For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

1. [5 marks] Asymptotic Notation I.

You may use `https://www.desmos.com/calculator` to look at the graph of the function in this question, *but NO other online resource is allowed* for any question on this test. Also, you still need to provide rigorous arguments for each proof: remember that *a graph is NOT a rigorous argument.*

Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n^2(\cos(n\pi) + 1) + 1.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(n^2)$

(b) **[3 marks]** $f \notin \Theta(n^2)$

Solution

(a) Since $-1 \leq \cos(n\pi) \leq 1$, we have that

$$n^2(\cos(n\pi) + 1) \leq 2n^2,$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 1$ and $c = 3$,

$$f(n) = n^2(\cos(n\pi) + 1) + 1 \leq 2n^2 + 1 \leq cn^2,$$

for all $n \geq n_0$.

(b) Since we already know that $f \in O(n^2)$, showing that $f \notin \Theta(n^2)$ is equivalent to showing that $f \notin \Omega(n^2)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = n^2(\cos(n\pi) + 1) + 1 < cn^2.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\lceil\sqrt{1/c}\rceil + 1, \lceil n_0 \rceil)$. Clearly, $1 < cn_1^2$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is even, and let $n = n_1$ is n_1 is odd. Since $\cos(n\pi) = -1$ whenever n is odd, we have that

$$f(n) = n^2(0) + 1 = 1.$$

Since $cn^2 > 1$, it follows that

$$f(n) = 1 < cn^2,$$

and we are done.

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1. [5 marks] Asymptotic Notation I.

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n^2(\cos(n\pi) + 1) + 1.$$

Prove that:

(a) **[2 marks]** $f \in \Omega(1)$

(b) **[3 marks]** $f \notin \Theta(1)$

Solution

(a) Since $-1 \leq \cos(n\pi) \leq 1$, we have that

$$n^2(\cos(n\pi) + 1) + 1 \geq 1, \tag{1}$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 0$ and $c = 1$,

$$f(n) = n^2(\cos(n\pi) + 1) + 1 \geq c, \tag{2}$$

for all $n \geq n_0$.

(b) Since we already know that $f \in \Omega(1)$, showing that $f \notin \Theta(1)$ is equivalent to showing that $f \notin O(1)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = n^2(\cos(n\pi) + 1) + 1 > c.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\sqrt{(c-1)/2} + 1, \lceil n_0 \rceil)$. Clearly, $2n_1^2 + 1 > c$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is odd, and let $n = n_1$ is n_1 is even. Since $\cos(n\pi) = 1$ whenever n is even, we have that

$$f(n) = n^2(2) + 1 = 2n^2 + 1.$$

Since $2n^2 + 1 > c$, it follows that

$$f(n) = 2n^2 + 1 > c,$$

and we are done.

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1. **[5 marks] Asymptotic Notation I.**

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n!(1 + (-1)^n) + n.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(n!)$

(b) **[3 marks]** $f \notin \Theta(n!)$

<u>Solution</u>

(a) Since $-1 \leq (-1)^n \leq 1$, we have that

$$n!(1 + (-1)^n) + n \leq 2n! + n, \tag{1}$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 1$ and $c = 3$,

$$f(n) = n!(1 + (-1)^n) + n \leq 2n! + n \leq cn!, \tag{2}$$

for all $n \geq n_0$.

(b) Since we already know that $f \in O(n!)$, showing that $f \notin \Theta(n!)$ is equivalent to showing that $f \notin \Omega(n!)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = n!(1 + (-1)^n) + n < cn!.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\lceil 1/c \rceil + 1, \lceil n_0 \rceil)$. Clearly, $n_1 < cn_1!$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is even, and let $n = n_1$ is n_1 is odd. Since $(-1)^n = -1$ whenever n is odd, we have that

$$f(n) = n!(0) + n = n.$$

Since $n < cn!$, it follows that

$$f(n) = n < cn!,$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

1. **[5 marks] Asymptotic Notation I.**

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n!(1 + (-1)^n) + n.$$

Prove that:

(a) **[2 marks]** $f \in \Omega(n)$

(b) **[3 marks]** $f \notin \Theta(n)$

Solution

(a) Since $-1 \leq (-1)^n \leq 1$, we have that

$$n!(1 + (-1)^n) + n \geq n, \tag{1}$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 0$ and $c = 1$,

$$f(n) = n!(1 + (-1)^n) + n \geq cn, \tag{2}$$

for all $n \geq n_0$.

