Putnam preparation

Induction, pigeonholing (Problem Sheet)

(1) The plane is divided into regions by finitely many straight lines. Show that it is always possible to color the regions with two colors so that adjacent regions are never the same color (like a checkerboard).

(2) People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate (this is commonly found in Chinese restaurants that specialize in banquets). Each person ordered a different entrée, and it turns out that no one has the correct entrée in front of him. Show that it is possible to rotate the platform so that at least *two* people will have the correct entrée.

(3) (A.1) Let k be a fixed positive integer. The n-th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

(4) (A.1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

(5) (A.2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

(6) (B.3) There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Disclaimer: Most (if not all) the material here is from outside sources, I am not claiming any originality.