UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2017 EXAMINATIONS – Solutions CSC148H1F

Duration: 3 hours
Instructors: Diane Horton and David Liu.
No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully.

- Please print your name and student number on the front of the exam.
- This examination has **10** questions. There are a total of **31 pages, DOUBLE-SIDED**.
- The last page is an aid sheet that may be detached.
- You may always write helper functions/methods unless explicitly asked not to.
- Docstrings are *not* required unless explicitly asked for.
- You must earn a grade of **at least 40% on this exam to pass this course**.

Take a deep breath.

This is your chance to show us

How much you've learned.

We **WANT** to give you the credit

That you've earned.

A number does not define you.

It's been a real pleasure teaching you this term.

Good luck!

Question	Grade	Out of
Q1		7
Q2		9
Q3		12
Q4		5
Q5		9
Q6		11
Q7		5
Q8		11
Q9		6
Q10		5
Total		80

1. **[7 marks]** Short answer questions. Point-form is acceptable; you do not need to write a lot for full marks.

   (a) **[1 mark]** Suppose we have a function with the following header, and we know it does no input or output:

   ```
   def mystery(number: int, bunch: tuple(str, float, str)) -> None:
   ```

   It is impossible to write a body for this function that would make this function useful. Explain why.

Solution

It can't have any effect that lasts after the call is over: (1) It can't change `number`, since `int` values are immutable. (2) It can't change `bunch` by adding or removing elements or by assigning one of them to a different object *and* it can't change the elements inside the `bunch`, since `str` and `float` values are immutable. And (3) it doesn't return anything.

   (b) **[2 marks]** Here's the beginning of a method inside a class `MyClass`:

   ```
   def silly(self, number: int) -> None:
       # Some code omitted
       this = that + number + self.thing
   ```

   What must be true when we reach the assignment statement so that this method runs without raising an error?

Solution

- `that` must have been assigned a value
- `self` must not be `None`
- `self.thing` must have been assigned a value
- The values of `that` and `self.thing` must be numbers. (The type contract allows us to assume that the value of `number` is an `int`. It is acceptable to assume that it remains an `int`.)

   (c) **[4 marks]** Consider this *incorrect* implementation of method `__getitem__` for class `LinkedList`:

   ```
   class LinkedList:
       def __getitem__(self, index: int) -> object:
           """Return the item at position <index> in this list.
           Raise an IndexError if <index> is out of bounds."""
           curr = self._first
           curr_index = 0
           while curr is not None and curr_index < index:
               curr = curr.next
               curr_index += 1

           if curr_index == index:
               return curr.item
           else:
               raise IndexError
   ```

What do you know will be true immediately after the while loop? Express it as an executable assertion.

Solution

```
assert curr is None or curr_index == index
```

It would also be acceptable to say:

```
assert curr is None or curr_index >= index
```

Give a doctest example that this method would fail.

Solution

```
>>> linky = LinkedList([])
>>> linky[0]
Traceback (most recent call last):
IndexError
```

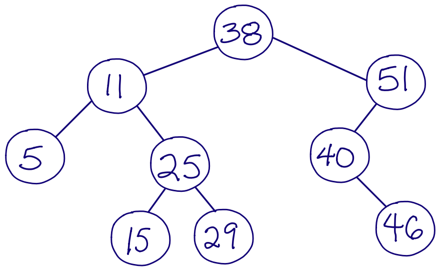
Describe the problem with this method.

Solution

It assumes that if `curr_index == index`, we have found the item we are looking for. But in fact, if `index` is 0, `curr_index` starts out as `index` and we never iterate. This could be because we are already at element 0 or because the list is empty. We could fix this by reversing the if and else bodies and checking instead for failure: `curr is None`.

2. **[9 marks]** Short answer questions. Point-form is acceptable; you do not need to write a lot for full marks.

   (a) **[2 marks]** Consider this tree:



   Suppose we traverse the tree using post-order traversal, and we push each value onto a stack as we go. If we pop those values out of the stack, write down the items in the order they will come out.

Solution

38 51 40 46 11 25 29 15 5

   (b) **[2 marks]** Explain the meaning of the term *composition*.

   Give one example of composition we have used in this course (*e.g.*, from lecture, a lab, exercise, assignment).

Solution

Composition occurs when a class has an attribute that refers to an instance of another class. One example is the `Block` contains an instance of `Game`.

   (c) **[2 marks]** Explain the meaning of the term *abstract class*.

   Give one example of an abstract class we have used in this course (*e.g.*, from lecture, a lab, exercise, assignment).

Solution

An abstract is one the should not be instantiated because it includes at least one method that is unimplemented and raises an error. One example is the `Player` class from assignment 2.

   (d) **[3 marks]** How do we indicate that an instance attribute is private?

Solution

We begin its name with an underscore.

What restriction on client code does that imply?

Solution

Client code should not access that instance attribute. It should certainly not change it, but it shouldn't even examine it.

