Learning Objectives

By the end of this worksheet, you will:

- Translate sentences between natural English and predicate logic.
- Use definitions of predicates to simplify or expand statements in predicate logic.
- Apply negation equivalence rules to simplify statements in predicate logic.

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1. **Translation with predicates**. Suppose we have a set P of computer programs that are each meant to solve the same task. Some of the programs are written in the Python programming language, and some are written in a different language. Some of the programs correctly solve the task, and others do not.

   Let's define the following predicates:

$$\begin{aligned} Python(x) &: \text{“}x \text{ is a Python program”}, \qquad \text{where } x \in P \\ Correct(x) &: \text{“}x \text{ solves the task correctly”}, \qquad \text{where } x \in P \end{aligned}$$

   Express each English statement below as a sentence in predicate logic.

   (a) Program *my_prog* is correct and is written in Python.

   Correct(my_prog) /\ Python(my_prog)

   (b) An incorrect program is written in Python.

   $\exists$ x $\in$ P, $\neg$Correct(x) $\wedge$ Python(x)

   (c) No Python program is correct.

   $\forall$ x $\in$ P, Python(x) $\wedge$ $\neg$Correct(x)

   (d) Every incorrect program is written in Python.

   $\forall$ x $\in$ P, $\neg$Correct(x) $\Rightarrow$ Python(x)

   Translate each predicate logic statement below into natural English.

   (a) $\exists x \in P,\ Python(x) \wedge Correct(x)$

   There's a program written in Python that's also correct.

   (b) $\forall x \in P,\ \neg Python(x) \wedge Correct(x)$

   All programs are not written in Python and are correct.

   (c) $\neg\big(\forall x \in P,\ Correct(x) \Rightarrow Python(x)\big)$ Distributing the negation: $\exists$ x $\in$ P, Correct(x) $\wedge$ $\neg$Python(x)

   A program is correct and not written in Python.

   (d) $\forall x \in P,\ \neg Python(x) \Leftrightarrow Correct(x)$

   All programs are not written in Python if and only if they are correct.

2. **Quantifiers in subformulas**. So far, we have seen quantifiers only as the leftmost components of our formulas. However, because all predicate statements have truth values (i.e., are either True or False), they too can be combined using the standard propositional operators. Let's see some examples of this (refer to the same predicates as Question 1).

   (a) Translate the following statement into English.

   $$\Big(\forall x \in P,\ \mathit{Python}(x) \Rightarrow \mathit{Correct}(x)\Big) \lor \Big(\forall y \in P,\ \mathit{Python}(y) \Rightarrow \neg \mathit{Correct}(y)\Big)$$

   All programs written in Python are correct, or all programs written in Python are incorrect.

   (b) Translate the following statement into predicate logic. "If at least one Python program is correct, then all Python programs are correct."

   $(\exists x \in P, \text{Python}(x) \land \text{Correct}(x)) \Rightarrow (\forall y \in P, \text{Python}(y) \Rightarrow \text{Correct}(y))$

   (c) Finally, consider the following two statements:

   $$\big(\exists x_1 \in \mathbb{N},\ x_1 \mid 165\big) \land \big(\exists x_2 \in \mathbb{N},\ 7 \mid x_2\big) \qquad \text{(Statement 1)}$$
   $$\exists x \in \mathbb{N},\ x \mid 165 \land 7 \mid x \qquad \text{(Statement 2)}$$

   What is the difference between these two statements? Are they True or False?

   S1 involves two different variables (x_1 and x_2, both naturals) which each have to satisfy their own, unrelated condition meanwhile S2 involves one variable (x, natural) which looks to satisfy both conditions from S1 at the same time.
   S1 is true. Take $x_1 = 5$, and $x_2 = 14$. Both are natural numbers which satisfy their respective conditions.
   S2 is false. Unpacking the definition of divisibility for an **ARBITRARY x**, we would have $165 = xj$ and $x = 7k$. This leads to $165 = 7(kj)$ and subsequently $165 \div 7 = kj$. Since 7 does not divide 165, the entire statement falls apart.

3. **Expanding definitions**. Consider the following statement:

   If m and n are odd integers, then mn is an odd integer.

   If we want to express this statement using predicate logic, we need to start with a definition of the term "odd". Let $n \in \mathbb{Z}$. We say that n is **odd** when $2 \mid (n+1)$, i.e., when $\exists k \in \mathbb{Z}, n+1 = 2k$.

   (a) Write the definition of a predicate over the integers named *Odd* that is True when its argument is odd.

   $\text{Odd}(x) : 2 \mid (x + 1)$, where $x \in Z$

   (b) Using the predicate *Odd* and the notation of predicate logic, express the statement:

   For every pair of odd integers m and n, mn is an odd integer.

   $\forall m, n \in Z, (\text{Odd}(m) \land \text{Odd}(n)) \Rightarrow \text{Odd}(mn)$

   (c) Repeat part (b) but fully expand the definitions of the predicates *Odd* and $\mid$.

   $\forall m, n \in Z, ((\exists i \in Z, m + 1 = 2i) \land (\exists j \in Z, n + 1 = 2j)) \Rightarrow (\exists k \in Z, mn + 1 = 2k)$

(d) Repeat parts (b) and (c) using the following statement (which states the converse of the original implication).

For every pair of integers m and n, if mn is odd, then m and n are odd.