(b) Since we already know that $f \in \Omega(n)$, showing that $f \notin \Theta(n)$ is equivalent to showing that $f \notin O(n)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = n!(1 + (-1)^n) + n > cn.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\lceil (c-1)/2 \rceil + 1, \lceil n_0 \rceil)$. Clearly, $2n_1! + n_1 > cn_1$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is odd, and let $n = n_1$ is n_1 is even. Since $(-1)^n = 1$ whenever n is even, we have that

$$f(n) = n!(1 + (-1)^n) + n = 2n! + n.$$

Since $2n! + n > cn$, it follows that

$$f(n) = 2n! + n > cn,$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

1. [5 marks] Asymptotic Notation I.

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = (1 + (-1)^n)/n + 1/n^2.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(1/n)$

(b) **[3 marks]** $f \notin \Theta(1/n)$

Solution

(a) Since $-1 \leq (-1)^n \leq 1$, we have that

$$(1 + (-1)^n)/n + 1/n^2 \leq 2/n + 1/n^2,$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 1$ and $c = 3$,

$$f(n) = (1 + (-1)^n)/n + 1/n^2 \leq 2/n + 1/n^2 \leq c/n,$$

for all $n \geq n_0$.

(b) Since we already know that $f \in O(1/n)$, showing that $f \notin \Theta(1/n)$ is equivalent to showing that $f \notin \Omega(1/n)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = (1 + (-1)^n)/n + 1/n^2 < c/n.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\lceil 1/c \rceil + 1, \lceil n_0 \rceil)$. Clearly, $1/n_1^2 < c/n_1$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is even, and let $n = n_1$ is n_1 is odd. Since $(-1)^n = -1$ whenever n is odd, we have that

$$f(n) = (0)/n + 1/n^2 = 1/n^2.$$

Since $1/n^2 < c/n$, it follows that

$$f(n) = 1/n^2 < c/n,$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

1. [5 marks] Asymptotic Notation I.

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = (1 + (-1)^n)/n + 1/n^2.$$

Prove that:

(a) **[2 marks]** $f \in \Omega(1/n^2)$

(b) **[3 marks]** $f \notin \Theta(1/n^2)$

Solution

(a) Since $-1 \leq (-1)^n \leq 1$, we have that

$$(1 + (-1)^n)/n + 1/n^2 \geq 1/n^2, \tag{1}$$

for all $n \in \mathbb{N}$. Thus, letting $n_0 = 0$ and $c = 1$,

$$f(n) = (1 + (-1)^n)/n + 1/n^2 \geq c/n^2, \tag{2}$$

for all $n \geq n_0$.

(b) Since we already know that $f \in \Omega(1/n^2)$, showing that $f \notin \Theta(1/n^2)$ is equivalent to showing that $f \notin O(1/n^2)$. This amounts to showing that, for each $c, n_0 \in \mathbb{R}^+$, there exists an $n \geq n_0$ such that

$$f(n) = (1 + (-1)^n)/n + 1/n^2 > c/n^2.$$

Suppose that $c, n_0 \in \mathbb{R}^+$. Let $n_1 = \max(\lceil (c-1)/2 \rceil + 1, \lceil n_0 \rceil)$. Clearly, $2/n_1 + 1/n_1^2 > c/n_1^2$ and $n_1 \geq n_0$. Now, let $n = n_1 + 1$ if n_1 is odd, and let $n = n_1$ is n_1 is even. Since $(-1)^n = 1$ whenever n is even, we have that

$$f(n) = (1 + (-1)^n)/n + 1/n^2 = 2/n + 1/n^2.$$

Since $2/n + 1/n^2 > c/n^2$, it follows that

$$f(n) = 2/n + 1/n^2 > c/n^2,$$

and we are done.

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **largest** number x such that $(x)_2$ is 4-digits long.

(b) The **smallest** number x such that $(x)_{16}$ is 5-digits long and contains exactly two A's and one E, with no leading 0's.

(c) The **smallest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution

(a) x is represented by $(1111)_2$, so

$$x = 2^3 + 2^2 + 2 + 1 = 15.$$

(b) x is represented by $(10AAE)_{16}$, so

$$x = 16^4 + 10 \cdot 16^2 + 10 \cdot 16 + 14 = 68270.$$

(c) x is represented by $(10201)_8$, so

$$x = 8^4 + 2 \cdot 8^2 + 1 = 4225.$$

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **smallest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's, with no leading 0's.

(b) The **largest** number x such that $(x)_8$ is 5-digits long and contains exactly two 2's and one 7.

