PSO

Assume $\exists k \in \mathbb{Z}^+$

$$3^{165x-1} = (\sqrt{k})^x$$

$$\log_3 3^{165x-1} = \log_3 (\sqrt{k})^x$$

$$165x - 1 = \frac{x}{2}\log_3 k$$

$$165x - \frac{x}{2}\log_3 k = 1$$

$$\left(165 - \frac{1}{2}\log_3 k\right)x = 1$$

$$x = \frac{1}{165 - \frac{1}{2}\log_3 k}$$

p	q	r	$(p \wedge q)$		$(\neg p \wedge r)$		$\neg(q \wedge r)$	$(p \wedge q) \Rightarrow (\neg p \wedge r) \vee \neg(q \wedge r)$
T	T	T	T	$\Rightarrow$	F	$\vee$	F	F
T	T	F	T	$\Rightarrow$	F	$\vee$	T	T
T	F	T	F	$\Rightarrow$				T
T	F	F	F	$\Rightarrow$				T
F	F	F	F	$\Rightarrow$				T
F	F	T	F	$\Rightarrow$				T
F	T	F	F	$\Rightarrow$				T
F	T	T	F	$\Rightarrow$				T

1.4) A finite string of $\{0,1\}$

→ sequence $b_0 b_1 b_2 \dots b_{k-1}$ → these terms are either 0 or 1.
k is length of the string

Operations on Sets

$|A|$ = cardinality, size of set ; # elements in the set.

Boolean set operators

$x \in A$ x is a element of A => returns True
x is not an element of A => returns False

$A \subset B$ every element of $A \subset B$ = returns True
& every element of $A \not\subset B$ = return false

$A \subseteq A$ and $\emptyset \subset A$ is always True

$A = B$ $A \subset B$ and $B \subset A$

Operations that return sets

$A \cup B$ → set containing elements of A or B or both $\{x \mid x \in A \text{ or } x \in B\}$

$A \cap B$ → A and B $\{x \mid x \in A \text{ and } x \in B\}$

$A \setminus B$ → are in A, not in B $\{x \mid x \in A \text{ and } x \notin B\}$

$A \times B$ → all pairs (a,b) where $a \in A$ and $b \in B$

$\{(x,y) \mid x \in A \text{ and } y \in B\}$

$P(A)$ → all subsets of A. $P(A) = 2^{|A|}$

if $A = \{1,2,3\}$ then

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$P(A) = \{S \mid S \subseteq A\}$

when $j > k$, $\sum_{i=j}^{k} f(i) = 0$

when $j > k$, $\prod_{i=j}^{k} f(i) = 1$

empty

$\forall c \in \mathbb{R}$, $\sum_{i=1}^{n} c = c \cdot n$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$\forall r \in \mathbb{R}$, if $r \neq 1$, then $\sum_{i=0}^{n-1} r^i = \frac{r^n - 1}{r-1}$

$\forall r \in \mathbb{R}$ if $r \neq 1$ then $\sum_{i=0}^{n-1} i \cdot r^i = \frac{n r^n}{r-1} - \frac{r(r^n - 1)}{(r-1)^2}$

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$$

$$\prod_{i=m}^{n} (a_i \cdot b_i) = \left(\prod_{i=m}^{n} a_i\right) \cdot \left(\prod_{i=m}^{n} b_i\right)$$

$$\sum_{i=m}^{n} c \cdot a_i = c \left(\sum_{i=m}^{n} a_i\right)$$

$$\prod_{i=m}^{n} c \cdot a_i = c^{n-m+1} \cdot \left(\prod_{i=m}^{n} a_i\right)$$

off by 1, index needs to be counted

$$\sum_{i=m}^{n} a_i = \sum_{i'=0}^{n-m} a_{i'+m}$$

$n = i' + m$

$i = i' + m$

$$\prod_{i=m}^{n} a_i = \prod_{i'=0}^{n-m} a_{i'+m}$$

$n = i' + m$

$i = i' + m$

$9^n - 1 = 8k$

$9^{n+1} - 1 = 8k$

$k_1 = 9^n + k$

$9^n \cdot 9 - 1 = 8k_1$

$8 \cdot 9^n + 9^n - 1 = 8k_1$

$8 \cdot 9^n + 8k = 8k_1$

$8(9^n + k) = 8k_1$

Every natural number has a binary rep.

$p' = p+1 \qquad b'_0 = 1 \qquad b'_i = b_{i-1}$

$n = \left(\sum_{i=0}^{p} b_i 2^{i+1}\right) + b'_0$

$i' = 1$

$= \sum_{i=1}^{p+1} b_{i-1} 2^i + b'_0$

$= \sum_{i=0}^{p'} b'_i 2^i$

$p' = p+1 \qquad b'_0 = 0 \qquad b'_i = b_{i-1}$

$n = 2k$

$n = 2\left(\sum_{i=0}^{p} b_i 2^i\right)$

$n = \sum_{i=0}^{p} b_i 2^{i+1}$

$n = \sum_{i=1}^{p'} b_{i-1} 2^i + b_0$

$= \sum_{i=0}^{p'} b'_i 2^i$

WS1

1. $\{x \mid x \in U \text{ and } 0 < x \leq 2\}$

$U \setminus A = A^c$
$U \setminus B = B^c$

$A^c \cap B^c = \{x \mid x \in U \text{ and } (0 \geq x \text{ and } x \geq 4)\}$
$A^c \cup B^c = \{x \mid x \in U \text{ and}$

$(A \cup B)^c \rightarrow$ de-morgan's laws

$\neg(A \cup B) \Rightarrow \neg A \cap \neg B \Rightarrow A^c \cap B^c$

b) $3(1) + 3(2) + 3(4) + 3(8)$
$3(2^0) + 3(2^1) + 3(2^2) + 3(2^3)$

$\sum_{i=0}^{5} 3(2^i)$

$\sum_{i=1}^{6} \frac{i^2}{3^i}$

$\sum_{i=0}^{5} (-1)^{i+1} i$

$\prod_{i=1}^{k} \frac{i}{i+1}$

$\prod_{i=1}^{3} \frac{i(i+1)}{(i+2)(i+3)}$

2.) 3, 6, 9   1, 4, 7
2, 5, 8   6, 12, 18

$\{T_0, T_1, T_2\}$ $T_3 \leq T_0$

↳ remainder when divided by 3.

integer remainders are 0, 1, 2

2.) $\{000, 001, 010, 011, 100, 101, 110, 111\}$

6. $\sum_{i=1}^{n} 5(2i-3)+(4-5i)$

$\sum_{i=1}^{n} -5+i$

$\prod_{i=1}^{n} \frac{i}{i+2}$

$S_1 = \{aa, ab, ac, bb, ba, bc, cc, ca, cb\}$

$S_1 \cap S_2 \{aa, bb, cc\}$

$S_1 \setminus S_2 = \{ab, ac, ba, bc, ca, cb\}$

$S_1 = (S_1 \setminus S_2) \cup (S_1 \cap S_2)$

$\sum_{i=10}^{15} 2i + \sum_{i=10}^{15} (i+90)$

$\sum_{i=10}^{15} 3i + 90$

$\sum_{i=101}^{106} 2(i-91) + (i-1)$

$2i - 182 + i - 1$

$3i - 183$

$3(i+101) - 183$

4. $\lfloor x_1 \rfloor = 6$ $\lfloor x_2 \rfloor = 0$ $\lfloor x_3 \rfloor = -3$
$\lceil x_1 \rceil = 7$ $\lceil x_2 \rceil = 1$ $\lceil x_3 \rceil = -2$

$\mathbb{R} \rightarrow \mathbb{Z}$

$\lfloor \frac{1}{2} + \frac{2}{3} \rfloor = \lfloor \frac{7}{6} \rfloor = 1$ $\lfloor \frac{1}{2} \rfloor + \lfloor \frac{2}{3} \rfloor = 0 + 0 = 0$

False above for example

$\lfloor 0.1 \rfloor + \lfloor 2.99 \rfloor = \lfloor 2.99 + 0.1 \rfloor$
$0 + 2 = \lfloor 3.09 \rfloor$
$2 = 3$

a) $(1+1)+(2+1)+(3+1)$

$\frac{1}{2^0} + \frac{1}{2^1}$

$((-1)^2 + 3) + (0^2 + 3) + (1^2 + 3) + (2^2 + 3)$

$(-1)^0 \frac{0}{0+1} + (-1)^1 \frac{1}{1+1} + (-1)^2 \frac{2}{2+1} + (-1)^3 \frac{3}{3+1} + (-1)^4 \frac{4}{4+1}$

$2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$\left(\frac{2(2+2)}{(2-1)(2+1)}\right)\left(\frac{3(3+2)}{(3-1)(3+1)}\right)\left(\frac{4(4+2)}{(4-1)(4+1)}\right)$

WS3

a) $Correct(my\text{-}prog) \land Python(my\text{-}prog)$

b) $\exists x \in P,\ \neg Correct(x) \Rightarrow Python(x)$

why $\neg Correct(x) \land Python(x)$

$\Rightarrow$ reserved for universals

c) $\forall x \in P,\ Python(x) \Rightarrow \neg Correct(x)$

$\neg(\exists x \in P,\ Python(x) \land Correct(x))$

d) $\forall x \in P,\ \neg Correct(x) \land Python(x)$

why $\forall x \in P,\ \neg Correct(x) \Rightarrow Python(x)$

$\neg Python(x) \Rightarrow Correct(x)$

2a) There is a program written in Python and is correct.

2b) Every program is not written in Python and correct.

2c) There is a program written correctly and not in Python. (Not every correct program is written in Python.)

2d. Every program is not written in Python if and only if it is correct.

2e) Every program is correct if it is written in Python or every program is incorrect if it is written in Python.

All Python Programs are correct, or all Python programs are incorrect

2b) $(\exists x \in P,\ Correct(x) \land Python(x)) \Rightarrow (\forall y \in P,\ Python(y) \Rightarrow Correct(y))$

2c) The first one is true

The second one is false, since $\underline{7 \nmid 165}$

- Scope of second one is different from scope in first

$\underline{x_1}\ \underline{|165}$ $\quad$ $\underline{7|x_2}$

0 is a natural number

$x|165 \land 7|x$

3 $Odd(n) = \exists k \in \mathbb{Z},\ n+1 = 2k$ where $n \in \mathbb{Z}$

$Odd(x) = 2 | (x+1)$ where $x \in \mathbb{Z}$

3b $(\forall m,n \in \mathbb{Z},\ 2|(m+1) \land 2|(n+1)) \Rightarrow (2|(mn+1))$

$\forall m,n \in \mathbb{Z},\ Odd(n) \land Odd(m) \Rightarrow Odd(mn)$

3c $\forall m,n \in \mathbb{Z}\ ((\exists k_1 \in \mathbb{Z}, n+1 = 2k_1) \land (\exists k_2 \in \mathbb{Z}, m+1 = 2k_2)) \Rightarrow (\exists k_3 \in \mathbb{Z},\ mn+1 = 2k_3)$

3a) $\forall m,n \in \mathbb{Z}$, $Odd(mn) \Rightarrow Odd(m) \land Odd(n)$

$\forall m,n \in \mathbb{Z}$, $(\exists k_1 \in \mathbb{Z}, mn = 2k_1 + 1) \Rightarrow ((\exists k_2 \in \mathbb{Z}, n = 2k_2 + 1) \land (\exists k_3 \in \mathbb{Z}, m = 2k_3 + 1))$

4a) $\neg((a \land b) \Leftrightarrow c)$

$(a \land b \land \neg c) \lor ((\neg a \lor \neg b) \land c)$

4b) $\neg(\forall x,y \in S, \exists z \in S, P(x,y) \land Q(x,z))$

$\exists x,y \in S, \neg(\exists z \in S, P(x,y) \land Q(x,z))$

$\exists x,y \in S, \forall z \in S, \neg(P(x,y) \land Q(x,z))$

$\exists x,y \in S, \forall z \in S, \neg P(x,y) \lor \neg Q(x,z)$

4c) $\neg((\exists x \in S, P(x)) \Rightarrow (\exists y \in S, Q(y)))$

$(\exists x \in S, P(x)) \land \neg(\exists y \in S, Q(y))$

$(\exists x \in S, P(x)) \land (\forall y \in S, \neg Q(y))$

5. $U = \mathbb{Z}$ (0, is usually an edge case)

$P(n): \exists n \in \mathbb{Z}, n > 3$

$Q(n): \exists n \in \mathbb{Z}, n < 3$

WS4

1a) $\exists n \in \mathbb{N}, \{n > 3 \wedge n^2 - 1.5n \geq 5\}$

existentially quatified ✓

1c) concrete ✓ $\qquad n(n-1.5) \geq 5$

1d let $n = 4$

then, $n > 3$ since $4 > 3$

and

$$n(n-1.5) \geq 5$$
$$4(4-1.5) \geq 5$$
$$4(2.5) \geq 5$$
$$10 \geq 5$$

□ ✓

e) $\forall n \in \mathbb{N}, n > 3 \Rightarrow n^2 - 1.5n > 4$ ✓

- ($\Rightarrow$) is a weaker statement

since it applies to all $n \in \mathbb{N}$

f) universally quantified ✓

g) arbitrary ✓

h) assume $n > 3$ as the hypothesis. ✓

let $n \in \mathbb{N}$, assume $n \geq 3$

i)
$$n > 3$$
$$n - 3 > 0$$
$$(n-3)(n+1.5) > 0(n+1.5)$$
$$n^2 - 1.5n - 4.5 > 0$$
$$n^2 - 1.5n > 4.5 > 4$$
$$n^2 - 1.5n > 4$$

□

$n > 3$

$n - 1.5 > 1.5$

$n^2 - 1.5n > 4.5$ (Multiplying inequality by $n > 3$)

$n^2 - 1.5n > 4$ (since $4.5 > 4$)

2a. $\forall n \in \mathbb{N}, n > 5 \Rightarrow 2 \mid n \vee 3 \mid n$

2b. $\exists n \in \mathbb{N}, n > 5 \wedge 2 \nmid n \wedge 3 \nmid n$

let $n = 7$

Concrete numbers do not need further explanation

$n > 5$ since $7 > 5$

and

$2 \nmid n$ since $7 = 2k$ and $k \notin \mathbb{Z}$

and

$3 \nmid n$ since $7 = 3k$ and $k \notin \mathbb{Z}$

3. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 165$

let $x \in \mathbb{R}$

let $y = 166 - x$

then $x + 166 - x > 165$

$166 > 165$ □

so $x + y > 165$

3b. $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}, x+y > 165$

let $y = 166$ $\quad$ $X+y = x+166$ $\quad$ since smallest $n \in \mathbb{N}$ is 0, $x \geq 0$

let $x \in \mathbb{N}$ $\quad$ $x + 166 > 165$

$x > 165 - 166$

$x > -1$ $\quad$ true for all $\mathbb{N}$.

$\square$

3c) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x+y > 165$

False so prove neg

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x+y \leq 165$

let $y \in \mathbb{R}$

let $x = 164 - y$ $\quad$ $164 - y + y \leq 165$

$164 \leq 165$

$\square$

let $x = -y$ $\quad$ $-y + y \leq 165$

$0 \leq 165$ $\square$

WS5

1. A direct proof

$\exists k \in \mathbb{Z}, n = 2k - 1$

$\forall n, m \in \mathbb{Z}, \ Odd(n) \wedge Odd(m) \Rightarrow Odd(mn)$

Let $m, n \in \mathbb{Z}$

Assume $Odd(n) \wedge Odd(m)$

meaning $\exists k_1 \in \mathbb{Z}, n = 2k_1 - 1$

$\exists k_2 \in \mathbb{Z}, m = 2k_2 - 1$

we want $\exists k_3 \in \mathbb{Z} \ mn = 2k_3 - 1$

$n = 2k_1 - 1$

$nm = (2k_1 - 1)(2k_2 - 1)$

$= 4k_2k_1 - 2k_1 - 2k_2 + 1 = 4k_2k_1 - 2k_1 - 2k_2 + 2 - 1$

Let $k_3 = 2k_2k_1 - k_1 - k_2 + 1$

$2k_3 - 1 = 2(2k_2k_1 - k_1 - k_2 + 1) - 1$

$= 4k_2k_1 - 2k_1 - 2k_2 + 2 - 1$

$= (2k_2 - 1)(2k_1 - 1)$

$= mn$

2. An incorrect proof

$\forall n, m \in \mathbb{Z}, \ Even(m) \wedge Odd(n) \Rightarrow m^2 - n^2 = m + n$

k is quantified twice

False

$\exists n, m \in \mathbb{Z} \ Even(m) \wedge Odd(n) \wedge \ m^2 - n^2 \neq m + n$

Let $m = 4, n = 1$

by definition of $Even(m)$ $\exists k_1 \in \mathbb{Z}, m = 2k \Rightarrow 4 = 2k \Rightarrow k = 2$

by definition of $Odd(n)$ $\exists k_2 \in \mathbb{Z}, n = 2k - 1 \Rightarrow 1 = 2k - 1 \Rightarrow 2 = 2k \Rightarrow k = 1$

$4^2 - 1^2 \neq 4 + 1$

$16 - 1 \neq 5$

$15 \neq 5$

$\square$

3. $Dom(f, g): \forall n \in \mathbb{N}, \ g(n) \leq f(n)$ where $f, g: \mathbb{N} \to \mathbb{R}^{\geq 0}$

b. $f(n) = 3n$ $g(n) = n$

Prove $Dom(f, g)$

Let $n \in \mathbb{N}$

then $g(n) \leq f(n)$

$n \leq 3n$

$1 \leq 3$ Always true

$\square$

c.) $f(n) = n^2 \quad g(n) = n + 165$

Prove $\neg Dom(f, g)$

$\exists n \in \mathbb{N},\ g(n) > f(n)$

Let $n = 1$

$n + 165 > n^2$

$166 > 1^2$

$166 > 1$ Always true

d) $\underbrace{\forall x \in \mathbb{R}^+}_{\text{general}}, \exists n \in \mathbb{N},\ n + x > n^2$

Let $x \in \mathbb{R}^+$ $\quad 0 + x > 0^2$

Let $n = 0$ $\quad x > 0$ since $x \in \mathbb{R}^+$ $\square$

$\exists \Rightarrow$

$\forall \wedge$

$\rightarrow$

$\lfloor x \rfloor := \exists k \in \mathbb{Z}, (k \leq x \wedge (\forall k_0 \in \mathbb{Z}, k_0 \leq k))$

$\lfloor x \rfloor \in \mathbb{Z} \wedge \lfloor x \rfloor \leq x \wedge \forall k \in \mathbb{Z}, k > x \Rightarrow \lfloor x \rfloor > k$

$\lfloor x \rfloor \in$ talk about directly

$\forall x, y \quad x \Leftrightarrow y$

$\forall k \in \mathbb{Z}, k \geq x \Rightarrow \lfloor x \rfloor \geq k \wedge$

4.) $\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1$ Assume $x \geq 4$

$\forall x \in \mathbb{R}^{\geq 0}, x \geq 4 \Rightarrow (\lfloor x \rfloor)^2 \geq \frac{1}{2}x^2$

$\varepsilon = x - \lfloor x \rfloor \quad \lfloor x \rfloor = x - \varepsilon$

$0 \leq x - (x - \varepsilon) \leq 1 \Rightarrow 0 \leq \varepsilon \leq 1$ By Fact

$\forall x \in \mathbb{R}^{\geq 0}, x \geq 4 \Rightarrow \frac{1}{2}x^2 \geq 2x$

$x \geq 4$

$\frac{1}{2}x \geq 2$

$\frac{1}{2}x^2 \geq 2x$

$\frac{1}{2}x^2 \geq 2x\varepsilon$ (since $\varepsilon < 1$)

$\square$

$(\lfloor x \rfloor)^2 = x^2 - 2x\varepsilon + \varepsilon^2$

$\geq x^2 - \frac{1}{2}x^2 + \varepsilon^2$ since $2x\varepsilon \leq \frac{1}{2}x^2$

$= \frac{1}{2}x^2 + \varepsilon^2$

$\geq \frac{1}{2}x^2$ since $\varepsilon^2 \geq 0$ $\square$

$(x - \varepsilon)^2 \geq \frac{1}{2}x^2 \geq 2x$

$(x - \varepsilon)^2 \geq 2x$

$x^2 - 2x\varepsilon + \varepsilon^2 \geq 2x$

$x^2 \geq 2x \quad (0 \leq \varepsilon \leq 1)$

$x \geq 2 \quad$ (since $x \geq 4$)

$x^2 - 2x + 1 \geq 2x$

$0 \leq \varepsilon < 1$

$x \geq 4 \Rightarrow (x - \varepsilon)^2 \geq \frac{1}{2}x^2$

$x \geq 4$

$\frac{1}{2}x \geq 2$

$\frac{1}{2}x^2 \geq 2x$

$\frac{1}{2}x^2 \geq 2x\varepsilon$ since $\varepsilon < 1$

$(\lfloor x \rfloor)^2 = x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - \frac{1}{2}x^2 + \varepsilon^2$ since $2\varepsilon x \leq \frac{1}{2}x^2$

$= \frac{1}{2}x^2 + \varepsilon^2 \quad -2\varepsilon x \geq \frac{1}{2}x^2$

$\geq \frac{1}{2}x^2$ since $\varepsilon^2 \geq 0$

Extra practice

general :

$\forall k \in \mathbb{R}^{\geq 0}, (k<1) \Rightarrow (\exists x_0 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{R}^{\geq 0}, x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2)$.

$P(k): \forall k \in \mathbb{R}^{\geq 0}\ k<1$

$Q(x,k): \exists x_0 \in \mathbb{R}^{\geq 0}, \forall x \in \mathbb{R}^{\geq 0}, x \geq x_0 \Rightarrow (\lfloor x \rfloor)^2 \geq kx^2$

Let $k \in \mathbb{R}^{\geq 0}$ Assume $P(k)$

prove $Q(x,k)$

Let $x_0 = 2k$

Let $x \in \mathbb{R}^{\geq 0}$

Assume $x \geq \frac{2}{1-k}$ $\quad k<1$

$x \geq \frac{2}{1-k}$

$(1-k)x^2 \geq$

$(1-k)x^2 \geq$

$(1-k)x^2 \geq$

$x^2 - kx^2$

$(\lfloor x \rfloor)^2 \geq kx^2$

$(x-\varepsilon)^2 \geq kx^2$ $\quad (1-k)x^2$

$x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - kx^2 + \varepsilon^2$

$\geq$

$\geq kx^2$

$\forall m,n \in \mathbb{Z}, Odd(m) \land Odd(n) \Rightarrow Odd(mn)$

$\exists k_1 \in \mathbb{Z}, m = 2k_1 - 1 \land \exists k_2 \in \mathbb{Z}, n = 2k_2 - 1$

prove $\exists k_3 \in \mathbb{Z}, mn = 2k_3 - 1$

$(2k_1 - 1)(2k_2 - 1) = 2k_3 - 1$

$4k_1k_2 - 2k_2 - 2k_1 + 1 = 2k_3 - 1$

$4k_1k_2 - 2k_2 - 2k_1 + 2 = 2k_3$

$2k_1k_2 - 2k_2 - 2k_1 + 1 = k_3$

WS6

1. Maximum number

$P(123) \land \forall x \in \mathbb{N}, P(x) \Rightarrow x \leq 123$

b. $IsCD(x,y,d)$: $d \mid x \land d \mid y$

$IsGCD(x,y,d)$: $(x=0 \land y=0 \Rightarrow d=0) \land$

$(x \neq 0 \lor y \neq 0 \Rightarrow IsCD(x,y,d) \land (\forall d_0 \in \mathbb{Z}, IsCD(x,y,d_0) \Rightarrow d_0 \leq d)$, where $x,y,d \in \mathbb{Z}$

c.) $\forall x \in \mathbb{Z}^+, IsGCD(x,0,x)$

$x=0 \land 0=0 \Rightarrow x=0 \land$

$x \neq 0 \lor y \neq 0 \Rightarrow IsCD(x,0,x) \land (\forall d_0 \in \mathbb{Z}, IsCD(x,0,d_0) \Rightarrow d_0 \leq x)$

Let $x \in \mathbb{Z}^+$ ↳ Vacuously True (since $x=0$ is false as $x \in \mathbb{Z}^+$)

then $x \neq 0$ Assume $x \neq 0 \lor y \neq 0$

Prove $IsCD(x,0,x)$

$x \mid x$ $x \mid 0$

$\exists k_1 \in \mathbb{Z}, x = dk_1$ $\exists k_2 \in \mathbb{Z}, 0 = xk_2$

$k_1 = 1$ $k_2 = 0$

Prove $\forall d_0 \in \mathbb{Z}, IsCD(x,0,d_0) \Rightarrow d_0 \leq x$

Let $d_0 \in \mathbb{Z}$, Assume $IsCD(x,0,d_0)$

$\Rightarrow d_0 \mid x$ $d_0 \mid 0$

$\Rightarrow d_0 \leq x$ By fact

$\Rightarrow \exists k_2 \in \mathbb{Z}, x = d_0 k_2$ $\exists k_3 \in \mathbb{Z}, 0 = d_0 k_3$

$x = d_0 k_2$

$x = d_0$ if $k_2 = 1$

else $x > d_0$

$\therefore x \geq d_0$

□

d) $\forall a,b \in \mathbb{Z}$ $a \neq 0 \lor b \neq 0 \Rightarrow \forall p,q \in \mathbb{Z}, (pa+qb = gcd(a,b)) \land (\exists k \in \mathbb{Z}^+, pa+qb = k \Rightarrow k \geq gcd(a,b))$ less/eq to every number that satisfies the predicate

$LinComb(a,b,c)$: "$\exists p,q \in \mathbb{Z}, c = pa+qb$" where $a,b,c \in \mathbb{Z}$

$\forall a,b \in \mathbb{Z}$ $a \neq 0 \lor b \neq 0 \Rightarrow LinComb(a,b,gcd(a,b)) \land (\forall d \in \mathbb{Z}^+, LinComb(a,b,d) \Rightarrow d \geq gcd(a,b))$

2. Proof by Cases

Let $n \in \mathbb{Z}$

Case 1. Assume $\exists k \in \mathbb{Z}, n = 2k$

prove $\exists k_0 \in \mathbb{Z}, n^2 - 3n = 2k_0$

Let $k_0 = kn - 3k$

$n^2 - 3n = 2(kn - 3k)$

$n(n-3) = 2k(n-3)$

$n = 2k$

→ use the assumption

Let $k_0 = 2k^2 - 3k$

$n^2 - 3n = (2k)^2 - 3(2k)$

$= 4k^2 - 6k$

$= 2(2k^2 - 3k)$

$= 2k_0$

Case 2. Assume $\exists k \in \mathbb{Z}, n = 2k - 1$ Let $k_0 = 2k^2 - 5k + 2$

$n^2 - 3n = (2k-1)^2 - 3(2k-1)$

$= 4k^2 - 4k + 1 - 6k + 3$

$= 4k^2 - 10k + 4$

$= 2(2k^2 - 5k + 2)$

$= 2k_0$

3. An indirect (contrapositive) proof use

$\forall a, b \in \mathbb{N}, 1 < gcd(a,b) \wedge gcd(a,b) < b \Rightarrow \neg Prime(b)$.

$\forall a, b \in \mathbb{N} \quad Prime(b) \Rightarrow 1 \geq gcd(a,b) \vee gcd(a,b) \geq b$

Let $a, b \in \mathbb{N}$, Assume b is prime

Case 1. Assume $b | a$ always max divisor $(b|b)$

$gcd(a,b) \Rightarrow b|b$ and $b|a$

b is cd $\Rightarrow$ b is gcd since every division of b is $\leq b$.

$\Rightarrow gcd(a,b) = b \geq b$

Case 2. Assume $b \nmid a$

$Prime(b) \Rightarrow 1|b \quad b|b$

$b \nmid a$ so b is not cd.

Since $1|a$, $gcd(a,b) = 1$, since it is the only positive common devisor of these numbers

$gcd(a,b) = 1 \leq 1$

WS7

1) $\forall n \in \mathbb{N},\ \neg \text{Prime}(n) \Rightarrow (n \leq 1 \vee (\exists a, b \in \mathbb{N},\ n \nmid a \wedge n \nmid b \wedge n \mid (ab)))$

Let $n \in \mathbb{N}$

Assume n is not prime, that $n \leq 1$ or $\exists d \in \mathbb{N}$ $d \mid n$, $d \neq 1$, $d \neq n$

Prove $n \leq 1$ or $\exists a, b \in \mathbb{N}$ st. $n \nmid a$, $n \nmid b$ and $n \mid ab$

Case 1. Assume $n \leq 1$

$n \leq 1$ true
or

$a = \frac{n}{2}$ $b = \frac{n}{2}$, then, $\frac{n}{2} = nk$

$\frac{1}{2} = k$ is not an integer

but $\frac{n}{2} \cdot \frac{n}{2} = nk \Rightarrow 1 = k$

Case 2. Assume $\exists d \in \mathbb{N}$ where $d \mid n \wedge d \neq 1 \wedge d \neq n$

$\exists k \in \mathbb{Z}$ where $n = dk$. B/c $n, d \in \mathbb{N}$ $k \in \mathbb{N}$.

$n \leq 1$ is handled $\Rightarrow$ $n > 1$

Let $a = d$, $b = k$

$n \mid ab$ divisibility

We know $n = dk$

$\exists k_2 \in \mathbb{Z}$ st. $dk = nk_2$

$\Downarrow$ $\Downarrow$

$d \mid n$ $k \mid n$

By Fact 2:

$1 \leq n \leq d$

$n < d$ since $d = n$

$1 \leq n \leq k$

$n < k$ since $n \neq k$ since $d \neq 1$

By CP Fact 2

$x < 1 \vee y < x \Rightarrow x \nmid y \vee y = 0.$

$n < 1 \vee a < n \Rightarrow \underbrace{n \nmid d}$ $\vee\ d = 0.$

$\hookrightarrow d \neq 0$ False

$n < 1 \vee b < n =$

WS8

1. Induction

$\forall n \in \mathbb{N}, n \leq 2^n$

$(\exists n_0 \in \mathbb{N},\ n_0 \leq 2^{n_0}) \wedge (\forall n \in \mathbb{N},\ n+1 \leq 2^{n+1})$

$0 \leq 2^0 \wedge (\forall k \in \mathbb{N}, k \leq 2^k \Rightarrow k+1 \leq 2^{k+1})$

b) Base Case.

Let $n_0 = 0$

$0 \leq 2^0$

$0 \leq 1$

Induction Step Let $k \in \mathbb{N}$

Assume $k \leq 2^k$

WTS $k+1 \leq 2^{k+1}$

$k+1 \leq 2^k + 2^k$

$1 \leq 2^k$

$1 = 1$

$1 = 2^0$

$1 \leq 2^k$

$k+1 \leq 2^k + 2^k$

$k+1 \leq 2^{k+1}$

$0 \leq k$

$1 \leq 2^k$ (raising 2 to the power of either side)

□

2. Induction (Summations)

$\sum_{i=1}^{n} i$

$T_0 = 0$ $T_4 = 10$

$T_1 = 1$ $T_5 = 15$

$T_2 = 3$

$T_3 = 6$

Proof that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\forall n \in \mathbb{N},\ t_n = \frac{n(n+1)}{2}$ ← triangular numbers

Prove $\forall n \in \mathbb{N},\ \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6}$

Base Case $n=0$

$\sum_{j=0}^{0} T_j \Rightarrow T_0 = \frac{0(0+1)(0+2)}{6} = 0$

Induction $\forall k \in \mathbb{N},\ \sum_{j=0}^{n} T_j = \frac{n(n+1)(n+2)}{6} \Rightarrow \sum_{j=0}^{n+1} T_j = \frac{(n+1)(n+2)(n+3)}{6}$

$\frac{n(n+1)(n+2)}{6}$

$T_{n+1} + \sum_{j=0}^{n} T_j = T_{n+1} + \frac{n(n+1)(n+2)}{6}$

$= \frac{(n+1)(n+2)}{2} + \frac{n(n+1)(n+2)}{6}$

$= \frac{3(n+1)(n+2) + n(n+1)(n+2)}{6}$

$= \frac{(n+1)(n+2)(n+3)}{6}$ □

3. Induction (Inequalities)

$\forall x \in \mathbb{R}^+, \forall n \in \mathbb{N}, (1+x)^n \geq (1+nx)$

Let $x \in \mathbb{R}^+$

Prove $\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$

$((1+x)^0 \geq 1+0x) \land \forall n \in \mathbb{N}, ((1+x)^n \geq 1+nx) \Rightarrow ((1+x)^{n+1} \geq 1+(n+1)x)$

Base Case: $n=0$

$(1+x)^0 \geq 1+0x$

$1 \geq 1$

Induction Step

Assume $(1+x)^n \geq 1+nx$

Pro tip: Always use the inequality with something else, multiple

Prove $(1+x)^{n+1} \geq 1+(n+1)x$

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x)\\ &\geq (1+x)^n \quad \text{since } x \in \mathbb{R}^+\\ &\geq 1+nx\end{aligned}$$

$$\begin{aligned}&\geq (1+nx)(1+x)\\ &= 1+nx+x+nx^2\\ &\geq 1+nx+x \quad \text{since } nx^2 \geq 0\\ &= 1+(n+1)x\end{aligned}$$

$\square$

It's a good idea to use k instead of n.

Extra practice

1. Would statement still be True w/ order of quantifiers reversed

$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx$

x isn't defined

2. If correct, how would it change the proof

Prove $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx$

Base Case $n=0$ Let $x \in \mathbb{R}^+$

$(1+x)^0 \geq 1+0x$

$1 \geq 1$

Inductive step: $P(n): \forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx$

$\Rightarrow P(n+1): \forall x \in \mathbb{R}^+, (1+x)^{n+1} \geq 1+(n+1)x$

Assume $\forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx$

Prove $\forall x' \in \mathbb{R}^+, (1+x')^{n+1} \geq 1+(n+1)x'$

Let $x' \in \mathbb{R}^+$,

$(1+x')^n \geq 1+(n+1)x'$

$(1+x')\cdot(1+x')^n \geq 1+nx'+x'$

$\geq$

$(1+x')^n(1+x') \geq (1+nx')(1+x')$

$= 1+nx'+x'+nx'^2$

$\geq 1+nx'+x'$

$P(n): \underline{\forall x \in \mathbb{R}^+, (1+x)^n \geq 1+nx}$

It has to be true for every x_2

4. Changing the starting number

$\forall n \in \mathbb{N},\ n \geq 8 \Rightarrow 30n \leq 2^n$

Base Case: $n = 8$

$240 \leq 256$

Induction Assume $30n \leq 2^n$

prove $30(n+1) \leq 2^{n+1}$

$30n + 30 \leq 2^n + 30$

$\leq 2^n + 2^n$ since $n \geq 8$

$= 2^{n+1}$

$8 \leq k$

$256 \leq 2^k$ ← raising 2 to the power of either side

$30k \leq 2^k$

$30k + 256 \leq 2^k + 2^k$ Adding inequalities

$30k + 30 \leq 2^{k+1}$ (Since $30 \leq 256$)

$30(k+1) \leq 2^{k+1}$

WS9 Induction on Sets

1. Insight: every set has a single corresponding natural number, its size

$\forall n \in \mathbb{N}$, every set of size n has $\frac{n(n-1)}{2}$ subsets of size 2

P(0) → every set S of size 0 has 0 subsets of size 2.

b.) then S has $\frac{0(0-1)}{2} = 0$ subsets of size 2.

c.) Induction. Assume P(n), prove P(n+1)

Every set S of size n+1 has $\frac{n(n+1)}{2}$ subsets of size 2.

Key idea: set of size k+1, and split it into a single element and another set of size k.

let $S' = \{s_1, s_2, \dots, s_k\}$, so that $S = S' \cup \{s_{k+1}\}$

↑ size k ← special element

Part 1. Subsets of S of size 2 that contain s_{k+1}

Every subset of S of size 2 must contain exactly 1 element from S'.

There are k choices of elements in S' since $|S'| = k$

so k subsets of S of size 2 that contain s_{k+1}

Part 2. Subsets of S of size 2 that don't contain s_{k+1}

every set S of size k has $\frac{k(k-1)}{2}$ subsets of size 2.

every subset of S of size must contain 2 elements from $\{s_1, \dots, s_k\} = S'$

Since S' has size k, the IH tells us that S' has exactly $\frac{k(k-1)}{2}$ subsets of size 2.

← only works on sets of size k.

The total number of subsets of S of size 2 is

$$\frac{k(k-1)}{2} + k$$

$$= \frac{k(k-1)+2k}{2} = \frac{k^2+k}{2} = \frac{k(k+1)}{2}$$ ⇒ which was what we wanted to show.

2. Every finite set S has exactly $\frac{|S|(|S|-1)(|S|-2)}{6}$ subsets of size 3.

$P(n)$: $\forall n \in \mathbb{N}$, every set S of size n has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3.

$P(0)$. Let $n=0$

$$\frac{0(0-1)(0-2)}{6} = 0$$

IH : Assume that every set of S of size n has exactly $\frac{n(n-1)(n-2)}{6}$ subsets of size 3.
prove $\frac{n(n+1)(n-1)}{6}$ subsets.

subsets that contain s_{k+1}
- contain exactly 2 elements from $S' = \{s_1, \dots, s_k\}$
$\Rightarrow$ we know from fact that
every set S' of size k has exactly $\frac{k(k-1)}{2}$ subsets of size 2

so there are $\frac{k(k-1)}{2}$ choices of elements from S'.
thus, there are $\frac{k(k-1)}{2}$ subsets of size 3 that contain s_{k+1}.

subsets that don't contain s_{k+1}
- must contain 3 elements from S'.
By IH, since S' has k elements,
it has exactly $\frac{k(k-1)(k-2)}{6}$ subsets of size 3.

adding $\frac{k(k-1)}{2} + \frac{k(k-1)(k-2)}{6}$

$$= \frac{3k(k-1) + k(k-1)(k-2)}{6} = \frac{k(3k-3+k^2-3k+2)}{6}$$

$$= \frac{k(k^2-1^2)}{6}$$

$$= \frac{k(k+1)(k-1)}{6} \quad \text{by difference of squares}$$

$\square$

3. Counting All Subsets

"{}" is a subset of every set, and every set is also a subset of itself.

$\forall n \in \mathbb{N}$, every set S of size n satisfies $|P(n)| = 2^n$

a) Intuition

Subsets that contain 3:

$\{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Subsets that don't contain 3:

$\{\{\}, \{1\}, \{2\}, \{1,2\}\}$

- They both contain 4 elements.

