Welcome back!

How was your reading week?

- PS0, TT0 remarks: done
- PS1: marks are out, remark requests due by March 5
- TT1: marks will be out this week
- PS2:
  - TA office hours today + regular office hours
  - lots of info on Piazza
  - remember academic integrity: submit only ideas & work of your group members; cite all external resources used

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Recap...

- propositional & predicate notation — for precise expression
- proof techniques, and connection to logical structure of statements being proved
- induction

Now: apply these tools to CS topics

Def: A binary representation of $x \in \mathbb{N}$ consists of $k \in \mathbb{N}$, $b_0, b_1, \ldots, b_{k-1} \in \{0,1\}$ ("bits") such that
$$x = \sum_{i=0}^{k-1} b_i \cdot 2^i = b_{k-1} \cdot 2^{k-1} + \cdots + b_1 \cdot 2^1 + b_0 \cdot 2^0$$

(usually written $(b_{k-1} \cdots b_1 b_0)_2$

e.g., $$(\underset{b_2}{1}\underset{b_1}{0}\underset{b_0}{1})_2 = \sum_{i=0}^{2} b_i \cdot 2^i = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$
$$= 4 + 0 + 1$$
$$= 5$$

Note 1: same idea applies to any base $b > 1$
$$x = \sum_{i=0}^{k-1} d_i \cdot b^i, \text{ where } d_i \in \underbrace{\{0, 1, \ldots, b-1\}}$$

Note 2: what about base 1 (unary)?

it's different: $x = \underbrace{111 \ldots 1}_{x \text{ copies}}$

Want to study relationship between values $x \in \mathbb{N}$ and number of bits required to express x in binary

Def: $B(n,x)$: "x can be expressed in binary using exactly n bits",

where $n, x \in \mathbb{N}$.

in pseudo-predicate notation

$$B(n,x): \ \exists \underline{b_0, b_1, \ldots, b_{n-1},} \ x = (b_{n-1} \cdots b_1 b_0)_2 = \sum_{i=0}^{n-1} b_i \cdot 2^i$$

e.g.:

- $B(3,5)$ = True because $5 = (101)_2$
- $B(4,5)$ = True because $5 = (0101)_2$
- $B(2,5)$ = False because largest value that can be expressed is 3

$(00)_2 = 0$
$(01)_2 = 1$
$(10)_2 = 2$
$(11)_2 = 3$

Q: General relationship between x, n?

Claim: $\forall n \in \mathbb{N}, \forall x \in \mathbb{N}, x \leq 2^n - 1 \Rightarrow B(n,x)$

Proof-idea 1: Let $n \in \mathbb{N}$, $x \in \mathbb{N}$. Assume $x \leq 2^n - 1$

WTS: $B(n,x)$ ...

... not clear how to proceed...

Idea 2: Try induction.

Q: which variable? Try one... pick n

- Predicate $P(n)$: $\underline{\forall x \in \mathbb{N}, x \le 2^n - 1 \Rightarrow B(n,x)}$

- <u>Base case</u>: WTS $P(0)$: $\forall x \in \mathbb{N}, x \le 2^0 - 1 \Rightarrow B(0,x)$

  Let $x \in \mathbb{N}$, and assume $x \le 2^0 - 1$.

  Then, $x = 0$.

  Does $B(0,0)$ hold? $n = 0$

  $\boxed{()_2 =}\ \boxed{\sum_{i=0}^{-1} b_i \cdot 2^i} \rightarrow$ empty sum $= 0 = x$

$\cancel{0}$ $\cancel{0}5$