CSC165H1: Problem Set 1 Sample Solutions

Due Tuesday **2** February *before* 17:00

Note: solutions may be incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[10 marks] Translating statements.** Trees can be grouped into forests. Some trees are pine trees, and some trees are oak trees (in this question, these are the only kinds of trees we will consider). Here are some sets and predicates to model trees and forests.

Symbol	**Definition**
T	the set of all trees
F	the set of all forests
$BelongsTo(t, f)$	"tree t is in forest f," where $t \in T$ and $f \in F$
$Oak(t)$	"tree t is an oak tree," where $t \in T$
$Pine(t)$	"tree t is a pine tree," where $t \in T$

Warning! In this question, the symbols T and F represent sets. If you need to denote truth values, please use the full names `True` and `False` in order to avoid any confusion!

Using these sets and predicates, the symbols $=$ and $\neq$, set notation ($\cap, \cup, \subseteq, \in$, etc.), and the standard propositional operators and quantifiers from lecture, translate each of the following English statements into predicate logic.

(a) **[2 marks]** There is exactly one tree that is not in a forest.

Solution

$\exists t \in T, \forall f \in F, \neg BelongsTo(t, f) \land \forall t_0 \in T, \neg BelongsTo(t_0, f) \Rightarrow t_0 = t$

(b) **[2 marks]** There is a forest of oak trees.

Solution

$\exists f \in F, \forall t \in T, BelongsTo(t, f) \Rightarrow Oak(t)$

(c) **[2 marks]** All pine trees are in the same forest.

Solution

$\exists f \in F, \forall t \in T, Pine(t) \Rightarrow BelongsTo(t, f)$

(d) **[2 marks]** No forest contains both pine and oak trees.

Solution

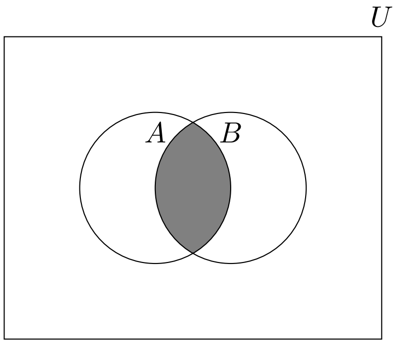
$\forall f \in F, \big(\forall t_1 \in T, BelongsTo(t_1, f) \Rightarrow Oak(t_1)\big) \lor \big(\forall t_2 \in T, BelongsTo(t_2, f) \Rightarrow Pine(t_2)\big)$

(e) **[2 marks]** Every group of trees is a forest.

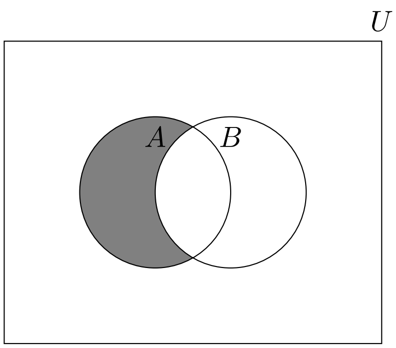
Solution

$\forall S \subseteq T, S \in F$, or equivalently, $\forall S \in \mathcal{P}(T), S \in F$

2. **[4 marks] Venn diagrams.** Sets can be represented by Venn diagrams. These are overlapping circles, with the different regions formed representing the result of different set operations. For example, the set $A \cap B$ is show below by the region shaded in grey:



   (a) **[1 mark]** Write an expression representing the set shown in grey in the following diagram:

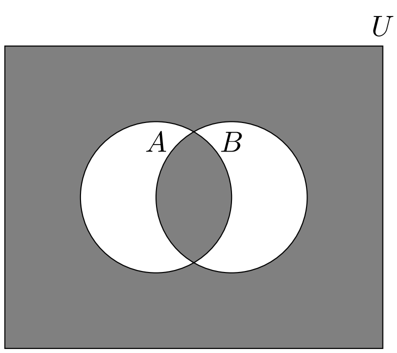


   Solution

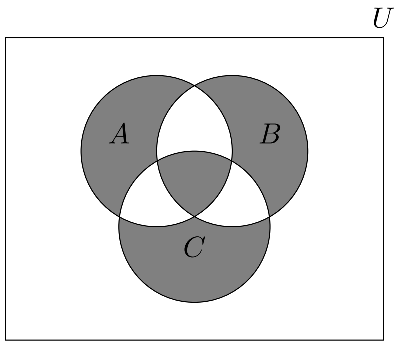
   $B^c \cap A$, or equivalently, $A \setminus B$

   (b) **[1 mark]** Draw a Venn diagram representing the set $(A \cap B) \cup (A \cup B)^c$

   Solution



   (c) **[1 mark]** Write an expression representing the set shown in grey in the following diagram:

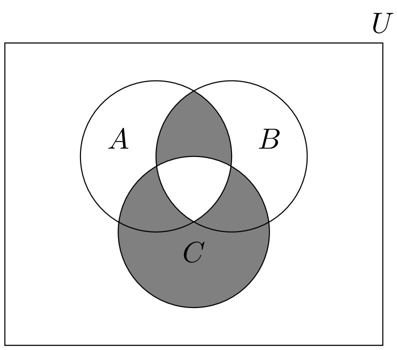


Solution

$((A \cup B \cup C) \setminus ((A \cap B) \cup (B \cap C) \cup (A \cap C)) \cup (A \cap B \cap C)$

(d) **[1 mark]** Draw a Venn diagram representing the set $(C \cup (A \cap B)) \setminus (A \cap B \cap C)$

Solution



3. **[8 marks] Choosing sets and predicates**— ***Updated at 08:45 on Tuesday 26 January.*** This question gets you to investigate some of the subtleties of variable scope and precedence rules that are discussed on pp. 29–31 in the Course Notes.

