Asymptotic Notation

CSC165 Week 8 - Part 1

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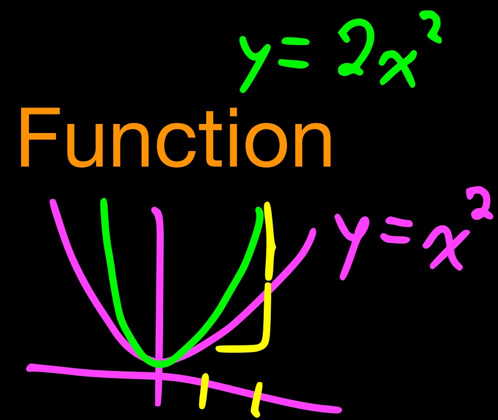
Upper, Lower, and Tight Bounds on a Function

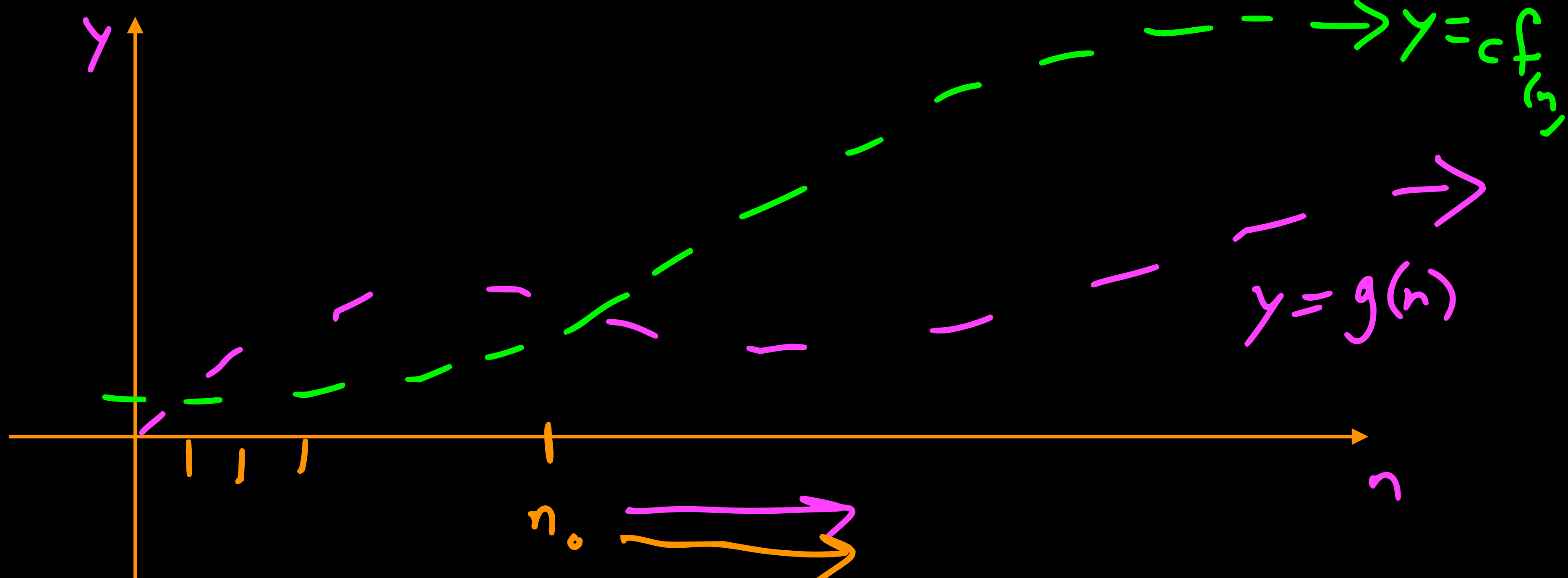


"Upper Bounds:"

$$\mathcal{O}(f) = \{g \mid g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq \boxed{c \cdot f(n)}\}.$$

"approximate number of steps"

"large input size"



