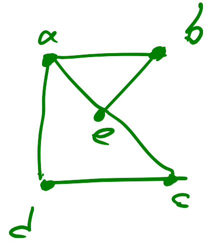
- PS4 office hours — check Quercus
  → FIRST: read PS4 FAQ on Piazza!

More graphs... examples of proofs...

[2] Notion of "connectivity"



G_1

For all $G = (V, E)$ and all $u, v \in V$

define $\boxed{\text{"}u, v \text{ are } \underline{\text{connected}} \text{ in } G\text{"}}$:

G contains a $\underline{\text{path}}$ between u and v

$\big(\exists k \in \mathbb{N}, \exists v_1, \ldots, v_k \in V, (u, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v) \in E \big)$

$k = 0$ is allowed! $\Rightarrow (u, v) \in E$

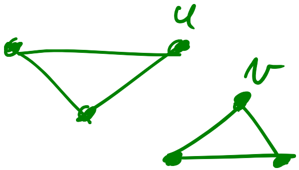
e.g.: a, b are connected because $(a, b) \in E$

b, c are connected because $(b, e), (e, c) \in E$

→ predicate of G, u, v

Special case: "u, u are connected in G" is always true for all $u \in V$ (where $G = (V, E)$).

Ex of G, u, v s.t. u, v are not connected in G:

G_2: 

Def: "G is connected": $\forall u, v \in V$, u, v are connected in G

e.g., G_1 is connected

G_2 is not connected

Example 3: study necessary and sufficient conditions on $|E|$ for G to be connected

- sufficient: condition $\Rightarrow$ G is connected

- necessary: $\neg$ condition $\Rightarrow$ G is <u>not</u> connected

  (G is connected $\Rightarrow$ condition)

<u>Necessary condition</u>: $|E| \geq |V| - 1$

$\boxed{\text{proof — see course notes}}$

<u>Sufficient condition</u>:

$$|E| \geq \frac{(|V|-1)(|V|-2)}{2} + 1$$

<u>Proof</u>: WTS: $\boxed{\forall G=(V,E),\ |E| \geq \frac{(|V|-1)(|V|-2)}{2}+1 \Rightarrow \underline{G \text{ is connected}}.}$

- <u>Idea 1</u>: direct proof

  Let $G=(V,E)$. Assume $|E| \geq \frac{(|V|-1)(|V|-2)}{2}+1$.

  WTS: G is connected.

... not clear how to proceed...

- **Idea 2**: indirect proof: might work better...
  but, we want to demonstrate another idea
- **Idea 3**: induction?
  Q: induction on what?
  - **Insight**: Introduce a new variable to do induction on. Generally, new variable = size of objects in the proof.

$$\forall n \in \mathbb{Z}^+, \boxed{\forall G=(V,E),\ |V| = n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2} + 1 \Rightarrow G \text{ is connected}\right)}$$

- how we express "size of $G = n$"
- not the only possibility

- **P(n)**:

- Base Case: WTS $P(1)$:

$$\forall G=(V,E),\ |V|=1 \Rightarrow \left(|E| \geq \frac{(1-1)(1-2)}{2}+1 \Rightarrow G \text{ is connected}\right)$$

Proof: exercise...

- IH: Let $n \in \mathbb{Z}^+$ and assume $P(n)$:

$$\forall G=(V,E),\ |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected}\right)$$

- I.S.: WTS $P(n+1)$:

$$\forall G_1=(V_1,E_1),\ |V_1|=n+1 \Rightarrow \left(|E_1| \geq \frac{(n+1-1)(n+1-2)}{2}+1 \Rightarrow G_1 \text{ is connected}\right)$$

Let $G_1=(V_1,E_1)$ and $|V_1|=n+1$, and assume $|E_1| \geq \frac{n(n-1)}{2}+1$.

WTS: G_1 is connected.

NOTE: start from graph of size $n+1$
(not n)