4, 4, 8)$	4	$(0.500, 0.0417)$	$(0.492, 0.0387)$
$(4, 4, 6, 8)$	4	$(0.250, 0.0724)$	$(0.241, 0.0694)$

Remarks

- δ^* and p^* are the Shannon thresholds
- $L = 30$, Single system length ≈ 30000

Numerical Results: Small Blocklength



Remarks

- $(L, w) = (30, 3)$, Single system length 1200, Shannon threshold of 0.5
- A total of 10^5 were attempted to encode
- No failures for $\delta < 0.43$

Concluding Remarks

Conclusion

- Spatially-coupled compound codes achieve the capacity of 2-write systems
- Coupling structure is also crucial
  - to achieve optimum thresholds
  - for encoding to succeed

Multi-Write Systems

- Will BPGD work for multi-write systems?

Outline

- SC-LDPC Lattices [C1]
- SC-Compound Codes
  - Side-Information Problems [C2]
  - Coding for WOM [C3]
- Sparse graph coding tools for solving sparse recovery problems
  - Regular bipartite sparse graphs for compressed sensing [C5]
  - Group testing*
  - Pattern matching*
- Uncoordinated multiple access
  - Universal schemes for massive uncoordinated multiple access [C4]
  - Optimal distributions for finite user multiple access*
- Coding for low latency requirements*

C1. A. Vem, Y. C. Huang, K. R. Narayanan and H. D. Pfister, "Multilevel lattices based on spatially-coupled LDPC codes with applications", in *Proc. IEEE. ISIT*, pp. 2336–2340, 2014.
C2. S. Kumar, A. Vem, K. R. Narayanan and H. D. Pfister, "Spatially-coupled codes for side-information problems", in *Proc. IEEE. ISIT*, pp. 516–520, 2014.
C3. S. Kumar, A. Vem, K. R. Narayanan and H. D. Pfister, "Spatially-coupled codes for write-once memories", in *Proc. Allerton. Conf*, pp. 125–131, 2015.
C4. A. Taghavi, A. Vem, J.-F. Chamberland and K. R. Narayanan "On the design of universal schemes for massive uncoordinated multiple access", in *Proc. IEEE. ISIT*, pp. 345–349, 2016.
C5. A. Vem, N. T. Janakiraman, and K. R. Narayanan, "Sub-linear time compressed sensing for support recovery using left and right regular sparse-graph codes", in *Proc. IEEE. ITW*, pp. 429–433, 2016.

*-To be submitted