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1. [5 marks] Asymptotic Notation I.

You may use <https://www.desmos.com/calculator> to look at the graph of the function in this question, *but NO other online resource is allowed* for any question on this test. Also, you still need to provide rigorous arguments for each proof: remember that *a graph is NOT a rigorous argument*.

Consider the function $f: \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n^2(\cos(n\pi) + 1) + 1.$$

Prove that:

(a) [2 marks] $f \in \mathcal{O}(n^2)$

(b) [3 marks] $f \notin \Theta(n^2)$

Solution

[Redacted solution content]

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Prove that:

(a) [2 marks] $f \in \Omega(1)$

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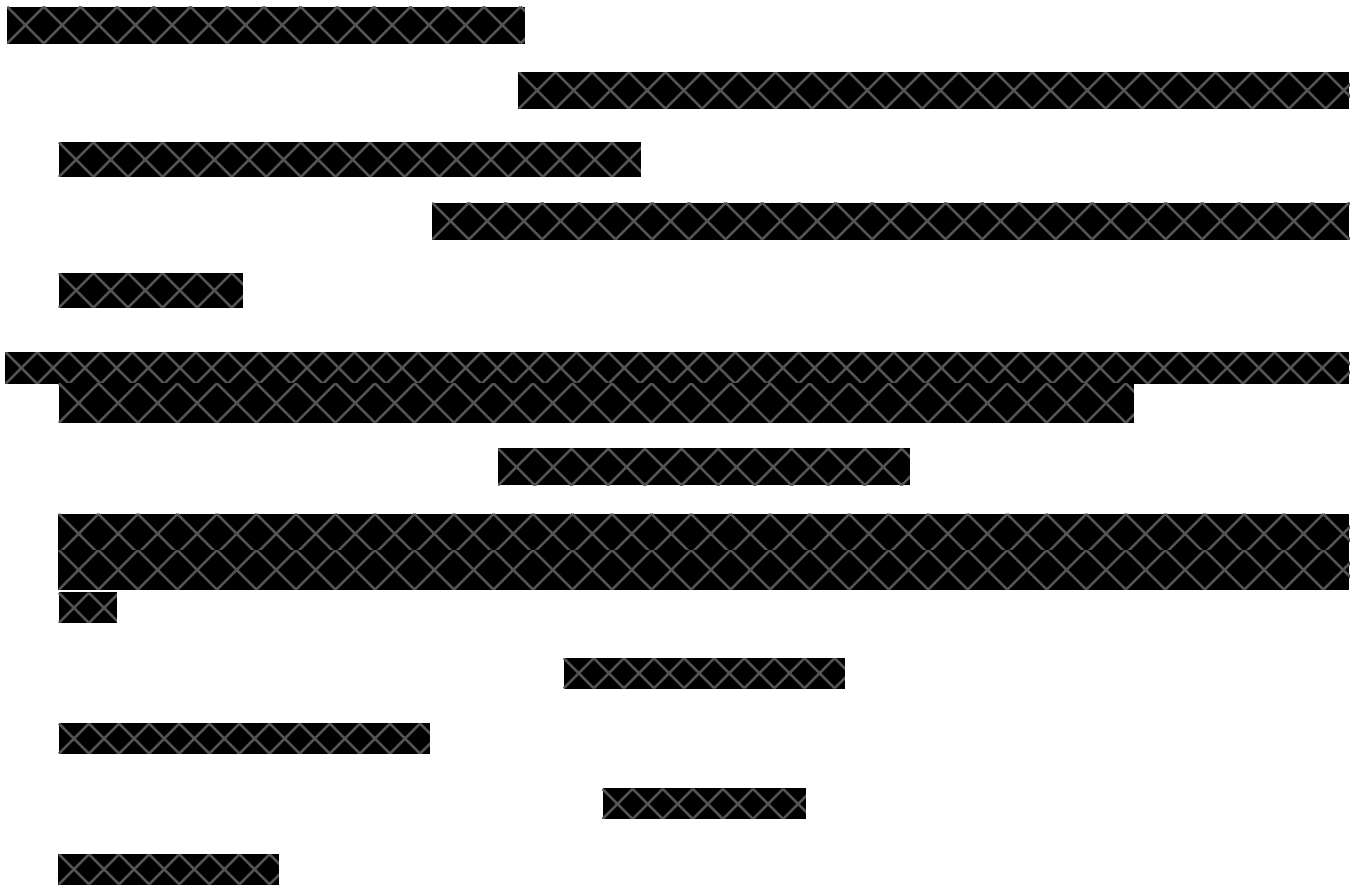
$$f(n) = n!(1 + (-1)^n) + n.$$

Prove that:

(a) [2 marks] $f \in \mathcal{O}(n!)$

(b) [3 marks] $f \notin \Theta(n!)$

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Consider the function $f: \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = (1 + (-1)^n)/n + 1/n^2.$$

Prove that:

(a) [2 marks] $f \in \mathcal{O}(1/n)$

(b) [3 marks] $f \notin \Theta(1/n)$

Solution

[Redacted]

[Redacted]

[Redacted]

[Redacted]

[Redacted]

[Redacted]

[Redacted]

[Redacted]

[Redacted]

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2. [3 marks] **Number Representations.**

Write the following natural numbers x . Feel free to write them as sums. *No proof is required for this question!*

- (a) The **largest** number x such that $(x)_2$ is 4-digits long.
- (b) The **smallest** number x such that $(x)_{16}$ is 5-digits long and contains exactly two A's and one E, with no leading 0's.
- (c) The **smallest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution













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2. [3 marks] Number Representations.

