

Learning Objectives

By the end of this worksheet, you will:

- Determine the exact number of iterations of loops with a variety of loop counter behaviours.
- Find the asymptotic running time of programs containing loops.

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1. **Loop variations.** Each of the following functions takes as input a *non-negative integer* and performs at least one loop. For each loop, determine the *exact* number of iterations that will occur (in terms of the function's input n), and then use this to determine the simplest Theta expression¹ for the running time of each function. You do *not* need to prove any " $g \in \Theta(f)$ " statements here.

Note: each loop body runs in $\Theta(1)$ time in this question. While this won't always be the case, such examples allow you to focus on just counting loop iterations here.

```
1 def f1(n: int) -> None:
2     i = 0
3     while i < n:
4         print(i)
5         i = i + 5
```

```
1 def f2(n: int) -> None:
2     i = 4
3     while i < n:
4         print(i)
5         i = i + 1
```

¹By "simplest," we mean ignoring constants and slower-growth terms. For example, write $\Theta(n)$ instead of $\Theta(2n + 0.3)$.

```
1 def f3(n: int) -> None:
2     """Precondition:  $n > 0$ ."""
3     i = 0
4     while i < n:
5         print(i)
6         i = i + (n / 10)
```

```
1 def f4(n: int) -> None:
2     i = 20
3     while i < n * n:
4         print(i)
5         i = i + 3
```

```
1 def f5(n: int) -> None:
2     i = 20
3     while i < n * n:
4         print(i)
5         i = i + 3
6
7     j = 0
8     while j < n:
9         print(j)
10        j = j + 0.01
```

2. **Multiplicative increments.** Consider the following function:

```
1 def f(n: int) -> None:
2     """Precondition:  $n > 0$ ."""
3     i = 1
4     while i < n:
5         print(i)
6         i = i * 2
```

Even though this looks similar to previous examples, the fact that the loop variable i changes by a multiplicative rather than additive factor requires a more principled approach in determining the number of loop iterations.

- (a) Let i_0 be the value of variable i when 0 loop iterations have occurred, i_1 be the value of i immediately after 1 loop iteration has occurred, and in general i_k be the value of i immediately after k loop iterations have occurred. For example, $i_0 = 1$ (the initial value of i), $i_1 = 2$, and $i_2 = 4$. Determine the values of i_3 , i_4 , and a general formula for i_k .²

- (b) Use your formula from part (a) to determine the exact number of loop iterations that occur, in terms of n .
HINT: Find the *smallest* value of k that makes the loop condition false.

- (c) Determine the Theta running time for the function f .

- (d) Why did we not initialize $i = 0$ in this function?

²Of course, if n is small then not a lot of loop iterations occur. More formally, i_k represents the value of i after k loop iterations, if k iterations occur.

3. **A more unusual increment.** Consider the following function:

```
1 def f(n: int) -> None:
2     """Precondition:  $n \geq 2$ ."""
3     i = 2
4     while i < n:
5         print(i)
6         i = i * i
```

Analyse the running time of this function using the same technique as the previous question.