Give two advantages of making an instance attribute private.

1.

2.

Solution

1. The programmer writing client code can be spared from having to think about how the code works, and focus instead on the only relevant thing: what it does. 2. Anyone working on the class has the freedom to change it in any way desired (for instance they can change the data structure to improve efficiency) without having any impact on client code.

3. **[12 marks]** Suppose you are working on a replacement for ROSI/ACORN. One small part of it is keeping records about undergraduate and graduate students. They are similar, but have several differences:

- An undergrad needs a grade of at least 50 to earn credit for a course, but a graduate student needs at least 70.
- A grad student has a supervisor, as well as a committee of professors (including the supervisor) who help guide his or her research.

The following code (on the left), and its expected output (on the right), demonstrate some features that we need:

Code:

```
# Construct two undergrads and a grad student.
# They start out with no courses.
s1 = Undergrad('Jacqueline Smith', 1234)
s2 = Undergrad('Paul Gries', 2345)
# Jen's supervisor is named Zemel.
s3 = Grad('Jen Campbell', 5555, 'Zemel')

# Record that they completed some courses.
s1.complete_course('CSC108', 80)
s1.complete_course('BIO350', 40)
s1.complete_course('BIO350', 90)  # Second try.
s3.complete_course('CSC2128', 80)
s3.complete_course('CSC3534', 60)

# Give Jen more committee members than just Zemel.
s3.add_committee_member('Urtasun')
s3.add_committee_member('Hinton')
s3.add_committee_member('Dickinson')

# Print everyone's basic info and their average.
# All grades count towards a student's average.
# Paul has completed no courses yet,
# so his average is reported as 0.0.
for s in [s1, s2, s3]:
    print(f'{str(s)}:')
    print(f'   {s.num_credits()} credits and ' +
          f'an average of {s.average()}')
# Print Jen's basic info and committee size.
print(f'{str(s3)} has {s3.committee_size()} members')
```

Output:

```
Jacqueline Smith (1234):
   2 credits and an average of 70.0
Paul Gries (2345):
   0 credits and an average of 0.0
Jen Campbell (5555):
   1 credits and an average of 70.0
Jen Campbell (5555) has 4 members
```

On the next two pages, write the **documentation** for the classes, including their methods, that are necessary for this code to run and produce the expected output. Your class design should use inheritance effectively to capture common attributes and methods.

Write excellent docstrings for classes and methods, but do not write doctests. **Do not spend time writing any method bodies**, except that if a method should be abstract, clearly mark it as such by raising the appropriate error in the method body. If a method overrides an inherited method, the docstring in the subclass method does not need to repeat information from the superclass method's docstring; it should just briefly explain why the method is being overridden.

Remember to include in each class:

- a class docstring describing instanc attributes and their representation invariants
- type declarations for all attributes
- the header (including type contracts) and docstring for each public method in the class

Don't forget: Except for abstract methods, **do not spend time writing any method bodies**.

Space for your class design. (Continue your answer on the next page.)

Space for your class design.

Solution

This solution includes method bodies, which were not required.

```
from typing import List, Tuple

class Student:
    """A university student.

    This class is abstract and should not be instantiated.

    === Public Attributes ===
    name:
        The full name of this student.
    student_number:
        The student number of this student.
    courses:
        The courses that this student has taken.  Each course has a course
        name and grade associated with it.

    === Representation Invariations ===
    - for each grade (course, grade) pair in courses,
          0 <= grade <= 100.
    """
    name: str
    student_number: int
    courses: List[Tuple[str, int]]

    def __init__(self, name: str, student_number: int) -> None:
        self.name = name
        self.student_number = student_number
        self.courses = []

    def __str__(self):
        return f'{self.name} ({self.student_number})'

    def complete_course(self, course, grade) -> None:
        """Record the fact that this student finished course <course>
        with grade <grade>.
        """
        self.courses.append((course, grade))

    def num_credits(self) -> int:
        """Return the number of credits this student has earned.
        """
        raise NotImplementedError

    def average(self) -> float:
        """Return this student's average grade across all courses completed.

        If this student has completed no courses, return 0.0.
        """
        if len(self.courses) == 0:
            return 0.0
        sum = 0
        num = 0
        for course, grade in self.courses:
            sum += grade
            num += 1
        return sum / num


class Undergrad(Student):
    """An undergraduate student.

    === Public Attributes ===
    No additional attributes.
    """
    def num_credits(self) -> int:
        """Return the number of credits this student has earned.
        """
        answer = 0
        for course, grade in self.courses:
            if grade >= 50:
                answer += 1
        return answer

class Grad(Student):
    """A graduate student.

    === Public Attributes ===
    supervisor:
        The name of the supervisor of this student.
    committee:
        The names of the committee members for this student.  A student's
        supervisor is one of their committee members.

    === Representation Invariations ===
    - supervisor in committee
    """
    supervisor: str
    committee: List[str]

    def __init__(self, name: str, student_number: int, supervisor: str) -> None:
        Student.__init__(self, name, student_number)
        self.supervisor = supervisor
        self.committee = [supervisor]

    def num_credits(self) -> int:
        """Return the number of credits this student has earned.
        """
        answer = 0
        for course, grade in self.courses:
            if grade >= 70:
                answer += 1
        return answer

    def add_committee_member(self, name) -> None:
        """Record the fact that faculty member <name> is a committee
        member for this student.
        """
        self.committee.append(name)

    def committee_size(self) -> int:
        """Return the number of members on this student's committee.
        """
        return len(self.committee)
```

