Propositions and Predicates

CSC165 Week 2 - Part 2

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Negating Logical Operators

NOT: $\neg(\neg p) \Longleftrightarrow p$

It is not the case that x is not positive $\Longleftrightarrow$ x is positive

OR: $\neg(p \vee q) \Longleftrightarrow (\neg p) \wedge (\neg q)$

AND: $\neg(p \wedge q) \Longleftrightarrow (\neg p) \vee (\neg q)$

Demorgan's Laws

p	q	¬(p ∨ q)	¬p ∧ ¬q
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

p	q	¬(p ∧ q)	¬p ∨ ¬q
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

IF: $\neg(p \Longrightarrow q) \Longleftrightarrow$ $p \wedge \neg q$

IF AND ONLY IF: $\neg(p \Longleftrightarrow q) \Longleftrightarrow$ $(\neg p \wedge q) \vee (p \wedge \neg q)$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\neg(\neg p \vee q)$

p	q	$\neg(p \implies q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

p	q	$p \iff q$	$(\neg p \vee q) \wedge (p \vee \neg q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

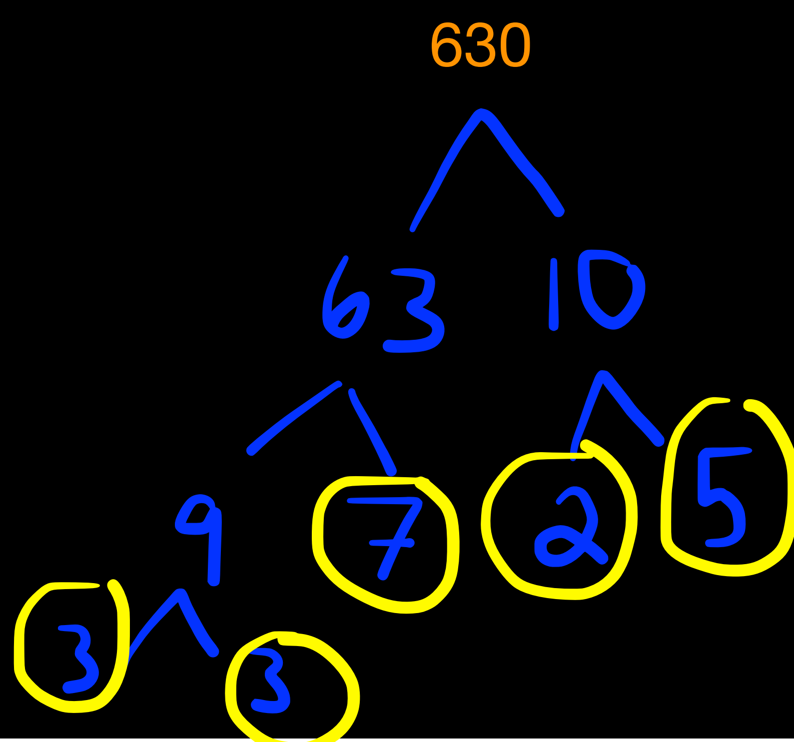
p	q	$\neg(p \iff q)$	$(\neg p \wedge q) \vee (p \wedge \neg q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

A useful domain: Factors

Definition 1.8. Let $n, d \in \mathbb{Z}$.[25] We say that d **divides** n, or n **is divisible by** d, when there exists a $k \in \mathbb{Z}$ such that $n = dk$. In this case, we use the notation $d \mid n$ to represent "d divides n."

630 is divisible by: $1, 630, 2, 3, 6, 9, 18, \ldots$

$$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$$



Using “|” or “=“ to describe divisibility

$2 \mid n$	“2 divides n” or “2 is a factor of n”
$\exists\, k \in \mathbb{Z},\ n = 2k$	“There exists an integer k such that n can be written as a product of 2 and k” or “n is a multiple of 2”

Both “$2 \mid n$” and “$n = 2k$” are predicates.

$x = x$

How do we symbolize “All multiples of 100 are divisible by 10”?

