Putnam preparation

Inequalities, maxima and minima

Finding the maximum or minimum of a function: In the Putnam, it is important that your justification of a max or a min is rigorous. For instance, it is not enough to solve the equation $f'(x) = 0$ to find a maximum of a function f. Here are some techniques:

• Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then the maximum value of $f(x)$ is $f(x_0)$.

• Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then f achieves both its maximuma and its minimum on $[a, b]$. These must occur at points x_0 where either $f'(x_0) = 0$, $f'(x_0)$ does not exist, or $x_0 = a$ or b.

• You may be able to show directly for instance that $f(x) \leq M$ for some M, and that $f(x_0) = M$, in which case M is the maximum.

Basic tools for inequalities:

• If $a \leq b$ and $b \leq c$, then $a \leq c$ with equality if and only if $a = b = c$.

• If $a_1 \leq b_1$ and $a_2 \leq b_2$, then $a_1 + a_2 \leq b_1 + b_2$, with equality if and only if $a_1 = b_1$ and $a_2 = b_2$.

• If $0 < a_1 \leq b_1$ and $0 < a_2 \leq b_2$, then $a_1 a_2 \leq b_1 b_2$, with equality if and only if $a_1 = b_1$ and $a_2 = b_2$.

• If $0 < a \leq b$, then $\frac{1}{a} \geq \frac{1}{b}$.

• If $0 < a \leq b$ and $\alpha > 0$, then $a^\alpha \leq b^\alpha$.

• $a^2 \geq 0$, with equality if and only if $a = 0$.

Classical inequalities:

• **Arithmetic mean - geometric mean inequality** [**AM-GM**] For $a_1, a_2, \ldots, a_n$ nonnegative,

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

• **Power mean inequality** For $a_1, a_2, \ldots, a_n$ nonnegative, and $\alpha \in \mathbb{R}$, let

$$M_\alpha(a_1, a_2, \ldots, a_n) := \begin{cases} \left(\frac{a_1^\alpha + a_2^\alpha + \ldots a_n^\alpha}{n}\right)^{\frac{1}{\alpha}} & \alpha \neq 0 \\ \sqrt[n]{a_1 \cdots a_n} & \alpha = 0 \end{cases}$$

Then M_α is an increasing function of α unless $a_1 = a_2 = \cdots = a_n$ in which case M_α is constant.

• **Cauchy-Schwartz inequality** For arbitrary real numbers $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$,

$$(a_1b_1 + \cdots + a_nb_n)^2 \le (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

Furthermore, equality holds if and only if the vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional. The Cauchy-Schwartz inequality is equivalent to the triangle inequality for the 2-norm.

• **Triangle inequality** For any two vectors x, y in $\mathbb{R}^n$,

$$\| x + y \|_2 \le \| x \|_2 + \| y \|_2 \quad .$$

Here $x = (x_1, x_2, \ldots, x_n)$ and $\| x \|_2 = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.

• **Definition** Suppose f is a continuous real-valued function defined on an interval, and for any points x, y in the interval

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2}.$$

Then f is *convex*.

Note that f is convex if and only if for any points x, y in the interval, and any $0 \le t \le 1$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

If f is twice differentiable with continuous second derivative, and $f'' > 0$, then f is convex.

Convex functions have useful properties.

• **Jensen's inequality** If $w_1, \ldots, w_n$ are positive numbers satisfying $w_1 + \cdots + w_n = 1$, and $x_1, \ldots, x_n$ are any n points in an interval where f is convex, then

$$f(w_1x_1 + \cdots + w_nx_n) \le w_1f(x_1) + \cdots + w_nf(x_n).$$

(The typical case is $w_1 = \cdots = w_n = \frac{1}{n}$.)

• **Points of maximum** If f is convex on $[a, b]$, then the maximum value of f is taken at one of the endpoints, i.e.

$$f(x) \le \max\{f(a), f(b)\}.$$

• **Weighted AM-GM inequality** If $x_1, \ldots, x_n$ are nonnegative real numbers and $w_1, \ldots, w_n$ are positive numbers satisfying $w_1 + \cdots + w_n = 1$, then

$$\prod_{i=1}^n x_i^{w_i} \le \sum_{i=1}^n w_ix_i.$$

Equality holds if and only if $x_1 = \cdots = x_n$.

• **Young's inequality** If a and b are nonnegative numbers and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $a^p = b^q$.

• **Hölder's inequality** Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be nonnegative real numbers, and let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}.$$

• **Minkowski's inequality** Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be nonnegative real numbers, and $p \geq 1$, then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p \right)^{\frac{1}{p}}.$$

• **Theorem (Hölder)** Let $X = (X_{ij})$ be an $m \times n$ matrix with nonnegative elements and let $w_1, \ldots, w_n$ be positive numbers satisfying $w_1 + \cdots + w_n = 1$, then

$$\sum_{i=1}^{m} \prod_{j=1}^{n} x_{ij}^{w_j} \leq \prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{w_j}.$$

• **Rearrangement inequality** Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be two sequences of real numbers and suppose $a_1 \leq a_2 \leq \cdots \leq a_n$. For each permutation π of $\{1, 2, \ldots, n\}$, let

$$\sum(\pi) := \sum_{k=1}^{n} a_k b_{\pi(k)}.$$

Then $\sum$ is largest when $b_{\pi(1)} \leq b_{\pi(2)} \leq \cdots \leq b_{\pi(n)}$, and smallest when $b_{\pi(1)} \geq b_{\pi(2)} \geq \cdots \geq b_{\pi(n)}$.

• **Chebychev's inequality** Let $a_1 \geq \cdots \geq a_n > 0$, and $b_1 \geq \cdots \geq b_n > 0$, then

$$\left(\frac{\sum_{i=1}^{n} a_i b_i}{n} \right) \geq \left(\frac{\sum_{i=1}^{n} a_i}{n} \right) \left(\frac{\sum_{i=1}^{n} b_i}{n} \right)$$

with equality if and only if all the a_i are equal or all the b_i are equal.

Some tricks:

• Use logarithms to change products to sums.
• Differentiate or integrate and use the fundamental theorem of calculus.
• Make choices, e.g. constants, for values in the classical inequalities.
• Use Taylor series.
• Look for symmetries.