Notice that the code for `num_credits` is very similar. We will accept this approach, even with the hard-coded values, for full marks. A better solution would have `num_credits` defined in the parent class with a cutoff parameter, and have each subclass pass the correct cutoff value, which would be defined as a constant in the subclass.

4. **[5 marks]** Short answer questions. Point-form is acceptable; you do not need to write a lot for full marks.

   (a) **[2 marks]** Suppose we have a Python list with 1000 items, and a linked list with 1000 items. We want to insert a new item at position 500 in each list. Is doing this faster in the Python list, faster in the linked list, or roughly the same amount of time in each? Circle one:

   faster in the Python list   faster in the linked list   roughly the same

   Explain your answer.

Solution

It takes roughly the same amount of time.

- In a Python list, we can go to position 500 in a constant amount of time, but then we must move about 500 items over by one spot.
  If we wanted to generalize this to inserting at position i in a `LinkedList` of length n, we could say that it is $O(n)$ in the worst case. But this question asked about a specific position, not the worst case.
- In a `LinkedList`, we must hop about 500 times to get to the right spot, and then do only a constant amount of work to insert the new value. If we wanted to generalize this to inserting at position i in a `LinkedList` of length n, we could say that it is $O(n)$ in the worst case. But this question asked about a specific position, not the worst case.

   (b) **[3 marks]** Here is a function which copies a list. **Hint**: It is quite similar to mergesort.

```python
def copy(lst: list) -> list:
    """Return a new list with the same elements as <lst>.

    >>> lst1 = [13, 4, 8, 7, 11, 25]
    >>> lst2 = copy(lst1)
    >>> lst2
    [13, 4, 8, 7, 11, 25]
    >>> lst2 is lst1
    False
    """
    if len(lst) < 2:
        return lst[:]
    else:
        mid = len(lst) // 2
        left = copy(lst[:mid])
        right = copy(lst[mid:])
        for item in right:
            left.append(item)
        return left
```

   What is the worst-case Big-Oh running time for this algorithm, in terms of n, the length of the input list?

Solution

This function is $O(nlogn)$ in the worst case.

Explain your answer.

Solution

The argument is exactly analogous to the one for mergesort:

- Until it reaches a list of length 0 or 1, this function splits its list in half and recurses on the two halves. So if we draw the tree of calls, it is a complete tree of height $O(logn)$.
- The non recursive parts of this function require $O(k)$ time for a list of length k, because:
  - each time we slice the list in half, that is $O(k)$ work.
  - appending is $O(1)$, but we do it $O(k)$ times, so the loop is $O(k)$.
  - in total that is $O(k)$ work.
- All the calls across a row in the tree of recursive calls, in total, deal with n elements, so in total they take $O(n)$ time.
- Since we do $O(n)$ work $O(logn)$ times, the total work is $O(nlogn)$.

Why is the algorithm used by this implementation of `copy` a poor choice for solving the problem?