$100k = 10m$

$$\boxed{\forall x \in \mathbb{Z}\ (\exists k \in \mathbb{Z},\ x = 100k)} \Rightarrow (10 \mid x)$$

$$\Rightarrow (\exists m \in \mathbb{Z},\ x = 10m)$$

$100x$

$x = 10k$

Precedence

1. $\neg$
2. $\vee$, $\wedge$
3. $\Rightarrow$, $\Leftrightarrow$
4. $\forall$, $\exists$

BEDMAS

So for example the expression

$$(p \vee \neg q) \wedge r \Rightarrow ((s \vee t) \wedge u) \vee (\neg v \wedge w)$$

$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}$

$$\forall x, y \in \mathbb{N},\ \exists z \in \mathbb{N},\ x + y = z \wedge x \cdot y = z \Rightarrow x = y$$

represents

$$\forall x, y \in \mathbb{N},\ \Big(\exists z \in \mathbb{N},\ \big((x + y = z \wedge x \cdot y = z) \Rightarrow x = y\big)\Big).$$

$f(g(x))$

Scope

✅ $\forall\, x \in \mathbb{N}, \exists\, k \in \mathbb{Z}, \exists\, m \in \mathbb{Z}, \exists\, n \in \mathbb{Z}, ((x = 2k) \wedge (x = 3m)) \implies (x = 6n)$

✅ $\forall\, x \in \mathbb{N}, ((\exists\, k \in \mathbb{Z}, x=2k) \wedge (\exists\, m \in \mathbb{Z}, x = 3m)) \implies (\exists\, n \in \mathbb{Z}, x = 6n)$

X $(\forall\, x \in \mathbb{N}, (\exists\, k \in \mathbb{Z}, x=2k) \wedge (\exists\, m \in \mathbb{Z}, x = 3m)) \implies (\exists\, n \in \mathbb{Z}, \boxed{x = 6n})$

X $(\forall\, x \in \mathbb{N}, (\exists\, k \in \mathbb{Z}, x=2k) \wedge (\exists\, m \in \mathbb{Z}, x = 3m))$
$\implies (\forall\, x \in \mathbb{N}, \exists\, n \in \mathbb{Z}, x = 6k)$

Negating the universal quantifier $\neg\forall\ldots$

Let U = {students in this course},
Let M(x) = "x is majoring in English Literature"

How do we symbolize:

"It is not the case that [all students in this course are majoring in English Literature]"

$$\neg\forall x \in U, M(x)$$

$$\exists x \in U, \neg M(x) \leftarrow$$ The x is a "counterexample".

Negating the existential quantifier

Let U = {students in this course},
Let M(x) = "x is majoring in English Literature"

How do we symbolize:

"No one is majoring in English Literature."
or
"It is not the case that someone in this course is majoring in English Literature."

$$\neg \exists x \in U, M(x)$$

$$\forall x \in U, \neg M(x)$$

Double Quantifiers

Let U = {students in this course}, and $x \in U$, $y \in U$

Let $S(x,y)$ = "x studies with y" Note: x = y is possible.

$\forall x \in U, \forall y \in U, S(x,y) =$ Everyone studies with everyone. (all possible pairs)

$\forall x \in U, \exists y \in U, S(x,y) =$ Everyone studies with someone.
(a study buddy for each person)

$\exists x \in U, \forall y \in U, S(x,y) =$ Someone studies with everyone. (the super studier)

$\exists x \in U, \exists y \in U, S(x,y) =$ Someone studies with someone. (at least one pair)

Negating Double Quantifiers

$\neg(\forall x \in U, \forall y \in U, S(x,y)) =$ $\neg$"Everyone studies with everyone."

$\exists x \in U, \exists y \in U, \neg S(x,y)$

$\neg(\forall x \in U, \exists y \in U, S(x,y)) = \exists x \in U, \forall y \in U, \neg S(x,y)$

$\neg(\exists x \in U, \forall y \in U, S(x,y)) = \forall x \in U, \exists y \in U, \neg S(x,y)$

It is not the case that there is someone who studies with everyone.

$\neg(\exists x \in U, \exists y \in U, S(x,y)) = \forall x \in U, \forall y \in U, \neg S(x,y)$

Negating Double Quantifiers

$\neg(\forall x \in U, \forall y \in U, S(x,y)) = \exists x \in U, \exists y \in U, \neg S(x,y)$

$\neg(\forall x \in U, \exists y \in U, S(x,y)) = \exists x \in U, \forall y \in U, \neg S(x,y)$

$\neg(\exists x \in U, \forall y \in U, S(x,y)) = \forall x \in U, \exists y \in U, \neg S(x,y)$

$\neg(\exists x \in U, \exists y \in U, S(x,y)) = \forall x \in U, \forall y \in U, S(x,y)$