Intro to Computational Complexity

CSC165 Week 7

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So far we have learned…

$\forall, \exists$

- How to symbolize logical statements precisely with quantifiers, predicates, and logical operators connecting them.

$P(x)$ $\quad \wedge \ \vee \ \Rightarrow \ \Leftrightarrow \ \neg$

- How to prove many of those statements using direct and indirect proof structures

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- How to symbolize logical statements precisely with quantifiers, predicates, and logical operators connecting them.

- How to prove many of those statements using direct and indirect proof structures

During weeks 7-10, we will look at…

- Ways to associate a function with part of a program to represent its complexity
- How to use these functions to decide which will run "faster"
- Proofs of precise logical statements about functions that will give us this information

How can we figure out the time it takes for a program to run?

- measure the time it takes to run
- counting the operations performed

loops? function calls?

Consider this program:

```
1  def print_items(lst: list) -> None:
2      for item in lst:
3          print(item)
```

→ Let n = the number of items in the list.

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How long will it take to run this program as n gets bigger?

Will this depend on the number of characters required to print each item?

Measuring runtime by timing the computer

- depends on hardware
- depends on whatever software is also running at the same time
- depends on definition of “basic operations”
- depends on the relative time it takes to run different each “basic operation”

Consider this program:

```
1  def print_items(lst: list) -> None:
2      for item in lst:
3          print(item)
```

Let n = the number of items in the list.

What counts as a basic operation?

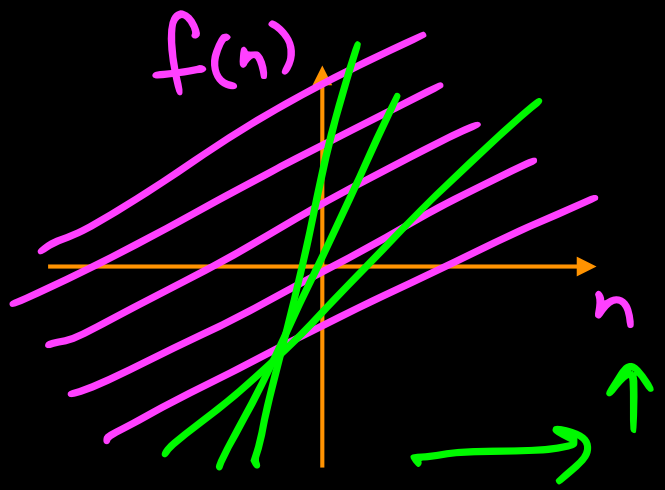
- The "print" call? n
- Reassigning the "item" variable during each iteration? n or 10n ...?
- Calling "print_items" in the first place? +1.5

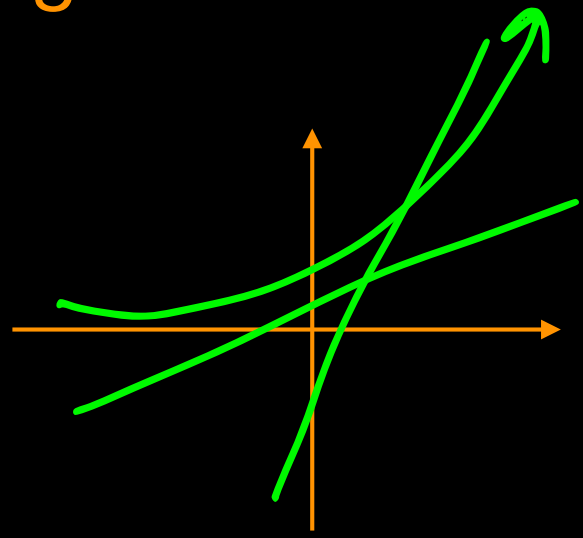
Total time to run the code snippet as a function of n:

Too much ambiguity!!

$$\underbrace{1 + \ldots + 1}_{n} + \underbrace{10 + \ldots + 10}_{n} + 1.5 = n + 10n + 1.5$$
$$= 11n + 1.5$$

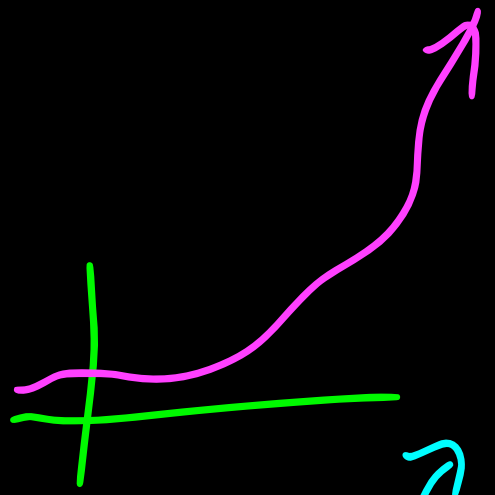
Solution: We can represent algorithms with a class of functions instead of using a single function.



