Putnam preparation

Recurrence, sequences, series, and progressions

(We follow notes and problems from other sources, do not assume the material is original.)

(1) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(2) If (a_n) is a sequence such that for $n \geq 1$,

$$(2 - a_n)a_{n+1} = 1 \ ,$$

prove that $\lim_{n\to\infty} a_n$ exists and is equal to 1.

(3) Show that

$$\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \cdots + \sin\left(\frac{(n-1)\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right) \ .$$

(4) (A1) Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are: 2, 3, 6, 14, 40, 152, 784, 5158, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

(5) Let x_n be a sequence of real numbers satisfying

$$x_{n+m} \leq x_n + x_m \ , \qquad n, m \geq 1 \ .$$

Prove that $\lim_{n\to\infty} \frac{x_n}{n}$ exists and is equal to $\inf_{n\geq 1} \frac{x_n}{n}$.

(6) (Morning 3) Evaluate

$$\lim_{n\to\infty} \sum_{j=1}^{n^2} \frac{n}{n^2 + j^2} \ .$$

(7) (B3) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in its simplest form.)

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.