$P(n)$: $\forall n \in \mathbb{N}$, every set S of size n satisfies $|P(S)| = 2^n$

Base Case $n = 0$

$|P(S)| = 2^0 = 1$

Induction Assume $|P(S)| = 2^n$ given any set S of size n

prove $|P(S)| = 2^{n+1}$

Subsets that contain s_{n+1}

we know that the number of elements that contain the number is half of the total subsets of S. $\frac{2^{n+1}}{2} = 2^n$.

subsets that do not contain s_{n+1}

by IH since our set has n elements,

$|P(S)| = 2^n$

Adding

$2^n + 2^n = 2^{n+1}$

WS9 Practice

1a. $P(0)$: every set S of size 0 has 0 subsets of size 2

$P(0)$: Let $n=0$, S is arbitrary, $|S|=0$

since $|S|=0$, $S=\emptyset$

thereby having no subsets of size 2.

Induction: Let $k \in \mathbb{N}$, assume $P(k)$

$P(k)$: every set S of size k has $\frac{k(k-1)}{2}$ subsets of size 2

$P(k+1)$: every set S of size $k+1$ has $\frac{(k+1)k}{2}$ subsets of size 2

Subsets of S of size 2 w/ s_{k+1}

k subsets since s_{k+1} must be an elem

$\binom{k}{1} = k$

Subsets of S of size 2 w/o s_{k+1}

$\frac{k(k-1)}{2}$ subsets

(

Exercise Break
3.8
Let $n \in \mathbb{N}$, and assume $n \geq 1$

Odd(x): if x shook w/ $2j+1$ people for some $j \in \mathbb{N}$
Even(x): if x shook w/ $2j$ people for some $j \in \mathbb{N}$.
Let $O = \{ x \in \mathbb{N} \mid x \leq n$ and Odd(x)$\}$
Let $E = \{ x \in \mathbb{N} \mid x \leq n$ and Even(x)$\}$
Prove $|O| = 2k$ for $k \in \mathbb{N}$.

Base Case: $n = 1$
1 shook w/ 0 people
Even(1) = 1 shook w/ 0 people
0 people w/ odd parity.
0 is even

Induction Assume P(K) where # of people w/ odd parity is even for k people

$2 \equiv 4 \mod 2$

Prove that for k+1 people, # of people w/ odd parity is even.

adding a handshake requires 2 people

/ count of Odd(x) = 2
count of even(x) = 0

/. count of odd(x) = 2
count of even(x) = 1

Δ count of Odd(x) = 0
count of even(x) = 3

∠ count of odd(x) = 2
count of even(x) = 1

WS10 Representing Natural Numbers

a) base 8

$$x = \sum_{i=0}^{k-1} d_i \cdot 8^i$$

$$x = (1 \cdot 8^0) + (1 \cdot 8^1)$$
$$= 1 + 8$$
$$= 9$$

$$(165)_8 = 1 \cdot 8^2 + 6 \; 8^1 + 5 \cdot 8^0$$
$$= 64 + 48 + 5$$
$$= 117$$

base-16

$$(B4)_{16} = (11 \;\; 4)_{16}$$
$$= 11 \cdot 16^1 + 4 \cdot 16^0$$
$$= 176 + 4$$
$$= 180$$

2. Converting between base

← Why are bits remainder.

a.

$357 \div 2 = 178 \quad 1$
$178 \div 2 = 89 \quad 0$
$89 \div 2 = 44 \quad 1$
$44 \div 2 = 22 \quad 0$
$22 \div 2 = 11 \quad 0$
$11 \div 2 = 5 \quad 1$
$5 \div 2 = 2 \quad 1$
$2 \div 2 = 1 \quad 0$
$1 \div 2 = 0 \quad 1$

$$(101100101)_2 = (545)_8 = (165)_{16}$$

b.) $8 = 2^3$ ⇒ every 3 binary bits correspond to one octal digit

$$2^0 + 2^1 + 2^2$$

$$n = \sum_{i=0}^{n} (a_i \cdot 8^i)$$

$$(76)_8 = (7 \cdot 8^1) + (6 \cdot 8^0)$$
$$= ((2^2 + 2^1 + 2^0) \cdot 2^3) + ((2^2 + 2^1 + 0(2^0)) \cdot (2^3)^0)$$
$$= (2^5 + 2^4 + 2^3 + 2^2 + 2^1)$$

3. Representing Fractions

a) $0.375 \times 2 = 0.75 + 0$
$0.75 \times 2 = 0.5 + 1$
$0.5 \times 2 = 0 + 1$

$(.011)_2 = (.375)_{10}$

$= \sum_{i=1}^{k} d_i \cdot \frac{1}{b^i} = 0 \cdot \frac{1}{2^1} + 1 \cdot \frac{1}{2^2} + 1 \cdot \frac{1}{2^3}$

$= \frac{1}{4} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} = \frac{3}{8} = 0.375$

b) $(0.1)_{10} = (.0\overline{0011})_2$

$0.1 \times 2 = 0.2 + 0$
$0.2 \times 2 = 0.4 + 0$
$0.4 \times 2 = 0.8 + 0$
$0.8 \times 2 = 0.6 + 1$
$0.6 \times 2 = 0.2 + 1$
$0.2 \times 2 = 0.4 + 0$

4. Infinite geometric series

$\sum_{i=1}^{\infty} 1 \cdot \frac{1}{2^i} = \frac{1 \cdot \frac{1}{2}}{1 - \frac{1}{2}} = \frac{1/2}{1/2} = 1$

$\left(0 \cdot \frac{1}{2^1}\right) + \left(0 \cdot \frac{1}{2^2}\right) + \left(0 \cdot \frac{1}{2^3}\right) + \left(1 \cdot \frac{1}{2^4}\right) + \left(1 \cdot \frac{1}{2^5}\right)$

$\sum_{i=1}^{\infty} \left(\frac{1}{2^0} + \frac{1}{2^1}\right) \frac{1}{(2^4)^i}$

$\sum_{i=1}^{\infty} \frac{3}{2} \left(\frac{1}{2^4}\right)^i = \frac{\frac{3}{2}\left(\frac{1}{2^4}\right)}{1 - \left(\frac{1}{2^4}\right)} = \frac{\frac{3}{2}\left(1/2^4\right)}{\frac{1}{2^4}\left(\frac{1}{1/2^4} - 1\right)}$

$= \frac{3/2}{2^4 - 1} = \frac{3/2}{15}$

$= \frac{3}{30} = \frac{1}{10}$ ✓

WS11

$\forall a,b \in \mathbb{R}^+, a \le b \Rightarrow n^a \in O(n^b)$ generalizes that $n \in O(n^2)$

$\forall a,b \in \mathbb{R}^+, a \le b \Rightarrow [\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^a \le c n^b)$ where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Let $a, b \in \mathbb{R}^+$.

Assume $a \le b$ why is this implication true?

Prove $n^a \in O(n^b)$ $a \Rightarrow b \Rightarrow c$

Let $c = 1$

$n_0 = 1$

Let $n \in \mathbb{N}$, Assume $n \geq n_0$

$a \le b$

$n^a \le n^b$ (since $n \geq 1$)

if $n < 1 \Rightarrow n = 0$

which wouldn't be useful

$n^a \le c n^b$ (since $c = 1$) f is an upper bound for g

2. Comparing Logarithms

$\forall a,b \in \mathbb{R}^+, a > 1 \land b > 1 \Rightarrow [\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \log_a n \le c \log_b n)$

Let $a, b \in \mathbb{R}^+$, Assume $a > 1 \land b > 1$.

Let $c = \frac{1}{\log_b a}$ $\forall a,b,x \in \mathbb{R}^+, a \neq 1 \land b \neq 1 \Rightarrow \log_a x = \frac{\log_b x}{\log_b a}$ — only valid when hypothesis is true

Let $n_0 = 1$ when do we not need to use assumptions

$$\log_a n = \frac{\log_b n}{\log_b a}$$

$$\log_a n = \frac{1}{\log_b a} \log_b n$$

$$\log_a n = c \log_b n$$

$\Rightarrow$ why can we conclude that $\log_a n \le c \log_b n$? is it because of the 'or'?

3. Sum of functions

$f + g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, $\forall n \in \mathbb{N}, (f+g)(n) = f(n) + g(n)$

Induction.

$\forall n \in \mathbb{N}, n \geq 1, g_1 + \dots + g_n \in O(f)$

$P = \bigcup_{i=1,\infty} O(n^i)$

$\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, g \in O(f) \Rightarrow f + g \in O(f)$

Assume $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \le c f(n)$

Prove $\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow (f+g)(n) \le c_1 f(n)$

Let $n_1 = n_0$ $\Rightarrow$ $n \geq n_1 = n_0$ by assumption we know,

Let $c_1 = c + 1$ $g(n) \le c f(n)$

$\Rightarrow g(n) + f(n) \le c f(n) + f(n)$

$\Rightarrow g(n) + f(n) \le (c+1) f(n)$

$\Rightarrow g(n) + f(n) \le c_1 f(n)$

$\square$

W12 (Why are we mapping $\mathbb{N} \to \mathbb{R}^{\geq 0}$)

(1) Constant functions

Practice $f(n) = 100$

a) $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ $\quad g \in O(1)$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq c$

b) $100 + \frac{77}{n+1} \in O(1)$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow 100 + \frac{77}{n+1} \leq c$

$n_0 = 76$ $\quad n \geq 76$

$c = 101$ $\quad n \geq \frac{77}{1} - 1$

$n \geq \frac{77}{101-100} - 1$

$n+1 \geq \frac{77}{c-100}$

$\frac{n+1}{77} \geq \frac{1}{c-100}$

$\forall x, y \in \mathbb{R}^+, \; x \geq y \Rightarrow \frac{1}{x} \leq \frac{1}{y}$ where $x = \frac{n+1}{77}$ since $n \in \mathbb{N}$

$\frac{77}{n+1} \leq c - 100$ $\quad y = \frac{1}{c-100}$ $\quad c = 101$

2. Omega $\quad g \in \Omega(f) : \exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \geq c f(n)$

$f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ if $g \in O(f)$, then $f \in \Omega(g)$

Assume $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \leq c f(n)$

Prove $\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_1 \Rightarrow f(n) \geq c_1 g(n)$

Let $n_1 = n_0$ $\quad$ we know $\Rightarrow$ $\quad n \geq n_0$

$c_1 = \frac{1}{c}$

$\Rightarrow g(n) \leq c f(n)$

$\Rightarrow g(n) \frac{1}{c} \leq f(n)$ since $c \in \mathbb{R}^+$

$\Rightarrow g(n) c_1 \leq f(n)$

$\square$

→ Practice

(3) Theta $\quad g \in \Theta(f) : g \in O(f) \land g \in \Omega(f)$ where $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$g \in \Theta(f) : \exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow c_1 f(n) \leq g(n) \leq c_2 f(n)$

Prove for all $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and all $a \in \mathbb{R}^{\geq 0}$, $\quad g \in \Omega(1) \Rightarrow a + g \in \Theta(g)$

Assume $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_0 \Rightarrow g(n) \geq c$

Prove $\exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \; n \geq n_1 \Rightarrow c_1 g(n) \leq a + g(n) \leq c_2 g(n)$

$n \geq n_0$ $\quad g(n) = g(n)$

Let $n_1 = n_0$ $\quad g(n) \geq c$ $\quad g(n) \leq g(n)$

$c_1 = 1$ $\quad \Rightarrow a g(n) \geq ac$ $\quad g(n) \leq g(n) + a$ since $a \in \mathbb{R}^{\geq 0}$

$c_2 = \frac{a}{c} + 1$ $\quad a g(n) \geq a$ $\quad c_1 g(n) \leq g(n) + a$ since $c_1 = 1$

$\frac{a}{c} g(n) \geq a$

$\frac{a}{c} g(n) + g(n) \geq a + g(n)$ $\quad a + g(n) \leq c_2 g(n)$

$(\frac{a}{c} + 1) g(n) \geq a + g(n)$ $\quad a \leq g(n)(c_2 - 1)$

$c_2 g(n) \geq a + g(n)$

WS12

3) $\forall g: \mathbb{N} \to \mathbb{R}^{\geq 0}, \forall a \in \mathbb{R}^{\geq 0}, \quad g \in \Omega(1) \Rightarrow a + g \in \Theta(g)$

Let $g: \mathbb{N} \to \mathbb{R}^{\geq 0}$

$a \in \mathbb{R}^{\geq 0}$

Assume $g \in \Omega(1)$: $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow g(n) \geq c$

$a+g \in \Theta(g)$: $\exists c_1, c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ n \geq n_1 \Rightarrow c_1 g(n) \leq a + g(n) \leq c_2 g(n)$

$n_1 = n_0$

$c_1 = \frac{a}{c}$

$g(n) \geq c$

$\frac{1}{g(n)} \leq \frac{1}{c}$

$c_1 g(n) \leq a + g(n)$

$(c_1 - 1) g(n) \leq a$

$c_1 - 1 \leq \frac{a}{g(n)}$

$c_1 \leq \frac{a}{g(n)} + 1$

$a + g(n) \leq c_2 g(n)$

$\frac{a+c}{c} \leq \frac{a+g(n)}{g(n)} \leq c_2$

$\frac{a}{c} + 1 \leq c_2$

4. Negating Big-O

a) $g \notin O(f)$

$\neg(\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq cf(n))$

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land g(n) > cf(n)$

b) Prove $\forall a, b \in \mathbb{R}^+, a > b \Rightarrow n^a \notin O(n^b)$

Let $a, b \in \mathbb{R}$. Assume $a > b$

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land n^a > cn^b$

Let $c, n_0 \in \mathbb{R}$

$n = \max(n_0, \lceil c^{1/a-b} \rceil + 1)$ $\quad a > b$

Then $n \geq n_0$

Choice for n

$n^a > cn^b$

$n^{a-b} > c$

$n > c^{1/a-b}$

WS13

1.) loop variations

```
def f1(n: int) -> None:
    i = 0
    while i < n:
        print(i)
        i = i + 5
```

$\mathbb{Z}^{\geq 0}$ Input

$\sum_{k=0}^{n/5} 5$

$\lceil \frac{n}{5} \rceil$ iterations

1. $0 < 13$
2. $5 < 13$
3. $10 < 13$
   $15 < 13$

$0 + 5 + 5 + 5 + 5 \ldots$

$\underbrace{0 < 5}_{+1} \underbrace{< 10}_{+1} \underbrace{< 13 < 15}_{+1}$

$\Theta(n)$

2.)

```
def f2(n: int) -> None:
    i = 4
    while i < n:
        print(i)
        i = i + 1
```

When proving theta bound what values of n do you care about?

$(4 + 1 + 1 + \cdots + 1)$ n times

$\sum_{k=4}^{n} 1$

$n - 4$ times $\Theta(n)$

why $\max(n-4, 0)$ loop iterations

3.)

```
def f3(n: int) -> None:
    i = 0
    while i < n:
        print(i)
        i = i + (n // 10)
```

$\sum_{k=0}^{n} \frac{n}{10}$

$\frac{n}{10} + \frac{n}{10} + \frac{n}{10}$

$i_k = k\left(\frac{n}{10}\right)$

$i_k \geq n$

$k\left(\frac{n}{10}\right) \geq n$

$k \geq 10$

exactly 10 loop iterations why?

4.)

```
def f4(n: int) -> None:
    i = 20
    while i < n * n:
        print(i)
        i = i + 3
```

$\sum_{k=20}^{\lceil \frac{n^2}{3} \rceil} 3$

$\underbrace{0 < 3}_{1} \underbrace{< 6}_{2} \underbrace{< 9}_{3} \underbrace{< 12}_{4} \underbrace{< 15}_{5} \underbrace{< 18}_{6}$

$\left\lceil \frac{n^2 - 20}{3} \right\rceil$

why $\max\left(\left\lceil \frac{n^2 - 20}{3} \right\rceil, 0\right)$

$\Theta(n^2)$

↳ we care about values ≥ 0.

5.) def f5(n: int) -> None:

```
i = 20
while i < n * n:
    print(i)
    i = i + 3
```

$\Theta(n^2)$

```
j = 0
while j < n:
    print(j)
    j = j + 0.01
```

$100n \Rightarrow \Theta(n)$

since $n \in O(n^2)$

$\Rightarrow \Theta(n^2)$

consequence of 1 version of "sum" of Big O theorem

6.) Multiplicative Increments

```
def f(n: int) -> None:
    i = 1
    while i < n:
        print(i)
        i = i * 2
```

$\prod_{k=0}^{n/2} 2 \Rightarrow \sum_{k=0}^{n-1} 2^k$

$i_0 = 1, \quad i_1 = 2, \quad i_2 = 4, \quad \ldots, \quad i_k = 2^k$

$i < n$

$i_k \geq n$ (false

$2^k \geq n$

$k \log_2 2 \geq \log_2 n$

$k \geq \log_2 n$

the smallest value k can be is $\lceil \log_2 n \rceil$

if k was $\lfloor \log_2 n \rfloor \Rightarrow \lfloor \log_2 n \rfloor \leq \log_2 n$.

* $\forall n \in \mathbb{Z}, \forall x \in \mathbb{R}, n \geq x \Rightarrow n \geq \lceil x \rceil$

c) $\Theta(\log_2 n)$

Why not initialize $i = 0$?

d) since it would run in constant time $\Theta(1)$

3.) def f(n : int) → None :

    i = 2
    while i < n :
        print(i)
        i = i * i

general formula for i_k

$i_0 = 2$, $i_1 = 4$, $i_2 = 16$, $i_3 = 256$

2^1 2^2 2^4 2^8

$i_k = 2^{2^k}$

$i < n$

$\Rightarrow i_k \geq n$ (false)

$2^{2^k} \geq n$

$2^k \log_2 2 \geq \log_2 n$

$k \log_2 2 \geq \log_2(\log_2 n)$

$k \geq \log_2(\log_2 n)$

loop runs $\lceil \log_2(\log_2 n) \rceil$ before it stops

$\Theta(\log_2(\log_2 n))$

WS14

1. Nested Loop Variations

a) f1(n: int) → None.

```
i = 0
while i < n :
    j = 0
    while j < n:
        j = j + 1
    i = i + 5
```

general formula: $i_k = 1$
$k \geq 1$

$i_k = k$
$k \geq n$

$i_k = 5k$
$5k \geq n$
$k \geq \lceil \frac{n}{5} \rceil \geq \frac{n}{5}$ $\Rightarrow$ $\lceil \frac{n}{5} \rceil$ iterations of outer loop.

$$\sum_{i=0}^{\lceil \frac{n}{5} \rceil - 1} n = n \sum_{i=0}^{\lceil \frac{n}{5} \rceil - 1} 1$$
$$= n \cdot \lceil \tfrac{n}{5} \rceil \in \Theta(n^2)$$

b) def f2(n: int) → None:

```
i = 4
while i < n:
    j = 1
    while j < n:
        j = j * 3
    k = 0
    while k < n:
        k = k + 2
    i = i + 1
```

Loop 2: $i_0 = 1$, $i_1 = 3$, $i_2 = 9$ $(k \in \mathbb{N})$ $n = 10$
$i_k = 3^k$
$3^k \geq n$
$k \geq \lceil \log_3 n \rceil \geq \log_3 n$ ← number of iterations

Loop 3: $i_0 = 0$, $i_1 = 2$, $i_2 = 4$, $i_3 = 6$
$i_k = 2k$
$2k \geq n$
property of ceil
$k \geq \frac{n}{2}$ $\Rightarrow$ $k \geq \lceil \frac{n}{2} \rceil$

Loop 1: $i_0 = 4$, $i_1 = 5$, $i_2 = 6$ k counts the iterations
$i_k = k + 4$ for some $k \in \mathbb{N}$
$k + 4 \geq n$
$k \geq n - 4$

$k \in \mathbb{N} \Rightarrow k \geq 0$ $\Rightarrow$ choose $\max(n-4, 0)$ → we don't care about values (-1, -2, -3, -4).

cost of inner loop is fixed for each outer loop iteration.
$$\max(n-4, 0) \cdot (\lceil \log_3 n \rceil + \lceil \tfrac{n}{2} \rceil)$$

expanding, $\lceil \log_3 n \rceil (n-4) + \lceil \frac{n}{2} \rceil (n-4)$ or 0
$= \lceil \log_3 n \rceil n - \lceil \log_3 n \rceil 4 + \lceil \frac{n}{2} \rceil n - \lceil \frac{n}{2} \rceil 4$

Theta theorem (sum law)
$n \log_3 n \in \Theta(n^2)$

$n \log_3 n \in \Theta(n^2)$
$n \in \Theta(n^2)$

c) def f3(n: int) → None:

```
i=0
while i<n:
    j=n
    while j>0:
        k=0
        while k<j:
            k = k+1
        j=j-1
    i=i+4
```

Loop 3:

$i_0 = 0,\ i_1 = 1$

$i_k = k$

$i_k < j$ $\qquad k<j$

False when $i_k \geq j$ $\qquad$ False when $k \geq j$

↳ j iterations

Loop 2: $i_0 = n,\ i_1 = n-1$

$i_{k_0} = n - k_0$

$n - k_0 > 0$

False when $n - k_0 \leq 0$

$n \leq k_0$

Loop 1:

$i_0 = 0,\ i_1 = 4$

$i_k = 4k$

False when $4k \geq n$

$k \geq \frac{n}{4}$

$k \geq \lceil \frac{n}{4} \rceil \geq \frac{n}{4}$ ← ceil property

$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ as cost depends on j

$\lceil \frac{n}{4} \rceil \cdot \frac{n(n+1)}{2} \in \Theta(n^3)$

d) def f4(n: int) → None:

```
i=1
while i<n:
    j=0
    while j<i:
        j=j+1
    i=i*2
```

Loop 2: $i_0 = 0,\ i_1 = 1$

$i_j = j < i$

False when $j > i$

Loop 1: $i_0 = 1,\ i_1 = 2,\ i_2 = 4$

$i_k = 2^k$

$2^k < n$

False when $2^k \geq n$

$\log_2 2^k \geq \log_2 n$

$k \geq \log_2 n$

$k \geq \lceil \log_2 n \rceil \geq \log_2 n$ $\Rightarrow$ $\sum_{i=0}^{\lceil \log_2 n \rceil - 1} 2^i = 2^{\lceil \log_2 n \rceil - 1}$

inner loop runs 1 cost per fixed iteration of loop i times

so outer loop depends on i

we know outer loop runs from 1 to $\lceil \log_2 n \rceil - 1$ which is exactly $\lceil \log_2 n \rceil$ iterations since our general formula is $i_k = 2^k$.

Extra: $\exists c_1, c_2, n_0 \in \mathbb{R}^+,\ \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow c_1 n \leq 2^{\lceil \log_2 n \rceil} \leq c_2 n$

$n_0 = 1$

$c_1 = 1$

$n = n$

$n \leq n$

$0 \leq \lceil x \rceil - x < 1$

$x \leq \lceil x \rceil$

$\log_2 n \leq \log_2 n \leq \lceil \log_2 n \rceil$ By fact

$\log_2 c_1 n \leq \lceil \log_2 n \rceil$

$c_1 n \leq 2^{\lceil \log_2 n \rceil}$

$n = n$

$n \leq n$

$1 + \log_2 n \leq 1 + \log_2 n$

$\lceil \log_2 n \rceil \leq 1 + \log_2 n$

$\lceil \log_2 n \rceil \leq \log_2 2 + \log_2 n$

$\lceil \log_2 n \rceil \leq \log_2 2n$

$2^{\lceil \log_2 n \rceil} \leq 2n$

$2^{\lceil \log_2 n \rceil} \leq c_2 n$

$\lceil x \rceil < 1 + x$ $\qquad c_2 = 2$

By fact (since $\lceil \log_2 n \rceil < 1 + \log_2 n$)

```
def max_subsequence_sum(lst: List[int]) -> int
    n = len(lst)
    max_so_far = 0
    for i in range(n):
        for j in range(i, n):
            s = 0
            for k in range(i, j + 1):
                s = s + lst[k]
            if max_so_far < s
                max_so_far = s
    return max_so_far
```

$k = i$

$k \geq j$

$i_k = i$

$i < j+1$

$i \geq j+1$

Loop 3: $\sum_{k=i}^{j} lst[k]$

$\sum_{k'=0}^{j-i} lst[k'+i] \Rightarrow j-i+1$ times

$= lst[i] + \cdots + lst[j]$

Loop 2: $\sum_{j=i}^{n-1} (j-i+1)$ $n-i$ times

$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \sum_{k=0}^{j-i} k$ n times

$$= \sum_{j'=1}^{n-i} j' = \frac{(n-i)(n-i+1)}{2}$$

↳ index is not 0 here

$$= \frac{n^2 - in + n - in + i^2 - i}{2} = \frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{2}i^2 - \frac{1}{2}i - in$$

$$= \left(\frac{1}{2}n^2 + \frac{1}{2}n\right) + \frac{1}{2}i^2 - \left(\frac{1}{2} + n\right)i$$

Loop 1: $\sum_{i=0}^{n-1}\left(\frac{1}{2}i^2 - \left(\frac{1}{2}+n\right)i + \left(\frac{n^2}{2} + \frac{n}{2}\right)\right)$

$$= \sum_{i=0}^{n-1} \frac{1}{2}i^2 - \sum_{i=0}^{n-1}\left(\frac{1}{2}+n\right)i + \sum_{i=0}^{n-1}\left(\frac{n^2}{2}+\frac{n}{2}\right)$$

$$= \frac{1}{2}\sum_{i=0}^{n-1} i^2 - \left(\frac{1}{2}+n\right)\sum_{i=0}^{n-1} i + \left(\frac{n^2}{2}+\frac{n}{2}\right)\sum_{i=0}^{n-1} 1$$

$$= \frac{1}{2}\cdot\frac{(n-1)(n)(2n-1)}{6} - \left(\frac{1}{2}+n\right)\frac{n(n-1)}{2} + \left(\frac{n^2}{2}+\frac{n}{2}\right)n$$

$$= \frac{(n-1)(2n-1)n}{12} - \frac{n(n-1)}{4} - \frac{n^2(n-1)}{2} + \frac{n^3}{2} + \frac{n^2}{2}$$

$$= \frac{(n-1)(2n-1)n}{12} - \frac{3(n-1)(1+2n)n}{12} + \frac{n^3}{2} + \frac{n^2}{2}$$

$$\frac{(n-1)n(2n-1-3-6n))}{12} + \frac{n^3}{2} + \frac{n^2}{2}$$

$$-\frac{(n-1)n(n+1)}{3} + \frac{n^3}{2} + \frac{n^2}{2}$$

$$-\frac{(n^3-n)}{3} + \frac{n^3}{2} + \frac{n^2}{2}$$

$$\frac{-2n^3+2n + 3n^3+3n^2}{6} = \frac{n^3}{6} + \frac{3n^2}{6} + \frac{2n}{6} = \frac{n^3}{6} + \frac{n^2}{6} + \frac{n}{3} \in \Theta(n^3)$$

Rep 2

$(p \Rightarrow q) \Leftrightarrow (q \vee r)$

F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	F
T	T	F	T
T	T	T	T

$p \Leftrightarrow q$

F	F	T
T	F	F
F	T	F
T	T	T

Q2)

p	q	r	$(p \Rightarrow q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
F	F	F	F	T

Rep 6

26	$\div$	2 =	13	0	
13	$\div$	2 =	6	1	
6	$\div$	2 =	3	0	
3	$\div$	2 =	1	1	
1	$\div$	2 =	0	1	

I don't understand this algorithm relating to binary sum $\sum_{i=0}^{n-1} d_i b^i$

$2 + 8 + 16 = 10 + 16 = 26$

I don't understand

$\forall k \in \mathbb{Z} \quad P(k) \Rightarrow P(k-1)$

is all integers $n \leq 0$.

Rep 7

$f(n) = n^3 - n^2 + n$
$g(n) = n^3/165$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow g(n) \leq c \cdot f(n)$

$$\frac{1}{165} n^3 \leq c \cdot f(n)$$

Lec 1

- About problem solving
  - → techniques
    - → Why?
      - → Apply to solve new problems
      - → Apply to something you haven't seen before
  - → Aim d in the right frame of mind

Online learning
- extra hassle to go to class for the purpose of class
- home is doing other things.

Review - terminology & Notation

CSC165 → express ourselves precisely
attention to detail

vagueness → priming

details → refining

Sets:
1. collection of elements
2. an object of its own

- unordered, no duplicates

$\underbrace{\{3,1,3,2,1,3\}}_{\text{diff. presentation}} = \{1,2,3\}$

$\{x \mid \underbrace{x \in \mathbb{Z} \text{ and } 1 \le x \le 3}_{\text{property}}\} = \{1,2,3\}$

x ↓ variable for individual elements

$\mid$ → such that

↳ the same set b/c what are its elements

every elem x s.t. $x \in \mathbb{Z}$ and $1 \le x \le 3$

- Empty set : $\{\}$ represented by $\emptyset$.
  ↳ same as nothing? No

$\emptyset$ is an elem. that contains nothing

$\emptyset$ =  ← empty box

array w/ assigned value = none.
als 0 pointers?

Operations on sets

- Size : $|A|$ = number of elements in set A.

$|A|$ = size (← set)

$|x|$ = abs (↑ value)

$|\emptyset| = 0$

$|\{\emptyset\}| = 1$

$\{1,2,3,1\}$ is a set $= \{1,2,3\}$

set w/ duplicates in math has no effect (ignored) in CS returns an error.

↦ misleading

$|\{3,1,2,1,1,1,3\}| = 3$

- $x \in A$: x is an element of A.

$3 \in \{1,2,3\}$

$3 \notin \{\text{'hi'}, \{1,3\}, \emptyset\} = \{\text{'hi'}, A, B\}$ ($A = \{1,3\}$, $B = \emptyset$)

↳ does not recurse into subsets

- $A \subseteq B$: A is a subset of B

↳ every element of A is also an element in B.

$\{1,3\} \subseteq \{1,2,3\}$
- $1 \in \{1,2,3\}$
- $3 \in \{1,2,3\}$

$\{1,3\} \not\subseteq \{\text{hi}, \{1,3\}, \emptyset\}$

$\{\{1,3\}\} \subseteq \{\text{hi}, \{1,3\}, \emptyset\}$

- for all sets A, $A \subseteq A$
- for all sets A, $\emptyset \subseteq A$

$\{\emptyset, 1, 2, 3\} \not\subseteq \{1,2,3\}$

↳ an element that does not belong to A

↳ counterexample to $\subseteq$ operation.

↳ we can't find an element in $\emptyset$ that is not in A.

"proper subset" : $A \subset B$ and $A \neq B$

↑ $\subsetneq$ $\quad\subset$ – ambiguous

(not $\not\subseteq$: not subset of) $\Rightarrow \neg(A \subseteq B)$

$A \not\subseteq B$: $\{1,2\} \not\subseteq \{3,4\}$

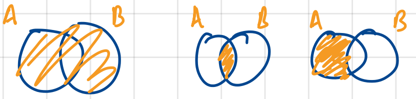
$A \subsetneq B$: $\{1,2\} \subsetneq \{1,2,3\}$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ (or both)

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

- all elems in A, remove $A \cap B$



- Power set

$P(A) = \{S \mid S \subseteq A\}$

all subsets of A.

$P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$|P(A)| = 2^{|P(A)|}$

$P(\emptyset) = \{\emptyset\}$

$P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

$P(\{x\}) = \{\emptyset, \{x\}\}$

$A = B$: $A \subseteq B$ and $B \subseteq A$

$\{3,1,2,1\} = \{3,1,2\}$

$\{3,1,2,1\} \subseteq \{3,1,2\}$

$\{3,1,2\} \subseteq \{3,1,2,1\}$

↳ every elem in $\{3,1,2,1\}$ is an elem in $\{3,1,2\}$

$A \times (B \times C) \neq (A \times B) \times C$

$A \times (B \times C) = A \times \{(b_1,c_1),(b_2,c_1),(b_1,c_2),(b_2,c_2)\}$
$= \{(a_1,(b_1,c_1)), (a_2,(b_1,c_1)),$
$(a_1,(b_2,c_1)), (a_2,(b_2,c_1)),$
$(a_1,(b_1,c_2)), (a_2,(b_1,c_2)),$
$(a_1,(b_2,c_2)), (a_2,(b_2,c_2))\}$

$(A \times B) \times C = \{(a_1,b_1),(a_2,b_1),(a_1,b_2),(a_2,b_2)\} \times C$
$= \{((a_1,b_1),c_1), ((a_1,b_1),c_2),$
$((a_2,b_1),c_1), ((a_2,b_1),c_2),$
$((a_1,b_2),c_1), ((a_1,b_2),c_2),$
$((a_2,b_2),c_1), ((a_2,b_2),c_2)\}$

Cartesian Product

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

pair with 1st member x
2nd member y

$\{1,2\} \times \{a,b,c\} = \{(1,a),(1,b),(1,c),(2,a),$
$(2,b),(2,c)\}$

$\{a,b,c\} \times \{1,2\} = \{(a,1),(a,2),(b,1),(b,2),(c,1),(c,2)\}$

$(1,a) \neq (a,1)$

$A_1 \neq A_2$

$\sum_{i=0}^{-1} i = 0$ (empty sum) $1 + 0 = 1$

$\prod_{i=0}^{-1} i = 1$ (empty product) $1(1) = 1$ → generalization of properties

we don't need to make exceptions → properties work with that value

→ identity element

$\{\emptyset, 1, 2\} \subseteq \{1, 2\}$?

A: $\emptyset$, 1, 2   B: 1, 2

$\emptyset \in A$ $\emptyset \subseteq A$

$\emptyset \notin B$ $\emptyset \subseteq B$

so $A \not\subseteq B$ since $A \not\subseteq B$ though $B \subseteq A$.

Vacuous truth

property that if there is an elem of that hypothesis satisfies, there is no elem that satisfies it so it is true by consequence.

$\boxed{\sum_{i=0}^{n-1}} 2i$ ↳ n things

$\underbrace{0 + 2 + \dots 2(n-1)}$

indexing always increments by 1

$|P(A)| = 2^{|A|}$

$\mathbb{R}^{\geq 0} = \{x \mid x \in \mathbb{R} \text{ and } x \geq 0\}$

$= \mathbb{R}^+ \cup \{0\}$

runtime = 3 times

$n = 2^3$

1. $1 < 8$, 2. $2 < 8$, 3. $4 < 8$

$n = 9$

1. $1 < 9$, 2. $2 < 9$, 3. $4 < 9$, 4. $8 < 9$, ~~5. 16 < 9~~

```
i=1
while i<n:
  j=0
  while j<i:   i times
    j=j+1
  i=i.2
```

non-inductive

$i = 2^0, 2^1, 2^2, \dots, 2^k, \dots, 2^{\log_2 n}$

stops when $2^k \geq n$ ← solve for k

$\lceil\ \rceil$ $\lfloor\ \rfloor$

$\log_2 2^k \geq \log_2 n \Rightarrow k \geq \log_2 n$

$k \log_2 2 \geq \log_2 n$

$k \geq 3$

$k \geq \log_2 9$

$k \geq 3.17$

```
j=0
while j<2
  j=j+1
```

1. $0 < 2$

2. $1 < 2$

2^k

$\boxed{\sum_{k=0}^{\lceil \log_2 n \rceil - 1} 2^k} = \underbrace{2^0 + 2^1 + 2^2 + \dots + 2^{\lceil \log_2 n \rceil - 1}}_{\text{formula}}$

$2^0 + 2^1 + 2^2 + \dots + 2^m = ?$ →

$\lg x = \underline{\log_2 x}$

Lec2

Functions, predicates, summation, product notation

Functions $f : A \to B$

Domain

codomain

name

"is a function"

must be one-to-one

Surjective : every element in domain must map to an element in codomain

- f maps elements of A to elements of B.
- for each $x \in A$, $f(x)$ denotes the image of x by f - the unique elem in B that corresponds to x

In practice : $f(x) = \dfrac{x^2}{x-3}$ domain : $\mathbb{R}$ codomain : $\mathbb{R}$

what about $x=3$? $f(3)$ is undefined

set difference

$f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R} \cup \{\text{undefined}\}$

$f : \mathbb{R} \to \mathbb{R}$

discrete math not continuous math!

predicates = function w/ codomain {True, False}

$P : A \to \{T, F\}$

eg. $P_1 : \mathbb{R} \to \{T, F\}$

$P(x) : x > 165$

$\exists$ goes with $\wedge$

$\forall$ goes with $\Rightarrow$

Predicates vs. sets (subsets)

A

$P : A \to \{T, F\}$

$\Downarrow$

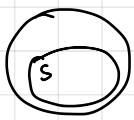
$S_p \subseteq A$ is the subset associated w/ P, defined as

$S_p = \{x \in A \mid P(x) = T\}$

Exercise:

- trace through this construction for an example

A



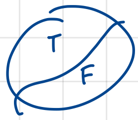
$S \subseteq A$

"characteristic predicate"

$P_s : A \to \{T, F\}$

$P_s(x) : x \in S = \begin{cases} T & \text{if } x \in S \\ F & \text{if } x \notin S \end{cases}$

Predicate $\equiv$ subset



subset is empty?

subset is = A ?

$P_0 \rightsquigarrow \underset{\subseteq A}{S_{P_0}} \rightsquigarrow P_{S_{P_0}} = P_0$ equivalent since its the exact same predicate

$\underbrace{\qquad}_{(T,F)}$ $\parallel$

$S_0 \rightsquigarrow P_{S_0}$

commas can be seen as conjunction

Summation / Product Notation

Ex: $\underbrace{\frac{2^2}{2-1} + \frac{3^2}{3-1} + \frac{4^2}{4-1} + \underbrace{\cdots}_{\text{pattern}} + \frac{165^2}{165-1}}_{\text{general term?}}$

$\sum_{i=2}^{165} \frac{i^2}{i-1} \leftarrow \frac{i^2}{i-1}$

165 → ending

2 → starting value = first

Summations are inclusive, limited

Q1) $\sum_{i=2}^{2} \frac{i^2}{i-1} = \frac{2^2}{2-1} = 4$ = just one term

step is always +1

Q2) $\sum_{j=3}^{2} \frac{j^3}{j-1} = \begin{cases} \frac{3^2}{3-1} + \frac{2^2}{2-1} & \text{X step is always +1} \\ \text{error / undefined} & \text{- breaks properties} \\ 0 & \checkmark \end{cases}$

"summation variable"

empty range for j

must be $\in \mathbb{Z}$

0 is identity for + : $\underbrace{0 + x = x}$

$\prod_{j=a}^{b} \text{expression}(j) = \text{expr.}(a) \times \underbrace{\text{expr.}(a+1)}_{\text{up by } 1} \times \cdots \times \text{expr.}(b)$

Q : $\prod_{j=3}^{1} j^3 = \underbrace{1}_{\text{identity for multiplication}}$

WRITE IT OUT !

propositional logic

Def: proposition = any statement that is True or False

1. it is sunny outside
2. there is intelligent life on Jupiter → can't tell → it is either
3. how are you?
4. Ike likes tomatoes.

