This document shows **all versions** of each question (or part of a question) on the test, along with their sample solution. Each individual test paper contained only one version of each question (or each part).

1. [6 marks] For both statements below:

(i) Write the negation of the original statement without using the $\neg$ symbol.

(ii) Write whether the original statement is true or false.

(iii) If the original statement is true, prove it. If the original statement is false, disprove it.

(NOTE: The notation $\mathbb{R}^{\geq 0}$ represents the set $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.)

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (6x < 2n + 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (3x > 5n + 2)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (5x < 2n + 1)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (4x > 3n + 6)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (7x < 4n - 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (2x > 3n + 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (6x < 2n + 1)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (3x > 7n - 4)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (5x < 3n - 1)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (3x + 1 > 3n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (4x + 6 > 5 - 2n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (2 - 3x < 3n^2 + 2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (8x + 1 > 4n^2 + 5)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (7 - 3x < 2n^2 + 1)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (6 + 10x > 3n^2 + 3)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (5x + 2 < 5 + 3n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (3x + 2 > n^2 - 4)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (3x + 4 > 5n^2 - 1)$

Solution

[redacted]

2. **[5 marks]** This question tests you on "proof by induction". Even if there is a simple proof using another technique, *you will receive* ***at most*** *half the marks if you do NOT use a proof by induction.*

In your answer, you may use the facts that $e = 2.71828\dots$ and $1/e = 0.367879\dots$ Also, if you want to look at the graph of any function in this question, you may use https://www.desmos.com/calculator—*but NO other online resource is allowed.*

Use induction to prove the following statement. As part of your answer, make sure to provide an explicit definition for your predicate $P(n)$, and to state clearly what you are proving in each section of your proof.

Version 1:

$$\forall n \in \mathbb{N}, (n \geq 2) \Rightarrow (e^{1-n} + 3 < n^2 + 2)$$

Version 2:

$$\forall n \in \mathbb{N}, (n \geq 3) \Rightarrow (e^{-n+2} + 4 < n^2 + 1)$$

Version 3:

$$\forall n \in \mathbb{N}, (n \geq 4) \Rightarrow (e^{3-n} + 6 < n^2 - 2)$$

Solution

[redacted]

[redacted]

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive* ***at most*** *half the marks if you do NOT use a proof by contradiction.*

Definition: Let $i \in \mathbb{N}$. A prime number p_i is said to be *balanced* if and only if $p_i = \dfrac{p_{i-1} + p_{i+1}}{2}$, where $p_0 < p_1 < p_2 < \cdots < p_i < \cdots$ are all the prime numbers ($p_0 = 2, p_1 = 3, p_2 = 5, \ldots$).

For example, 5 is *balanced* because $5 = \dfrac{3+7}{2}$.

Give a proof by contradiction that 7 (or 11, or 13, or 17) is NOT a *balanced* prime.

Solution

[redacted]

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive* ***at most*** *half the marks if you do NOT use a proof by contradiction.*

Definition: A *Sophie-Germain* prime p is a prime number such that $2p + 1$ is also a prime number.

For example, 2 is a *Sophie-Germain* prime because $2(2) + 1 = 5$ which is also a prime number.

Give a proof by contradiction that 7 (or 13, or 17, or 19) is NOT a *Sophie-Germain* prime.

Solution

[redacted]

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive* ***at most*** *half the marks if you do NOT use a proof by contradiction.*

Definition: A *Pythagorian* prime p is a prime number for which $\exists d \in \mathbb{N}, p = 4d + 1$.

For example, 5 is a *Pythagorean* prime because $5 = 4(1) + 1$.

Give a proof by contradiction that 7 (or 11, or 19, or 23) is NOT a *Pythagorean* prime.

Solution

[redacted]

Don't forget: this test contains ***four*** *separate questions (plus the Academic Integrity statement)!*

[redacted]

4. [5 marks] In this question, you **must** use the following definition of absolute value:

$$\forall z \in \mathbb{R}, \quad |z| = \begin{cases} z, & \text{if } z \geq 0, \\ -z, & \text{if } z < 0. \end{cases}$$

Prove that every solution to

$$|x - 6| \leq b - 2x$$

belongs to the set $(-\infty, b-6]$, where b is the *last non-zero digit* in your student number. (Here, "last" means furthest to the right; for example, if your student number is 1000305070, the last non-zero digit is "7".)

<u>Solution</u>

[redacted]