Write the following natural numbers x . Feel free to write them as sums. *No proof is required for this question!*

- (a) The **smallest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's, with no leading 0's.
- (b) The **largest** number x such that $(x)_8$ is 5-digits long and contains exactly two 2's and one 7.
- (c) The **smallest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

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- (c) The **largest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, where each digit appears at most twice.

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- (c) The **smallest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

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Solution













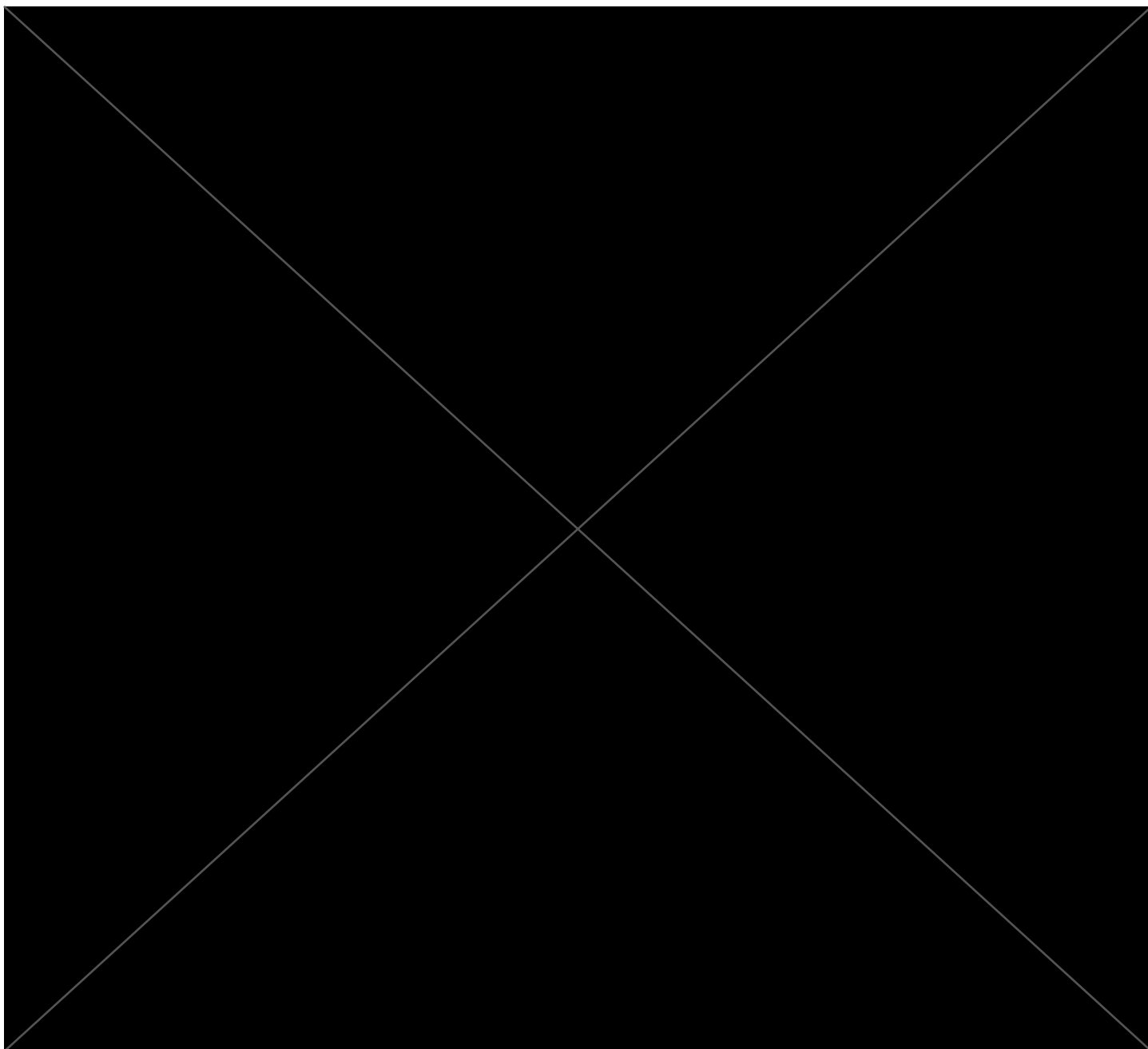
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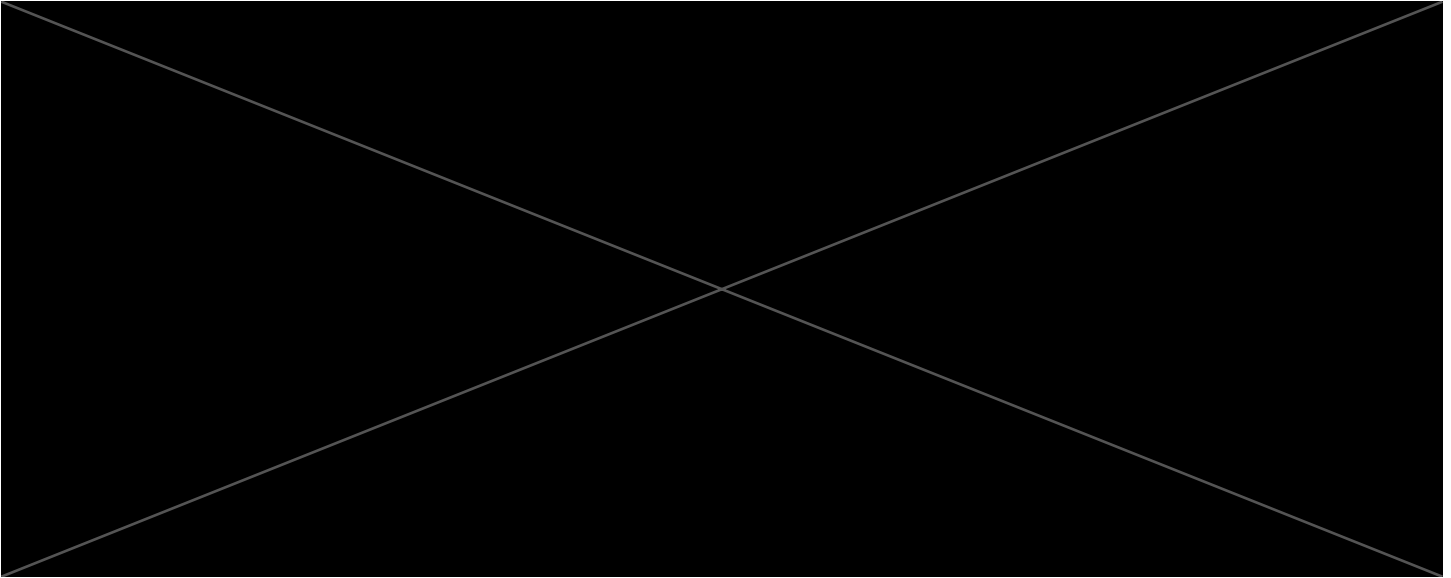
3. [4 marks] Induction.