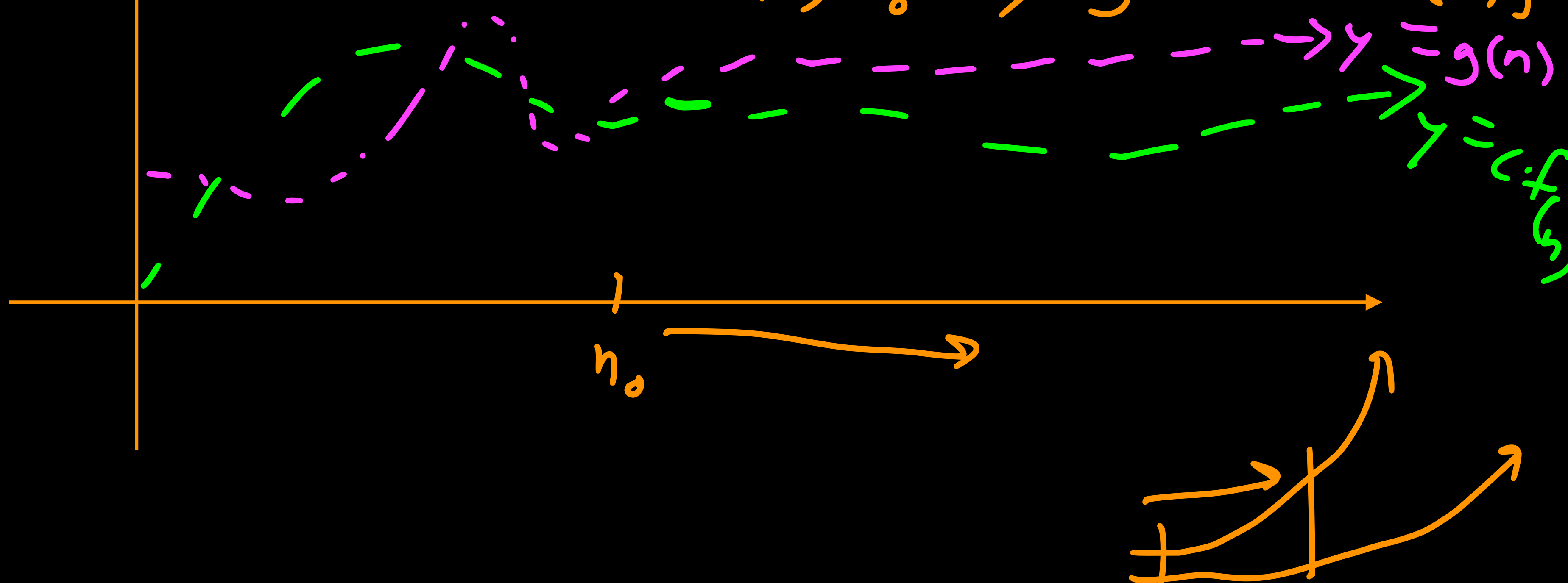
$n + 10n = 11n$

Upper, Lower, and Tight Bounds on a Function

"Lower Bounds:"

Definition 5.5. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is Omega of** f if and only if there exist constants $c, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$, then $g(n) \geq c \cdot f(n)$. In this case, we can also write $g \in \Omega(f)$.

