Binary Representations of Numbers — a Proof

CSC165 Week 6 - Part 1

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Proof by Induction — General Structure

We want to prove that the statement $P(n)$ is true for all natural numbers n.

In other words, we want to prove $\forall n \in \mathbb{N}, P(n)$.

Step 1: Base case

Prove $P(0)$ (or any other case that should be verified first.

Step 2: Induction hypothesis

Since we want to show that $P(k) \Longrightarrow P(k+1)$,

We assume $P(k)$ is true. (let $n = k$)

Step 3: Let $n = k+1$ and Prove $P(k+1)$

This is where we use the induction hypothesis $P(k)$ to prove $P(k+1)$ must also be true.

Proof by Induction — Summary

We want to show that P(n) is true for all $x \in \mathbb{N}$ and $x \geq a$.

Strategy: Prove P(a) is true and then that $P(k) \implies P(k+1)$.

Goal for this week

We want to prove: $\forall\, n \in \mathbb{N},\ 0 \leq x \leq 2^n - 1 \iff B(n,x)$

Strategy: Prove the $\implies$ direction using ____________________________

Then prove the $\impliedby$ direction using ____________________________

Decimal (Base 10) Numbers

Possible digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9

Where do we get the digits from?

10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0

Binary (Base 2) Numbers

Example: $(46)_{10} = (101110)_2$

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Example: $(46)_{10} = (101110)_2$

$= \Sigma\, b_i\, 2^i$

where $b_0 = 0$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 1$

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	1	0

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

A binary representation of $x \in \mathbb{N}$ can be written like this:

$\exists\ k \in \mathbb{Z}+$, $b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that

$$x = \Sigma\ b_i\ 2^i = b_{k-1}\ 2^{k-1} + b_{k-2}\ 2^{k-2} + \ldots + b_1\ 2^1 + b_0\ 2^0$$

We can then write $x = (b_{k-1}b_{k-2}\ldots b_1b_0)_2$

For example, $5 = (101)_2$ or $(5)_{10} = (101)_2$

Definition of Predicate B(n,x)

$\forall\, n \in \mathbb{N}, \forall\, x \in \mathbb{N}$, B(n,x) is true if and only if

$\exists\, b_0, b_1, \ldots, b_{n-1} \in \{0,1\}$ such that $x = (b_{n-1}b_{n-2}\ldots b_1 b_0)_2$

$= \Sigma\, b_i\, 2^i$

In other words, B(n,x) is true when x can be written in binary using exactly n bits.

True or False?

$B(3,5) =$

$B(4,5) =$

$B(2,5) =$

B(n,x) is true when x can be written in binary using exactly n bits.

We want to prove that:

$$\forall\ n \in \mathbb{N},\ 0 \leq x \leq 2^n - 1 \iff B(n,x)$$

We want to prove that:

$\forall\ n \in \mathbb{N},\ 0 \leq x \leq 2^n - 1 \Longrightarrow B(n,x)$

Base Case: Let n =

We want to prove that:

$\forall\ n \in \mathbb{N},\ 0 \leq x \leq 2^n - 1 \iff B(n,x)$

Induction Step: Let $n = k \in \mathbb{N}$

Assume

Case of $n = k+1$:

Want to show that $\forall\ x \in \mathbb{N},\ 0 \leq x \leq 2^{k+1}-1 \implies B(k+1,x)$

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Want to show that $\forall\, x \in \mathbb{N},\ 0 \leq x \leq 2^{k+1}-1 \implies B(k+1,x)$

B(k+1,x) is true when x can be written in binary using exactly k+1 bits.

Case 1: Assume $x \leq 2^k-1$

By Induction Hypothesis, we know that B(k,x) is true. Therefore:

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that $x = (b_{k-1}b_{k-2}\ldots b_1b_0)_2$

$= \Sigma\ b_i\ 2^i$

We can append a 0 to the left side of the number without changing its value. So $x = (0b_{k-1}b_{k-2}\ldots b_1b_0)_2$

$= (0)\ 2^k + \Sigma\ b_i\ 2^i$

Thus B(k+1, x) is true.

Case 2: Assume $x > 2^k-1$

Then, $2^k \leq x \leq 2^{k+1} - 1$

Subtract 2k from all parts to get:

$2^k - 2^k \leq x - 2^k \leq 2^{k+1} - 1 - 2^k$

$0 \leq x - 2^k \leq 2^k(2-1) - 1$

By the Induction hypothesis, we know $B(k, x-2^k)$

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that

$x - 2^k = (b_{k-1}b_{k-2}\ldots b_1 b_0)_2 = \Sigma\ b_i\ 2^i$

Therefore,

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that

$x = (1b_{k-1}b_{k-2}\ldots b_1b_0)_2 = (1)2^k + \Sigma\ b_i\ 2^i$

$B(k+1, x)$ is true.

Next time: $\forall\, x \in \mathbb{N},\ 0 \leq x \leq 2^{k+1}-1 \Longleftarrow B(k+1,x)$