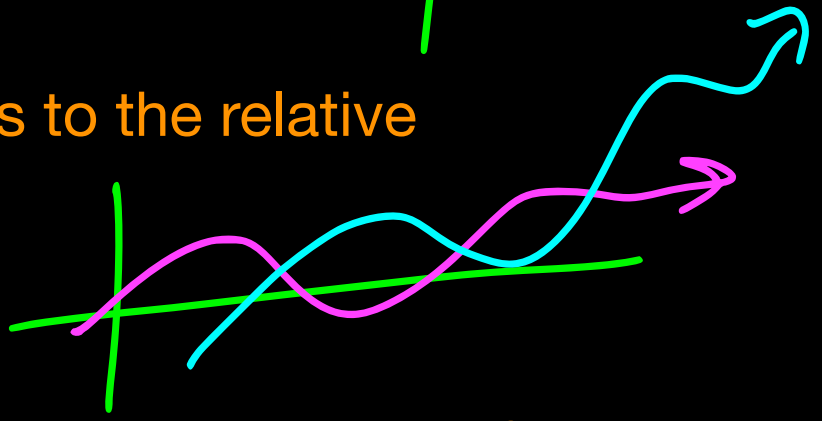


Asymptotic Growth

As n gets bigger, what happens to f(n)?



As n gets bigger, what happens to the relative positions of f(n) and g(n)?

