Due before 17:00 (EST) on ***Wednesday 17 March 2021*** — (a 24-hour extention)

Note: **solutions may be incomplete, and meant to be used as guidelines only**. We encourage you to ask follow-up questions on the course forum or during office hours.

1. [14 marks] Proofs by induction.

(a) [**4 marks**] Prove each of the following identities by induction.

i. $\displaystyle\sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2$

ii. $\displaystyle\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1$

Solution

i. *Proof.* **Base case**: let $n = 1$. Clearly,

$$(1)^3 = \Big(\frac{1 \cdot (1+1)}{2}\Big)^2,$$

so the base case holds.
Inductive step: Suppose that

$$\sum_{k=1}^{n} k^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

We want to show that

$$\sum_{k=1}^{n+1} k^3 = \Big(\frac{(n+1)(n+2)}{2}\Big)^2.$$

We begin by observing that

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3.$$

By the induction hypothesis,

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + \frac{(4n+4)\cdot(n+1)^2}{4} \\ &= \frac{(n^2+4n+4)(n+1)^2}{4} \\ &= \frac{(n+2)^2(n+1)^2}{4} \\ &= \left(\frac{(n+2)(n+1)}{2}\right)^2,\end{aligned}$$

and we are done.

□

ii. *Proof.* **Base case**: let $n = 1$. Clearly,

$$1! \cdot 1 = (1+1)! - 1,$$

so the base case holds.

Inductive step: Suppose that

$$\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1.$$

We want to show that

$$\sum_{k=1}^{n+1} k! \cdot k = (n+2)! - 1.$$

We first observe that

$$\sum_{k=1}^{n+1} k! \cdot k = \sum_{k=1}^{n} k! \cdot k + (n+1)! \cdot (n+1).$$

By the induction hypothesis,

$$\begin{aligned}\sum_{k=1}^{n+1} k! \cdot k &= (n+1)! - 1 + (n+1)! \cdot (n+1) \\ &= (n+1)! \cdot (n+2) - 1 \\ &= (n+2)! - 1,\end{aligned}$$

and we are done.

□

(b) [**4 marks**] Prove that any integer $n \geq 10$ is expressible as a sum of 3's and/or 5's. For example, 21 is expressible as a sum of 3's and/or 5's because $21 = 5 + 5 + 5 + 3 + 3$.

Hint: Let $P(n)$ denote the predicate that n is expressible as a sum of 3's and/or 5's. Start by showing that $P(10)$, $P(11)$, and $P(12)$ are all true (so your proof will have more than one base case). Then, show that $P(k) \Rightarrow P(k+3)$. It is also possible to do this with a different argument and a "standard" inductive step ($P(k) \Rightarrow P(k+1)$).

Solution

Proof. First, we prove that $P(n)$ is true for all $n = 10, 10+3, 10+6, \ldots$.
Base case: Since $10 = 5+5$, $P(n)$ is true for $n = 10$.
Inductive step: Suppose that $P(k)$ is true, and n is expressible as a sum of 3's and/or 5's. Clearly then, $n+3$ is expressible as a sum of 3's and/or 5's, so $P(k+3)$ is true.
By induction, $P(n)$ is true for all $n = 10, 10+3, 10+6, \ldots$.

Next, we prove that $P(n)$ is true for all $n = 11, 11+3, 11+6, \ldots$.
Base case: Since $11 = 5+3+3$, $P(n)$ is true for $n = 11$.
Inductive step: Same as before.
By induction, $P(n)$ is true for all $n = 11, 11+3, 11+6, \ldots$.

Finally, we prove that $P(n)$ is true for all $n = 12, 12+3, 12+6, \ldots$.
Base case: Since $12 = 3+3+3+3$, $P(n)$ is true for $n = 12$.
Inductive step: Same as before.
By induction, $P(n)$ is true for all $n = 12, 12+3, 12+6, \ldots$.

Combining all three proofs, we find that $P(n)$ is true for all natural numbers $n \geq 10$.

□

Alternative proof

Proof. **Base case**: $P(10)$ is true because $10 = 5+5$.
Inductive step: Let $k \in \mathbb{N}$ and assume $k \geq 10$. Also assume $P(k)$, i.e., $\exists a, b \in \mathbb{N}, k = 5a + 3b$. Then, either $a = 0$ or $a > 0$.