Solution

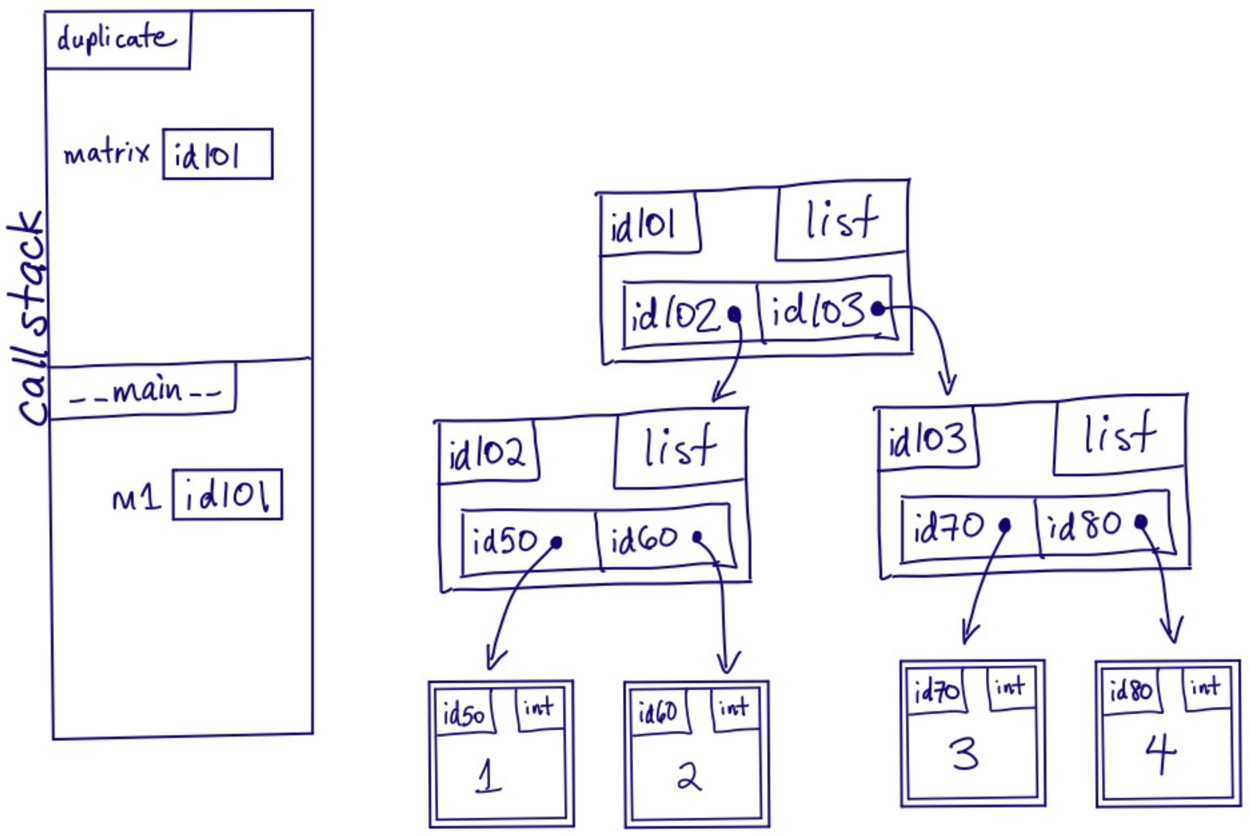
This is a poor choice of algorithm because we can accomplish the same thing in $O(n)$ times. Slicing the entire list using `lst[:]` is $O(n)$, or if we do it by hand, we can start with an empty list and append (not prepend!) one element at a time. Appending in a Python list is $O(1)$, and we would be doing it n times, so that's $O(n)$.

5. **[9 marks]** The code below defines and calls a function.

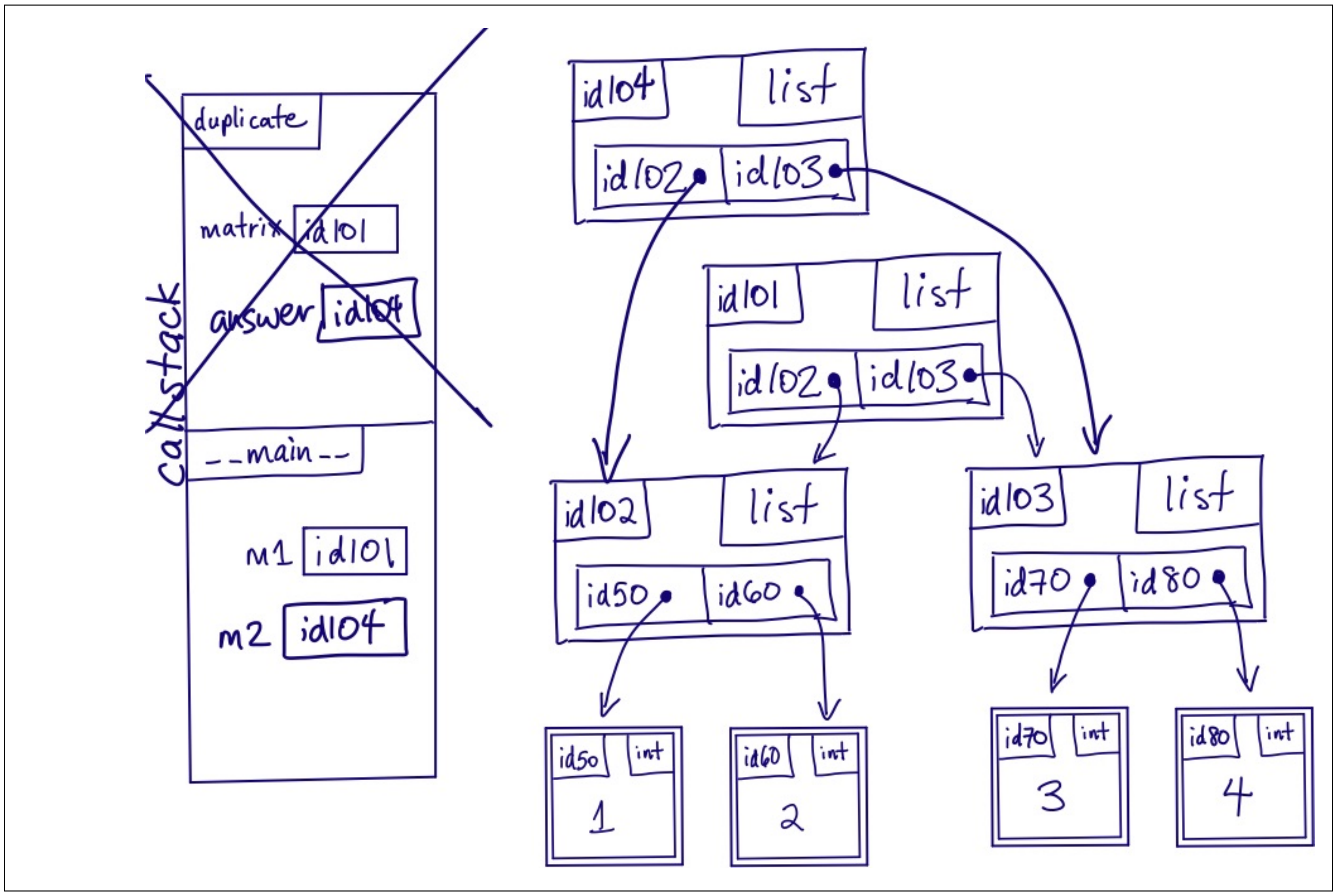
```
def duplicate(matrix: List[List[int]]) -> List[List[int]]:
    answer = []
    for item in matrix:
        answer.append(item)
    return answer

if __name__ == '__main__':
    m1 = [[1, 2], [3, 4]]
    m2 = duplicate(m1)
```