Exercise Break

1.2) Tauto

de morgan

p	q	r	$((p \Rightarrow q) \wedge (p \Rightarrow r)) \Leftrightarrow (p \Rightarrow (q \wedge r))$					$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$			$(\neg(p \vee q)) \Leftrightarrow (\neg p \wedge \neg q)$	
T	T	T	T	T	T	T	T	T	T	T	F	F
T	T	F	T	F	F	F	T	T	T	T	F	F
T	F	T	F	T	F	F	T	F	F	T	F	F
T	F	F	F	F	F	F	T	F	F	T	F	F
F	F	F	T	T	T		T	T	T	T	T	T
F	T	F	T	T	T		T	T	T	T	F	F
F	F	T	T	T	T		T	T	T	T	T	T
F	T	T	T	T	T		T	T	T	T	F	F

1.3c $\forall x,z \in S, \exists y \in C, T(x,y) \Leftrightarrow T(z,y)$

↳ (x,z) $S \times S$ all combination of x,z

Every person has travelled to at least one country

iff

every person has travelled to at least one country

1.4 If all birds fly and if Tweety is a bird, then Tweety flies

$D = \{\text{all animals}\}$

$B(x)$: x is a bird

$S = \{x \mid x \in D \wedge (B(x) \Rightarrow F(x)) \wedge B(\text{Tweety})\}$

$\forall x \in S, F(\text{Tweety})$

$\exists$ → works for all things often

← depends on all things

$S = \{x \mid x \in D \wedge (B(x) \Rightarrow F(x))\}$

$\forall x \in S, B(\text{Tweety}) \Rightarrow F(\text{tweety})$

$\forall x \in S, \neg F(\text{tweety}) \Rightarrow \neg B(\text{Tweety})$

$\forall x \in S, F(\text{tweety}) \Rightarrow B(\text{tweety})$

$B = \{x \mid x \text{ is a bird}\}$

$F(x)$: x can fly → $\{T, F\}$

$((\forall x \in B, F(x)) \wedge B(\text{Tweety})) \Rightarrow F(\text{tweety})$

b.) If it does not rain or it is not foggy, then the sailing race will be held and registration will go on.

$(\neg \text{Rain} \vee \neg \text{Foggy}) \Rightarrow (\text{Sailing Race} \wedge \text{Registration})$

$(\neg \text{Sailing} \vee \neg \text{Registration}) \Rightarrow (\text{Rain} \wedge \text{Foggy})$

$(\text{Sailing Race} \wedge \text{Registration}) \Rightarrow (\neg \text{Rain} \vee \neg \text{Foggy})$

c.) If rye bread is for sale at Ace Bakery, then rye bread was baked that day

$\text{ForSale}(x)$: x is for sale at Ace Bakery

$\text{Baked}(x)$: x was baked that day

$\text{ForSale}(\text{rye bread}) \Rightarrow \text{Baked}(\text{rye bread})$

$\text{Baked}(\text{rye bread}) \Rightarrow \text{ForSale}(\text{rye bread})$

$\neg \text{Baked}(\text{rye bread}) \Rightarrow \neg \text{ForSale}(\text{rye bread})$

1.4 a) $A = \{\text{set of all animals}\}$ — Tweety is a global var

$Bird(x) : x$ is a bird where $x \in D$

$Flies(x) : x$ flies

$$((\forall x \in A,\ Bird(x) \Rightarrow Flies(x)) \land Bird(Tweety)) \Rightarrow Flies(Tweety)$$

$A = \{x \in A\}$

$B = \{\text{set of all birds}\} \iff B = \{x \in A \mid B(x) = T\}$

$$(\forall x \in B,\ F(x)) \land (\exists x \in B,\ x = Tweety) \Rightarrow F(Tweety)$$

Characteristic Function

$Bird(x): x \in \{x \in A \mid Bird(x)\} \iff Bird(x)$

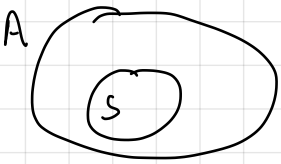
$P : A \to \{T, F\}$

$x \in \{x \in A \mid x \in \{x \in A \mid P(x) = T\} = T\} \iff P(x)$

$S = \{x \mid x \in A \text{ and } P(x) = True\}$

set of elements that satisfy P.

$S \subseteq A \longrightarrow$ "characteristic predicate"



$P_S : A \to \{T, F\}$

$P_S(x) : x \in S = \begin{cases} T & \text{if } x \in S \\ F & \text{if } x \notin S \end{cases}$

$S = \{x \mid x \in A \text{ and } P_S(x) = True\}$

$P_0 \rightsquigarrow S_{P_0} \to P_{S_{P_0}}$

$P_{S_p}(x) : x \in S_p = \{ T \text{ if } x \in S_p$

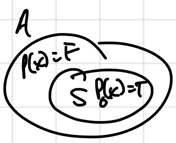
$P : A \to \{T, F\}$

$S = \{x \in A \mid P(x) = T\}$

$P_0 : A \to \{T, F\}$

$P_0(x) = x \in S$

$S_{P_0} = \{x \in A \mid P_0(x) = T\}$



$P_0(x) = \begin{cases} T & \text{if } x \in S \\ F & \text{if } x \notin S \end{cases}$

$P_{S_{P_0}}(x) = x \in S_{P_0}$

$P_{S_{P_0}}(x) = x \in S_{P_0} \begin{cases} T & \text{if } x \in S_{P_0} \\ F & \text{if } x \in S_{P_0} \end{cases}$

WS2

onto(x) : $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y$

2c) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \neq y$

$y = -1$

$x^2 \neq -1$

3) $\forall n \in \mathbb{N}, n > 3 \Rightarrow n > 1$

$S = \{x \mid x > 3 \wedge x \in \mathbb{N}\}$

$P(n)$: $n > 3$ where $n \in \mathbb{N}$

4) $P(x)$: $x > 10 \vee x < -40$ where $x \in \mathbb{Z}$

$\forall x \in \mathbb{Z}, P(x) \Rightarrow n \neq 0$.

$S = \{x \mid (x > 10 \vee x < -40) \wedge x \in \mathbb{Z}\}$

$\forall x \in S, n \neq 0$.

$P(x)$: SameDept(x, Dong) , $x \in E$

$\forall x \in E, P(x) \Rightarrow$ Rich(x)

$S = \{x \mid$ Samedept(x, Dong and $x \in E\}$

Ch 2.1 test

1. T
2. F
3. same truth tables (values)
4. or , negated $\sim p \vee \sim q$
   and , negated $\sim p \wedge \sim q$

5. T
6. F

Exercises

1.a) then the number X → written in prefix notation
All algebraic exp →
Therefore X

$x = (a + 2b)(a^2 - b)$

1b)

p	q	$p \vee (p \wedge q)$
T	T	T
T	F	T
F	T	F
F	F	F

17.) $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

18.)

p	t	$p \vee t$
T	T	T
T	F	T
F	T	T
F	F	F

19

p	t	$p \wedge t$
T	T	
T	F	
F	T	
F	F	

22 Test

1. T, F
2. $p \wedge \neg q$
3. if q, then p
4. if not q, then not p
5. if not p, then not q
6. logically equiv
7. logically equiv
8. R, S
9. $\neg r, \neg S$ S, R
10. S, r R, S R only if S

p : you paid full price
q : you bought at crown books

p	q	$p \to \neg q$	$\neg q \vee p$
T	T	F	T
T	F	T	
F	T	T	
F	F	T	

Lec3

$\forall x \in D, P(x)$ $\exists x \in D, P(x)$

Given $S \subseteq D$

$\forall x \in S, P(x)$ $\forall x \in D, x \in S \Rightarrow P(x)$

Same meaning

$\exists x \in S, P(x)$ $\exists x \in D, x \in S \land P(x)$

$\forall x \in D, P(x)$ $\forall x, x \in D \Rightarrow P(x)$

Non-Standard in CSC165

preference:
always use a "common" domain D (eg. $D = \mathbb{Z}, \mathbb{R}, \ldots$)
and use operators to restrict further

eg. $\forall x \in \mathbb{Z}, x > 10 \Rightarrow P(x)$ is preferred

over $\forall x \in S, P(x)$, where $S = \{x \in \mathbb{Z} \mid x > 10\}$

$P(\varepsilon, \varepsilon)$
→ similar to assigning same value to diff var.

$P(\varepsilon, \varepsilon)$ ⇒ if left undefined ⇒ exceptions
⇒ problems if defined False
⇒ True

Empty sum

edge cases

d divides n iff $n = d \cdot k$ for some $k \in \mathbb{Z}$

body meaning

expression, function, predicate ... being defined

where $d, n \in \mathbb{Z}$

domain

PS1

1a) $p \Rightarrow \neg q \equiv \neg p \vee \neg q$ (implication)
$\equiv \neg q \vee \neg p$ (commutativity)
$\equiv q \Rightarrow \neg p$ (implication)

b) $(p \wedge q) \Rightarrow r \equiv \neg(p \wedge q) \vee r$
$\equiv \neg p \vee \neg q \vee r$
$\equiv \neg p \vee (q \Rightarrow r)$
$\equiv p \Rightarrow (q \Rightarrow r)$

$(p \vee q) \Rightarrow r \equiv \neg(p \vee q) \vee r$ implication
$\equiv \neg p \wedge \neg q \vee r$ de morgans
$\equiv r \vee (\neg p \wedge \neg q)$ associativity
$\equiv (r \vee \neg p) \wedge (r \vee \neg q)$ distributivity
$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ commutativity
$\equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$ implication

p	q	r	$(p \Rightarrow q) \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$(p \vee q) \Rightarrow r$
F	F	F	F	T	T
F	F	T	T	T	T
F	T	F	F	T	F
F	T	T	T	T	T
T	F	F	T	T	F
T	F	T	T	T	T
T	T	F	F	F	F
T	T	T	T	T	T

Q2)

a) "always"

$\forall d \in D$, Holiday(d) $\Rightarrow$ Weather(d, windy)

$d \in D$, windy $\in C$

b) "Some" ? ask about set size!

$\forall c \in C, \exists d \in D$, $\boxed{|C| > 1}$ $\Rightarrow$ Weather(d, c)

$\exists c_1, c_2 \in C, \exists d \in D$, $c_1 \neq c_2 \wedge$ Weather$(d, c_1) \wedge$ Weather(d, c_2)

There are at least 2 conditions that occur on same days

c) $\forall d \in D, \forall c \in C$, $\neg$Weather(d, c)

There are at least two days, where snowy occurred on d,

d) $\exists d_1, d_2 \in D$, (Weather$(d_1$, snowy) $\vee$ Weather$(d_2$, snowy)) $\wedge$ ($\forall d_1', d_2' \in D$, Weather$(d_1'$, snowy) $\vee$ Weather$(d_2'$, snowy) $\Rightarrow d_1' = d_1 \vee d_2' = d_2$

e) $\exists d \in D$, (Weekend(d) $\Rightarrow$ Weather(d, rainy)) $\vee$ (Weekend(d) $\Rightarrow$ Weather(d, sunny))

$\Downarrow$

$\exists d \in D$, Weather(weekend(d), sunny) $\vee$ (Weather(weekend(d), rainy))

f) $\forall d \in D$, Weather(d, snowy) $\wedge$ Weather(d, rainy) $\Rightarrow \neg$Holiday(d)

Holiday $\Rightarrow \neg$ Weather(d, snowy) $\wedge \neg$

g) $\exists c \in C$, ($\forall d \in D$, Weekend(d) $\Rightarrow$ Weather(d, c))

$\forall d \in D$, snowy $\vee$ rainy $\Rightarrow \neg$ Holiday

h) $\forall c \in C, \exists d \in D$, Holiday(d) $\Rightarrow$ Weather(d, c)

or

$\forall y \in S, \exists x \in T$ Weather(Holiday(d), c)

all x

$\exists y, \forall x$ applys to 1 value of y

stronger than $\forall x \exists y$

$\exists$ is like $\vee$

$\forall$ is like $\wedge$

Equivalences are symmetric

$\forall y \in D$, ($P(y) \Rightarrow Q$) P(y)=False P(y)=True $\Rightarrow Q(y)$

vs ($\forall y \in D$, $P(y)$) $\Rightarrow Q(x)$ P(y)=True $\Rightarrow Q(y)$

$\hookrightarrow$ Quantifiers apply last

apply to everything after

3a)(6) $P(y) = T \wedge Q(x) = F$ ✓

(7) $\forall y \in S, P(y) = F$ or $\forall y \in S, P(y) = T \wedge \exists x \in T, Q(x) = T$

$\forall y \in S, \exists x \in T$

3b) $P(y) = F$ or $P(y) = T \wedge Q(x) = T$

$\Leftrightarrow$

(7) $\forall y \in S, P(y) = T$ and $\exists x \in T, Q(x) = F$

$\forall y \in \{0,1\}, \exists x \in \{0,1\}, P(y) \Rightarrow Q(x)$ T

P(x): x is an integer

Q(x): x is an integer > 2 outputs truth values are fixed

$\forall y \in S, P(x) \Rightarrow \exists x \in T, Q(x)$ F

$\exists x \in \{0,1\}, Q(x)$

Prove Q is true/false so that whenever P is true, Q is true

3a) $\forall y \in S, \exists x \in T, P(y) \Rightarrow Q(x)$

P(x) = x is an integer Q(x) = even

$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}^+, P(y) \Rightarrow Q(x)$

False

$\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}^+, P(y) \wedge \neg Q(x)$

Let y = 3

Let $x \in \mathbb{Z}^+$,

P(3) = True

3 = not even = True

$\forall y \in \mathbb{Z}, P(y) \Rightarrow \exists x \in \mathbb{Z}^+, Q(x)$

T $\Rightarrow$ 2

True

$\forall y \in \{0,1\} \exists x \in \mathbb{Z}^+$, $P(y) \Rightarrow Q(x)$ ($\forall y \in \{0,1\}, P(x)) \Rightarrow \exists x \in \{0,1,3\}, Q(x)$)

P(x): x is integer

$\exists x \in \{0,1,3\}$

Q(x): x is integer > 2

False

Q4)

a) $\exists x \in \mathbb{Z}, (x+10 \mid x^3+100) \wedge (\forall x_0 \in \mathbb{Z}, (x_0+10 \mid x^3+100) \Rightarrow x_0 \le x)$

what is meant by largest?

$\exists$

b) $\forall x \in \mathbb{Q} \wedge x$

$\neg \exists x \in \mathbb{Q}, P(x)$ largest negative rational number

$P(x): (x \in \mathbb{Q}) \wedge (x < 0) \wedge (\forall x_0 \in \mathbb{Q}, x_0 < 0 \Rightarrow x_0 \le x)$

$\forall x \in \mathbb{Q}, \neg P(x)$

$x \notin \mathbb{Q} \vee (x \ge 0) \vee (\exists x_0 \in \mathbb{Q}, x_0 < 0 \wedge x_0 > x)$

c) $\exists x \in \mathbb{N}, P(x)$ → meaning it is natural?

smallest odd prime number

$P(x): (\exists k \in \mathbb{N}, x = 2k) \wedge (\text{Prime}(x)) \wedge (\forall x_0 \in \mathbb{N}, (\text{Prime}(x_0) \wedge \exists k_2 \in \mathbb{N}, x_0 = 2k_2) \Rightarrow x_0 \ge x)$

different entry

" $\Rightarrow$ 'a'

vacuous truth

" is in there

At most 2

$0 \le x \le 2$

$\exists x, y \in \mathbb{R}$

↳ can be same / different elems in a domain

$\forall y \in S, \exists x \in T, P(y): x$ is even $\Rightarrow Q(x): x$ is even T

$(\forall y \in S, x \text{ is even}) \Rightarrow (\exists x \in \text{even}, x \text{ is even})$ T

↑ truth cannot be deduced

← Stronger

a) $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}, P(x) \Rightarrow Q(x)$

Let $y \in \mathbb{Z}$, Let $x = 1$

If $y = 2 \Rightarrow x = 1$

$\forall y \in \mathbb{Z}, P(y) \to \exists x \in \mathbb{Z}, Q(x)$

Assume $y \in \mathbb{Z}, y = 2$

Let $x = 1$

$x = 1$

$\forall y \in \mathbb{Z}, P(y) \wedge \neg(\exists x \in \mathbb{Z}, Q(x))$

Let $y \in \mathbb{Z}$

$y = 2$

$\forall y \exists x$ If $P(y)$, then $Q(x)$ weak

$(\forall y\, P(y))$, then $\exists x\, Q(x)$ Stronger

$\forall y, \exists x, \neg p \vee q$

$\forall y\, p \to \exists x q$

$\exists x\; p(x) \Rightarrow q$

$\exists x\; p(x) \wedge q(x)$

$\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, y \mid 5 \Rightarrow x \mid 0$ F

$(\forall y \in \mathbb{N}, x \mid 5) \Rightarrow (\exists x \in \mathbb{N}, x < 0)$ T

$\forall y \in \mathbb{N}, x \mid 5 \wedge \forall x \in \mathbb{N}, x \ge 0$ F

3b) $\forall y \in S, \exists x \in T, \underbrace{P(y) \Rightarrow Q(x)}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad T \quad\quad\quad T$

$(\forall y \in S, P(y)) \Rightarrow (\exists x \in T, Q(x)) \quad = F$
$\quad\quad T \quad\quad\quad\quad\quad\quad\quad\quad F$

$\forall y \in S$

Note that in both cases $\exists x \in T$ always works

$\forall y \in S$
if $(\forall y \in S, y = 3)$

$\exists x \in T, P(y) \wedge Q(x)$ (6)

if $P(y) = T$
All y's have to be T
↳ a single y that's F $\underline{\Rightarrow F}$. Thus it is impossible to find a $Q(x)$ which is both T and F.

$\exists x \in T, Q(x)$ (7)

$(\forall y \in S, P(y)) = F$
but
$\forall y \in S \ P(y) = T$

↳ Not all y's satisfy $P(y)$ T (6)
$\exists y [y \in S \wedge \neg P(y)]$
$\neg \forall y [y \in S \Rightarrow P(y)]$

All y's satisfy $P(y)$ F (7)
$\forall y [y \in S \wedge P(y)]$
$\exists y [y \in S \Rightarrow \neg P(y)]$

Lec 5

$\forall z \in \mathbb{Z},\ z \mid 10 \Rightarrow z \mid 100$ ← for the ones that do divide 10, also divide 100

$\forall y \in \mathbb{Z},\ y \mid 10 \wedge y \mid 100$ X ← too strong

"Some positive integer has a square less than -ve 4.

$\exists x \in \mathbb{Z},\ x > 0 \wedge x^2 < -4$ ← correct (does really have both properties)

$\underbrace{\exists s \in \mathbb{Z},\ s > 0 \Rightarrow s^2 < -4}$ ← too weak

True : pick $s = -1$:

$\underbrace{\text{vacuously true}}$ since $s > 0 \Rightarrow s^2 < -4$

↳ too weak

logical strength

→ $\underbrace{x > 20 \text{ is stronger than } x > 10}_{\text{less true outcomes}}$

→ $\underbrace{x > 10 \text{ is weaker than } x > 20}_{\text{more true outcomes}}$

Proofs = any convincing argument

$\forall n \in \mathbb{N},\ n > 20 \Rightarrow n^2 - 165 \geq 4$

↳ just the ones greater than twenty, don't say every $\forall$

$(\forall n \in \mathbb{N},\ n > 20) \Rightarrow n^2 - 165 \geq 4$

fix : restrict the domain

Start from what we want or what we know

$n^2 - 165 \geq 4$ slow	$n \in \mathbb{N}$
⋮	$n > 20$
this is true $n^2 - 165 \geq 4$	⋮
if $n^2 \geq 169$	
if $n \geq 13$	$n^2 > 400$
know	$n^2 - 165 > 235 \geq 4$

Let $n \in \mathbb{N}$ and assume $n > 20$

Then, $n \geq 13$ by hypothesis

$\left.\begin{array}{l} n^2 \geq 169 \\ n^2 - 165 \geq 4 \end{array}\right\}$ okay to leave out for simple arithmetic

1a) $\exists n \in \mathbb{N}, n > 3 \wedge n^2 - 1.5n \geq 5$

exist

concrete

$n = 4$ $\qquad 4^2 - 1.5 \cdot 4 \geq 5$

$16 - 6 \geq 5$

$10 \geq 5$

$\forall n \in \mathbb{N}, n > 3 \Rightarrow n^2 - 1.5n > 4.$

weaker consequence, allows for more true statements

f) universal

g) arbitrary

h) Assume $n > 3$

1.5

Let $n \in \mathbb{N}$, Assume $n > 3$

$n^2 > 9$

$n^2 - 1.5n > 9 - 4.5$

$n^2 - 1.5n > 4.5 > 4.5$

$\square$

induction

$n > 3$

$n^2 > 9 \quad \{ \quad n^2 > 3n$

$n^2 + n^2 > 9 + 3n$

$2n^2 > 9 + 3n$

$2n^2 - 3n > 9$

$n^2 - 1.5 > 4.5$

fix mistake where if $a < b$, $-a > -b$

$x \in \mathbb{R}, x \geq 0 \Rightarrow$ ____

2. $\forall n \in \mathbb{N}, n > 5 \Rightarrow n | 2 \vee n | 3$

$\exists n \in \mathbb{N}, n > 5 \wedge n \nmid 2 \wedge n \nmid 3$

$n = 7$, then $2 \nmid 7 \wedge 3 \nmid 7$

3. $y = 166$

$x \in \mathbb{N}$

$x + 166 > 165$

c.) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 165$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \leq 165$

Let $y \in \mathbb{R}$

$x = 165 - y$

$165 - y + y \leq 165$

Lec6

$\forall x \in \mathbb{Z}, (x \mid x+5) \Rightarrow (x \mid 5)$

Let $x \in \mathbb{Z}$. Assume $\exists k \in \mathbb{Z}$, $x+5 = xk$

Prove $\exists k_2 \in \mathbb{Z}$, $5 = xk_2$

Let $k_2 = k-1$

then, $x+5 = xk$

$5 = xk - x$

$5 = x(k-1)$

$5 = xk_2$ since $k_2 = k-1$

$x \mid x+5 \Rightarrow x \mid 5$

$x+5 \equiv 5 \pmod{x}$

$x \equiv 0 \pmod{x}$

$x \mid x == 0$

$x \mid 0 == 0$

$x = kx$

$= k - 1$

$0 = k_2$

$\overset{\text{is a factor}}{x \mid x+5}$ is a multiple

$\overset{\text{divide}}{x \mid x+5}$ is divisible by

cannot choose $k_2 = \frac{x}{5}$ since not clear $k_2 \in \mathbb{Z}$

$k_2 \in \mathbb{Q}$

Recap:

$\forall n \in \mathbb{N},\ n > 1 \wedge \text{Atomic}(n) \Rightarrow \text{Prime}(n)$

Indirect proof (contrapositive)

Let $n \in \mathbb{N}$. Assume $\neg\text{Prime}(n)$ - by def.

disjunction (proof by cases)

$\underline{n \leq 1 \vee \exists d \in \mathbb{Z}^+, d \mid n \wedge d \neq 1 \wedge d \neq n}$ we know

$\neg(\forall d \in \mathbb{Z}^+, d \mid n \Rightarrow (d = 1 \vee d = n))$

WTP: $n \leq 1 \vee \exists a, b \in \mathbb{Z}, n \nmid a \wedge n \nmid b \wedge n \mid ab$

not meant to be mutually exclusive (both parts of or could both be true)

Case 1: Assume $n \leq 1$.

Then, $\underbrace{n \leq 1}_{\text{common, not always work}} \vee \exists a, b \in \mathbb{Z}, n \nmid a \wedge n \nmid b \wedge n \mid ab$

Case 2: Assume $\exists d \in \mathbb{Z}^+, d \mid n \wedge d \neq 1 \wedge d \neq n$

$\exists k \in \mathbb{Z}, n = kd$

Know: $n \geq 0$ ($n \in \mathbb{N}$)

~~$n \leq 1$ prob~~ $n = kd \quad d \neq 1 \quad d \neq n$

Could $n = 0$? ← subcase

positive divisor $d > 0$

pick $k = 0$ → then $n \leq 1$

could $n = 1$?

No: $n = kd$ with $d > 1$ is impossible $\qquad n = 1 \cdot d \Rightarrow n = d$ but $n \neq d$

could $n > 1$? Sure ...

subcase when $n \leq 1$: already covered

so consider subcase when $n > 1$.

$n = kd \wedge d > 1 \wedge d \neq n \qquad n, k, d \in \mathbb{Z}^+$

$\Rightarrow k > 1 \wedge k < n$

$1 < k < n \qquad$ "obvious"

Want:

lemma: $n = kd \wedge d > 1$

$\Rightarrow k < n$

if $k \geq n$, then

$kd > n \cdot 1 = n$

WTP: $\exists a, b \in \mathbb{Z}^+, n \nmid a \wedge n \nmid b \wedge \underbrace{n \mid ab}_{\exists l \in \mathbb{Z}, ab = ln}$

try: $a = k$, $b = d$, $l = 1$

then $ab = kd = n = 1 \cdot n = ln$ ✓

still want to show

$n \nmid a \wedge n \nmid d$

Know: $1 < d < n \wedge 1 < k < n$

Want: $n \nmid d \wedge n \nmid k$

intuition: $\forall x, y \in \mathbb{Z}^+, x \mid y \Rightarrow x \leq y \quad (*)$

if $(*)$ is true, then $\Rightarrow n > d \Rightarrow n \nmid d$

$n > k \Rightarrow n \nmid k$

Prove: $\forall n \in \mathbb{N}, \text{Prime}(n) \Rightarrow n > 1 \wedge \text{Atomic}(n)$

Let $n \in \mathbb{N}$. Assume $\text{Prime}(n)$:

$n > 1 \wedge \forall d \in \mathbb{Z}^+, d \mid n \Rightarrow d = 1 \vee d = n$

(WTP: $n > 1 \wedge \forall a, b \in \mathbb{Z}, n \nmid a \wedge n \nmid b \Rightarrow n \nmid ab$

then, $n > 1$ (by assumption)

Let $a, b \in \mathbb{Z}$. Assume $n \nmid a \wedge n \nmid b$

WTP: $n \nmid ab$

Note: connection not obvious

easiest approach involves using <u>linear combinations</u>

external fact:

$\forall n \in \mathbb{N}, \forall a \in \mathbb{N}, \text{Prime}(n) \Rightarrow$

$(n \nmid a \Leftrightarrow \exists r, s \in \mathbb{Z}, rn + sa = 1)$

WS7

2b) Fact 5 and Fact 2

$\forall n \in \mathbb{N}, \neg Prime(n) \Rightarrow (n \leq 1 \vee (\exists a,b \in \mathbb{N}, n \nmid a \wedge n \nmid b \wedge n \mid (ab)))$ $\Rightarrow$

Assume $n \leq 1$ or $\exists d \in \mathbb{N}, d \mid n \wedge d \neq 1 \wedge d \neq n$

$Prime(n) \to (n > 1 \wedge (\forall a,b \in \mathbb{N}, n \mid a \vee n \mid b \vee n \nmid ab))$
$(\forall a,b \in \mathbb{N}, n \nmid a \wedge n \nmid b \Rightarrow n \nmid ab)$

Case 1: $n \leq 1$

then $n \leq 1$ and we're done

Case 2: Assume $\exists d \in \mathbb{N}, d \mid n \wedge d \neq 1 \wedge d \neq n$

if $n = 0$

$0 = dk$, since $d \neq 1$, $k = 0$

if $n = 1$

$1 = dk$, since $d > 1$, $k \not< 1$ $\Rightarrow$ impossible

if $n > 1$, assume $n = dk$ for some k $\qquad 1 < d < n$

let $a = d$, $b = k$, prove $n \nmid a$, $n \nmid b$, $n \mid ab$

let $l = 1$ $\qquad n \mid dk = n \mid n$

$ab = dk = n = n \cdot l$ $\qquad$ by fact 1

Fact 2

$x < 1 \vee y < x \Rightarrow y < 1 \vee x \nmid y$

$n \nmid d$, $n \mid d \Rightarrow n \leq d$

$n > d \Rightarrow n \nmid d$

we know, $d \mid n$ or $n = dk$, $\Rightarrow n \geq d$ since if $k = 1$ $n = d$

$k > 1$ $n > d$

so $n \nmid d$

$n \nmid k$, we know $k \mid n$

$n = kd$

so $n > k$

and $n \nmid k$.

Facts

1. Every integer divides itself.
2. every natural number that divides every natural number greater / equal to 1 is leq than that number and geq 1.
3. Every prime number that does not divide every integer has a gcd w/ that number = 1.
4. Every non-zero natural number or every non zero natural number have a gcd ≥ 1. (when gcd(0,0) = 0)
   if gcd(0,0) $\neq$ 0:

5. All natural numbers have a gcd which divides their lin comb for ~~some~~ two integers

6. All natural numbers have a gcd which equal their lin comb.

2. $\forall n \in \mathbb{N},\ \mathrm{Prime}(n) \Rightarrow (n > 1 \wedge (\forall a, b \in \mathbb{N},\ n \nmid a \wedge n \nmid b \Rightarrow n \nmid ab))$

a) $\forall n, m \in \mathbb{N},\ \mathrm{Prime}(n) \wedge n \nmid m \Rightarrow (\exists r, s \in \mathbb{Z},\ rm + sm = 1)$

midterm 1

1a) $\{aaa, aab, aac, bbb, bba, bbc, ccc, cca, ccb\}$

1b)

p	q	r	$p \vee q \Rightarrow \neg r$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	T
T	T	T	F

1c) Let $x \in \mathbb{N}$

Let $y =$ ____

We will prove that, $P(x,y)$ or $Q(x,y)$ ~~= True~~ False

2a) $\exists x \in P$, $Student(x) \wedge Attends(x)$

2b) $\forall x \in P, \exists y \in P$, $Attends(y) \Rightarrow Loves(x,y)$ $\forall x \in P, (\exists y \in P, Student(y) \wedge Attends(y) \wedge Loves(x,y))$ why

2c) $\forall x \in P$, $Attends(x) \Rightarrow Loves(x,x)$ $\wedge Student(x)$

2d) $\forall x, y \in P$ $x \neq y \wedge Loves(x,y) \wedge Loves(y,x) \Rightarrow \neg Attends(x) \vee \neg Attends(y)$

3a) $\forall a,b,c \in \mathbb{Z}$, $(\exists k_1 \in \mathbb{Z}, b = ak_1) \wedge (\exists k_2 \in \mathbb{Z}, c = bk_2) \Rightarrow (\exists k_3 \in \mathbb{Z}, c = ak_3)$

Pf:

Let $a, b, c \in \mathbb{Z}$,

Assume $\exists k_1 \in \mathbb{Z}, b = ak_1$ and $\exists k_2 \in \mathbb{Z}, c = bk_2$

Prove $\exists k_3 \in \mathbb{Z}, c = ak_3$

Let $k_3 = k_1 k_2$

Then, $c = bk_2$ Second assumption

$c = ak_1k_2$ By assumption

$c = ak_3$ By choice of k_3

□

4. $\forall x, y \in \mathbb{R}, \lfloor x+y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$

Let $x, y \in \mathbb{R}$ We know $\exists \varepsilon_1, \varepsilon_2 \in \mathbb{R}, 0 \leq \varepsilon_1 < 1 \wedge 0 \leq \varepsilon_2 < 1$

$\lfloor x+y \rfloor = \lfloor \lfloor x \rfloor + \varepsilon_1 + \lfloor y \rfloor + \varepsilon_2 \rfloor$ By Fact 1

$= \lfloor \lfloor x \rfloor + \lfloor y \rfloor + \varepsilon_1 + \varepsilon_2 \rfloor$

$= \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \varepsilon_1 + \varepsilon_2 \rfloor$ By Fact 2

$\geq \lfloor x \rfloor + \lfloor y \rfloor$

□

By definition of floor

these are integers

midterm 1 v2

a) $S_2 = \{x \mid x \in \mathbb{N} \text{ and } x \mid 30\}$

$\{2, 3, 5\}$ $\{1, 10, 15, 30, 6\}$

definition of prime means (>1) → 1 is not prime

b)

p	q	r	$(\neg p \Leftrightarrow q) \Rightarrow r$
F	F	F	T
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

True

c) Let $x \in \mathbb{N}$

Assume $P(x)$

We want to prove $(\exists y \in \mathbb{N}, Q(x,y))$

Let $y \in \mathbb{N}$

We want to prove $Q(x,y)$ is true.

2. $\forall x \in P,\ Cat(x) \Rightarrow Loves(x,x)$

b) $\forall x \in P, \exists y \in P,\ Cat(x) \land Cute(y) \overset{why?}{\land} Loves(x,y)$ $\quad \forall x \in P,\ Cat(x) \Rightarrow (\exists y \in P,\ Cute(y) \land Loves(x,y))$

c) $(\exists x \in P,\ Cat(x) \land Cute(x)) \Rightarrow (\forall y \in P,\ Cat(y) \Rightarrow Cute(y))$

d) $\forall x, y \in P,\ x \neq y \land Loves(x,y) \land Loves(y,x) \Rightarrow \boxed{(\forall x_0 \in P,\ Cat(x_0) \Rightarrow x_0 = x \lor x_0 = y)}$

$(\neg Cat(x) \Leftrightarrow Cat(y))$

$(Cat(x) \land \neg Cat(y)) \lor (\neg Cat(x) \land Cat(y))$

3. $\exists n \in \mathbb{N},\ n > 1 \land \forall x \in \mathbb{R}^+ \Rightarrow \lfloor nx \rfloor = n \cdot \lfloor x \rfloor$

Disproof

$\forall n \in \mathbb{N}, \exists x \in \mathbb{R}^+,\ n \leq 1 \lor \lfloor nx \rfloor \neq n \cdot \lfloor x \rfloor$

Let $n \in \mathbb{N}$

Let $x = \frac{1}{2}$ $\quad \lfloor \frac{n}{2} \rfloor \neq n \lfloor \frac{1}{2} \rfloor$

$n - 1 \neq 0$

$x = \frac{1}{n}$

$\lfloor \frac{n}{n} \rfloor = \lfloor 1 \rfloor = 1$

$n \lfloor \frac{1}{n} \rfloor = n \cdot 0 = 0$

$1 \neq 0$

4. $\forall a, b \in \mathbb{N},\ b \mid a \land b \mid (a+2) \Rightarrow b = 1 \lor b = 2$

Let $a, b \in \mathbb{N}$

Assume $\exists k_1 \in \mathbb{Z},\ a = bk_1$ and $\exists k_2 \in \mathbb{Z},\ a + 2 = bk_2$ Definition

prove $b = 1 \lor b = 2$

choose $k_3 = k_2 - k_1$

$a + 2 = bk_2$

$bk_1 + 2 = bk_2$

$2 = bk_2 - bk_1$

$2 = b(k_2 - k_1)$

$2 = bk_3$

then $b \mid 2$ since $b \in \mathbb{N}$

by primality of 2, the only divisors are $b = 1 \lor b = 2$.

midterm 1 v3

$U = \{a, b\}$

$\{aaa, aba, bab, bbb\}$

b)

p	q	r	$(p \Rightarrow q) \Leftrightarrow \neg r$
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	F

Disprove

c) Let $x \in \mathbb{N}$

Let $y = \_$

Assume $P(x)$

Prove $Q(x,y)$ or $Q(x, y+1)$ is false

$P(x)$ is true

$Q(x,y) \wedge Q(x,y+1)$ are false

2. $\forall x \in T$, Canadian$(x) \Rightarrow$ Star(x)

b. $\forall x \in T$, Canadian$(x) \Rightarrow (\forall y \in T, \neg$Canadian$(y) \Rightarrow$ Defeated$(x,y))$

c.) $(\exists x \in T$, Canadian$(x) \wedge$ Star$(x)) \Rightarrow (\forall y \in T$, Canadian$(y) \Rightarrow \exists z \in T, y \neq z \wedge$ Defeated$(y,z))$

d.) $\exists x \in T$, Canadian$(x) \wedge$ Star$(x) \wedge (\forall y \in T, y \neq x \wedge$ Canadian$(y) \Rightarrow \neg$Star$(y))$

don't include brackets

3. $\forall n \in \mathbb{N}, n > 1 \wedge (\exists k \in \mathbb{N}, n = 2k+1) \Rightarrow (\exists p, q \in \mathbb{Z}^+, n = p^2 - q^2)$

Let $n \in \mathbb{N}$

Assume $n > 1 \wedge \exists k \in \mathbb{N}, n = 2k+1$ (since $n > 1$, $k > 0$) ($k = 0$ means $n = 1$)

Let $p = k+1$ $\qquad (k+1)^2 = k^2 + 2k + 1$

$q = k$

$n = 2k+1$

$n + k^2 = k^2 + 2k + 1$

$n + k^2 = (k+1)^2$ By fact

$n = (k+1)^2 - k^2$

$n = p^2 - q^2$

$\square$

$\mathbb{Z}^+$ since $d = 0$ or $n = 0$

4. $\forall d, n \in \mathbb{N}$, $d | n \wedge d \neq n \Rightarrow d \leq \frac{n}{2}$

Let $d, n \in \mathbb{N}$

Assume $\exists k \in \mathbb{Z}, n = dk \wedge d \neq n$

we know $k | n$

by Fact 1 we have that

$1 < k \leq n$ since $n \neq d$

and $1 \leq d < n$ since $n \neq d$

however, k is an integer so,

$2 \leq k$

$2d \leq kd$ $\quad$ × both sides by d since $d \geq 1$.

$2d \leq n$

$d \leq \frac{n}{2}$

$2 \leq k \leq n$

$1 \leq d < n$

$2 \leq k$

$2d \leq kd$

$2d \leq n$

$d \leq \frac{n}{2}$

$\square$

3.1) $\forall n \in \mathbb{N},\ 8 \mid 9^n - 1$

Base Case $n=0$

$k=0$

$9^0 - 1 = 8(0)$

$0 = 0$

Inductive Step $8 \mid 9^n - 1 \Rightarrow 8 \mid 9^{n+1} - 1$

Assume $8k_0 = 9^n - 1$

Prove $8k_1 = 9^{n+1} - 1$

Let $k_1 = 9^n + k_0$

$8k_1 = 8 9^n + 8k_0$

$8k_1 = 8 \cdot 9^n + 9^n - 1$ By inductive hypothesis.

