

# Propositions and Predicates

CSC165 Week 2 - Part 2

Lindsey Shorser, Winter 2021

# Negating Logical Operators

NOT:  $\neg(\underline{\neg p}) \Leftrightarrow \underline{p}$

It is not the case that x is not positive  $\Leftrightarrow$  x is positive

OR:  $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

AND:  $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

Demorgan's Laws

p	q	$\neg(p \vee q)$	<u>¬P ∧ ¬q</u>
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

→

p	q	$\neg(p \wedge q)$	<u>¬P ∨ ¬q</u>
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

IF:  $\neg(p \Rightarrow q) \Leftrightarrow$

$$P \wedge \neg q$$

IF AND ONLY IF:  $\neg(p \Leftrightarrow q) \Leftrightarrow (\neg P \wedge q) \vee (P \wedge \neg q)$

$\wedge \vee \neg \Rightarrow \Leftarrow$

p	q	<u><math>p \Rightarrow q</math></u>	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T



$$\neg(\neg p \vee q)$$



p	q	$\neg(p \Rightarrow q)$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	F	F

p	q	$p \Leftrightarrow q$	
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

$$(\neg p \vee q) \wedge (\underline{p} \vee \neg \underline{q})$$

$$(\neg p \wedge q) \vee (p \wedge \neg q)$$

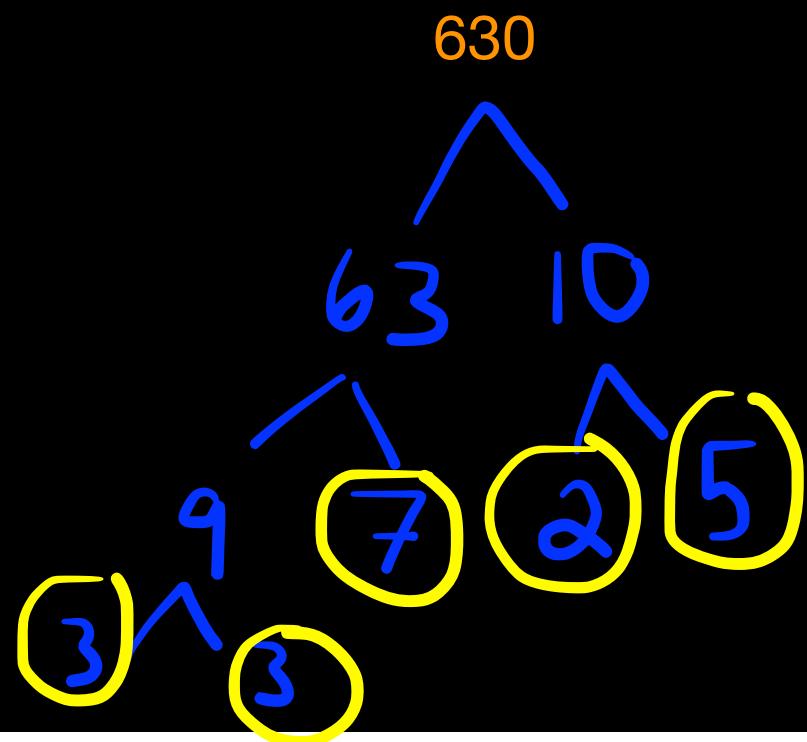
p	q	$\neg(p \Leftrightarrow q)$	
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

## A useful domain: Factors

**Definition 1.8.** Let  $n, d \in \mathbb{Z}$ .<sup>25</sup> We say that  $d$  divides  $n$ , or  $n$  is divisible by  $d$ , when there exists a  $k \in \mathbb{Z}$  such that  $n = dk$ . In this case, we use the notation  $d | n$  to represent “ $d$  divides  $n$ .”

630 is divisible by:  $1, 630, 2, 3, 6, 9, 18, \dots$

$$630 = 2 \cdot 3^2 \cdot 5 \cdot 7$$



# Using “|” or “=” to describe divisibility

$$2 | n$$

“2 divides n” or “2 is a factor of n”

$$\exists k \in \mathbb{Z}, n = 2k$$

“There exists an integer k such that n can be written as a product of 2 and k” or “n is a multiple of 2”

Both “ $2 | n$ ” and “ $n = 2k$ ” are predicates.

$\equiv$

$\equiv$

$\equiv$

$x = x$

How do we symbolize ‘All multiples of 100 are divisible by 10’?

$$100k = 10m$$

$$\forall x \in \mathbb{Z} (\exists k \in \mathbb{Z}, x = \underline{100k}) \Rightarrow (10 | x)$$

$$\Rightarrow (\exists m \in \mathbb{Z}, x = \underline{10m})$$

$$\begin{array}{l} 100x \\ x = 10k \end{array}$$

# Precedence

- 1.  $\neg$
- 2.  $\vee, \wedge$
- 3.  $\Rightarrow, \Leftrightarrow$
- 4.  $\forall, \exists$

B  
E  
N  
D  
O  
M  
A  
S

So for example the expression

$$(p \vee (\neg q)) \wedge r \Rightarrow ((s \vee t) \wedge u) \vee (\neg v \wedge w)$$

$\forall x \in \mathbb{N}, \forall y \in \mathbb{N}$

$$\underline{\forall x, y \in \mathbb{N}, \exists z \in \mathbb{N}, (x + y = z) \wedge (x \cdot y = z)} \Rightarrow (x = y)$$

represents

$$\rightarrow \forall x, y \in \mathbb{N}, \left( \exists z \in \mathbb{N}, ((x + y = z) \wedge (x \cdot y = z)) \Rightarrow x = y \right).$$

$f(g(x))$

# Scope

✓  $\forall x \in \mathbb{N}, \exists k \in \mathbb{Z} \exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, ((x = 2k) \wedge (x = 3m)) \Rightarrow (x = 6n)$

The diagram shows the scope of the existential quantifier  $\exists k \in \mathbb{Z}$  highlighted with a purple oval. A purple arrow points from this oval to the term  $x = 2k$ , indicating that  $k$  is bound by this quantifier. Another purple arrow points from  $x = 2k$  to the final conclusion  $x = 6n$ , showing the transitivity of the scope.