(a) **[2 marks]** Below, we have drawn the state of memory when we call `duplicate(m1)`, right before the first line of the function body, `answer = []`, has been executed. Modify the diagram to show the state of memory after the function has returned and `m2` has been initialized. You may show references using arrows, id values, or both.



Solution


callstack
duplicate
matrix
id101
answer
id104
__main__
m1
id101
m2
id104
id104
list
id102
id103
id101
list
id102
id103
id102
list
id50
id60
id103
list
id70
id80
id50
int
1
id60
int
2
id70
int
3
id80
int
4

(b) [**2 marks**] For each assertion below, circle YES or NO to indicate whether it would succeed if we ran it immediately after the line `m2 = duplicate(m1)`. Explain your answers.

`assert m1 == m2` YES NO

Explain:

Solution

Yes, this assertion succeeds. These are equivalent objects. Each one is equivalent to `[[1, 2], [3, 4]]`

`assert m1 is m2` YES NO

Explain:

Solution

No, this assertion fails. These are two different objects, with different ids.

(c) [**2 marks**] Suppose that after `m2` is initialized, we change `m2` by running the line of code:

```
m2[1][1] = 99
```

For each assertion below, circle YES or NO to indicate whether it would succeed if we ran it immediately after running `m2[1][1] = 99`. Explain your answers.

`assert m2 == [[1, 2], [3, 99]]` YES NO

Explain:

Solution

Yes, this assertion succeeds. The assignment statement did make this change to `m2`

`assert m1 == [[1, 2], [3, 4]]` YES NO

Explain:

Solution

No, this assertion fails. Although m1 and m2 are different objects, they contain aliases to the same two sublists. So when we change something in one of the sublists of m2, we have changed the same thing in a sublist of m1 – they are the same list object!

(d) **[3 marks]** Suppose we call `duplicate` on an input list of that contains `n` lists, where each inner list contains `m` items. Use Big-Oh notation to describe how long this function takes to run, in terms of `n` and/or `m`. Do not assume anything about the relationship between `n` and `m`.

Solution

$O(n)$. Notice that the running time does not depend in any way on m.

Explain your answer.

Solution

- The work before the loop is constant time.
- There are $O(n)$ iterations.
- Each iteration is $O(1)$ or constant time because appending at the end of a list is constant time. It doesn't matter that we are appending a list; it is just an id that is being moved into place.
- Since we do $O(n)$ iterations and constant work per iteration, the whole thing is $O(n)$.

6. **[11 marks]** Consider the following linked list method:

```
class LinkedList:
    def bisect(self, i: int) -> LinkedList:
        """Remove from this linked list the nodes at position i and beyond,
        and return them in a new linked list.

        The nodes in the new linked list should be in the same order as they
        were in <self>.

        Raise an IndexError if i < 0 or i >= the length of this linked list.

        Note: this is a mutating method because it modifies the contents
        of <self>.
        """
```

(a) **[3 marks]** Write two doctests examples for this method, using `str` to display the values of the linked list. The examples must illustrate *different* cases for the method behaviour. We have started the first doctest for you.

