Learning Objectives

By the end of this worksheet, you will:

- Analyse the average-case running time of an algorithm.

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1. **Average-case analysis.** Consider the following algorithm that we studied a few weeks ago. The input is an array A of length n, containing a list of n integers.

```python
def has_even(lst: List[int]) -> int:
    n = len(lst)
    for i in range(n):
        if lst[i] % 2 == 0:
            return i

    return -1
```

We proved that the worst-case running time of this algorithm is $\Theta(n)$. In this problem we will analyse its *average-case* running time.

For this analysis, we will consider the sets of *binary* lists *lst* of length n, for each $n \in \mathbb{Z}^+$. That is, *lst* is a list of n integers, where each integer is either 0 or 1.

(a) For each $n \in \mathbb{Z}^+$, let $\mathcal{I}_n$ be the set of all binary lists of length n. Find an expression (in terms of n) for $|\mathcal{I}_n|$, the size of $\mathcal{I}_n$.

(b) For each $n \in \mathbb{Z}^+$ and each $i \in \{0, 1, \dots, n-1\}$, let $S_{n,i}$ denote the *set* of all binary lists lst of length n where the first 0 occurs in position i. More precisely, every list lst in $S_{n,i}$ satisfies the following two properties:

(i) $lst[i] = 0$.

(ii) for all $j \in \mathbb{N}$, if $j < i$ then $lst[j] = 1$.

For each i, $0 \leq i \leq n$, find an expression for $|S_{n,i}|$.

(c) Also, for each $n \in \mathbb{Z}^+$, let $S_{n,n}$ denote the set of binary lists of length n that do not contain a 0 at all. Find an expression for $|S_{n,n}|$.

(d) Give a brief argument (informal proof) that for every $n \in \mathbb{Z}^+$, each binary list of length n is in exactly one set $S_{n,i}$ (for some $i \in \{0, 1, \dots, n\}$). That is, you're arguing that $S_{n,0}, S_{n,1}, \dots, S_{n,n}$ form a *partition* of $\mathcal{I}_n$.

(e) Assume that we calculate the running time of `has_even` by counting just the costs of Lines 4 and 7. Find an exact expression for the average runtime of this algorithm for this input set $\mathcal{I}_n$, in terms of n.
You should get a summation; do not simplify the summation in this part.

(f) Show that the average running time expression that you calculated is in $\mathcal{O}(1)$. You may use the fact that for all $x \in \mathbb{R}$, if $|x| < 1$, then $\displaystyle\sum_{i=1}^{\infty} ix^i = \frac{x}{(1-x)^2}$.