University of Toronto
Faculty of Arts and Science

CSC165H1S Midterm 1, Version 2

Date: February 6, 2019 Duration: 75 minutes Instructor(s): David Liu, François Pitt

No Aids Allowed

Name:

Student Number:

- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- All statements predicate logic must have negations applied directly to propositional variables or predicates.
- You may not define your own propositional operators, predicates, or sets, unless asked to do so in the question. Please work with the symbols we have introduced in lecture, and any additional definitions provided in the questions.
- Proofs should follow the guidelines used in the course (e.g., explicitly introduce all variables, clearly state all assumptions, justify every deduction in your proof body, etc.)
- In your proofs, you may always use definitions from the course. However, you may **not** use any external facts about these definitions unless the yare given in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

Question	Grade	Out of
Q1		8
Q2		7
Q3		6
Q4		5
Total		26

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

1. **[8 marks] Short answers questions.**

(a) **[2 marks]** Let S_1 be the set of all prime numbers, and $S_2 = \{x \mid x \in \mathbb{N} \text{ and } x \mid 30\}$. Write down all the elements of $S_2 \backslash S_1$.

Solution

$S_2 \backslash S_1 = \{1, 6, 10, 15, 30\}$

Note: the definition of prime requires that the number be > 1, and so 1 is *not* a prime number.

(b) **[3 marks]** Write down a truth table for the following expression in propositional logic. Rough work (e.g., intermediate columns of the truth table) is **not** required, but can be included if you want.

$$(\neg p \Leftrightarrow q) \Rightarrow r$$

Solution

p	q	r	$(\neg p \Leftrightarrow q) \Rightarrow r$
False	False	False	True
False	False	True	True
False	True	False	False
False	True	True	True
True	False	False	False
True	False	True	True
True	True	False	True
True	True	True	True

(c) **[3 marks]** Consider the following statement (assume predicates P and Q have already been defined):

$$\forall x \in \mathbb{N},\ P(x) \Rightarrow \big(\exists y \in \mathbb{N},\ Q(x, y)\big)$$

Suppose we want to **prove** this statement. Write the complete *proof header* for a proof; you may write statements like "Let $x =$ _____" without filling in the blank. The last statement of your proof header should be "We will prove that..." where you clearly state what's left to prove, in the same style as the lectures or the Course Notes.

You do not need to include any other work (but clearly mark any rough work you happen to use).

Solution

Let $x \in \mathbb{N}$. Assume that $P(x)$ is true. Let $y =$ _____. We will prove that $Q(x, y)$ is True.

2. **[7 marks] Translations.** Let P be the set of all pets, and suppose we define the following predicates:

- $Cat(x)$: "x is a cat", where $x \in P$
- $Cute(x)$: "x is cute", where $x \in P$
- $Loves(x, y)$: "x loves y", where $x, y \in P$ (note that $Loves(x, y)$ does *not* mean the same thing as $Loves(y, x)$)

Translate each of the following statements into predicate logic. No explanation is necessary. Do not define any of your own predicates or sets. You may use the $=$ and $\neq$ symbols to compare whether two pets are the same.

(a) **[1 mark]** Every cat loves itself.

Solution

$\forall x \in P,\ Cat(x) \Rightarrow Loves(x, x)$

(b) **[2 marks]** Every cat loves at least one pet that is cute.

Solution

$\forall x \in P,\ Cat(x) \Rightarrow \big(\exists y \in P,\ Cute(y) \land Loves(x, y)\big)$

(c) **[2 marks]** If at least one cat is cute, then every cat is cute.

Solution

$\big(\exists x \in P,\ Cat(x) \land Cute(x)\big) \Rightarrow \big(\forall y \in P,\ Cat(y) \Rightarrow Cute(y)\big)$

(d) **[2 marks]** For every two distinct (i.e., not equal) pets, if the two pets love each other, then exactly one of them is a cat.

Solution

$\forall x, y \in P,\ x \neq y \land Loves(x, y) \land Loves(y, x) \Rightarrow (\neg Cat(x) \Leftrightarrow Cat(y))$

Or,

$\forall x, y \in P,\ x \neq y \land Loves(x, y) \land Loves(y, x) \Rightarrow (Cat(x) \land \neg Cat(y)) \lor (\neg Cat(x) \land Cat(y))$

3. **[6 marks] A proof about numbers.** Consider the following statement: "For some natural number n greater than 1, every positive real number x satisfies the equation $\lfloor nx \rfloor = n \cdot \lfloor x \rfloor$."

(a) **[2 marks]** Translate the above statement into predicate logic. You may use $\mathbb{R}^+$ to represent the set of all positive real numbers.

Solution

$\exists n \in \mathbb{N},\ n > 1 \land \big(\forall x \in \mathbb{R}^+,\ \ \lfloor nx \rfloor = n \cdot \lfloor x \rfloor\big)$

(b) **[4 marks]** Prove or disprove the above statement. If you choose to disprove the statement, you must start by writing its negation. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

Solution

This statement is false, so we'll prove its negation: $\forall n \in \mathbb{N},\ n \leq 1 \lor \big(\exists x \in \mathbb{R}^+,\ \ \lfloor nx \rfloor \neq n \cdot \lfloor x \rfloor\big)$.

Proof. Let $n \in \mathbb{N}$. We'll divide up our proof into two cases.

Case 1: Assume $n \leq 1$. In this case, the first part of the OR is true.

Case 2: Assume $n > 1$. We'll prove the second part of the OR.
Let $x = \frac{1}{n}$. Then $\lfloor nx \rfloor = \lfloor n \cdot 1/n \rfloor = \lfloor 1 \rfloor = 1$. But since $n > 1$, we know that $0 < x < 1$, and so $\lfloor x \rfloor = 0$. This means that $n \cdot \lfloor x \rfloor = 0$. So then $\lfloor nx \rfloor \neq n \cdot \lfloor x \rfloor$. ☐

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

4. **[5 marks] Divisibility**. Prove the following statement.

$$\forall a, b \in \mathbb{N},\ b \mid a \wedge b \mid (a+2) \Rightarrow b = 1 \vee b = 2$$

Clearly state where you use any definition from class in your proof. We have left you space for rough work here and on the next page, but write your formal proof in the box below.

<u>Solution</u>

Proof. Let $a, b \in \mathbb{N}$. Assume that $b \mid a$ and $b \mid (a+2)$, i.e. (by the definition of divisibility), that there exist $k_1, k_2 \in \mathbb{Z}$ such that $a = k_1 b$ and $a + 2 = k_2 b$.

We'll first prove that $b \mid 2$. Let $k_3 = k_2 - k_1$, so that subtracting the two equations from our assumptions gives:

$$\begin{aligned}(a+2) - a &= k_2 b - k_1 b \\ 2 &= (k_2 - k_1) b \\ 2 &= k_3 b\end{aligned}$$

So then $b \mid 2$. Since $b \in \mathbb{N}$, and the only positive divisors of 2 are 1 and 2, we know that $b = 1$ or $b = 2$.*

□

*Indeed, we could replace 2 with any prime number to generalize this proof.

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