1

(a)
False
Negation:

$$\forall z \in \mathbb{Z}, \exists k \in \mathbb{Z}, z(z+1)(z+2)(z+3) = 12k$$

Let $z \in \mathbb{Z}$
If z is an odd number,

$$\exists k_1 \in \mathbb{Z}, z = 2k_1 + 1$$

Then,

$$\begin{aligned} z(z+1)(z+2)(z+3) &= (2k_1+1)(2k_1+2)(2k_1+3)(2k_1+4) \\ &= 4(2k_1+1)(k_1+1)(2k_1+3)(k_1+2) \end{aligned}$$

Therefore $z(z+1)(z+2)(z+3)$ is divisible by 4 when z is an odd number
If z is an even number,

$$\exists k_2 \in \mathbb{Z}, z = 2k_2$$

Then,

$$\begin{aligned} z(z+1)(z+2)(z+3) &= (2k_2)(2k_2+1)(2k_2+2)(2k_2+3) \\ &= 4k_2(2k_2+1)(k_2+1)(2k_2+3) \end{aligned}$$

Therefore $z(z+1)(z+2)(z+3)$ is divisible by 4 when z is an even number
Then, $z(z+1)(z+2)(z+3)$ is always divisible by 4

If $3 \mid z$, which means

$$\exists a_1 \in \mathbb{Z}, z = 3a_1$$

Then

$$z(z+1)(z+2)(z+3) = 3a_1(3a_1+1)(3a_1+2)(3a_1+3)$$

Then $z(z+1)(z+2)(z+3)$ is divisible by 3 when $3 \mid z$
If $3 \mid z+1$, which means

$$\exists a_2 \in \mathbb{Z}, z = 3a_2 - 1$$

Then

$$\begin{aligned} z(z+1)(z+3)(z+3) &= (3a_2-1)(3a_2)(3a_2+1)(3a_2+2) \\ &= 3(3a_2-1)(a_2)(3a_2+1)(3a_2+2) \end{aligned}$$

Then $z(z+1)(z+2)(z+3)$ is divisible by 3 when $3 \mid z+1$
If $3 \mid z+2$, which means

$$\exists a_3 \in \mathbb{Z}, z = 3a_3 - 2$$

Then

$$\begin{aligned} z(z+1)(z+2)(z+3) &= (3a_2-2)(3a_2-1)(3a_2)(3a_2+1) \\ & 3(3a_2-2)(3a_2-1)(a_2)(3a_2+1) \end{aligned}$$

Then, $z(z+1)(z+2)(z+3)$ is divisible by 3 when $3 \mid z+2$
Then, $z(z+1)(z+2)(z+3)$ is always divisible by 3

We have shown that both 3 and 4 divide $z(z+1)(z+2)(z+3)$

Since the greatest common divisor of 3 and 4 is 1 and $4 \times 3 = 12$,

$$12 \mid z(z+1)(z+2)(z+3)$$

Then we have proven that for all 4 consecutive integers whose product is divisible by 12 as needed

(b)
True

$$\forall x \in \mathbb{R}, x \geq 6 \implies 4x^2 - 3\lfloor x \rfloor^2 \geq 9$$

Let $x \in \mathbb{R}$
Assume $x \geq 6$

$$x \geq \lfloor x \rfloor \geq 6$$
$$x^2 \geq \lfloor x \rfloor^2 \geq 36$$
$$3x^2 \geq 3\lfloor x \rfloor^2 \geq 108$$

$$4x^2 - 3x^2 = x^2 \geq 36$$
$$4x^2 - 3\lfloor x \rfloor^2 \geq 4x^2 - 3x^2 = x^2 \geq 36$$
$$4x^2 - 3\lfloor x \rfloor^2 \geq 36 \geq 9$$
$$4x^2 - 3\lfloor x \rfloor^2 \geq 9 \blacksquare$$

(c)
Disprove
Negation:

$$\forall g, h, (g : \mathbb{R} \to \mathbb{R}) \wedge (h : \mathbb{R} \to \mathbb{R}) \wedge (\forall x_1 \in \mathbb{R}, g(-x_1) = -g(x_1)) \wedge (\forall x_2 \in \mathbb{R}, h(-x_2) = -h(x_2)) \wedge$$
$$(f(x) = g(x) - h(x)) \wedge (\forall x_3 \in \mathbb{R}, f(-x_3) = f(x_3) \implies \exists k \in \mathbb{R}, \forall x_4 \in \mathbb{R}, f(x_4) = k$$

Let g, h be functions
Assume $(g : \mathbb{R} \to \mathbb{R}) \wedge (h : \mathbb{R} \to \mathbb{R}) \wedge (\forall x_1 \in \mathbb{R}, g(-x_1) = -g(x_1)) \wedge (\forall x_2 \in \mathbb{R}, h(-x_2) = -h(x_2)) \wedge (f(x) = g(x) - h(x)) \wedge (\forall x_3 \in \mathbb{R}, f(-x_3) = f(x_3))$

Let $x \in \mathbb{R}$

$$f(x) = g(x) - h(x)$$
$$f(-x) = g(-x) - h(-x)$$

Since g, h are odd functions,

$$f(-x) = -g(x) + h(x)$$

Since $f(x) = g(x) - h(x)$,

$$f(-x) = -f(x)$$

Then f is an odd function
Since we assumed $\forall x_3 \in \mathbb{R}, f(-x_3) = f(x_3)$,

$$f(-x) = -f(x) = f(x)$$

$$2f(x) = 0$$

$$f(x) = 0$$

We haved concluded that $f(x) = 0$
Let $k = 0$
Let $x_4 \in \mathbb{R}$

$$f(x_4) = k$$

Therefore, f is a constant function
Then for all odd functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) - h(x)$ and f is an even function is a constant function
We have proven the negation of the original predicate