$8k_1 = 9 \cdot 9^n - 1$

$8k_1 = 9^{n+1} - 1$

3.2) $\forall n \in \mathbb{N},\ 6 \mid 5^{2n} - 1$

Base Case: $n=0$

$k=0$

$5^{2(0)} - 1 = 6k$

$0 = 0$

Inductive step: $6 \mid 5^{2n} - 1 \Rightarrow 6 \mid 5^{2(n+1)} - 1$

$5^{2n} - 1 = 6k_0$

Let $4 \cdot 5^{2n} + k_0 = k_1$

$24 \cdot 5^{2n} + 6k_0 = 6k_1$

$24 \cdot 5^{2n} + 5^{2n} - 1 = 6k_1$

$25 \cdot 5^{2n} - 1 = 6k_1$

$5^{2n+2} - 1 = 6k_1$

$5^{2n} - 1 = 5^{2n+2} - 1$
$= 25 \cdot 5^{2n} - 1$
$= 24 \cdot 5^{2n} + 5^{2n} - 1$
$= 24 \cdot 5^{2n} + 6k_0$
$= 6(4 \cdot 5^{2n} + k_0)$
$= 6k_1$

3.3) $\forall n \in \mathbb{N},\ (x-y) \mid x^n - y^n$

Base Case: $n=0$ $k=0$

$x^0 - y^0 = (x-y)k$

$1 - 1 = 0$

Inductive Step: Assume $x^n - y^n = (x-y)k_0$

Prove $x^{n+1} - y^{n+1} = (x-y)k_1$

$x^n = (x-y)k_0 + y^n$

$x^n - (x-y)k_0 = y^n$

Let $k_1 = x^n + yk_0$

$x^{n+1} - y^{n+1} = x^n x - y^n y$
$= x^n x - x^n y + (x-y)k_0 y$ By inductive hypothesis
$= (x-y)x^n + (x-y)yk_0$
$= (x-y)(x^n + yk_0)$
$= (x-y)k_1$

3.4) $\forall n \in \mathbb{N},\ n \geq 6 \Rightarrow 5n+5 \leq n^2$

Base Case $n=6$

$5(6)+5 \leq 6^2$

$35 \leq 36$

Inductive assume $n \geq 6 \Rightarrow 5n+5 \leq n^2$

prove $n \geq 6 \Rightarrow 5(n+1)+5 \leq (n+1)^2$

$5n+5 \leq n^2$ $\qquad$ $5n \leq n^2$

$5n+5+5 \leq n^2+2n+1$ $\qquad$ $5 < 2n+1$ since, $n \geq 6$

$5(n+1)+5 \leq (n+1)^2$

35 $\forall n \in \mathbb{N},\ n \geq 1 \Rightarrow$ at least n distinct primes $\mid 2^{2^n}-1$

$\forall n \in \mathbb{N},\ n \geq 1 \Rightarrow \forall i,j \in \mathbb{N},\ i \neq j$, Prime( ) $\wedge$ Prime(p_j)

$\Rightarrow$

Proof techniques

- Logical structure of statements
  $\rightsquigarrow$ proof structure

how to prove: $\forall, \exists, \land, \lor$ $\Rightarrow$ (direct, indirect)
$\Leftrightarrow, \neg$

how to use: $\forall$, $\exists$, $\lor$ (proof by cases), $\land$, $\Rightarrow$, $\Leftrightarrow$, $\neg$
elem from domain $\uparrow$ — true if H: False, C: True

external fact: $\lor$, proof by cases

Contradiction – Not contrapositive!

Ex 1
$\forall n,k,d \in \mathbb{Z}^+,\ n = kd \Rightarrow k \le n \land d \le n$

Let $n, k, d \in \mathbb{Z}^+$. Assume $n = kd$

For a contradiction, assume
$k > n \lor d > n$ ← Cases

Case 1: Assume $k > n$
$d \in \mathbb{Z}^+ \Rightarrow d \geq 1$
$\Rightarrow kd > n \cdot 1$
$\Rightarrow n > n$ ← contradiction

$n = kd$

Case 2: → WLOG → means: other case is the same
assume $k \geq n$, ...

Generally, proof by contradiction works b/c

$$p \Rightarrow \text{False} \equiv \neg p$$

$(p \land \neg p)$

In a PBC, prove p is true by proving $\neg p$ is False — $\neg p$ cannot be true.

Intuitively,
(everything I know) $\Rightarrow p$
is the same as
$\neg p \Rightarrow \neg$(everything I know)
$\Rightarrow$ something I know is False

Ex 2: There are infinitely many primes

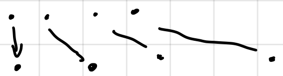
1. How to express "infinitely many"

A. Express "size of $\{n \in \mathbb{N} \mid \text{Primes}(n)\}$ is $\neq$ every $\mathbb{N}$."

to complicated

B. $\forall n \in \mathbb{N}$, $\text{Prime}(n) \Rightarrow \exists m \in \mathbb{N}, m > n \wedge \text{Prime}(m)$

$n \in \mathbb{N}$



C. $\forall n \in \mathbb{N}$

$\forall m \in \mathbb{N}$, $m \leq n \vee \neg\text{Prime}(m) \Rightarrow \neg\text{Prime}(n)$

Prove (C): $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}, m < n \wedge \text{Prime}(m)$

For a contradiction

$\neg(C)$: $\exists n \in \mathbb{N}, \forall m \in \mathbb{N}, m > n \Rightarrow \neg\text{Prime}(m)$

$\equiv \exists n \in \mathbb{N}, \forall m \in \mathbb{N}, \text{Prime}(m) \Rightarrow m \leq n$ ← finitely many

Trying to get a contradiction ...

<u>insight</u>: find a prime $> n$ - upper bound

- consider $N = 1 + n! = 1 + \prod_{i=1}^{n} i$

Remainder Theorem: Consequence

$$(*) \begin{cases} 2 \nmid N : N = 2 \cdot (3 \cdot 4 \cdots n) + 1 \\ 3 \nmid N : N = 3 \cdot (2 \cdot 4 \cdots n) + 1 \\ \vdots \\ n \nmid N : N = (2 \cdot 3 \cdots) n + 1 \end{cases}$$

- Fact: every integer > 1 has at least 1 prime divisor

$1 \leq k \leq n \Rightarrow \text{Prime}(n)$

$(*) \Rightarrow p > n$

↳ $\text{prime}(p) \wedge p > n$

This contradicts our assumption

PS1

$\exists t \in T, \forall f \in F, \neg BelongsTo(t,f) \wedge \forall t_0 \in T, \neg BelongsTo(t_0,f) \Rightarrow t_0 = t$

b) $\exists f \in F, \forall t \in T, BelongsTo(t,f) \Rightarrow Oak(t)$

Cannot define our own predicates
|C|

c) $\exists f \in F, \forall t \in T, Pine(t) \Rightarrow BelongsTo(t,f)$

d) $\exists f \in F, \forall t \in T, BelongsTo(t,f) \wedge Pine(t) \wedge Oak(t)$

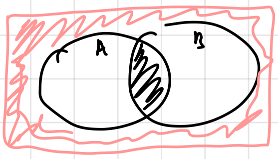
$\forall f \in F, \exists t \in T, \neg BelongsTo(t,f) \vee Pine(t) \vee Oak(t)$

$\exists f \in F, (\forall t \in T, BelongsTo(t,f) \Rightarrow Oak(t))$

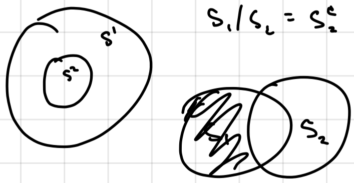
e) $\forall t \in T, t \in F$

2a) $A / (A \cap B)$ A/B or $B^c \cap A$

2b) $(A \cap B) \cup (A \cup B)^c$ $\neg A \cap \neg B$

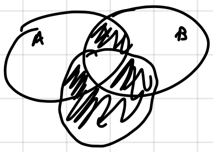


$S_1 / S_2 = S_2^c \cap S_1$



2c) $A \cup B \cup C \setminus ((A \cap B) \cup (B \cap C) \cup (A \cap C))$ $\cup (A \cap B \cap C)$

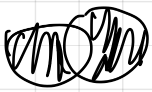
2d)



3. $\forall x \in S, \exists y \in T, (P(x) \vee Q(x,y)) \wedge (\neg P(x) \vee \neg Q(x,y))$
T F F T

$\forall y \in T, \exists x \in S \quad (\neg Q(x,y) \vee P(x)) \wedge (Q(x,y) \vee \neg P(x))$
F T T F

$P(x) \oplus Q(x,y)$



$\forall x \in \{1,2,3,4\}, \exists y \in \{2,3,4\}$ $x = y$ F

$\forall y \in \{2,3,4\} \exists x \in \{1,2,3,4\}$ $x = y$ T

1a) $S = \{1, 3, 5, 7, 9\}$

$|S_1| = 6$

$\forall x \in S, \exists y \in S_1, \ y = x - 1$

$\{0, 2, 4, 6, 8, 10\}$

1b) $T = \{1, 3, 5, 7, 9\}$

$|T_2|^2 = |T_1| \cdot 5 \ - 1$

$\frac{1 + x^2}{5} = 1$

$|t_1| > 1$

$T_2 = \{1, 3, 5\} \quad T_1 = \{1, 3\}$

1c)

p	q	r	$(p \Rightarrow \neg q) \Leftrightarrow r$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	F

1d) Let $x \in \mathbb{N}$

Assume $\exists y \in \mathbb{N}, P(x, y)$ is true

Let $z =$ ___

we will prove that $Q(x, z)$ is true

$\exists x \in P, (\forall p_1 \in P, Loves(p_1, p)) \wedge$

2. $\forall x \in P, Doctor(x) \Rightarrow \forall y \in P, Loves(x, y)$

$\exists p \in P, (\forall p_1 \in P, Loves(p_1, p)) \wedge (\forall q \in P, (\forall p_2 \in P, Loves(p_2, q)) \Rightarrow p = q)$

b. $\exists x \in P, \forall y \in P, Loves(y, x) \wedge (\forall x_0 \in P, Loves(y, x_0)) \rightarrow x_0 = x$ X

c. $(\forall x \in P, \exists y \in P, Loves(x, y) \wedge Doctor(y)) \Rightarrow (\forall z \in P, \neg Doctor(z))$

d.) $\exists x \in P, \neg Doctor(x) \wedge Loves(x, x) \wedge \underbrace{(\forall z \in P, Loves(x, z) \Rightarrow x = z)}_{x \neq z \ \Rightarrow \neg Loves(x, z)}$

3a) $\forall n \in \mathbb{Z}^+, \left(\exists k \in \mathbb{Z}, \sum_{i=1}^{(n^2-1)} i = 2k\right)$

3b) Let $n \in \mathbb{Z}^+$

Case 1: n is odd: $\exists k_0 \in \mathbb{Z}, n = 2k_0 + 1$

$1 + \ldots + 4k_0^2 = 2k$

$4k_0^2(4k_0^2 + 1) = 4k$

Case 2: n is even: $\exists k_1 \in \mathbb{Z}, n = 2k_1$

$1 + \ldots + (4k_1^2 - 1) = 2k$

$(4k_1^2 - 1)4k_1^2 = 4k$

4. $\forall k \in \mathbb{Z}^+, \exists x \in \mathbb{Z}^+, x - (\lfloor \sqrt{x} \rfloor)^2 > k$

Let $k \in \mathbb{Z}^+$

$k \in \mathbb{Z}^+$

Let $x = k^2 + 2k$

$\lfloor \sqrt{x} \rfloor = k$

$\lfloor \sqrt{x} \rfloor = b$

$x + b^2 > b + b^2$

$\forall a, b \in \mathbb{Z}^+,\ b^2 \leq a < (b+1)^2 \Rightarrow \lfloor \sqrt{a} \rfloor = b$

$b^2 \leq x < b^2 + 2b + 1$

$b^2 \leq b^2 + 2b < b^2 + 2b + 1$

$$
\begin{aligned}
x - (\lfloor \sqrt{x} \rfloor)^2 &= x - k^2 \\
&= k^2 + 2k - k^2 \\
&= 2k \\
&> k
\end{aligned}
$$

$\exists c_1, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n + 10 \leq c_1 n^2$

$n_0 = 1$

$1 \leq n$

$n \leq n^2$

$n + 10 \leq n^2 + 10$

$\forall c_2, n_1 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_1 \wedge n + 10 < c_2 n^2$

Let $c_2, n_1 \in \mathbb{R}^+$

Let $n = \lceil n_1 + \frac{2}{c_2} + \sqrt{\frac{20}{c_2}} + 1 \rceil$ $\quad n \in \mathbb{N}$

$\frac{2}{c_2} < n \wedge \sqrt{\frac{20}{c_2}} < n$

$\Rightarrow \frac{n}{c_2} < \frac{n^2}{2} \wedge \frac{10}{c_2} < \frac{n^2}{2}$

$\Rightarrow \frac{n}{c_2} + \frac{10}{c_2} < \frac{1}{2}n^2 + \frac{1}{2}n^2$

$\Rightarrow n + 10 < c_2 n^2$

midterm 1 v2

a) $S_1 = \{1, 2, 3, 4\}$

b) $T_1 = \{4, 6\}$ $T_2 = \{8, 10\}$

$$2 \cdot x^2 = 4 \cdot y$$
$$8 = 8$$

c)

p	q	r	$(\neg p \vee q) \Leftrightarrow r$
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

d) Let $x \in \mathbb{N}$

Assume $\exists y \in \mathbb{N}, P(x,y)$ is true

Let $z = \_\_$

We will prove that $Q(x,z)$ is true

$\forall x \in P$, Teacher$(x) \Rightarrow$ Remembers$(x,x) \wedge$ $(\forall y \in P$, Remembers$(x,y) \Rightarrow y = x)$

$\forall x \in P$, Teacher$(x) \Rightarrow$

2. $\exists x \in P, \exists y \in P$, Student$(x) \wedge$ Teacher$(y) \wedge$ Remembers(x,y)

b. $\forall x \in P$, teacher$(x) \wedge$ Remembers$(x,x) \wedge (\forall x_0 \in P$, Remembers$(x_0, x) \Rightarrow x_0 = x)$

$\exists x \in P, (\forall s \in P$, Student$(s) \Rightarrow$ Remembers$(x,s)) \wedge (\forall y \in P, (\forall s \in P$, Student$(s)$ $\Rightarrow$ Remembers$(y,s)) \Rightarrow y = x)$

c.) $(\exists x \in P$, teacher$(x)) \Rightarrow (\forall y \in P, \neg$ Student$(y))$

d.) $\exists x \in P, \forall y \in P$, (Student$(y) \Rightarrow$ Remembers$(x,y)) \wedge (\forall x_0 \in P, (\forall y_0 \in P$, Student$(y_0) \Rightarrow$ Remembers$(x_0, y_0)) \Rightarrow x_0 = x)$

3a) $\sum_{i=0}^{n-1} (x+i)$. for some $x \in \mathbb{Z}$

$$\forall n \in \mathbb{Z}^+, (\exists k \in \mathbb{Z}, n = 2k+1) \Rightarrow (\forall x \in \mathbb{Z}, \exists k_0 \in \mathbb{Z} \ \sum_{i=0}^{n-1} (x+i) = nk_0)$$

Let $n \in \mathbb{Z}^+$. Assume $\exists k \in \mathbb{Z}, n = 2k+1$

Let $x \in \mathbb{Z}$

Let $k_0 = x + \frac{n-1}{2} \rightarrow x + k$

We will show that $\sum_{i=0}^{n-1} (x+i) = nk_0$

$$\sum_{i=0}^{n-1} (x+i) = nx + \frac{n(n-1)}{2} \quad \text{By the formula}$$

$$= n\left(x + \frac{2k+1-1}{2}\right) \quad \text{By our choice of } k_0$$

$$= nk_0$$

4a. $\forall y \in \mathbb{R}^{\geq 0}, (y > 20) \Rightarrow (\exists x \in \mathbb{R}^{\geq 0}, x^{\lfloor x \rfloor} = y)$

$\exists y \in \mathbb{R}^{\geq 0}, \ y > 20 \land \forall x \in \mathbb{R}^{\geq 0}, \ x^{\lfloor x \rfloor} \neq y$

$y = 26 = 3^3 - 1$

Case 1 : $x \geq 3$

y

Let $\lfloor x \rfloor \geq 3 \Rightarrow x^{\lfloor x \rfloor} \geq x^3$

$x^{\lfloor x \rfloor} \geq 3^3 = 27 \neq 26$

Case 2 : $x < 3$

$\lfloor x \rfloor < 3$

$\lfloor x \rfloor \leq 2$

$x^{\lfloor x \rfloor} \leq x^2$

$x^{\lfloor x \rfloor} \leq 3^2 = 9 \neq 26$

TT1

1a) $6 > 5 > 4$

$\{5, 10, 15, 25, 35\}$

1a) $xy < 5^2$

$T_1 = \{1\}$ $T_2 = \{1\}$

$\{(1,1)\} = \{(1,1)\}$

1b) $\{\emptyset, \{\emptyset\}\}$

$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

1b) $\{\{\{0\}, \emptyset\}, \{\{0\}\}, \{\emptyset\}, \emptyset\}$

1d) Let $n \in \mathbb{N}$

Assume $n > 2$

Let $x = \_\_$, $y = \_\_$

We want to prove that Prime(x) and Prime(y) and $x+y=n$ are true

1d) Let $n = \_\_$

Let $n_0 \in \mathbb{N}$

We will prove that $n_0 > n$ and Prime(n_0) and Prime(n_0+2) are false.

2a) $\exists c \in C, (\forall s \in P, \text{Enrolled}(s,c))$

2b) $\left(\exists x \in P, (\forall y \in P, \text{Teaches}(x,y)) \wedge (\forall x_0 \in P, \forall y_0 \in P, \text{Teaches}(x_0, y_0) \Rightarrow x_0 = x\right)$

2c) $\forall x \in P, \exists x_0 \in P, \exists y \in C, (\text{Teaches}(x, x_0) \wedge \text{Enrolled}(x_0, y)) \Rightarrow \neg\text{Enrolled}(x,y)$ $\forall x \in P, \forall s \in P, \text{Teaches}(x,s) \Rightarrow \forall c \in C, \text{Enrolled}(s,c) \Rightarrow \neg\text{Enrolled}(x,c)$

2d) $\forall x \in P, (\forall y \in P, \neg\text{Teaches}(x,y)) \Rightarrow (\exists c \in C, \text{Enrolled}(x,c))$

2e) $\exists s \in P, \forall c \in C, \text{Enrolled}(s,c)$

2f) $\forall s \in P, \exists y \in P, \neg\text{Teaches}(s,y)$

2g) $\forall x \in P, (\exists y \in P, \text{Teaches}(x,y)) \Rightarrow (\forall c \in C, \neg\text{Enrolled}(x,c))$

2h) $\forall x_1 \in P, (\exists x_2 \in P, \text{Teaches}(x_1, x_2)) \Rightarrow (\exists c \in C, \text{Enrolled}(x_1, c))$

2i) $\exists x_1, x_2 \in P, \exists c_1, c_2 \in C, c_1 \neq c_2 \wedge \text{Enrolled}(x_1, c_1) \wedge \text{Enrolled}(x_2, c_1) \wedge \text{Enrolled}(x_1, c_2) \wedge \text{Enrolled}(x_2, c_2)$

2j) $\forall x_1, x_2 \in P, \exists c \in C, (\text{Enrolled}(x_1, c) \wedge \text{Enrolled}(x_2, c)) \Rightarrow (\exists y \in P, \text{Teaches}(y, x_1) \wedge \text{Teaches}(y, x_2))$

2k) $\forall x \in P, \neg\text{Teaches}(x,x)$

2l) $\exists x \in P,\ \text{Teaches}(x,x) \wedge (\forall x_0 \in P,\ \text{Teaches}(x_0,x) \Rightarrow x_0 = x)$

different to winter 2020 uniqueness

2m) $\forall x \in P,\ \text{Teaches}(x,x) \wedge (\exists y \in P,\ \text{Teaches}(x,y))$

$\wedge$ not an $\Rightarrow$

2n) $\exists c \in C, \exists x_1, x_2 \in P,\ \text{Enrolled}(x_1,c) \wedge \text{Enrolled}(x_2,c) \wedge (\forall y \in P,\ \text{Enrolled}(y,c) \Rightarrow y = x_1 \vee y = x_2)$

2o) $\forall c \in C, \exists x \in P,\ \text{Enrolled}(x,c)$

2p) $\exists x \in P, \exists c \in C, (\text{Enrolled}(x,c)) \Rightarrow (\exists z \in P,\ z \neq x \wedge \text{Teaches}(z,x))$

3.) $\exists x,y \in \mathbb{Z},\ x \mid y \wedge y \mid x \wedge x \neq y$

Let $x = -1$
$y = 1$

Let $k_1 = -1$
$-1 = -1$
$-1 = 1(-1)$
then $y \mid x$

Let $k = -1$
$1 = 1$
$1 = (-1)(-1)$ sign law
then $x \mid y$

but $x \neq y$ $\square$

3a) $\exists x,y \in \mathbb{R},\ x > 0 \wedge y > 0 \wedge \lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$

Let $x = 0.5$
$y = 0.5$

$\lfloor 0.5 + 0.5 \rfloor = 1$
$\lfloor 0.5 \rfloor + \lfloor 0.5 \rfloor = 0$

3a) $\exists x,y \in \mathbb{R},\ x > 0 \wedge y > 0 \wedge \sqrt{4x} + \sqrt{2y} \neq \sqrt{4x+2y}$

Let $y = 2$
$x = 1$

$\sqrt{4(1)} + \sqrt{2(2)} = 4 \neq \sqrt{8} = \sqrt{4(1)+2(2)}$

$\forall x \, \exists y \, \forall z \, , \, P(z,y) \Rightarrow Q(z,x)$

$\forall x \, \forall z \, \exists y \, , \, P(z,y) \Rightarrow Q(z,x)$

$\exists y \, \forall x \, \forall z \, , \, P(z,y) \Rightarrow Q(z,x)$

$\exists y \, \forall z \, \forall x \, , \, P(z,y) \Rightarrow Q(z,x)$

p	q	$p \to q$	$\neg p \vee q$		$\neg p \wedge \neg q$	$\neg p \wedge q$	$p \wedge \neg q$	$p \wedge q$
F	F	T	T	$\neg p \wedge \neg q$	T	F	F	F
F	T	T	T	$\neg p \wedge q$	F	T	F	F
T	F	F	F	$p \wedge \neg q$	F	F	T	F
T	T	T	T	$p \wedge q$	F	F	F	T

$\forall x [Px \to Qx]$

PS2

1a) $\forall n \in \mathbb{Z},\ \gcd(7n+1, 15n+2) = 1$

By fact 3, $\gcd(7,1) = 1$

$\gcd(2,15) = 1$

It suffices to show

$\gcd(\gcd(7,1), \gcd(15,2)) = 1$

Let $n \in \mathbb{Z}$, $Prime(7n+1) \wedge 7n+1 \neq 15n+2$

$7n+1 = \gcd(7,1) = 1$

$15n+2 = \gcd(15,2) = 1$

1b) $\exists p \in \mathbb{N}, (\exists n \in \mathbb{Z}, p = n^3 - 1) \wedge Prime(p) \wedge (\forall p_0 \in \mathbb{N}, Prime(p_0) \wedge (\exists n_1 \in \mathbb{Z}, p_0 = n_1^3 - 1) \Rightarrow p_0 = p)$

Let $p = 7$, let $n = 2$

$7 = 2^3 - 1$

$7 = 7$

$prime(7) = True$

Let $p_0 \in \mathbb{N}$

Assume $\exists n_1 \in \mathbb{Z}, p_0 = n_1^3 - 1$

Use n from definition of $\exists n_1 \in \mathbb{Z}, p_0 = n_1^3 - 1$

prove $n_1 = n$?

$n_1^3 = n^3$

$n_1^3 - 1 = n^3 - 1$

$p_0 = p$

$(a-b)(a^2 + 2ab + b^2)$

prime

$p = ab$

$a = 1, b = p$

$n^3 - 1 = (n-1)(n^2+n+1)$

$Prime(n^3-1) : n^3 - 1 > 1 \wedge (\forall d \in \mathbb{Z}, d \mid x \Rightarrow d = 1 \vee d = n^3 - 1)$

Case 1. $(n-1) = 1$

$n^2 + n + 1 = p$

Case 2. $(n^2 + n + 1) = 1$

$(n-1) = p$

Q2)

$\forall x, y \in \mathbb{R}, x \leq y \Rightarrow \lceil x \rceil \leq \lceil y \rceil$

Let $x, y \in \mathbb{R}$

Assume $x \leq y \Rightarrow 0 \leq y - x$

by definition,

$\lceil y \rceil - 1 < y \leq \lceil y \rceil$

$\lceil x \rceil - 1 < x \leq \lceil x \rceil$

so, $y \leq \lceil x + y - x \rceil$

$y \leq \lceil x + 0 \rceil \leq \lceil x + y - x \rceil$ (since $0 \leq y - x$)

$\lceil x \rceil \leq \lceil y \rceil$

$\exists x, y \in \mathbb{R}, \lceil x \rceil \leq \lceil y \rceil \wedge x > y$

Let $x = 0.7$

$y = 0.5$ $\quad \lceil 0.7 \rceil \leq \lceil 0.5 \rceil$

$1 = 1$

$\Rightarrow 0.7 > 0.5$

Let $x \in \mathbb{Z}$

x is odd

$\frac{x+1}{2} = \lceil \frac{x}{2} \rceil$

$\frac{x}{2} + \frac{1}{2} = \lceil \frac{x}{2} \rceil$

$\frac{1}{2} = \lceil \frac{x}{2} \rceil - \frac{x}{2}$ and by fact 2, $0 \leq \lceil \frac{x}{2} \rceil - \frac{x}{2} < 1$

but, $\lfloor \frac{x+1}{2} \rfloor = \frac{x+1}{2}$

so, $\lfloor \frac{x+1}{2} \rfloor = \frac{x+1}{2} = \lceil \frac{x}{2} \rceil$

x is even

$\lfloor \frac{x+1}{2} \rfloor = \frac{x}{2}$

$\lfloor \frac{x+1}{2} \rfloor + \frac{1}{2} = \frac{x+1}{2}$

$\frac{1}{2} = \frac{x+1}{2} - \lfloor \frac{x+1}{2} \rfloor$ by fact 1, $0 \leq \frac{x+1}{2} - \lfloor \frac{x+1}{2} \rfloor < 1$

but $\frac{x}{2} = \lfloor \frac{x}{2} \rfloor$

so, $\lfloor \frac{x+1}{2} \rfloor = \frac{x}{2} = \lfloor \frac{x}{2} \rfloor$

Q3)

$\frac{1}{2} \times \dots \times \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n+3}}$

$\frac{1}{9} < \frac{1}{2}$

$\frac{1}{2} \times \dots \times \frac{2n-1}{2n} \leq \frac{1}{\sqrt{4}} \leq \frac{1}{\sqrt{3n}}$

$\frac{1}{2} \times \dots \times \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n}} \cdot \frac{2n+1}{2n+2} \leq \frac{1}{\sqrt{3n+3}}$

Base Case $n=1$

$\frac{1}{2} \leq \frac{1}{2}$

$\frac{1}{\sqrt{4}} \leq \frac{1}{\sqrt{3n}}$ $n=1$

Inductive Step

$= \frac{1}{2} \times \dots \times \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} \leq \underbrace{\frac{1}{2} \cdot \frac{2n+1}{2n+2}} \leq \underbrace{\frac{1}{2} \leq \frac{1}{\sqrt{3n+3}}}$

$\frac{1}{2} \times \dots \times \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} \leq \frac{1}{2} \frac{2n+1}{2n+2} \leq \frac{1}{2} \frac{2n+1}{(2n+1)}$

Wait

$\frac{1}{4n+4} \leq \frac{1}{4n+2} \leq \frac{1}{\sqrt{3n+3}}$

$\frac{1}{\sqrt{\ }} \leq \frac{1}{\sqrt{3n}} \Rightarrow \frac{1}{\sqrt{\ }} \leq \frac{1}{\sqrt{3n+3}}$

$\frac{1}{2n+1} \leq \frac{1}{\sqrt{3n+3}}$

$3n+3 \leq (2n+1)^2$

$2 \leq 4n^2 + n$ $n=1$

$2 \leq 4(1)^2 + 1$

$2 \leq 5$

$\exists k \in \mathbb{Z},\ 2^{2n+1} + 1 = 3k$

Base Case $n=0$ $k=1$

$2+1 = 3k$

$3=3$

Inductive step

Assume $2^{2n+1} + 1 = 3k_0$ prove $2^{2n+3} + 1 = 3k_1$

$2^2 \cdot 2^{2n+1} + 1 = 3k_1$

$3 \cdot 2^{2n+1} + 2^{2n+1} + 1 = 3k_1$

$3 \cdot 2^{2n+1} + 3k_0 = 3k_1$

$3(2^{2n+1} + k_0) = 3k_1$

$\frac{1}{\sqrt{3n+1}} \leq \frac{1}{\sqrt{3n}} \Rightarrow \frac{1}{3n+1} \leq \frac{1}{3n}$

$\frac{1}{\sqrt{9}} \leq \frac{1}{\sqrt{4}} \leq \frac{1}{(3n+1)} \cdot \frac{2n+1}{(2n+2)} \leq \frac{1 \cdot 2n+1}{(3n+4)(2n+1)} \leq \frac{1}{3n+3}$

$\frac{2n+1}{6n^2 + 8n + 2} \leq \frac{2n+1}{6n^2 + 8n + 3n + 2 + 2}$

$\frac{2n+1}{6n^2 + 8n + 2} \leq \frac{2n+1}{6n^2 + 8n + 2 + 3n + 2}$

$\frac{1}{2} \times \dots \times \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$

Base Case $n=1$

$3n+1$

Inductive step $P(n) \Rightarrow P(n+1)$

Assume $\frac{1}{2} \times \dots \times \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n-1}}$

$\frac{1}{(3n-1)} \cdot \frac{2n+1}{(2n+3)} \leq \frac{1}{3n+2} \frac{2n+1}{2n+2}$

$\frac{1}{(3n+1)} \cdot \frac{(2n+1)^2}{(3n+2)^2} \leq \frac{1}{\sqrt{3n+4}}$

$(3n+1)(4n^2+8n+4) \leq \frac{(2n+1)^2}{(3n+4)(2n+1)^2}$

$(3n+4)(4n^2+4n+1)$

$\frac{1}{12n^3 + 24n^2 + 12n + 4n^2 + 8n + 4} \leq \frac{1}{12n^3 + 12n^2 + 3n + 16n^2 + 16n + 4}$

$\cancel{12n^3} + \cancel{12n^2} + 3n + \cancel{16n^2} + 16n + 4 \leq \cancel{12n^3} + \cancel{24n^2} + 12n + \cancel{4n^2} + 8n + 4$

$19n \leq 20n$

$\frac{1}{3n} \cdot \frac{(2n+1)^2}{(2n+2)^2} \leq \frac{1}{3n+3}$

$\frac{1}{(3n+2)}$

$3n+3$

0

$0 \leq 5n+3$

$0 \leq 9n+6$

$(3n+4)\cdot(2n+1)^2 \leq (3n+1)(2n+2)^2$

$(3n+4)(4n^2+4n+1) \leq (3n+1)(4n^2+8n+4)$

$\cancel{12n^3} + \cancel{12n^2} + 3n + \cancel{16n^2} + 16n + \cancel{4} \leq \cancel{12n^3} + \cancel{24n^2} + 12n + \cancel{4n^2} + 8n + \cancel{4}$

$19n \leq 20n$

$0 \leq 20n - 19n$

$0 \leq n$

Q4) $g_1 : \mathbb{Z} \to \mathbb{Z}$

$g_1(x) = x - 4$

$\forall a_1, a_2 \in \mathbb{Z},\ (a_1 - 4) = (a_2 - 4) \Rightarrow a_1 = a_2$

Assume $a_1 \cancel{-4} = a_2 \cancel{-4}$

$a_1 = a_2$

$\forall b \in \mathbb{Z}, \exists a \in \mathbb{Z},\ a - 4 = b$

Let $b \in \mathbb{Z}$

Let $a = b + 4$

$(b+4) - 4 = b$

$b = b$

$g_2 : \mathbb{R} \to \mathbb{R}$

$g_2(x) = |x| + x$

$\exists a_1, a_2 \in \mathbb{R},\ |a_1| + a_1 = |a_2| + a_2 \wedge a_1 \neq a_2$

$|-1| + -1 = |0| + 0 \qquad -1 \neq 0$

$1 + -1 = 0 + 0$

$0 = 0$

$\exists b \in \mathbb{R}, \forall a \in \mathbb{R},\ |a| + a \neq b$

If $a \geq 0$

$|a| + a = 2a$

If $a < 0$

$|a| + a = a - a = 0$

$a \in [0, \infty) \neq b$

4b) $f_1 : \mathbb{Z} \to \mathbb{Z}^+$

$-e^x$

$\forall a_1, a_2 \in \mathbb{Z},\ -e^{a_1} = -e^{a_2} \Rightarrow a_1 = a_2$

Assume $-e^{a_1} = -e^{a_2}$

$a_1 \cancel{\ln(e)} = \cancel{\ln(e)}\, a_2$

$a_1 = a_2$

$\exists b \in \mathbb{Z}^+, \forall a \in \mathbb{Z},\ -e^a \neq b$

$-e^a \neq 1$

$f_2 : \mathbb{Z} \to \mathbb{Z}^+$

$|x|$

$\exists a_1, a_2 \in \mathbb{Z},\ |a_1| = |a_2| \wedge a_1 \neq a_2$

$|1| = |-1| \qquad 1 \neq -1$

$1 = 1$

$\forall b \in \mathbb{Z}^+, \exists a \in \mathbb{Z},\ |a| = b$

Let $a = b \qquad |b| = b$

$b = b$

4c) $g \circ f$ is 1-1 $\Rightarrow f$ is 1-1

$\forall a \in A, (g \circ f)(a) = g(f(a)) \wedge \forall a_1, a_2 \in A,\ g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$

$\Rightarrow \exists a_3, a_4 \in A,\ f(a_3) = f(a_4) \wedge a_3 \neq a_4$

Let $a_3, a_4 =$

$g(f(a_1)) = g(f(a_1)) \Rightarrow a_1 = a_2$

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$\forall b \in C, \exists a \in A,\ g(f(a)) = b \Rightarrow \forall b_0 \in C, \exists a_0 \in B,\ g(a_0) = b_0$

$\exists b_0 \in C, \forall a_0 \in B,\ g(a_0) \neq b_0$

$b = g(a_0)$

$g(f(a)) \neq g(a_0)$

4c) $(\forall a_1, a_2 \in A, g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2) \wedge (\forall c \in C, \exists a \in A, g(f(a)) = c)$

$\Rightarrow [(\forall a_3, a_4 \in A, f(a_3) = f(a_4) \Rightarrow a_3 = a_4) \wedge (\forall b \in B, \exists a \in A, f(a) = b)] \wedge$

$((\forall b_1, b_2 \in B, g(b_1) = g(b_2) \Rightarrow b_1 = b_2) \wedge (\forall c \in C, \exists b \in B, g(b) = c)$

p	q	$p \wedge q$	$p \vee q$	$\neg p \vee \neg q$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	T	F

p : f is 1-1 $\neg p$: f is not 1-1

q : f is onto $\neg q$: f is

$A \to B \to C$

$g \circ f$ is both 1-1 $\wedge$ onto

g is not 1-1 or onto   f is not 1-1 or onto

$(g \circ f)(1) = g(f(1))$

$-e^{|x|}$

$X = \{0\}$ $Y = \{0, 1\}$

$f : X \to Y$

f(

$g : X \to Y$

$g(1) = g(0) = 0$

Exercise Break 3.13

$\forall n, m \in \mathbb{N}$, $|A| = n$, $|B| = n$

$$|A \times B| = |A| \times |B| = n \cdot m$$

Base case $n=0$

$0m = 0$

Induction: $P(n,m) \Rightarrow P(n+1, m)$

Assume $|A \times B| = n \cdot m$

Prove $|A \cup \{s_{n+1}\} \times B| = (n+1)m$

$= nm + m$

$= |A \times B| + |B|$

Part 1: cartesian product that contains s_{n+1}

$\forall a,b \in \mathbb{R}^+,\ an+b \in O(n^2)$

Let $a,b \in \mathbb{R}^+$
Let $c =$ ___ , $n_0 =$ ___
Let $n \in \mathbb{N}$. assume $n \geq n_0$
WTP: $an+b \leq c \cdot n^2$

Approaches:
Idea 1: solve for c in $cn^2 - an - b \geq 0$ — avoid when possible
[exercise]

Idea 2: focus on c
Try $n_0 = 1 \Rightarrow n \geq 1$

$an = an \cdot 1 \leq an \cdot n = n^2$

$1 \leq n$
$1 \leq n^2$ multiply by itself
$b \leq bn^2$

$an+b \leq an^2 + bn^2 = (a+b)n^2$

Idea 3: $c=1$, focus on n_0
$an+b \leq n^2$
$\Leftarrow an+b \leq \frac{n^2}{2} + \frac{n^2}{2}$

$\Leftarrow an \leq \frac{n^2}{2} \land b \leq \frac{n^2}{2}$

$2a \leq n \land \sqrt{2b} \leq n$

so pick $n_0 = \max(2a, \sqrt{2b})$ — $\max(0,n)$ — we don't care for negative values

Then $n_0 \in \mathbb{R}$

Then $an+b \leq an^2+bn^2$ (since $n \geq 1$)
$\leq cn^2$

Ex. Prove $\forall a, b \in \mathbb{R}^+$, $an + b \notin \Omega(n^2)$

WS 11

$\forall a,b \in \mathbb{R}^+,\ a \leq b \Rightarrow n^a \in O(n^b)$

Let $a,b \in \mathbb{R}$.

