• PS2 marking: almost done, but not quite...
• TT2 marking: in progress – hard to say when it will be done

---

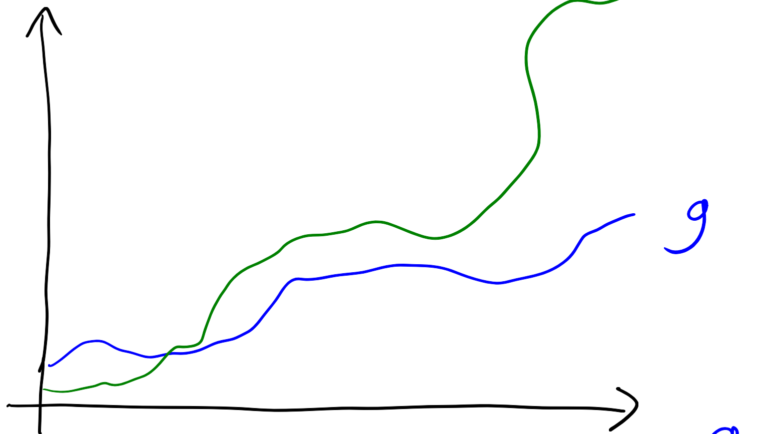
Recap: O, Ω, Θ (asymptotic notation)

For all functions $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

• $g \in O(f)$

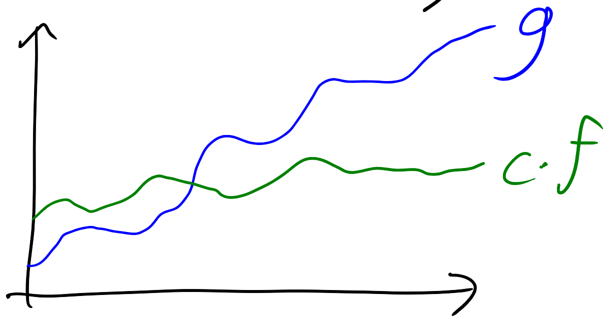
$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},$
$n \geq n_0 \Rightarrow g(n) \leq c f(n)$

"f is an upper bound on g"



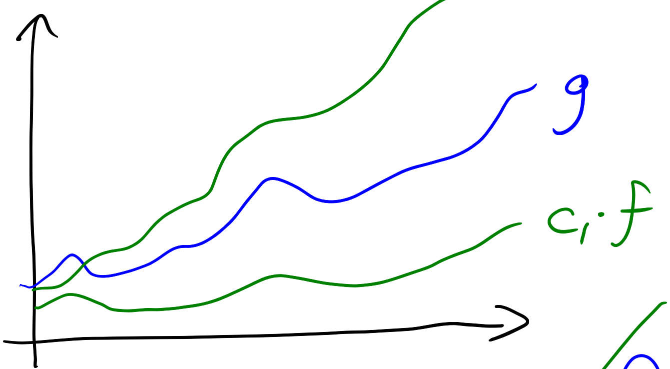
• $g \in \Omega(f)$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},$
$n \geq n_0 \Rightarrow g(n) \geq c \cdot f(n)$



"f is a <u>lower bound</u> on g"

<u>Note:</u> $g \in \Omega(f) \iff f \in O(g)$

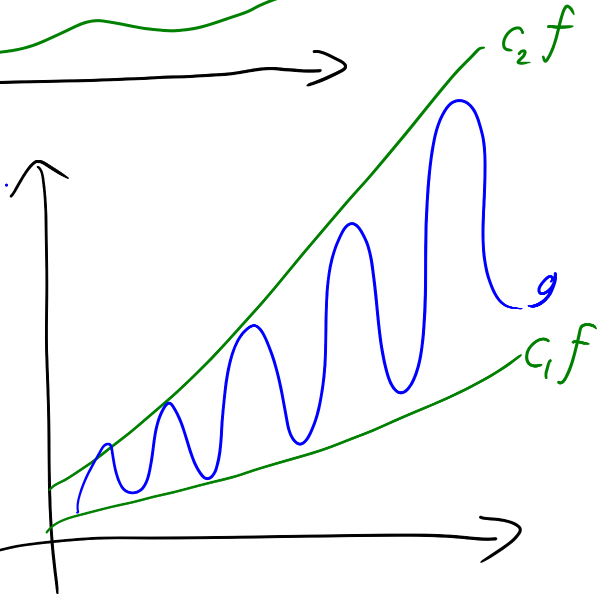


- $g \in \Theta(f)$

$g \in O(f) \wedge g \in \Omega(f)$

"f is a <u>tight bound</u> on g"

<u>Note:</u> $g \in \Theta(f) \not\Rightarrow g = f$



Ex: Prove that $\forall a,b \in \mathbb{R}^+$, $an+b \notin \Omega(n^2)$.