- Assume $a = 0$. Then,

$$\begin{aligned} k+1 &= 5a + 3b + 1 \\ &= 3b + 1 && \text{(since } a = 0\text{)} \\ &= 3b - 9 + 9 + 1 \\ &= 3(b-3) + 5(2) && \text{(since } k \geq 10 \Leftrightarrow 3b \geq 10 \Leftrightarrow b \geq 3\text{)} \end{aligned}$$

  Thus, $k+1$ can be expressed as a sum of 3's and/or 5's, i.e., $P(k+1)$ holds.
- Assume $a > 0$. Then,

$$\begin{aligned} k+1 &= 5a + 3b + 1 \\ &= 5a - 5 + 5 + 1 + 3b \\ &= 5(a-1) + 3(b+2) \end{aligned}$$

  Thus, $k+1$ can be expressed as a sum of 3's and/or 5's, i.e., $P(k+1)$ holds.

□

(c) **[6 marks]** Sometimes it's possible to use induction "backwards", proving things from k to $k-1$

instead of vice versa! Consider the statement

$$P(n) : \forall x_1, x_2, \ldots, x_n \in \mathbb{R}^{\geq 0}, x_1 \cdots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n}\right)^n$$

where $n \in \mathbb{N}$ and $n \geq 1$.

i. Prove that $P(2)$ is true.
Hint: Think about the quantity $(x_1 + x_2)^2 - (x_1 - x_2)^2$.

ii. Prove that, for each $n \geq 2$, if $P(2)$ and $P(n)$ are true, then $P(2n)$ is also true.
Use this to prove that $P(2^m)$ is true for all $m \in \mathbb{N}$, where $m \geq 1$.

iii. Prove that $P(k) \Rightarrow P(k-1)$ for all $k \geq 2$. (*Hint*: Set $x_k = (x_1 + \cdots + x_{k-1})/(k-1)$.)

iv. Why is $P(n)$ is true for all $n \in \mathbb{N}$, where $n \geq 1$? An informal argument here is fine (you do NOT have to provide a rigorous proof).

Solution

i. *Proof.* Suppose that $x_1, x_2 \geq 0$. Since $(x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2$, it follows that $4x_1x_2 = (x_1 + x_2)^2 - (x_1 - x_2)^2 \leq (x_1 + x_2)^2$. Thus,

$$x_1x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2,$$

so $P(2)$ is true.

□

ii. *Proof.* Suppose that $P(2)$ and $P(n)$ are true. Suppose further that $x_1, x_2, \ldots, x_{2n} \geq 0$. Then, since $P(n)$ is true, we have that

$$x_1 \cdots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n}\right)^n,$$

and

$$x_{n+1} \cdots x_{2n} \leq \left(\frac{x_{n+1} + \cdots + x_{2n}}{n}\right)^n.$$

Thus,

$$x_1 \cdots x_{2n} \leq \left(\frac{(x_1 + \cdots + x_n)(x_{n+1} + \cdots + x_{2n})}{n^2}\right)^n.$$

Since $P(2)$ is true, then

$$(x_1 + \cdots + x_n)(x_{n+1} + \cdots + x_{2n}) \leq \left(\frac{x_1 + \cdots + x_{2n}}{2}\right)^2.$$

Therefore,

$$x_1 \cdots x_{2n} \leq \left(\frac{x_1 + \cdots + x_{2n}}{2n}\right)^{2n},$$

so $P(2n)$ is true. Since $P(2)$ is true (the **base case**), and, if $P(2^k)$ is true, then $P(2^{k+1})$ is true (the **inductive step**), it follows that $P(2^n)$ is true for all $n \in \mathbb{N}$. Therefore, $P(n)$ is true for infinitely many n.

□

iii. *Proof.* Suppose that $P(k)$ is true. Let $x_1, x_2, \ldots, x_{k-1} \geq 0$, and set $x_k = (x_1 + \cdots + x_{k-1})/(k-1)$. Then

$$x_1 \cdots x_k \leq \left(\frac{x_1 + \cdots + x_k}{k}\right)^k,$$

which means that

$$x_1 \cdots x_k \leq \left(\frac{x_1 + \cdots + x_{k-1} + \frac{x_1 + \cdots + x_{k-1}}{k-1}}{k}\right)^k,$$

so

$$x_1 \cdots x_k \leq \left(\frac{x_1 + \cdots + x_{k-1}}{k-1}\right)^k.$$

Thus, dividing both sides by x_k,

$$x_1 \cdots x_{k-1} \leq \left(\frac{x_1 + \cdots + x_{k-1}}{k-1}\right)^{k-1},$$

so $P(k-1)$ is true.

$\square$

iv. *Proof.* Suppose that $P(m)$ is true for some $m \in \mathbb{N}$ (the **base case**). Then, since $P(k) \Rightarrow P(k-1)$ (the **inductive step**), by induction $P(n)$ is true for all $1 \leq n \leq m$. By part ii, $P(m)$ is true for infinitely many $m \in \mathbb{N}$, so there is no largest $m \in \mathbb{N}$ for which $P(m)$ is true. Thus, $P(n)$ is true for all $n \in \mathbb{N}$, and we are done.

