

# Binary Representations of Numbers – a Proof

CSC165 Week 6 - Part 1

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# Proof by Induction – General Structure

We want to prove that the statement P(n) is true for all natural numbers n.

In other words, we want to prove  $\forall n \in \mathbb{N}, P(n)$ .

$n \geq a$

Step 1: Base case

Prove P(0) (or any other case that should be verified first.)

$P(a)$

Step 2: Induction hypothesis

Since we want to show that  $P(k) \implies P(k+1)$ ,  
We assume  $P(k)$  is true. (let  $n = k$ )



$A \Rightarrow B$

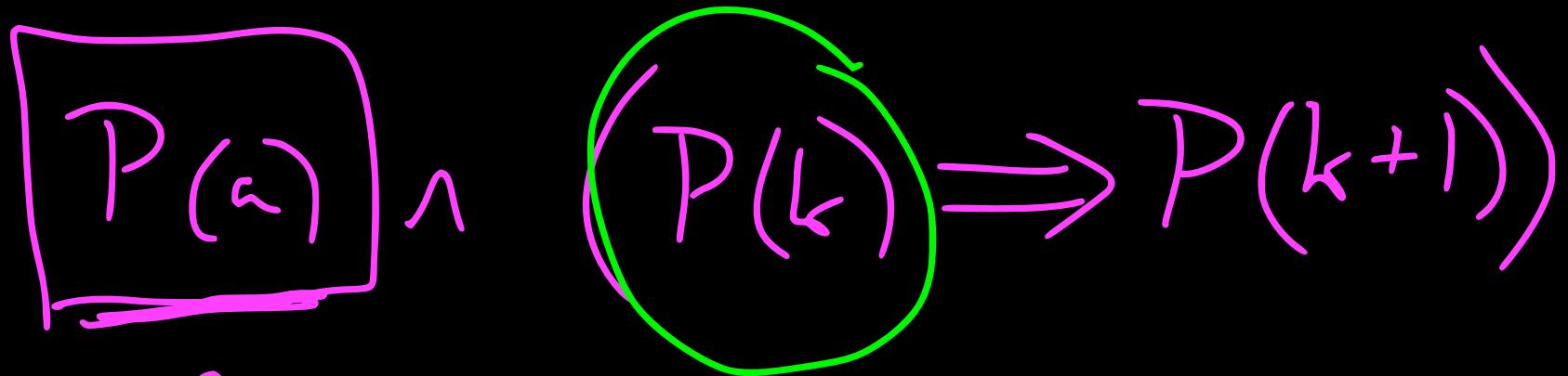
Step 3: Let  $n = k+1$  and Prove  $P(k+1)$

This is where we use the induction hypothesis  $P(k)$  to prove  $P(k+1)$  must also be true.

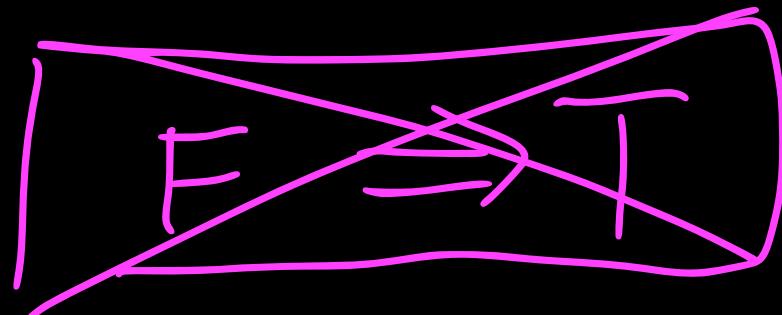
# Proof by Induction – Summary

We want to show that  $P(n)$  is true for all  $x \in \mathbb{N}$  and  $x \geq a$ .

Strategy: Prove  $P(a)$  is true and then that  $P(k) \Rightarrow P(k+1)$ .



$A$	$B$	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T



## Goal for this week

We want to prove:  $\forall n \in \mathbb{N}, 0 \leq x \leq 2^n - 1 \iff B(n, x)$

Strategy: Prove the  $\implies$  direction using induction on  $n$

Then prove the  $\impliedby$  direction using \_\_\_\_\_

# Decimal (Base 10) Numbers

Possible digits: {0, 1, 2, 3, 4, 5, 6, 7, 8, and 9}

Where do we get the digits from?

$$00300402 = 3 \times 10^5 + 0 \times 10^4 \\ + 0 \times 10^3 + 4 \times 10^2 \\ + 0 \times 10^1 + 2 \times 10^0$$