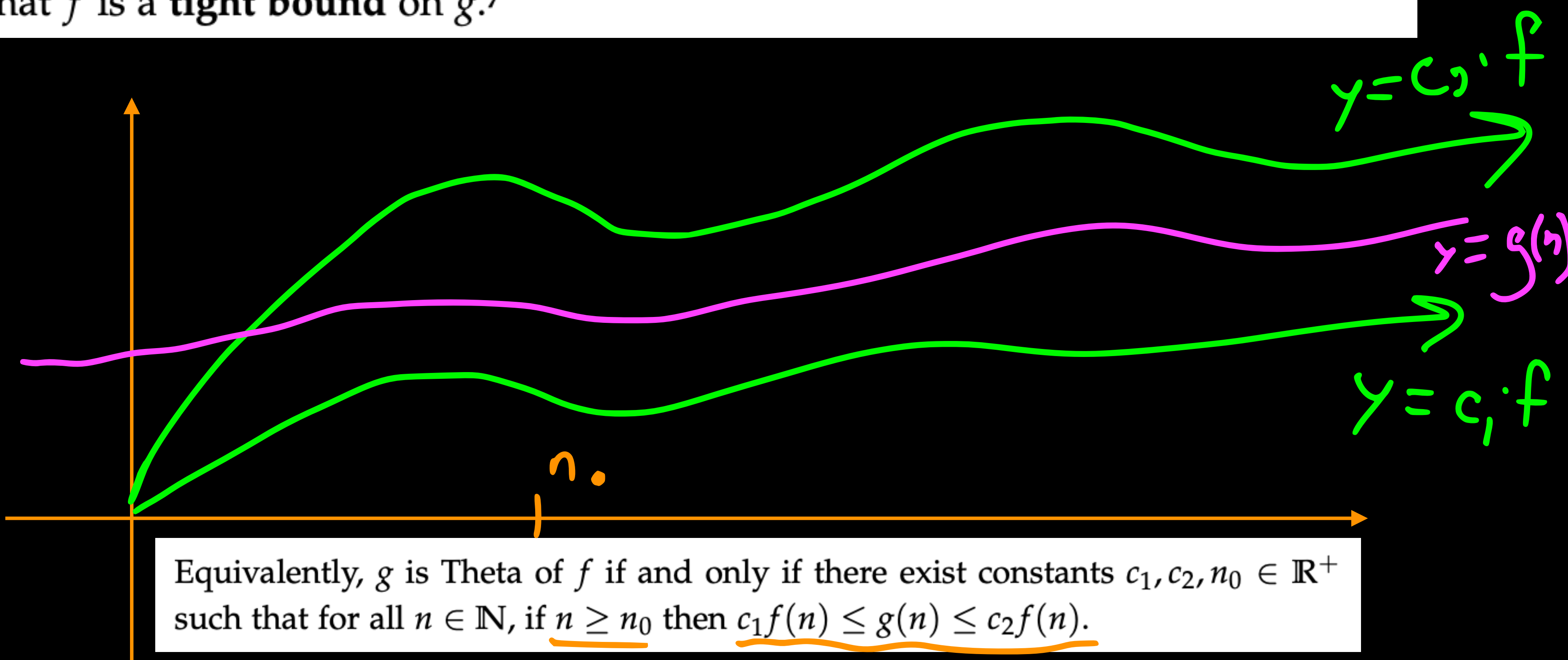
$$\Omega(f) = \{ g \mid g : \mathbb{N} \to \mathbb{R}^{\geq 0}, \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \geq c \cdot f(n) \}$$



Upper, Lower, and Tight Bounds on a Function

"Tight Bounds:"

Definition 5.6. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is Theta of** f if and only if g is both Big-O of f and Omega of f. In this case, we can write $g \in \Theta(f)$, and say that f is a **tight bound** on g.[7]



Equivalently, g is Theta of f if and only if there exist constants $c_1, c_2, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$ then $c_1 f(n) \leq g(n) \leq c_2 f(n)$.

$\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}$

Upper, Lower, and Tight Bounds on a Function

$$\mathcal{O}(f) = \{g \mid g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ \exists c, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c \cdot f(n)\}.$$

Definition 5.5. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is Omega of** f if and only if there exist constants $c, n_0 \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, if $n \geq n_0$, then $g(n) \geq c \cdot f(n)$. In this case, we can also write $g \in \Omega(f)$.

$\to \Omega(f) = \{g \ldots\}$

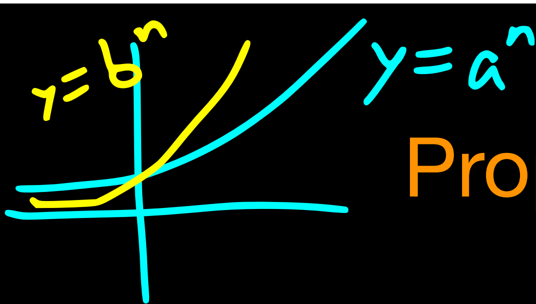
Definition 5.6. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is Theta of** f if and only if g is both Big-O of f and Omega of f. In this case, we can write $g \in \Theta(f)$, and say that f is a **tight bound** on g.[7]

$$g \in \Theta(f) \iff \underline{g \in \mathcal{O}(f)} \wedge \underline{g \in \Omega(f)}$$

Example: an+b $\in \mathcal{O}(n^2)$

Example: $an+b \notin \Omega(n^2)$

[redacted]



