University of Toronto
Faculty of Arts and Science

CSC165H1S Midterm 1, Version 3

Date: February 6, 2019 Duration: 75 minutes Instructor(s): David Liu, François Pitt

No Aids Allowed

Name: ______________________

Student Number: ____________

- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- All statements predicate logic must have negations applied directly to propositional variables or predicates.
- You may not define your own propositional operators, predicates, or sets, unless asked to do so in the question. Please work with the symbols we have introduced in lecture, and any additional definitions provided in the questions.
- Proofs should follow the guidelines used in the course (e.g., explicitly introduce all variables, clearly state all assumptions, justify every deduction in your proof body, etc.)
- In your proofs, you may always use definitions from the course. However, you may **not** use any external facts about these definitions unless the yare given in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

Question	Grade	Out of
Q1		8
Q2		7
Q3		6
Q4		5
Total		26

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

1. **[8 marks] Short answers questions.**

(a) **[2 marks]** Let $U = \{a, b\}$. Let S_1 be the set of strings over U that start and end with different letters, and let S_2 be the set of strings over U with length 3. Write down all the elements of $S_2 \backslash S_1$.

Solution

$S_2 \backslash S_1 = \{aaa, aba, bab, bbb\}$

(b) **[3 marks]** Write down a truth table for the following expression in propositional logic. Rough work (e.g., intermediate columns of the truth table) is **not** required, but can be included if you want.

$$(p \Rightarrow q) \Leftrightarrow \neg r$$

Solution

p	q	r	$(p \Rightarrow q) \Leftrightarrow \neg r$
False	False	False	True
False	False	True	False
False	True	False	True
False	True	True	False
True	False	False	False
True	False	True	True
True	True	False	True
True	True	True	False

(c) **[3 marks]** Consider the following statement (assume predicates P and Q have already been defined):

$$\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ P(x) \Rightarrow Q(x, y) \vee Q(x, y + 1)$$

Suppose we want to **disprove** this statement. Write the complete *proof header* for a disproof; you may write statements like "Let $x =$ _____" without filling in the blank. The last statement of your proof header should be "We will prove that..." where you clearly state what's left to prove, in the same style as the lectures or the Course Notes.

You do not need to include any other work (but clearly mark any rough work you happen to use).

Solution

Let $x =$ _____. Let $y \in \mathbb{N}$. We will prove that $P(x)$ is True and that $Q(x, y)$ and $Q(x, y + 1)$ are False.

2. **[7 marks] Translations.** Let T be the set of all hockey teams, and suppose we define the following predicates:

- $Canadian(x)$: "x is a Canadian team", where $x \in T$
- $Star(x)$: "x has at least one player who is a superstar", where $x \in T$
- $Defeated(x, y)$: "x defeated y at least once", where $x, y \in T$ (note that $Defeated(x, y)$ does *not* mean the same thing as $Defeated(y, x)$)

Translate each of the following statements. No explanation is necessary. Do not define any of your own predicates or sets. You may use the $=$ and $\neq$ symbols to compare whether two teams are the same.

(a) **[1 mark]** Every Canadian team has at least one superstar player.

Solution

$\forall x \in T,\ Canadian(x) \Rightarrow Star(x)$

(b) **[2 marks]** Every Canadian team has defeated every non-Canadian team.

Solution

$\forall x \in T,\ Canadian(x) \Rightarrow \big(\forall y \in T,\ \neg Canadian(y) \Rightarrow Defeated(x, y)\big)$

Or equivalently:

$\forall x, y \in T,\ Canadian(x) \land \neg Canadian(y) \Rightarrow Defeated(x, y)$

(c) **[2 marks]** If at least one Canadian team has a superstar, then every Canadian team has defeated at least one other team. (Note: "other" means that the second team must be different from the first.)

Solution

$\Big(\exists x \in T,\ Canadian(x) \land Star(x)\Big) \Rightarrow \Big(\forall x \in T,\ Canadian(x) \Rightarrow \big(\exists y \in T,\ y \neq x \land Defeated(x, y)\big)\Big)$

(d) **[2 marks]** There is a Canadian team with a superstar, but every other Canadian team does not have a superstar. (Note: "other" means that the second team must be different from the first.)

Solution

$\exists x \in T,\ Canadian(x) \land Star(x) \land \big(\forall y \in T,\ x \neq y \land Canadian(x) \Rightarrow \neg Star(y)\big)$

3. **[6 marks] A proof about numbers.** Consider the following statement about natural numbers: "Every odd natural number greater than one is a difference of squares." (Recall that natural number n is *odd* when $n = 2k+1$ for some $k \in \mathbb{N}$, and n is a *difference of squares* when $n = p^2 - q^2$ for some $p, q \in \mathbb{Z}^+$.)

(a) **[2 marks]** Translate the above statement into predicate logic. Do *not* use the *Odd* or *DifferenceOfSquares* predicates in your answer (instead, use the definitions provided above).

Solution

$\forall n \in \mathbb{N},\ n > 1 \land \big(\exists k \in \mathbb{N},\ n = 2k+1\big) \Rightarrow \big(\exists p, q \in \mathbb{Z}^+,\ n = p^2 - q^2\big)$

(b) **[4 marks]** Prove or disprove the above statement. If you choose to disprove the statement, you must start by writing its negation. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

HINT: Remember that $\forall m \in \mathbb{N}, (m+1)^2 = m^2 + 2m + 1$.

Solution

Proof. Let $n \in \mathbb{N}$. Assume $n > 1$, and that $\exists k \in \mathbb{N}, n = 2k+1$. (Since $n > 1$, we know that $k > 0$.)
Let $p = k+1$ and $q = k$.
Then, $p^2 - q^2 = (k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = n$. $\square$

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

4. [**5 marks**] **Divisibility**. Recall the following fact about divisibility.

$$\forall d, n \in \mathbb{N},\ d \mid n \Rightarrow 1 \leq d \leq n \tag{Fact 1}$$

Use this fact to prove the following statement.

$$\forall d, n \in \mathbb{N},\ d \mid n \land d \neq n \Rightarrow d \leq n/2$$

Clearly state where you use the fact in your proof. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

Solution

NOTE: we made a correction on the test: in *both* of the above statements, the domain of the variables d and n should be $\mathbb{Z}^+$, not $\mathbb{N}$.

Solution

Proof. Let $d, n \in \mathbb{Z}^+$. Assume that $d \mid n$, i.e., that there exists a $k \in \mathbb{Z}$ such that $n = kd$, and assume that $d \neq n$.

By Fact 1, since $d \mid n$ we know that $1 \leq d \leq n$. Since we assumed that $d \neq n$, we can conclude that $d < n$. Then since $n = kd$ and $d < n$, we know that $k > 1$. And since k is an integer, we conclude that $k \geq 2$.

So then multiplying both sides by d, we get:

$$\begin{aligned} k &\geq 2 \\ kd &\geq 2d \\ n &\geq 2d && \text{(since } n = kd\text{)} \\ d &\leq \frac{n}{2} \end{aligned}$$

□

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