UNIVERSITY OF TORONTO
Faculty of Arts & Science

DECEMBER 2019 EXAMINATIONS

CSC 165 H1F
Instructor(s): Danny Heap

Duration: 3 hours

Aids Allowed: Sorry, no aids.

Do **not** *turn this page until you have received the signal to start.*
In the meantime, write your name, student number, and UTORid below (please do this now!) and *carefully* read *all* the information on the rest of this page.

First (Given) Name(s):

Last (Family) Name(s):

10-Digit Student Number:

UTORid (e.g., `pitfra12`):

- This final examination consists of 8 questions on 18 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the examination is complete.*
- Answer each question directly on the examination paper, in the space provided, and use a "blank" page for rough work. If you need more space for one of your solutions, use one of the "blank" pages and *indicate clearly the part of your work that should be marked.*
- *Remember that, in order to pass the course, you must achieve a grade of at least* ***40%*** *on this final examination.*
- As a student, you help create a fair and inclusive writing environment. If you possess an unauthorized aid during an exam, you may be charged with an academic offence.
- You may reference and use external facts only from course notes, problem set solutions, lecture examples, and worksheet solutions, provided the external facts have not been prohibited.

MARKING GUIDE

Nº 1: ______/ 8

Nº 2: ______/ 8

Nº 3: ______/ 8

Nº 4: ______/ 4

Nº 5: ______/ 8

Nº 6: ______/ 8

Nº 7: ______/ 8

Nº 8: ______/ 8

TOTAL: ______/60

STUDENTS MUST HAND IN ALL EXAMINATION MATERIALS AT THE END

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
Clearly label each such solution with the appropriate question and part number.

December 2019 Examinations

CSC 165 H1F — Danny Heap

Duration: **3 hours**

Question 1. (short answer) [8 marks]

Part (a) (binary representation) [1 mark]

Give the binary representation of 61 that has no leading 0s on the left.

Part (b) (predicates) [1 mark]

Define predicates $P(n)$ and $Q(n)$ so that one of the statements below is true, and the other is false.

$$(\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)) \Rightarrow (\exists m \in \mathbb{N}, P(m) \wedge Q(m))$$
$$(\exists n \in \mathbb{N}, P(n) \Rightarrow Q(n)) \Rightarrow (\forall m \in \mathbb{N}, P(m) \wedge Q(m))$$

Part (c) (moduli) [1 mark]

What number $0 \leq i < 15$ satisfies $i \equiv 2 \pmod 3$ and $i \equiv 4 \pmod 5$?

Part (d) (run-time) [1 mark]

How many times does the the loop iterate for **doubler(17)**?

```
def doubler(n):
    i =  1
    while i * i < n:
        i = 2 * i
    return loops
```

Part (e) (*i* for iterations?) [1 mark]

Give a formula for $i(s)$, the value of i at the end of s iterations of the loop in **doubler**.

Part (f) (iterations for *n*?) [1 mark]

Give a general formula for the number of iterations of the loop for **doubler(n)**, where $n \in \mathbb{N}^+$.

Part (g) (dogs...) [1 mark]

Define the set of dogs as D, and the predicate $A(d)$: "d lives in the arctic." and $F(d)$: "d has fleas." Write a predicate logic statement that says there is exactly one dog that has fleas and lives in the arctic. You may **not** use the short-form $\exists!$

Part (h) units digit... [1 mark]

What is the units digit (ones-place digit) of 3983^{12}

Question 2. **(number theory)** [8 MARKS]

Part (a) **(moduli)** [4 MARKS]

Prove that for any $a, b \in \mathbb{Z}$ if $a \equiv b \pmod{17}$ and $a \equiv b \pmod{19}$ then $a \equiv b \pmod{17 \times 19}$. You may use, without proof, the following theorem (Example 2.16 in notes):

$$\forall i, j, p \in \mathbb{N}, Prime(p) \wedge p \mid ij \Rightarrow p \mid i \vee p \mid j$$

You may **not** use the result of Q2(a) in Problem Set #2.