   Consider the following statements:

$$\forall x \in S, \exists y \in T, (P(x) \lor Q(x,y)) \land (\neg P(x) \lor \neg Q(x,y))$$

$$\exists y \in T, \forall x \in S, (P(x) \lor Q(x,y)) \land (\neg P(x) \lor \neg Q(x,y))$$

   (a) [**5 marks**] Provide one definition of sets S and T, and predicates P and Q, that makes the first statement true and the second statement false. Your sets must be non-empty, and your predicates may not be constant functions (i.e., always True or always False). Briefly justify your response, but no formal proofs are necessary.

   Solution

   Take $S = \{0, 1, 2, 3\}$, $T = \{4, 5, 6\}$, $P(x) : x < 1$, and $Q(x, y) : x + y = 7$. The first statement is true and the second is false.

   Note that the expression $(P(x) \lor Q(x,y)) \land (\neg P(x) \lor \neg Q(x,y))$ is $P(x) \oplus Q(x,y)$ and $P(x)$ is only true when $x = 0$. For any other possible value of x, we can pick a y from T such that $x + y = 7$.

   In the second expression, fix an arbitrary y. At most one possible x results in $x + y = 7$, so $Q(x, y)$ is always false. And since $P(x)$ is not a constant function, $P(x)$ is not always true, so the second statement is false.

   (b) [**3 marks**] Is it possible to provide one definition of sets S and T, and predicates P and Q, that makes the first statement false and the second statement true? Your sets must be non-empty, and your predicates may not be constant functions (i.e., always True or always False).

   If it is possible, describe the sets and briefly justify your response; if not, give a brief explanation as to why it is not possible. No formal proofs are necessary.

   Solution

   Suppose the second statement is true, and we've picked $y \in T$ which makes the expression true for all possible x. Then in the first statement, for any x we could pick the same y to make the expression true. Therefore it is not possible to provide one definition of sets S and T, and predicates P and Q, that makes the first statement false and the second statement true.

4. **[13 marks] Working with infinity.** Sometimes when dealing with predicates over $\mathbb{N}$, we don't care so much about which numbers satisfy the given predicate, or even exactly how many numbers satisfy it. Instead, we care about whether the predicate is satisfied by **infinitely many** numbers. For example:

"There are infinitely many even numbers."

One way we can express the idea of "infinitely many" is by saying that for every natural number n_0, there is a number greater than n_0 that satisfies a predicate. For example:

$$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land Even(n)$$

You may use the Even and | ("divides") predicates in your solutions for all parts of this question.

(a) **[2 marks]** Express the following statement in predicate logic: "There are **finitely** many odd natural numbers."

Solution

$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \leq n_0 \lor Even(n)$

(b) **[3 marks]** We might want to say something stronger about even natural numbers. In particular, given one even number, we now how to find infinitely many other even numbers.

Express the following statement in predicate logic: "Zero is an even number, and given any even number, two more than that number is also an even number."

Solution

$Even(0) \land \forall n \in \mathbb{N},\ Even(n) \Rightarrow Even(n+2)$

(c) **[4 marks]** Consider the following definition.

Definition 1. Let $a \in \mathbb{N}$. We say that a is a **perfect number** when it is equal to the sum of all of its divisors, excluding itself. For example, 6 is a perfect number since $6 = 1 + 2 + 3$.

Translate the following statement into predicate logic: "There are infinitely many even perfect numbers."

You may *not* define your own predicate for "*Perfect*" in this part. Here, we're looking for you to translate the above definition and embed that translation into a larger logical statement. You may use the **indicator function**, $\mathbb{I}(p)$, which is equal to 1 when p is true, and 0 when p is false, for any proposition p. For example, you could write the sum of the first 100 even natural numbers as:

$$\sum_{i=1}^{100} i \cdot \mathbb{I}(Even(i))$$

Solution

$$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land Even(n) \land n = \sum_{i=1}^{n-1} i \cdot \mathbb{I}(i|n)$$

(d) **[4 marks]** Here is another variation of combining predicates with infinity. Let $P : \mathbb{N} \to \{\texttt{True}, \texttt{False}\}$. We say that P is **eventually true** when there exists a natural number n_0 such that all natural numbers greater than n_0 satisfy P.

Use this idea to express the following statement in predicate logic (no justification is required here):

Eventually, all perfect numbers are odd.

Solution

$$\exists n_0 \in \mathbb{N}\ , \forall n \in \mathbb{N},\ \left(n > n_0 \land n = \sum_{i=1}^{n-1} i \cdot \mathbb{I}(i|n)\right) \Rightarrow \neg Even(n)$$