Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements using the definition of Big-Oh.
- Investigate properties of Big-Oh for some common families of functions.

Note: In Big-Oh expressions, it will be convenient to just write down the "body" of the functions rather than defining named functions all the time. We'll always use the variable n to represent the function input, and so when we write "$n \in \mathcal{O}(n^2)$," we really mean "the functions defined as $f(n) = n$ and $g(n) = n^2$ satisfy $f \in \mathcal{O}(g)$."

As a reminder, here is the formal definition of Big-Oh:

$$g \in \mathcal{O}(f): \ \exists c, n_0 \in \mathbb{R}^+, \ \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow g(n) \leq cf(n) \qquad \text{where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$$

1. **Comparing polynomials.** Our first step in comparing different families of functions is looking at different powers of n. Consider the following statement, which generalizes the fact that $n \in \mathcal{O}(n^2)$:

$$\forall a, b \in \mathbb{R}^+, \ a \leq b \Rightarrow n^a \in \mathcal{O}(n^b)$$

(a) Rewrite the above statement by expanding the definition of Big-Oh.

(b) Prove the above statement. **Hint**: you can actually pick c and n_0 to both be 1. Even though this is pretty simple, take the time to write the formal proof as a good warm-up for the rest of this worksheet.

2. **Comparing logarithms.** One slight oddity about the definition of Big-Oh is that it treats all logarithmic functions "the same". Your task in this question is to investigate this by proving the following statement:[1]

$$\forall a, b \in \mathbb{R}^+,\ a > 1 \land b > 1 \Rightarrow \log_a n \in \mathcal{O}(\log_b n)$$

We won't ask you to expand the definition of Big-Oh, but if you aren't quite sure, then we highly recommend doing so before attempting even your rough work!

Hint: use the "change of base rule" for logarithms.

[1] If you are concerned by the fact that $\log n$ is not defined at $n = 0$, you can replace $\log_a n$ with $\log_a(1 + n)$ in the above, and similarly with $\log_b$. We usually don't worry about this subtlety, since our concern is with the value of the functions for larger values of n. Picking an $n_0 > 0$ avoids the evaluation worry.

3. **Sum of functions.** Now let's look at one of the most important properties of Big-Oh: how it behaves when adding functions together. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We define the **sum of f and g** as the function $f + g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ such that $\forall n \in \mathbb{N},\ (f+g)(n) = f(n) + g(n)$. For example, if $f(n) = 2n$ and $g(n) = n^2 + 3$, then $(f+g)(n) = 2n + n^2 + 3$.

   Consider the following statement:

$$\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ g \in \mathcal{O}(f) \Rightarrow f + g \in \mathcal{O}(f)$$

   In other words, if g is Big-Oh of f, then $f + g$ is no bigger than just f itself, asymptotically speaking.

   Your task for this question is to prove this statement. Keep in mind this is an implication: you're going to *assume* that $g \in \mathcal{O}(f)$, and you want to *prove* that $f + g \in \mathcal{O}(f)$. It will likely be helpful to write out the full statement (with the definition of Big-Oh expanded), and use subscripts to help keep track of the variables.