(b) ∀m, n ∈ Z, Odd(mn) ⇒ (Odd(m) ∧ Odd(n))

(c) ∀m, n ∈ Z, (∃i ∈ Z, mn + 1 = 2i) ⇒ ((∃j ∈ Z, m + 1 = 2j) ∧ (∃k ∈ Z, n + 1 = 2k))

4. **Simplifying negated formulas**. Recall the rules governing how to simplify negations of predicate formulas:

- $\neg(\neg p)$ becomes p.
- $\neg(p \lor q)$ becomes $\neg p \land \neg q$.
- $\neg(p \land q)$ becomes $\neg p \lor \neg q$.
- $\neg(p \Rightarrow q)$ becomes $p \land \neg q$.
- $\neg(p \Leftrightarrow q)$ becomes $(p \land \neg q) \lor (\neg p \land q)$.
- $\neg(\exists x \in S,\ P(x))$ becomes $\forall x \in S,\ \neg P(x)$.
- $\neg(\forall x \in S,\ P(x))$ becomes $\exists x \in S,\ \neg P(x)$.

Using these rules, simplify each of the following formulas so that the negations are applied directly to predicates/propositional variables. Note: this is a pretty mechanical exercise, but an extremely valuable one: once we get to the next chapter, we will be assuming you can take negations of statements very quickly as a first step in some proofs.

(a) $\neg\Big((a \land b) \Leftrightarrow c\Big)$

(a ∧ b ∧ ¬c) ∨ ((¬a ∨ ¬b) ∧ c)

(b) $\neg\Big(\forall x, y \in S,\ \exists z \in S,\ P(x, y) \land Q(x, z)\Big)$

∃x, y ∈ S, ∀z ∈ S, ¬P(x, y) ∨ ¬Q(x, z)

(c) $\neg\Big(\big(\exists x \in S,\ P(x)\big) \Rightarrow \big(\exists y \in S,\ Q(y)\big)\Big)$

(∃x ∈ S, P(x)) ∧ (∀y ∈ S, ¬Q(y))

5. **Choosing a universe and predicates**. Consider the statement

$$\Big[\big(\exists x \in U,\ P(x)\big) \wedge \big(\exists y \in U,\ Q(y)\big)\Big] \Rightarrow \Big[\exists z \in U,\ P(z) \wedge Q(z)\Big].$$

Define a non-empty domain U and predicates P and Q for which this statement is False.

Hint: The statement says: "If some $x \in U$ makes $P(x)$ True and some $y \in U$ makes $Q(y)$ True, then some $z \in U$ makes both $P(z)$ and $Q(z)$ True." Think about how this statement could be False, and use this to construct a U, P and Q.

At the very base, we see this is an implication statement. In order to falsify any implication, we want to make the conclusion false, but while the hypothesis holds true. In this case, the conclusion is composed of an AND ($\wedge$), and we know in order to make AND false, at least one of the constituents must be false. With this in mind, we look to show the first part ($\exists$z $\in$ U, P(z)) is false:

Suppose we have:

U = {set of naturals} = N
P(x) : 2 | x
Q(y) : 2 | y + 1

The hypothesis is composed of an AND which connects two existentials together. In order to have a true statement with an existential, we only need to find one example such that the predicate is true.

For $\exists$x $\in$ U, P(x):

Let x be 10.
We know that 2 | 10 because $\exists$k $\in$ Z, 10 = 2k, and in this case k = 5.

For $\exists$y $\in$ U, P(y):

Let y be 27.
This means y + 1 = 27 + 1 = 28.

Similarly to our work for $\exists$x $\in$ U, P(x), we know that 2 | 28 (recall that Q(y) : 2 | y + 1, but we've already done the y + 1 step) because $\exists$k $\in$ Z, 28 = 2k, and in this case k = 14.

We've shown the hypothesis is true, and now we must show the conclusion is false for the entire statement to be false. The conclusion states that there exists a natural number z (recall we set U = N) such that 2 divides z, and 2 divides z + 1.

EASY EXPLANATION:
We know the naturals are {0, 1, 2, ...}, and the conclusion is trying to assert that there is a number that is both even and odd. Off our intuition, we know this is impossible. Take 0 for example (an edge case with most existential problems). We can have 2 | 0 (which satisfies P(z)), but then we cannot satisfy Q(z) because 2 | 0 + 1 is not true; 2 does not divide 1.

RIGOROUS EXPLANATION:
If we want to show an existential is false, we must show its negation is true. Taking the negation of $\exists$z $\in$ U, P(z) $\wedge$ Q(z), we have ¬($\exists$z $\in$ U, P(z) $\wedge$ Q(z)) which is equivalent to $\forall$z $\in$ U, ¬P(z) $\vee$ ¬Q(z). We now look to prove the truth of the inverse.

We want to show that ALL natural numbers z (recall again that U = N) are not divisible by 2, or the next natural above z (z + 1) is not divisible by 2 either. Take an arbitrary z $\in$ U. If z is not divisible by 2, it must be that z is an odd number by definition. Similarly, if z + 1 is not divisible by 2, it must be that z is an even number by definition.

Case 1 (z is odd):
Suppose z is odd, we then have z = 2m + 1 for some natural number m. In this case, 2 does not divide z, however, 2 does divide z + 1. That is, 2 | z + 1 which means 2 | 2(m + 1).

Case 2 (z is even):
Suppose z is even, we then have z = 2n for some natural number n. In this case, 2 does not divide z + 1, however, 2 does divide z. That is, 2 | z which means 2 | 2n.

Case 3 (z is 0):
Suppose z = 0. In this edge case, 2 divides 0, that is, 2 | 0. However, 2 does not divide z + 1 which would be to say 2 | 1, a false statement.

Throughout our cases, we find that ¬P(z) $\vee$ ¬Q(z) holds: any natural number z either fails to be divided by 2, or z + 1 fails to be divided by 2. By showing this is true, we have shown that $\exists$z $\in$ U, P(z) $\wedge$ Q(z) is false, thus showing our choices of U, P, and Q were correct in falsifying the entire statement.