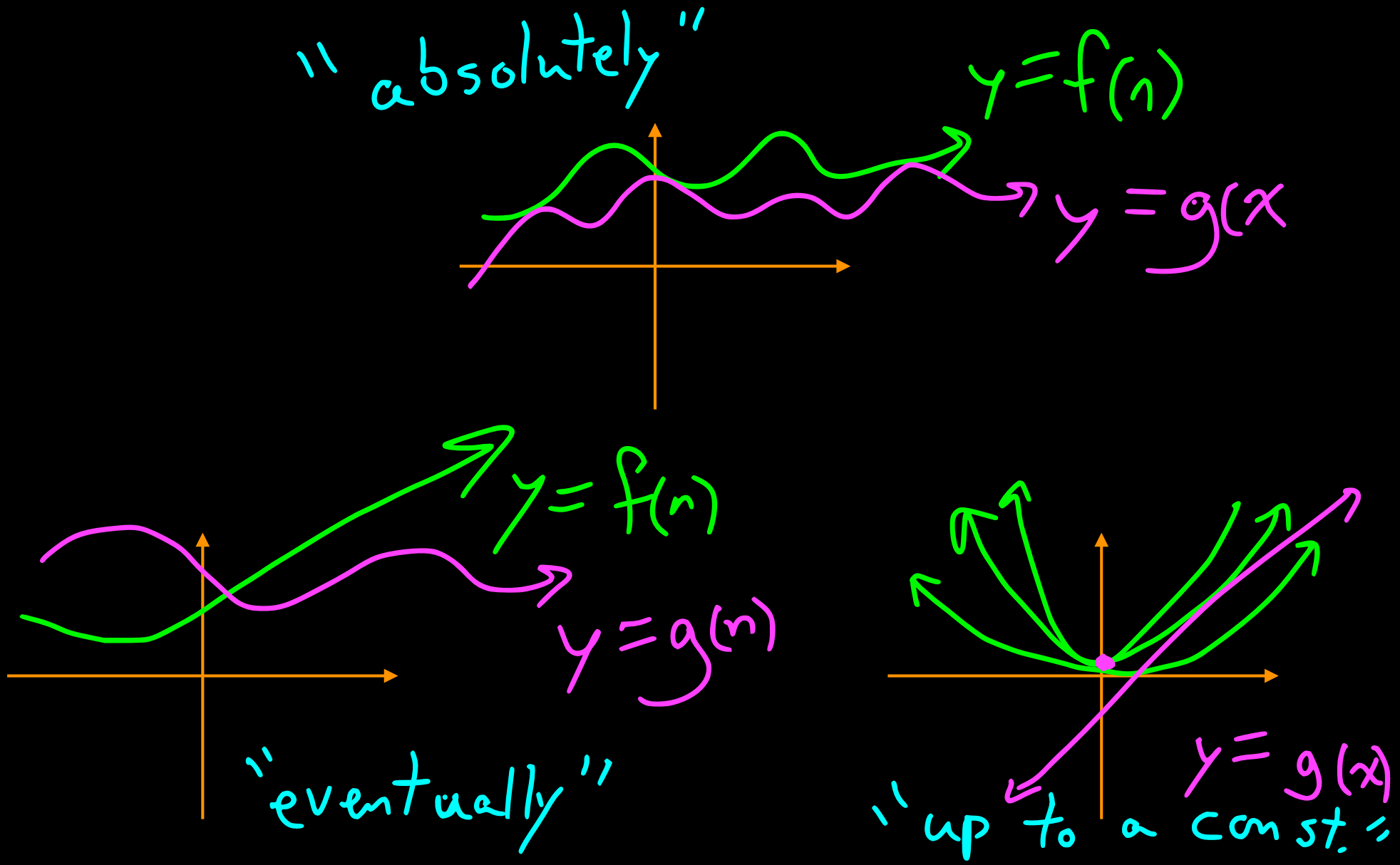


How can we describe functions that "grow together"?

$f(x) \ll g(x)$ $\lim_{x \to \infty} f(x)/g(x)$