Assume $a \leq b$

prove $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow n^a \leq c \cdot n^b$

Let $c = \underline{1}$ $\quad c = 1$ $\quad n_0 = 1$

$n_0 = \underline{\text{any}}$ $\quad n^a \leq n^b$ $\quad n \geq 1$

Let $n \in \mathbb{N}$ $\quad a \leq b$ $\quad 1 \leq n$

Assume $n \geq n_0$ $\quad n^a \leq cn^b$

2. logs

$\forall a,b \in \mathbb{R}^+,\ a > 1 \wedge b > 1 \Rightarrow \log_a n \in O(\log_b n)$

Let $a,b \in \mathbb{R}^+$

Assume $a > 1 \wedge b > 1$

Prove $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow \log_a n \leq c \cdot \log_b n$

Let $c = \dfrac{1}{\log_b a}$

$n_0 = \underline{\quad}$ $\quad 1 \leq n$ $\quad \log_a n \leq c \log_b n$

Let $n \in \mathbb{N}$

Assume $n \geq n_0$ $\quad \leq c \log_b a$

$\log_a n = \dfrac{\log_b n}{\log_b a}$

$\log_a n = \dfrac{1}{\log_b a} \log_b n$

$\log_a n = c \log_b n$

$\log_a n \leq c \log_b n$

3. Sum $\quad f+g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$\forall n \in \mathbb{N},\ (f+g)(n) = f(n) + g(n)$

$\forall f,g : \mathbb{N} \to \mathbb{R}^{\geq 0},\ g \in O(f) \Rightarrow f+g \in O(f)$

Let $f,g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Assume $g \in O(f)$, that is

$\exists c_1, n_0, \forall n \in \mathbb{N},\ n \geq n_0 \Rightarrow g(n) \leq c_1 f(n)$

Prove $\exists c_2, n_1, \forall n \in \mathbb{N},\ n \geq n_1 \Rightarrow f(n) + g(n) \leq c_2 f(n)$

Let $c_2 = \underline{c_1 + 1}$

$n_1 = n_0$ $\quad$ then $\quad n_0 \leq n \Rightarrow g(n) \leq c_1 f(n)$

$g(n) + f(n) \leq c_1 f(n) + f(n)$

$g(n) + f(n) \leq (c_1 + 1) f(n)$

$g(n) + f(n) \leq c_2 f(n)$

ps3

q1: $(b_{K-1}\cdots b_0 a)_2 = 2(b_{k-1}\cdots b_0) + a$

$$(b_{k-1}\cdots b_0)_2 = \sum_{i=0}^{k-1} b_i 2^i$$

$$2(b_{k-1}\cdots b_0)_2 = \sum_{i=0}^{k-1} b_i 2^{i+1}$$

$$= \sum_{i=1}^{k} b_{i-1} 2^i + a2^0$$

$$= \sum_{i=1}^{k} b_{i-1} 2^i + b_0 2^0$$

$$= \sum_{i=0}^{k} b_i 2^i$$

$b_0 = a$

$$\sum_{i=0}^{k} b_i 2^i$$

$$\sum_{i=0}^{k-1} b_i 2^i + b_k 2^k$$

$$+ a2^k$$

q2) Let $n \in \mathbb{N}$, $(b_{2n-1}\cdots b_0)_2$

prove $3 \mid (b_{2n-1}\cdots b_0)_2 - (b_1 b_0)_2 - (b_3 b_2)_2 - \cdots - (b_{2n-1} b_{2n-2})_2 = 3k$

$\forall n \in \mathbb{N}$, $3 \mid 4^n - 1$

$(b_{2n-1}\cdots b_0)_2 = 3k +$

Base Case $n=0$

$(b_{-1} b_0)_2 - (b_{-1} b_{-2}) = 3k$

$\sum_{i=0}^{-1} b_i 2^i = 0$ empty sum

$\Rightarrow 3 \mid 0$

Induction Step $P(n) \Rightarrow P(n+1)$

Assume $(b_{2n-1}\cdots b_0)_2 - (b_1 b_0)_2 - (b_3 b_2)_2 - \cdots - (b_{2n-1} b_{2n-2})_2 = 3k$

prove $(b_{2n+1}\cdots b_0)_2 - (b_1 b_0)_2 - (b_3 b_2)_2 \cdots - (b_{2n+1} b_{2n})_2 = 3k_0$

$k_0 = d(b_{2n+1} b_{2n}) + k$

$= (b_{2n+1} b_{2n} b_{2n-1}\cdots b_0)_2 \underbrace{- (b_1 b_0)_2 - (b_3 b_2)_2 - \cdots - (b_{2n-1} b_{2n-2})_2}- (b_{2n+1} b_{2n})_2$

$= 2^{2n+1} \cdot b_{2n+1} + 2^{2n} \cdot b_{2n} + (b_{2n-1}\cdots b_0) -$ " $- (b_{2n+1} b_{2n})_2$

$= 2^{2n}(2^1 b_{2n+1} + 2^0 b_{2n}) + 3k - (b_{2n+1} b_{2n})_2$

$= (4^n - 1)(b_{2n+1} b_{2n})_2 + 3k$

$= 3d(b_{2n+1} b_{2n})_2 + 3k$

$= 3(d(b_{2n+1} b_{2n})_2 + k)$

$= 3k_0$

1c) $x \in \mathbb{N}$

$= (b_{k-1} \cdots b_0)_2$

$=$ # 1 bits (even index) $-$ # 1 bits (odd index) $= 3k_0$

if K is even,

$|\{b_0, b_2, b_4, \dots, b_{k-2}\}| - |\{b_1, b_3, \dots, b_{k-1}\}| = 3k_0$

K is odd

$|\{b_0, \dots, b_{k-1}\}| - |\{b_1, \dots, b_{k-2}\}| = 3k_0$

$(10101)_2 = 3 - 0 = 3 \leftarrow K = 1$

$(1010111)_2 = 4 - 1 = 3$

$(101010)_2 = 0 - 3 = -3 = 3(-1)$

$b_5 b_4 b_3 b_2 b_1 b_0$

$b_4 b_3 b_2 b_1 b_0$

$(10101)_2 = 3k_1$

$1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 =$

$b_8 \quad b_0$

0101110111

$5 - 2$

$-(11)_2 - (01)_2 - (11)_2 - (01)_2 - (01)_2$

$(0101110111)_2 - 2(11)_2 - 3(01)_2 = 3k$

$(0101010011)_2 \quad (0101010111)$

or

$(11)_2 \quad 3(10)_2$

$10 \quad -2 \quad -1$

$K = 6$

$\sum_{i=0}^{(k-1)/2} b_{2i} \cdot 2^{2i}$ if $K-1$ even

$\sum_{i=0}^{(k-2)/2} b_{2i} \cdot 2^{2i}$ if $K-1$ odd

$\sum_{i=0}^{(k-1)/2} b_{2i-1} \cdot 2^{2i-1}$

$\sum_{i=0}^{(k-2)/2} b_{2i+1} \cdot 2^{2i+1}$

$(101010)_2 - 3(10)_2 = 3K$

$2^3 + 1, \ 2^2 - 1, \ 2^1 + 1$

$(01010111)_2 - 1(11)_2 - 3(01)_2 = 3K$

$(1113)_4 - (3)_4 - 3(1)_4 = 3k$

$(2223)-$

$b_0 2^0 - 2^0 + b_1 2^1 + 2^0 \quad + b_2 2^2 - 2$

$(b_0 - 1)_2 + (b_1 1)_2 + (b_2(-1))_2$

$(123)_4 = 4^0 \cdot 3 + 4^1 \cdot 2 + 4^2 \cdot 1$

$2^2 - 1$

$|E - O| = 3k$

$(0)_4 = (00)_2$

$(1)_4 = (01)_2$

$(2)_4 = (10)_2$

$(3)_4 = (11)_2$

#E : number of even indices, #O : ... odd indices

Assume $(x)_2 \rightarrow \#E - \#O = 3K$ for some $K \in \mathbb{Z}$

by 1b we see that

$(x)_2 - (\alpha E + \beta OE) = 3k_1$ for some $\alpha, \beta \in \mathbb{N}$

Hence we want to show

$(x)_2 = 3k_0$

Let $k_0 = k_1 + K + \beta$

$(10)_2 \rightarrow +1$

$(01)_2 \rightarrow -1$

$(11)_2$

$(00)_2$

by assumption, $\alpha = 3k$ since $\#E - \#O$ is a multiple of 3.

$$
\begin{aligned}
(x)_2 &= 3k_1 + (\alpha E + \beta OE) \\
&= 3k_1 + (3kE + 3\beta) \\
&= 3(k_1 + K + \beta) \\
(x)_2 &= 3k_0
\end{aligned}
$$

thus,

$x = 3k_0$

Assume $\#E - \#O = 3k$ for some $(x)_2$

from 1b) $k_1 = k_0 + \#O + k$

$(x)_2 - (\#O(10)_2 + \#E(01)_2) = 3k_0$

$(x)_2 = 3k_0 + \#O(10)_2 + \#E(01)_2$

$= 3k_0 + \#O(10)_2 + (\#O + 3k)(01)_2$

$= 3k_0 + \#O(10)_2 + \#O(01)_2 + 3k$

$= 3k_0 + \#O(11)_2 + 3k$

$= 3(k_0 + \#O + k)$

$= 3k_1$

$(101010)_2 = 2^1 + 2^3 + 2^5 = 2 + 8 + 32 =$

$(10101)_2 = 2^0 + 2^2 + 2^4 = 1 + 4 + 16 =$

Assume

$$\frac{(n+m)!}{(n+1)!\,(m-1)!} \wedge \frac{(n+m)!}{n!\,m!}$$

$P(2,1) = 2$

$P(3,1) = 3$

$P(2,2) = 3$

$P(3,2) = 6$

$n+1 = x_1 + \cdots + x_m$

$n = x_1 + \cdots + x_m + x_{m+1}$

# of ways to write

$n+1 = x_1 + \cdots + x_{m+1} + 1$

$n+1 = \underbrace{x_1 + \cdots}_{} + x_{m+1}$

$\# \; n+1 \to x_{m+1}$ for fixed $x_1 \ldots x_m$

$n+1$

$x_1 \ldots x_{m+1}$

$\# 1 \to x_1 + \cdots + x_{m+1}$ for fixed n.

$$\frac{(n+m-1)!}{n!\,(n-1)!}$$

$\Rightarrow P(m+1,n) + P(m,n+1)$

$$\Rightarrow \frac{(n+m)!}{n!\,m!} + \frac{(n+m)!}{(n+1)!\,(m-1)!}$$

$P(2,1)$ 3

$n = x_1 + \cdots + x_{m+1}$

$1 = 0 + 0 + 1$

$P(3,1)$ 6

$n+1 = x_1 + \cdots + x_m$

$2 =$ $+1$

$$\Rightarrow \frac{(n+m+1)(n+m)!}{(n+1)!\,m(m-1)!}$$

$$\Rightarrow \frac{(n+m+1)!}{(n+1)!\,m!}$$

$\Rightarrow P(m+1, n+1)$

Comb w/ rep

$$\binom{n+m-1}{n} = \binom{n+m-1}{m-1}$$

$$P(m+1,n+1) \Rightarrow \binom{n+m+1}{n+1} = \binom{n+m+1}{m}$$

$$P(m+1,n) \Rightarrow \binom{n+m}{m} = \binom{n+m}{n}$$

$$P(m,n+1) \Rightarrow \binom{n+m}{n+1} = \binom{n+m}{m-1}$$

$P(m+1,n)$, $P(m,n+1)$

$n = x_1 + x_2 + \cdots + x_m$

~~$P(m+1,n) \wedge P(m,n+1)$~~

$n = x_1 + \; + x_2 + \cdots + x_{m+1}$

$n+1 = x_1 + \cdots + x_{m+1}$

$\downarrow$

0

$P(m,n+1) \Rightarrow n+1 = x_2 + \cdots + x_{m+1}$

$\to \geq 1$

$n+1 = x_1 + \cdots + x_{m+1}$

$n = (x_1 - 1) + x_2 + \cdots + x_{m+1}$ $\Leftarrow P(m+1,n)$

$$\binom{n+1+m-1}{n+1} + \binom{n+m+1-1}{n}$$

$$\binom{n+m}{n+1} + \binom{n+m}{n} \Rightarrow \binom{n+m+1}{n+1}$$

b) i) $P(1,2)$

ii) $P(2,1)$

iii) $P(1,2) \wedge P(2,1) \rightarrow P(2,2)$

i) $P(1,3)$

ii) $P(3,1)$

iii) $P(1,3) \wedge P(3,1) \Rightarrow P(3,3)$

c) $t \in \mathbb{Z}^+$ w/ $t \geq 2$ $\quad Q(t): \forall m,n \in \mathbb{Z}^+, m+n=t \Rightarrow P(m,n)$

$\forall t \in \mathbb{Z}^+, t \geq 2 \Rightarrow Q(t)$

" $\quad \Rightarrow (\forall m,n \in \mathbb{Z}^+, m+n=t \Rightarrow P(m,n))$

Let $t \in \mathbb{Z}^+$

Assume $t \geq 2$

Let $m,n \in \mathbb{Z}^+$

Assume $m+n=t$

$m+n \geq 2 \Rightarrow m+n \geq 1+1$

Prove $P(m,n)$

Base Case 1: $n=1$, $m \in \mathbb{Z}^+$

$P(m,1)$ by ii holds

Base Case 2: $m=1$, $n \in \mathbb{Z}^+$

$P(1,n)$ by i holds

Prove

$P(m,n) \Rightarrow P(m+1,n) \wedge P(m,n+1)$

$\Rightarrow P(m+1,n+1)$

Assume $P(m,n)$

$n = x_1 + \dots + x_{m+1}$

$n = (x_1 + x_2) + \dots + x_{m+1}$

By I.H

$P(m+1,n)$

WLOG

$n+1 = x_1 + \dots + x_m$

$n = (x_1 - 1) + \dots + x_m$

By I.H

$P(m,n+1)$

By 2a iii, $P(m+1,n) \wedge P(m,n+1) \Rightarrow P(m+1,n+1)$

d) Let $m,n \in \mathbb{Z}^+$

$\forall t \in \mathbb{Z}, \forall m,n \in \mathbb{Z}^+, Q(t)$

Algorithms w/ a single input

```
def twisty (n: int):
    while n > 1:
        if n % 2 == 0
            n = n // 2
        else:
            n = 2 * n - 2
```

→ # iterations

body: $\Theta(1)$

$\max\{RT(x) \mid$

g is an upper bound on WC $(WC(n) \in O(g(n)))$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \forall x \in I_n, RT(x) \leq c \cdot g(n)$

f is a lower bound on WC $(WC(n) \in \Omega(f(n)))$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow WC(n) \geq c \cdot f(n)$

$\exists x \in I_n, RT(n) \geq c \cdot f(n)$

is_pal()

Upper bound: overestimate

Talk about all inputs because an input could be worse.

- Let $n \in \mathbb{N}$
- For any input s w/ len$(s) = n$, the loop performs at most n iterations
- $RT(s) \leq n$

$\Rightarrow WC(n) \in O(n)$

- Lower-Bound: underestimate.
  - Let $n \in \mathbb{N}$ → input must have n
    it takes at least this long (not at most)
    - concrete w/ arbitrary length
    Let $s_0 = \underbrace{\text{"a...a"}}_{n \text{ times}}$
  - Then, loop completes every iteration because if condition is always false.

- $RT(s_0) \geq n$

$\Rightarrow WC(n) \in \Omega(n)$

$\Rightarrow WC(n) \in \Theta(n)$

Induction Practice

a) $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

Base Case $n = 1$

$1(1+1) = \frac{1(1+1)(1+2)}{3}$

$2 = 2$

Inductive Step $P(n) \Rightarrow P(n+1)$

Assume $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$

Prove $\sum_{i=1}^{n+1} i(i+1) = \frac{(n+1)(n+2)(n+3)}{3}$

$$\frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$

$$= \sum_{i=1}^{n} i(i+1) + (n+1)(n+2) \quad \text{I.H}$$

$$= \sum_{i=1}^{n} i(i+1) + T_{n+1}$$

$$= \sum_{i=1}^{n+1} i(i+1)$$

b) $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Base Case $n = 0$

$\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$

$2^0 = 1$

$1 = 1$

Inductive Step $P(n) \Rightarrow P(n+1)$

Assume $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

Prove $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$

$$2 \cdot 2^{n+1} - 1 = 2^{n+1} + 2^{n+1} - 1$$

$$= 2^{n+1} + \sum_{i=0}^{n} 2^i \quad \text{IH}$$

$$= T_{n+1} + \sum_{i=0}^{n} 2^i$$

$$= \sum_{i=0}^{n+1} 2^i$$

c) $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \quad (r \neq 1)$

Base Case $n = 0$

$r^0 = \frac{1-r^1}{1-r}$

$1 = 1$

Inductive Step $P(n) \Rightarrow P(n+1)$

Assume $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$

Prove $\sum_{i=0}^{n+1} r^i = \frac{1-r^{n+2}}{1-r}$

$$\frac{1-r^{n+1}-(r-1)r^{n+1}}{1-r} = \frac{1-r^{n+1}}{1-r} - \frac{(r-1)r^{n+1}}{1-r}$$

$$= \sum_{i=0}^{n} r^i + r^{n+1} \quad \text{IH}$$

$$= \sum_{i=0}^{n+1} r^i$$

d) $\sum_{i=0}^{n} i!i = (n+1)! - 1$

Base Case $n=0$

$0!0 = 1! - 1$

$0 = 0$

Inductive Step $P(n) \to P(n+1)$

Assume $\sum_{i=0}^{n} i!i = (n+1)! - 1$

Prove $\sum_{i=0}^{n+1} i!i = (n+2)! - 1$

$$
\begin{aligned}
(n+2)! - 1 &= (n+2)(n+1)! - 1 \\
&= (n+1)!(n+1) + (n+1)! - 1 \\
&= T_{n+1} + \sum_{i=0}^{n} i!i \quad IH \\
&= \sum_{i=0}^{n+1} i!i
\end{aligned}
$$

e) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{3}{n(n+1)(n+2)}$

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \frac{1}{5\cdot6}$$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots + \frac{1}{n(n+1)}$$

$+4$ $+6$ $+8$ $+10$

Base Case $n = 1$

$$\frac{1}{1(1+1)} = \frac{\cancel{3}}{1(1+1)(\cancel{1+2})}$$

$$\frac{1}{2} = \frac{1}{2}$$

Inductive $P(n) \to P(n+1)$

$$\sum_{i=0}^{n} \frac{1}{i(i+1)} = \frac{3}{n(n+1)(n+2)}$$

$$\sum_{i=0}^{n+1} \frac{1}{i(i+1)} = \frac{3}{(n+1)(n+2)(n+3)}$$

$$\frac{3}{n(n+1)(n+2) + 3(n+1)(n+2)} = \frac{3}{n(n+1)(n+2)} + \frac{X}{3(n+1)(n+2)}$$

$$\left(\frac{3}{(n+1)(n+2)(n+3)} - \frac{3}{n(n+1)(n+2)}\right) 3(n+1)(n+2) = X$$

$$\frac{9}{n+3} - \frac{9}{n} = X$$

$$\frac{27}{n(n+3)} = X$$

$$\Rightarrow \frac{3}{n(n+1)(n+2)} + \frac{\left(\frac{27}{n(n+3)}\right)}{3(n+1)(n+2)}$$

$$= \frac{3}{n(n+1)(n+2)} + \frac{9}{n(n+1)(n+2)(n+3)}$$

Inequalities

a) $2^n > n$

Base Case : $2^n = n + 1$

$2^n - 1 = n$

Base Case : $2^0 - 1 = 0$

$0 = 0$

Induction : $2^n = n+1 \Rightarrow 2^{n+1} = n+2$

$2 \cdot 2^n = n + 1 + 1$

$2(n+1) = n+1+1$

$2n + 2 = n + 2$

$2n = n$

$n = 0$

Inductive Step $2^n > n \Rightarrow 2^{n+1} > n+1$

$2 \cdot 2^n > 2 \cdot n$

$2^n + 2^n > n + n$

$> n + 1$

$\square$

b) $2^n \geq n^2 \quad (n \geq 4)$

Base case $n = 4$

$2^4 \geq 4^2$

$16 \geq 16$

Induction $2^n \geq n^2 \Rightarrow 2^{n+1} \geq (n+1)^2$

$2^n \geq n^2$

$2 \cdot 2^n \geq 2n^2$

$2^n + 2^n \geq n^2 + n^2$

$\geq n^2 + 2n + 1$

$n^2 \geq 2n + 1$ $\qquad$ $n \geq 2$

$4n \geq 2n + 1$ $\qquad$ since $n^2 \geq 4n$

$2n \geq 1$

$n \geq \frac{1}{2}$

$n \geq 4 \geq \frac{1}{2}$

so, $n^2 + 2n + 1$

$WC_{is\_pal}(n) \in \Theta(n)$ $\quad (n = len(s))$

Ex : def pal_prefix(s: str) → int

```
    """Return the length of a longest prefix of s
    that is a palindrome."""
    for p in range(len(s), 0, -1):
        if is_pal(s[0:p]):
            return p
```

$WC_{pal\_prefix}(n) = ?$ $\quad (n = len(s))$

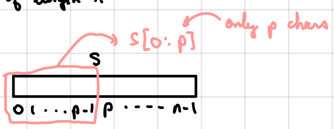
- Upper Bound : (on the worst case) * Let $n \in \mathbb{N}$, let s be a string of length n.

Approach : show $RT(s) \leq$ ___

$\forall$ inputs s of length n

- overestimate

- loop body :



input size for is_pal is p

$RT_{is\_pal} \leq c \cdot p$ (b.c. $WC_{is\_pal} \in \Theta(n)$)

$\leq c \cdot n$ (c comes from ↗) $\quad$ could be $\leq p-1 \leq p$ ← upper bound

(Note : simplify : assume $c = 1$)

- # iterations $\leq n$ $\quad$ $1 \leq p \leq n$

- total is $\leq n \cdot c \cdot n = cn^2$

$\Rightarrow WC_{pal\_prefix}(n) \in O(n^2)$

Lower Bound : why is it $\exists$?

Approach : Let $n \in \mathbb{N}$

Pick input s_n for which

we can show $RT_{pal\_prefix}(s) \geq$ ___

- underestimate

- if s_n is a palindrome,
  - is_pal($s_n[0:n]$) take time $\geq n$
  - loop will stop w/ $p = n$
- if s_n is not a palindrome

$\underbrace{s_n = abcd....}_{\text{all different}}$

- loop will iterate over $p = n, n-1, ..., 1$
- each iteration takes time 1.

How many values of p do we need to try

$p = \underbrace{n, n-1, ...,}_{\text{fail}} \lceil \frac{n}{2} \rceil$ ↳ succeed

Total time

$\lceil \frac{n}{2} \rceil$ $\quad + \quad \sum_{p=\lceil \frac{n}{2} \rceil + 1}^{n} (p - \lceil \frac{n}{2} \rceil)$ $\quad$ ↗ $1 + 2 + \dots + (n - \lceil \frac{n}{2} \rceil)$

last call that succeed

$\geq \frac{n}{2} + \underbrace{(1 + 2 + \dots + (n - \lceil \frac{n}{2} \rceil))}_{\frac{\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor + 1)}{2}}$ $\quad = \lfloor \frac{n}{2} \rfloor$ $\quad = \frac{\lfloor \frac{n}{2} \rfloor^2 + \lfloor \frac{n}{2} \rfloor + n}{2}$

$\geq \frac{n^2}{8} \Rightarrow WC(n) \in \Omega(n^2)$

<u>insight</u>

not a palindrome → looks like a palindrome

let $s_n = \underbrace{aa...a}_{\lceil \frac{n}{2} \rceil} b \underbrace{a...a}_{\lfloor \frac{n}{2} \rfloor - 1}$ $\quad$ aaba
aaaba

$\frac{n}{2} \leq \lceil \frac{n}{2} \rceil < \frac{n}{2} + 1$

eg.: $n = 10$ $\quad s_{10} = aaaaabaaaa$ $\quad \frac{n}{2}$ X ← is_pal time

$p = 10$ $\quad \frac{n}{2} - 1$ X

$p = 9$

$\vdots$

$p = 6$ $\quad 1$

$p = 5$ $\quad n/2$

WS15

3a)
```
def f(n: int) -> None:
    x = n
    while x > 1:
        if x % 2 == 0:
            x = x // 2
        else
            x = 2 * x - 2
```

$\forall x \in \mathbb{Z}^+$, $\exists$ loop $\Rightarrow$ x: $y \mid 2$ — y is a factor of 2.

Assume $\exists$ loop iterations occur — $2 = yk$

Let x_0 be the initial value of x.

Let y be a factor of 2 which satisfies $y \mid 2$

then $y = \frac{2}{k}$ for some $k \in \mathbb{Z}$, $k \neq 0$

note, $k = 2$ or $k = 1 \Rightarrow y = 1$ or $y = 2$.

$x_3 \leq \frac{1}{1} x_0$ or $x_3 \leq \frac{1}{2} x_0$ $\Rightarrow y = 1$ is trivial and doesn't decrease.

Case 1: Assume x_0 is even, that is $2 \mid x_0$

After iteration 1: $x_1 = \lfloor \frac{x_0}{2} \rfloor$

Case 1: x_1 is even

after iteration 2: $x_2 = \lfloor \frac{x_1}{2} \rfloor$

Case 1: x_2 is even

after iteration 3: $x_3 = \lfloor \frac{x_2}{2} \rfloor$

$x_3 = \lfloor \frac{1}{2} \lfloor \frac{1}{2} \lfloor \frac{1}{2} x_0 \rfloor \rfloor \rfloor$

$x_3 = \lfloor \frac{1}{8} x_0 \rfloor \leq \frac{1}{8} x_0$

$x_3 \leq \frac{1}{8} x_0 \leq \frac{1}{2} x_0$

Case 2: x_2 is odd

$$\begin{aligned} x_3 &= 2x_2 - 2 \\ &= 2\lfloor \tfrac{x_1}{2} \rfloor - 2 \\ &= 2\lfloor \tfrac{1}{2} \lfloor \tfrac{1}{2} x_0 \rfloor \rfloor - 2 \\ &\leq \lfloor \tfrac{1}{2} x_0 \rfloor - 2 \\ &\leq \tfrac{1}{2} x_0 - 2 \\ &\leq \tfrac{1}{2} x_0 \end{aligned}$$

Case 2. x_1 is odd

After iteration 2: $x_2 = 2x_1 - 2$

Case 1: x_2 is even: $x_3 = \lfloor \frac{x_2}{2} \rfloor$

$$\begin{aligned} &= \left\lfloor \frac{2x_1 - 2}{2} \right\rfloor \\ &= \left\lfloor \frac{2\lfloor \frac{x_0}{2} \rfloor - 2}{2} \right\rfloor \\ &= \lfloor \lfloor \tfrac{x_0}{2} \rfloor - 1 \rfloor \\ &\leq \tfrac{x_0}{2} - 1 \\ &\leq \tfrac{x_0}{2} \end{aligned}$$

Case 2: x_0 is odd: $x_1 = 2x_0 - 2$

Case 1 x_1 is even: $x_2 = \lfloor \frac{x_1}{2} \rfloor = \lfloor \frac{2x_0 - 2}{2} \rfloor$

$= \lfloor x_0 - 1 \rfloor$

x_2 is even: $x_3 = \lfloor \frac{\lfloor x_0 - 1 \rfloor}{2} \rfloor \leq \lfloor \frac{x_0 - 1}{2} \rfloor \leq \frac{x_0}{2} - \frac{1}{2} \leq \frac{x_0}{2}$

x_2 is odd: $x_3 = 2(\lfloor x_0 - 1 \rfloor) - 2$

$\leq 2x_0 - 2 - 2$

$\leq 2x_0 - 4 \leq \frac{x_0}{2}$?

$\frac{3}{2} x_0 \leq 4$

$\frac{x_0}{2} \leq \frac{4}{3}$

WS16

1.)
```
def has_duplicates(lst: list) -> bool:
    n = len(lst)
    for i in range(n):
        for j in range(i+1, n):
            if lst[i] == lst[j]:
                return True
    return False
```

Loop 2: $i+1 \to n-1$

$j_k = i+k+1$

$i_k \geq n$

$1+i+k \geq n$

$k \geq n-i-1$

Loop 1: $0 \to n-1$

$i_k = k$

$i_k \geq n$

$k \geq n$

$O(n^2)$

Upper Bound: $\forall n \in \mathbb{N}, \forall x \in I_{func,n}, RT_{func(x)} \leq f(n)$

Let $n \in \mathbb{N}$, $x \in I_{has\_duplicates,n}$

lst is an arbitrary list of length n.

has_duplicates(lst)

Lower Bound: $\forall n \in \mathbb{N}, \exists x \in I_{func,n}, RT_{func(x)} \geq f(n)$

Let $n \in \mathbb{N}$

$x =$

If x has duplicates

loop 2: $x[i:n]$ take time $n-i-1$

If x doesn't have duplicates

loop 1: take time $\sum_{i=0}^{n} n-i-1$

$x = [1, 2, \ldots, n]$

$\in \Theta(n^2)$

extra: $x = [1, 2, \ldots, n-1, n-1]$

$$\sum_{i=0}^{n-1} (n-i-1)$$

$$= \sum_{i=0}^{n-1} n - \sum_{i=0}^{n-1} i - \sum_{i=0}^{n-1} 1$$

$$= n^2 - \frac{n(n-1)}{2} - n$$

2. def binary_search(lst: List[int], x: int) → Bool:

```
    i = 0
    j = len(lst)
    while i < j:
        mid = (i + j) // 2
        if lst[mid] == x:
            return True
        elif lst[mid] < x:
            i = mid + 1
        else:
            j = mid
    return False
```

$r = j - i$ (size of search range) → # loop iterations

a) $n = \text{len}(lst)$

$r = n$

b) $i \geq j \iff r \leq 0$

c) $\forall$ loop iteration, if x is not found → then $r_{k+1} \leq \frac{1}{2} r_k$ ⇒ means $r_{k+1} \leq \frac{1}{n} r_k \leq \dots \leq \frac{1}{3} r_k \leq \frac{1}{2} r_k$

Let k be an arbitrary loop iteration

Assume x is not found

if lst[mid] < x: $i_{k+1} = \left\lfloor \frac{j_k + i_k}{2} \right\rfloor + 1 \qquad j_{k+1} = j_k$

$r_k = j_k - i_k \qquad r_{k+1} = j_{k+1} - i_{k+1}$

$r_{k+1} = j_{k+1} - \left(\left\lfloor \frac{i_k + j_k}{2} \right\rfloor + 1 \right)$

$-\lfloor x \rfloor = \lceil -x \rceil$

$= j_{k+1} - \left\lfloor \frac{i_k + j_k}{2} \right\rfloor - 1$

$= j_{k+1} + \left\lceil \frac{-i_k + j_k}{2} \right\rceil - 1$

$=$

$\frac{2j_k}{2} - \left\lfloor \frac{j_k}{2} \right\rfloor = -\left\lfloor \frac{-j_k}{2} \right\rfloor$

if lst[mid] > x:

$r_{k+1} = \left\lfloor \frac{i_k + j_k}{2} \right\rfloor - i_{k+1}$

$\leq \frac{i_k + j_k}{2} - \frac{2 i_{k+1}}{2}$

$\leq \frac{j_k - i_k}{2} \qquad (i_{k+1} = i_k)$

$r_{k+1} \leq \frac{1}{2} r_k$

d) we know that $r_{k+1} \le \frac{1}{2} r_k$ $\quad r_0 = n$ $\quad \forall n \in \mathbb{N}, \forall x \in \mathcal{X}_{binary\_search, n}, RT_{binary\_search}(x) \le f(n)$

UBWC :

$r_k \le \frac{1}{2} r_{k-1}$

$r_k \le \frac{1}{2^k} r_0$

$r_k \le \frac{n}{2^k}$

$r_k \le \lfloor \frac{n}{2^k} \rfloor \le \frac{n}{2^k}$

$\lfloor \frac{n}{2^k} \rfloor \le 0$

$\log_2 \lfloor \frac{n}{2^k} \rfloor \le \log_2 0$

$\log_2 \lfloor \frac{n}{2^k} \rfloor \le 1$

$\lfloor \frac{n}{2^k} \rfloor \le 2$

loop terminates.

$\lfloor \frac{n}{2^k} \rfloor = 0$ floor rule $m \le x < m+1$

$\lfloor \frac{n}{2^k} \rfloor \le \frac{n}{2^k} < \lfloor \frac{n}{2^k} \rfloor + 1$

$0 \le \frac{n}{2^k} < 1$

$n < 2^k$

$\lfloor \frac{n}{2^k} \rfloor \le 0$

$\frac{n}{2^k} - 1 < \lfloor \frac{n}{2^k} \rfloor \le \frac{n}{2^k}$ floor rule $x - 1 < m \le x$

$0 \le \frac{n}{2^k}$

$\log_2 n < k$

$\log_2 n < \lfloor \log_2 n \rfloor + 1$

$\log_2 n < k$

$\log_2 n + 1 \le k$

$k = \lfloor \log_2 n \rfloor + 1 = f(n)$

$RT_{binary\_search}(x) \le \lfloor \log_2 n \rfloor + 1$

$O(\log_2 n)$

e) Lower bound on WC

$\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{binary\text{-}search, n}, RT_{binary\text{-}search}(x) \geq \log n$

$\tilde{\mathcal{I}}_{func, n}$:= the set of allowed inputs to func of size n.

Let $n \in \mathbb{N}$,

Let lst = $[0, 1, \dots, n-1]$ of length n

$x = -1$

By our choice of lst and x, the else clause will always evaluate to true.

By c) we know $r_k \leq \lfloor \frac{n}{2^k} \rfloor$

and that $r_k \leq 0$

$$\lfloor \frac{n}{2^k} \rfloor \leq 0$$

$$\frac{n}{2^k} < 1$$

$$n < 2^k$$

$$\log_2 n < k$$

$RT_{binary\text{-}search}(lst, x) = k \geq \log n$

If $a > 1$ and $b > 1$
then $\log_a n \in \Theta(\log_b n)$

$\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 \log_b n \leq \log_a n \leq c_2 \log_b n$

$c_1 \log_b n \leq \log_a n$

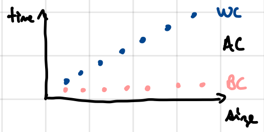
$\forall a, b, n \in \mathbb{R}^+, a \neq 1 \wedge b \neq 1 \Rightarrow \log_a x = \dfrac{\log_b x}{\log_b a}$ $\quad$ or $\quad \dfrac{\log_a n}{\log_a b} = \log_b n$

$\log_a n = \dfrac{1}{\log_b a} \log_b n$

$c \log_a n \leq \log_b n$

$\log_a n \leq c \log_b n$

Avg Case



$$AC(n) \neq \frac{BC(n) + WC(n)}{2}$$

Want: $AC(n)$ = average of $RT(x)$ over all inputs of size n.

In general:

$$AC(n) = \frac{1}{|\mathcal{X}_n|} \sum_{x \in \mathcal{X}_n} RT(x)$$

e.g.: if $\mathcal{X}_n = \{x_0, x_1, \dots, x_{m-1}\}$

$$AC(n) = \frac{RT(x_0) + RT(x_1) + \dots + RT(x_{m-1})}{m}$$ ← all inputs are equally likely

"naive average"

Ex: $AC_{is\_in}(n)$?

- need exact expression for $RT(x)$, for each $x \in \mathcal{X}_n$
- need complete def^n of $\mathcal{X}_n$

For is_in(), $\mathcal{X}_n$ = ?

Insights: don't need "all conceivable inputs" — we have to represent all possible behaviour of the algorithm.

Idea 1: (Course Notes)

$\mathcal{X}_n = \{(x, L) \mid x = 1$ and L is a permutation of $[1, 2, \dots, n]\}$

eg. $\mathcal{X}_3 = \{(1, [1,2,3]), (1, [1,3,2]), (1, [2,1,3]) \dots\}$

Note: no input w/ x not in L.

Idea 2: instead, vary x

$\mathcal{X}'_n = \{(x, L) \mid L = [1, 2, \dots, n]$ and $x \in \{0, 1, \dots, n\}\}$

eg. $\mathcal{X}'_3 = \{(0, [1,2,3]), (1, [1,2,3]), (2, [1,2,3]), (3, [1,2,3])\}$

lecture

```
def is_in(x, L):
    for item in L:
        if item == x:
            return True
    return False
```

$AC(n) = ?$

$I'_3 = \{(0,[1,2,3]),(1,[1,2,3]),(2,[1,2,3]),(3,[1,2,3])\}$

$I'_n = \{(x,L) \mid L = [1,2,\dots,n],\ x \in \{0,1,\dots,n\}\}$

$AC(n) = \frac{1}{|I'_n|} \sum_{(x,L)\in I'_n} RT(x,L)$ need: expression for $RT(x,L)$

$= \frac{1}{n+1}\left(RT(0,[1,2,\dots n]) + \sum_{x=1}^{n} RT(x,[1,\dots,n])\right)$

+1 for extra "return False"

$= \frac{1}{n+1}\left((n+1) + \sum_{x=1}^{n} x\right)$ → # iterations

$= 1 + \frac{1}{n+1}\frac{n(n+1)}{2} = \frac{n}{2} + 1$

so $AC(n) \in \Theta(n)$ → we're not doing probability / expected values

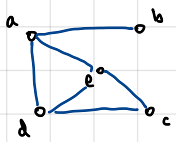
Q: what if we want case "x not in L" to be more likely?

A: use different set of inputs, eg.:

$I''_n = \{(x,L) \mid L = [1,2,\dots,n],\ x \in \{1,2,\dots,n^2\}\}$

Avg case is never bigger than the worse case

Graphs lec



$V = \{a, b, c, d, e\}$

$E = \{\{a,b\}, \{a,d\}, \{a,e\}, \{c,d\}, \{c,e\}, \{d,e\}\}$

"undirected" graphs : edge $\{u, v\} = \{v, u\}$

Ex1: Prove $|E| \leq \frac{|V|(|V|-1)}{2}$

($|E|$ = size of set E, etc.)

Q: How to write "for all graphs" symbolically?

A: $\forall G = (V, E), \ldots$

$\underbrace{\exists G = (V, E)}, \ldots$

introduces 3 related variables:

- graph G
- vertex set V
- edge set E

Prove $|E| \leq \frac{|V|(|V|-1)}{2}$

Proof: Let $G = (V, E)$ be a graph

Each element of E is a subset of V of size 2

(Note: $\{a, a\} = \{a\}$ Not an edge!)

Worksheet #9

↳ # subsets of size $2 = \frac{n(n-1)}{2}$

when set has size n.

Here, V has size $|V|$

So, $|E| \leq \frac{|V|(|V|-1)}{2}$ $\circ$

Def^n: $u, v \in V$ are "adjacent"

iff $\{u, v\} \in E$

Notation: (u, v) means $\{u, v\}$ for edges!

eg, a is adjacent to b

b is not adjacent to c

Def^n: $u, v \in V$ are "connected"

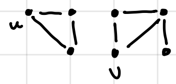
iff $u = v$ or G contains some path between u and v,

$\exists k \in \mathbb{N}, \exists v_1, \ldots, v_k \in V, (u, v_1) \in E \wedge (v_1, v_2) \in E \wedge \ldots \wedge (v_k, v) \in E$

Note: case $k = 0$ is allowed! meaning: $(u, v) \in E$ eg. b is connected to c

Q) can $u, v \in V$ be not connected?

A:



Next Time ...

