Sets, Functions, and Predicates

CSC165 Week 1 - Part 2

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From last class:

$x \in S$	"x is an element of A"
$A \subset S$	"A is a subset of S"
$A \subseteq S$	"A is a subset of S or A is equal to S"

$A \subseteq A$	is always true
$\varnothing \subseteq A$	always true (but not always an element)

$\varnothing \overset{?}{\in} A$

$\varnothing$ = the empty set = $\{\ \}$

Let $A = \{1, x, \varnothing\}$

$$P(A) = \{A, \varnothing, \ldots\}$$

Let $B = \varnothing = \{\ \}$

Let $C = \{\ \varnothing\ \} = \{\ \{\ \}\ \}$

$P(\varnothing) = P(\{\ \}) = \{\varnothing\}$

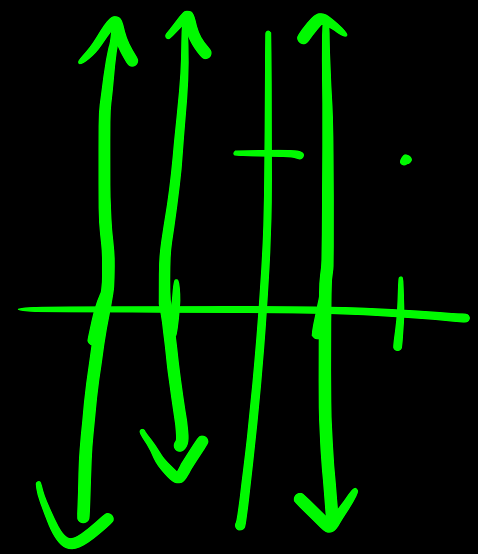
$\{\varnothing\}$

$P(\{\varnothing\}) = P(\{\{\}\}) = \{\ \{\varnothing\},\ \varnothing\}$

$$\boxed{A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}}$$

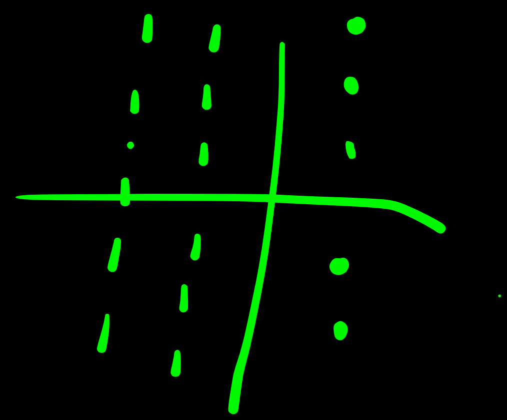
Example of Cartesian Product:

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$



- $\mathbb{Z} \times \mathbb{R} = \{ (n,y) \mid n \in \mathbb{Z} \text{ and } y \in \mathbb{R} \}$

- $\mathbb{Z} \times \mathbb{Z} = \{ (n,m) \mid n \in \mathbb{Z} \text{ and } m \in \mathbb{Z} \}$



- $R \times \varnothing = \varnothing$

$(r, \cancel{x})$

Functions

A function is a mapping from its Domain to its Codomain.

$f: A \longrightarrow B$ means $\boxed{f(a) = b}$ where $a \in A$ and $b \in B$

$a \mapsto b$

Example: $f(x) = \dfrac{1}{|x|}$ $\boxed{f : \mathbb{R} \setminus \{0\} \longrightarrow (0,\infty)}$ $[0,\infty)$

For $k \in \mathbb{Z}^+$, we say f is a k-ary function if

$$f: A_1 \times A_2 \times \ldots \times A_k \longrightarrow B$$

$$f(a_1,a_2,\ldots,a_k) = b$$

3-2

Summation Notation

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \sum_{i=1}^{10} i$

$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{i=1}^{100} i$

$1 + 4 + 9 + \ldots + 81 + 100 = \sum_{i=1}^{10} i^2$

$1 \quad 2 \quad 3$

$3 + 9 + 19 + \ldots + 163 + 201 = \sum_{i=1}^{10} \left(2i^2 + 1\right)$

Summation Notation

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \sum_{i=1}^{10} i$$

$$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{i=1}^{10} i$$

$$4 + 9 + \ldots + 81 + 100 + 121 = \sum_{i=2}^{11} i^2$$

$$3 + 9 + 19 + \ldots + 163 + 201 = \sum_{i=1}^{10} 2i^2 + 1$$

$$\sum_{i=1}^{3} 3\sin(2i) = 3\sin(2) + 3\sin(4) + 3\sin(6)$$

$i=1$ $i=2$ $i=3$

$$\sum_{i=1}^{5} \left(3i + i^2\right) = (4) + (6+4) + (9+9) + (12+16) + (15+25)$$

