Binary Representations of Numbers — a Proof

CSC165 Week 6 - Part 1

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Proof by Induction — General Structure

We want to prove that the statement P(n) is true for all natural numbers n.

In other words, we want to prove $\forall\, n \in \mathbb{N}, P(n)$.

$n \geq a$

Step 1: Base case

Prove P(0) (or any other case that should be verified first.

$P(a)$

Step 2: Induction hypothesis

$A \Rightarrow B$

Since we want to show that $\boxed{P(k) \implies P(k+1)}$,

We assume P(k) is true. (let n = k)

Step 3: Let n = k+1 and Prove P(k+1)

This is where we use the induction hypothesis P(k) to prove P(k+1) must also be true.

Proof by Induction — Summary

We want to show that P(n) is true for all $x \in \mathbb{N}$ and $x \geq a$.

Strategy: Prove P(a) is true and then that $P(k) \implies P(k+1)$.

$$\boxed{P(a)} \wedge \left(P(k) \implies P(k+1) \right)$$

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T

$$\cancel{\boxed{F \Rightarrow T}}$$

Goal for this week

We want to prove: $\forall\, n \in \mathbb{N}, \boxed{0 \le x \le 2^n - 1} \Longleftrightarrow \boxed{B(n,x)}$

Strategy: Prove the $\Longrightarrow$ direction using induction on n

Then prove the $\Longleftarrow$ direction using ______________________

Decimal (Base 10) Numbers

Possible digits: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, \text{ and } 9\}$

$00300402 = 3\times10^5 + 0\times10^4 + 0\times10^3 + 4\times10^2 + 0\times10^1 + 2\times10^0$

Where do we get the digits from?

10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0
0	0	3	0	0	4	0	2

$\{0,1\}$

Binary (Base 2) Numbers

$\longrightarrow \sum_{i=0}^{5} b_i \times 2^i$

Example: $(46)_{10} = (101110)_2$

$= 1\times 2^5 + 1\times 2^3 + 1\times 2^2 + 1\times 2^1$

$$\begin{array}{r} 46 \\ -32 \\ \hline 14 \\ -8 \\ \hline 6 \\ -4 \\ \hline 2 \\ -2 \\ \hline 0 \end{array}$$

2^7 $=128$	2^6 $=64$	2^5 $=32$	2^4 $=16$	2^3 $=8$	2^2 $=4$	2^1 $=2$	2^0 $=1$
0	0	1	0	1	1	1	0

Example: $(46)_{10} = (101110)_2$

$$= \sum_{i=0}^{5} b_i \, 2^i$$

where $b_0 = 0$, $b_1 = 1$, $b_2 = 1$, $b_3 = 1$, $b_4 = 0$, $b_5 = 1$

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	1	0

$$(51)_{10} = (110011)_2$$

$$= \sum_{i=0}^{5} b_i \, 2^i$$

$$\begin{array}{r} 51 \\ -32 \\ \hline 19 \\ -16 \\ \hline 3 \\ -2 \\ \hline 1 \\ -1 \\ \hline 0 \end{array}$$

2^8

$128 = 2^7$	$64 = 2^6$	$32 = 2^5$	$16 = 2^4$	$8 = 2^3$	$4 = 2^2$	$2 = 2^1$	$1 = 2^0$
0	0	1	1	0	0	1	1
						$= b_1$	$= b_0$

$$\{0, 1, \ldots 9, \overset{10}{\overset{\shortparallel}{A}}, B, \ldots F\}$$

$$(4AC1)_{16} = 4\times 16^3 + 10\times 16^2 + 12\times 16^1 + 2\times 16^0$$

2^7	2^6	2^5	2^4	16^3	16^2	16^1	$\underline{\underline{16^0}}$
				4	A	C	1

1 0

A binary representation of $x \in \mathbb{N}$ can be written like this:

$\exists\, k \in \mathbb{Z}+$, $\boxed{b_0, b_1, \ldots, b_{k-1}} \in \{0,1\}$ such that

$$x = \sum_{i=0}^{k-1} b_i 2^i = (b_{k-1}) 2^{k-1} + (b_{k-2}) 2^{k-2} + \ldots + (b_1) 2^1 + (b_0) 2^0$$

We can then write $x = (b_{k-1} b_{k-2} \ldots b_1 b_0)_2$

$$5 = (1) \times 2^2 + (0) \times 2^1 + (1) \times 2^0$$

For example, $5 = (101)_2$ or $(5)_{10} = (101)_2$

Definition of Predicate B(n,x)