- necessary and sufficient conditions on $|E|$ for G to be connected
  - sufficient : condition $\Rightarrow G$ connected
  - necessary : $\neg$ condition $\Rightarrow \neg$ (G connected)

    (G connected) $\Rightarrow$ condition

"G is connected":

u, v are connected $\forall u, v \in V$

Worst / Best / Avg Case

Worst Case

Best Case

Runtime functions

$WC : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$BC : \mathbb{N} \to \mathbb{R}^{\geq 0}$

variable : input size

$\mathcal{X}_n = \{$ all inputs of size $n\}$

$WC(n) = \max\{RT(x) \mid x \in \mathcal{X}_n\}$

$BC(n) = \min\{RT(x) \mid x \in \mathcal{X}_n\}$

Note : Instead of trying to find "the" worst or best input – and to prove that it is worst / best – we find bounds on the runtime

Upper bounds

$WC(n) \in O(g(n))$:

$\forall x \in \mathcal{X}_n,\ RT(x) \leq c \cdot g(n)$

(for some $c \in \mathbb{R}^+$ and for all n "large enough")

$BC(n) \in O(g(n))$:

$\exists x \in \mathcal{X}_n,\ RT(x) \leq c \cdot g(n)$ ← only care about the min

(for some $c \in \mathbb{R}^+$ and for all n large enough)

$\max(a,b) \leq c \iff a \leq c \wedge b \leq c$

↳ universal

$\min(a,b) \leq c \iff a \leq c \vee b \leq c$

↳ existential

$a \leq \max(a,b) \wedge b \leq \max(a,b)$

$\min(a,b) \leq a \wedge \min(a,b) \leq b$

Lower bounds

$\max(a,b) \geq c \iff a \geq c \vee b \geq c$

$\min(a,b) \geq c \iff a \geq c \wedge b \geq c$

$WC(n) \in \Omega(g(n))$:

$\exists x \in \mathcal{X}_n,\ RT(x) \geq c \cdot g(n)$

(for some $c \in \mathbb{R}^+$ and for all n large enough)

$BC(n) \in \Omega(g(n))$:

$\forall x \in \mathcal{X}_n,\ RT(x) \geq c \cdot g(n)$

(for some $c \in \mathbb{R}^+$ and for all n large enough)

Example :

```
def is_in(x: int, L: list) -> bool:
    for item in L:          — # iterations
        if item == x:
            return True
    return False
```

→ RT depends only on how many times it iterate.

$\Theta(1) = 1$

$WC(n) \in O(n)$: where $n = len(L)$

look at every possible list

$\forall x \in \mathcal{X}_n$, loop iterates $\leq n$ times

each step is $\Theta(1)$

$WC(n) \in \Omega(n)$:

loop iterates n times when x not in L.

$BC(n) \in O(1)$:

if $x == L[0]$:

then $RT \leq 1$

[] of arbitrary length?

no, len([]) == 0 not n.

$BC(n) \in \Omega(1)$:

trivial ← why?

Q: $2^x \in \Omega(x^2)$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow 2^x \geq c x^2$

$n_0 = 1 \qquad c = \frac{1}{2}$

$2^n \geq c n^2 \qquad\qquad 2^n \geq 2n^2$

Base: $n = 1$

$2 \geq \frac{1}{2}$

$4 \geq 1$

Inductive $\quad 2^n \geq \frac{1}{2}n^2 \Rightarrow 2^{n+1} \geq \frac{1}{2}(n+1)^2$

$2^{n+1} \geq c(n+1)^2$

$2 \cdot 2^n \geq c(n^2 + 2n + 1)$

$\frac{2cn^2}{n^2+2n+1} \geq c$

$\frac{2n^2}{n^2+2n+1} \geq c$

$\frac{1}{2} \geq c \qquad$ Since $n \geq 1$

WS17

```python
def substring(s1: str, s2: str) -> bool:
    for i in range(len(s2) - len(s1)):
        match = True
        for j in range(len(s1)):
            if s1[j] != s2[i+j]:
                match = False
                break
        if match:
            return True
    return False
```

$s1 = n \qquad s2 = n^2$

Upper bound on WC:

$\forall n \in \mathbb{N}, \forall x \in \mathcal{I}_{substring,n}, \ RT_{substring}(x) \leq f(n)$

Let $n \in \mathbb{N}$

Let $x \in \mathcal{I}_{substring,n}$

For fixed loop 1 iteration,

Loop 2's if condition will always be false unless $s1[n-1] \;!= s2[i+n-1]$

$RT_{loop\_2}(x) = k$

$k \geq n$

$RT_{loop\_1}(x) = k$

$k \geq n^2 - n$

$$RT_{substring}(x) = \sum_{i=0}^{n^2-n-1} (n+1) + 1$$

$$= (n+1)(n^2-n) + 1$$

$$= n^3 - n^2 + n^2 - n + 1$$

$$= n^3 - n + 1$$

$RT_{substring}(x) \leq n^3$

$\Rightarrow O(n^3)$

aab

aaaaab

$x - 1 < f \leq x \leq c < x + 1$

Lower Bound on WC:

$\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{substring,n}, \ RT_{substring}(x) \geq f(n)$

$n \in \mathbb{N}$, $\quad s1 = \underbrace{aaa\ldots a}_{n\text{-times}}$

$s2 = \underbrace{bbb\ldots b}_{n^2 \text{ times}}$

WTS: $RT_{substring}(s1, s2) \geq n^3$ $\qquad i = 0, 1, \ldots, n^2 - n - 1$

$j = 0, \ldots, n-1$

$s2[i+n-1] = b$

$s1[n-1] \;!= s2[i+n-1]$

aaa

aabbb

a a a

$n - 1$ $\qquad 3+2+1$

aabaab

Let $s1 = aaa\ldots a$ $\qquad i = 0$ $\quad$ aaaa

$s2[i] = \begin{cases} b & \text{if } n \mid i+1 \\ a & \text{else} \end{cases}$ $\qquad$ aaabaa

Loop 2 iterations

For a fixed input of Loop 1 $\qquad$ aaab

$RT_{loop\_2}(x) = i$

$RT_{loop\_1}(x) = n(n-1)$ $\qquad$ aaaaaaa

$$= \sum_{i=1}^{n-1} i + \overset{n \text{ times}}{\ldots} + \sum_{i=1}^{n-1} i$$

$$= n \sum_{i=0}^{n-1} i$$

$$= n \frac{(n-1)n}{2}$$

$$= n \frac{n^2 - n}{2}$$

$$= \frac{n^3 - n^2}{2} = \frac{1}{2}n^3 - \frac{1}{2}n^2$$

$\Rightarrow \Omega(n^3)$

midterm 2 v1

a) $\sum_{i=n}^{2n-1} 2^i$

b) $n^2 \in O(1.2^n)$ T $\qquad n\log_2 n \in O(n)$ F $\qquad 10 + \log_2 n \in \Theta(\log_{10} n)$ T

$\frac{n^3 - n + 10}{n^2 + 3} \in \Omega(n^2)$ F $\qquad 10 \in \Omega(\log_3 n)$ F $\qquad 10^n + 11^n \in O(10^{n+11})$ F

c) S_k : value of s after k loop iterations

Assume k iterations occur.

Stopping condition:

$S_k \geq 2n$

$i_j = j + j$

$i_j = 2j$

$S_k = \underbrace{i_j + i_j + \ldots + i_j}_{k \text{ times}}$

$S_k = \sum_{j=0}^{k-1} i_j \qquad = \sum_{j=0}^{k-1} 2j = k^2 - k$

2.) Induction

$\forall n \in \mathbb{Z}^+, \sum_{i=1}^{n} i2^i = 2^{n+1}(n-1) + 2$

Base case: $n = 1$

$1 \cdot 2^1 = 2^{1+1}(1-1) + 2$

$2 = 2$

Inductive step: $\sum_{i=1}^{n} i2^i = 2^{n+1}(n-1) + 2 \Rightarrow \sum_{i=1}^{n+1} i2^i = 2^{n+2}(n) + 2$

$$\begin{aligned}\sum_{i=1}^{n} i2^i + (n+1)2^{n+1} &= 2^{n+1}(n-1) + 2^{n+1}(n+1) + 2 \\ &= 2^{n+1}(n-1+n+1) + 2 \\ &= 2^{n+1}(2n) + 2 \\ &= 2^{n+2}(n) + 2\end{aligned}$$

$\square$

3.) $\forall f,g,h : \mathbb{N} \to \mathbb{R}^{\geq 0}, f \in O(g) \Rightarrow f+h \in O(g+h)$

Let $f,g,h : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Assume $f \in O(g)$: $\exists c_1, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq c_1 g(n)$

Prove $f+h \in O(g+h)$: $\exists c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(n) + h(n) \leq c_2(g(n) + h(n))$

$n_1 = n_0$

$c_2 = \max\{1, c_1\}$

$f(n) \leq c_1 g(n) \qquad f(n) \leq c_2 g(n) + c_2 h(n)$

$f(n) + h(n) \leq c_1 g(n) + h(n)$

$f(n) + h(n) \leq c_1 g(n) + h(n) \leq c_1 g(n) + c_1 h(n)$ (since $c_1 \geq 1$)

$f(n) + h(n) \leq c_2(g(n) + h(n))$

4) Running-time analysis ✗

Loop 2: $j_k = k\left(\frac{n}{i}\right)$

$j_k \geq n$

$k\left(\frac{n}{i}\right) \geq n$

$\frac{kn}{i} \geq n$

$kn \geq in$

$k \geq i_k$

Loop 1: $i_k = 2^k$

$i_k \geq n$

$2^k \geq n$

$k \geq \log_2 n$

$i = 1, 2, 4, 8, 16$

$= 2^0$

$$\sum_{i=0}^{\log_2 n} 2^i = -(1 - 2^{\log_2 n + 1}) = 2^{\log_2 n + 1} - 1$$

b)
```
my_alg(lst: List[int]) -> None:
    for i in range(len(lst)):
        if lst[i] % (i+1) == 0:
            for j in range(i+1, min(len(lst), lst[i])):
                lst[j] = lst[j] + 2
```

UBWC : $\forall n \in \mathbb{N}, \forall x \in \mathcal{I}_{my\text{-}alg,\, n},\ RT_{my\text{-}alg}(x) \leq f(x)$

Let $n \in \mathbb{N}$

Let x be an arbitrary input of size n.

Loop 2: $j_k = k + i + 1$

$j_k \geq \min(n, lst[i])$

$k + i + 1 \geq \min(n, lst[i])$

$k \geq \min(n, lst[i]) - i - 1$

$k = n - i - 1$ or $k = lst[i] - i - 1$

Loop 1: $\sum_{i=0}^{n-1} n - i - 1$ or $\sum_{i=0}^{n-1} lst[i] - i - 1$ $\qquad lst[i] = (i+1)k_0 = ik_0 + k_0$

$= n^2 - \frac{(n-1)n}{2} - n$ $\qquad = \sum_{i=0}^{n-1} ik_0 + k_0 - i - 1$

$= \frac{2n^2 - n^2 + n - 2n}{2}$ $\qquad = k_0 \sum_{i=0}^{n-1}(i+1) - \sum_{i=0}^{n-1} i - \sum_{i=0}^{n-1} 1$

$= \frac{n^2 - n}{2}$ $\qquad = k_0 \frac{(n-1)n}{2} + k_0 n - \frac{(n-1)n}{2} - n$

$= \frac{n(n-1)}{2}$ $\qquad = \frac{(n-1)n}{2}(k_0 - 1) + n(k_0 - 1)$

$\Rightarrow O(n^2)$ $\qquad = (k_0 - 1)\left(\frac{n^2 - n}{2}\right)$

$\Rightarrow O(n^2)$

LBWC : $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{my\text{-}alg, n}, RT_{my\text{-}alg}(x) \geq n^2 \rightarrow \Omega(n^2)$

Let $n \in \mathbb{N}$

$lst[i] = (i+1)k_0$

$x = \_\_\_\_$

midterm 2 v2

1. Short answer

a) $\sum_{i=0}^{\lfloor \frac{2n-1}{2} \rfloor} 2^{2i+1}$ $1010 = 2^3 + 2^1 = 10$

101010

b) $1.01^n \in O(n^2)$ F $\forall m \in \mathbb{Z}^+, \sum_{i=1}^{m} n^i \in \Theta(n^m)$ T $10 + \log_2 n \in (\log_{10} n)$ T

$n + n^2 + \ldots + n^m \leftarrow$

$\frac{n^2+n}{n+3} \in \Omega(\sqrt{n})$ T $10 \in O(\log_3 n)$ T $10^n \in O(9^{n+1})$ F

c) s_k : value of s after k loop iterations

assume k loop iterations occur

$i_j = 1 + 2j$

$s_k = \underbrace{i_j + \ldots + i_j}_{k \text{ times}}$

$s_k = \sum_{j=1}^{k-1} i_j = \sum_{j=1}^{k-1} (1+2j)$

d) induction

$\forall n \in \mathbb{Z}^+, \sum_{i=1}^{n} (-1)^i \cdot i^2 = \frac{(-1)^n \cdot n(n+1)}{2}$

Let $n \in \mathbb{Z}^+$

Base Case : $n = 1$

$\sum_{i=1}^{1} (-1)^i \cdot i^2 = -1 = \frac{-1 \cdot 2}{2} = -1$

Inductive step : $\sum_{i=1}^{n} (-1)^i \cdot i^2 = \frac{(-1)^n \cdot n(n+1)}{2} \Rightarrow \sum_{i=1}^{n+1} (-1)^i \cdot i^2 = \frac{(-1)^{n+1} \cdot (n+1)(n+2)}{2}$

$\sum_{i=1}^{n} (-1)^i \cdot i^2 + (-1)^{n+1} \cdot (n+1)^2$

$= \frac{(-1)^n \cdot n(n+1)}{2} + (-1)^{n+1} \cdot (n+1)^2$

$= \frac{(-1)^n \cdot n(n+1) + 2(n+1)^2 \cdot (-1)^{n+1}}{2}$

$= \frac{(-1)^n (n+1)(n-2n-2)}{2}$

$= \frac{(-1)^{n+1} (n+1)(n+2)}{2}$

3.) $\forall f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}, \ f \in \Omega(g) \Rightarrow f + h \in \Omega(g+h)$

Let $f, g, h, : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Assume $f \in \Omega(g)$; $\exists c_1, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ n \geq n_0 \Rightarrow f(n) \geq c_1 g(n)$

Prove $\exists c, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, \ n \geq n_1 \Rightarrow f(n) + h(n) \geq c(g(n) + h(n))$

$n_1 = n_0$

$c = \min(c_1, 1)$ $f(n) \geq c_1 g(n)$

$f(n) + h(n) \geq c_1 g(n) + h(n)$

$\geq c_2 g(n) + c_2 h(n)$ since $c_2 \leq 1$

$= c_2 (g(n) + h(n))$

$\square$

4. Running-Time Analysis

```
def f(n: int) -> None:
    i = 2
    while i < n:
        j = 0
        while j < n:
            j = j + (n/i^2)
        i = i + 2
```

Let $n \in \mathbb{N}$ and $n \geq 2$

Assume k iterations occur $\Rightarrow$ k is arbitrary

Loop 2: $j_k = k\left(\frac{n}{i^2}\right)$

$\Rightarrow j_k \geq n$ (loop is false)

$k\left(\frac{n}{i^2}\right) \geq n \Rightarrow k \geq i^2$

Loop 1: $i_k = 2k$

$i_k \geq n$ (loop is false)

$2k \geq n$

$k \geq \lceil \frac{n}{2} \rceil$

$$\sum_{i=1}^{\lceil \frac{n}{2} \rceil} (2i)^2 = 4\sum_{i=1}^{\lceil \frac{n}{2} \rceil} i^2 = 4\,\frac{\lceil \frac{n}{2}\rceil(\lceil \frac{n}{2}\rceil + 1)(n+2)}{6}$$

b)

```
def my_alg(lst: List[int]) -> None:
    for i in range(len(lst)):
        if 0 < lst[i] <= len(lst):
            for j in range(i+1, lst[i])
                lst[j] = lst[j] // lst[i]
```

UBWC: $\forall n \in \mathbb{N}, \forall x \in \mathcal{X}_{my\_alg,n}, RT_{my\_alg}(x) \leq f(n) \Rightarrow O(n^2)$

Let $n \in \mathbb{N}$, let x be an arbitrary input of size n.

Let k be an arbitrary number of iterations.

Loop 2: $j_k = i + 1 + \sum_{i=1}^{k} 1$

$j_k = i + 1 + k$

$j_k \geq lst[i]$

$k \geq lst[i] - i - 1$

Loop 1: $i_k = 0 + \sum_{j=1}^{k} 1$

$= k$

$k \geq n$

$$RT_{my\_alg}(x) \leq \sum_{i=0}^{n-1} lst[i] - i - 1 \leq \sum_{i=0}^{n-1} n - i - 1$$

$$= n^2 - \frac{n^2 - n}{2} - n$$

$$= \frac{2n^2 + n - n^2 - 2n}{2}$$

$$= \frac{n^2 - n}{2} \leq n^2 \Rightarrow O(n^2)$$

LBwc: $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{my\_alg,n}, RT_{my\_alg}(x) \geq g(n) \Rightarrow \Omega(g(n)) = \Omega(n^2)$

let $n \in \mathbb{N}$. let $lst = [n, n^2, \dots, n^{n+1}]$ ← n+1 items?

$$RT_{my\_alg}(x) \geq \sum_{i=0}^{n-1} lst[i] - i - 1$$

$$= \sum_{i=0}^{n-1} lst[i] - \frac{n+n^2}{2}$$

$$= \sum_{i=0}^{n-1} n - \frac{n+n^2}{2} \qquad (\text{since } lst[i] = n)$$

$$= n^2 - \frac{n+n^2}{2}$$

$$= \frac{n^2-n}{2} \Rightarrow \Omega(n^2)$$

$[2^n, 2^{n-1}, \dots, 2^1, 2^0]$ ← doesn't work

look at max iterations for diff loops

$lst = [1, \dots, n]$

want: $\overbrace{1, \lfloor\frac{2}{1}\rfloor, \lfloor\frac{3}{1}\rfloor, \lfloor\frac{4}{1}\rfloor, \dots, \lfloor\frac{n+1}{1}\rfloor}^{n+1 \text{ times}}$

$\lfloor\frac{3}{2}\rfloor, \lfloor\frac{4}{2}\rfloor, \dots, \lfloor\frac{n+1}{2}\rfloor$

$, \lfloor\frac{4}{3}\rfloor, \dots, \lfloor\frac{n+1}{2}\rfloor$

$\vdots$

$\lfloor\frac{n+1}{n}\rfloor$

Have: 1

$\lfloor\frac{3}{2}\rfloor$

$\lfloor\frac{4}{3}\rfloor$

$\ddots$

$\lfloor\frac{n+1}{n}\rfloor$

if condition specifies $0 < lst[i] \leq len(lst) = n$

$$lst_0 = \left[\frac{n^1}{n^0}, \frac{n^2}{n^0}, \frac{n^3}{n^0}, \frac{n^4}{n^0}, \dots, \frac{n^{n+1}}{n^0}\right]$$

$$lst_1 = \left[n, \frac{n^2}{n^1}, \frac{n^3}{n^1}, \frac{n^4}{n^1}, \dots, \frac{n^{n+1}}{n^1}\right]$$

$$lst_2 = \left[n, n, \frac{n^3}{n^2}, \frac{n^4}{n^2}, \dots, \frac{n^{n+1}}{n^2}\right]$$

$$\vdots$$

$$lst_{n+1} = \left[n, n, \dots, \frac{n^{n+1}}{n^n}\right]$$

midterm 2 2019 v1

1) $2^6 + 2^5 + 2^2$

$= (1100100)_2$

b) $1 = \sum_{i=0}^{n-1} (-1) \cdot 3^i = - \sum_{i=0}^{n-1} 3^i = \frac{1-3^n}{2}$

c) $f(n) = n^2 + 10n + 2$

$g(n) = 100 \log_2 n$

$f(n) \in \Omega(n)$ T $g(n) \in \Omega(n)$ F $f(n) \in O(g(n))$ F

$f(n) \in \Theta(g(n))$ F $g(n) \in \Theta(\log_3 n)$ T $f(n) + g(n) \in \Theta(f(n))$ T

d) $i_k = (3 \times 3)(3 \times 3) \dots (3 \times 3)$

$i_k^2 \geq n$ $i_k = 3^2 \cdot 9^2 \cdot 81^2$

$(3^{2^k})^2 \geq n$ $= 3^2 \cdot (3^2)^2 \cdot ((3^2)^2)^2$

$3^{2k} \geq n^{1/2}$

$2^k \geq \log_3 n^{1/2}$

$k \geq \log_2 \log_3 n^{1/2}$

$= \log_2 \left(\frac{\log_3 n}{2}\right)$

$\leq \lceil \log_2 \log_3 n - 1 \rceil = k$

2. Induction

$\forall n \in \mathbb{N}, n \geq 3 \Rightarrow 5^n + 50 < 6^n$

let $n \in \mathbb{N}$, assume $n \geq 3$

Base case $n = 3$

$5^3 + 50 < 6^3$

$175 < 216$

Induction: $5^n + 50 < 6^n \Rightarrow 5^{n+1} + 50 < 6^{n+1}$

$5^n + 50 < 6^n$

$5^{n+1} + 50 < 6^n + 4 \cdot 5^n$ $4 < 5$ and $5^n < 6^n$

$\quad " \quad < 6^n + 5 \cdot 6^n$

$\quad " \quad < 6^{n+1}$

3. Asymptotic Analysis

$\forall a \in \mathbb{R}^+, an + 1 \notin \Theta(n^3)$

let $a \in \mathbb{R}^+$

$an + 1 \notin \Theta(n^3)$: $an + 1 \notin O(n^3) \wedge an + 1 \notin \Omega(n^3)$

$an + 1 \notin \Omega(n^3)$:

let $c_1, n_0 \in \mathbb{R}^+$

let $n = \lceil \max(n_0, \sqrt{\frac{2a}{c_1}}, (\frac{2}{c_1})^{1/3}) \rceil + 1$ $\frac{c_1}{2} n^3 + \frac{c_1}{2} n^3 > an + 1$

$\frac{c_1}{2} n^2 > a \quad \wedge \quad \frac{c_1}{2} n^3 > 1$

$n > \sqrt{\frac{2a}{c_1}} \qquad n^3 > \frac{2}{c_1}$

$\qquad n > \left(\frac{2}{c_1}\right)^{1/3}$

4.) Running time analysis

Loop 2: $j_k = 3k$

$3k \geq i$

$k \geq \frac{i}{3}$

Loop 1: $\sum_{i=0}^{n^2-1} \frac{i}{3} = \frac{1}{3} \sum_{i=0}^{n^2-1} i = \frac{1}{3} \frac{(n^2-1)n^2}{2}$

$= \frac{(n^2-1)n^2}{6}$

$= \frac{n^4 - n^2}{6}$

b) WBWC :

Let $n \in \mathbb{N}$, x be an arbitrary input of size n.

Loop 2: $j_k = i + 1 + k$

$k \geq n - i - 1$

$k = n - i - 1$

Loop 1: only runs 1 time since 2 | lst[i+1 : n]

$k = n - i - 1 \leq n \Rightarrow O(n)$

LBWC :

Let $n \in \mathbb{N}$

lst = [1, 2, 3, ..., n]

Loop 1 $i_k = k$

$k \geq n$

n iterations

Loop 2 $j_k = i + 1$

$i + 1 \geq n$

$\Rightarrow \Omega(n)$

midterm v2

a) 108 - 8

$3^4 + 3^3 - 3^2 + 3^0$

$(11-101)_3$

b) $\sum_{i=0}^{n-1} 2^i$

c) T, F, F, F, T, T

d) $i_k = 2^{3^k}$

$i_k + i_k \geq n$

$2^{3^k+1} \geq n$

$3^k + 1 \geq \log_2 n$

$3^k \geq \log_2 n - 1$

$k \geq \lceil \log_3(\log_2 n - 1) \rceil$

2.) Induction

$\forall n \in \mathbb{N}, n \geq 2 \Rightarrow \prod_{i=1}^{n} \frac{2i-1}{2i} \geq \frac{1}{2n}$

Let $n \in \mathbb{N}$

$n = 2 \Rightarrow \prod_{i=1}^{2} \frac{2i-1}{2i} = \frac{2-1}{2} \cdot \frac{4-1}{4} = \frac{1}{2} \cdot \frac{3}{4} \geq \frac{1}{4}$

$= \frac{3}{8} \geq \frac{1}{4}$

$= \frac{3}{8} \geq \frac{2}{8}$

Induction $\prod_{i=1}^{n} \frac{2i-1}{2i} \geq \frac{1}{2n} \Rightarrow \prod_{i=1}^{n+1} \frac{2i-1}{2i} \geq \frac{1}{2n+2}$

$\prod_{i=1}^{n} \frac{2i-1}{2i} \geq \frac{1}{2n}$

$\frac{2n+1}{2n+2} \prod_{i=1}^{n} \frac{2i-1}{2i} \geq \frac{1}{2n} \frac{2n+1}{2n+2}$

$\geq \frac{2n+1}{4n^2+4n}$

$\geq \frac{2n+1}{2n(2n+2)}$

$\geq \frac{\cancel{2n}}{\cancel{2n}(2n+2)} = \frac{1}{2n+2}$ $\square$

Asymptotic Analysis

$\forall a \in \mathbb{R}^+,\ an^2+1 \notin O(n^4) \vee an^2+1 \notin \Omega(n^4)$

$an^2+1 \notin O(n^4)$

$: \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N},\ n \geq n_0 \wedge an^2+1 > cn^4$

$c, n_0 \in \mathbb{R}^+$

$n = \lfloor \min(n_0, \sqrt{\frac{2a}{c}}, (\frac{2}{c})^{1/4}) \rfloor$

$an^2+1 > \frac{c}{2}n^4 + \frac{c}{2}n^4$

$an^2 > \frac{c}{2}n^4 \wedge 1 > \frac{c}{2}n^4$

$\sqrt{\frac{2a}{c}} > n^2 \wedge (\frac{2}{c})^{1/4} > n$

lst = [0, 1, 2, 3, 4, ..., n-1]

2 3 4 5, ..., n

4 5 6, ..., n+1

6 7, ...,

8

$n-i-1$

$(n-1)+(n-2)+(n-3)+\dots+0 = \sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}$

lst = [1, 2, 3, ..., n]

1 2 3 4 . n-1

1 2 3 . n-2

1 2 . n-3

lst = [n, n+1, n+2, ..., 2n]

n, n+1, ... 2n-1

⋮

$2n - n = n$

$i_k = i_{k-1}^2$

$i_0 = 3$ $\quad i_1 = 3^2$ $\quad i_2 = i_1^2$ $\quad i_3 = i_2^2$

$(3^2)^2$ $\quad ((3^2)^2)^2$

$= 3^{2^k}$

midterm v3

1a) 128 + 32 + 10 = 160 $\quad 2^7+2^5+2^2+2^1$

$(10100110)_2$

b) $\sum_{i=0}^{n-1} 1 \cdot 3^i$

c) $f(n) \in O(n)$ T $\quad g(n) \in \Omega(n)$ T $\quad f(n) \in O(g(n))$ T

$f(n) \in \Theta(g(n))$ F $\quad g(n) \in \Theta(n)$ F $\quad f(n)+g(n) \in \Theta(g(n))$ T

$i_{k+1} = i_k^2$

$i_0 = 3 \quad i_1 = i_0^2 \quad i_2 = i_1^2$

$i_1 = 3^2 \quad i_2 = 3^{2^2}$

$i_k = 3^{2^k}$

$3^{2^k} \geq n^3$

$2^k \geq \log_3 n^3$

$k \geq \log_2(3\log_3 n)$

$k \geq \lceil \log_2(3\log_3 n) \rceil$

2. induction

$\forall n \in \mathbb{N}, n \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1$

Let $n \in \mathbb{N}$

$n=1$

$\frac{1}{\sqrt{1}} > \sqrt{1} - 1$

$1 > \sqrt{1} - 1$

$2 > \sqrt{1}$

$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1 \Rightarrow \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} > \sqrt{n+1} - 1$

$\frac{1}{\sqrt{n+1}} + \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n} - 1 + \frac{1}{\sqrt{n+1}}$

$> \frac{(\sqrt{n+1})(\sqrt{n}-1)+1}{\sqrt{n+1}}$

$> \frac{\sqrt{n+1}\sqrt{n} - \sqrt{n+1} + 1}{\sqrt{n+1}}$

$> \frac{1}{\sqrt{n+1}}$ $\quad$ since $\sqrt{n+1}\sqrt{n} > \sqrt{n+1}$

Asymptotic Analysis

$\exists a \in \mathbb{R}^+,\ a > 1 \wedge (a^n + 3 \notin O(2^n) \vee a^n + 3 \notin \Omega(2^n))$

$a = 3 \Rightarrow a > 1$

$2^n + 3 \notin \Omega(2^n)$

$: \forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge 2^n + 3 < c\,2^n$

let $c, n_0 \in \mathbb{R}^+$

$n = \lceil \max(n_0, \log_{2/3} \frac{2}{c}, \log_2 \frac{6}{c}) \rceil + 1$ $\quad 3^n + 3 < \frac{c}{2} 2^n + \frac{c}{2} 2^n$

$3^n < \frac{c}{2} 2^n \qquad 3 < \frac{c}{2} 2^n$

$\frac{2}{c} 3^n < 2^n \qquad \frac{6}{c} < 2^n$

$\log_2 \frac{6}{c} < n$

$\frac{2}{c} < \left(\frac{2}{3}\right)^n$

$\lceil \log_{2/3} \frac{2}{c} \rceil < n$

Running time

Loop 2: $j_{k+1} = j_k + 2$

$j_0 = 0 \quad j_1 = 2 \quad j_2 = 4$

$j_k = 2k \geq i$

$k \geq \left\lceil \frac{i}{2} \right\rceil$

Loop 1: $i_{k+1} = i_k + 1$

$i_0 = 0 \quad i_1 = 1$

$i_k = k \quad i_k^2 \geq n$

$k \geq \sqrt{n}$

$$\sum_{k=0}^{\sqrt{n}-1} \left\lceil \frac{i_k}{2} \right\rceil \geq \frac{1}{2} \sum_{k=0}^{\sqrt{n}-1} i_k$$

$$= \frac{(\sqrt{n}-1)(\sqrt{n})}{4}$$

$$= \frac{n - \sqrt{n}}{4}$$

UBWC

Loop 1: n

Loop 2: $n - i - 1 \leq n$

$\leq n^2$

LBWC

$lst[i]_{k+1} = lst[i]_k - 1$ for all $i \in \{0, \dots, n-1\}$

$lst[i]_1 = lst[i]_0 - 1$

$\leftarrow$ denotes values of the list after 1 iteration of the outer loop.

$lst[i]_n > i + n$

n

$lst[j]_n = lst[j]_0 + n$

$$S_k = S_{k-1} + i_{k-1}$$

$$\vdots$$

$$= S_0 + \sum_{j=0}^{k-1} i_j$$

$$= S_0 + \sum_{j=0}^{k-1} 2j+1$$

$$= \sum_{j=1}^{k} (2j-1) = k^2$$

$$i_j = i_{j-1} + 2$$
$$= 2j + 1$$

i_{k-1} since S_k depends on the previous value of i_k

PS4 :

1a) $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{twisty\text{-}too, n}, RT_{twisty\text{-}too}(x) \geq f(n)$

Let $n \in \mathbb{N}$

Let $x = n$

$$i_k = i_{k-1} - 4$$
$$= -4k$$
$$-4k < 0$$

1c) Loop 2: $S_k = S_{k-1} + 2$
$$= 2k - 1$$

$$j_k = j_{k-1} + S_{k-1}$$

$$j_k = j_0 + \sum_{i=0}^{k-1} S_i$$

$$= j_0 + \sum_{i=0}^{k-1} 2 - 1$$

$$= j_0 + \sum_{i=0}^{k-1} (2i - 1)$$

$$= k^2 - 2k \geq i_k$$
$$\geq n - k$$
$$k^2 \geq n + k$$
$$k^2 - k \geq n$$
$$k \geq \sqrt{n}$$

$$i_k =$$

Loop 3: $i_k = i_{k-1} - 1$
$$= i_0 - k$$
$$= (i_{j-1} - 1) - k$$

$$n - j - k \ \% \ 4 \leq 0$$

$$n - j - k = 41$$

Loop 1: $i_j = i_0 - j$
$$i_j = n - j$$

$$RT_{twisty\text{-}too}(x) \leq n + \sqrt{n} + 3n$$

$$RT_{twisty\text{-}too}(x) \geq n + \sqrt{n}$$

$$\Rightarrow \Theta(n)$$

2a) $\Theta(n^{4/3})$ $n^{4/3} = n \cdot n^{1/3}$ $lst = [2, 2, ..., 2]$, $t = 2^{\sqrt[3]{n}}$ $2^{k+1} > 2^{\sqrt[3]{n}}$

Loop 2: $p_k = p_{k-1} \cdot lst[j]$ $j_k = j_{k-1} - 1$ $k+1 > \sqrt[3]{n}$

$p_k = p_0 \cdot lst[j]^k$ $= j_0 - k$ $k > \sqrt[3]{n} - 1 \Rightarrow k \geq \sqrt[3]{n}$

$= lst[j]^k$ $= i_\alpha - 1 - k$

stopping $j_k < 0$ or $p_k \cdot lst[j] > t$ $k = \lceil \log_{lst[j_k]} t \rceil = n^{1/3}$

$i_\alpha - 1 - k < 0$ or $lst[j]^{k+1} > t$

$i_\alpha - 1 < k$ $k > \log_{lst[j]} t - 1$ $= n^{1/3} \geq \log_{lst[j]} t$

$i_\alpha \leq k$ $k \geq \log_{lst[j]} t$ $lst[j]^{\sqrt[3]{n}} \geq t$

$\lceil \log_{lst[j]} t \rceil$ $\lfloor lst[j]^{\sqrt[3]{n}} \rfloor$

Loop 1: $= \sum_{i=1}^{n}$ # Loop 2 iterations

$= \sum_{i=1}^{n} i$ or $\sum_{i=1}^{n} \lceil \log_{lst[j]} t \rceil$

By our choice of input L1

$= \frac{n(n+1)}{2}$ iterates $n(\lceil \log_{lst[j]} t \rceil) = n(n^{1/3}) = n^{4/3}$

2b) $\forall n \in \mathbb{N}, \forall x \in \mathcal{I}_{long\text{-}prod\, lst,t,n}, RT_{long\text{-}prod}(x) \leq f(n)$

Let $n \in \mathbb{N}$

Let $x \in \mathcal{I}_{long\text{-}prod\, lst,t,n}$

from a)

$$RT_{long\text{-}prod}(x) \leq \frac{n(n+1)}{2} + 1 + 1$$

2c) $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{long\text{-}prod\, lst,t,n}, RT_{long\text{-}prod}(x) \geq f(n)$

Let $n \in \mathbb{N}$

$x = lst = [-1, ..., -1]$

$t = 2$

Then p is either 1 or -1 < 2

and L2 runs i iterations $\rightarrow RT_{long\text{-}prod}(x) \geq \frac{(n+1)n}{2}$

2d) $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{long\text{-}prod\, lst,t,n}, RT_{long\text{-}prod}(x) < f(n)$

Let $n \in \mathbb{N}$

$x = lst = [1, ..., 1]$

$t = 0$

then L2 never runs

L1 runs $n-1$ iterations $\leq n$

$\forall n \in \mathbb{N}, \exists$

Q3) Avg Case

lst[i] == 1

a) let $n \in \mathbb{N}$, $n \geq 2$

K is the value returned by alpha_min

$$RT_{alpha\_min}(x) = 1 + 1 + RT_{loop1}(x)$$
$$= 1 + 1 + (1 + \text{index of b in } s - 1)$$ ← True since loop 1 is false when $s[i-1] > s[i]$ or when we find 'b'.

$$= 2 + \sum_{k=0}^{n-1} k$$

$$RT_{loop}(x) = i_j = i_{j-1} - 1$$
$$= i_0 - j$$

$$= 2 + \frac{(n-1)n}{2} = \frac{4 + n^2 - n}{2}$$

$i_0 - j < 0$ $\qquad$ $s[i-1] > s[i]$

$n - 1 \leq j - 1$ $\qquad$ or $\qquad$ $j \geq n - K$ ← How do I prove this

$n \leq j$

b) $$Avg_{alpha\_min}(4) = \frac{1}{|\mathcal{I}_4|} \sum_{s \in \mathcal{I}_4} RT_{alpha\_min}(s)$$

$$= \frac{1}{\binom{4}{2}} \sum_{s \in \mathcal{I}_4} \frac{4 + 4^2 - 4}{2}$$

$2 + n - K$

$$= \frac{1}{6} \cdot \frac{16}{2} \sum_{s \in \mathcal{I}_4} 1$$

$$= \frac{4}{3} \sum_{s \in \mathcal{I}_4} 1$$ ← not really sure how to compute this

c) $n \in \mathbb{N}$, where $\mathcal{I}_n$ is defined and k

$|\{s \in \mathcal{I}_n \mid \text{alpha\_min}(s) \text{ returns } k\}| \neq |\mathcal{I}_n|$?

$\mathcal{I}_2 = \{bb\}$ (0) $\qquad$ 1, 0

$\mathcal{I}_3 = \{abb, bba, bab\}$ (0, 2, 1) $\qquad$ 1, 1, 1

$\mathcal{I}_4 = \{aabb, abab, abba, baab, baba, bbaa\}$ (0, 2, 3, 1, 3, 2) $\qquad$ 1, 1, 2, 2

$\mathcal{I}_5 = \{aaabb, aabba, abbaa, bbaaa, aabab, abaab, baaab, baaba, babaa, ababa\}$ (0, 4, 3, 2, 3, 2, 1, 4, 3, 4) $\qquad$ 1, 1, 2, 3, 3

P(s): $\forall s \in \mathcal{I}_n$, alpha_min(s) returns k

$S_{n,k} = \{s \in \mathcal{I}_n \mid \text{alpha\_min}(s) \text{ return } k\}$

$|S_{n,k}| = K(n-k+1) - 1$

	k = 0	1	2	3	4	5	6
n = 2	1	0	0	0	0	0	0
n = 3	1	1	1	0	0	0	0
4	1	1	2	2	0	0	0
5	1	1	2	3	3	0	0
6	1	1	2	3	4	4	0
7	1	1	2	3	4	5	5

$S_{4,0} = \{aabb\}$

$S_{4,1} = \{baab\}$ $\qquad$ $n - k$

$S_{4,2} = \{abab, bbaa\}$

$$\binom{K}{2} = \frac{K(K-1)}{2}$$

$= 1 \qquad = 1 \qquad$ if $K = 0$

$= \binom{k-1}{1} + 1 \qquad = k \qquad$ if $n > k+1$

or

$= \binom{k-1}{1} \qquad = k - 1 \qquad$ if $n = k+1$

$= 0 \qquad = 0 \qquad$ if $n < k+1$

$\forall n \in \mathbb{Z}^+$, each string of length n is in exactly 1 set $S_{n,k}$ (for some $k \in \{0, \dots, n\}$)

For each input,

3d) $\forall$

$$Avg_{alpha\_min}(n) = \frac{1}{|\mathcal{I}_n|} \sum_{s \in \mathcal{I}_n} RT_{alpha\_min}(s)$$

$$= \frac{1}{|\mathcal{I}_n|} \sum_{k=0}^{n-1} \sum_{s \in S_{n,k}} RT_{alpha\_min}(s)$$

$$= \frac{1}{\binom{n}{2}} \sum_{k=0}^{n-1} |S_{n,k}|(n-k)$$

$$= \frac{1}{\binom{n}{2}} \sum_{k=0}^{n-2} |S_{n,k}|(n-k) + |S_{n,n-1}|(n-(n-1))$$

$$= \frac{1}{\binom{n}{2}} \sum_{k=1}^{n-2} k(n-k) + 1 + (n-2)$$

$$= \frac{1}{\binom{n}{2}} \left(- \frac{(n-2)(n-1)(2n-3)}{6} + n\frac{(n-2)(n-1)}{2} + 1 + (n-2) \right)$$

$$= \frac{1}{\binom{n}{2}} \left((n-2)(n-1)\left(\frac{1}{6}n+3\right) + 1 + (n-2) \right)$$

$$= \frac{2(n-2)(\frac{1}{6}n+3)}{n} + \frac{2 + 2(n-2)}{n(n-1)}$$

$$= \frac{(n-2)(n+3)}{3n} + \frac{2n-2}{n(n-1)}$$

$$\frac{2 \cdot 7}{12} + \frac{6}{12} = \frac{20}{12} = \frac{10}{6} = \frac{5}{3}$$

$|S_{n,n-1}| = k-1$

$|S_{n,k}| = k$

$|S_{n,0}| = 1$

$\hookrightarrow$ each form a partition of $\mathcal{I}_n$

148 can be retaken exactly once

$\exists x \in T,\ retake\_148(x) \wedge (\forall y \in T,\ retake\_148(y) \Rightarrow x = y)$

WS18 Avg Case Analysis

a) $n \in \mathbb{Z}^+$, $\mathcal{I}_n :=$ the set of all binary lists of length n.