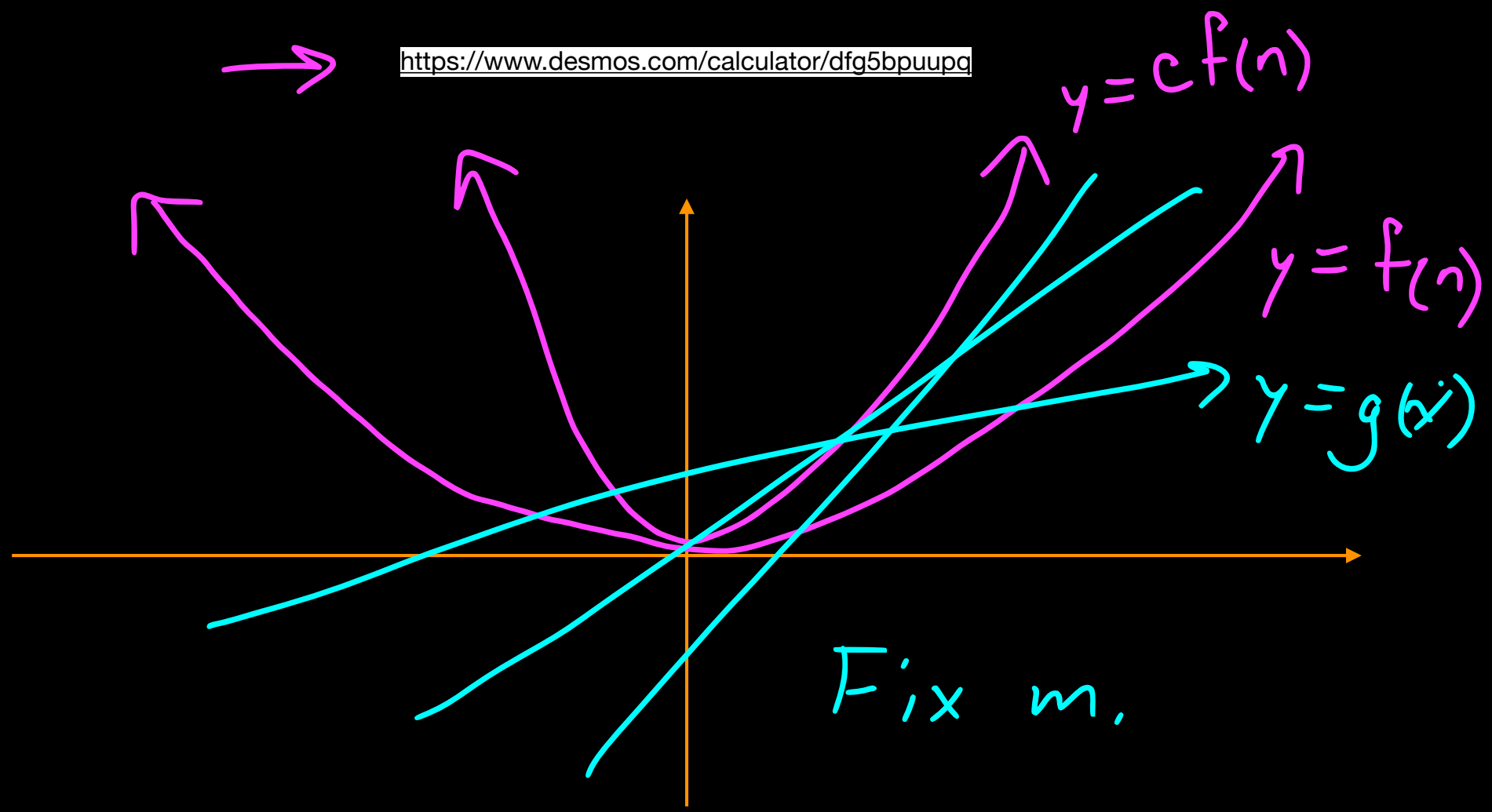
How can we describe the worst case, best case, and average case scenarios for the runtime of an algorithm?

