Learning objectives

By the end of this worksheet, you will:

- Understand and apply definitions about sets, strings, and common mathematical functions.
- Simplify summation and product expressions.

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1. **Set complement**. Let A and U be sets, and assume that $A \subseteq U$. The **complement of A in U**, denoted A^c, is defined to be set of elements that are in U but not A. A^c depends on the choice of both U and A!

   (a) Let U be the set of natural numbers between 1 and 6, inclusive. Let $A = \{2, 5\}$. What is A^c?

   Solution

   $A^c = \{1, 3, 4, 6\}$.

   (b) Given an arbitrary A and U, write an expression for A^c in terms of A, U, and the set difference operator $\setminus$.

   Solution

   $A^c = U \setminus A$.

   (c) Let $U = \mathbb{R}$, $A = \{x \mid x \in U \text{ and } 0 < x \leq 2\}$, and $B = \{x \mid x \in U \text{ and } 1 \leq x < 4\}$. Find each of the following: $A^c \cap B^c$, $A^c \cup B^c$, $(A \cap B)^c$ and $(A \cup B)^c$. Drawing number lines may be helpful. What relationships do you notice between these sets?

   Solution

$$
\begin{aligned}
A^c \cap B^c &= \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\} \\
A^c \cup B^c &= \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\} \\
(A \cap B)^c &= \{x \mid x \in U \text{ and } x < 1 \text{ or } x > 2\} \\
(A \cup B)^c &= \{x \mid x \in U \text{ and } x \leq 0 \text{ or } x \geq 4\}
\end{aligned}
$$

   There are two equalities we observe: $A^c \cap B^c = (A \cup B)^c$ and $A^c \cup B^c = (A \cap B)^c$ (de Morgan's laws for sets).

2. **Set partitions.** Let A be a set. A (finite or infinite) collection of nonempty sets $\{A_1, A_2, A_3, \dots\}$ is called a **partition** of A when (1) A is the union of all of the A_i,[1] and (2) the sets $A_1, A_2, A_3, \dots$ do not have any element in common.[2]

   (a) Recall that $\mathbb{Z}^+$ is the set of all positive integers. Let

$$
\begin{aligned}
T_0 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k, \text{ for some integer } k\}, \\
T_1 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+1, \text{ for some integer } k\}, \\
T_2 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 3k+2, \text{ for some integer } k\}, \\
T_3 &= \{n \mid n \in \mathbb{Z}^+ \text{ and } n = 6k, \text{ for some integer } k\}.
\end{aligned}
$$

   Write the smallest three elements of T_0, of T_1, of T_2, and of T_3.

[1] We say the A_i are **exhaustive**.

[2] We say the A_i are **mutually disjoint** (or **pairwise disjoint** or **non-overlapping**) when no two distinct sets A_i and A_j have any element in common.

Solution

$T_0 = \{3, 6, 9, \dots\}$, $T_1 = \{1, 4, 7, \dots\}$, $T_2 = \{2, 5, 8, \dots\}$, $T_3 = \{6, 12, 18, \dots\}$.

(b) Write down a partition of $\mathbb{Z}^+$ using T_0, T_1, T_2, and/or T_3. Why can't you use all four sets?

Solution

The set $\{T_0, T_1, T_2\}$ is a partition of $\mathbb{Z}^+$, since, when any positive integer is divided by 3, the possible integer remainders are 0, 1, and 2. The sets T_0, T_1, T_2 list the numbers whose remainder when divided by 3 are 0, 1, or 2, respectively.

$T_3 \subseteq T_0$, so we can't use both T_0 and T_3 in our partition (they have elements in common).

3. **Strings.** An **alphabet** A is a set of symbols like $\{0, 1\}$ or $\{a, b, c\}$. We define a **string over alphabet** A as an ordered sequence of elements from A; the **length** of a finite string is its number of elements.

   For example, 011 is a string over $\{0, 1\}$ of length three, and *abbbacc* is a string over $\{a, b, c\}$ of length seven.

   (a) Write down all strings over the alphabet $\{0, 1\}$ of length three (you should have eight in total).