Let $a,b \in \mathbb{R}^+$.

WTS: $an+b \notin \Omega(n^2)$, i.e.,

$\hookrightarrow \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge an+b < c \cdot n^2$

Let $c, n_0 \in \mathbb{R}^+$.

Let $n =$ ___?___

WTS: $n \geq n_0 \wedge an+b < cn^2$

ROUGH WORK

KNOW

$a, b, c, n_0 \in \mathbb{R}^+$

$an+b < cn^2$

- idea 1: solve for n

WANT

$n \in \mathbb{N}$

$n \geq n_0$

$an+b < cn^2$

- idea 2: $an + b < cn^2 \iff \frac{a}{c}n + \frac{b}{c} < n^2$

if $\frac{a}{c}n < \frac{n^2}{2}$ and $\frac{b}{c} < \frac{n^2}{2}$, then

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$n > \frac{2a}{c} \qquad\qquad n > \sqrt{\frac{2b}{c}}$$

(back to proof)

let $n = \max\left(\lceil n_0 \rceil, \left\lceil \frac{2a}{c} \right\rceil, \left\lceil \sqrt{\frac{2b}{c}} \right\rceil\right) + 1$

[$n = \left\lceil n_0 + \frac{2a}{c} + \sqrt{\frac{2b}{c}} \right\rceil$ would also work]

Then, $n > \lceil n_0 \rceil \implies n \geq n_0$

and $n > \frac{2a}{c} \implies \frac{n^2}{2} > \frac{a}{c}n$

and $n > \sqrt{\frac{2b}{c}} \implies \frac{n^2}{2} > \frac{b}{c}$

$$\left.\begin{array}{l} \frac{n^2}{2} > \frac{a}{c}n \\ \frac{n^2}{2} > \frac{b}{c} \end{array}\right\} \implies n^2 > \frac{a}{c}n + \frac{b}{c}$$

$\implies cn^2 > an + b$. $\square$

Properties of O, Ω, Θ

(Theorems 5.1 — 5.8 in the course notes.)

Example: Prove that $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$,

$$g \in O(f) \Rightarrow f+g \in \Theta(f).$$

(where $(f+g)(n) = f(n) + g(n)$, $\forall n \in \mathbb{N}$).

Proof: Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$.

Assume $g \in O(f)$: $\exists c_0, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)$

(note: this introduces variables c_0, n_0 in the proof)

WTS: $f+g \in \Theta(f)$: $\exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n' \in \mathbb{N}$,

$$n' \geq n_1 \Rightarrow c_1 f(n') \leq (f+g)(n') \leq c_2 f(n')$$

Let $c_1 = 1$, $c_2 = c_0 + 1$, $n_1 = n_0$

want: $c_1 f(n') \le f(n') + g(n')$

know: $g(n') \ge 0 \implies f(n') + g(n') \ge f(n')$

so pick $c_1 = 1$

want: $(f+g)(n') \le c_2 f(n')$

know: $g(n') \le c_0 f(n')$ as long as $n' \ge n_0$

$\implies f(n') + g(n') \le (c_0 + 1) f(n')$ so pick $c_2 = c_0 + 1$

only one condition on n' $(n' \ge n_0)$, so pick $n_1 = n_0$

EXERCISE: finish writing the proof...

For reference, here is one way to finish the proof.

Let $c_1 = 1$, $c_2 = c_0 + 1$, $n_1 = n_0$. Then $c_1, c_2, n_1 \in \mathbb{R}^+$

Let $n' \in \mathbb{N}$ and assume $n' \geq n_1$.

Then $f(n') \leq f(n') + g(n')$ (because $g(n') \geq 0$)

Also, $f(n') + g(n') \leq f(n') + c_0 f(n') = (c_0 + 1) f(n')$

(because $n \geq n_1 = n_0$ and by the assumption that $g \in O(f)$)

So $c_1 f(n') \leq (f+g)(n') \leq c_2 f(n')$. $\square$