✓  $\forall x \in \mathbb{N}, ((\exists k \in \mathbb{Z}, x=2k) \wedge (\exists m \in \mathbb{Z}, x = 3m)) \Rightarrow (\exists n \in \mathbb{Z}, x = 6n)$

The diagram shows the scopes of the existential quantifiers  $\exists k \in \mathbb{Z}$  and  $\exists m \in \mathbb{Z}$  highlighted with purple underlines. Purple arrows point from each underline to its corresponding term ( $x = 2k$  and  $x = 3m$ ). A final purple arrow points from the conjunction of these terms to the conclusion  $x = 6n$ .

✗  $(\forall x \in \mathbb{N}, (\exists k \in \mathbb{Z}, x=2k) \wedge (\exists m \in \mathbb{Z}, x = 3m)) \Rightarrow (\exists n \in \mathbb{Z}, x = 6n)$

✗  $(\forall x \in \mathbb{N}, (\exists k \in \mathbb{Z}, x=2k) \wedge (\exists m \in \mathbb{Z}, x = 3m)) \Rightarrow (\forall x \in \mathbb{N}, \exists n \in \mathbb{Z}, x = 6k)$

The diagram shows two versions of the same formula. In the first version, the existential quantifiers  $\exists k \in \mathbb{Z}$  and  $\exists m \in \mathbb{Z}$  are underlined in purple, with arrows pointing to their respective terms ( $x = 2k$  and  $x = 3m$ ). In the second version, the existential quantifier  $\exists n \in \mathbb{Z}$  is underlined in purple, with an arrow pointing to its term ( $x = 6k$ ). A purple bracket groups the entire formula in both cases.

# Negating the universal quantifier $\neg \forall . .$

Let  $U = \{\text{students in this course}\}$ ,  
Let  $M(x) = \text{"}x \text{ is majoring in English Literature"}$

How do we symbolize:

“It is not the case that all students in this course are majoring in English Literature”

$$\neg \forall x \in U, M(x)$$

$$\exists x \in U, \neg M(x) \leftarrow \text{The } x \text{ is a "counterexample".}$$

## Negating the existential quantifier

Let  $U = \{\text{students in this course}\}$ ,

Let  $M(x) = "x \text{ is majoring in English Literature}"$

How do we symbolize:

"No one is majoring in English Literature."

or

"It is not the case that someone in this course is majoring in English Literature."

$$\neg \exists x \in U, M(x)$$

$$\forall x \in U, \neg M(x)$$



# Double Quantifiers

Let  $U = \{\text{students in this course}\}$ , and  $x \in U, y \in U$

Let  $S(x,y) = \text{"}x \text{ studies with } y\text{"}$  Note:  $x = y$  is possible.

$\forall$   $x \in U, \underline{\forall} y \in U, S(x,y) =$  Everyone studies with everyone. (all possible pairs)

$\forall$   $x \in U, \underline{\exists} y \in U, S(x,y) =$  Everyone studies with someone.  
(a study buddy for each person)

$\exists$   $x \in U, \underline{\forall} y \in U, S(x,y) =$  Someone studies with everyone. (the super studier)

$\exists$   $x \in U, \underline{\exists} y \in U, S(x,y) =$  Someone studies with someone. (at least one pair)

# Negating Double Quantifiers

- $\neg (\underline{\forall} x \in U, \underline{\forall} y \in U, S(x,y)) = \neg " \text{Everyone studies with everyone.}"$
- $$\exists x \in U, \exists y \in U, \neg S(x,y)$$
- $\neg (\underline{\forall} x \in U, \exists y \in U, S(x,y)) = \exists x \in U, \forall y \in U, \neg S(x,y)$
- $\neg (\underline{\exists} x \in U, \forall y \in U, S(x,y)) = \forall x \in U, \exists y \in U, \neg S(x,y)$
- It is not the case that there is someone who studies with everyone.
- $$\forall x \in U, \forall y \in U, \neg S(x,y)$$
- $\neg (\underline{\exists} x \in U, \underline{\exists} y \in U, S(x,y)) = \forall x \in U, \forall y \in U, \neg S(x,y)$

# Negating Double Quantifiers

$$\neg (\forall x \in U, \forall y \in U, S(x,y)) = \exists x \in U, \exists y \in U, \neg S(x,y)$$

$$\neg (\forall x \in U, \exists y \in U, S(x,y)) = \exists x \in U, \forall y \in U, \neg S(x,y)$$

$$\neg (\exists x \in U, \forall y \in U, S(x,y)) = \forall x \in U, \exists y \in U, \neg S(x,y)$$

$$\neg (\exists x \in U, \exists y \in U, S(x,y)) = \forall x \in U, \forall y \in U, \neg S(x,y)$$