$10^7$	$10^6$	$10^5$	$10^4$	$10^3$	$10^2$	$10^1$	$10^0$
0	0	3	0	0	4	0	2

{0,1}

## Binary (Base 2) Numbers

$$\rightarrow \sum_{i=0}^s b_i \times 2^i$$

Example:  $(46)_{10} = \underline{(101110)_2}$

$$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2$$

$$\begin{array}{r} 46 \\ - 32 \\ \hline 14 \\ - 8 \\ \hline 6 \\ - 4 \\ \hline 2 \\ - 2 \\ \hline 0 \end{array}$$

$128$ $2^7$	$64$ $2^6$	$32$ $2^5$	$16$ $2^4$	$8$ $2^3$	$4$ $2^2$	$2$ $2^1$	$1$ $2^0$
0	0	1	0	1	1	1	0

Example:  $(46)_{10} = (101110)_2$

$$= \sum_{i=0}^5 b_i 2^i$$

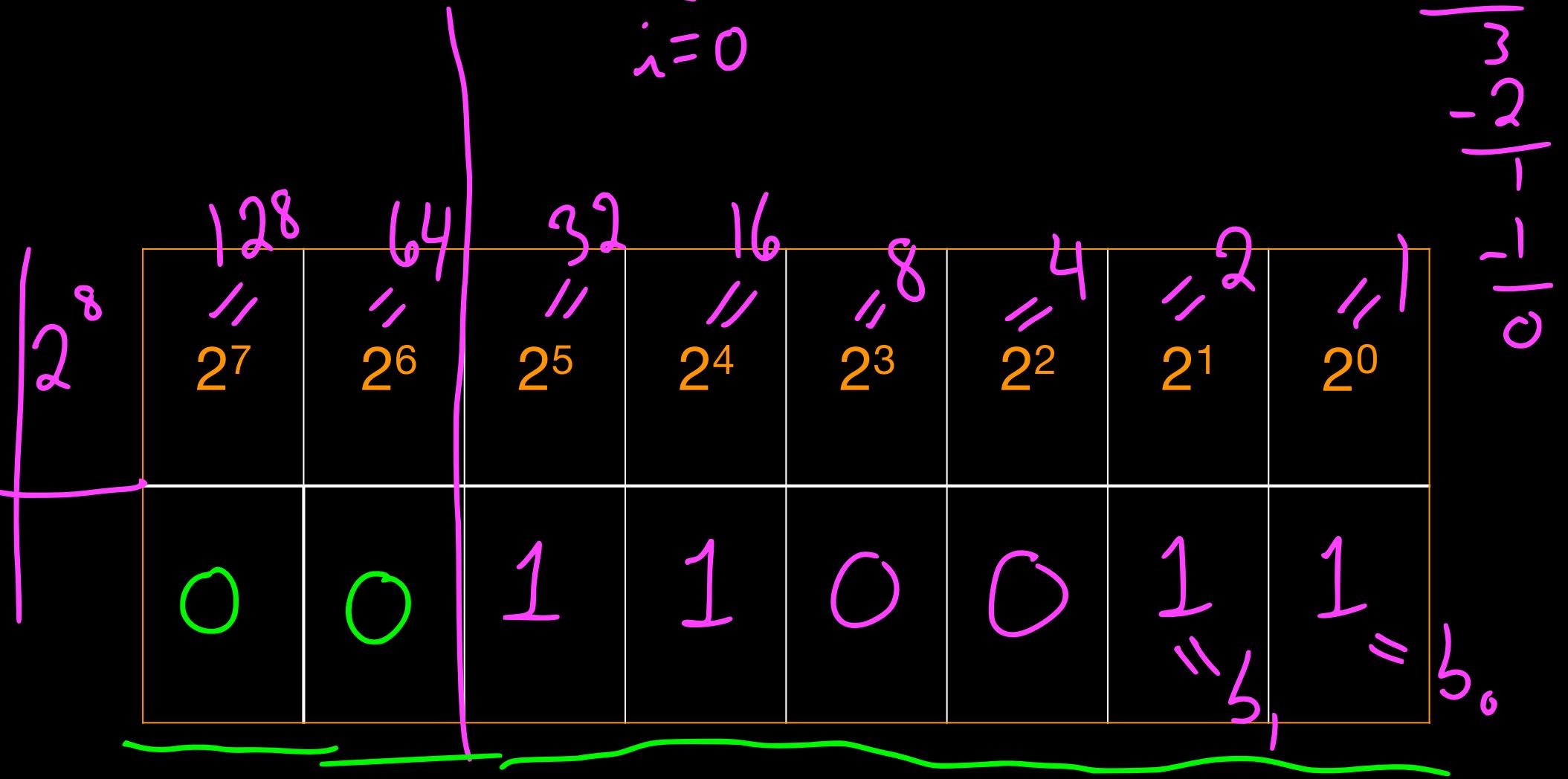
where  $b_0 = 0$ ,  $b_1 = 1$ ,  $b_2 = 1$ ,  $b_3 = 1$ ,  $b_4 = 0$ ,  $b_5 = 1$