$\Longleftrightarrow$

$\forall\, n \in \mathbb{N}, \forall\, x \in \mathbb{N}$, $B(n,x)$ is true if and only if

$\exists\, b_0, b_1, \ldots, b_{n-1} \in \{0,1\}$ such that $x = (b_{n-1}b_{n-2}\ldots b_1 b_0)_2$

$= \sum_{i=0}^{n-1} b_i 2^i$

In other words, $\boxed{B(n,x)}$ is true when x can be written in binary using exactly n bits. $\in \{0,1\}$

True or False?

$$5_{10} = 101_2$$

$$= 0101_2$$

$$\begin{array}{r} 5 \\ -4 \\ \hline 1 \\ -1 \\ \hline 0 \end{array}$$

B(3,5) = True

B(4,5) = True

4 digits

B(2,5) = False

B(n,x) is true when x can be written in binary using exactly n bits.

We want to prove that:

$$\forall\, n \in \mathbb{N},\ \underline{0 \leq x \leq 2^n - 1} \iff \underline{B(n,x)}$$

$\Rightarrow$

$\Leftarrow$

We want to prove that: (using induction on n)

$\forall x \in \mathbb{N}$

$\forall \underline{n} \in \mathbb{N}, \boxed{0 \leq x \leq 2^n - 1} \Longrightarrow \boxed{B(n,x)}$ $(A \Rightarrow B)$

Base Case: Let n = 0 ? (zero digits)

$2^0 - 1 = 0 \Rightarrow 0 \leq x \leq 0$

Let $n=1$. Assume $0 \leq x \leq 2^1 - 1 = 1$

$$\left.\begin{array}{l} \text{Case 1}: x = 0 = 0_2 \\ \text{Case 2}: x = 1 = 1_2 \end{array}\right\} \Rightarrow B(1,x) \text{ is true}$$

We want to prove that:

$\forall\, n \in \mathbb{N},\ 0 \leq x \leq 2^n - 1 \Longleftrightarrow B(n,x)$

Induction Step: Let $n = k \in \mathbb{N}$

Assume $0 \leq x \leq 2^k - 1 \Rightarrow B(k,x)$

Case of $n = k+1$:

Want to show that $\forall\, x \in \mathbb{N},\ \boxed{0 \leq x \leq 2^{k+1} - 1} \Longrightarrow \boxed{B(k+1,x)}$

Case of n = k+1:

Want to show that $\forall x \in \mathbb{N}$, $\boxed{0 \leq x \leq 2^{k+1}-1} \Longrightarrow B(k+1,x)$

Either $\underbrace{0 \leq x \leq 2^k - 1}_{\text{I.H.}}$ or $2^k \leq x \leq 2^{k+1} - 1$

B(k+1,x) is true when x can be written in binary using exactly k+1 bits.

0 ___ 1 ___

Case 1: Assume $x \leq 2^k - 1$

By Induction Hypothesis, we know that $B(k,x)$ is true. Therefore:

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that $x = (b_{k-1}b_{k-2}\ldots b_1b_0)_2$

$$= \sum_{i=0}^{k-1} b_i 2^i$$

We can append a 0 to the left side of the number without changing its value. So $x = (0b_{k-1}b_{k-2}\ldots b_1b_0)_2$

$= (0)\ 2^k + \Sigma\ b_i\ 2^i$

Thus $B(k+1, x)$ is true.

Case 2: Assume $x > \underline{2^k - 1}$

Then, $\boxed{2^k \leq x \leq 2^{k+1} - 1}$

Subtract 2^k from all parts to get:

$$\left(2^k - 2^k\right) \leq x - 2^k \leq \left(2^{k+1} - 1 - 2^k\right)$$

$$\underline{0} \leq \underline{x - 2^k} \leq 2^k(\cancel{2-1}) - 1$$

$$0 \leq x - 2^k \leq 2^k - 1$$

By the Induction hypothesis, we know $\underline{B(k, x-2^k)}$

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that

$$x - 2^k = \underline{(b_{k-1}b_{k-2}\ldots b_1 b_0)_2} = \sum_{i=0}^{k-1} b_i\, 2^i$$

Therefore,

$\exists\ b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ such that

$x = (1b_{k-1}b_{k-2}\ldots b_1b_0)_2 = \boxed{(1)2^k} + \sum_{i=0}^{k-1} b_i\, 2^i = \sum_{i=0}^{k} b_i 2^i$

B(k+1, x) is true.

$\boxed{\text{Let } b_k = 1}$

QED

Next time: $\forall\, x \in \mathbb{N},\ 0 \leq x \leq 2^{k+1}-1 \Longleftarrow B(k+1,x)$