   Solution

   $\{000, 001, 010, 011, 100, 101, 110, 111\}$

   (b) Let S_1 be the set of all strings over $\{a, b, c\}$ that have length two, and S_2 be the set of all strings over $\{a, b, c\}$ that start and end with the same letter. Find $S_1 \cap S_2$ and $S_1 \setminus S_2$.

   Solution

   $S_1 \cap S_2 = \{aa, bb, cc\}$.
   $S_1 \setminus S_2 = \{ab, ac, ba, bc, ca, cb\}$.
   Note that S_2 is actually an infinite set, but both $S_1 \cap S_2$ and $S_1 \setminus S_2$ are finite.

   (c) What is the relationship between S_1, $S_1 \cap S_2$, and $S_1 \setminus S_2$?

   Solution

   Hint: look at $(S_1 \cap S_2) \cup (S_1 \setminus S_2)$.

4. **The floor and ceiling functions.** Let $x \in \mathbb{R}$. We define the **floor of** x, denoted $\lfloor x \rfloor$, to be the largest integer that is less than or equal to x. Similarly, we define the **ceiling of** x, denoted $\lceil x \rceil$, to be the smallest integer that is greater than or equal to x.

   (a) Compute $\lfloor x \rfloor$ and $\lceil x \rceil$ for each of the following values of x: $x = \dfrac{25}{4}$, $x = 0.999$, and $x = -2.01$.

   Solution

   $\left\lfloor \dfrac{25}{4} \right\rfloor = \lfloor 6.25 \rfloor = 6$, $\left\lceil \dfrac{25}{4} \right\rceil = \lceil 6.25 \rceil = 7$, $\lfloor 0.999 \rfloor = 0$, $\lceil 0.999 \rceil = 1$, $\lfloor -2.01 \rfloor = -3$, $\lceil -2.01 \rceil = -2$.

   (b) What is the domain and codomain of the floor and ceiling functions?

   Solution

   The domain is $\mathbb{R}$ and the codomain is $\mathbb{Z}$.

   (c) Consider the following statement: For all real numbers x and y, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$. Is this statement True or False? Why?

   Solution

   The statement is False, since, for example, $\left\lfloor \dfrac{1}{2} + \dfrac{2}{3} \right\rfloor = \left\lfloor \dfrac{7}{6} \right\rfloor = 1$, while $\left\lfloor \dfrac{1}{2} \right\rfloor + \left\lfloor \dfrac{2}{3} \right\rfloor = 0 + 0 = 0$.

5. **Sum and product notation**. Recall that the notation $\sum_{i=j}^{k} f(i)$ gives us a short form for $f(j) + f(j+1) + \cdots + f(k-1) + f(k)$, and that $\prod_{i=j}^{k} f(i)$ gives us a short form for $f(j) \times f(j+1) \times \cdots \times f(k-1) \times f(k)$.

(a) Expand the following expressions into their long sum/product form. Do *not* evaluate the resulting expressions.

Solution

$$\sum_{k=1}^{3}(k+1) = (1+1)+(2+1)+(3+1) \qquad \sum_{m=0}^{1} \frac{1}{2^m} = \frac{1}{2^0} + \frac{1}{2^1}$$

$$\sum_{k=-1}^{2}(k^2+3) = (1+3)+(0+3)+(1+3)+(4+3) \qquad \sum_{j=0}^{4}(-1)^j \frac{j}{j+1} = 0 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5}$$

$$\sum_{k=1}^{5}(2k) = 2+4+6+8+10 \qquad \prod_{i=2}^{4} \frac{i(i+2)}{(i-1)(i+1)} = \frac{2 \cdot 4}{1 \cdot 3} \cdot \frac{3 \cdot 5}{2 \cdot 4} \cdot \frac{4 \cdot 6}{3 \cdot 5}$$

(b) Rewrite each of the following expressions by using sum or product notation.