Warning! This question does not require deep insight but it is longer to write up (you may need more than 1 page). You should keep it for last. Also, *you will receive **at most** half the marks* if you do NOT use induction.

Let $a_0, a_1, a_2, \dots \in \mathbb{R}$ and $b_0, b_1, b_2, \dots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$, if $n \geq 1$, then

$$\sum_{k=0}^{n-1} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1} (b_{k+1} - b_k).$$

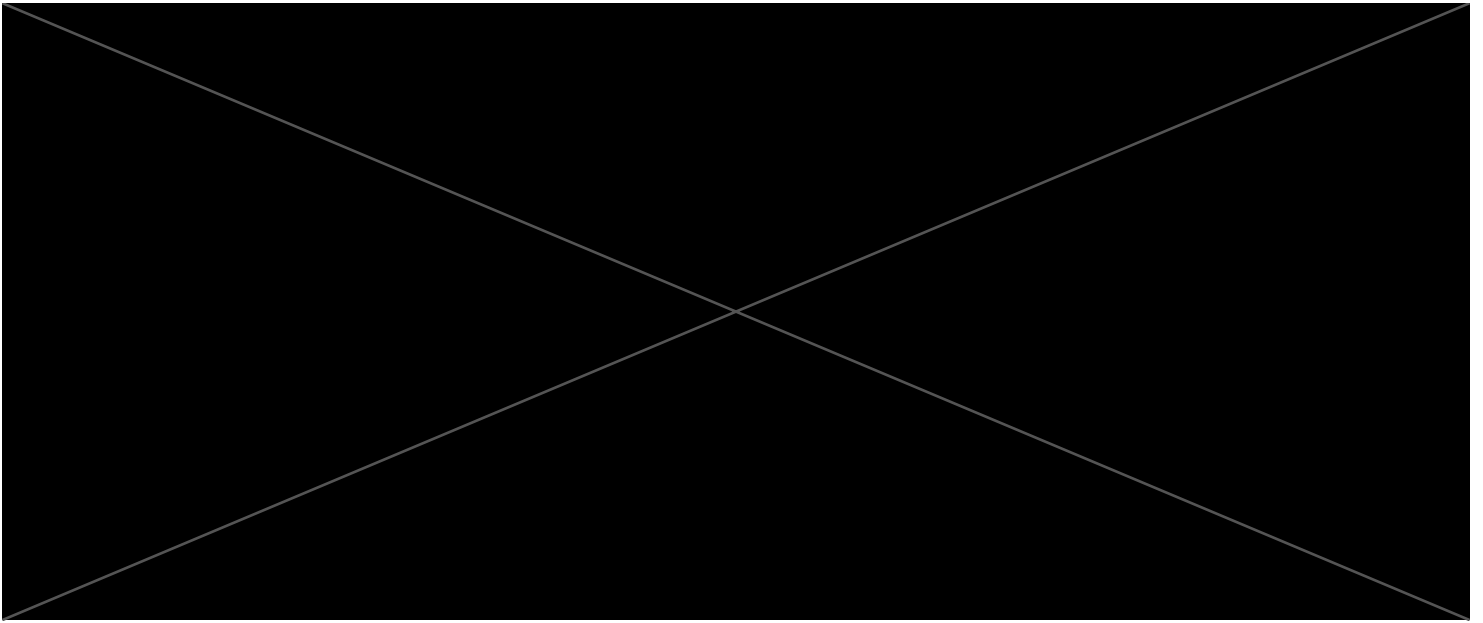