Doctest 1

```
>>> linky1 = LinkedList([1, 2, 3, 4])
>>> str(linky1)
'[1 -> 2 -> 3 -> 4]'
```

Doctest 2:

Solution

Doctest 1:

```
>>> linky1 = LinkedList([1, 2, 3, 4])
>>> str(linky1)
'[1 -> 2 -> 3 -> 4]'
>>> linky2 = linky1.bisect(2)
>>> str(linky1)
'[1 -> 2]'
>>> str(linky2)
'[3 -> 4]'
```

Doctest 2: many possibilities for a *different* case, including:

- Input linked list of length 0 or 1.
- Input `i` which is equal to 0 or `len(self) - 1`.

(b) **[2 marks]** You will need a `while` loop to iterate to the `i`-th node in the linked list (or to discover that `i` is out of bounds). Suppose we have a counter `index` that begins at 0, and a variable `curr` that begins by referencing the first node of the linked list.

Under what conditions should the loop stop, in terms of `index` and `curr`?

Under what conditions should the loop continue, in terms of `index` and `curr`?

Solution

Stop: `index == i or curr is None`.
Continue: `index < i and curr is not None`.

(c) [**6 marks**] Implement the `bisect` method, including the `while` loop as described above.

You may not use any `LinkedList` or `_Node` methods other than their initializers; you may access all attributes (public or private) of both the `LinkedList` and `_Node` classes.

```
def bisect(self, i: int) -> LinkedList:
    """Remove from this linked list the nodes at position i and beyond,
    and return them in a new linked list.

    The nodes in the new linked list should be in the same order as they
    were in <self>.

    Raise an IndexError if i < 0 or i >= the length of this linked list.

    Note: this is a mutating method because it modifies the contents
    of <self>.
    """
```

Solution

```
    if i < 0:
        raise IndexError

    index = 0
    curr = self._first

    # We really need to iterate to the (i-1)-th node.
    while index < i - 1 and curr is not None:
        index += 1
        curr = curr.next

    if curr is None or curr.next is None:
        raise IndexError
    elif i == 0:
        result = LinkedList([])
        result._first = self._first
        self._first = None
        return result
    else:
        # Note: this still works if curr.next is None, but this case
        # is specified in the docstring.
        result = LinkedList([])
        result._first = curr.next
        curr.next = None
        return result
```

7. **[5 marks]** Recall from Assignment 2 the definition of the Block class. We've provided the full docstring as a reference, but you won't need most of it for this question.

```
class Block:
    """A square block in the Blocky game.

    === Public Attributes ===
    position:
        The (x, y) coordinates of the upper left corner of this Block.
        Note that (0, 0) is the top left corner of the window.
    size:
        The height and width of this Block.  Since all blocks are square,
        we needn't represent height and width separately.
    colour:
        If this block is not subdivided, <colour> stores its colour.
        Otherwise, <colour> is None and this block's sublocks store their
        individual colours.
    level:
        The level of this block within the overall block structure.
        The outermost block, corresponding to the root of the tree,
        is at level zero.  If a block is at level i, its children are at
        level i+1.
    max_depth:
        The deepest level allowed in the overall block structure.
    highlighted:
        True iff the user has selected this block for action.
    children:
        The blocks into which this block is subdivided.  The children are
        stored in this order: upper-right child, upper-left child,
        lower-left child, lower-right child.
    parent:
        The block that this block is directly within.

    === Representation Invariations ===
    - len(children) == 0 or len(children) == 4
    - If this Block has children,
        - their max_depth is the same as that of this Block,
        - their size is half that of this Block,
        - their level is one greater than that of this Block,
        - their position is determined by the position and size of this Block,
          as defined in the Assignment 2 handout, and
        - this Block's colour is None
    - If this Block has no children,
        - its colour is not None
    - level <= max_depth
    """
    position: Tuple[int, int]
    size: int
    colour: Optional[Tuple[int, int, int]]
    level: int
    max_depth: int
    highlighted: bool
    children: List['Block']
    parent: Optional['Block']
```

Implement the new `Block` method `colour_valid` according to its docstring. Your solution must be recursive; you may not use *any* `Block` methods, but you may access all `Block` attributes.

```
def colour_valid(self) -> bool:
    """Return whether this Block and all Blocks contained in this block
    satisfy these representation invariants:

        - If a Block has children, it has exactly four children and its own colour is None.
        - If a Block has no children, its colour is not None.

    Note: normally we assume that representation invariants are always satisfied.
    But the purpose of this method is to write an explicit check for these invariants,
    so don't simply write 'return True'.
    """
```

Solution

```
def colour_valid(self) -> bool:
    """Return whether or not this Block satisfies its representation
    invariants concerning colour:
        - If this Block has children, it has exactly four children and its own colour is None.
        - If this Block has no children, its colour is not None.

    >>> from renderer import BOARD_SIZE
    >>> block = Block(0)
    >>> board = block.random_init(0, 4)
    >>> board.update_block_locations((0, 0), BOARD_SIZE)
    >>> board.colour_valid()
    True
    """
    if len(self.children) == 0:
        return self.colour is not None
    else:
        if len(self.children) != 4:
            return False
        for kid in self.children:
            if not kid.colour_valid():
                return False
        return self.colour is None
```

8. **[11 marks]** Please refer to the documentation for the `BinarySearchTree` class for this question.

