Proof by Contradiction and Proof by Induction

CSC165 Week 5

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Proof Techniques so far:

- Direct Proof of an implication
- Proof of a $\forall$ statement
- Proof of a $\exists$ statement
- Proof of the contrapositive
- Proof by cases

New Proof Techniques:

- Proof by contradiction
- Proof by Induction

Proof by Contradiction — General Structure

We want to show that statement A is true.

Assume $\neg A$ is true.

Show that $\neg A \implies$ a contradiction.

Given a correct implication, this must mean that $\neg A$ is false.

Therefore, A is true. Q.E.D.

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$$\neg\ (\ \forall x \in \mathbb{N},\ \exists n \in \mathbb{N},\ \mathrm{Prime}(n) \wedge (x < n)\ )$$

Proof: We want to show that there are infinitely many prime numbers in a proof by contradiction.

Assume that there are not infinitely many prime numbers.

So $\exists x \in \mathbb{N}, \forall n \in \mathbb{N}, \neg \mathrm{Prime}(n) \vee (x \geq n)$

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We will try proof that the last bracket is true and thereby get a contradiction.

Proof: We want to show that there are infinitely many prime numbers in a proof by contradiction.

Assume that there are not infinitely many prime numbers.

So $\exists x \in \mathbb{N}, \forall n \in \mathbb{N}, \neg Prime(n) \vee (x \geq n)$

Let $p_1, p_2, \ldots p_n$ be a complete list of all prime numbers.

Let $x = (p_1)(p_2)(p_3)\ldots(p_n) + 1$

What do we know about $x = (p_1)(p_2)(p_3)\ldots(p_n) + 1$?

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This contradicts the idea that $p_1,\ldots,p_n$ is a complete list of prime numbers.

Therefore, $\forall x \in \mathbb{N}, \exists n \in \mathbb{N}, \text{Prime}(n) \wedge (x < n)$ Q.E.D.

Proof by Induction — General Structure

We want to prove that the statement P(n) is true for all natural numbers n.

In other words, we want to prove $\forall\, n \in \mathbb{N},\ P(n)$.

Step 1: Base case

Prove P(0) (or any other case that should be verified first.

Step 2: Induction hypothesis

Since we want to show that $P(k) \Longrightarrow P(k+1)$,

We assume P(k) is true. (let n = k)

Step 3: Let n = k+1 and Prove P(k+1)

This is where we use the induction hypothesis P(k) to prove P(k+1) must also be true.

Why do we need to prove the base case?

Example: $\forall\ n \in \mathbb{N},\ n \geq 3 \implies 2n+1 < 2^n$

$P(n) = 2n+1 < 2^n$

$Q(n) = n \geq 3 \implies 2n+1 < 2^n$

Example: $\forall\ n \in \mathbb{N},\ n \geq 3 \implies 2n+1 < 2^n$ (Method 1)

We want to show that $\forall n \in \mathbb{N}$, $P(n)$ is true.

Base Case: $P(3)$

Induction Hypothesis: Assume $P(k)$

Proof of $P(k+1)$:

Example: $\forall\ n \in \mathbb{N},\ n \geq 3 \implies 2n+1 < 2^n$ (Method 2)

We want to show that $\forall n \in \mathbb{N}$, Q(n) is true.

Three cases:

- $n < 3$
- $n = 3$
- $n > 3$

Induction Hypothesis: Assume Q(k)

Proof of Q(k+1):