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$$\sum_{k=1}^{n-1} (a_{k+1} - a_k) b_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1} (b_{k+1} - b_k).$$



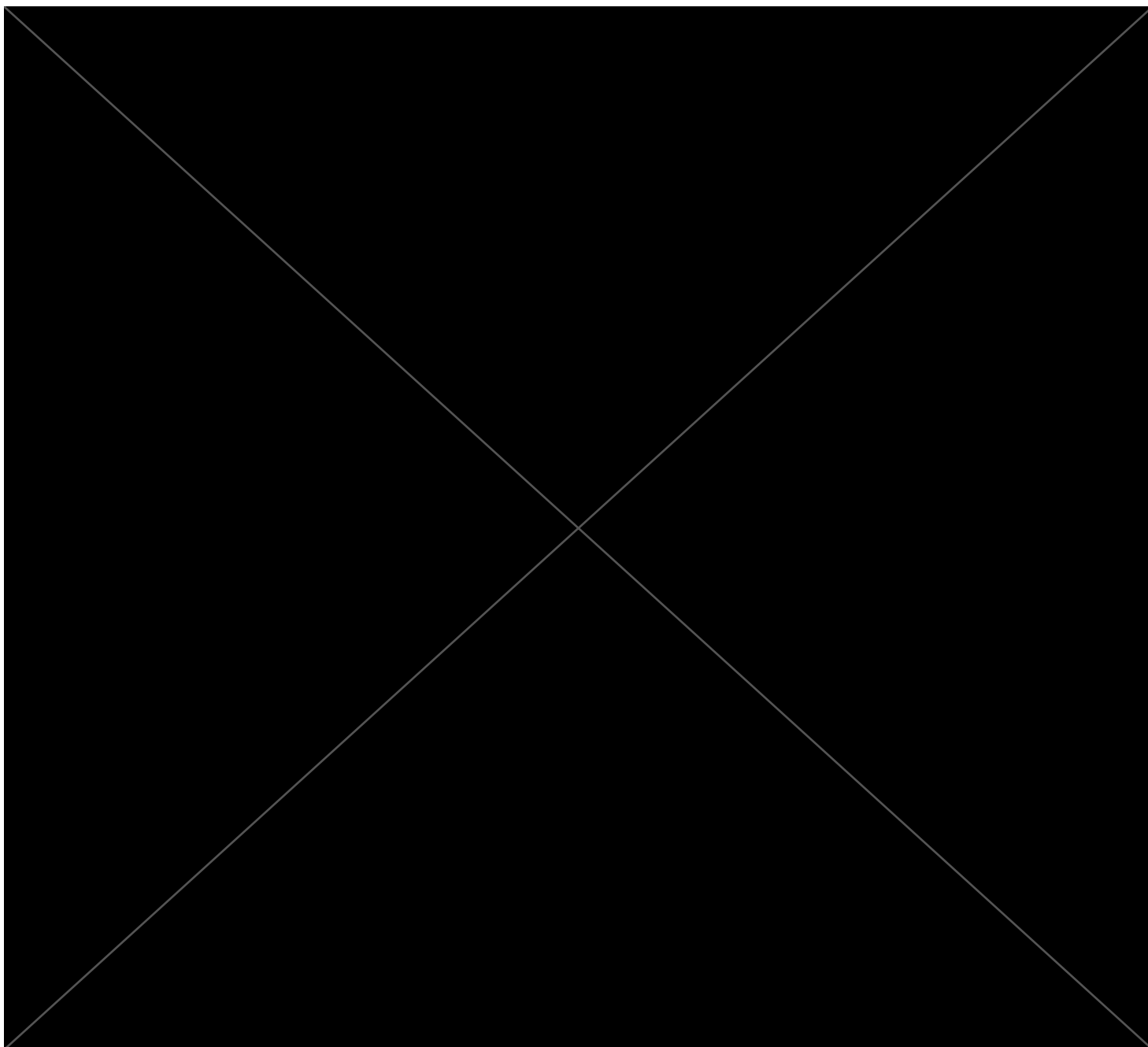
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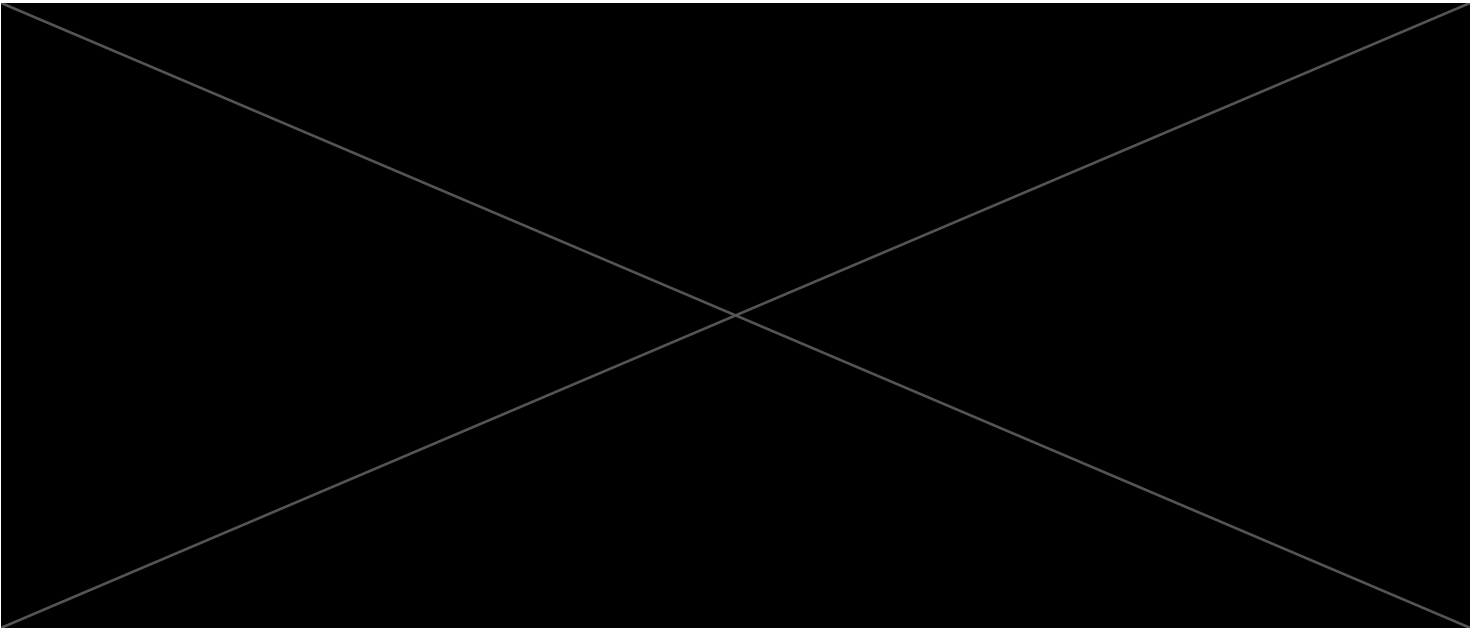
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$$\sum_{k=1}^n (a_{k+1} - a_k) b_k = a_{n+1} b_{n+1} - a_1 b_1 - \sum_{k=1}^n a_{k+1} (b_{k+1} - b_k).$$