$2^7$	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
0	0	1	0	1	1	1	0

$$(51)_{10} = (110011)_2$$

$$\begin{array}{r}
 51 \\
 -32 \\
 \hline
 19 \\
 -16 \\
 \hline
 3 \\
 -2 \\
 \hline
 1
 \end{array}$$

$$= \sum_{i=0}^5 b_i \cdot 2^i$$



10  
" {0, 1, ..., 9, A, B, ..., F}

$$(4AC1)_{16} = 4 \times 16^3 + 10 \times 16^2 + 12 \times 16^1 + 1 \times 16^0$$

$2^7$	$2^6$	$2^5$	$2^4$	$16^3$	$16^2$	$16^1$	$16^0$	=
				4	A	C	1	0

A binary representation of  $x \in \mathbb{N}$  can be written like this:

$\exists k \in \mathbb{Z}_+, \underbrace{\{b_0, b_1, \dots, b_{k-1}\}}_{\in \{0,1\}}$  such that

$$x = \sum_{i=0}^{k-1} b_i 2^i = (b_{k-1})2^{k-1} + (b_{k-2})2^{k-2} + \dots + (b_1)2^1 + (b_0)2^0$$

We can then write  $x = \underbrace{(b_{k-1}b_{k-2}\dots b_1b_0)_2}$

$$5 = (1) \times 2^2 + (0) \times 2^1 + (1) \times 2^0$$

For example, 5 =  $(101)_2$  or  $(5)_{10}$  =  $(101)_2$

## Definition of Predicate $B(n,x)$



$\forall n \in \mathbb{N}, \forall x \in \mathbb{N}, B(n,x)$  is true if and only if

$$\exists b_0, b_1, \dots, b_{n-1} \in \{0,1\} \text{ such that } x = (b_{n-1}b_{n-2}\dots b_1b_0)_2$$
$$= \sum_{i=0}^{n-1} b_i 2^i$$

In other words,  $B(n,x)$  is true when  $x$  can be written in binary using exactly  $n$  bits.  $\in \{0,1\}$

$$\begin{array}{r}
 5 \\
 -4 \\
 \hline
 1 \\
 -1 \\
 \hline
 0
 \end{array}$$

True or False?

$$\boxed{5_{10} = 101_2} \\
 = 0101_2$$

$$\underline{\underline{B(3,5)}} = \text{True}$$

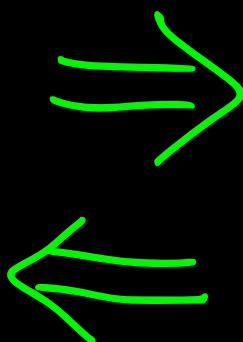
$$\begin{array}{r}
 \xrightarrow{4 \text{ digits}} B(4,5) = \text{True}
 \end{array}$$

$$B(2,5) = \text{False}$$

B(n,x) is true when x can be written in binary using exactly n bits.

We want to prove that:

$$\forall n \in \mathbb{N}, 0 \leq x \leq 2^n - 1 \iff \underline{B(n,x)}$$



We want to prove that:

(using induction on  $n$ )

$\forall x \in \mathbb{N}$

$\forall n \in \mathbb{N}, \boxed{0 \leq x \leq 2^n - 1} \Rightarrow \boxed{B(n, x)}$  ( $A \Rightarrow B$ )

Base Case: Let  $n = 0$ ? (zero digits)

$$2^0 - 1 = 0 \Rightarrow 0 \leq x \leq 0$$

Let  $n=1$ . Assume  $0 \leq x \leq 2^1 - 1 = 1$

Case1 :  $x = 0 = 0_2 \quad \left. \right\} \Rightarrow B(1, x)$  is true  
Case2 :  $x = 1 = 1_2$

We want to prove that:

$$\forall n \in \mathbb{N}, 0 \leq x \leq 2^n - 1 \Rightarrow B(n, x)$$

Induction Step: Let  $n = k \in \mathbb{N}$

Assume  $0 \leq x \leq 2^k - 1 \Rightarrow B(k, x)$

Case of  $n = k+1$ :

