Due before 17:00 on Tuesday 23 February 2021

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Hand-written submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set2.pdf**.
- Each problem set may be completed in groups of up to three—**except for Problem Set 0**. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student-Guide` for a brief explanation of how to create a group on MarkUs.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with one or more partner(s), you must form a group on MarkUs, and make one submission per group.
- Your submitted file(s) should not be larger than **5MB**. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF; in that case, you should look into PDF compression tools to make your PDF smaller, but please make sure that your PDF is still legible!
- Submissions must be made *before* the due date on MarkUs. Please see the Assessment section on the course website for details on how late submissions will be handled.
- MarkUs is known to be slow when many students try to submit right before a deadline. **Aim to submit your work at least one hour before the deadline. It is your responsibility to meet the deadline.** You can submit your work more than once; the most recent version submitted within the deadline (or within the late submission period) is the version that will be marked.
- The work you submit must be that of your group; you may not use or copy from the work of other groups, or external sources like websites or textbooks. Please see the section on Academic Integrity in the course syllabus for further details.

Additional instructions

- You may not define your own propositional operators, predicates, or sets for this problem set. Work with the symbols we have introduced in lecture, and any definitions provided in the questions.
- In your proofs you can always use *definitions* we have covered in the course. You may **not** use any external facts about these definitions unless they are explicitly stated in the question.

  However, you *may* make statements about these definitions without proof when they concern only specific integers, e.g., "7 is odd" or "13 is prime".
- You are *not* required to submit translations of statements you're proving in predicate logic unless we explicitly ask for it. However, if you are doing a **disproof**, then you *must* submit a translation of the negation of the statement in predicate logic (following the same guidelines as Problem Set 1).

1. [12 marks] Number theory. For this question, you may use any of the following definitions or facts *without proof*. In other words, you may refer to any of these definitions, and any of these facts may appear as a justification for some deduction in your proofs, but *do NOT prove* these facts as part of your justification.

- All definitions, facts, and statements proven in worksheets 1–8.
- Theorem 2.1 (Quotient-Remainder Theorem) from the Course Notes.
- *Fact 1:* $\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1$.
- *Definition 1:* An *odd function* is any function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
- *Definition 2:* An *even function* is any function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

Now, for each statement below:

(i) Write down whether the statement is true or false.

(ii) If the statement is true, write it in predicate notation and then write a detailed proof of the statement.

(iii) If the statement is false, write its *negation* in predicate notation, and then write a detailed proof of its negation.

State all definitions you use in your proofs (other than the ones above).

(a) [**4 marks**] There exist 4 consecutive integers whose product is **not** divisible by 12. (Integers are *consecutive* when their difference is 1, e.g., $2, 3, 4, 5$ are consecutive.)

(b) [**4 marks**] For all real numbers x greater than or equal to 6, $4x^2 - 3\lfloor x \rfloor^2 \geq 9$. (Hint: Try to find a lower bound for $(x - 3)^2$ first.)

(c) [**4 marks**] There exist odd functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) - h(x)$ is a non-constant even function. (A function $f : \mathbb{R} \to \mathbb{R}$ is *constant* when $\exists k \in \mathbb{R}, \forall x \in \mathbb{R},\ f(x) = k$.)

2. [8 marks] Sets of real numbers. Consider the following predicates, defined for arbitrary sets $S, T \subseteq \mathbb{R}$:

$$G_0(S,T) : \forall x \in S, \forall y \in T, x > y$$
$$G_1(S,T) : \forall x \in S, \exists y \in T, x > y$$
$$G_2(S,T) : \exists x \in S, \forall y \in T, x > y$$
$$G_3(S,T) : \exists x \in S, \exists y \in T, x > y$$

(a) [**2 marks**] For fixed (but arbitrary) **non-empty** sets S and T, do any of these predicates imply each other?

To answer this question, for each predicate, write any true implications for which it can be the hypothesis. For example, you would write "$G_0(S,T) \Rightarrow G(S,T)$" if $G(S,T)$ was on the list and was true whenever $G_0(S,T)$ is true.

For each implication you write, **explain** in 1–3 sentences why the implication is true.

(Note: this will require you to consider all 16 possible implications between the four predicates.)

(b) [**6 marks**] Prove or disprove each of the following statements. State clearly whether you are attempting a proof or a disproof.

- $G_0([3,5],[0,2])$
- $G_0(\mathbb{R},\mathbb{R})$
- $G_1((0,1],(0,1))$
- $G_2(\mathbb{Z},(-\infty,0))$
- $G_2(\{10\},\varnothing)$, where $\varnothing$ is the empty set.
- $G_3([0,1] \cap \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$

3. [10 marks] Pascal's Triangle. Pascal's Triangle is the following arrangement of numbers:

$$\begin{array}{ccccccccccc} & & & & & \binom{0}{0} & & & & & \\ & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\ & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\ & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\ \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ \kern-1em\cdot^{\cdot^{\cdot}} & & & & & \vdots & & & & & \ddots \end{array} = \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \cdot^{\cdot^{\cdot}} & & & & & \vdots & & & & & \ddots \end{array}$$

where $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \leq n \Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$, and $\forall n \in \mathbb{N}, n! = \prod_{i=1}^{n} i = (n)(n-1)\cdots(2)(1)$.
(In particular, note that $0! = 1$.)

Binomial Theorem: $\forall a, b \in \mathbb{R}, \forall n \in \mathbb{N}, (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

Note that the n^{th} row of Pascal's Triangle contains the coefficients in the simplified expansion of any binomial of the form $(a+b)^n$. (Where we number rows starting at the top, and counting from 0, so the top row is the 0^{th} row, the second row from the top is the 1^{st} row, and so on.)

The three statements below are true and can be proven by at least one of these methods:

- indirect proof (proof by contrapositive);
- direct proof.

Prove each statement so that you use *each* type of proof listed above *at least once*.

(a) [**4 marks**] Consider a function $f(x) = \left(x + \frac{c}{x}\right)^b$, where $b, c \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.
Then $\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*$, if $f(x)$ has no constant term, then b is an odd number.
(A *constant term* is one whose value does not depend on x.)

(b) [**3 marks**] Let $f(n,k) = \binom{n}{k}$, where $n, k \in \mathbb{N}$. Then $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}^+$ is an onto function.

(c) [**3 marks**] $\forall n, k \in \mathbb{Z}^+, k < n \Rightarrow \binom{n}{k} \div \binom{n-1}{k} > 1$