This document shows **all versions** of each question (or part of a question) on the test, along with their sample solution. Each individual test paper contained only one version of each question (or each part).

1. **[6 marks]** For both statements below:

(i) Write the negation of the original statement without using the $\neg$ symbol.

(ii) Write whether the original statement is true or false.

(iii) If the original statement is true, prove it. If the original statement is false, disprove it.

(NOTE: The notation $\mathbb{R}^{\geq 0}$ represents the set $[0, \infty) = \{x \in \mathbb{R} \mid x \geq 0\}$.)

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (6x < 2n + 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (3x > 5n + 2)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (5x < 2n + 1)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (4x > 3n + 6)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x > n) \Rightarrow (7x < 4n - 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (2x > 3n + 3)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (6x < 2n + 1)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (3x > 7n - 4)$

(a) $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}, (x < n) \Rightarrow (5x < 3n - 1)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (3x + 1 > 3n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (4x + 6 > 5 - 2n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (2 - 3x < 3n^2 + 2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (8x + 1 > 4n^2 + 5)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \geq n) \wedge (7 - 3x < 2n^2 + 1)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (6 + 10x > 3n^2 + 3)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (5x + 2 < 5 + 3n^2)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (3x + 2 > n^2 - 4)$

(b) $\forall x \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{Z}, (x \leq n) \wedge (3x + 4 > 5n^2 - 1)$

Solution

[We show solutions for the first version of each part; the others are similar.]

(a) (i) $\exists x \in \mathbb{R}, \exists n \in \mathbb{Z}, (x > n) \land (6x \geq 2n + 3)$
 (ii) False
 (iii) Let $x = 1$ and $n = 0$. Then, $x > n$ and $6x = 6 \geq 3 = 2(0) + 3$.

(b) (i) $\exists x \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{Z}, (x < n) \lor (3x + 1 \leq 3n^2)$
 (ii) True
 (iii) Let $x \in \mathbb{R}^{\geq 0}$. Let $n = 0$. Then $x \geq n$ and $3x + 1 \geq 3(0) + 1 > 0 = 3(0)^2 = 3n^2$.

Here are representative solutions for some of the versions of part (b), that were slightly different.

(b') (i) $\exists x \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{Z}, (x \leq n) \Rightarrow (6 + 10x \leq 3n^2 + 3)$
 (ii) False
 (iii) Let $x = 2.2$. Then for all $n \geq 2.2$, $3n^2 + 3 \geq 3(3)^2 + 3 = 30 \geq 28 = 6 + 10(2.2)$.
 For the other parts, $x = 3.3$ works with $3x + 2 \leq n^2 - 4$ and $x = 1.1$ works with $3x + 4 \leq 5n^2 - 1$.

(b") (i) $\exists x \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{Z}, (x \leq n) \Rightarrow (5x + 2 \geq 5 + 3n^2)$
 (ii) True
 (iii) Let $x \in \mathbb{R}^{\geq 0}$. Let $n = \lceil 2n \rceil$. If $x < 1$, then $5x + 2 < 7 < 8 \leq 5 + 3n^2$. If $x \geq 1$, then $5 + 3n^2 \geq 5 + 3(2x)^2 = 5 + 12x^2 > 2 + 5x$.

2. [5 marks] This question tests you on "proof by induction". Even if there is a simple proof using another technique, *you will receive* ***at most*** *half the marks if you do NOT use a proof by induction.*

In your answer, you may use the facts that $e = 2.71828\ldots$ and $1/e = 0.367879\ldots$ Also, if you want to look at the graph of any function in this question, you may use https://www.desmos.com/calculator—*but NO other online resource is allowed.*

Use induction to prove the following statement. As part of your answer, make sure to provide an explicit definition for your predicate $P(n)$, and to state clearly what you are proving in each section of your proof.

Version 1:

$$\forall n \in \mathbb{N}, (n \geq 2) \Rightarrow (e^{1-n} + 3 < n^2 + 2)$$

Version 2:

$$\forall n \in \mathbb{N}, (n \geq 3) \Rightarrow (e^{-n+2} + 4 < n^2 + 1)$$

Version 3:

$$\forall n \in \mathbb{N}, (n \geq 4) \Rightarrow (e^{3-n} + 6 < n^2 - 2)$$

Solution

[We show a solution for version 1; the others are similar.]

Predicate: $P(n) : e^{1-n} + 3 < n^2 + 2$, where $n \in \mathbb{N}$

Base Case: We prove $P(2)$:

$$\begin{aligned} e^{1-2} + 3 &= 1/e + 3 \\ &< 1 + 3 \end{aligned}$$

$$\begin{aligned} &< 6 \\ &= 2^2 + 2 \end{aligned}$$

Ind. Hyp.: Let $n \in \mathbb{N}$ and assume $n \geq 2$. Further assume $P(n) : e^{1-n} + 3 < n^2 + 2$.