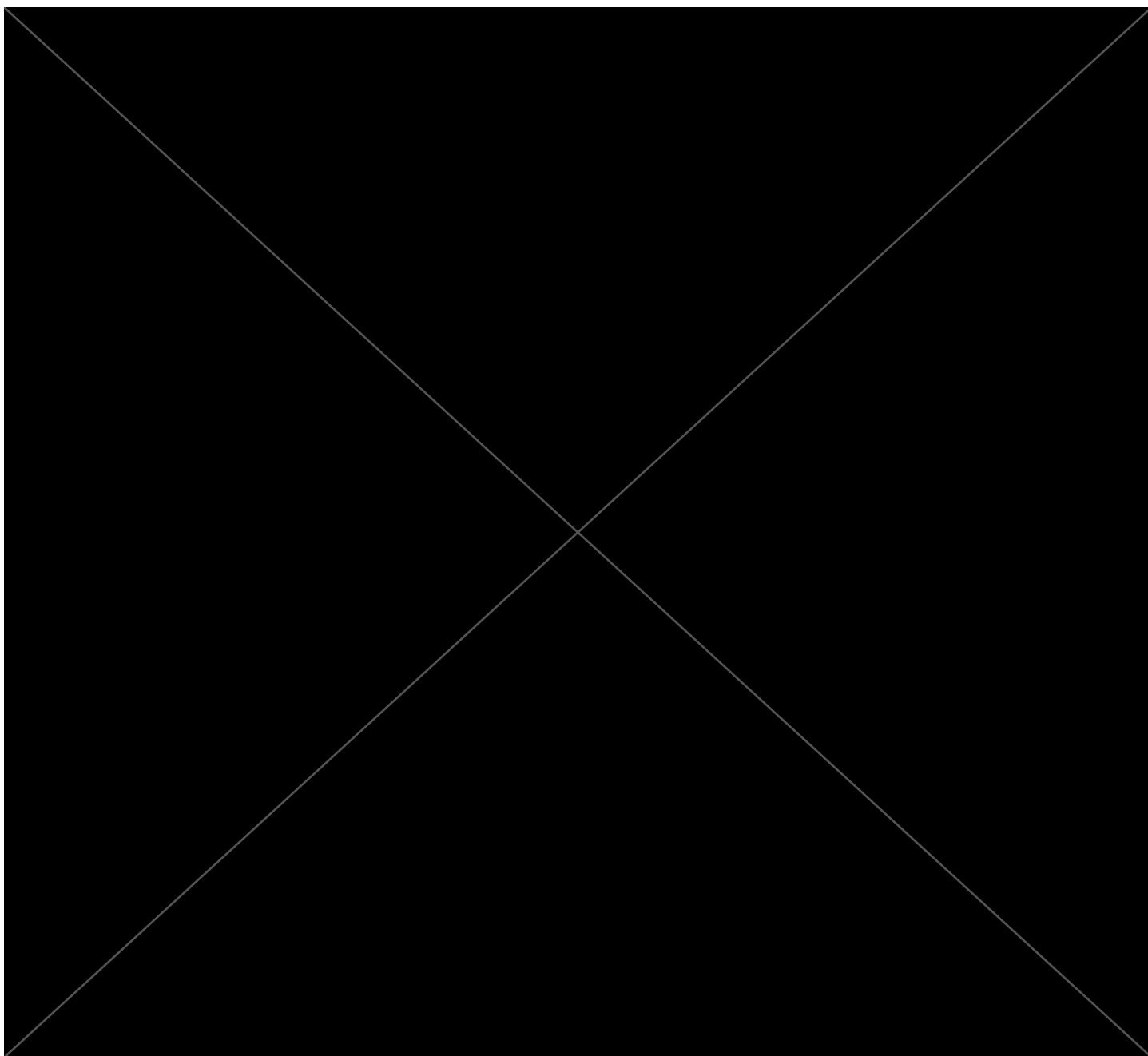
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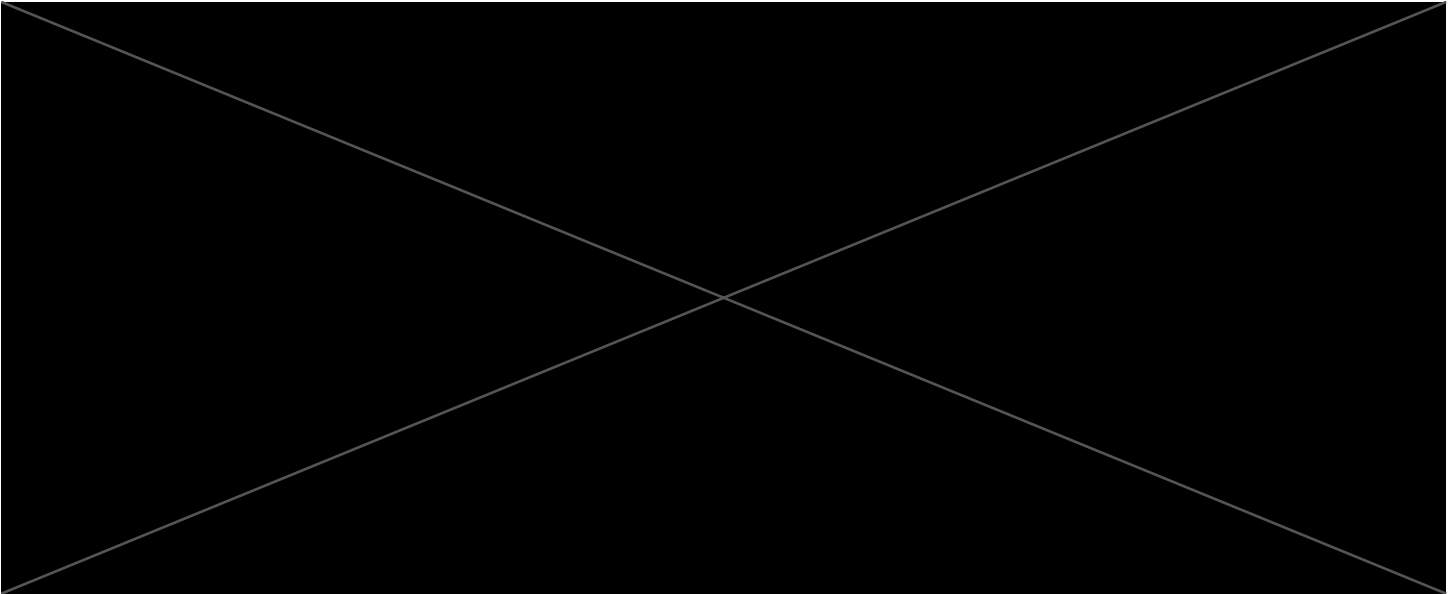
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$$\sum_{k=1}^{n+1} (a_{k+1} - a_k) b_k = a_{n+2} b_{n+2} - a_1 b_1 - \sum_{k=1}^{n+1} a_{k+1} (b_{k+1} - b_k).$$