Part (b) (prime free?) [4 MARKS]

Recall that for natural number n, the quantity $n!$ (AKA "n factorial") is defined:

$$n! = \begin{cases} \prod_{i=1}^{i=n} i & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases} \qquad \text{(informally, } n! = n(n-1)(n-2)\cdots 1\text{)}$$

Assume m is some natural number bigger than 1. Prove that there does not exist a prime number p such that $(m! + 2) \leq p \leq (m! + m)$.

Question 3. big-omega/oh [8 MARKS]

In what follows use the following definitions for $f \in \Omega(g)$ and $f \in \mathcal{O}(g)$:

$$f \in \Omega(g) \,:\, \exists c_\Omega, n_\Omega \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_\Omega \Rightarrow f(n) \geq c_\Omega g(n)$$

$$f \in \mathcal{O}(g) \,:\, \exists c_\mathcal{O}, n_\mathcal{O} \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_\mathcal{O} \Rightarrow f(n) \leq c_\mathcal{O} g(n)$$

Define $f(n) = n^4$ and $g(n) = 2^n$. You **may** not use techniques of calculus such as limits, and you may **not** use external facts from the notes or elsewhere stating $n^a \notin \Omega(b^n)$ or that $n^a \in \mathcal{O}(b^n)$. You may assume, without proof, that for any integer k greater than 5, $2^k > 8k$ (although you are not required to use this). You may also assume, without proof, that $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1$.

Part (a) (big-omega) [4 MARKS]

Prove that $f \notin \Omega(g)$.

Part (b) **(big-oh)** [4 MARKS]

Prove that $f \in \mathcal{O}(g)$.

Question 4. (runtime) [4 MARKS]

Read the code for **has_dominator** below:

```
def has_dominator(integer_list):
    n = len(integer_list)
    for i in range(n):
        for j in range(i + 1, n):
            if integer_list[i] < integer_list[j]:
                return True
    return False
```

If it helps, in the questions below you may assume without proof:

$$\sum_{i=0}^{i=m} i = \frac{m(m+1)}{2}$$

Part (a) (upper bound) [1 MARK]

State and prove a "good" upper bound, $U(n)$, on the worst-case (i.e. maximum) run-time for **has_dominator** on inputs of length n. By "good" I mean it should have the same asymptotic complexity as the lower bound.

Part (b) (lower bound) [3 MARKS]

State and prove a "good" lower bound, $L(n)$ on the worst-case (i.e. maximum) run-time for **has_dominator** on inputs of length n. By "good" I mean it should have the same asymptotic complexity as the upper bound.

Use the space on this "blank" page for scratch work, or for any solution that did not fit elsewhere.
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Question 5. (hunting primes) [8 marks]

In the code below **integer_list** contains numbers from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, with duplicates allowed. The code is intended to return **True** if **integer_list** contains one of the primes in $\{2, 3, 5, 7\}$, and **False** otherwise.

In what follows, if **has_prime** returns **True** right after examining k entries in **integer_list**, count this as k steps. If **has_prime** returns **False** after examining all entries in **integer_list**, count this as the length of the list + 1 steps.

```
def has_prime(integer_list) -> bool:
  for i in range(len(integer_list)):
      if integer_list[i] in [2, 3, 5, 7]:
          return True
  return False
```

Part (a) (length 3) [4 marks]

Calculate the average number of steps **has_prime** takes on lists of length 3. Be sure to show the number of lists that have their first prime in the first, second, or third position, and how many have no primes at all.

