- TT1 marks are out!
- TT2 Details available; extra office hours this week

... in the middle of proving

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{N}, \quad x \leq 2^n - 1 \Rightarrow B(n, x)$$

where $B(n,x): \exists b_0, b_1, \dots, b_{n-1} \in \{0,1\}, x = (b_{n-1} \cdots b_0)_2$

- $P(n): \forall x \in \mathbb{N}, x \leq 2^n - 1 \Rightarrow B(n, x)$
- B.C.: we have proved $P(0)$
- Ind. Hyp.: Let $n \in \mathbb{N}$. Assume $P(n)$:

  $\forall x_1 \in \mathbb{N}, x_1 \leq 2^n - 1 \Rightarrow B(n, x_1)$;
- Ind. Step: WTS $P(n+1)$: $\forall x \in \mathbb{N}, x \leq 2^{n+1} - 1 \Rightarrow B(n+1, x)$

ROUGH WORK:

KNOW	WANT
$n \in \mathbb{N}$	$\underbrace{\forall x \in \mathbb{N},\ x \leq 2^{n+1}-1 \Rightarrow B(n+1, x)}_{\text{write down proof header}}$
$\forall x_1 \in \mathbb{N},\ x_1 \leq 2^n - 1 \Rightarrow B(n, x_1)$	

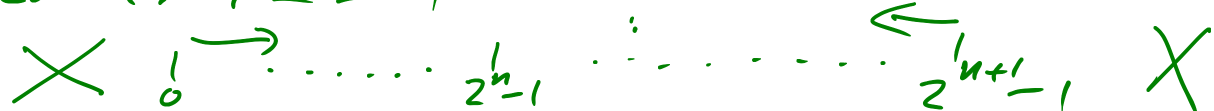
Let $x \in \mathbb{N}$. Assume $x \leq 2^{n+1} - 1$.

$x \in \mathbb{N}$	$B(n+1, x)$
$x \leq 2^{n+1} - 1$	

Idea: either $x \leq 2^n - 1$ or $x > 2^n - 1$.

Case 1: Assume $x \leq 2^n - 1$

case 1: $x \leq 2^n - 1$



By I.H. (with $x_1 = x$), we know

$B(n,x)$: $\exists\, b_0, b_1, \ldots b_{n-1} \in \{0,1\}$, $x = (b_{n-1} \cdots b_0)_2$

$$\parallel$$

$$\sum_{i=0}^{n-1} b_i \cdot 2^i$$

Then, $x = (0\, b_{n-1} \cdots b_0)_2$

$$\parallel$$

$$\sum_{i=0}^{n} b_i \cdot 2^i = b_n \cdot 2^n + \sum_{i=0}^{n-1} b_i \cdot 2^i$$

$$\boxed{(b_n = 0)} \quad = 0 + x$$

So $B(n+1, x)$ holds.

KNOW

$\exists\, b_0, b_1, \ldots, b_{n-1} \in \{0,1\}$,

$x = (b_{n-1} \cdots b_0)_2$

$$\longrightarrow$$

WANT

$\exists\, b_0', b_1', \ldots, b_n' \in \{0,1\}$,

$x = (b_n' \cdots b_0')_2$

let $b_0' = b_0, b_1' = b_1, \ldots, b_{n-1}' = b_{n-1}, \underline{b_n' = 0}$

then ....

Case 2: Assume $x > 2^n - 1$

Case 2: $x > 2^n - 1$ $\vdots$

NOTE: case 1 assumption $(x \le 2^n - 1)$ no longer true

Idea: $x > 2^n - 1 \iff x \ge 2^n$

$$\iff \underbrace{x - 2^n \ge 0}_{x - 2^n \in \mathbb{N}}$$

and

$$x \le 2^{n+1} - 1 \iff \underbrace{x - 2^n} \le 2^{n+1} - 1 - 2^n = \underbrace{2^n - 1}$$

let $x' = x - 2^n$. Then, $x' \in \mathbb{N}$ and $x' \leq 2^n - 1$

so by I.H. (applied to $x_1 = x'$), we know

$$B(n, x') : \exists\, b_0, b_1, \ldots, b_{n-1} \in \{0,1\},\ \underbrace{x' = (b_{n-1} \cdots b_0)_2}$$

$$x - 2^n = \sum_{i=0}^{n-1} b_i 2^i \longleftarrow$$

$$\Leftrightarrow \quad x = 1 \cdot 2^n + \sum_{i=0}^{n-1} b_i 2^i = (1\, b_{n-1} \cdots b_0)_2$$

So $B(n+1, x)$ holds (by picking $b_0' = b_0, \ldots, b_{n-1}' = b_{n-1}, b_n' = 1$).

$\square$

We have proved

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{N}, \quad x \le 2^n - 1 \Rightarrow B(n, x)$$

Turns out

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{N}, \quad B(n,x) \Rightarrow x \le 2^n - 1$$

is also true!

<u>EXERCISE</u> — Try to prove this!

Hint: you don't need induction!

but you will need to know

$$(*) \quad \forall n \in \mathbb{N}, \sum_{i=0}^{n-1} 2^i = 2^n - 1$$