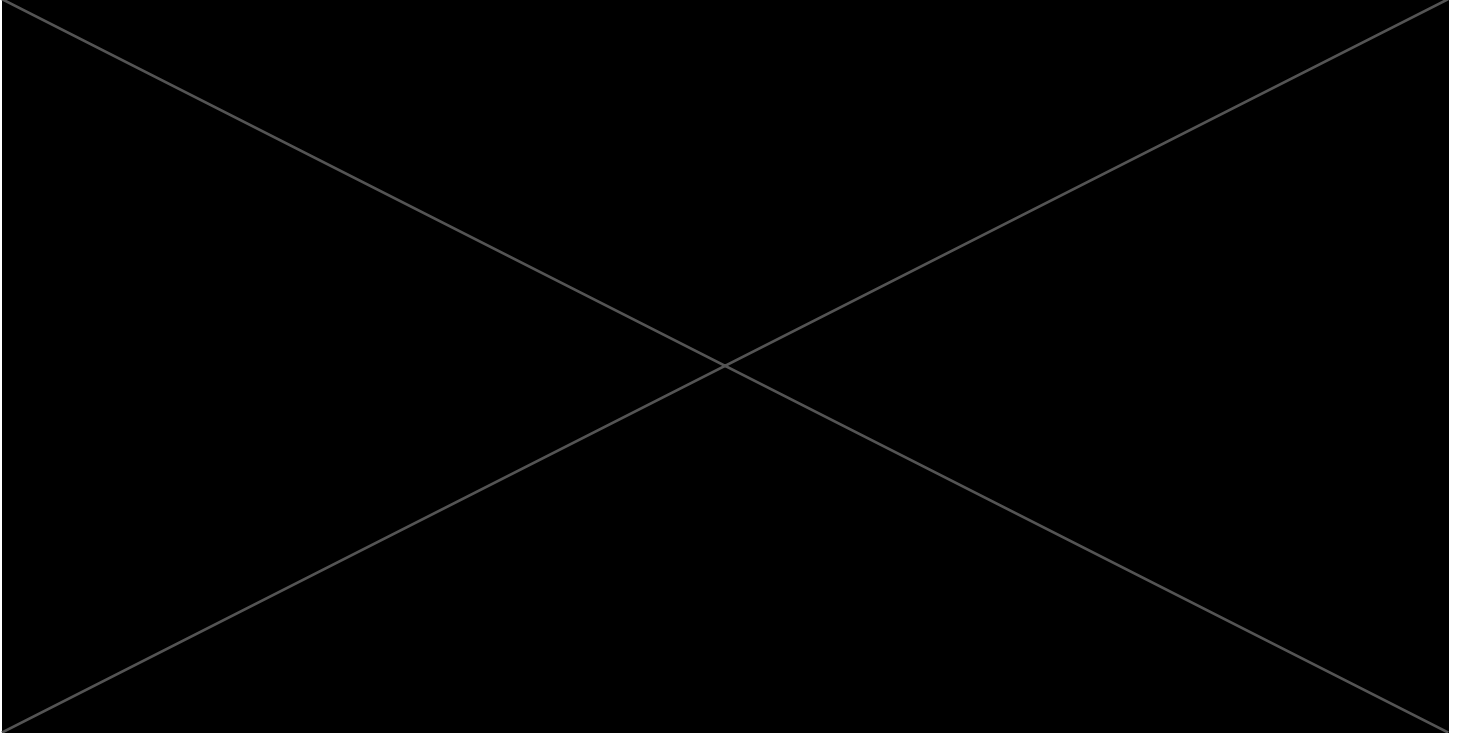


*Don't forget: this test contains **four** separate questions (plus the Academic Integrity statement)!*

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4. [4 marks] **Asymptotic Notation II.**

