# Learning objectives

By the end of this worksheet, you will:

- Understand and apply definitions about sets, strings, and common mathematical functions.
- Simplify summation and product expressions.

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1. **Set complement**. Let $A$ and $U$ be sets, and assume that $A \subseteq U$. The **complement of** $A$ **in** $U$, denoted $A^c$, is defined to be set of elements that are in $U$ but not $A$. $A^c$ depends on the choice of both $U$ and $A$!

   (a) Let $U$ be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is $A^c$?

   (b) Given an arbitrary $A$ and $U$, write an expression for $A^c$ in terms of $A$, $U$, and the set difference operator $\setminus$.

   (c) Let $U = \mathbb{R}$, $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$, and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. What relationships do you notice between these sets?

2. **Set partitions.** Let $A$ be a set. A (finite or infinite) collection of nonempty sets $\{A_1, A_2, A_3, \dots\}$ is called a **partition** of $A$ when (1) $A$ is the union of all of the $A_i$,[1] and (2) the sets $A_1, A_2, A_3, \dots$ do not have any element in common.[2]

   (a) Recall that $\mathbb{Z}^+$ is the set of all positive integers. Let

   $$\begin{aligned}
   T_0 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\},\\
   T_1 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+1, \text{ for some integer } k\},\\
   T_2 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+2, \text{ for some integer } k\},\\
   T_3 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}.
   \end{aligned}$$

   Write the smallest three elements of $T_0$, of $T_1$, of $T_2$, and of $T_3$.

   (b) Write down a partition of $\mathbb{Z}^+$ using $T_0$, $T_1$, $T_2$, and/or $T_3$. Why can't you use all four sets?

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[1] We say the $A_i$ are **exhaustive**.

[2] We say the $A_i$ are **mutually disjoint** (or **pairwise disjoint** or **non-overlapping**) when no two distinct sets $A_i$ and $A_j$ have any element in common.

3. **Strings.** An **alphabet** $A$ is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. We define a **string over alphabet** $A$ as an ordered sequence of elements from $A$; the **length** of a finite string is its number of elements.

   For example, 011 is a string over $\{0, 1\}$ of length three, and *abbbacc* is a string over $\{a, b, c\}$ of length seven.

   (a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

   (b) Let $S_1$ be the set of all strings over $\{a, b, c\}$ that have length two, and $S_2$ be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

   (c) What is the relationship between $S_1$, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

4. **The floor and ceiling functions.** Let $x \in \mathbb{R}$. We define the **floor of** $x$, denoted $\lfloor x \rfloor$, to be the largest integer that is less than or equal to $x$. Similarly, we define the **ceiling of** $x$, denoted $\lceil x \rceil$, to be the smallest integer that is greater than or equal to $x$.

   (a) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of $x$: $x = \dfrac{25}{4}$, $x = 0.999$, and $x = -2.01$.

   (b) What is the domain and codomain of the floor and ceiling functions?

   (c) Consider the following statement: For all real numbers $x$ and $y$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Is this statement is True or False? Why?

5. **Sum and product notation**. Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for $f(j) + f(j+1) + \cdots + f(k-1) + f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for $f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k)$.

(a) Expand the following expressions into their long sum/product form. Do *not* evaluate the resulting expressions.

$$\sum_{k=1}^{3}(k+1)$$

$$\sum_{m=0}^{1} \frac{1}{2^m}$$

$$\sum_{k=-1}^{2} (k^2+3)$$

$$\sum_{j=0}^{4}(-1)^j \frac{j}{j+1}$$

$$\sum_{k=1}^{5}(2k)$$

$$\prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)}$$

(b) Rewrite each of the following expressions by using sum or product notation.

$$3 + 6 + 12 + 24 + 48 + 96$$

$$\frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729}$$

$$0 + 1 - 2 + 3 - 4 + 5$$

$$\left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right)$$

$$\left(\frac{1 \cdot 2}{3 \cdot 4}\right) \times \left(\frac{2 \cdot 3}{4 \cdot 5}\right) \times \left(\frac{3 \cdot 4}{5 \cdot 6}\right)$$

6. **Sum and product laws**. It is possible to prove properties that help us manipulate sums and products. Let $m, n \in \mathbb{Z}$, and let $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ be sequences of real numbers, and let $c \in \mathbb{R}$. Then the following equations hold:[3]

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right) \qquad \text{(separating sums)}$$

$$\prod_{i=m}^{n} (a_i \cdot b_i) = \left(\prod_{i=m}^{n} a_i\right) \cdot \left(\prod_{i=m}^{n} b_i\right) \qquad \text{(separating products)}$$

$$\sum_{i=m}^{n} c \cdot a_i = c \cdot \left(\sum_{i=m}^{n} a_i\right) \qquad \text{(pulling out constant)}$$

$$\sum_{i=m}^{n} a_i = \sum_{i'=0}^{n-m} a_{i'+m} \qquad \text{(changing index)}$$

$$\prod_{i=m}^{n} a_i = \prod_{i'=0}^{n-m} a_{i'+m} \qquad \text{(changing index)}$$

Using these laws, rewrite each of the following as a single sum or product, but do not evaluate your final sum/product.[4]

$$3 \cdot \sum_{i=1}^{n} (2i - 3) + \sum_{i=1}^{n} (4 - 5i)$$

$$\left(\prod_{i=1}^{n} \frac{i}{i+1}\right) \left(\prod_{i=1}^{n} \frac{i+1}{i+2}\right)$$

$\displaystyle\sum_{i=10}^{15} 2i + \sum_{i=101}^{106} (i - 1)$ (change the indexes to match)

[3] Because of how we've defined the *empty sum* and *empty product*, these equations hold even when $n < m$!

[4] We'll cover some formulas for evaluating common sums and products throughout this course.