Learning Objectives

By the end of this worksheet, you will:

- Prove statements using the technique of simple induction.
- Write proofs using simple induction with different starting base cases.
- Write proofs using simple induction within the scope of a larger proof.

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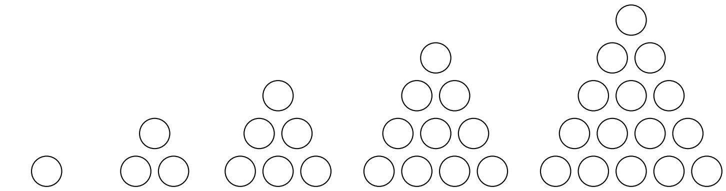
1. **Induction**. Consider the following statement:

$$\forall n \in \mathbb{N}, n \leq 2^n$$

(a) Suppose we want to prove this statement using induction. Write down the full statement we'll prove (it should be an **AND** of the base case and induction step). Consult your notes if you aren't sure about this!

(b) Prove the statement using induction. We strongly recommend reviewing the induction proof template from lecture before working on a proof here.
Hint: $2^{k+1} = 2^k + 2^k$.

2. **Induction (summations).** If marbles are arranged to form an equilateral triangle shape, with n marbles on each side, a total of $\sum_{i=1}^{n} i$ marbles will be required. (For convenience, we also define $T_0 = 0$.)



$T_1 = 1$ $T_2 = 3$ $T_3 = 6$ $T_4 = 10$ $T_5 = 15$

In the course notes, we prove that $\sum_{i=1}^{n} i = n(n+1)/2$. For each $n \in \mathbb{N}$, let $T_n = n(n+1)/2$; these numbers are usually called the *triangular numbers*. Use induction to prove that

$$\forall n \in \mathbb{N},\ \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$$

3. **Induction (inequalities).** Consider the statement:

   For every positive real number x and every natural number n, $(1 + x)^n \geq (1 + nx)$.

   We can express the statement using the notation of predicate logic as:

$$\forall x \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ (1 + x)^n \geq 1 + nx$$

   Notice that in this statement, there are two universally-quantified variables: n and x.[1] Prove this statement is True using the following approach:

   (a) Use the standard proof structure to introduce x.
   (b) When proving the $\big(\forall n \in \mathbb{N}, (1 + x)^n \geq 1 + nx\big)$, do induction on n.[2]

[1]For extra practice, think about the following questions. First, would the statement still be True with the order of the quantifiers reversed: $\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}^+,\ (1 + x)^n \geq 1 + nx$? Second, if this variation is correct, how would this change the proof?

[2]Your predicate $P(n)$ that you want to prove will also contain the variable x—that's okay, since when we do the induction proof, x has already been defined.

4. **Changing the starting number.** Recall that you previously proved that $\forall n \in \mathbb{N},\ n \leq 2^n$ using induction.

   (a) First, use trial and error to fill in the blank to make the following statement true—try finding the *smallest natural number* that works!

   $$\forall n \in \mathbb{N}, n \geq \underline{\hspace{3cm}} \Rightarrow 30n \leq 2^n$$

   (b) Now, prove the completed statement using induction. Be careful about how you choose your base case!