Properties of Elementary Functions

Theorem 5.1. For all $a, b \in \mathbb{R}^+$, the following statements are true:

1. If $a > 1$ and $b > 1$, then $\log_a n \in \Theta(\log_b n)$.
2. If $a < b$, then $n^a \in \mathcal{O}(n^b)$ and $n^a \notin \Omega(n^b)$.
3. If $a < b$, then $a^n \in \mathcal{O}(b^n)$ and $a^n \notin \Omega(b^n)$.
4. If $a > 1$, then $1 \in \mathcal{O}(\log_a n)$ and $1 \notin \Omega(\log_a n)$.
5. $\boxed{\log_a n \in \mathcal{O}(n^b)}$ and $\boxed{\log_a n \notin \Omega(n^b)}$.
6. If $b > 1$, then $\boxed{n^a} \in \mathcal{O}(b^n)$ and $\boxed{n^a} \notin \Omega(b^n)$.

$\log_a n = \frac{\log_b n}{(\log_b a)}$

$\log_a n$ is not bound above.

- and $b^n \notin \mathcal{O}(n^a)$
- Also, $\forall a,b \in \mathbb{R}^+$, $\log_b(n) \in \mathcal{O}(a^n)$ and $a^n \notin \mathcal{O}(\log_b(n))$

[redacted]

Properties of Elementary Functions

Theorem 5.2. For all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $f \in \Theta(f)$.

$f \in O(f), f \in \Omega(f)$

Theorem 5.3. For all $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $g \in \mathcal{O}(f)$ if and only if $f \in \Omega(g)$.

Theorem 5.4. For all $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$:

- If $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(h)$, then $f \in \mathcal{O}(h)$.
- If $f \in \Omega(g)$ and $g \in \Omega(h)$, then $f \in \Omega(h)$.
- If $f \in \Theta(g)$ and $g \in \Theta(h)$, then $f \in \Theta(h)$.[10]

h

g

f

cg

dh

$f = f$

$\Leftrightarrow f \geq f$

$f \leq f$

$g \leq cf$

$f \geq \frac{1}{c} g$

Properties of Elementary Functions

Definition 5.7. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We can define the **sum of** f **and** g as the function $f + g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ such that

$$\forall n \in \mathbb{N},\ (f+g)(n) = \boxed{f(n) + g(n)}.$$

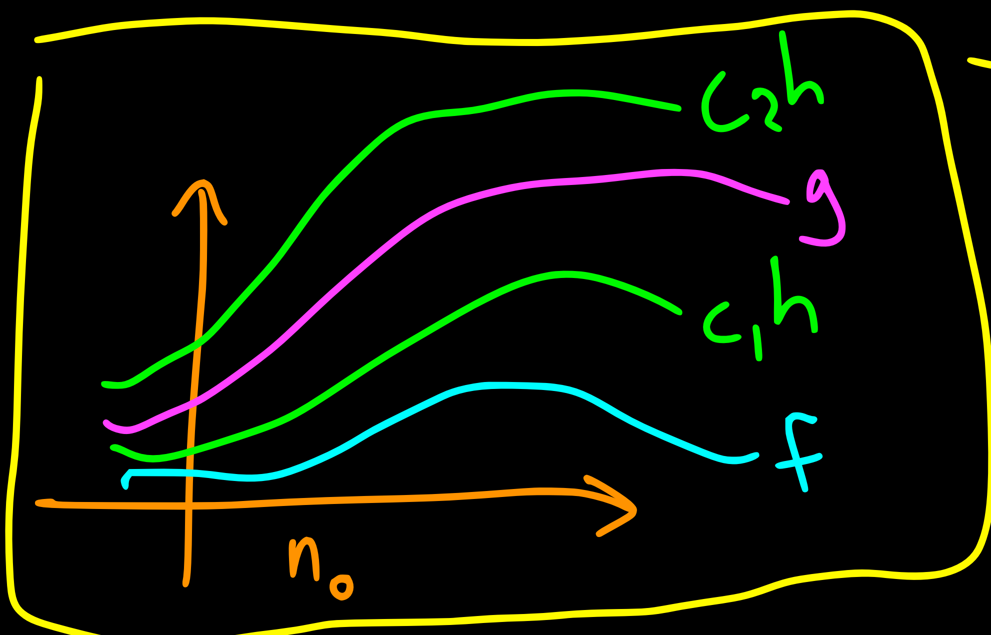
Theorem 5.5. For all $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$, the following hold:

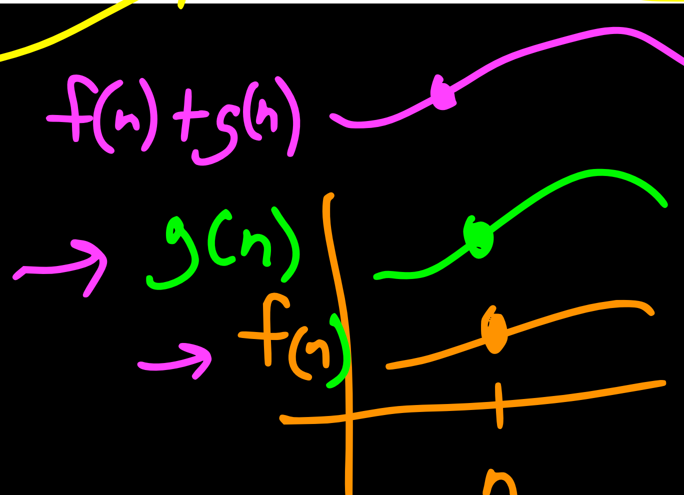
1. If $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$, then $f + g \in \mathcal{O}(h)$.
2. If $f \in \Omega(h)$, then $f + g \in \Omega(h)$.
3. If $f \in \Theta(h)$ and $g \in \mathcal{O}(h)$, then $f + g \in \Theta(h)$.[11]

$f \leq c_1 h$, $g \leq c_2 h$

$f + g \leq c_1 h + c_2 h$

$= (c_1 + c_2) h$





$f \geq ch \Rightarrow f + g \geq f \geq ch$

Example: g $\in \mathcal{O}$(f) $\implies$ f+g $\in \Theta$(f) $\leftarrow$

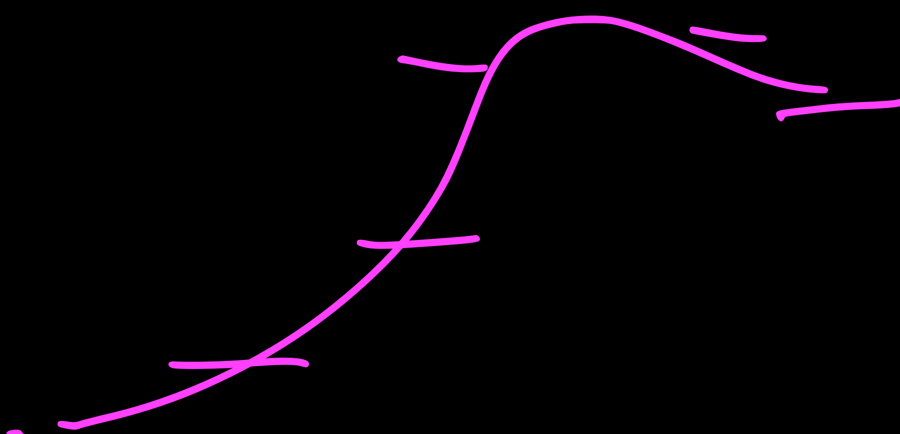
Properties of Elementary Functions

Theorem 5.6. For all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and all $a \in \mathbb{R}^{+}$, $a \cdot f \in \Theta(f)$.

Theorem 5.7. For all $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $g_1 \in \mathcal{O}(f_1)$ and $g_2 \in \mathcal{O}(f_2)$, then $g_1 \cdot g_2 \in \mathcal{O}(f_1 \cdot f_2)$. *Moreover*, the statement is still true if you replace Big-O with Omega, or if you replace Big-O with Theta.

Theorem 5.8. For all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f(n)$ is eventually greater than or equal to 1, then $\lfloor f \rfloor \in \Theta(f)$ and $\lceil f \rceil \in \Theta(f)$.

$$(a \geq c) \wedge (b \geq d) \implies (ab \geq cd)$$



Example: $\forall a \in \mathbb{R}, \; af \in \Theta(f)$

Moral of the story:

1. “Ignore lower order terms”

2. “Ignore constant factors”