(c) The **smallest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution

(a) x has the number representation $(10011)_2$, so

$$x = 2^4 + 2 + 1 = 19.$$

(b) x has the number representation $(76622)_8$, so

$$x = 7 \cdot 8^4 + 6 \cdot 8^3 + 6 \cdot 8^2 + 2 \cdot 8 + 2 = 32146.$$

(c) x has the number representation $(10201)_{16}$, so

$$x = 16^4 + 2 \cdot 16^2 + 1 = 66049.$$

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **largest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's.

(b) The **largest** number x such that $(x)_{16}$ is 4-digits long, and no digit appears more than once.

(c) The **largest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, where each digit appears at most twice.

Solution

(a) x has the number representation $(11100)_2$, so

$$x = 2^4 + 2^3 + 2^2 = 28.$$

(b) x has the number representation $(FEDC)_{16}$, so

$$x = 15 \cdot 16^3 + 14 \cdot 16^2 + 13 \cdot 16 + 12 = 65244.$$

(c) x has the number representation $(76567)_8$, so

$$x = 7 \cdot 8^4 + 6 \cdot 8^3 + 5 \cdot 8^2 + 6 \cdot 8 + 7 = 32119.$$

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **largest** number x such that $(x)_2$ is 4-digits long.

(b) The **smallest** number x such that $(x)_8$ is 3-digits long, with no leading 0's, and no digit appears more than once.

(c) The **largest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, where each digit appears at most twice.

Solution

(a) x is represented by $(1111)_2$, so

$$x = 2^3 + 2^2 + 2 + 1 = 15.$$

(b) x has the number representation $(102)_8$, so

$$x = 8^2 + 2 = 66.$$

(c) x has the number representation $(FEDEF)_{16}$, so

$$x = 15 \cdot 16^4 + 14 \cdot 16^3 + 13 \cdot 16^2 + 14 \cdot 16 + 15 = 1043951.$$

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **smallest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's, with no leading 0's.

(b) The **largest** number x such that $(x)_{16}$ is 4-digits long, and no digit appears more than once.

(c) The **smallest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution

(a) x has the number representation $(10011)_2$, so

$$x = 2^4 + 2 + 1 = 19.$$

(b) x has the number representation $(FEDC)_{16}$, so

$$x = 15 \cdot 16^3 + 14 \cdot 16^2 + 13 \cdot 16 + 12 = 65244.$$

(c) x is represented by $(10201)_8$, so

$$x = 8^4 + 2 \cdot 8^2 + 1 = 4225.$$

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **largest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's.

(b) The **largest** number x such that $(x)_8$ is 5-digits long and contains exactly two 2's and one 7.

(c) The **largest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, where each digit appears at most twice.

Solution

(a) x has the number representation $(11100)_2$, so

$$x = 2^4 + 2^3 + 2^2 = 28.$$

(b) x has the number representation $(76622)_8$, so

$$x = 7 \cdot 8^4 + 6 \cdot 8^3 + 6 \cdot 8^2 + 2 \cdot 8 + 2 = 32146.$$

(c) x has the number representation $(FEDEF)_{16}$, so

$$x = 15 \cdot 16^4 + 14 \cdot 16^3 + 13 \cdot 16^2 + 14 \cdot 16 + 15 = 1043951.$$

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

3. [4 marks] Induction.

Warning! This question does not require deep insight but it is longer to write up (you may need more than 1 page). You should keep it for last. Also, *you will receive* ***at most*** *half the marks* if you do NOT use induction.

Let $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_0, b_1, b_2, \ldots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$, if $n \geq 1$, then

$$\sum_{k=0}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

Solution

Base case: let $n = 1$. Clearly,

$$\sum_{k=0}^{0}(a_{k+1} - a_k)b_k = (a_1 - a_0)b_0$$

and

$$\sum_{k=0}^{0} a_{k+1}(b_{k+1} - b_k) = a_1(b_1 - b_0).$$

Thus,

$$\begin{aligned}
&\sum_{k=0}^{0}(a_{k+1} - a_k)b_k = (a_1 - a_0)b_0 \\
&= a_1 b_0 - a_0 b_0 \\
&= a_1 b_1 - a_0 b_0 - (a_1 b_1 - a_1 b_0) \\
&= a_1 b_1 - a_0 b_0 - a_1(b_1 - b_0) \\
&= a_1 b_1 - a_0 b_0 - \sum_{k=0}^{0} a_{k+1}(b_{k+1} - b_k),
\end{aligned}$$

so the base case holds.