Find $|\mathcal{I}_n|$

$|\mathcal{I}_n|$ = Permutations w/ repetition = 2^n

b) $n \in \mathbb{Z}^+$, $i \in \{0, 1, \dots, n-1\}$, $S_{n,i} :=$ set of all binary lists of length n where the first 0 is in ith index

$$|S_{n,i}| = 2^{n-1} + 2^{n-2} + 2^{n-3}$$
$$= 2^{n-1-i}$$

$S_{n,i} := \{ lst \in S_n \mid lst[i] = 0 \land \forall j \in \mathbb{N},\ j < i \Rightarrow lst[j] = 1 \}$

$\mathcal{I}_1 = \{[0], [1]\}$ 1, 0 0

$\mathcal{I}_2 = \{[0,1], [0,0], [1,1], [1,0]\}$ 2, 1 0

$\mathcal{I}_3 = \{[0,0,0], [0,0,1], [0,1,0], [1,0,0], [1,1,0], [1,0,1], [0,1,1], [1,1,1]\}$ 4, 2, 1

c) $|S_{n,n}| = |[1, 1, \dots, 1]| = 1$

d) $\forall n \in \mathbb{Z}^+$, $\exists \mathcal{I}_n \in S_{n,i} \land \forall \mathcal{I}_n^i \in S_{n,i} \Rightarrow \mathcal{I}_n^i = \mathcal{I}_n$

$$\begin{matrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & & \\ 2 & 1 & 0 & 0 & & \vdots \\ 4 & 2 & 1 & 0 & & \\ 8 & 4 & 2 & 1 & & \\ \vdots & & & & \ddots & \\ 2^{n-1} & 2^{n-2} & 2^{n-3} & 2^{n-4} & \cdots & 1 \quad 0 \end{matrix}$$

Graph Theory

When is the graph connected

- Necessary condition?

  ($\neg$ condition $\Rightarrow \neg$(G connected))

  $|E| \geq |V| - 1$

- Sufficient condition

  (condition $\Rightarrow$ G connected)

  $|E| \geq \frac{(|V|-1)(|V|-2)}{2} + 1$

r is suff for s

$r \to s$ or $\neg r \vee s$ $\qquad$ $s \to r$ or $\neg s \vee r$

r	s	$\neg r \vee s$
F	F	T
F	T	T
T	F	
T	T	

we don't know $r \to s$ if r is false $\qquad$ we don't know $s \to r$ if r is true

Proof of the sufficient condition

$\forall G = (V, E)$, $|E| \geq \frac{(|V|-1)(|V|-2)}{2} + 1 \to$ G is connected

- Direct proof

  Let $G = (V, E)$. Assume $|E| \geq \frac{(|V|-1)(|V|-2)}{2} + 1$

  WTP: G is connected, i.e.

  $\forall u, v \in V$, $\exists$ path between u and v.

- Indirect proof

  Let $G = (V, E)$. Assume G is not connected

  ie, $\exists u, v \in V$, G contains no path between u and v.

  WTP: $|E| < \frac{(|V|-1)(|V|-2)}{2} + 1$

- Induction?

  - need an integer / nat. number
  - try induction on $|V|$

  WTP:

  $\forall n \in \mathbb{N}$, $\forall G = (V, E)$, $\overbrace{|V| = n}^{\text{induction on } |V|} \Rightarrow \underbrace{(|E| \geq \frac{(n-1)(n-2)}{2} + 1 \Rightarrow G \text{ is connected})}_{\text{same as before}}$ $\leftarrow$ P(n) for induction

Base Case: $n = 0$:

WTP: $\forall G = (V, E)$, $|V| = 0 \Rightarrow (|E| \geq \frac{(0-1)(0-2)}{2} + 1 \Rightarrow G$ connected$)$

Let $G = (V, E)$. Assume $|V| = 0$

$|E| = 0$ so implication is vacuously true

$n = 1$ is also vacuously true

Ind. Step: Let $n \in \mathbb{N}$ and $n \geq 1$

Assume P(n):

$\forall G = (V, E)$, $|V| = n \Rightarrow (|E| \geq \frac{(n-1)(n-2)}{2} + 1 \Rightarrow G$ is connected$)$

WTP: P(n+1)

$\forall G_1 = (V_1, E_1)$, $|V_1| = n+1 \Rightarrow (|E_1| \geq \frac{(n+1-1)(n+1-2)}{2} + 1 \Rightarrow G$ is connected$)$

Do not start w/ assumption

$\to$ does not give you an object of that type that you can work w/.

Start w/ what you WTP.

Let $G_1 = (V_1, E_1)$ w/ $|V_1| = n+1$

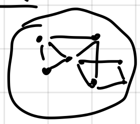
Assume $|E_1| \geq \frac{n(n-1)}{2} + 1$

WTP: G_1 is connected. Let $u, v \in V_1$.

G_1 contains a path between u, v

Rough Work

G_1 :



$|V_1| = n+1$

Idea: pick some $v_0 \in V_1$

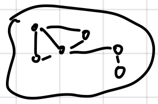
consider

$G' = (V', E')$, where

$V' = V_1 - \{v_0\}$

$E' = E - \{\text{every edge in } E_1 \text{ that contains } v_0\}$

G' :

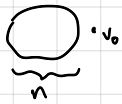


- v_0 belongs to at most n edges in G

  degree$(v_0) \leq n$

  so $|E'| \geq |E_1| - n$

G_1 :



- could degree$(v_0) = 0$?

  If that were the case, then

  $|E_1| \leq \frac{n(n-1)}{2}$ — contradicts the assumption $|E_1| \geq \frac{n(n-1)}{2} + 1$

  so $|E'| \leq |E_1| - 1$

- back to first observation:

  $|E'| \geq |E_1| - n$

  $\geq \left(\frac{n(n-1)}{2} + 1\right) - n = \frac{(n-1)(n-2)}{2}$ → missing $+1$

Idea: pick another "v_0" with degree $< n$

Fact: Either $\exists v_0 \in V_1$, degree$(v_0) < n$ or

$\forall v_0 \in V_1$, degree$(v_1) = n$

Case 1: $\exists v_0 \in V$, degree$(v_0) < n$

let $G' = "G_1 - v_0"$

Then, $|E'| \geq \frac{(n-1)(n-2)}{2} + 1$ (see above)

so G' is connected by I.H

so G_1 is connected (v_0 is connected to G')

Case 2: $\forall v_0 \in V_1$, degree$(v_0) = n$

Then G_1 is connected

$(\forall u, v \in V_1, (u,v) \in E_1)$

Simpler Argument

Proof of $\forall n \in \mathbb{N}, \forall G=(V,E), |V|=n \Rightarrow (|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G$ is connected$)$

Base Case: $n=0$ and $n=1$

statement is vacuously true

(see last time)

This is okay b.c. of the extra bc.

Inductive Step: Let $n \in \mathbb{N}$ w/ $n \geq 1$

Assume $\underbrace{\forall G=(V,E), |V|=n \Rightarrow (|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected})}_{\text{I.H}}$

Let $G_1=(V_1,E_1)$ w/ $|V_1|=n+1$

Assume $|E_1| \geq \frac{n(n-1)}{2}+1$

WTP: G_1 is connected

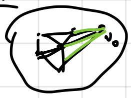
We cannot start w/ graph w/ n vertices

→ we don't have a graph w/ n vertices.

In my proof I have to start w/ a graph w/ n+1 vertices.

Intuition

G_1:



$G' = G_1$ w/o v_0 an its edges

- when I pick v_0 it must have at least 1 edge connected to another vertex.

  cannot have a vertex v_0 which is not connected to another vertex.

- exactly this many edges

  when I remove v_0 I will have too few edges.

W19

1. Terminology Review

a. vertices = 7

b. edges = 11

c. vertices adj to G = {B, C, D, F}

d. length = 2

Yes paths = {(A, D, G), (A, B, G), (A, C, G)}

e. Path that goes through all vertices:

A, D, G, F, C, E, B

2. Vertex degree

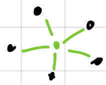
Def^n: degree := Let $G = (V, E)$ be a graph
v be a vertex in V.
$d(v)$ is # neighbours of v.

a. $d(D) = 3$

b. B, C, and G

c. $G = (V, E)$, assume $\forall v \in V,\ d(v) \leq 5$

prove a good upper bound on $|E|$ in terms of $|V|$ $\qquad d(v) = 5$



$\forall G = (V, E), (\forall v \in V, d(v) \leq 5) \Rightarrow |E| \leq 5|V|$

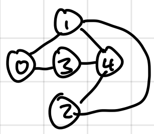
in total, all of the vertices can be an endpoint for at most $5|V|$ edges. Since every edge must have an endpoint,
$|E| \leq 5|V|$

Note: every edge must have exactly two distinct endpoints
since $|E| \leq 5|V|$ possible spots, meaning
$|E| \leq \frac{5}{2}|V|$

3. Adjacency Matrix

$G = (V, E)$, vertex set := {0, 1, ..., n-1} for some $n \in \mathbb{N}$.
A[i][j] is the set to 1 if i and j are adjacent and 0 if they aren't.

a.



$$A = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 1 & 1 & 1 & 0 \end{array}$$

b.
```
def degree(A: List[List[int]], i: int) -> int

    return sum(A[i])
```

WS20

1. Bipartite Graphs

Let $G = (V, E)$

G is bipartite when it satisfies:

- $\exists V_1, V_2 \subset V$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and V_1 and V_2 form a partition of V.
- $\forall e \in E$, has exactly one endpoint in V_1 and one in V_2.

$\Leftrightarrow$ (No two vertices in V_1 are adjacent and no two vertices in V_2 are adjacent)

When G is bipartite, V_1 and V_2 are a bipartition of G.

a. $V = \{1,2,3,4,5,6\} \wedge E = \{(1,2),(1,6),(2,3),(3,4),(4,5),(5,6)\}$

Choose $V_1 = \{1,3,5\}$ $\qquad$ $\forall e \in E$, $\exists$ endpoint $\in V_1 \wedge \forall$ endpoint, $\in V_1$

$V_2 = \{2,4,6\}$

then $V_1 \wedge V_2 = V$

each edge has a vertex that is odd and a vertex that is even.

b. Let $m, n \in \mathbb{Z}^+$. A complete bipartite graph on (m,n) vertices is a graph w/ the following properties

- G is bipartite, w/ bipartition V_1, V_2
- $|V_1| = m$ and $|V_2| = n$
- $\forall u \in V_1$, $\forall w \in V_2$, u and w are adjacent

Let $G = (V,E)$ be a CBPGraph on (m,n) vertices w/ bipartition V_1, V_2 and $|V_1| = m$ and $|V_2| = n$

Then each vertex $u \in V_1$ appears as an endpoint in n edges in E, since it has an edge to each of the n vertices in V_2.

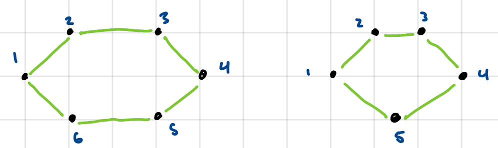
As there are m vertices in V_1, the previous statement is true for each of them, we know that there are at least mn edges in E.

But, since there are no edges between vertices in V_1 and no edges between vertices in V_2, there are no edges to count.

$|E|$ in a bipartite graph $= mn$

c. Cycle := sequence of vertices $v_0, v_1, \dots, v_k$, s.t. $k \geq 3$, $v_k = v_0$

and G contains every edge between consecutive vertices: $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$



Conjecture. The lengths of every cycle in a bipartite graph is even.

Pf: $G = (V,E)$ and assume G is bipartite, w/ Bipartition V_1, V_2. $\quad$ proof is identical if V_1 and V_2 were switched.

Let $C = v_0, \dots, v_k$ be a cycle in G. WLOG, assume $v_0 \in V_1$

k is even.

$k \geq 3 \Rightarrow \forall$ paths

Cont.

c. $\forall k \in \mathbb{N},\ k \geq 3 \Rightarrow \forall$ paths in G (bipartite graph), $v_0 \in V_0$, $0 \leq i \leq k$, Even$(i) \Rightarrow v_k \in V_1$ and Odd$(i) \Rightarrow v_k \in V_2$

Let $k \in \mathbb{N}$

Assume $k \geq 3$

Let p be an arbitrary path in G

$v_0 \in V_1$

Base Case ($k=3$)

$v_0 \in V_1$, $v_1 \in V_2$, $v_2 \in V_1$, $v_3 \in V_2$

then this path is not Bipartite.

Base Case ($k=4$)

Exercise Break 2.1

Prove that $\forall a,b,c \in \mathbb{Z}$, $a \mid b \land a \mid c \rightarrow a \mid b+c$

TT2

1. $\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x\geq n) \wedge 6x \geq 2n+3$ $T \to F$

let $x = 2$

$n = 1$

then, $6(2) \geq 2(1)+3$

$12 \geq 5$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x> n) \wedge 3x \leq 5n+2$ $T \to F$

let $x=2$ $\quad 6 \leq$

$n=1$ $\quad 2>1$

then, $3(2) \leq 5(1)+2$

$6 \leq 7$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x>n) \wedge (5x \geq 2n+1)$ $T \to F$

let $x = 2$

$n=1$

then $2>1$

and $5(2) \geq 2(1)+1$

$10 \geq 3$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x>n) \wedge (4x \leq 3n+6)$ $T \to F$

let $x=2$

$n=1$

then $2>1$

and $4(2) \leq 3(1)+6$

$8 \leq 9$ □

$\exists x\in\mathbb{Z}, \exists n\in\mathbb{Z}, (x>n) \wedge (7x \geq 4n-3)$ $T \to F$

let $x=2$

$n=1$

then $2>1$

and $7(2) \geq 4(1)-3$

$14 \geq 1$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x<n) \wedge (2x \leq 3n+3)$ $T \to F$

let $x=1$

$n=2$

then $1<2$

$2(1) \leq 3(2)+3$

$2 \leq 9$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x<n) \wedge (6x \geq 2n+1)$ $T \to F$

let $x=1$

$n=2$ $\quad 1<2$ and $6(1) \geq 2(2)+1$

$6 \geq 5$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x<n) \wedge (3x \leq 7n-4)$ $T \to F$

let $x=1$ $\quad 3(1) \leq 7(2)-4$

$n=2$ $\quad 3 \leq 10$ □

$\exists x\in\mathbb{R}, \exists n\in\mathbb{Z}, (x<n) \wedge (5x \geq 3n-1)$ $T \to F$

let $x=1$

$n=2$ $\quad 1<2$ $\quad 5(1) \geq 3(2)-1$

$5 \geq 5$ □

b)

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x<n) \vee (3x+1 \leq 3n^2)$ $F \to T$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\geq n) \wedge (3x+1 > 3n^2)$

let $x\in\mathbb{R}^{\geq 0}$ $\quad 3x+1 > 0$

$n=0$ $\quad 3x > -1$

$x > -\frac{1}{3}$ $\quad$ true since $x \geq 0$

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x<n) \vee (4x+6 \leq 5-2n^2)$ $F \to T$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\geq n) \wedge (4x+6 > 5-2n^2)$

let $x\in\mathbb{R}^{\geq 0}$ $\quad 1\geq 0$ $\quad 4x+1 > -2n^2$

$n=0$ $\quad 4x+1 > -2(0)^2$

$4x+1 > 0$

$x > -\frac{1}{4}$ $\quad$ since $x \geq 0$

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x\geq n) \vee (2-3x < 3n^2+2)$ $F \to T$

let $x\in\mathbb{R}^{\geq 0}$ $\quad -3x < 3n^2$

$n=0$ $\quad x\geq 0 \wedge -x < n^2$

$-x < 0^2$ $\quad$ since $-x \leq -n$

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x<n) \vee (8x+1 \leq 4n^2+5)$ $T \to F$

let $x = 1$

$8(1)+1 \leq 4n^2+5$

$-4 \leq 4n^2+5$

$-1 \leq n^2$ $\quad$ true $n^2 \geq 0$

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x<n) \vee (7-3x \geq 2n^2+1)$ $F \to T$

(?) $\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\geq n) \wedge (7-3x < 2n^2+1)$

let $x\in\mathbb{R}^{\geq 0}$ $\quad \sqrt{\frac{6-3x}{2}} < n$ $\quad x \geq \left\lceil\sqrt{\frac{6-3x}{2}}\right\rceil$

$n = \left\lceil\sqrt{\frac{6-3x}{2}}\right\rceil + 1$ $\quad 2x^2 > 6-3x$

$7-3x < 2n^2+1$ $\quad 2x^2+3x-6 > 0$ □

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x>n) \vee (6+10x \leq 3n^2+3)$ $F \to T$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\leq n) \wedge (6+10x > 3n^2+3)$

$x\in\mathbb{R}^{\geq 0}$ $\quad \sqrt{\frac{3+10x}{3}} > n$

$n = \left\lfloor\sqrt{\frac{3+10x}{3}}\right\rfloor - 1$

$\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x>n) \vee (5x+2 \geq 5+3n^2)$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\leq n) \wedge (5x+2 < 5+3n^2)$

let $x\in\mathbb{R}^{\geq 0}$ $\quad \sqrt{\frac{5x-3}{3}} < n$

$n = \left\lceil\sqrt{\frac{5x-3}{3}}\right\rceil + 1$

(?) $\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x>n) \vee (3x+2 \leq n^2-4)$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\leq n) \wedge (3x+2 > n^2-4)$

let $x\in\mathbb{R}^{\geq 0}$ $\quad \sqrt{3x-2} > n$

$n = \lfloor\sqrt{3x-2}\rfloor$ $\quad x \leq \lfloor\sqrt{3x-2}\rfloor$

(?) $\exists x\in\mathbb{R}^{\geq 0}, \forall n\in\mathbb{Z}, (x>n) \vee (3x+4 \leq 5n^2-1)$

$\forall x\in\mathbb{R}^{\geq 0}, \exists n\in\mathbb{Z}, (x\leq n) \wedge (3x+4 > 5n^2-1)$

let $x\in\mathbb{R}^{\geq 0}$

$n =$

Induction

1. $\forall n \in \mathbb{N}, (n \geq 2) \Rightarrow (e^{1-n} + 3 < n^2 + 2)$

Let $n \in \mathbb{N}$ Assume $n \geq 2$

Base Case: $n = 2$

$e^{1-2} + 3 < 2^2 + 2$

$e^{-1} + 3 < 6$

$\frac{1}{e} + 3 < 6$

$\frac{1}{e} < 3$ since $\frac{1}{n}$ for some n in $0 \leq \frac{1}{n} \leq 1$

Inductive Step: $P(n) \Rightarrow P(n+1)$

Assume $(e^{1-n} + 3 < n^2 + 2)$, prove $e^{1-(n+1)} + 3 < (n+1)^2 + 2$

$e^{-n} + 3 < n^2 + 2n + 3$

$$\frac{1}{e^n} + 3 < \frac{e^1}{e^n} + 3$$

$$< n^2 + 2$$

$$< n^2 + 2n + 3$$ since $2 < 2n + 3$

$2 < 7$

2. $\forall n \in \mathbb{N}, (n \geq 3) \Rightarrow (e^{-n+2} + 4 < n^2 + 1)$

Let $n \in \mathbb{N}$. Assume $n \geq 3$

Base Case: $n = 3$

$e^{-3+2} + 4 < 3^2 + 1$

$e^{-1} + 4 < 10$

$\frac{1}{e} + 4 < 10$

$\frac{1}{e} < 6$ true

Inductive Step: $P(n) \Rightarrow P(n+1)$

Assume $e^{-n+2} + 4 < n^2 + 1$. Prove $e^{-(n+1)+2} + 4 < (n+1)^2 + 1$

$e^{1-n} + 4 < n^2 + 2n + 2$

I.H: $$\frac{e}{e^n} + 4 < \frac{e^2}{e^n} + 4$$

$$< n^2 + 1$$

$$< n^2 + 2n + 2$$ since $1 < 2n + 2$

3. $\forall n \in \mathbb{N}, (n \geq 4) \Rightarrow (e^{3-n} + 6 < n^2 - 2)$

Let $n \in \mathbb{N}$. Assume $n \geq 4$.

Base Case: $n = 4$

$e^{3-4} + 6 < 4^2 - 2$

$e^{-1} < 8$ true

Inductive Step: $P(n) \Rightarrow P(n+1)$

Assume $e^{3-n} + 6 < n^2 - 2$. Prove $e^{3-(n+1)} + 6 < (n+1)^2 - 2$

$e^{2-n} + 6 < n^2 + 2n - 1$

I.H $e^{2-n} + 6 < e^{3-n} + 6$

$< n^2 - 2$

$< n^2 + 2n - 1$

$-2 < 2n - 1$

$-1 < 2n$

Proof By Contradiction

1. 7 (or 11, or 13, or 17) is not a balanced prime

Assume 7, 11, 13, 17 are balanced.

meaning it satisfies $p_i = \frac{p_{i-1} + p_{i+1}}{2}$

but $7 = \frac{5+11}{2} = 8$

$11 = \frac{7+13}{2} = 10$

$13 = \frac{11+17}{2} = 14$

$17 = \frac{13+19}{2} = 16$ this is a contradiction $\Rightarrow\Leftarrow$

2. 7 (or 13, or 17, or 19) is NOT a Sophie-Germain Prime

Assume 7, 13, 17, 19 are SG primes

which satisfies $2p+1$ = a prime number

but, $2(7)+1 = 15$ $(3 \mid 15)$

$2(13)+1 = 27$ $(3 \mid 27)$

$2(17)+1 = 35$ $(5 \mid 35)$

$2(19)+1 = 39$ $(3 \mid 39)$ which is a contradiction.

3. 7 (or 11, or 19, or 23) is not a P-prime

Assume 7, 11, 19, 23 are P-primes

$\exists d \in \mathbb{N}, p = 4d+1$

but $7 = 4d+1 \Rightarrow 6 = 4d$

$19 = 4d+1 \Rightarrow 18 = 4d$

$23 = 4d+1 \Rightarrow 22 = 4d$

$11 = 4d+1 \Rightarrow 10 = 4d$

all not divisible by 4

contradiction

Absolute value

$\forall z \in \mathbb{R}, |z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z < 0 \end{cases}$

$|x-6| \leq b-2x$

$\Rightarrow x-6 \leq b-2x$ if $x-6 \geq 0$ and $6-x \leq b-2x$ if $x-6 < 0$

$3x \leq b+6$ and $6-b \leq -x$

$x \leq \frac{b+6}{3}$ $\qquad b-6 \geq x$

thus $x \leq b-6 \leq \frac{b+6}{3}$

and the domain of x is

$\in (-\infty, b-6]$

Case 1: $x-6 \geq 0$

$x \geq 6$

$x-6 \leq b-2x$

$x \leq \frac{b+6}{3}$

$(-\infty, \frac{b+6}{3}] \cap [6, \infty) = \emptyset \subseteq (-\infty, b-6]$

Case 2: $x-6 < 0$

$x < 6$

$x \leq b-6$

$(-\infty, 6) \cap (-\infty, b-6] = (-\infty, b-6]$

TT3 Asymptotic Notation

1. $f(n) = n^2(\cos(n\pi)+1)+1$

$f \in O(n^2)$

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^2(\cos(n\pi)+1)+1 \leq cn^2$

$c = 4$

$n_0 = \sqrt{\frac{1}{2}}$

$\cancel{n^2}(\cos(n\pi)+1) \leq \frac{c}{2}\cancel{n^2}$ or $1 \leq \frac{c}{2}n^2$

$\cos(n\pi)+1 \leq \frac{c}{2}$

$2\cos(n\pi)+2 \leq c$

$\frac{2}{n^2} \leq c$

$\sqrt{\frac{2}{c}} \leq n$

$\sqrt{\frac{1}{2}}$

$-1 \leq \cos(n\pi) \leq 1$

$2\cos(n\pi)+2 \leq 4$

$\cos(n\pi)+1 \leq \frac{4}{2}$

$n^2(\cos(n\pi)+1 \leq \frac{4}{2}n^2$

$\sqrt{\frac{1}{2}} \leq n$

$\frac{1}{2} \leq n^2$

$1 \leq 2n^2$

$n^2(\cos(n\pi)+1)+1 \leq \frac{4}{2}n^2 + \frac{4}{2}n^2$

$\leq cn^2$

b. $f \notin \Theta(n^2) \rightarrow f \notin \Omega(n^2)$

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^2(\cos(n\pi)+1)+1 < cn^2$

Let $c, n_0 \in \mathbb{R}^+$

$\sqrt{\frac{2}{c}}$

$\cos(n\pi)+1 < c$

$\cos(n\pi) < c-1$

$-1 \leq \cos(n\pi) \leq 1$

Number Representations

1. $(x)_2$ is four digits long   largest
   $= (1111)_2$
   $(x)_{16}$ is five digits long and contains   smallest
   exactly 2 A's and 1 E w/ no leading 0s
   $= (11AAE)_{16}$ $(10AAE)_{16}$
   $(x)_8$ is a five digit long palindrome   smallest
   $= (10001)_8$ $(10201)_8$
2. $(x)_2$ is five digits long contains exactly 2 0's 3 1's   smallest
   $= (10011)_2$
   $(x)_8$ is five digits long w/ 2 2's, 1 7's   largest
   $(76622)_8$
   $(x)_{16}$ is five digit palindrome   smallest
   $(21012)_{16}$ $(10201)_{16}$
3. $(x)_2$ w/ 2 0's, 3 1's
   $(11100)_2$
   $(x)_{16}$
   $(FEDC)_{16}$
   $(x)_8$
   $(76567)_8$
4. $(x)_2$ 4 digits   L
   $(1111)_2$
   $(102)_8$ S   3 digits no more than 1
   $(FEDEF)_{16}$ L   5 digits no more than 2.
5. $(10011)_2$
   $(FEDC)_{16}$
   $(10201)_8$
6. $(11100)_2$
   $(76622)_8$
   $(FEDEF)_{16}$

Induction

1. $\forall n \in \mathbb{N}, n \geq 1 \Rightarrow \sum_{k=0}^{n-1} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1} (b_{k+1} - b_k)$

Let $n \in \mathbb{N}$. Assume $n \geq 1$

Base case $n = 1$

$$(a_1 - a_0) b_0 = a_1 b_1 - a_0 b_0 - a_1 (b_1 - b_0)$$

$$a_1 b_0 - a_0 b_0 = a_1 b_1 - a_1 b_1 + a_1 b_0 - a_0 b_0$$

$$\cancel{-a_0 b_0} = \cancel{-a_0 b_0}$$

$$1 = 1$$

Inductive step: $P(n) \Rightarrow P(n+1)$

Assume $\sum_{k=0}^{n-1} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1} (b_{k+1} - b_k)$

Prove

$$\sum_{k=0}^{n} (a_{k+1} - a_k) b_k = a_{n+1} b_{n+1} - a_0 b_0 - \sum_{k=0}^{n} a_{k+1} (b_{k+1} - b_k)$$

$$(a_{n+1} - a_n) b_n + \sum_{k=0}^{n-1} (a_{k+1} - a_k) b_k = a_{n+1} b_{n+1} - a_0 b_0 - a_{n+1} (b_{n+1} - b_n) - \sum_{k=0}^{n-1} a_{k+1} (b_{k+1} - b_k))$$

$$(a_{n+1} - a_n) b_n + a_n b_n - a_0 b_0 - \sum_{k=0}^{n-1} a_{k+1} (b_{k+1} - b_k) = a_{n+1} b_{n+1} - a_{n+1} b_{n+1} + a_{n+1} b_n - a_0 b_0$$

$$-a_0 b_0 = -a_0 b_0$$

$$1 = 1$$

Asymptotic Notation II

1. $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$f(n) \in \Theta(\sqrt{n}) \wedge g(n) \in O(n^2)$

$\Rightarrow g(f(n)) \in O(n)$

Assume $\exists c_1, c_2, n_0 \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1\sqrt{n} \leq f(n) \leq c_2\sqrt{n}$

and $\exists c_3, n_1 \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow g(n) \leq c_3 n^2$

Prove $\exists c_4, n_2 \in \mathbb{R}^{\geq 0}, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow g(f(n)) \leq c_4 n$

$n_2 = n_1 + \left(\frac{n_0}{c_1}\right)^2$

$c_4 = c_3 c_2^2$

$n \geq n_1 \Rightarrow g(n) \leq c_3 n^2$

$f(n) \geq n_1$ when $n \geq \left(\frac{n_0}{c_1}\right)^2$

$f(n) \geq n_1 \Rightarrow g(f(n)) \leq c_3 f(n)^2$

$\leq c_3 c_2^2 n$

$\leq c_4 n$

2. $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$f(n) \in \Theta(n^2) \wedge g(n) \in O(n^2)$

$\Rightarrow g(f(n)) \in O(n^4)$

Assume $\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 n^2 \leq f(n) \leq c_2 n^2$

and $\exists c_3, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow g(n) \leq c_3 n^2$

Prove $\exists c_4, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow g(f(n)) \leq c_4 n^4$

$n_2 = n_1 + \sqrt{\frac{n_0}{c_1}}$

$c_4 = c_3 c_2^2$

$n \geq n_1 \Rightarrow g(n) \leq c_3 n^2$

$f(n) \geq n_1$ when $n \geq \sqrt{\frac{n_0}{c_1}}$

$f(n) \geq n_1 \Rightarrow g(f(n)) \leq c_3 f(n)^2$

$\Rightarrow \leq c_3 (c_2 n^2)^2$

$\leq c_3 c_2^2 n^4$

3. $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$f(n) \in \Omega(n^2) \wedge g(n) \in O(\frac{1}{n})$

$\Rightarrow g(f(n)) \in O(\frac{1}{n^2})$

Assume $\exists c_1, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \geq c_1 n^2$

and $\exists c_2, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow g(n) \leq c_2 \frac{1}{n}$

Prove $\exists c_3, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow g(f(n)) \leq c_3 \frac{1}{n^2}$

$n_2 = n_1 + \sqrt{\frac{n_0}{c_1}}$

$c_4 = c_3 \frac{1}{c_1}$

$n \geq n_1 \Rightarrow g(n) \leq c_2 \frac{1}{n}$

$f(n) \geq n_1$ when $n \geq \sqrt{\frac{n_0}{c_1}}$

$f(n) \geq n_1 \Rightarrow g(f(n)) \leq c_2 \frac{1}{f(n)}$

$\leq c_3 \frac{1}{c_1 n^2}$ since $\frac{1}{f(n)} \leq \frac{1}{c_1 n^2}$

$= c_3 \frac{1}{c_1} \cdot \frac{1}{n^2}$

$= c_4 \frac{1}{n^2}$

4. $f : \mathbb{N} \to \mathbb{N}$, $g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$f(n) \in \Theta(n^{10}) \wedge g(n) \in O(\log n)$

$\Rightarrow g(f(n)) \in O(\log n)$

Assume $\exists c_1, c_2, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow c_1 n^{10} \leq f(n) \leq c_2 n^{10}$

and $\exists c_3, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow g(n) \leq c_3 \log n$

Prove $\exists c_4, n_2 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_2 \Rightarrow g(f(n)) \leq c_4 \log n$

$n_2 = n_1 + \sqrt[10]{\frac{n_0}{c_1}}$

$c_4 = 11 c_3$

$n \geq n_1 \Rightarrow g(n) \leq c_3 \log n$

$f(n) \geq n_1$ when $n \geq \sqrt[10]{\frac{n_0}{c_1}}$

$f(n) \geq n_1 \Rightarrow g(f(n)) \leq c_3 \log(f(n))$

$\leq c_3 \log(c_2 n^{10})$

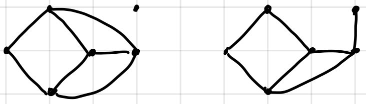
$\leq c_3 \log(n^{11})$

$\leq c_3 11 \log(n)$

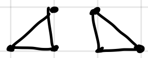
$\leq c_4 \log(n)$

TT4 Short Answer Q's

1. Give 2 graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ where $|V_1| = |V_2| = 6$ and $|E_1| = |E_2| = 7$
   G_1 is not connected / G_2 is connected.



2. $\exists$ some non-empty $G = (V, E)$, $\forall e \in E$ belongs to some cycle in G, and G contains at least two different cycles.
   Let G =



and every edge belongs to a cycle in G and G contains 2 cycles.

3. $n + \frac{1}{n} \in \Theta(n)$

$\exists c_1, c_2, n_0 \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow c_1 n \leq n + \frac{1}{n} \leq c_2 n$

$n \leq \frac{c}{2} n$ and $\frac{1}{n} \leq \frac{c}{2} n$

Let $c_2 = 2$ $\quad$ $1 \leq \frac{c}{2}$ $\quad$ $\sqrt{\frac{2}{c}} \leq n$

$n_0 = \sqrt{\frac{2}{c}}$ $\quad$ $2 \leq c$

then $c \geq 2$ $\quad$ $n \geq n_0$

$\frac{c}{2} \geq 1$ and $n \geq \sqrt{\frac{2}{c}}$

$n \frac{c}{2} \geq n$ $\quad$ $\frac{c}{2} n \geq \frac{1}{n}$

$n + \frac{1}{n} \leq \frac{c}{2} n + \frac{c}{2} n$

$\leq cn$

$\square$

4. $(12)_8 + (40)_{16}$ $\quad\quad$ $(888)_8 = (2)$

$(12)_8 + 001000000$

$(12)_8 + (100)_8$

$(112)_8$

Asymptotic Notation

$f_0 : \mathbb{N} \to \mathbb{R}^{\geq 0}$

for each $k \in \mathbb{N}$, $f_{k+1} : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$f_{k+1}(n) = (f_k(n))^{k+1}$ for all $n \in \mathbb{N}$

$f_0 \in \Omega(n)$

prove $\forall m \in \mathbb{N}$, $f_m \in \Omega(n^{m!})$

Let $m \in \mathbb{N}$

Base Case : $m = 0$

$f_0 \in \Omega(n^{0!}) = \Omega(n)$ true by assumption

Inductive Step $P(m) \Rightarrow P(m+1)$

Assume $f_m \in \Omega(n^{m!})$

Prove $f_{m+1} \in \Omega(n^{m+1!})$

$$f_{m+1}(n) \geq n^{m+1!}$$

$$(f_m(n))^{m+1} \geq n^{m+1!}$$

I.H. $(f_m(n))^{m+1} \geq (n^{m!})^{m+1} \geq n^{m+1!}$

Algorithm Analysis: Counting Iterations

1.

```
def woohoo(n: int) -> int:
    i = 0
    while i < n:    # L1
        for j in range(i)  # L2
            print(j)
        if i % 5 == 0:
            if i % 3 == 0:
                i = i + 1
            else:
                i = i + 2
        else:
            i = i + 4
```

a) Lower bound

Let $n \in \mathbb{N}$.

Loop 1 iterates # L2 + RT if

$i_k = i_{k-1} + 4$

$i_k = 4k \geq n$

$k \geq \lceil \frac{n}{4} \rceil$

b) an upper bound

Let $n \in \mathbb{N}$.

Loop 1 iterates # L2 + RT if

$i_k = i_{k-1} + 1$

$i_k = k$

$k \geq n$

c

Algorithm Analysis : WC / BC

1a. UBWC

Let $n \in \mathbb{N}$. Let (lst, t) be an arbitrary input family for big_short.

Loop 2 : SC : $s \geq t$ or $j \geq n$

$A: s_k = s_{k-1} + lst[j]$ $\qquad j_k = j_0 + k$

$= s_0 + lst[j] \cdot k$ $\qquad j_k = i + k$

$= lst[j] \cdot k$

SC: $s_k \geq t$ or $j_{k-1} \geq n$

$lst[j] \cdot k \geq t$ or $i + k \geq n$

$k \geq \left\lceil \frac{t}{lst[j]} \right\rceil$ or $k \geq n - i$

Loop 1: $k \geq n$

$$\sum_{i=0}^{n-1} n - i \quad \text{or} \quad \sum_{i=0}^{n-1} \left\lceil \frac{t}{lst[j]} \right\rceil$$

$$= n^2 - \frac{(n-1)n}{2} \quad \text{or} \quad = n \left\lceil \frac{t}{lst[j]} \right\rceil$$

Therefore $RT_{big\_short}(lst, t) \leq n^2 - \frac{(n-1)n}{2}$

$= \frac{2n^2 - n^2 - n}{2}$

$= \frac{n^2 - n}{2}$

$WC_{big\_short}(x) \in O(n^2)$

b. Let $n \in \mathbb{N}$. Let $lst = [1, 1, ..., 1]$

$t = n$

then while will run until

$s \geq n$ or $j \geq n$

the first stopping condition stops after n iterations.

the second stopping condition stops after $n - i$ iterations.

c.) $\left\lceil \frac{t}{lst[j]} \right\rceil = \log_2(n)$

$\left\lceil \frac{t}{lst[j]} \right\rceil = n$

$\frac{t}{lst[j]} \leq n$

$t \leq n \cdot lst[j]$

$\lfloor n \cdot lst[j] \rfloor$

Algorithm Analysis : AC

1. def max_alpha(s: str) -> int :

```
    i = 1
    while i < len(s) and s[i-1] <= s[i] :
        i = i + 1
    return i
```

Loop 1 :

SC: $i_k \geq n$ or $s[i_k - 1] > s[i_k]$

$i_k = i_{k-1} + 1$ $\quad$ $s[i_k - 1] > s[i_k]$

$= i_0 + k$ $\quad$ Find out on saturday

$= 1 + k$ $\quad$ or

$1 + k \geq n$

$k \geq n - 1$

Thus the exact steps = $n - k$

b. $|\{s \in \mathcal{I}_n \mid \text{max\_alpha}(s) \text{ returns } k\}| = |S_{n,k}|$

$|S_{n,k}| =$ 0 0 1

1 1 1

c. $\text{Avg}_{\text{max\_alpha}}(4) = \frac{1}{\binom{4}{2}} \sum_{s \in \mathcal{I}_4} \text{RT of max\_alpha}(s)$

$= \frac{1}{\binom{4}{2}} \sum_{k=0}^{n} \sum_{s \in \mathcal{I}_{4,k}} \text{RT of max\_alpha}(s)$

$= \frac{1}{\binom{4}{2}} \sum_{k=0}^{n} \sum_{s \in \mathcal{I}_{4,k}} (n-k)$

$= \frac{1}{\binom{4}{2}} \sum_{k=0}^{n} |S_{4,k}|(n-k) = \frac{1}{\binom{4}{2}} (4 + 2 + 3 + 1 + 1 + 2)$

$= \frac{13}{6}$

d) $\text{Avg}_{\text{max\_alpha}}(n) = \frac{1}{\binom{n}{2}} \sum_{k=0}^{n} |S_{n,k}|(n-k)$

$=$

Exercise Break 4.1

Why did we need the restriction that $n \geq 2$?

proof fails for $n=0$ or $n=1$. → binary rep for n is rational / doesn't matter

Dividing by 2 lemma

Let $n \in \mathbb{N}$, assume $n \geq 2$.