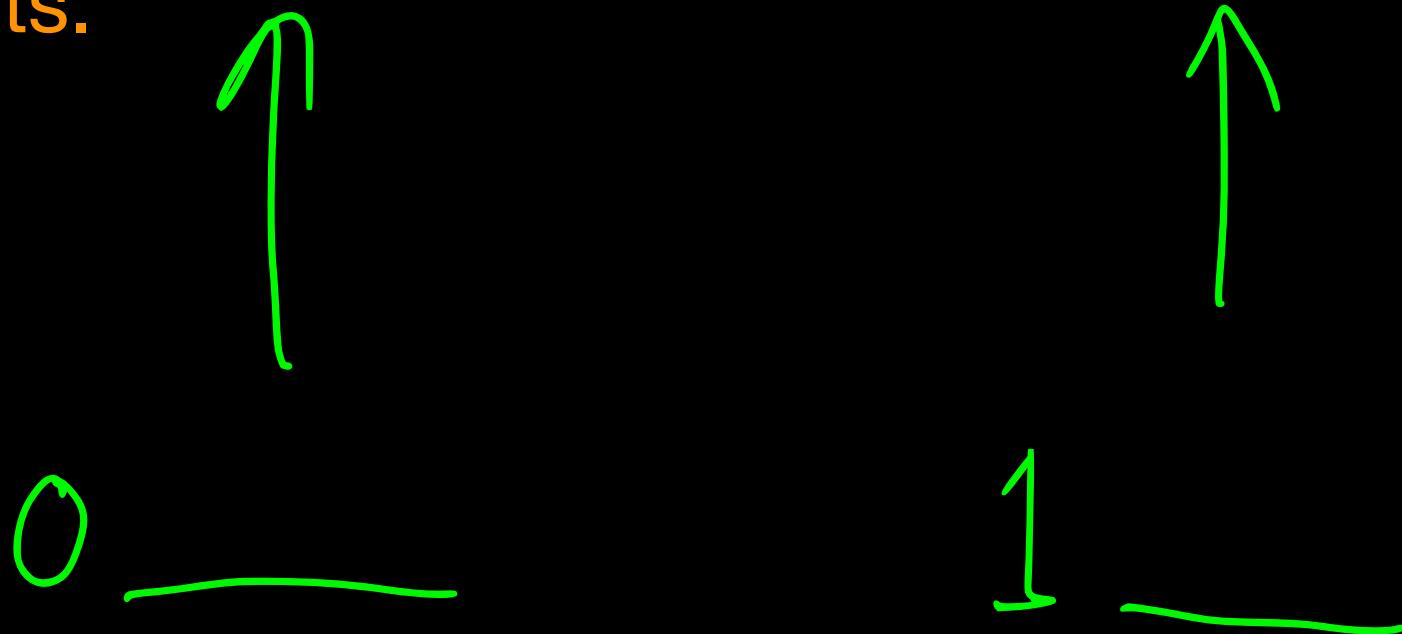
Want to show that  $\forall x \in \mathbb{N}, 0 \leq x \leq 2^{k+1} - 1 \Rightarrow B(k+1, x)$

Case of  $n = k+1$ :

Want to show that  $\forall x \in \mathbb{N}$ ,  $0 \leq x \leq 2^{k+1}-1 \Rightarrow B(k+1, x)$

Either  $0 \leq x \leq 2^k-1$  or  $2^k \leq x \leq 2^{k+1}-1$

$B(k+1, x)$  is true when  $x$  can be written in binary using exactly  $k+1$  bits.



Case 1: Assume  $x \leq 2^{k-1}$

By Induction Hypothesis, we know that  $B(k, x)$  is true.  
Therefore:

$\exists b_0, b_1, \dots, b_{k-1} \in \{0,1\}$  such that  $x = (b_{k-1}b_{k-2}\dots b_1b_0)_2$

$$\begin{aligned} &= \sum_{i=0}^{k-1} b_i 2^i \\ &\quad \because 0 \end{aligned}$$

We can append a 0 to the left side of the number without changing its value. So  $x = (0b_{k-1}b_{k-2}\dots b_1b_0)_2$

$$= (0) 2^k + \sum b_i 2^i$$

Thus  $B(k+1, x)$  is true.

Case 2: Assume  $x > \underline{2^k - 1}$

Then,  $2^k \leq x \leq 2^{k+1} - 1$

Subtract  $2^k$  from all parts to get:

$$(2^k - 2^k) \leq x - 2^k \leq (2^{k+1} - 1 - 2^k)$$

$$0 \leq x - 2^k \leq 2^k(2^1 - 1) - 1 \quad \text{or} \quad 0 \leq x - 2^k \leq 2^k - 1$$

By the Induction hypothesis, we know  $B(k, x-2^k)$

$\exists b_0, b_1, \dots, b_{k-1} \in \{0, 1\}$  such that

$$x - 2^k = (\underbrace{b_{k-1}b_{k-2}\dots b_1b_0}_i)_2 = \sum_{i=0}^{k-1} b_i 2^i$$

Therefore,

$\exists b_0, b_1, \dots, b_{k-1} \in \{0,1\}$  such that

$$x = (1b_{k-1}b_{k-2}\dots b_1b_0)_2 = (1)2^k + \sum_{i=0}^{k-1} b_i 2^i = \sum_{i=0}^k b_i 2^i$$

$B(k+1, x)$  is true.

Let  $b_k = 1$

QED



Next time:  $\forall x \in \mathbb{N}, 0 \leq x \leq 2^{k+1}-1 \Leftarrow B(k+1, x)$