Definition 5.1. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g is **absolutely dominated by** f if and only if for all $n \in \mathbb{N}$, $g(n) \leq f(n)$.



Definition 5.2. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is dominated by f up to a constant factor** if and only if there exists a positive real number c such that for all $n \in \mathbb{N}$, $g(n) \leq c \cdot f(n)$.

https://www.desmos.com/calculator/dfg5bpuupq



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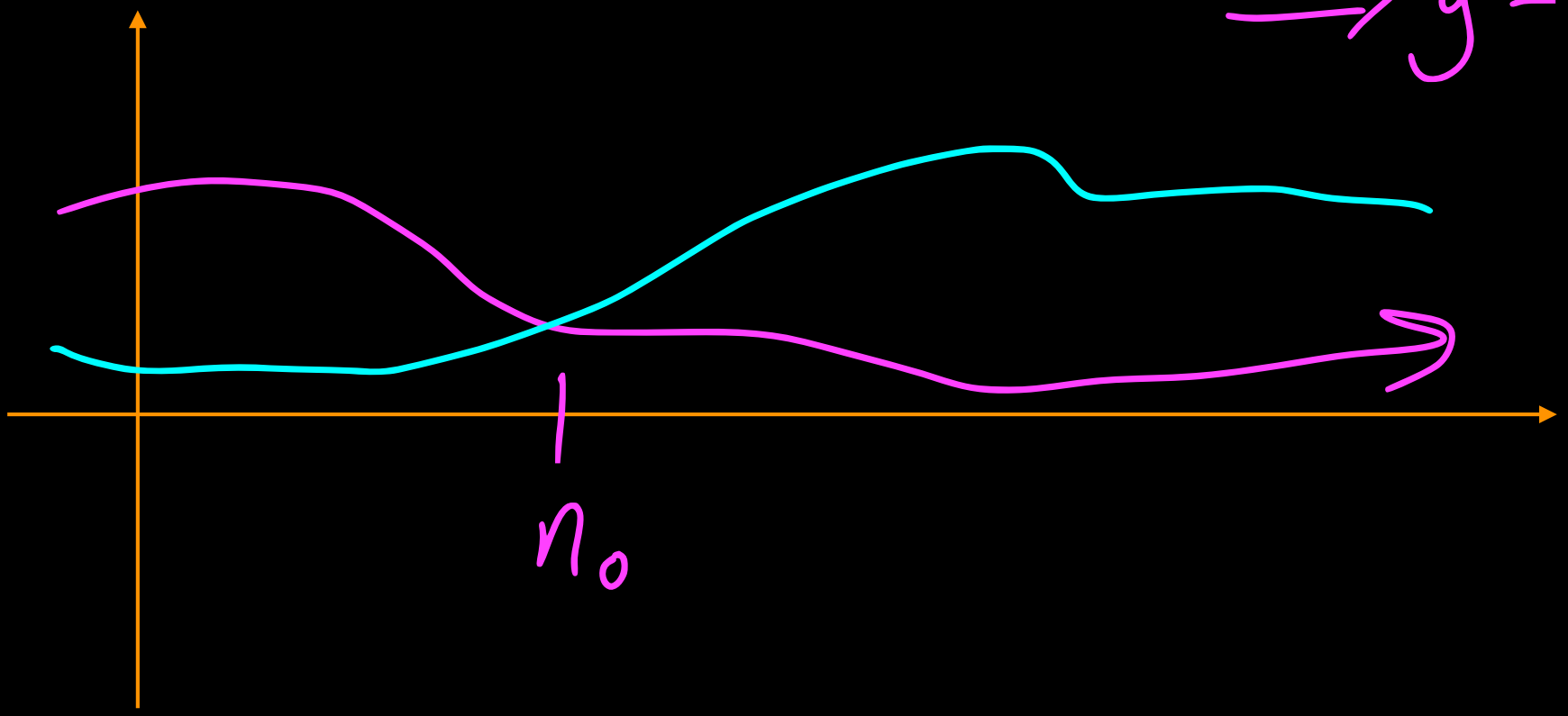
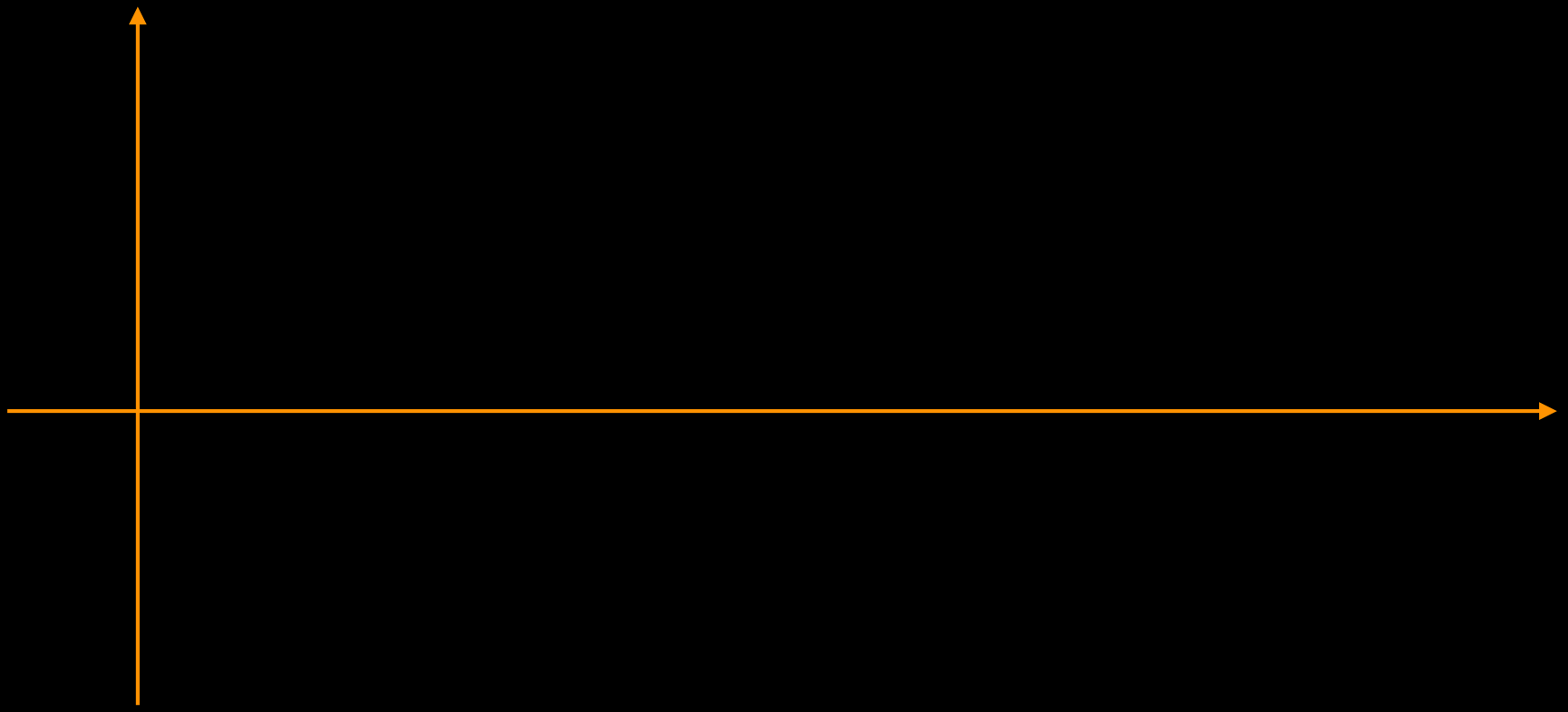
"eventually dominated by f"