Let binary rep of n be $b_p b_{p-1} \dots b_0$ $\qquad 7 = (1 \cdot 2 + 1)2 + 1$

where $b_p = 1$

BR of $\lfloor \frac{n}{2} \rfloor$ is $b_p b_{p-1} \dots b_1$ $\qquad 7 = q3 + r$

Pf: Let $n \in \mathbb{N}$, assume $n \geq 2$ $\qquad 2 \cdot 3 + 1$

Let $p \in \mathbb{N}$ and $b_0, b_1, \dots, b_p \in \{0,1\}$

be s.t. $n = \sum_{i=0}^{p} b_i 2^i$ and $b_p = 1$

Case 1: Assume n is even $b_0 = 0$

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$$

$$= \frac{\sum_{i=0}^{p} b_i 2^i}{2}$$

$$= \frac{\sum_{i=1}^{p} b_i 2^i}{2} \quad \text{since } b_0 = 0$$

$$= \sum_{i=1}^{p} b_i 2^{i-1}$$

$$= \sum_{i=0}^{p-1} b_{i+1} 2^i$$

Exercise break 4.2

$\forall n \in \mathbb{N}$, the binary rep of n w/ exactly 1 leading zero can be turned into a br of $n+1$ by flipping exactly 1 bit from 0 to 1, and some number of bits from 1 to 0.

Pf: Let $n \in \mathbb{N}$, let a br of $n = (0b_pb_{p-1}\dots b_0)$ where $b_0, \dots, b_p \in \{0,1\}$

n is even: $n = (0b_pb_{p-1}\dots b_1 0)$

Exercise Book 5.2

5.5 Let $f(n) = n^2$ and $g(n) = n + 10$

$g \in O(f)$, but $g \notin \Theta(f)$.

$\neg(g \in O(f)$ and $g \in \Omega(f))$

$(g \notin O(f)$ or $g \notin \Omega(f))$ and $g \in O(f)$

$g \in O(f)$:

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n + 10 \leq cn^2$

$n_0 = 1$ Always true

$c = 22$ $2 \leq 22$ $20 \leq 22$

$2n \leq 22n$ $n \geq 1$ $10 \leq 11n^2$ $n \geq 1$

$n \leq 11n^2$ $n \leq n^2$

$n + 10 \leq 11n^2 + 11n^2$

$n + 10 \leq 22n^2$

$\square$

$g \notin \Omega(f)$:

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n + 10 > cn^2$

$n + 10 >$

Exercise Break 5.3

Claim: $\forall x$, $x > 2 \Rightarrow$ after 2 iterations of the loop, the value of x decreases by at least 1. $x_2 \leq x_0 - 1$

b. WS15

Case 1: Assume $4 \mid x_0$, ie., $\exists k \in \mathbb{Z}$, $x_0 = 4k$

x_0 is even, if executes in first loop iter.

$x_1 = \frac{x_0}{2} = 2k$.

x_1 is also even so if executes again.

$x_2 = \frac{x_1}{2} = k$

$x_2 = \frac{1}{4} x_0 \leq x_0 - 1$ (since $x_0 \geq 4$)

Case 2: Assume $4 \mid x_0 - 1$, ie., $\exists k \in \mathbb{Z}$, $x_0 = 4k + 1$

x_0 is odd, else executes

$x_1 = 2x_0 - 2 = 8k$

x_1 is even so

$x_2 = \frac{x_1}{2} = 4k$

$x_2 = 4k = x_0 - 1$

Case 3: Assume $4 \mid x_0 - 2$, $\exists k \in \mathbb{Z}$, $x_0 = 4k + 2$

x_0 is even, if executes

$x_1 = \frac{x_0}{2} = 2k + 1$

x_1 is odd, else executes

$x_2 = 2 \cdot (2k+1) - 2 = 4k$

$= x_0 - 2 \leq x_0 - 1$

Case 4: Assume $4 \mid x_0 - 3$, $\exists k \in \mathbb{Z}$, $x_0 = 4k + 3$

x_0 is odd, else executes

$x_1 = 2(4k+3) - 2 = 8k + 4$

x_1 is even

$x_2 = \frac{8k+4}{2} = 4k + 2$

$x_2 = x_0 - 3 + 2 = x_0 - 1$

d.

Exercise Beah 5.4

$\mathcal{J}_n$: (lst, x) has length n

x and elements of lst are between 1 and 10

$|\mathcal{J}_n| = 10^n$ PwIR

$|S_{n,i}|$ is the partition where x appears at index i.

$= 10^{n-1-i}$

$$\mathrm{Avg}_{search}(n) = \frac{1}{|\mathcal{J}_n|} \sum_{(lst,x) \in \mathcal{J}_n} RT_{search}(lst, x)$$

$$= \frac{1}{10^n} \sum_{x=1}^{10} \sum_{lst \in S_n} RT_{search}(lst, x)$$

$$= \frac{1}{10^n} \sum_{x=1}^{10} \sum_{i=0}^{n} \sum_{lst \in S_{n,i}} (1+i)$$

x is not in lst

$$= \frac{1}{10^n} \sum_{x=1}^{10} \left(\sum_{i=0}^{n-1} |S_{n,i}|(1+i) + |S_{n,n}|(1+n) \right)$$

$$= \frac{1}{10^n} \sum_{x=1}^{10} \left(\sum_{i=0}^{n} 10^{n-1-i}(i+1) + \frac{1+n}{10^n} \right.$$

$$= \frac{1}{10^n} \sum_{x=1}^{10} \left(\sum_{i'=1}^{n} 10^{n-i'} i' \right) + \frac{\sum_{x=1}^{10} (1+n)}{10^n}$$

$$= \frac{1}{10^{1-n}} \frac{n\left(\frac{1}{10^n}\right)^n}{\left(\frac{1}{10^n}\right) - 1} + \frac{10+10n}{10^n}$$

Exercise Break 6.1

$|E| = O|V|$

6.2 Let $n \in \mathbb{Z}^+$.

$$|E| \leq \frac{|V|(|V|-1)}{2}$$

Thus, $|E| - 1 \leq \binom{|V|-1}{2}$

TTI v1

a) False

b) True

b. $A = \{\phi, \{x\}\}$

$B = \{S \mid S \subseteq A \land |S| < 2\}$

$= \{\phi, \{\phi\}, \{\{x\}\}\}$

c.) $\forall x \in D, P(x) \land Q(x)$

$\forall x \in D, \neg P(x) \lor Q(x)$

$D = \{0, 1\}$

$P(x)$: x is 2

$Q(x)$: x is 0 or 1

dn 2, if $P(x)$ is false then the entire statement is vacuously true

dn 1 $P(x)$ is always false thus the entire statement is False.

$p \iff q \land r$

$p \iff (q \land r)$ the same!

d.) $\neg(p \iff q \land \neg r)$

$\neg(p \Rightarrow q \land \neg r) \lor \neg(q \land \neg r \Rightarrow p)$

$(p \land \neg(q \land \neg r)) \lor ((q \land \neg r) \land \neg p)$

$(p \land (\neg q \lor r)) \lor (q \land \neg r \land \neg p)$

2. $\forall x \in R, Tasty(x) \Rightarrow (\exists y_1, y_2 \in I, Contains(x, y_1) \land Contains(x, y_2))$

$\forall x \in R, \exists y \in I, (\forall y_0 \in I, Expensive(y_0) \land Contains(x, y_0) \Rightarrow y_0 = y)$

$\exists x \in I, \forall y \in R, Contains(y, x)$

$\forall y_1, y_2 \in I, Contains(chocolate\_cake, y_1) \land Contains(spaghetti\_sauce, y_2) \Rightarrow y_1 \neq y_2$

$(\forall x \in R, Contains(x, salt)) \Rightarrow (\exists x_0 \in R, Contains(x_0, sugar))$

3a. $\forall a, b, n \in \mathbb{Z}, a \neq 0, b \neq 0, n \neq 0 \Rightarrow a \mid b \iff na \mid nb$

Let $a, b, n \in \mathbb{Z}$, Assume $a, b, n \neq 0$

Case 1: '$\Rightarrow$' Assume $a \mid b$

$b = ak$ for some $k \in \mathbb{Z}$

$nb = nak$ since $n \neq 0$

$na \mid nb$

Case 2: '$\Leftarrow$' Assume $na \mid nb$

$\cancel{n}b = \cancel{n}ak_1$ for some $k_1 \in \mathbb{Z}$

$b = ak_1$ since $n \neq 0$

$a \mid b$

4. $\neg(\forall x \in \mathbb{R}, \lfloor x^2 \rfloor = \lceil x \rceil^2)$

$\exists x \in \mathbb{R}, \lfloor x^2 \rfloor \neq \lceil x \rceil^2$

6. Let $x = \frac{1}{2}$

$$\begin{aligned} \lfloor \tfrac{1}{2}^2 \rfloor &= \lfloor \tfrac{1}{4} \rfloor \\ &= 0 \\ &\neq 1 \\ &= 1^2 \\ &= \lceil \tfrac{1}{2} \rceil^2 \end{aligned}$$

5. $\forall a,b,c \in \mathbb{Z}, \gcd(a,b) = 1 \wedge a \mid bc \Rightarrow a \mid c$

Let $a,b,c \in \mathbb{Z}$, Assume $\gcd(a,b) = 1$

and $bc = ak$ for some k.

prove $c = ak_1$ for some k_1

Let $k_1 = cp + kq$

$ap + bq = \gcd(a,b) = 1$

$1 \mid a$ and $1 \mid b$

$$\begin{aligned} ap + bq &= 1 \\ cap + cbq &= c \\ a(cp + bq) &= c \\ ak_1 &= c \end{aligned}$$

TT1 v2

1a. True

False

b. $|A \setminus B| = x$, $|A \cap B| = y$, and $|A \cup B| = z$ $\quad |(B \setminus A) \times A| =$

$$A \cup B - A \cap B = A^c \cap B^c$$
$$z - y = q + x$$
$$z - y - x = q$$

$$A = A \setminus B + A \cap B$$
$$|A| = |A \setminus B| + |A \cap B|$$
$$= (x + y)(z - (y + x))$$

c. $\forall x \in D, P(x) \vee Q(x)$

$(\forall x \in D, P(x)) \vee (\forall y \in D, Q(y))$

$D = \{0, 1\}$

$P(x)$: x is 1

$Q(x)$: x is 0

In 1, both are true

In 2, both are false.

d. $\neg(p \wedge (\neg p \vee \neg q \vee r))$

$\neg p \vee \neg(\neg p \vee \neg q \vee r)$ negation

$\neg p \vee (p \wedge q \wedge \neg r)$ negation / double

$(\neg p \vee p) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg r)$ distributivity

$\neg p \vee q \wedge \neg p \vee \neg r$ identity

$\neg p \vee (q \wedge \neg r)$ distributivity

$p \Rightarrow q \wedge \neg r$ implication

2. $\neg(\exists x \in C, \forall y \in P, \text{Quiet}(y) \Rightarrow \text{Supports}(y, x))$ $\neg(\exists x \in C, \forall y \in P, \text{Supports}(y, x) \Rightarrow \text{Quiet}(y))$

$\forall x \in C, \exists y \in P, \text{Quiet}(y) \wedge \neg \text{Support}(y, x)$

b. $\exists x \in C, \neg(\exists y \in P, \text{Quiet}(y) \wedge \text{Supports}(y, x))$

$\exists x \in C, \forall y \in P, \neg \text{Quiet}(y) \vee \neg \text{Supports}(y, x)$

$\exists x \in C, \forall y \in P, \text{Supports}(y, x) \Rightarrow \neg \text{Quiet}(y)$

c. $\forall x \in P, \exists y \in C, \text{Support}(x, y)$

d. $(\exists x \in C, \forall y \in P, \text{Support}(y, x)) \Rightarrow (\forall y \in C \ \neg \text{Supports}(\text{greta}, y))$

e. $\exists x, y \in P, \text{Supports}(x, \text{Climate\_change}) \wedge \text{Supports}(y, \text{Climate\_Change}) \wedge x \neq y$

3. $\forall x, y \in \mathbb{Z}^+, x | y \wedge y | x \Rightarrow x = y$

$\forall x, y \in \mathbb{Z}^+, (\exists k_1 \in \mathbb{Z}, y = x k_1) \wedge (\exists k_2 \in \mathbb{Z}, x = y k_2) \Rightarrow x = y$

Let $x, y \in \mathbb{Z}^+$. Assume $\exists k_1 \in \mathbb{Z}, y = x k_1 \wedge \exists k_2 \in \mathbb{Z}, x = y k_2$

$x = x k_1 k_2$

$1 = k_1 k_2 \Rightarrow k_1 = 1$ and $k_2 = 1 \Rightarrow y = x$ and $x = y$ $\square$

4. $\exists x, y \in \mathbb{R}.\ \lceil xy \rceil \neq \lfloor x \rfloor \lfloor y \rfloor$

Let $x = \frac{1}{2}$

$y = 1$ $\quad \lceil \frac{1}{2} \rceil = 1$

$\neq 0$

$= \lfloor \frac{1}{2} \rfloor \lfloor 1 \rfloor$

$= \lfloor x \rfloor \lfloor y \rfloor$

5. $\forall a, b, d \in \mathbb{Z},\ a \neq 0 \land b \neq 0 \land d \mid a \land d \mid b \Rightarrow d \mid \gcd(a,b)$

Let $a, b, d \in \mathbb{Z}$. Assume $a, b \neq 0$

$d \mid a \land d \mid b$

$k_3 = pk_1 + qk_2$

$pa + qb = \gcd(a,b)$

$pa + qb = \gcd(a,b)$

$pdk_1 + qdk_2 = \gcd(a,b)$

$d(pk_1 + qk_2) = \gcd(a,b)$

$dk_3 = \gcd(a,b)$

TT1 v3

a. False

True   010 →

b. $A = \{1\}$

$B = \{\{2, S\} \mid S \subseteq A\}$

$= \{\{2,1\}, \{2, \emptyset\}\}$ → dont forget $\emptyset$!

c. $\exists x \in D,\ P(x) \land Q(x)$

$\exists x \in D,\ P(x) \lor Q(x)$

$D = \{0,1\}$

$P(x): x = 0$

$Q(x): x = 1$

In 1, 1 is true, 1 is false so the statement is false

In 2. the statement is true.

d. $\neg p \iff (\neg q \land \neg r)$

$\neg p \Rightarrow (\neg q \land \neg r) \land (\neg q \land \neg r) \Rightarrow \neg p$

$(p \lor (\neg q \land \neg r)) \land ((q \lor r) \lor \neg p)$

$(q \lor r \Rightarrow p) \land (p \Rightarrow q \lor r)$

$(q \lor r) \iff p$

2. $\neg(\exists x \in S, \forall y \in A,\ Good(y) \Rightarrow Practices(y, s))$

$\forall x \in S, \exists y \in A,\ Good(y) \land \neg Practices(y, s)$

b. $\exists x \in S,\ Easy(x) \land (\forall y \in A,\ Practices(y, x))$

c. $\forall x \in A, \exists y_1, y_2 \in S,\ Practices(x, y_1) \land Practices(x, y_2) \land y_1 \neq y_2$

d. $\neg(\exists x, y \in A$

$\exists x \in A, \forall y \in A, (Practices(y, Bobsleigh) \Rightarrow x = y)$

e) $(\exists x \in A,\ Practices(x, Bobsleigh)) \Rightarrow (\forall y \in A, \neg Practices(y, slalom))$

3. $\forall a, b, c \in \mathbb{Z},\ (a \mid b \land a \mid c) \Rightarrow a \mid bc$

$\forall a, b, c \in \mathbb{Z}, ((\exists k_1 \in \mathbb{Z}, b = ak_1) \land (\exists k_2 \in \mathbb{Z}, c = ak_2)) \Rightarrow (\exists k_3 \in \mathbb{Z},\ bc = ak_3)$

Let $a, b, c \in \mathbb{Z}$. Assume $k_3 = k_1 k_2$

$bc = ak_1 ak_2$

$= ak_1k_2$

$= ak_3$

4. $\exists x, y \in \mathbb{R},\ \lfloor x + y \rfloor \neq \lceil x \rceil + \lceil y \rceil$

Let $x = \frac{1}{2}$ $\lfloor \frac{1}{2} + \frac{1}{2} \rfloor = \lfloor 1 \rfloor$

$y = \frac{1}{2}$ $= 1$

$\neq 2$

$= \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil$

$= \lceil x \rceil + \lceil y \rceil$

5. $\forall a, b \in \mathbb{Z},\ a \neq 0 \land b \neq 0 \Rightarrow gcd(a+b, b) \geq gcd(a, b)$

Let $a, b \in \mathbb{Z}$. Assume $a \neq 0$, $b \neq 0$

$ap + bq = gcd(a, b)$

$(a+b)p + bq = gcd(a+b, b)$

$ap + bp + bq = gc$

TT2 v1

i. $\forall k \in \mathbb{N}, P(2^k)$
ii $\forall k \in \mathbb{N}, P(k+1)$

b) $\forall n \in \mathbb{N}, n \geq 1 \Rightarrow 3 \mid n^3 - 7n$

Base Case $P(1)$ $k = -2$

$3 \mid n^3 - 7n$: $1^3 - 7 = 3k$

$-6 = 3k$

Inductive Step $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

Let $n \in \mathbb{N}$, Assume $3 \mid n^3 - 7n$

Prove $3 \mid (n+1)^3 - 7n$: $(n+1)^3 - 7n - 7 = 3k_0$ for some k_0

Let $k_0 = k + n^2 + n - 2$

$(n+1)(n+1)(n+1) - 7n = (n^2 + 2n + 1)(n+1) - 7n + 7$

$= n^3 + 2n^2 + n + n^2 + 2n + 1 - 7n - 7$

$= n^3 + 3n^2 + 3n - 7n - 6$

$= 3k + 3n^2 + 3n - 6$ IH

$= 3(k + n^2 + n - 2)$

$= 3k_0$

2. T F F F

b. $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f \in O(n^2) \Rightarrow f \circ g \in O(g \circ g)$

Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

Assume $f \in O(n^2)$, that is

$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f(n) \leq cn^2$

Prove $f \circ g \in O(g \circ g)$:

$\exists c_1, n_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow f(g(n)) \leq g(n)^2 c_1$

Let $n_1 = n_0$ then $f(n) \leq cn^2$

$c_1 = c$ $f(g(n)) \leq c\, g(n)^2$ when

since $g(n) \in \mathbb{N}$ $= g \cdot g(n)\ c_1$

c. $\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow n^3 - 3n + 1 \leq cn^3$

Let $c = 3$ $n^3 - 3n + 1 \leq c_1 n^3 + c_2 n^3 + c_3 n^3$

$n_0 = 1$

$n^3 \leq n^3$

$-3n \leq n^3$ then $n^3 - 3n + 1 \leq n^3 + n^3 + n^3$

$1 \leq n^3$ $= 3n^3$

$= cn^3$

d. $\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \land n^3 - 3n + 1 > cn^2$

Let $c, n_0 \in \mathbb{R}^+$

$n = \lceil \min($ $\frac{n^3}{2} - 3n + 1 > \frac{c}{2}n^2 + \frac{c}{2}n^2 - \frac{n^3}{2}$

3. def blah(n: int) -> None:

```
i = 1
while i < n + 3:
    j = 1
    while j > i // 2
```

Proving a logical equivalence

$(p \land \neg q) \lor (q \land \neg r) \equiv (p \lor q) \land \neg(q \land r)$

$(p \lor q) \land (p \lor \neg r) \land (\neg q \lor q) \land (\neg q \lor \neg r)$ dist...

$(p \lor (q \land \neg r)) \land (\neg q \lor \neg r)$ dist...

$(p \land (\neg q \lor \neg r)) \lor ((q \land \neg r) \land (\neg q \lor \neg r))$ dist...

$(p \land (\neg q \lor \neg r)) \lor ((q \land \neg q) \lor (q \land \neg r)) \land ((\neg r \land \neg q) \lor (\neg r \lor \neg r))$ dist...

" $\lor$ $(q \land \neg r) \land \neg r$ absorption

$(p \land (\neg q \lor \neg r)) \lor (q \land \neg r)$ "

$(p \land (\neg q \lor \neg r)) \lor ((q \land \neg q) \lor (q \land \neg r))$ iden...

$(p \land (\neg q \lor \neg r)) \lor (q \land (\neg q \lor \neg r))$ dist...

$(p \lor q) \land (\neg q \lor \neg r)$ dist...

$(p \lor q) \land \neg(q \land r)$ neg...

$f(n^4) \notin \Omega(2^n)$

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N}, n \geq n_0 \wedge n^4 < c2^n$

Let $c, n_0 \in \mathbb{R}^+$

$n = \left\lceil \max\left(20, n_0, \frac{1}{2^{10}c}\right) \right\rceil$ Using the hint:

by def^n, $n \geq n_0$. $\quad n \geq 5 \Rightarrow 2^n > 8n$

but, $n \geq 20 \Rightarrow 2^{\frac{n}{5}} > 8\frac{n}{5}$

Q5 : Avg Case (hunting primes)

$RT_{has\_prime}(x) = i_{k-1} + 1$

$|\mathcal{I}_3| = 10^3$

$S_{3,i}$: lst[i] in {2, 3, 5, 7}

$|S_{3,0}| = 6^0 \cdot 4 \cdot 10^2$

$|S_{3,1}| = 6^1 \cdot 4 \cdot 10^1$

$|S_{3,2}| = 6^2 \cdot 4 \cdot 10^0$

$6^i \cdot 10^{n-i-1} \cdot 4 = |S_{n,i}|$

$|S_{3,3}| = 6^3$

$$= \frac{1}{10^3} \sum_{lst \in \mathcal{I}_3} RT_{has\_prime}(lst) \text{ ; lst[i] is Prime}$$

$$= \frac{1}{10^3} \sum_{i=0}^{n-1} \sum_{\substack{lst \in \mathcal{I}_3 \\ lst[i] \text{ is prime}}} RT_{has\_prime}(lst)$$

$$= \frac{1}{10^3} \sum_{i=0}^{n-1} |S_{n,i}| (i+1)$$

$$= \frac{1}{10^3} \sum_{i=0}^{2} 6^i \cdot 10^{3-i-1} \cdot 4 (i+1) + |S_{3,3}| (3+1)$$

$= 2.176$

Q6 : sets

Base case omitted

Inductive step. Let $k \in \mathbb{N}$. Assume $P(k)$

$P(k)$: $\forall$ sets S, $|S| = k \Rightarrow S$ has $\frac{k(k-1)(k-2)(k-3)(k-4)}{120}$ subsets of size 5.

prove $P(k+1)$: $|S| = k+1$ " $\frac{(k+1)(k)(k-1)(k-2)(k-3)}{120}$ "

Let S be an arbitrary set of size $k+1$

$S = S' \cup \{s_{k+1}\}$ where S' does not contain s_{k+1}

Subsets of size 5 w/ s_{k+1}

$\frac{k(k-1)(k-2)(k-3)}{24}$ subsets $\binom{k}{4}$

Subsets of size 5 w/o s_{k+1}

I.H. : $\frac{k(k-1)(k-2)(k-3)(k-4)}{120}$ subsets $\binom{k}{5}$

$$\frac{k(k-1)(k-2)(k-3)}{24} + \frac{k(k-1)(k-2)(k-3)(k-4)}{120}$$

$$= \frac{k(k-1)(k-2)(k-3)(5+k-4)}{120}$$

$$= \frac{(k+1)(k)(k-1)(k-2)(k-3)}{120} \qquad \binom{k+1}{5}$$

$\square$

Q7: graphs and connectedness

Assume $G=(V,E)$, $\forall v \in V$

$d(v) \geq |V|-5$ and $|V| \geq 9$

$\forall u,v \in V$, G contains a path b/t u and v.

Let $u,v \in V$

$\exists$ path b/t u and v

Assume G is not connected.

$\exists u,v \in V$, G does not contain a path between u and v.

Loop 2 : $k \in \mathbb{N}$

$$j_k = j_{k-1} + 1$$
$$= j_0 + k$$
$$= k \text{ iterations}$$

stopping : $2^{j_k} > n$

$$j_k > \log_2 n$$
$$k > \log_2 n$$

$$k = \lfloor \log_2 n \rfloor + 1 > \log_2 n$$

Loop 1 : $l \in \mathbb{N}$

$$n_l = n_{l-1} - 2^{(j_k - 1)}$$ for l iterations

$$= n_{l-1} - 2^{(\lfloor \log_2 n_{l-1} \rfloor + 1 - 1)}$$

$$n_l = n_{l-1} - 2^{\lfloor \log_2 n_{l-1} \rfloor}$$
$$\leq n_{l-1} - 2^{\log_2 n_{l-1}}$$
$$= n_{l-1} - n_{l-1} = 0$$

$$i_k = i_{k-1} + j_{k-1}$$

$$= i_0 + \sum_{i=0}^{k-1} j_i$$

$$i_0 + \sum_{i'=1}^{k} i'$$

$$i_k = \frac{k(k+1)}{2}$$

$$i_k \geq n$$

$$\frac{k(k+1)}{2} \geq n$$

$$k^2 + k - 2n \geq 0$$

$$k - \frac{-1 - \sqrt{8n-1}}{}$$

If a graph G is disconnected, then G^c is connected.

Let G be a disconnected graph

$$V(G) = V(\overline{G})$$

Take distinct vertices $u, v \in V(G)$

Suppose

Dec 2019

F r e d e r i c k
M e n e s e s
m e n e s e 2 3

1) IsSquareNumber(n): $\forall n \in \mathbb{N}, \exists a \in \mathbb{N}, a^2 = n$

2) $\forall m, n \in \mathbb{N}$, IsSquareNumber(n) $\wedge$ IsSquareNumber(m) $\Rightarrow$ IsSquareNumber(mn)

3) $\forall n \in \mathbb{N}$, (Odd(n) $\Rightarrow$ ($\exists m_1, m_2 \in \mathbb{N}$, IsSquareNumber($m_1$) $\wedge$ IsSquareNumber(m_2) $\wedge$ $n = m_2 - m_1$))

4) $\exists m \in \mathbb{N}$, ($\forall n \in \mathbb{N}$, $n \le m$) $\wedge \neg$ IsSquareNumber(m)

2.) Bounded(f) : $\forall f: \mathbb{N} \to \mathbb{R}^{\ge 0}, \exists y \in \mathbb{R}^{\ge 0}, \forall x \in \mathbb{N}, f(x) \le y$

3.) Let $f_1, f_2 \in F$
Assume Bounded(f_1) $\wedge$ Bounded(f_2)
that is $\exists y_1 \in \mathbb{R}^{\ge 0}, \forall x \in \mathbb{N}, f_1(x) \le y_1$
$\exists y_2 \in \mathbb{R}^{\ge 0}, \forall x \in \mathbb{N}, f_2(x) \le y_2$
show $\exists y_3 \in \mathbb{R}^{\ge 0}, \forall x \in \mathbb{N}, (f_1 f_2)(x) \le y_3$
Let $y_3 = y_2 y_1$, Let $x \in \mathbb{N}$
By assumption, $f_1(x) \le y_1$ and $f_2(x) \le y_2$
$f_1(x) \cdot f_2(x) \le y_2 y_1$
$(f_1 f_2)(x) \le y_3$ D

c) $\exists f_1, f_2 \in F$, Bounded($f_1 f_2$) $\wedge$ ($\neg$ Bounded(f_1) $\vee \neg$ Bounded(f_2))
Let $f_1 : \forall x \in \mathbb{N}, f_1(x) = \frac{1}{x}$
$f_2 : \forall x \in \mathbb{N}, f_2(x) = x$

then let $y_1 = 1$, let $x \in \mathbb{N}$ $\to$ $x \frac{1}{x} \le 1$
$1 \le 1$

and $\forall y_2 \in \mathbb{R}^{\ge 0}, \exists x \in \mathbb{N}$ $\to$ $x \le y_2$
$x = y_2 - 1$
$\forall y_3 \in \mathbb{R}^{\ge 0}, \exists x \in \mathbb{N}$ $\to$ $\frac{1}{y_3} \le y_3$
$x = y_3 - 1$

3) $2^2, 2^3, 2^5, 2^7, 2^{11}$
$\exists x, y \in S$, $x \cdot y \in S$
Let $x = 4$
$y = 8$ $\Rightarrow$ $4 \cdot 8 = 32 \in S$.
Let $x \in S$
$2^{2^x} \to$ Prime(2^x) is False
Contradiction

c) $\forall n \in \mathbb{N}$, $n \ne 1 \Rightarrow \exists x \in S, \exists m \in \mathbb{N}, 2^n = x^m$

Hint: $\forall l \in \mathbb{N}$, $l > 1 \to \exists p \in \mathbb{N}$, Prime(p) $\wedge$ $p \mid l$

Let $n \in \mathbb{N}$. Assume $n \ne 1$
from hint we know $\exists p \in \mathbb{N}$, Prime(p) $\wedge$ $p \mid n$
let $x = 2^p$
$m = \frac{n}{p}$
then $x^m = (2^p)^{n/p} = 2^n$

4. $a_n = \begin{cases} 1 & n=0 \\ \left(\frac{1}{2}\right)^{a_{n-1}}, & n \geq 1 \end{cases}$

$a_0 = \frac{1}{2^0}$ $a_1 = \frac{1}{2^{1/1}}$ $a_2 = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$a_2 = \frac{1}{2^{1/2}}$

$P(n): \frac{1}{2} < a_n \leq \frac{1}{\sqrt{2}}$

$\forall n \in \mathbb{N}, P(n)$

d) Base Case: $n = 2$

$\frac{1}{2} < a_2 = \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{2}}$

Induction: $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$

Let $n \in \mathbb{N}$. Assume $P(n): \frac{1}{2} < a_n \leq \frac{1}{\sqrt{2}}$

Prove $P(n+1): \frac{1}{2} < a_{n+1} \leq \frac{1}{\sqrt{2}}$

$\frac{1}{2^{1/1}} < \frac{1}{2^{1/n}} \leq \frac{1}{2^{1/n+1}} < \frac{1}{2^{1/2}}$

5. $\exists f: \mathbb{N} \to \mathbb{R}^{\geq 0}, \forall k \in \mathbb{N}, f(n) \notin O(n^k) \wedge f(n) \notin \Omega(n^k)$

$f(n): 165$

$\forall c_1, n_0 \in \mathbb{R}^{\geq 0}, \exists n \in \mathbb{N}, n \geq n_0 \wedge f(n) > c_1 n^k$

$n = \lfloor \min(n_0, \sqrt[k+1]{\frac{165}{c_1}}) \rfloor$

$\left(165\left(\frac{1}{c_1}\right)\right)^{1/k} > n$

$\left(\frac{1}{c_1}\right)^{1/k} > \frac{n}{f(n)^{1/k}}$

$f(n) > c_1 n^k$

$\forall c_2, n_1 \in \mathbb{R}, \exists n \in \mathbb{N}, n \geq n_1 \wedge f(n) < c_2 n^k$

$n = \lceil \max(n_1, \sqrt[k]{\frac{165}{c_2}}) \rceil + 1$

$\left(165\left(\frac{1}{c_2}\right)\right)^{1/k} < n$

$\left(\frac{1}{c_2}\right)^{1/k} < \frac{n}{f(n)^{1/k}}$

$f(n) < c_2 n^k$

6a) UBBC $\forall n \in \mathbb{N}, \exists x \in \mathcal{I}_{func, n}, RT_{func}(x) \leq f(n)$

Let $n \in \mathbb{N}$

$lst = [1, \ldots, 1]$

$j_k = j_{k-1} \cdot 2$
$= j_0 \cdot 2^k$
$= 2^k$

$i_k = k$

Loop 1 runs until $i_k \geq n$ k iterations

Loop 2 runs until $j_k \geq 1$ $\log_2 1$ iterations

$2^k \geq 1$

$k \geq \lceil \log_2 1 \rceil$

LBBL

Let $n \in \mathbb{N}$

Let lst be an arbitrary list of size n.

if executes

Loop always iterates n times

UBWC

Let $n \in \mathbb{N}$

$lst \in \mathcal{I}_{my\text{-}algo,\, n}$

else executes always

Loop2 runs $\lceil \log_2 n \rceil$ iterations

Loop1 runs n iterations

$$\sum_{i=0}^{n-1} \lceil \log_2 n \rceil = n \lceil \log_2 n \rceil$$

LBWC

$lst = [0, \ldots, 0]$

then 'else' always executes

for a total of $n \lceil \log_2 n \rceil$ iterations

$$RT_{my\,alg}(lst) = n \lceil \log_2 n \rceil$$

$$\rightarrow \Omega(n \log n)$$

7. $O(n)$

5, 6, 11 are basic $\Rightarrow +3$

if never executes

$i_k = i_{k-1} + 1 = k$

n iterations

$n + 3 \Rightarrow O(n)$

: $\Omega(\sqrt{n})$

5, 6, 11 are basic $\rightarrow +3$

if always executes $\qquad j_k = j_{k-1} + 1 = k$

$i_k = i_{k-1} + j_k$

$i_k = i_{k-1} + k$

$$i_k = i_0 + \sum_{m=1}^{k} m$$

$$= \sum_{m=1}^{k} m = \frac{k(k+1)}{2}$$

stops when $\dfrac{k(k+1)}{2} \geq n$

$$k^2 + k - 2n \geq 0$$

$$-\frac{1}{2} \pm \sqrt{\left(-\frac{1}{2}\right)^2 + 2n}$$

$$k = \left\lceil \frac{-1 + \sqrt{8n+1}}{2} \right\rceil \text{ iterations}$$

$$\Rightarrow \Omega(\sqrt{n})$$

8.) $O(n \log n)$ $\Omega(n)$ $\Big\}$ $\Theta(\log(n^n))$ $\Omega(n \log n)$

9. $|\mathcal{I}_n| = 2^n$

$S_{n,i} := \{ lst \in \mathcal{I}_n \mid lst[i] \text{ is } 0 \}$

$|S_{n,0}| = 2^{n-1}$ $|S_{n,1}| = 2^{n-1}$

$$Avg_{my\text{-}alg}(n) = \frac{1}{2^n} \sum_{lst \in \mathcal{I}_n} RT_{my\text{-}alg}(lst)$$

$$= \frac{1}{2^n} \Big(\sum_{\substack{lst \in \mathcal{I}_n \\ lst[i] \text{ is } 0}} n + \sum_{\substack{lst \in \mathcal{I}_n \\ lst[i] \text{ is } 1}} n \log n$$

$$= \frac{1}{2^n} \Big(n(2^{n-1} \; n) + (2^{n-1} \; n \log n) n \Big)$$

$$= \frac{2^{n-1}}{2^n} (n^2 + n^2 \log n)$$

$$= \frac{n + n \log n}{2} \implies$$

9) $|V| = 5$ $|E| = 6$ $deg(v_2) = 2$

b) $V_n = \{v_1, \dots v_n\}$

$E_n = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)\}$

c) $V_n = \{v, \dots, v_n\}$

$E_n = \{ (v_1, v_2), (v_1, v_3), \dots, (v_1, v_n), (v_2, v_3), \dots, (v_2, v_n), \dots, (v_{n-1}, v_n) \}$

TT2 v3

i. $P(2) \wedge P(3)$

$P(4)$ $P(5)$ $P(6)$

$P(6)$ $P(7)$ $P(8)$ $P(9)$

$\forall k \in \mathbb{N},\ k \neq 1 \Rightarrow P(k)$

ii) $P(0) \wedge P(1)$

$P(1)$ $P(2)$

$\forall k \in \mathbb{N},\ P(k)$

b) Base case $n=2$

$2^2 < 4!$

$4 < 4 \cdot 3 \cdot 2 \cdot 1 = 24$

Let $n \in \mathbb{N}$. Assume $2^n < (n+2)!$

$2^{n+1} < (n+3)!$

$2^n < (n+2)!$

$2 \cdot 2^n < (n+2)! \cdot 2 < (n+2)! \cdot (n+3)$

since $2 < n+3$

$-1 < n$

a)

a) Loop 2: $j_k = j_{k-1} - 2$

$= j_0 - 2k$

$j_k = i_{k-1} - 2k \leq 5$

$= \left\lceil \frac{i_{k-1} - 5}{2} \right\rceil \leq k$

Loop 1: $j_k = i_{k-1} - 2k$

$i_k = i_{k-1} + j_k$

$= i_{k-1} + i_{k-1} - 2k$

$= 2(i_{k-1} - k) \geq n^2$

$2(2(i_{k-2} - (k-1)) - k$

$2^2(i_{k-2} - (k-1) - 2k)$

$i_{k-3} - (k-2)$

$k_l = k_{l-1} + 1$

$= (k_{l-2} + 1) + 1$

$= k_0 + \sum_{m=0}^{l-1} 1$

$= k_0 + l$

$= i + l$

stops when $k_l \geq j+1$

$i + l \geq j+1$

$l \geq j + 1 - i$

$\sum_{n=0}^{n-1} k$

$x - 1 < f \le x \le c < x+1$

$k \geq 2$ or

$j \geq \sqrt{n}$

$$s[j] = \begin{cases} a & \text{if } j = \lceil \sqrt{n} \rceil \text{ or } \sqrt{n} - 1 \\ b & \text{otherwise} \end{cases}$$