Inductive step: Suppose that $n \in \mathbb{N}^+$ and

$$\sum_{k=0}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

Then

$$\sum_{k=0}^{n-1}(a_{k+1} - a_k)b_k + (a_{n+1} - a_n)b_n = a_n b_n - a_0 b_0 + (a_{n+1} - a_n)b_n - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k),$$

so

$$\sum_{k=0}^{n}(a_{k+1} - a_k)b_k = a_{n+1}b_n - a_0b_0 - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

Thus,

$$\sum_{k=0}^{n}(a_{k+1} - a_k)b_k = a_{n+1}b_n - a_0b_0 + a_{n+1}(b_{n+1} - b_n) - \sum_{k=0}^{n} a_{k+1}(b_{k+1} - b_k),$$

so

$$\sum_{k=0}^{n}(a_{k+1} - a_k)b_k = a_{n+1}b_{n+1} - a_0b_0 - \sum_{k=0}^{n} a_{k+1}(b_{k+1} - b_k),$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

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Let $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_0, b_1, b_2, \ldots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$, if $n \geq 2$, then

$$\sum_{k=1}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

Solution

Base case: let $n = 2$. Clearly,

$$\sum_{k=1}^{1}(a_{k+1} - a_k)b_k = (a_2 - a_1)b_1$$

and

$$\sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k) = a_2(b_2 - b_1).$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{1}(a_{k+1} - a_k)b_k = (a_2 - a_1)b_1 \\
&= a_2 b_1 - a_1 b_1 \\
&= a_2 b_2 - a_1 b_1 - (a_2 b_2 - a_2 b_1) \\
&= a_2 b_2 - a_1 b_1 - a_2(b_2 - b_1) \\
&= a_2 b_2 - a_1 b_1 - \sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k),
\end{aligned}$$

so the base case holds.

Inductive step: Suppose that $n \in \mathbb{N}$, $n \geq 2$, and

$$\sum_{k=1}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

Then

$$\sum_{k=1}^{n-1}(a_{k+1} - a_k)b_k + (a_{n+1} - a_n)b_n = a_n b_n - a_1 b_1 + (a_{n+1} - a_n)b_n - \sum_{k=1}^{n-1} a_{k+1}(b_{k+1} - b_k),$$

so

$$\sum_{k=1}^{n}(a_{k+1}-a_k)b_k = a_{n+1}b_n - a_1b_1 - \sum_{k=1}^{n-1} a_{k+1}(b_{k+1}-b_k).$$

Thus,

$$\sum_{k=1}^{n}(a_{k+1}-a_k)b_k = a_{n+1}b_n - a_1b_1 + a_{n+1}(b_{n+1}-b_n) - \sum_{k=1}^{n} a_{k+1}(b_{k+1}-b_k),$$

so

$$\sum_{k=1}^{n}(a_{k+1}-a_k)b_k = a_{n+1}b_{n+1} - a_1b_1 - \sum_{k=1}^{n} a_{k+1}(b_{k+1}-b_k),$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

3. [4 marks] Induction.

Warning! This question does not require deep insight but it is longer to write up (you may need more than 1 page). You should keep it for last. Also, *you will receive* ***at most*** *half the marks* if you do NOT use induction.

Let $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_0, b_1, b_2, \ldots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$, if $n \geq 1$, then

$$\sum_{k=1}^{n} (a_{k+1} - a_k) b_k = a_{n+1} b_{n+1} - a_1 b_1 - \sum_{k=1}^{n} a_{k+1}(b_{k+1} - b_k).$$

Solution

Base case: let $n = 1$. Clearly,

$$\sum_{k=1}^{1} (a_{k+1} - a_k) b_k = (a_2 - a_1) b_1$$

and

$$\sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k) = a_2(b_2 - b_1).$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{1} (a_{k+1} - a_k) b_k = (a_2 - a_1) b_1 \\
&= a_2 b_1 - a_1 b_1 \\
&= a_2 b_2 - a_1 b_1 - (a_2 b_2 - a_2 b_1) \\
&= a_2 b_2 - a_1 b_1 - a_2(b_2 - b_1) \\
&= a_2 b_2 - a_1 b_1 - \sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k),
\end{aligned}$$

so the base case holds.