$\exists n_0, \forall n \in \mathbb{N} \quad n \geq n_0 \Rightarrow g \leq f$



Definition 5.3. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is eventually dominated by** f if and only if there exists $n_0 \in \mathbb{R}^+$ such that $\forall n \in \mathbb{N}$, if $n \geq n_0$ then $g(n) \leq f(n)$.



Big-Oh Notation

$\mathcal{O}(f)$

A way to describe the set of functions g that will eventually dominate f up to a constant factor

\mathcal{O}

Big-Oh Notation

A way to describe the set of functions g that will

eventually dominate f up to a constant factor

$$\mathcal{O}(f) = \{g \mid g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \exists c, n_0 \in \mathbb{R}^{+},\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c \cdot f(n)\}.$$

We can talk about the runtime of an algorithm that is represented by function g(n) by finding f(n) so that

$g \in \mathcal{O}(f)$

Definition 5.3. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is eventually dominated by** f if and only if there exists $n_0 \in \mathbb{R}^+$ such that $\forall n \in \mathbb{N}$, if $n \geq n_0$ then $g(n) \leq f(n)$.

$\exists n_0 \in \mathbb{R} \quad \forall n \in \mathbb{N}$

c

"by a const."

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up to a constant

Example: Let $f(n) = n^3$ and $g(n) = n^3 + 100n + 500$

Strategy:

If $a \leq x$, $b \leq y$, and $c \leq z$, then $a+b+c \leq x+y+z$

$$n^3 \leq c_1 n^3$$
$$100n \leq c_2 n^3$$
$$5000 \leq c_3 n^3$$

$c_1 = 1$

find n_0 or c_2=?

" or c_3=?

Approach 1: focus on choosing n_0.

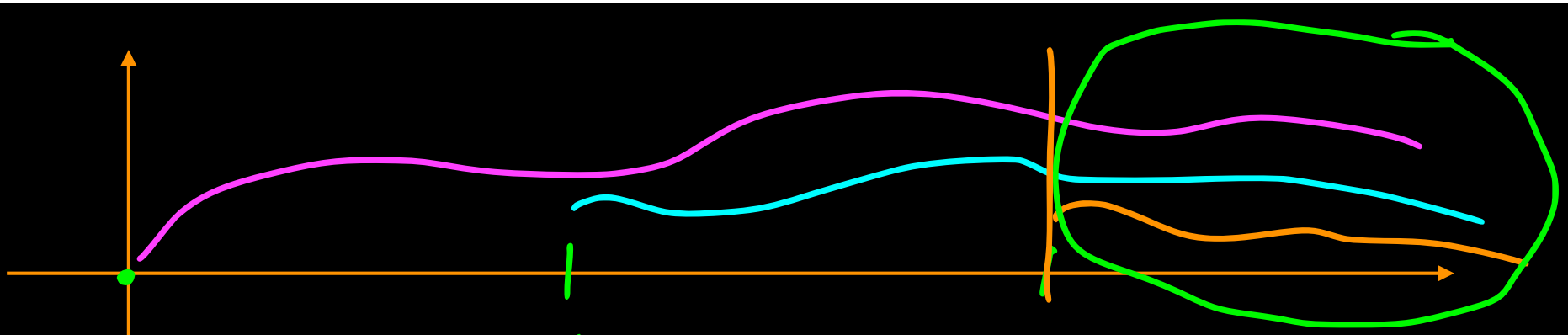
It turns out we can satisfy the three inequalities even if $c_1 = c_2 = c_3 = 1$:

- $n^3 \leq n^3$ is always true (so for all $n \geq 0$).
- $100n \leq n^3$ when $n \geq 10$.
- $5000 \leq n^3$ when $n \geq \sqrt[3]{5000} \approx 17.1$

Why?