Ind. Step: We prove $P(n+1)$:

$$\begin{aligned} e^{1-(n+1)} + 3 &= e^{1-n}/e + 3 \\ &< e^{1-n} + 3 && \text{(because } 1/e < 1\text{)} \\ &< n^2 + 2 && \text{(by the I.H.)} \\ &< (n^2 + 2n + 1) + 2 && \text{(because } n \geq 2\text{)} \\ &= (n+1)^2 + 2 \end{aligned}$$

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive **at most** half the marks if you do NOT use a proof by contradiction.*

Definition: Let $i \in \mathbb{N}$. A prime number p_i is said to be *balanced* if and only if $p_i = \dfrac{p_{i-1} + p_{i+1}}{2}$, where $p_0 < p_1 < p_2 < \cdots < p_i < \cdots$ are all the prime numbers ($p_0 = 2, p_1 = 3, p_2 = 5, \ldots$).

For example, 5 is *balanced* because $5 = \dfrac{3+7}{2}$.

Give a proof by contradiction that 7 (or 11, or 13, or 17) is NOT a *balanced* prime.

Solution

[We show a solution for 7; other proofs are similar.] For a contradiction, assume that 7 is a balanced prime. By definition, this means $7 = \dfrac{5+11}{2} = 8$. Since this is a contradiction, 7 is not a balanced prime.

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive **at most** half the marks if you do NOT use a proof by contradiction.*

Definition: A *Sophie-Germain* prime p is a prime number such that $2p + 1$ is also a prime number.

For example, 2 is a *Sophie-Germain* prime because $2(2) + 1 = 5$ which is also a prime number.

Give a proof by contradiction that 7 (or 13, or 17, or 19) is NOT a *Sophie-Germain* prime.

Solution

[We show a solution for 19; other proofs are similar.] For a contradiction, assume that 19 is a Sophie-Germain prime. By definition, this means $2(19) + 1 = 39$ is also prime. However, $39 = 3 \times 13$ is not prime. Since this is a contradiction, 19 is not a Sophie-Germain prime.

3. **[3 marks]** This question tests you on "proof by contradiction". Even if there is a simple proof using another technique, *you will receive **at most** half the marks if you do NOT use a proof by contradiction.*

Definition: A *Pythagorian* prime p is a prime number for which $\exists d \in \mathbb{N}, p = 4d + 1$.

For example, 5 is a *Pythagorean* prime because $5 = 4(1) + 1$.

Give a proof by contradiction that 7 (or 11, or 19, or 23) is NOT a *Pythagorean* prime.

Solution

[We show a solution for 11; other proofs are similar.] For a contradiction, assume that 11 is a Pythagorean prime. By definition, this means $\exists d \in \mathbb{N}, 11 = 4d + 1$. But then, $d = 5/2 \notin \mathbb{N}$. Since this is a contradiction, 11 is not a Sophie-Germain prime.

4. [5 marks] In this question, you **must** use the following definition of absolute value:

$$\forall z \in \mathbb{R}, \quad |z| = \begin{cases} z, & \text{if } z \geq 0, \\ -z, & \text{if } z < 0. \end{cases}$$

Prove that every solution to

$$|x - 6| \leq b - 2x$$

belongs to the set $(-\infty, b-6]$, where b is the *last non-zero digit* in your student number. (Here, "last" means furthest to the right; for example, if your student number is 1000305070, the last non-zero digit is "7".)

Solution

Consider two cases: either $x - 6 \geq 0$ or $x - 6 < 0$.

Case 1: $(x \geq 6)$

On this domain, the original inequality is equivalent to: $x - 6 \leq b - 2x \Leftrightarrow 3x \leq 6 + b \Leftrightarrow x \leq 2 + b/3$.

Therefore, the solutions are in the set $[6, \infty) \cap \left(-\infty, 2 + \frac{b}{3}\right] = \varnothing \subseteq (-\infty, b-6]$, since $b \leq 9 \Rightarrow 2 + \frac{b}{3} \leq 5$.

Case 2: $(x < 6)$

On this domain, the original inequality is equivalent to $-x + 6 \leq b - 2x \Leftrightarrow x \leq b - 6$

Therefore, the solutions are in the set $(-\infty, b-6] \cap (-\infty, 6) = (-\infty, b-6]$, since $b \leq 9 \Rightarrow b - 6 < 3 < 6$.