Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the definition of Big-O and its negation.
- Represent constant functions in Big-O expressions.
- Understand and use the definition of Omega and Theta to compare functions.

For your reference, here is the formal definition of Big-O:

$$g \in \mathcal{O}(f):\ \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq cf(n) \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

1. **Constant functions**. As we discussed in class, *constant functions*, like $f(n) = 100$, will play an important role in our analysis of running time next week. For now let's get comfortable with the notation.

   (a) Let $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Show how to express the statement $g \in \mathcal{O}(1)$ by expanding the definition of Big-O.[1]

   Solution

   $g \in \mathcal{O}(1) : \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c.$

   (b) Prove that $100 + \dfrac{77}{n+1} \in \mathcal{O}(1)$.

   Note: this proof isn't too mathematically complex; treat this as another exercise in making sure you understand the definition of Big-O.

   Hint: one algebraic property of inequalities is that $\forall x, y \in \mathbb{R}^+,\ x \geq y \Rightarrow \dfrac{1}{x} \leq \dfrac{1}{y}$.

   Solution

   We want to prove that $\exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow 100 + \dfrac{77}{n+1} \leq c$.

   There are many possible choices of c and n_0 here. One possibility is $c = 101$ and $n = 76$. We leave the calculation as an exercise.

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[1] Remember that we often abbreviate Big-O expressions to just show the function bodies. "$\mathcal{O}(1)$" is really shorthand for "$\mathcal{O}(f)$, where f is the constant function $f(n) = 1$."

2. **Omega**. Recall that we can think of Big-O notation as describing an *upper bound* on the rate of growth of a function: saying "$g \in \mathcal{O}(f)$" is like saying "g grows at most as fast as f." Sometimes we care just as much about a *lower bound* on the rate of growth and for this, we have the symbol Ω (the Greek letter Omega), which is defined analogously to Big-O:

$$g \in \Omega(f) : \ \exists c, n_0 \in \mathbb{R}^+, \ \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow g(n) \geq cf(n) \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

Using this definition, prove that for all $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $g \in \mathcal{O}(f)$, then $f \in \Omega(g)$.

Solution

Proof. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume that $g \in \mathcal{O}(f)$, i.e., that there exist $c_1, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $g(n) \leq c_1 f(n)$. We want to prove that there exist $c_2, n_2 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_2$ then $f(n) \geq c_2 g(n)$.

Let $c_2 = \dfrac{1}{c_1}$, and $n_2 = n_1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_2$. We want to prove that $f(n) \geq c_2 g(n)$.

Since $n_2 = n_1$, we know from our assumption that $n \geq n_1$. So then by our first assumption (that $g \in \mathcal{O}(f)$), we know that $g(n) \leq c_1 f(n)$. Dividing both sides by c_1 yields $\dfrac{1}{c_1} g(n) \leq f(n)$, and so $c_2 g(n) \leq f(n)$. $\square$

3. **Theta**. Both Big-O and Omega are limited in the same way as inequalities on numbers. "$2 \leq 10^{10}$" is a true statement, but not very insightful; similarly, "$n + 1 \in \mathcal{O}(n^{10})$" and "$2^n + n^2 \in \Omega(n)$" are both true, but not very precise.

   Our final piece of asymptotic notation is Θ (the Greek letter Theta), which we define as:

$$g \in \Theta(f) : \; g \in \mathcal{O}(f) \wedge g \in \Omega(f) \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

   Or equivalently,

$$g \in \Theta(f) : \; \exists c_1, c_2, n_0 \in \mathbb{R}^+, \; \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n) \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

   When we write $g \in \Theta(f)$, what we mean is "g grows at most as quickly as f and g grows at least as quickly as f"—in other words, that f and g have the *same* rate of growth. In this case, we call f a **tight bound** on g, since g is essentially squeezed between constant multiples of f.

   Prove that for all functions $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and all numbers $a \in \mathbb{R}^{\geq 0}$, if $g \in \Omega(1)$, then $a + g \in \Theta(g)$.[2]

   (Or in other words, for such functions g, shifting them by a constant amount does not change their "Theta" bound.)

   Solution

   Proof. Let $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and let $a \in \mathbb{R}^{\geq 0}$. Assume that $g \in \Omega(1)$, i.e., that there exist $c_0, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$ then $g(n) \geq c_0$. We want to prove that $a + g \in \Theta(g)$, i.e., that there exist $c_1, c_2, n_1 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_1$ then $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

   Let $c_1 = 1$, $c_2 = \dfrac{a}{c_0} + 1$, and $n_1 = n_0$. Let $n \in \mathbb{N}$, and assume that $n \geq n_1$. We want to prove that $c_1 g(n) \leq a + g(n) \leq c_2 g(n)$.

   [We leave the calculation as an exercise. The trickiest part was figuring out how to choose c_2; the intuition is that we need to take the assumed inequality $g(n) \geq c_0$ and turn the right-hand side into a instead of c_0.] ☐

[2] Here we use $a + g$ to denote the function g_1 defined as $g_1(n) = a + g(n)$ for all $n \in \mathbb{N}$.

4. **Negating Big-O**. So far, we have only looked at proving that a function *is* Big-O of another function. In this question, we'll investigate what it means to show that a function *isn't* Big-O of another.

   (a) Express the statement $g \notin \mathcal{O}(f)$ in predicate logic, using the expanded definition of Big-O. (As usual, simplify so that all negations are pushed as far "inside" as possible.)

   Solution

   $g \notin \mathcal{O}(f) : \forall c, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ n \geq n_0 \wedge g(n) > cf(n)$

   (b) Prove that for all positive real numbers a and b, if $a > b$ then $n^a \notin \mathcal{O}(n^b)$.

   Solution

   Discussion. In the proof below, we need to find a value of $n \in \mathbb{N}$ that satisfies two different inequalities: $n \geq n_0$ and $n^a > cn^b$. A general technique to approach this is to find values n_1 and n_2 that satisfy each inequality separately, and then let $n = n_1 + n_2$ or $n = \max(n_1, n_2)$, so that the chosen n will satisfy both.

   Proof. Let $a, b \in \mathbb{R}^+$, and assume that $a > b$. We want to show the following:

   $$\forall c, n_0 \in \mathbb{R}^+,\ \exists n \in \mathbb{N},\ n \geq n_0 \wedge n^a > cn^b$$

   Let $c, n_0 \in \mathbb{R}^+$. Let $n = \left\lceil n_0 + c^{1/(a-b)} \right\rceil$.* We want to prove that $n \geq n_0$ and $n^a > cn^b$.

   [*We leave the rest of the proof as an exercise.*] □

   *The ceiling function in the choice of n is used to ensure that n is a natural number.