Putnam preparation

Number Theory and Congruences (Problem Sheet)

(1) Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

(2) The algebraic sum of any number of irreducible fractions whose denominators are relatively prime to each other cannot be an integer. That is, if $(a_i, b_i) = 1$ for $i = 1, 2, ..., n$, and $(b_i, b_j) = 1$ for $i \neq j$, show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}$$

is not an integer.

(3) Prove that the equation

$$x^2 + y^2 + z^2 = 2xyz$$

has no solutions in the integers except for $x = y = z = 0$.

(4) (IMO.1) If p, q are positive integers, then

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... - \frac{1}{1318} + \frac{1}{1319} = \frac{p}{q} \Longrightarrow 1979|p\ .$$

(5) (Putnam, morning 2) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

(6) (B2) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

(7) (IMO.1) Find the smallest natural number n which has the following properties:

  (i) its decimal representation has a 6 as its last digit, and
  (ii) if the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number number n.

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.