$\square$

2. **[7 marks] Number representations.** On Worksheet #10, we looked at representing rational numbers in binary notation. Here, we'll also consider representations in two other bases that appear often in computer science. Recall that a **binary representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_2,$$

where $a_i, b_i \in \{0,1\}$ and

$$x = \sum_{i=0}^{k-1} a_i 2^i + \sum_{i=1}^{m} b_i 2^{-i}.$$

Likewise, an **octal (base 8) representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_8,$$

where $a_i, b_i \in \{0,1,2,3,4,5,6,7\}$ and

$$x = \sum_{i=0}^{k-1} a_i 8^i + \sum_{i=1}^{m} b_i 8^{-i}.$$

Finally, a **hexadecimal (base 16) representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_{16},$$

where $a_i, b_i \in \{0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F\}$ (with $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, and $F = 15$) and

$$x = \sum_{i=0}^{k-1} a_i 16^i + \sum_{i=1}^{m} b_i 16^{-i}.$$

(a) **[3 marks]** In each of the following equalities, find the missing representation. To get full credit, you MUST show your work.

i. $(EA)_{16} = (x)_8$

ii. $(755)_8 = (x)_2$

iii. $(9009)_{10} = (x)_{16}$

Solution

i. $(EA)_{16} = (352)_8$. This follows from observing that

$$\begin{aligned}(EA)_8 &= 14 \cdot 16 + 10 \cdot 1\\ &= 3 \cdot 64 + 32 + 10\\ &= 3 \cdot 64 + 5 \cdot 8 + 2\\ &= (352)_8.\end{aligned}$$

ii. $(755)_8 = (111101101)_2$. This follows from observing that

$$\begin{aligned}(755)_8 &= 7 \cdot 8^2 + 5 \cdot 8 + 5 \cdot 1\\ &= (2^2+2+1)\cdot 2^6 + (2^2+0+1)\cdot 2^3 + (2^2+0+1)\cdot 1\\ &= 2^8 + 2^7 + 2^6 + 2^5 + 0\cdot 2^4 + 2^3 + 2^2 + 0 \cdot 2 + 1\\ &= (111101101)_2.\end{aligned}$$

iii. $(9009)_{10} = (2331)_{16}$. First, we find the largest power of 16 that is less than 9009. We observe that

$$\begin{aligned}16^2 &= 256,\\ 16^3 &= 4096,\\ 16^4 &= 65536.\end{aligned}$$

Thus, we see the hexadecimal representation has 3 digits, and that

$$9009 = 2 \cdot 4096 + 817.$$

The representation therefore has the form $(2\_\_\_)_{16}$. Next, since

$$817 = 3 \cdot 256 + 49,$$

the representation has the form $(23\_\_)_{16}$. Since

$$49 = 3 \cdot 16 + 1,$$

we see that the representation is $(2331)_{16}$, and we are done.

(b) [**4 marks**] On Worksheet #10, we encountered representations of fractional numbers for which the representations have repeating digits after the decimal point. For example, 1/3 has the representation $(0.\overline{3})_{10}$, where the overline indicates that the 3 repeats. Likewise, 1/3 has the representation $(0.\overline{01})_2$ in binary notation. Prove that every fraction p/q (where $p, q \in \mathbb{N}$, $q \neq 0$, and $\gcd(p, q) = 1$) has a base-b representation *without* repeating digits if and only if there exists an $m \in \mathbb{N}$ such that $q \mid b^m$. **You can use the fact that every natural number has a base-b representation.**

Solution

Proof. First, suppose that $(p/q)_b$ does not have any repeating digits. Then, there exists some $k, m \in \mathbb{N}$ such that

$$\frac{p}{q} = \sum_{i=0}^{k-1} c_i b^i + \sum_{i=1}^{m} d_i b^{-i},$$

where $c_i, d_i \in \mathbb{N}$ and $0 \leq c_i, d_i < b$. Multiplying both sides by b^m,

$$b^m \cdot \frac{p}{q} = \sum_{i=0}^{k-1} c_i b^{m+i} + \sum_{i=1}^{m} d_i b^{m-i}.$$

Thus,

$$b^m p = q\Big(\sum_{i=0}^{k-1} c_i b^{m+i} + \sum_{i=1}^{m} d_i b^{m-i}\Big),$$

from which we see that $q \mid b^m$ (since $\gcd(q, p) = 1$). Now, suppose that $q \mid b^m$. Then there exists some $j \in \mathbb{N}$ such that $b^m = jq$. Thus, $b^m \cdot p/q = jp$. Since $jp \in \mathbb{N}$, and we know that every natural number has a base-b representation, there exists some $k \in \mathbb{N}$ such that

$$jp = \sum_{i=0}^{k-1} c_i b^i,$$

where $c_i \in \mathbb{N}$ and $0 \le c_i < b$. Dividing both sides by b^m,

$$\frac{jp}{b^m} = \sum_{i=0}^{k-1} c_i b^{i-m},$$

but this is just the representation of p/q,

$$\frac{p}{q} = \sum_{i=0}^{k-1} c_i b^{i-m}.$$

Thus, $(p/q)_b$ does not have any repeating digits.