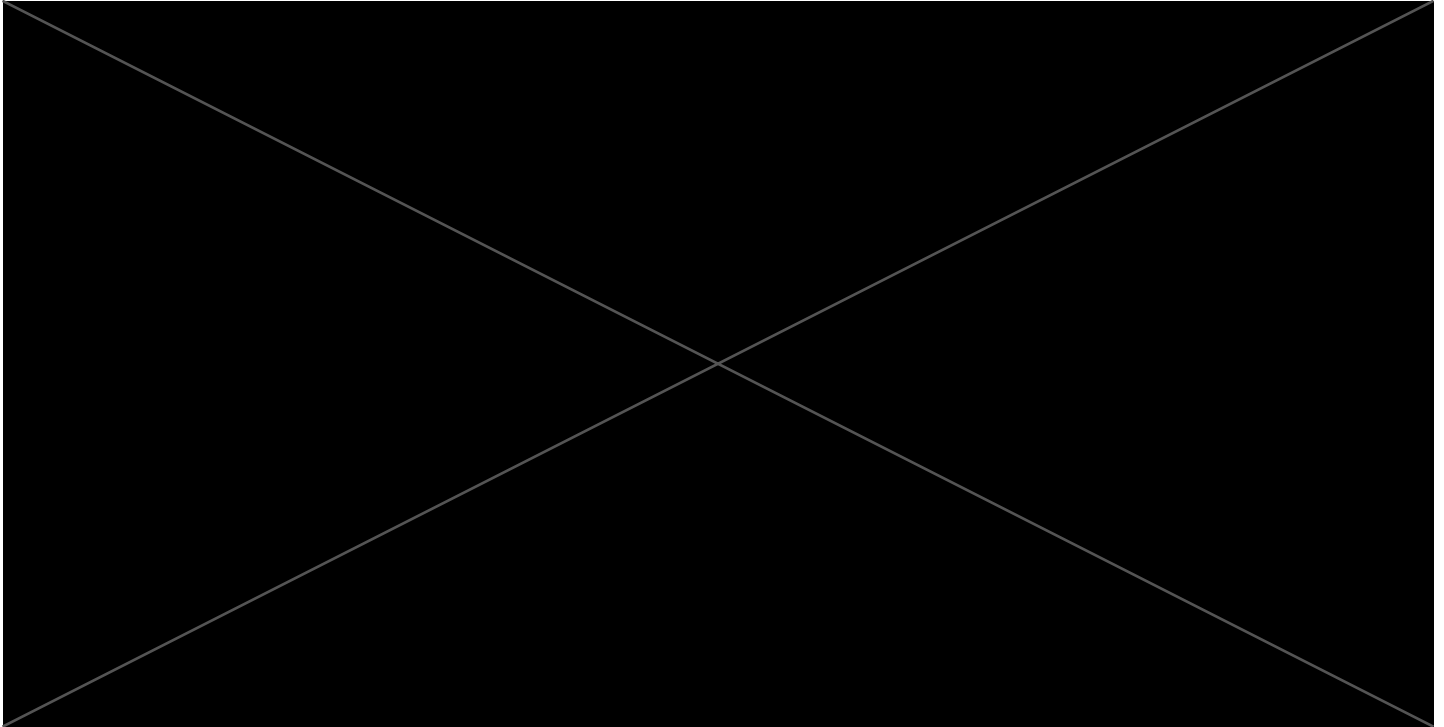
Let $f: \mathbb{N} \rightarrow \mathbb{N}$ and $g: \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Theta(\sqrt{n})$ and $g(n) \in \mathcal{O}(n^2)$, then $g(f(n)) \in \mathcal{O}(n)$.



