Learning Objectives

By the end of this worksheet, you will:

- Prove and disprove statements about numbers and functions.
- Use mathematical definitions of predicates to simplify or expand formulas.
- Identify errors in an incorrect proof.

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1. **A direct proof**. Let $n \in \mathbb{Z}$. Recall that we say n is **odd** when $\exists k \in \mathbb{Z},\ n = 2k - 1$. Prove the following statement:

For every pair of odd integers, their product is odd.

Be sure to first translate the statement into predicate logic. You can use the predicate $Odd(n)$ in your formula without expanding the definition, but you'll need to use the definition in your proof.

Solution

Translation:

$$\forall m, n \in \mathbb{Z},\ \ Odd(m) \wedge Odd(n) \Rightarrow Odd(mn).$$

Discussion. Like the proof we saw in lecture, we'll need to use the definition of odd to introduce new variables to write $m = 2k_1 - 1$ and $n = 2k_2 - 1$; the rest should be straightforward algebra.

Proof. Let $m, n \in \mathbb{Z}$, and assume they are both odd. That is, we assume there exist $k_1, k_2 \in \mathbb{Z}$ such that $m = 2k_1 - 1$ and $n = 2k_2 - 1$. We need to prove that mn is odd, i.e., there exists k_3 such that $mn = 2k_3 - 1$. Let $k_3 = 2k_1k_2 - k_1 - k_2 + 1$.*

Then we can calculate:

$$\begin{aligned} 2k_3 - 1 &= 2(2k_1k_2 - k_1 - k_2 + 1) - 1 \\ &= 4k_1k_2 - 2k_1 - 2k_2 + 1 \\ &= (2k_1 - 1)(2k_2 - 1) \\ &= mn \end{aligned}$$

□

*We actually did the calculation in reverse to find the value of k_3; this was our rough work!

2. **An incorrect proof.** Consider the following claim:

For every even integer m and odd integer n, $m^2 - n^2 = m + n$.

(a) Using the predicates $Even(n)$ and $Odd(n)$ (which return whether an integer n is even or odd, respectively), express the above statement in predicate logic.

Solution

$$\forall m, n \in \mathbb{Z},\ \ Even(m) \wedge Odd(n) \Rightarrow m^2 - n^2 = m + n.$$

(b) The following was submitted as a proof of the statement:

Proof. Let m and n be arbitrary integers, and assume m is even and n is odd. By the definition of even, $\exists k \in \mathbb{Z},\ m = 2k$; by the definition of odd, $\exists k \in \mathbb{Z},\ n = 2k - 1$. We can then perform the following algebraic manipulations:

$$\begin{aligned} m^2 - n^2 &= (2k)^2 - (2k-1)^2 \\ &= 4k^2 - 4k^2 + 4k - 1 \\ &= 4k - 1 \\ &= 2k + (2k-1) \\ &= m + n \end{aligned}$$

□

The given argument is not a correct proof. What is the flaw?[1]

Solution

The author has assumed that $m = 2k$ and that $n = 2k - 1$, using the same variable k to express both m and n. In this way, the author has unwittingly assumed that m and n are consecutive integers. But the statement is about an arbitrary m and an arbitrary n, and so it is wrong to assume that they are consecutive numbers. The author should have let $m = 2k_1$ and $n = 2k_2 - 1$, for some integers k_1 and k_2. And of course, k_1 is not necessarily equal to k_2!

3. **Comparing functions.** Consider the following definition:[2]

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. We say that g **is dominated by** f (or f **dominates** g) when for every natural number n, $g(n) \leq f(n)$.

(a) Express this definition symbolically by defining the following predicate:

$$Dom(f, g) : \underline{\hspace{6cm}}, \text{ where } f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}.$$

Solution

$Dom(f, g) : \forall n \in \mathbb{N},\ g(n) \leq f(n)$.

(b) Let $f(n) = 3n$ and $g(n) = n$. Prove that g is dominated by f.

