Putnam preparation

Recurrence, sequences, series, and progressions

(We follow notes and problems from other sources, do not assume the material is original.)

A *sequence* is a function f defined for every nonnegative integer n. Usually one writes $x_n = f(n)$, or a_n instead of x_n.

Arithmetic progressions. An *arithmetic progression* is a sequence (a_n) (finite or infinite) with the property that, for a fixed *common difference* d and all indices n, $a_{n+1} - a_n = d$. An equivalent property is that $a_n = \frac{a_{n-1}+a_{n+1}}{2}$.

The kth term of an arithmetic progression with initial value a_1 and common difference d is

$$a_k = a_1 + (k-1)d \ .$$

The sum of the first n terms of an arithmetic progression is

$$a_1 + a_2 + \cdots + a_n = n\frac{a_1 + a_n}{2} \ .$$

Geometric progressions. An *geometric progression* is a sequence (a_n) (finite or infinite) with the property that, for a fixed *common ratio* r and all indices n, $a_{n+1} = ra_n$.

The kth term of a geometric progression with initial value a_1 and common ratio r is

$$a_k = a_1 r^{k-1} \ .$$

The sum of the first n terms of a geometric progression is

$$a_1 + a_2 + \cdots + a_n = \begin{cases} \frac{a_1(1-r^n)}{1-r} & \text{if } r \neq 1 \\ a_1 n & \text{if } r = 1 \ . \end{cases}$$

An infinite geometric series with $|r| \geq 1$ diverges. The sum of an infinite geometric series with common ratio r, $|r| < 1$, and initial term a_1 is

$$\sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r} \ .$$

Summing powers of integers. Consider

$$S_r(n) := \sum_{k=1}^{r} k^r \ .$$

Then

$$\begin{aligned} S_1(n) &= \tfrac{n(n+1)}{2} \\ S_2(n) &= \tfrac{n(n+1)(2n+1)}{6} \\ S_3(n) &= \left(\tfrac{n(n+1)}{2}\right)^2 \ , \end{aligned}$$

and in general,

$$S_r(n) = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{r+1} \ ,$$

where $B_m(x)$ is the mth Bernoulli polynomial. Bernoulli polynomials satisfy the relation $B_m(x+1) - B_m(x) = mx^{m-1}$.

However, often it is enough to remember the fact that $S_r(n)$ is a polynomial in n of degree $(r+1)$, and thus then one can solve for its coefficients using the first $(r+2)$ values of n. E.g. knowing that

$$1 + 2 + \cdots + n = an^2 + bn + c,$$

one can solve for a, b, c just by solving the equations one gets by plugging in the cases $n = 1, 2, 3$.

Linear recursive sequences. A linear difference equation of order 2 is an equation of the form

$$x_n = px_{n-1} + qx_{n-2}, \ (q \neq 0) \tag{1}$$

where p and q are constants.

To find the general solution to (1), we first try to find a solution of the form $x_n = \lambda^n$ for a suitable number λ. To find λ, plug λ^n into (1) and get $\lambda^n = p\lambda^{n-1} + q\lambda^{n-2}$, or

$$\lambda^2 - p\lambda - q = 0 \ . \tag{2}$$

This is the *characteristic equation* of (1). For distinct roots λ_1 and λ_2,

$$x_n = a\lambda_1^n + b\lambda_2^n$$

is the general solution. The numbers a and b can be found from the initial values x_0 and x_1.

If $\lambda_1 = \lambda_2 = \lambda$, the general solution has the form

$$x_n = (a + bn)\lambda^n \ .$$

In general, a kth order linear recurrence with constant coefficients is a relation of the form

$$x_n = a_1x_{n-1} + a_2x_{n-2} + \cdots + a_kx_{n-k}, \quad n \geq k \tag{3}$$

where the numbers a_i are constants. The sequence is completely determined by $x_0, x_1, \ldots, x_{k-1}$ (the initial condition.) To find the solution, first find the roots of the characteristic equation

$$\lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_k = 0 \ . \tag{4}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the roots of the characteristic equation. If these roots are distinct (which is the most frequent case), then the solution is given by

$$x_n = b_1\lambda_1^n + b_2\lambda_2^n + \cdots + b_k\lambda_k^n \ , \text{ for } n \geq 0 \ , \tag{5}$$

and the numbers $b_1, b_2, \ldots, b_n$ can be found from the initial condition.

If the roots of the characteristic equation have multiplicities greater than 1, for each root λ_t which is repeated m_t times, substitute the terms in (5) corresponding to λ_t by

$$\left(c_1 + c_2 n + \cdots + c_t n^{m_t - 1}\right) \lambda_t^n \ . \tag{6}$$

To be completely precise and accurate, one considers the matrix

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{k-1} & a_k \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{7}$$

and finds its Jordan canonical form. Then you do (6) for each Jordan block of size $m_t \times m_t$ with eigenvector λ_t, i.e. each Jordan block of the form

$$J_{m_t}(\lambda_t) = \begin{pmatrix} \lambda_t & 1 & 0 & \dots & 0 \\ 0 & \lambda_t & 1 & \dots & 0 \\ 0 & 0 & \lambda_t & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_t \end{pmatrix} \tag{8}$$

To find the general term of an inhomogeneous linear recurrence

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k} + f(n), \quad n \geq 1 \ , \tag{9}$$

one has to find a particular solution to the recurrence, and then add it to the general term of the associated homogeneous recurrence equation (i.e. the equation (3).)