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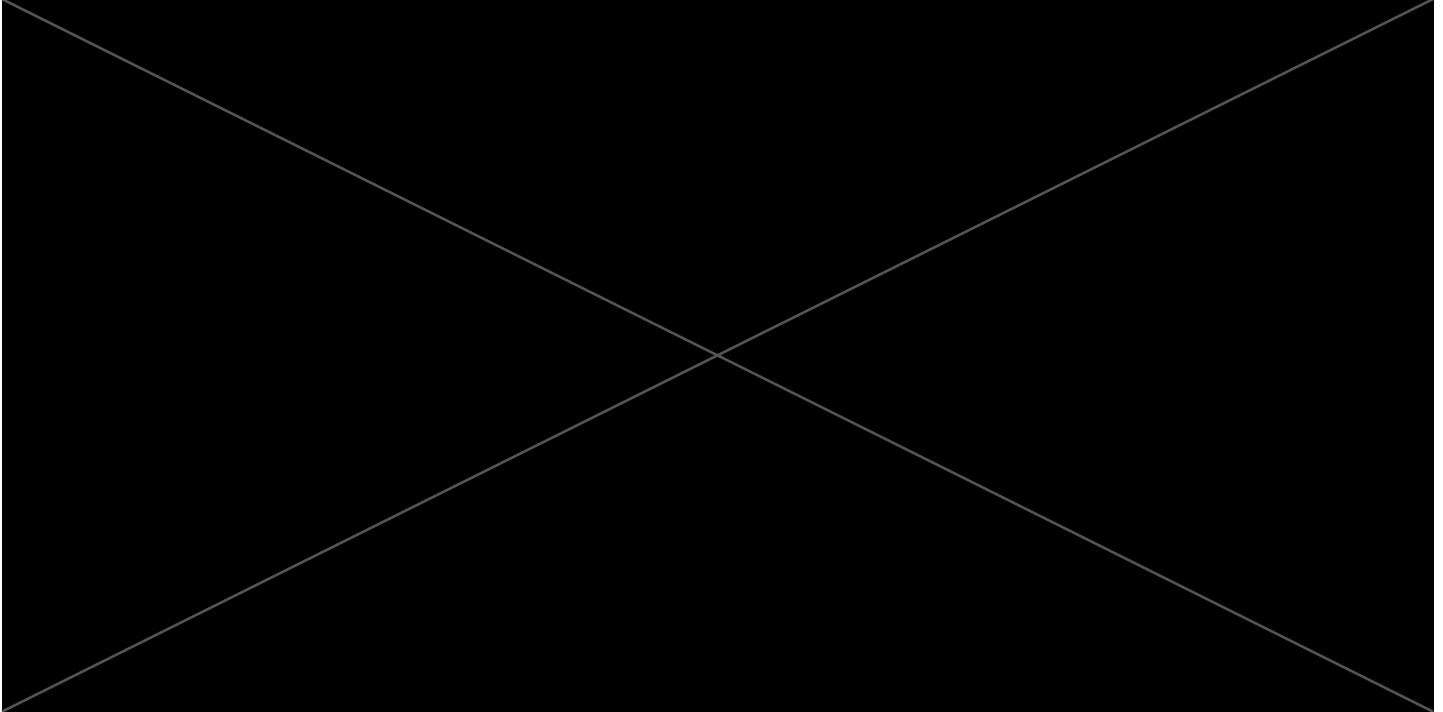
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