We can pick n_0 to be the largest of the lower bounds on n, $\sqrt[3]{5000}$, and then these three inequalities will be satisfied!

$a \leq x$, $b \leq y$ $\Rightarrow a+b \leq x+y$



$\forall n \in \mathbb{N}$, $n > 17.2 \Rightarrow (n^3 + 100n + 500) \leq n^3 + n^3 + n^3 = 3n^3$

Approach 2: focus on choosing c.

Another approach is to pick c_1, c_2, and c_3 to make the right-hand sides large enough to satisfy the inequalities.

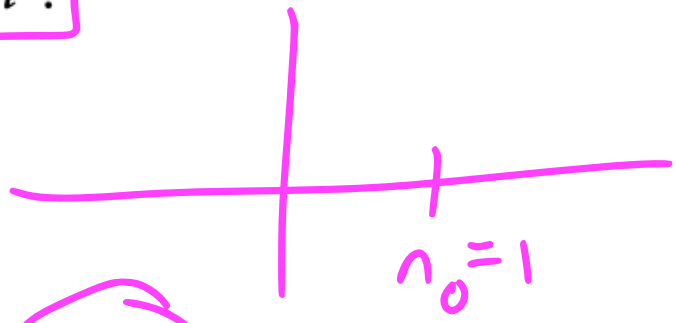
- $n^3 \leq c_1 n^3$ when $c_1 = 1$.
- $100n \leq c_2 n^3$ when $c_2 = 100$.
- $5000 \leq c_3 n^3$ when $c_3 = 5000$, *as long as* $n \geq 1$.

→ $c_3 n^3$ so we can collect like terms

→ $n = 0$ will not work for $5000 \leq c_3 n^3$

Proof. (*Using Approach 2*) Let $c = 5101$ and $n_0 = 1$. Let $n \in \mathbb{N}$, and assume that $n \geq n_0$. We want to show that $n^3 + 100n + 5000 \leq cn^3$.

First, we prove three simpler inequalities:



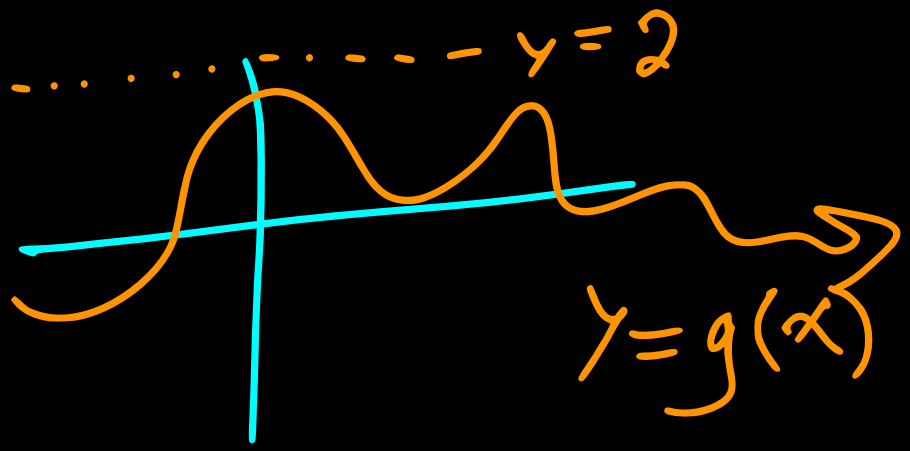
- $n^3 \leq n^3$ (since the two quantities are equal).
- Since $n \in \mathbb{N}$, we know that $n \leq n^3$, and so $100n \leq 100n^3$.
- Since $1 \leq n$, we know that $1 \leq n^3$, and then multiplying both sides by 5000 gives us $5000 \leq 5000n^3$.

Adding these three inequalities gives us:

$$n^3 + 100n + 5000 \leq n^3 + 100n^3 + 5000n^3 = 5101n^3 = cn^3.$$

What does it mean if g ∈ $\mathcal{O}(1)$?

$$\mathcal{O}(f) = \{ g \mid g : \mathbb{N} \to \mathbb{R}^{\geq 0}, \exists c, n_0 \in \mathbb{R}^+ \quad \forall n \in \mathbb{N}, n \geq n_0 \implies g(n) \leq c f(n) \}$$



$g(n) \leq c$ ← not dep. on n.