Putnam preparation

Combinatorics and Probability

(I follow outside sources, these are not my exposition or my problems)

The binomial coefficient $\binom{n}{k}$ counts the number of ways one can choose k objects from given n objects (the order among the chosen objects is not relevant.) Binomial coefficients show up in Newton's binomial expansion

$$(x+1)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}x + \binom{n}{n} .$$

Explicitly,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} ,$$

if $0 \le k \le n$.

Relations:

$$\binom{n}{k} = \binom{n}{n-k}$$

and

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} .$$

This last relation allows binomial coefficients to be arranged in Pascal's triangle, where every entry is obtained by summing the two entries just above it.

The inclusion-exclusion principle. This principle concerns the counting of elements in a union of sets $A_1 \cup A_2 \cup \cdots \cup A_n$. Let $|A|$ denote the number of elements of the set A.

If we simply wrote

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| ,$$

we would be overcounting the elements in the intersections $A_i \cap A_j$. So we have to subtract $|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$. But then the elements in the triple intersections $A_i \cap A_j \cap A_k$ are both added and subtracted, so we have to put them back. Therefore we must add $|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|$. And so on. The final formula is

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \cdots + (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n| .$$

Probability. Remember that the probability of an event A and the complimentary event A^c (i.e. the event that A does not occur) add up to 1, i.e. $P(A) + P(A^c) = 1$, and $0 \le P(A) \le 1$.

If we consider experiments with finitely many outcomes each of which can occur with equal probability, the probability of an event A is

$$P(A) = \frac{\text{number of favorable outcomes}}{\text{total number of possible outcomes}}.$$

Relations among probabilities. $P(A \cap B)$ is the probability that the events A and B occur simultaneously. $P(A \cup B)$ is the probability that either the events A or B occur, $P(A - B)$ is the probability that A occurs, but B does not occur, and $P(A|B)$ is the probability that A occurs, given that B also occurs.

The classical formulas are:

• Addition formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \ .$$

• Multiplication formula:

$$P(A \cap B) = P(A)P(B|A) \ .$$

• Total probability formula: if $B_i \cap B_j = \emptyset$, for $i, j = 1, 2, \ldots, n$ (meaning that they are independent), and $A \subset B_1 \cup B_2 \cup \cdots \cup B_n$, then

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n) \ .$$

• Bayes' formula: with the same hypothesis,

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)} \ .$$

In particular, if $B_1, B_2, \ldots, B_n$ cover the entire probability field, then

$$P(B_i|A) = \frac{P(B_i)}{P(A)} P(A|B_i) \ .$$

The Bernoulli scheme. As a result of an experiment, either the event A occurs with probability p or the contrary event A^c occurs with probability $q = 1 - p$. We repeat the experiment n times. The probability that A occurs exxactly m times is $\binom{n}{m} p^m q^{n-m}$.

The Poisson scheme. We perform n independent experiments. For each k, $1 \leq k \leq n$, in the kth experiment the event A can occur with probability p_k, or A^c can occur with probability $q_k = 1 - p_k$. The probability that A occurs exactly m times while the n experiments are performed is the coefficient of x^m in the expansion of

$$(p_1 x + q_1)(p_2 x + q_2) \ldots (p_n x + q_n) \ .$$

Expectation. If an experiment has as outcome a real number x_i (e.g. the number appearing when you roll a die) which occurs with probability p_i, the expectation (or expected value) of the experiment is

$$\sum_i x_i p_i.$$

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.