Solution

$$3+6+12+24+48+96 = \sum_{i=0}^{5} 3 \cdot 2^i \qquad \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \frac{16}{81} + \frac{25}{243} + \frac{36}{729} = \sum_{j=1}^{6} \frac{j^2}{3^j}$$

$$0+1-2+3-4+5 = \sum_{j=0}^{5}(-1)^{j+1} \cdot j \qquad \left(\frac{1}{1+1}\right) \times \left(\frac{2}{2+1}\right) \times \left(\frac{3}{3+1}\right) \times \cdots \times \left(\frac{k}{k+1}\right) = \prod_{j=1}^{k}\left(\frac{j}{j+1}\right)$$

$$\left(\frac{1 \cdot 2}{3 \cdot 4}\right) \times \left(\frac{2 \cdot 3}{4 \cdot 5}\right) \times \left(\frac{3 \cdot 4}{5 \cdot 6}\right) = \prod_{j=1}^{3} \frac{j \cdot (j+1)}{(j+2) \cdot (j+3)}$$

6. **Sum and product laws**. It is possible to prove properties that help us manipulate sums and products. Let $m, n \in \mathbb{Z}$, and let $a_m, a_{m+1}, a_{m+2}, \dots$ and $b_m, b_{m+1}, b_{m+2}, \dots$ be sequences of real numbers, and let $c \in \mathbb{R}$. Then the following equations hold:[3]

$$\sum_{i=m}^{n}(a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right) \qquad \text{(separating sums)}$$

$$\prod_{i=m}^{n}(a_i \cdot b_i) = \left(\prod_{i=m}^{n} a_i\right) \cdot \left(\prod_{i=m}^{n} b_i\right) \qquad \text{(separating products)}$$

$$\sum_{i=m}^{n} c \cdot a_i = c \cdot \left(\sum_{i=m}^{n} a_i\right) \qquad \text{(pulling out constant)}$$

$$\sum_{i=m}^{n} a_i = \sum_{i'=0}^{n-m} a_{i'+m} \qquad \text{(changing index)}$$

$$\prod_{i=m}^{n} a_i = \prod_{i'=0}^{n-m} a_{i'+m} \qquad \text{(changing index)}$$

Using these laws, rewrite each of the following as a single sum or product, but do not evaluate your final sum/product.[4]

$$3 \cdot \sum_{i=1}^{n}(2i-3) + \sum_{i=1}^{n}(4-5i)$$

Solution

$$\begin{aligned} 3 \cdot \sum_{i=1}^{n}(2i-3) + \sum_{i=1}^{n}(4-5i) &= \sum_{i=1}^{n}(6i-9) + \sum_{i=1}^{n}(4-5i) \\ &= \sum_{i=1}^{n}\big((6i-9)+(4-5i)\big) \\ &= \sum_{i=1}^{n}(i-5) \end{aligned}$$

$$\left(\prod_{i=1}^{n} \frac{i}{i+1}\right)\left(\prod_{i=1}^{n} \frac{i+1}{i+2}\right)$$

Solution

$$\begin{aligned} \left(\prod_{i=1}^{n} \frac{i}{i+1}\right)\left(\prod_{i=1}^{n} \frac{i+1}{i+2}\right) &= \prod_{i=1}^{n}\left(\frac{i}{i+1} \cdot \frac{i+1}{i+2}\right) \\ &= \prod_{i=1}^{n} \frac{i}{i+2} \end{aligned}$$

$\displaystyle\sum_{i=10}^{15} 2i + \sum_{i=101}^{106} (i-1)$ (change the indexes to match)

[3] Because of how we've defined the *empty sum* and *empty product*, these equations hold even when $n < m$!

[4] We'll cover some formulas for evaluating common sums and products throughout this course.

Solution

$$\begin{aligned}
\sum_{i=10}^{15} 2i + \sum_{i=101}^{106} (i-1) &= \sum_{i=0}^{5} 2(i+10) + \sum_{i=0}^{5} (i+101-1) \\
&= \sum_{i=0}^{5} 2(i+10) + \sum_{i=0}^{5} (i+101-1) \\
&= \sum_{i=0}^{5} \big(2(i+10) + (i+101-1)\big) \\
&= \sum_{i=0}^{5} (3i+120)
\end{aligned}$$

Note: it is also possible to change the first summation to go from $i = 101$ to 106, or the second summation to go from $i = 10$ to 15.