Putnam preparation

Continuity, Derivatives and Integrals

(1) (USAMO) Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2 ,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that the triangle ABC is similar to a triangle T whose sidelengths are all positive integers with no common divisors and determine these integers.

(2) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and

$$f(2x) - f(x) = x ,$$

for all $x \in \mathbb{R}$.

(3) A cross-country runner runs a six-mile course in 30 minutes. Prove that somewhere along the course the runner ran a mile in exactly 5 minutes.

(4) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function. Assume that there is no point x in $[0,1]$ such that $f(x) = 0 = f'(x)$. Show that f has only a finite number of zeros in $[0,1]$.

(5) (B1) Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} \left(n^2 + i^2\right)^{\frac{1}{n}} .$$

(6) (Morning 2) If $a(x), b(x), c(x)$ and $d(x)$ are polynomials in x, show that

$$\int_1^x a(x)c(x)\ dx \int_1^x b(x)d(x)\ dx - \int_1^x a(x)d(x)\ dx \int_1^x b(x)c(x)\ dx$$

is divisible by $(x-1)^4$.

(7) (Afternoon 3) Give an example of a continuous real-valued function f from $[0,1]$ to $[0,1]$ which takes on every value in $[0,1]$ an infinite number of times.

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.