$\square$

3. [9 marks] Asymptotic notation.

For each part of this question, you *may* (but are not required to) use any of the following facts.

- Fact 1: $\forall n \in \mathbb{Z}^+, n \leq 2^n$
- Fact 2: $\forall x, y \in \mathbb{R}^{\geq 0}, x \leq y \Leftrightarrow \log_2(x) \leq \log_2(y)$
- Fact 3: $\forall x, y \in \mathbb{R}^{\geq 0}, x \leq y \Leftrightarrow 2^x \leq 2^y$

(a) [**3 marks**] Prove each of the following statements. You do NOT have to use induction on n. (Keep it simple!)

i. $n \in \mathcal{O}(n^{1+\epsilon})$, for any real number $\epsilon > 0$.
ii. $\log_2(n) \in \mathcal{O}(n)$
iii. $2^n \in \mathcal{O}(n!)$

Solution

i. *Proof.* Let $\epsilon > 0$. We know that $n^\epsilon > 1$ for all $n > 1$. Let $n_0 = 2$. Then

$$1 < n^\epsilon,$$

for all $n \geq n_0$. Multiplying both sides by n,

$$n < n^{1+\epsilon},$$

for all $n \geq n_0$. Thus,

$$n \leq cn^{1+\epsilon},$$

for all $n \geq n_0$, where $c = 1$.

$\square$

ii. *Proof.* We know from Fact 1 that $n \leq 2^n$ for all $n \geq 1$. Using Fact 2, we have that $\log_2(n) \leq \log_2(2^n) = n$, for all $n \geq 1$. Thus, $\log_2(n) \leq cn$, for all $n \geq n_0$, where $n_0 = 1$ and $c = 1$.

$\square$

iii. *Proof.* We observe that $2^n = 2 \cdot 2 \cdot 2 \cdots 2$, where the product is taken n times. Thus $2^n = 2 \cdot 2 \cdot 2 \cdots 2 \leq 2 \cdot (1 \cdot 2 \cdot 2 \cdots 2) \leq 2 \cdot (1 \cdot 2 \cdot 3 \cdots n) = 2 \cdot n!$, for all $n \geq 1$. Thus, $2^n \leq c \cdot n!$, for all $n \geq n_0$, where $n_0 = 1$ and $c = 2$.

$\square$

(b) [**3 marks**] Suppose that $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$, and that

$$f(n) \in \mathcal{O}(\log_2(n)).$$

Prove that there exists a constant $c > 0$ such that

$$2^{f(n)} \in \mathcal{O}(n^c).$$

Solution

Proof. Since

$$f(n) \in \mathcal{O}(\log_2(n)).$$

there exists a $c > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$f(n) \leq c \log_2(n).$$

Thus,

$$2^{f(n)} \leq 2^{c \log_2(n)},$$

so

$$2^{f(n)} \leq 2^{\log_2(n^c)},$$

which is equivalent to

$$2^{f(n)} \leq n^c,$$

for all $n \geq n_0$. □

(c) **[3 marks]** The mathematical function $e^x : \mathbb{R} \to \mathbb{R}$ can be represented as series by the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Suppose that $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and that $f(n)$ is eventually dominated by 1.
Prove that

$$e^{f(n)} - 1 \in \mathcal{O}(f(n)).$$

<u>Solution</u>

Proof. By the series representation of e^x, we have

$$e^{f(n)} - 1 = f(n) + \frac{f(n)^2}{2!} + \frac{f(n)^3}{3!} + \frac{f(n)^4}{4!} + \cdots$$

Since, $f(n)$ is eventually dominated by 1, there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $f(n) \leq 1$. Thus, for each $m \in \mathbb{N}$, $f(n)^m \leq f(n)$ for all $n \geq n_0$. Therefore,

$$e^{f(n)} - 1 \leq f(n)\Big(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\Big) = f(n)(e - 1),$$

for all $n \geq n_0$. Thus,

$$e^{f(n)} - 1 \in \mathcal{O}(f(n)).$$

□