2

Our explanations include the fact that S and T are non-empty sets.
(a)

$$G_0(S,T) \implies G_0(S,T)$$

$$G_1(S,T) \implies G_1(S,T)$$

$$G_2(S,T) \implies G_2(S,T)$$

$$G_3(S,T) \implies G_3(S,T)$$

Each predicate implies itself.

$$G_0(S,T) \implies G_1(S,T)$$

Assume $G_0(S,T)$. Let $x \in S$ and $y \in T$ such that $x > y$. Since we have assumed $G_0(S,T)$, it is valid and the implication is true.

$$G_0(S,T) \implies G_2(S,T)$$

Assume $G_0(S,T)$. Let $x \in S$ such that $\forall y_1 \in S, x > y_1$ and let $y_2 \in S$. Since we have assumed $G_0(S,T)$, it is valid and the implication is true.

$$G_0(S,T) \implies G_3(S,T)$$

Assume $G_0(S,T)$. Let $x \in S$ and $y \in T$. Since we have assumed $G_0(S,T)$, it is valid and the implication is true.

$$G_1(S,T) \implies G_3(S,T)$$

Assume $G_1(S,T)$. Let $x \in S$ and $y \in T$ such that $x > y$. Since we haved assuemd $G_1(S,T)$, it is valid and the implication is true.

$$G_2(S,T) \implies G_3(S,T)$$

Assume $G_2(S,T)$. Let $x \in S$ such that $\forall y_1 \in T, x > y_1$ and $y_2 \in S$ such that $x > y_2$. Since we have assumed $G_2(S,T)$, it is valid and the implication is true.

(b)
Prove $G_0([3,5],[0,2])$
Let $x \in [3,5]$
Let $y \in [0,2]$

$$x \geq 3$$
$$y \leq 2$$
$$x \geq 3 > 2 \geq y$$
$$x > y \blacksquare$$

Disprove $G_0(\mathbb{R},\mathbb{R})$

$$\neg G_0(\mathbb{R},\mathbb{R}) \equiv \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x \leq y$$

Let $x = 0$
Let $y = 1$

$$x \leq y \blacksquare$$

Prove $G_1((0,1],(0,1))$
Let $x \in (0,1]$
Let $y = \dfrac{x}{2}$

$$x > y > 0 \blacksquare$$

Prove $G_2(\mathbb{Z},(-\infty,0))$
Let $x = 1$
Let $y \in (-\infty,0)$

$$x > 0 > y \blacksquare$$

Prove $G_2(\{10\},\varnothing)$
Let $x = 10$
Let $y \in \varnothing$
Then it is vacuously true
Prove $G_3([0,1] \cap \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$

Let $x = 1$
Let $y = -\sqrt{2}$

$$x > y \blacksquare$$

3

(a)
Indirect proof(Proof by contrapositive)
WTS

$$\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*, f(x) \text{ has no constant term } \implies 2 \mid b + 1$$

Using contrapostive, WTS

$$\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*, 2 \nmid b + 1 \implies f(x) \text{ has a constant term}$$

Let $b, c \in \mathbb{Z}^+$
Let $x \in \mathbb{R}^*$
Assume $2 \nmid b + 1$
Then $2 \mid b$, according even and odd number properties.
According to binomial theorem we can represent $f(x)$ as

$$\sum_{k=0}^{b} \binom{b}{k} x^k \left(\frac{c}{x}\right)^{b-k}$$

where $k \in \mathbb{N}$
Since $2 \mid b$, it can also be represented as

$$f(x) = \sum_{k=0}^{\frac{b}{2}-1} \binom{b}{k} x^k \left(\frac{c}{x}\right)^{b-k} + \binom{b}{\frac{b}{2}} x^{\frac{b}{2}} \left(\frac{c}{x}\right)^{b-\frac{b}{2}} + \sum_{k=\frac{b}{2}+1}^{b} \binom{b}{k} x^k \left(\frac{c}{x}\right)^{b-k}$$

The term,

$$\binom{b}{\frac{b}{2}} x^{\frac{b}{2}} \left(\frac{c}{x}\right)^{b-\frac{b}{2}} = \binom{b}{\frac{b}{2}} x^{\frac{b}{2}} \cdot \frac{c^{\frac{b}{2}}}{x^{\frac{b}{2}}} = \binom{b}{\frac{b}{2}} c^{\frac{b}{2}}$$

is not depdendent on the value of x
Therefore it has a constant form

(b)
WTS

$$\forall y \in \mathbb{Z}^+, \exists n, k \in \mathbb{N}, f(n, k) = y$$

Let $y \in \mathbb{Z}^+$
Let $n = y$

Let $k = y - 1$

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{y!}{(y-1)!(y-y+1)!} \\ &= \frac{y!}{(y-1)!} \\ &= \frac{\prod_{i=1}^{y} i}{\prod_{i=1}^{y-1} i} \\ &= y \blacksquare\end{aligned}$$

(c)
Let $n, k \in \mathbb{Z}^+$
Assume $k < n$

$$\begin{aligned}\binom{n}{k} \div \binom{n-1}{k} &= \frac{n!}{k!(n-k)!} \div \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{n!k!(n-1-k)!}{k!(n-k)!(n-1)!} \\ &= \frac{n!(n-1-k)!}{(n-k)!(n-1)!} \\ &= \frac{n(n-1-k)!}{(n-k)!} \\ &= \frac{n}{n-k}\end{aligned}$$

Since $n, k \in \mathbb{Z}^+$ and $k < n$,

$$\begin{gathered} n > n-k \\ \frac{n}{n-k} > 1 \\ \binom{n}{k} \div \binom{n-1}{k} > 1 \blacksquare \end{gathered}$$