Solution

We want to prove the following statement:

$$\forall n \in \mathbb{N},\ n \leq 3n$$

Proof. Let $n \in \mathbb{N}$.
We start with the inequality $1 \leq 3$. Since $n \geq 0$, we can multiply both sides by n to obtain the desired inequality, $n \leq 3n$. □

(c) Let $f(n) = n^2$ and $g(n) = n + 165$. Prove that g is *not* dominated by f. Make sure to write the statement you'll prove in predicate logic, in fully simplified form (negations moved all the way inside).

Solution

The statement we want to prove is the negation of $Dom(f, g)$:

$$\exists n \in \mathbb{N},\ n + 165 > n^2.$$

We leave the proof as an exercise.

(d) Now let's *generalize* the previous statement. Translate the following statement into predicate logic (expanding the definition of Dom) and then prove it!

For every positive real number x, $g(n) = n + x$ is *not* dominated by $f(n) = n^2$.

[1] For extra practice, determine whether the given statement is actually True or False, and write a correct proof or disproof.
[2] Recall that $\mathbb{R}^{\geq 0} = \{x \mid x \in \mathbb{R} \land x \geq 0\}$.

Solution

Translation:

$$\forall x \in \mathbb{R},\ x > 0 \Rightarrow \big(\exists n \in \mathbb{N},\ n + x > n^2\big)$$

We leave the proof as an exercise.

4. **More with floor.** Recall that the **floor** of a number x, denoted $\lfloor x \rfloor$, is the largest integer less than or equal to x. Here is a **fact** you may use about floor: for every $x \in \mathbb{R}$, $0 \le x - \lfloor x \rfloor < 1$.
Prove the following statement:[3]

$$\forall x \in \mathbb{R}^{\ge 0},\ x \ge 4 \Rightarrow (\lfloor x \rfloor)^2 \ge \frac{1}{2}x^2$$

Hints: First introduce a variable $\epsilon = x - \lfloor x \rfloor$ and rewrite $\lfloor x \rfloor$ as $x - \epsilon$. What can you conclude about ϵ given the above **fact**? Then, prove the following simpler statement, and use it in your proof: $\forall x \in \mathbb{R}^{\ge 0},\ x \ge 4 \Rightarrow \frac{1}{2}x^2 \ge 2x$.

Solution

Proof. Let $x \in \mathbb{R}^{\ge 0}$, and assume that $x \ge 4$. As noted in the question, we let $\epsilon = x - \lfloor x \rfloor$, and so we know that $0 \le \epsilon < 1$. Then we can calculate:

$$\begin{aligned}(\lfloor x \rfloor)^2 &= (x - \epsilon)^2 \\ &= x^2 - 2x\epsilon + \epsilon^2 \qquad (1)\end{aligned}$$

We now want to show that $2x\epsilon < \frac{1}{2}x^2$, which we can do using our assumption $x \ge 4$:

$$\begin{aligned}4 &\le x \\ 4x &\le x^2 && \text{(multiplying both sides by } x\text{)} \\ 2x &\le \frac{1}{2}x^2 \\ 2x\epsilon &\le \frac{1}{2}x^2 && \text{(since } \epsilon < 1\text{)}\end{aligned}$$

So then we can use this inequality in equation (1) to get:

$$\begin{aligned}(\lfloor x \rfloor)^2 &= x^2 - 2x\epsilon + \epsilon^2 \\ &\ge x^2 - \frac{1}{2}x^2 + \epsilon^2 && \text{(since } 2x\epsilon \le \frac{1}{2}x^2\text{)} \\ &= \frac{1}{2}x^2 + \epsilon^2 \\ &\ge \frac{1}{2}x^2 && \text{(since } \epsilon^2 \ge 0\text{)}\end{aligned}$$

□

[3] For extra practice, try proving the following generalization of this statement: $\forall k \in \mathbb{R}^{\ge 0},\ k < 1 \Rightarrow \left(\exists x_0 \in \mathbb{R}^{\ge 0},\ \forall x \in \mathbb{R}^{\ge 0},\ x \ge x_0 \Rightarrow (\lfloor x \rfloor)^2 \ge kx^2\right)$.