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1. [5 marks] Asymptotic Notation I.

You may use `https://www.desmos.com/calculator` to look at the graph of the function in this question, *but NO other online resource is allowed* for any question on this test. Also, you still need to provide rigorous arguments for each proof: remember that *a graph is NOT a rigorous argument.*

Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n^2(\cos(n\pi) + 1) + 1.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(n^2)$

(b) **[3 marks]** $f \notin \Theta(n^2)$

Solution

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[redacted]

[redacted]

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$$f(n) = n^2(\cos(n\pi) + 1) + 1.$$

Prove that:

(a) **[2 marks]** $f \in \Omega(1)$

(b) **[3 marks]** $f \notin \Theta(1)$

Solution

[redacted]

[redacted]

[redacted]

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = n!(1 + (-1)^n) + n.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(n!)$

(b) **[3 marks]** $f \notin \Theta(n!)$

Solution

[redacted]

[redacted]

[redacted]

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Solution

[redacted]

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Consider the function $f\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ defined by the formula

$$f(n) = (1 + (-1)^n)/n + 1/n^2.$$

Prove that:

(a) **[2 marks]** $f \in \mathcal{O}(1/n)$

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Solution

[redacted]

[redacted]

[redacted]

[redacted]

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **largest** number x such that $(x)_2$ is 4-digits long.

(b) The **smallest** number x such that $(x)_{16}$ is 5-digits long and contains exactly two A's and one E, with no leading 0's.

(c) The **smallest** number x such that $(x)_8$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution

[redacted]

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2. [3 marks] Number Representations.

Write the following natural numbers x. Feel free to write them as sums. *No proof is required for this question!*

(a) The **smallest** number x such that $(x)_2$ is 5-digits long and contains exactly two 0's and three 1's, with no leading 0's.

(b) The **largest** number x such that $(x)_8$ is 5-digits long and contains exactly two 2's and one 7.

(c) The **smallest** number x such that $(x)_{16}$ is a 5-digit long palindrome, by which we mean a number that reads the same forward and it does backward, i.e., 737 or 24542, with no leading 0's, where each digit appears at most twice.

Solution

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Solution

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3. [4 marks] Induction.

Warning! This question does not require deep insight but it is longer to write up (you may need more than 1 page). You should keep it for last. Also, *you will receive* ***at most*** *half the marks* if you do NOT use induction.

Let $a_0, a_1, a_2, \ldots \in \mathbb{R}$ and $b_0, b_1, b_2, \ldots \in \mathbb{R}$ be arbitrary. Prove the following statement by induction: for all $n \in \mathbb{N}$, if $n \geq 1$, then

$$\sum_{k=0}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

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$$\sum_{k=1}^{n-1}(a_{k+1} - a_k)b_k = a_n b_n - a_1 b_1 - \sum_{k=1}^{n-1} a_{k+1}(b_{k+1} - b_k).$$

<u>**Solution**</u>

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4. [4 marks] Asymptotic Notation II.

Let $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ be two functions. Prove the following statement: if $f(n) \in \Theta(\sqrt{n})$ and $g(n) \in \mathcal{O}(n^2)$, then $g(f(n)) \in \mathcal{O}(n)$.

[redacted]

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[redacted]

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[redacted]