Inductive step: Suppose that $n \in \mathbb{N}^+$ and

$$\sum_{k=1}^{n} (a_{k+1} - a_k) b_k = a_{n+1} b_{n+1} - a_1 b_1 - \sum_{k=1}^{n} a_{k+1}(b_{k+1} - b_k).$$

Then

$$\sum_{k=1}^{n} (a_{k+1} - a_k) b_k + (a_{n+2} - a_{n+1}) b_{n+1} = a_{n+1} b_{n+1} - a_1 b_1 + (a_{n+2} - a_{n+1}) b_{n+1} - \sum_{k=1}^{n} a_{k+1}(b_{k+1} - b_k),$$

so

$$\sum_{k=1}^{n+1}(a_{k+1}-a_k)b_k = a_{n+2}b_{n+1} - a_1 b_1 - \sum_{k=1}^{n} a_{k+1}(b_{k+1}-b_k).$$

Thus,

$$\sum_{k=1}^{n+1}(a_{k+1}-a_k)b_k = a_{n+2}b_{n+1} - a_1 b_1 + a_{n+2}(b_{n+2}-b_{n+1}) - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1}-b_k),$$

so

$$\sum_{k=1}^{n+1}(a_{k+1}-a_k)b_k = a_{n+2}b_{n+2} - a_1 b_1 - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1}-b_k),$$

and we are done.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

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Let $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_0, b_1, b_2, \ldots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n+1}(a_{k+1} - a_k)b_k = a_{n+2}b_{n+2} - a_1b_1 - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1} - b_k).$$

Solution

Base case: let $n = 0$. Clearly,

$$\sum_{k=1}^{1}(a_{k+1} - a_k)b_k = (a_2 - a_1)b_1$$

and

$$\sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k) = a_2(b_2 - b_1).$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{1}(a_{k+1} - a_k)b_k = (a_2 - a_1)b_1 \\
&= a_2b_1 - a_1b_1 \\
&= a_2b_2 - a_1b_1 - (a_2b_2 - a_2b_1) \\
&= a_2b_2 - a_1b_1 - a_2(b_2 - b_1) \\
&= a_2b_2 - a_1b_1 - \sum_{k=1}^{1} a_{k+1}(b_{k+1} - b_k),
\end{aligned}$$

so the base case holds.

Inductive step: Suppose that $n \in \mathbb{N}$ and

$$\sum_{k=1}^{n+1}(a_{k+1} - a_k)b_k = a_{n+2}b_{n+2} - a_1b_1 - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1} - b_k).$$

Then

$$\sum_{k=1}^{n+1}(a_{k+1} - a_k)b_k + (a_{n+3} - a_{n+2})b_{n+2} = a_{n+2}b_{n+2} - a_1b_1 + (a_{n+3} - a_{n+2})b_{n+2} - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1} - b_k),$$

so

$$\sum_{k=1}^{n+2}(a_{k+1}-a_k)b_k = a_{n+3}b_{n+2} - a_1b_1 - \sum_{k=1}^{n+1} a_{k+1}(b_{k+1}-b_k).$$

Thus,

$$\sum_{k=1}^{n+2}(a_{k+1}-a_k)b_k = a_{n+3}b_{n+2} - a_1b_1 + a_{n+3}(b_{n+3}-b_{n+2}) - \sum_{k=1}^{n+2} a_{k+1}(b_{k+1}-b_k),$$

so

$$\sum_{k=1}^{n+2}(a_{k+1}-a_k)b_k = a_{n+3}b_{n+3} - a_1b_1 - \sum_{k=1}^{n+2} a_{k+1}(b_{k+1}-b_k),$$

and we are done.

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4. [4 marks] Asymptotic Notation II.

Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Theta(\sqrt{n})$ and $g(n) \in \mathcal{O}(n^2)$, then $g(f(n)) \in \mathcal{O}(n)$.

Solution

Since $g(n) \in \mathcal{O}(n^2)$, there exist $c, n_0 \in \mathbb{R}^+$ such that

$$g(n) \leq cn^2,$$

for all $n \geq n_0$. Likewise, since $f(n) \in \Theta(\sqrt{n})$, it follows that there exist $c_1, c_2, n_0' \in \mathbb{R}^+$ such that

$$c_1\sqrt{n} \leq f(n) \leq c_2\sqrt{n},$$

for all $n \geq n_0'$. Let $n_0'' = \max((n_0/c_1)^2, n_0')$. Then, $f(n) \geq n_0$ whenever $n \geq n_0''$. Thus,

$$g(f(n)) \leq cf(n)^2,$$

for all $n \geq n_0''$. Since $n_0'' \geq n_0'$, it follows that

$$g(f(n)) \leq cf(n)^2 \leq c(c_2\sqrt{n})^2 = c \cdot c_2^2 n,$$

for all $n \geq n_0''$. Thus, $g(f(n)) \in \mathcal{O}(n)$.