Part (b) (general case) [4 marks]

Derive a closed formula for the average number of steps **has_prime** takes on lists of length n. Explain how your formula is derived. You may use, without proof, the following summation (although you are not required to use it):

$$\sum_{i=1}^{m} ir^i = \frac{(m+1)r^{m+1}}{r-1} + \frac{r - r^{m+2}}{(r-1)^2}$$

Question 6. **(sets)** [8 MARKS]

Part (a) **(subsets) of size 5** [5 MARKS]

Define:

$$P(n) : \forall \text{ sets } S, |S| = n \Rightarrow S \text{ has exactly } \frac{n(n-1)(n-2)(n-3)(n-4)}{120} \text{ subsets of size } 5$$

Use induction on n to prove $\forall n \in \mathbb{N}, P(n)$. You may assume, without proof, that any set with n elements has $\frac{n(n-1)(n-2)(n-3)}{24}$ subsets of size 4.

Part (b) (reverse quantification?) [3 MARKS]

Suppose we modify your proof by changing the introduction of S and n as follows:

Let S be an arbitrary set. Define:

$$P(n) : |S| = n \Rightarrow S \text{ has exactly } \frac{n(n-1)(n-2)(n-3)(n-4)}{120} \text{ subsets of size } 5$$

... followed by the base case and inductive step you have in the previous part. Is your proof still valid? Explain why, or why not.

Question 7. (connected?) [8 marks]

Part (a) (enough vertices?) [3 marks]

Assume $G = (V, E)$ is a finite, undirected graph where every vertex $v \in V$ has degree at least $|V| - 5$, and $|V| \geq 9$. Prove that G is connected.

Part (b) (too few vertices?) [2 MARKS]

Assume $G = (V, E)$ is a finite, undirected graph where every vertex $v \in V$ has degree at least $|V| - 5$, and $|V| = 8$. Prove that G is not necessarily connected.

Part (c) (what's wrong with this?) [3 MARKS]

Read the following "proof." What is the smallest n for which $P(n)$ does not imply $P(n+1)$? Explain how the argument breaks down in this case, and why the proof is invalid.

Define $P(n) : \forall G = (V, E), |E| = |V| - 1 \Rightarrow G$ is connected. I will "prove" by induction that $\forall n \in \mathbb{N}^+, P(n)$.

base case $P(1)$: Any graph with $|V| = 1$ vertex and $1 - 1 = 0$ edges, has its sole vertex connected to itself, which verifies $P(1)$.

inductive step: Let $n \in \mathbb{N}^+$ and assume $P(n)$. Let $G = (V, E)$ be an arbitrary graph with $|V| = n$ and $|E| = n - 1$. By the inductive hypothesis G is connected. Add an arbitrary vertex v, and an arbitrary edge (v, u) connecting v to an arbitrary vertex in V, so $G' = (V', E')$, where $V' = V \cup \{v\}$ and $E' = E \cup \{(v, u)\}$. Now $|V'| = n + 1$, $|E'| = n$, and v is connected to u and hence, by transitivity, to every vertex in V'. So G' has $n + 1$ vertices, n edges, and is connected. $\blacksquare$

Question 8. (more graphs...) [8 MARKS]

Part (a) (connected?) [5 MARKS]

Assume $G = (V, E)$ is a finite, undirected, bipartite graph, that is $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and $(u, v) \in E \Rightarrow (u \in V_1 \wedge v \in V_2) \vee (v \in V_1 \wedge u \in V_2)$. Also assume that G is complete, that is $u \in V_1 \wedge v \in V_2 \Rightarrow (u, v) \in E$ — every possible edge between partitions is present. Finally, assume $|V| \geq 2$, $|V_1| = \lfloor |V|/2 \rfloor$, and $|V_2| = \lceil |V|/2 \rceil$. Prove that G is connected.

Part (b) (paths) [3 MARKS]

Suppose $k \in \mathbb{N}$ and path $P = (V, E)$, where $V = \{v_i : 0 \le i \le k\}$ and $E = \{(v_i, v_{i+1}) : 0 \le i < k\}$ is a path from v_0 to v_k. Recall that P is connected means that for any pair of vertices u, v in P, there is a path from u to v. Prove that P is connected. You may **not** use the result from Problem Set #4 that says a cycle with one edge removed is connected, nor Example 6.8 from the course notes that says a connected graph containing a cycle remains connected if an edge is removed from the cycle.

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