Putnam preparation

Inequalities, maxima and minima

(1) Show that for real a, b, c,

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

(2) (Bernoulli's inequality) Prove that for $n \in \mathbb{N}$ and $x \geq -1$, then $(1+x)^n \geq 1 + nx$.

(3) Assuming the AM-GM inequality, prove $M_{-1} \leq M_0$ in the power mean inequality. (See the notes for the names of the inequalities.)

(4) (Iosevich) Let $x_1, \ldots, x_n$ and $a_1, \ldots, a_n$ be positive real numbers. Prove that

$$x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \leq \frac{(x_1 + \cdots + x_n)^{a_1 + \cdots + a_n}}{a_1 \ldots a_n} a_1^{a_1} \cdot a_2^{a_2} \cdots a_n^{a_n}.$$

Is it possible to choose the a_js in such a way that this inequality reduces to AM-GM or Hölder's inequality? If so, demonstrate it. If the answer is no, explain why not.

(5) (A1) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

(6) (A2) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 \cdots a_n)^{\frac{1}{n}} + (b_1 \cdots b_n)^{\frac{1}{n}} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{\frac{1}{n}}$$

(7) (B1) For which real numbers c is $\frac{(e^x + e^{-x})}{2} \leq e^{cx^2}$ for all real x?

(8) (A2) The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.