Putnam preparation

Combinatorics and Probability (Problem Sheet)

(1) For an arithmetic sequence $a_1, a_2, \ldots, a_n, \ldots$, let $S_n = a_1 + a_2 + \cdots + a_n$, $n \geq 1$. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}.$$

(2) Prove that

$$1 \cdot 2\binom{n}{2} + 2 \cdot 3\binom{n}{3} + \cdots + (n-1) \cdot n\binom{n}{n} = n(n-1)2^{n-2} \ .$$

(3) Consider n indistinguishable balls randomly distributed in m boxes. What is the probability that exactly k boxes remain empty?

(4) (Romanian Math Olympiad, 1975) Given the independent events $A_1, A_2, \ldots, A_n$ with probabilities $p_1, p_2, \ldots, p_n$, find the probability that an odd number of these events occurs.

(5) What is the probability that 3 randomly chosen points on a circle form an acute triangle?

(6) (A.2) You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

(7) (A.3) Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.