Putnam preparation

Polynomials, Abstract Algebra

(1) Given the polynomial $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with integral coefficients $a_0, a_1, \ldots, a_{n-1}$, and given also that there exist four distinct integers a, b, c, d such that $F(a) = F(b) = F(c) = F(d) = 5$, show that there is no integer k such that $F(k) = 8$.

(2) If $x + y + z = 0$, prove that

$$\frac{x^2+y^2+z^2}{2} \cdot \frac{x^5+y^5+z^5}{5} = \frac{x^7+y^7+z^7}{7}\ .$$

(3) Suppose that G is a set and $*$ is a binary operation such that

(i) *Associative property.* $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.

(ii) *Right identity.* There exists an element $e \in G$ such that $a * e = a$ for all $a \in G$.

(iii) *Right inverse.* For each $a \in G$ there exists an element $a^{-1} \in G$ such that $a*a^{-1} = e$.

Prove that G is a group.

(4) If G is a finite group and m is a positive integer relatively prime to the order of G, then for each $a \in G$, there is a unique $b \in G$ such that $b^m = a$.

(5) (Morning 1) Prove that if $F(x)$ is a polynomial with integral coefficients, and there exists an integer k such that none of the integers $F(1), F(2), \ldots, F(k)$ is divisible by k, then $F(x)$ has no integral zero.

(6) (A1) Consider a set S and a binary operation $*$ on S. Assume that $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

(7) (A3)

(a) Find all pairs (m, n) of positive integers such that $|3^m - 2^n| = 1$.

(b) Find all pairs (m, n) of integers larger than 1 such that $|p^m - q^n| = 1$, where p and q are primes.

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.