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4. [4 marks] Asymptotic Notation II.

Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Theta(n^2)$ and $g(n) \in \mathcal{O}(n^2)$, then $g(f(n)) \in \mathcal{O}(n^4)$.

Solution

Since $g(n) \in \mathcal{O}(n^2)$, there exist $c, n_0 \in \mathbb{R}^+$ such that

$$g(n) \leq cn^2,$$

for all $n \geq n_0$. Likewise, since $f(n) \in \Theta(n^2)$, it follows that there exist $c_1, c_2 > 0, n_0' \in \mathbb{R}^+$ such that

$$c_1 n^2 \leq f(n) \leq c_2 n^2,$$

for all $n \geq n_0'$. Let $n_0'' = \max(\sqrt{n_0/c_1}, n_0')$. Then, $f(n) \geq n_0$ whenever $n \geq n_0''$. Thus,

$$g(f(n)) \leq cf(n)^2,$$

for all $n \geq n_0''$. Since $n_0'' \geq n_0'$, it follows that

$$g(f(n)) \leq cf(n)^2 \leq c(c_2 n^2)^2 = c \cdot c_2^2 n^4,$$

for all $n \geq n_0''$. Thus, $g(f(n)) \in \mathcal{O}(n^4)$.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

4. [4 marks] Asymptotic Notation II.

Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Omega(n^2)$ and $g(n) \in \mathcal{O}(1/n)$, then $g(f(n)) \in \mathcal{O}(1/n^2)$.

Solution

Since $g(n) \in \mathcal{O}(1/n)$, there exist $c, n_0 \in \mathbb{R}^+$ such that

$$g(n) \leq c/n,$$

for all $n \geq n_0$. Likewise, since $f(n) \in \Omega(n^2)$, it follows that there exist $c_1, n_0' \in \mathbb{R}^+$ such that

$$c_1 n^2 \leq f(n),$$

for all $n \geq n_0'$. Let $n_0'' = \max(\sqrt{n_0/c_1}, n_0')$. Then, $f(n) \geq n_0$ whenever $n \geq n_0''$. Thus,

$$g(f(n)) \leq c/f(n),$$

for all $n \geq n_0''$. Since $n_0'' \geq n_0'$, it follows that

$$g(f(n)) \leq c/f(n) \leq c/(c_1 n^2) = (c/c_1) \cdot 1/n^2,$$

for all $n \geq n_0''$. Thus, $g(f(n)) \in \mathcal{O}(1/n^2)$.

For all questions in this test, "proof" means a *formal* proof that includes a header, and a proof body with justifications for each deduction. Each question can be answered correctly in less than one (1) page. You will NOT be penalized directly if you use more space for your answer, but longer answers increase the chance of errors... Remember that we are looking for evidence that you understand the conventions for writing correct proofs, so pay attention to the *structure* of your answers in addition to their content!

4. [4 marks] Asymptotic Notation II.

Let $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Theta(n^{10})$ and $g(n) \in \mathcal{O}(\log(n))$, then $g(f(n)) \in \mathcal{O}(\log(n))$.

Solution

Since $g(n) \in \mathcal{O}(\log(n))$, there exist $c, n_0 \in \mathbb{R}^+$ such that

$$g(n) \leq c \log(n),$$

for all $n \geq n_0$. Likewise, since $f(n) \in \Theta(n^{10})$, it follows that there exist $c_1, c_2, n_0' \in \mathbb{R}^+$ such that

$$c_1 n^{10} \leq f(n) \leq c_2 n^{10},$$

for all $n \geq n_0'$. Let $n_0'' = \max((n_0/c_1)^{1/10}, n_0', c_2)$. Then, $f(n) \geq n_0$ whenever $n \geq n_0''$. Thus,

$$g(f(n)) \leq c \log(f(n)),$$

for all $n \geq n_0''$. Since $n_0'' \geq n_0'$ and $n_0'' \geq c_2$, it follows that

$$g(f(n)) \leq c \log(f(n)) \leq c \log(c_2 n^{10}) \leq c \log(n^{11}) = 11 \cdot c \log(n),$$

for all $n \geq n_0''$. Thus, $g(f(n)) \in \mathcal{O}(\log(n))$.