(a) **[3 marks]** Here is the code for the `BinarySearchTree` method `height`.

```
1   def height(self) -> int:
2       """Return the height of this BST.
3       """
4       if self.is_empty():
5           return 0
6       else:
7           return 1 + max(self._left.height(), self._right.height())
```

What would happen if either `self._left` or `self._right` were `None` at line 7?

Solution

There would be an `AttributeError` raised.

How do we know that neither of them can be `None` when we reach this line? Be specific.

Solution

At this line, we know the tree is non-empty. Then because of the representation invariant for the `BinarySearchTree` class, we know `self._left` and `self._right` are `BinarySearchTree`s, not `None`.

(b) **[2 marks]** Suppose we have a binary search tree with n items and height h. What is the Big-Oh running time of this method, in terms of n and/or h? Be as specific as possible, and explain your reasoning.

Solution

The runtime would be $O(n)$. Each call to `height` always recurses on both subtrees; this means that there will be a recursive call made for each item in the tree.

(c) [**6 marks**] The `BinaryTree` class is the same as the `BinarySearchTree` class, except that its instances do not necessarily satisfy the BST property. You are given the following documentation for the class:

```
class BinaryTree:
    # === Private Attributes ===
    _root: Optional[object]
    _left: Optional['BinaryTree']
    _right: Optional['BinaryTree']

    def is_empty(self) -> bool:
        """Return whether this binary tree is empty."""

    def min_max(self) -> Tuple[int, int]:
        """Return the minimum and maximum value stored in this binary tree.
        The returned tuple contains (min value, max value), in that order.

        Precondition: this binary tree contains only integers.
        """
```

Implement the following `BinaryTree` method according to its docstring. You may access all `BinaryTree` attributes, but the only `BinaryTree` methods you may use are `is_empty` and `min_max` (assume they're implemented correctly already). Your solution must be recursive.

```
    def is_bst(self) -> bool:
        """Return whether this binary tree is a binary *search* tree.

        An empty binary tree satisfies the binary search tree property.
        """
```

Solution

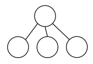
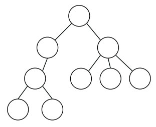
```
    if self.is_empty():
        return True
    elif not (self._left.is_bst() and self._right.is_bst()):
        return False
    else:
        # Check BST property for root.
        if not self._left.is_empty() and self._left.min_max()[1] > self._root:
            return False
        elif not self._right.is_empty() and self._right.min_max()[0] < self._root:
            return False
        else:
            return True
```

9. **[6 marks]** The following `Tree` method contains multiple errors.

```
class Tree:
    def num_at_or_below(self, depth: int) -> int:
        """Return the number of items in this tree at depth >= the given depth.
        The tree's root has depth 1.
        """
        sum = 0
        if depth <= 1:
            # Count the root
            sum = sum + 1
        for subtree in self._subtrees:
            # Count the relevant items for each subtree
            subtree.num_at_or_below(depth - 1)
        return sum
```

(a) **[2 marks]** For each test case in the table below, write what the method *should* return (according to its docstring) for the given tree and value of `depth`. Note that we have drawn only the tree structure; the actual values stored in the tree do not affect the behaviour of this method.

Solution

Tree	Expected result for `depth = 1`	Expected result for `depth = 3`
empty tree	Test case 1 0	Test case 2 0
	Test case 3 1	Test case 4 0
	Test case 5 4	Test case 6 0
	Test case 7 9	Test case 8 6

(b) **[4 marks]** Identify two different code errors in the given implementation of `num_at_or_below`, and for each one explain how it would cause one of the test cases from the previous part to have an incorrect *actual* result.

Problem 1:

Problem 2:

Solution

- Problem 1: when `d == 1`, the code always counts the root, even if the tree is empty. This would cause Test Case 1 to fail.

- Problem 2: the code does not use the recursive calls to update `sum`, and so `sum` will always be 0 or 1 (i.e., never counts any nodes in the subtrees). This would cause Test Cases 5, 7, or 8 to fail.

10. **[5 marks]** At McDonalds, it is possible to buy chicken nuggets ("McNuggets") in packs of 4, 6, or 25. Consider the following function, which takes a number and returns whether it is possible to buy *exactly* that many chicken nuggets from McDonalds.

```
def buyable(n: int) -> bool:
    """Return whether one can buy exactly <n> McNuggets.

    It is considered possible to buy exactly 0 McNuggets.

    Precondition: n >= 0

    >>> buyable(6)
    True
    >>> buyable(35)
    True
    >>> buyable(5)
    False
    >>> buyable(13)
    False
    """
```

(a) **[1 mark]** Suppose you know that it's possible to buy exactly 1231 McNuggets. Is it possible to buy 1235 McNuggets? How do you know?

