Last time...

Proof of

$P(n)$

- $\forall n \in \mathbb{Z}^+$, $\boxed{\forall G=(V,E),\ |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected}\right)}$

- induction on n

- IH: Let $n \in \mathbb{Z}^+$, and assume $P(n)$:

  $\forall G=(V,E),\ |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected}\right)$

- I.S.: WTS $P(n+1)$.

  Let $G_1=(V_1,E_1)$ and assume $|V_1|=n+1$.

  Also assume $|E_1| \geq \frac{n(n-1)}{2}+1$.

  WTS G_1 is connected.

ROUGH WORK

KNOW

$n \in \mathbb{Z}^+$

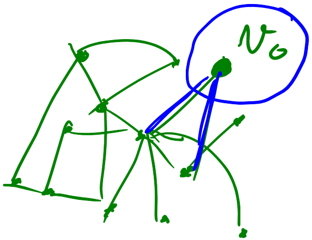
$P(n)$

$G_1 = (V_1, E_1)$

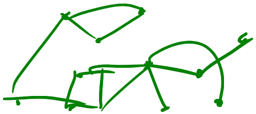
$|V_1| = n+1$

$|E_1| \geq \frac{n(n-1)}{2} + 1$

G_1:



G':



WANT

G_1 is connected:

$(\forall u,v \in V_1,$
$\exists$ path between u and v
in $G_1)$

- pick $v_0 \in V_1$
- let $G' = (V', E')$ with
  $V' = V_1 - \{v_0\}$
  $E' = E_1 - \{(u, v_0) \mid u \in V_1\}$
- $|V'| = |V_1| - 1 = n+1-1 = n$
- $|E'|$?

- how many edges could be connected to v_0?

let $E_0 = \{(u, v_0) \mid u \in V_1 \text{ and } (u, v_0) \in E_1\}$

- $|E_0| \geq 1$: max number of edges on V' is $\frac{n(n-1)}{2}$ (because $|V'| = n$)

but $|E_1| \geq \frac{n(n-1)}{2} + 1 \Rightarrow$ at least one edge in E_1 is adjacent to v_0

- $|E_0| \leq n$ because there are $|V_1| - 1$ many vertices other than v_0

So $|E'| = |E_1| - |E_0|$

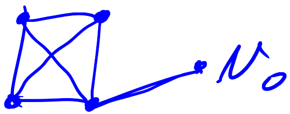
$$\geq |E_1| - n \geq \frac{n(n-1)}{2} + 1 - n = \frac{(n-1)(n-2)}{2}$$

- Uh, oh! not big enough to apply $P(n)$

- Note: we fall short only if v_0 is adjacent to <u>every</u> other vertex in G_1
- Introduce cases: either some $v_2 \in V_1$ is <u>not</u> adjacent to all others in V_1, or not
  - if v_2 exists, removing it from G_1 removes <u>fewer</u> than n edges from E_1, so "$G_1 - v_2$" contains more than $\frac{(n-1)(n-2)}{2}$ edges and is connected by I.H.
    so <u>G_1 is also connected.</u>
  - if no such v_2 exists, then every vertex in V_1 is adjacent to every other vertex
    — <u>G_1 is connected.</u>

$n=4$

$|E| \geq \frac{4(3)}{2}+1 = 7$



$\forall n \in \mathbb{Z}^+, \; P(n) \Rightarrow P(n+1)$

KNOW

$n \in \mathbb{Z}^+$

$P(n)$

$\left(\forall G=(V,E), \dots\right)$

$G_1=(V_1,E_1)$

$|V_1| = n+1$

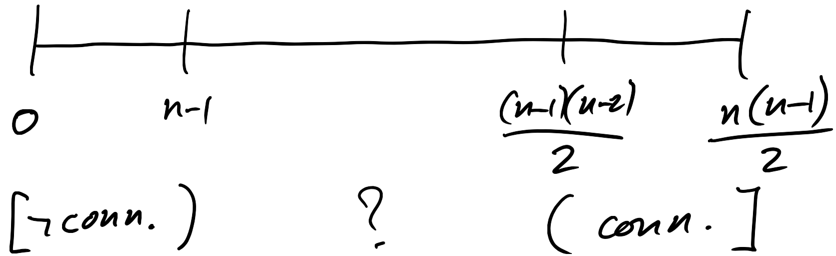
...

WANT

$P(n+1)$

$\left(\forall G=(V,E), \dots\right)$

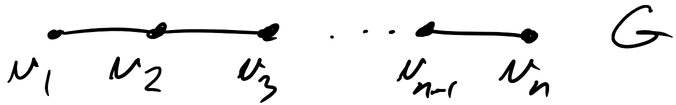
From the conditions so far:



$|E|$ related to $n = |V|$

Is there a connected graph with n-1 edges? Yes

Is there an unconnected graph with $\frac{(n-1)(n-2)}{2}$ edges? Yes



Def: A tree is any graph that is

connected and acyclic (does not contain a cycle).

Cycle: path from u to itself (with at least 3 edges)