Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

---

1. **A direct proof**. Let $n \in \mathbb{Z}$. Recall that we say n is **odd** when $\exists k \in \mathbb{Z},\ n = 2k - 1$. Prove the following statement:

For every pair of odd integers, their product is odd.

Be sure to first translate the statement into predicate logic. You can use the predicate $Odd(n)$ in your formula without expanding the definition, but you'll need to use the definition in your proof.

2. **An incorrect proof.** Consider the following claim:

For every even integer m and odd integer n, $m^2 - n^2 = m + n$.

(a) Using the predicates $Even(n)$ and $Odd(n)$ (which return whether an integer n is even or odd, respectively), express the above statement in predicate logic.

(b) The following was submitted as a proof of the statement:

Proof. Let m and n be arbitrary integers, and assume m is even and n is odd. By the definition of even, $\exists k \in \mathbb{Z},\ m = 2k$; by the definition of odd, $\exists k \in \mathbb{Z},\ n = 2k - 1$. We can then perform the following algebraic manipulations:

$$\begin{aligned} m^2 - n^2 &= (2k)^2 - (2k-1)^2 \\ &= 4k^2 - 4k^2 + 4k - 1 \\ &= 4k - 1 \\ &= 2k + (2k - 1) \\ &= m + n \end{aligned}$$

$\square$

The given argument is not a correct proof. What is the flaw?[1]

3. **Comparing functions.** Consider the following definition:[2]

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is dominated by** f (or f **dominates** g) when for every natural number n, $g(n) \leq f(n)$.

(a) Express this definition symbolically by defining the following predicate:

$$Dom(f, g) : \underline{\hspace{6cm}}, \text{ where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}.$$

(b) Let $f(n) = 3n$ and $g(n) = n$. Prove that g is dominated by f.

(c) Let $f(n) = n^2$ and $g(n) = n + 165$. Prove that g is *not* dominated by f. Make sure to write the statement you'll prove in predicate logic, in fully simplified form (negations moved all the way inside).

[1] For extra practice, determine whether the given statement is actually True or False, and write a correct proof or disproof.

[2] Recall that $\mathbb{R}^{\geq 0} = \{x \mid x \in \mathbb{R} \wedge x \geq 0\}$.

(d) Now let's *generalize* the previous statement. Translate the following statement into predicate logic (expanding the definition of *Dom*) and then prove it!

For every positive real number x, $g(n) = n + x$ is *not* dominated by $f(n) = n^2$.

4. **More with floor.** Recall that the **floor** of a number x, denoted $\lfloor x \rfloor$, is the largest integer less than or equal to x. Here is a **fact** you may use about floor: for every $x \in \mathbb{R}$, $0 \leq x - \lfloor x \rfloor < 1$.

Prove the following statement:[3]

$$\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$$

Hints: First introduce a variable $\epsilon = x - \lfloor x \rfloor$ and rewrite $\lfloor x \rfloor$ as $x - \epsilon$. What can you conclude about ϵ given the above **fact**? Then, prove the following simpler statement, and use it in your proof: $\forall x \in \mathbb{R}^{\geq 0},\ x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$.

[3]For extra practice, try proving the following generalization of this statement: $\forall k \in \mathbb{R}^{\geq 0},\ k < 1 \Rightarrow \big(\exists x_0 \in \mathbb{R}^{\geq 0},\ \forall x \in \mathbb{R}^{\geq 0},\ x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2\big)$.