For example, the Fibonacci sequence satisfies the recurrence relation $F_{n+1} = F_n + F_{n-1}$, which has characteristic equation $\lambda^2 - \lambda - 1 = 0$, which has roots $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. Writing $F_n = b_1 \lambda_1^n + b_2 \lambda_2^n$ and solving the system

$$\begin{gathered} b_1 + b_2 = F_0 = 0 \\ b_1 \lambda_1 + b_2 \lambda_2 = F_1 = 1 \ , \end{gathered}$$

we obtain $b_1 = -b_2 = -\frac{1}{\sqrt{5}}$, and we rediscover the well-known Binet formula

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right] \ .$$

Limits of sequences. Recall the definition of limit: A sequence $(x_n)_n$ converges to a limit L if and only if for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n > n(\varepsilon)$, $|x_n - L| < \varepsilon$.

A sequence $(x_n)_n$ tends to infinity if for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n > n(\varepsilon)$, $x_n > \varepsilon$.

The squeeze principle. If $a_n \leq b_n \leq c_n$ for all n, and if $(a_n)_n$ and $(c_n)_n$ converge to the finite limit L, then $(b_n)_n$ also converges to the finite limit L.

If $a_n \leq b_n$ for all n, and if $(a_n)_n$ tends to infinity, then $(b_n)_n$ also tends to infinity.

The following are two criteria for proving that a sequence is convergent without actually computing the limit.

Weierstrass' theorem. A monotonic bounded sequence of real numbers is convergent.

Cauchy's criterion for convergence. A sequence $(x_n)_n$ of points in $\mathbb{R}^n$ is convergent if and only if, for every $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that whenever $n, m > n(\varepsilon)$, $\| x_n - x_m \| < \varepsilon$.

The following fixed point theorem can be proved as a direct application of Cauchy's criterion for convergence.

Theorem. Let X be a closed subset of $\mathbb{R}^n$ and $f : X \to X$ a function with the property that $\| f(x) - f(y) \| \leq c \| x - y \|$ for any $x, y \in X$, where $0 < c < 1$ is a constant. Then f has a unique fixed point in X (i.e. a point x such that $f(x) = x$.)

Such a function is called a contraction.

The Stolz Theorem. Let $(x_n)_n$ and $(y_n)_n$ be two sequences of real numbers with $(y_n)_n$ strictly positive, increasing, and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L \ ,$$

then the limit

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

exists and is equal to L.

Another sometimes useful fact is that, given a sequence $(a_n)_{n\geq 1}$, its Cesáro means are

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \ , \quad n \geq 1 \ .$$

Theorem If $(a_n)_{n\geq 1}$ converges to L, then $(s_n)_{n\geq 1}$ also converges to L.

A couple of other useful facts are the following.

Cantor's nested intervals theorem. Given a decreasing sequence of closed intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \ldots$ with lengths converging to zero, the intersection $\cap_{n=1}^{\infty} I_n$ consists of exactly one point.

Heine-Borel theorem. A closed and bounded interval of real numbers is compact.

Remember that if a set A is compact, and if $(x_n)_n$ is a sequence contained in A, then there exists a subsequence $(x_{n_j})_j$ that converges to a point $x \in A$.

Series. A series is a sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$$

The first question asked about a series is whether it converges. Convergence can be decided using Cauchy's $\varepsilon - \delta$ criterion, or by comparing it with another series. For comparison, two families of series are most useful:

(i) geometric series

$$1 + x + x^2 + \cdots + x^n + \ldots,$$

which converges if $|x| < 1$ and diverges otherwise, and

(ii) p-series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p} + \ldots,$$

which converges if $p > 1$ and diverges otherwise.

The p-series corresponding to $p = 1$ is the harmonic series. Its truncation to the nth term approximates $\log(n)$, the natural logarithm of n.

The comparison with a geometric series gives rise to **d'Alembert's ratio test:** $\sum_{n=1}^{\infty} a_n$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ and diverges if $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$.

The comparison with a geometric series also gives rise to **the root test:** $\sum_{n=1}^{\infty} a_n$ converges if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$ and diverges if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$.

It might also be useful to remember **Stirling's formula:**

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot e^{\frac{\theta_n}{12n}}\ , \text{ for some } 0 < \theta_n < 1\ .$$

The telescopic method for summing series. Given a series $\sum_{k=1}^{n} a_k$, if one can find b_k such that $a_k = b_{k+1} - b_k$, for $k \geq 1$, then

$$\sum_{k=1}^{n} a_k = b_{n+1} - b_1\ .$$

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.