- TT1 marks!
- TT2: details & office hours

Last time...

proving $\forall n \in \mathbb{N}, \forall x \in \mathbb{N},\ x \leq 2^n - 1 \Rightarrow B(n,x)$

where $B(n,x)$: $\exists b_0, b_1, \ldots, b_{n-1} \in \{0,1\},\ x = (b_{n-1} \cdots b_0)_2$

(recall, $(b_{n-1} \cdots b_0)_2 = \sum_{i=0}^{n-1} b_i 2^i$).

- $P(n)$: $\forall x \in \mathbb{N},\ x \leq 2^n - 1 \Rightarrow B(n,x)$
- <u>BC:</u> we proved $P(0)$ [<u>idea:</u> $0 = ()_2$]
- <u>Ind. Hyp.:</u> Let $n \in \mathbb{N}$. Assume $P(n)$:

$$\underline{\forall x \in \mathbb{N},\ x \leq 2^n - 1 \Rightarrow B(n,x)}$$

- Ind. Step: WTS $P(n+1)$: $\forall x \in \mathbb{N},\ x \leq 2^{n+1}-1 \Rightarrow B(n+1, x)$

KNOW	WANT
$n \in \mathbb{N}$	$\underbrace{\forall x \in \mathbb{N},\ x \leq 2^{n+1}-1 \Rightarrow B(n+1, x)}_{\text{set up proof headers}}$
$\forall x_0 \in \mathbb{N},\ x_0 \leq 2^n - 1 \Rightarrow B(n, x_0)$	

Let $x \in \mathbb{N}$. Assume $x \leq 2^{n+1}-1$.

$x \in \mathbb{N}$	$B(n+1, x)$
$x \leq 2^{n+1}-1$	

$$\underbrace{0 \quad \cdots \quad 2^n-1}_{\text{IH applies}} \quad \underbrace{\cdots \quad 2^{n+1}-1}_{\text{IH does not apply}}$$

Insight: either $x \leq 2^n - 1$ or $x > 2^n - 1$

Then, either $x \le 2^n - 1$ or $x > 2^n - 1$.

<u>Case 1</u>: Assume $x \le 2^n - 1$

Added: $x \le 2^n - 1$

By the IH (with $x_0 = x$), we can conclude $B(n, x)$:

(KNOW) $\exists b_0, b_1, \dots, b_{n-1} \in \{0,1\},\ x = (b_{n-1} \cdots b_0)_2$

WTS: $B(n+1, x)$: $\underbrace{\exists b_0', b_1', \dots, b_n' \in \{0,1\},\ x = (b_n' \cdots b_0')_2}_{\text{(WANT)}}$

<u>e.g.</u>: $x = (101)_2 = 5$

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Let $b'_0 = b_0,\ b'_1 = b_1,\ \ldots,\ b'_{n-1} = b_{n-1},\ b'_n = 0$

Then, $(b'_n \cdots b'_0)_2 = (0\, b_{n-1} \cdots b_0)_2 = x$.

So $B(n+1, x)$ holds.

Recall: $(\underset{\substack{\downarrow \\ b'_n}}{0}\, b_{n-1} \cdots b_0)_2 = \sum_{i=0}^{n} b'_i 2^i = \underline{0 \cdot 2^n} + \underbrace{\sum_{i=0}^{n-1} b_i 2^i}$

Case 2: Assume $x > 2^n - 1$

done with $x \leq 2^n - 1$ (case 1)

new assumption: $x > 2^n - 1$

$\left(\begin{array}{c} 2^n - 1 < x \\ \Leftrightarrow\ 2^n \leq x \end{array} \right)$

Then, $2^n \leq x \leq 2^{n+1} - 1 = 2^n + 2^n - 1$

So $0 \leq x - 2^n \leq 2^n - 1$

By the IH (with $x_0 = x - 2^n$)
we know $B(n, x-2^n)$

$\exists\, b_0, \ldots, b_{n-1} \in \{0,1\}, \quad x - 2^n = \underbrace{(b_{n-1} \cdots b_0)_2}$

$$\Leftrightarrow \quad x - 2^n = \sum_{i=0}^{n-1} b_i 2^i$$

$$\Leftrightarrow \quad x = 1 \cdot 2^n + \sum_{i=0}^{n-1} b_i 2^i = (1\, b_{n-1} \cdots b_0)_2$$

So $B(n+1, x)$ holds (by picking
$b'_0 = b_0, \ldots, b'_{n-1} = b_{n-1}, \; b'_n = 1$).

$\square$

We have proved:

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{N},\ x \leq 2^n - 1 \Rightarrow B(n,x)$$

Turns out the converse is also true:

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{N},\ B(n,x) \Rightarrow x \leq 2^n - 1$$

<u>EXERCISE:</u> prove this!

<u>Hint</u>: you don't need induction...

but you will need

Fact: $\forall n \in \mathbb{N},\ \sum_{i=0}^{n-1} 2^i = 2^n - 1$