$i=1$ 2 3 4

$$\sum_{i=8}^{2} \left((i+3)^{1/2} - 6\right) = 0$$

Product Notation

$$\prod_{k=-1}^{2} (3k + 2) = (-3+2)(0+2)(3+2)(6+2)$$

$$\prod_{i=2}^{5} (x - i)^2 = (x-2)^2(x-3)^2(x-4)^2(x-5)^2$$

$$\prod_{i=10}^{9} (i - 5) = 1$$

Propositional Logic

A proposition is a statement that can be true or false.

Examples:

$S = 3 < 4$ ← T

$T = 3 \geq 4$ ← F

U = “It is raining right now.” ←

V = “This loop will terminate in a finite number of iterations.” ←

NOT (negation)

Ex: Let $\boxed{q = \neg (x < 3)}$

(pred next)

"not p"

p	$\neg p$
TRUE	F
FALSE	T

AND (conjunction)

Ex: $(x < 2) \wedge (x > 3)$ ✗

Ex: It is January and it is sunny.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

OR inclusive (disjuction)

Ex: $(x < 2) \vee (x > 3)$

Ex: It is January or it is sunny.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

✓

XOR

Exclusive OR is used here:

"I want a red car or a blue car"

We understand here that the person is not looking for two cars: one of each.

T T F

Implies (implication) (conditional) $\boxed{\sim p \vee q}$

$p \Longrightarrow q$ "if p is true, then q must be true"

p is the hypothesis

q is the conclusion

Ex: If it rains, then I bring an umbrella

Ex: $(n \in \mathbb{N}) \Longrightarrow (n \in \mathbb{Z})$

$\not\Longleftarrow$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

iff (if and only if) (biconditional) $(p \implies q) \wedge (q \implies p)$

p is true if and only if q is true

Ex: $(n \in \mathbb{N}) \iff (n \in \mathbb{Z})$ X

$\not\Leftarrow$

Ex: It is Monday if and only if it is the first CSC165 lecture of the week

Ex: A number is a natural number if and only if it is a positive integer.

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

All definitions are biconditional statements.

Example of Evaluating a Truth Table:

$$(p \vee q) \implies ((r \implies (p \wedge q)))$$

p	q	r	$(p \vee q)$	$(p \wedge q)$	$r \implies (p \wedge q)$	$(p \vee q) \Rightarrow (r \Rightarrow (p \wedge q))$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	F	F	F
T	F	F	T	F	T	T
F	T	T	T	F	F	F
F	T	F	T	F	T	T
F	F	T	F	F	F	T
F	F	F	F	F	T	T

Two statements are said to be logically equivalent if they are true for the same input and false for the same input:

$p \iff q$ means that p and q are logically equivalent to each other.

Example: $(x = 2) \iff ((x \in \mathbb{Z}) \wedge (x \geq 0))$

$(x = 2)$	$((x \in \mathbb{Z}) \wedge (x \geq 0))$	$(x = 2) \iff ((x \in \mathbb{Z}) \wedge (x \geq 0))$
T	T	T
~~T~~	~~F~~	impossible (F)
F	T	F
F	F	T

Converse

The original implication ~~Inverse~~

$p \implies q$? $q \implies p$

?

?

$\neg q \implies \neg p$? $\neg p \implies \neg q$

Contrapositive

Inverse

The original statement is “All cats are animals”.

This is logically equivalent to the implication:

“If it is a cat, then it is an animal” ← Original Statement

Let A = “it is a cat” and B = “it is an animal”

$A \Rightarrow B$ = the original statement

(contrapositive) $\neg B \Rightarrow \neg A$ = "If it is not an animal, then it is not a cat."

(converse) $B \Rightarrow A$ = "If it is an animal, then it is a cat."

(inverse) $\neg A \Rightarrow \neg B$ = "If it is not a cat, then it is not an animal."

The original implication

$$\boxed{p \implies q}$$

$$\Updownarrow$$

$$\boxed{\neg q \implies \neg p}$$

Contrapositive

Converse

~~Inverse~~

$$\boxed{q \implies p}$$

$$\Updownarrow$$

$$\boxed{\neg p \implies \neg q}$$

Inverse