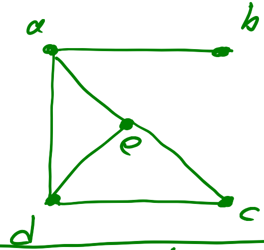
• PS4 office hours — check details on Quercus

FIRST — read PS4 FAQ on Piazza!

More proofs with graphs

"Connectivity"



$G_1 = (V_1, E_1)$

$V_1 = \{a, b, c, d, e\}$

$E_1 = \{\{a,b\}, \{a,d\}, \{a,e\}, \{c,d\}, \{c,e\}, \{d,e\}\}$

• $u, v \in V$ are "adjacent" iff $(u,v) \in E$

• $u, v \in V$ are "connected" iff G contains some path between u and v, i.e.,

$$\boxed{\exists k \in \mathbb{N}, \exists v_1, \ldots, v_k \in V, (u, v_1) \in E \land (v_1, v_2) \in E \land \ldots \land (v_{k-1}, v_k) \in E \land (v_k, v) \in E}$$

e.g., b, a, e is a path in G_1

NOTE: $k=0$ is possible — $(u,v) \in E$

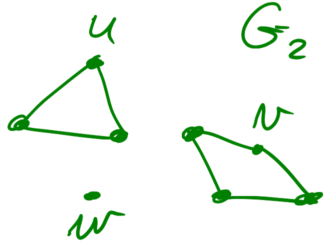
e.g., a,b are connected because $(a,b) \in E$
b,c are connected because $(b,a),(a,d),(d,c) \in E$

Q: v,v connected?

- v cannot be adjacent to v ((v,v) not an edge)
- special case: v is connected to v

NOTE: "u,v are connected in G" is a predicate of G,u,v

Q: u,v not connected? e.g.



Definition: "G is connected" iff $\forall u,v \in V$, u,v are connected in G

e.g., G_1 is connected, G_2 is not

Next, let's study <u>necessary</u> and <u>sufficient</u> conditions on $|E|$, for G to be connected.

- sufficient: condition $\Rightarrow$ G is connected
- necessary: $\neg$ condition $\Rightarrow$ G is <u>not</u> connected
  (G is connected $\Rightarrow$ condition)

- <u>Necessary condition</u>: $|E| \geq |V| - 1$
  proof: see notes...
- <u>Sufficient condition</u>: $|E| \geq \dfrac{(|V|-1)(|V|-2)}{2} + 1$
- Proof of sufficient condition:

WTS: $\boxed{\forall G=(V,E),\ |E| \geq \dfrac{(|V|-1)(|V|-2)}{2}+1 \Rightarrow G \text{ is connected}}$

- **Idea 1:** Let $G=(V,E)$. Assume $|E| \geq \frac{(|V|-1)(|V|-2)}{2}+1$

  WTS: G is connected...

  (direct proof) $\longrightarrow$ no obvious connection...

- **Idea 2:** indirect proof: assume G is <u>not</u> connected, try to prove $|E| < \frac{(|V|-1)(|V|-2)}{2}+1$.

- **Idea 3:** try to use induction

  Q: induction on what? need a natural number...

Insight: introduce a new variable to do induction on — typically, this is the <u>size</u> of the objects in the proof.

To prove:

choice: measure $|V|$

$\forall n \in \mathbb{Z}^+, \left[\forall G=(V,E), |V| = n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2} + 1 \Rightarrow G \text{ is connected} \right) \right]$

same as before

- $P(n)$: *
- Base Case: WTP $P(1)$:

  $\forall G=(V,E), |V|=1 \Rightarrow \left(|E| \geq \frac{(1-1)(1-2)}{2} + 1 \Rightarrow G \text{ is connected} \right)$

  Proof: vacuously true...

- I.H.: Let $n \in \mathbb{Z}^+$ and assume $P(n)$:

  $\forall G=(V,E), |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2} + 1 \Rightarrow G \text{ is connected} \right)$

- I.S.: WTP $P(n+1)$

- $\forall G_1=(V_1,E_1), |V_1|=n+1 \Rightarrow \left(|E_1| \geq \frac{(n+1-1)(n+1-2)}{2}+1 \Rightarrow G_1 \text{ is connected}\right)$

Let $G_1=(V_1,E_1)$.
Assume $|V_1|=n+1$.
Assume $|E_1| \geq \frac{n(n-1)}{2}+1$

WTP: G_1 is connected.

IMPORTANT:
we do NOT
start with some
graph of size n...