Solution

Yes, it's possible. Take the McNegget combination for 1231, and then add one pack of 4.

(b) **[4 marks]** Implement the `buyable` function. Your solution must be recursive, and you may not write any helper functions for this question.

Solution

```
def buyable(n: int) -> bool:
    if n == 0:
        return True
    elif n >= 4 and buyable(n - 4):
        return True
    elif n >= 6 and buyable(n - 6):
        return True
    elif n >= 25 and buyable(n - 25):
        return True
    else:
        return False
```

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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List methods

```
lst = [1, 2, 3]
len(lst)             # 3
lst[0]               # 1
lst[0:2]             # [1, 2]
lst[0] = 'howdy'     # lst == ['howdy', 2, 3]
lst.append(29)       # lst == ['howdy', 2, 3, 29]
lst.pop()            # lst == ['howdy', 2, 3], returns 29
lst.pop(1)           # lst == ['howdy', 3], returns 2
lst.insert(1, 100)   # lst == ['howdy', 100, 3]
lst.extend([4, 5])   # lst == ['howdy', 100, 3, 4, 5]
3 in lst             # returns True
```

Dictionary methods

```
d = {'hi': 4, 'bye': 100}
d['hi']            # 4
d[100]             # raises KeyError!
'hi' in d          # True
4 in d             # False
d['howdy'] = 15    # adds new key-value pair
d['hi'] = -100     # changes a key-value pair
```

Classes

```
class Point:
    x: int
    y: int

    def __init__(self, x: int, y: int) -> None:
        self.x = x
        self.y = y

    def size(self) -> float:
        return (self.x ** 2 + self.y ** 2) ** 0.5

p = Point(3, 4)    # initializer
p.x                # attribute access: returns 3
p.size()           # method call: returns 5.0

class MyWeirdClass(Point):  # inheritance
    pass
```

Exceptions

```
class MyCustomError(Exception):
    pass

raise MyCustomError
```

Stack and Queues

```
s = Stack()
s.is_empty()
s.push(10)
s.pop()  # Raises an EmptyStackError if the stack is empty.

q = Queue()
q.is_empty()
q.enqueue(10)
q.dequeue()  # Returns None if the Queue is empty
```

Linked List

```
class _Node:
    """A node in a linked list.

    === Attributes ===
    item: The data stored in this node.
    next:
        The next node in the list, or None if there are
        no more nodes in the list.
    """
    item: object
    next: Optional['_Node']

    def __init__(self, item: object) -> None:
        """Initialize a new node storing <item>,
        with no 'next' node.
        """

class LinkedList:
    """A linked list implementation of the List ADT.
    """
    # === Private Attributes ===
    # _first:
    #     The first node in the linked list,
    #     or None if the list is empty.
    _first: Optional['_Node']

    def __init__(self, items: list) -> None:
        """Initialize a linked list with the given items.

        The first node in the linked list contains the
        first item in <items>.
        """
```

Tree

```
class Tree:
    # === Private Attributes ===
    # The item stored at this tree's root, or None if the tree is empty.
    _root: Optional[object]
    # The list of all subtrees of this tree.
    _subtrees: List['Tree']

    # === Representation Invariants ===
    # - If self._root is None then self._subtrees is an empty list.
    #   This setting of attributes represents an empty Tree.
    # - self._subtrees may be empty when self._root is not None.
    #   This setting of attributes represents a tree consisting of just one
    #   node.

    def __init__(self, root: object, subtrees: List['Tree']) -> None:
        """Initialize a new Tree with the given root value and subtrees.

        If <root> is None, the tree is empty.
        Precondition: if <root> is None, then <subtrees> is empty.
        """

    def is_empty(self) -> bool:
        """Return True if this tree is empty."""
```

BinarySearchTree

```
class BinarySearchTree:
    # === Private Attributes ===
    # The item stored at the root of the tree, or None if the tree is empty.
    _root: Optional[object]
    # The left subtree, or None if the tree is empty
    _left: Optional['BinarySearchTree']
    # The right subtree, or None if the tree is empty
    _right: Optional['BinarySearchTree']

    # === Representation Invariants ===
    #  - If _root is None, then so are _left and _right.
    #    This represents an empty BST.
    #  - If _root is not None, then _left and _right are BinarySearchTrees.
    #  - (BST Property) All items in _left are <= _root,
    #    and all items in _right are >= _root.

    def __init__(self, root: Optional[object]) -> None:
        """Initialize a new BST with the given root value and no children.

        If <root> is None, make an empty tree, with subtrees that are None.
        If <root> is not None, make a tree with subtrees are empty trees.
        """

    def is_empty(self) -> bool:
        """Return True if this BST is empty."""
```