$A \wedge B \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$

Last time...

- Prove $\forall n \in \mathbb{Z}^+$, $\boxed{\forall G=(V,E),\ |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected}\right)}$ $P(n)$
- induction on n
- I.H.: Let $n \in \mathbb{Z}^+$ and assume $P(n)$:

  $\forall G=(V,E),\ |V|=n \Rightarrow \left(|E| \geq \frac{(n-1)(n-2)}{2}+1 \Rightarrow G \text{ is connected}\right)$
- I.S.: WTP $P(n+1)$

  Let $G_1=(V_1,E_1)$ be an arbitrary graph with $|V_1|=n+1$.

  Assume $|E_1| \geq \frac{n(n-1)}{2}+1$.

  WTS: G_1 is connected.

ROUGH WORK

KNOW

$n \in \mathbb{Z}^+$

$P(n)$

$G_1=(V_1,E_1)$ is a graph

$|V_1| = n+1$

$|E_1| \geq \frac{n(n-1)}{2}+1$

WANT

G_1 is connected

($\forall u,v \in V_1$, $\exists$ a path in G_1 between u and v)

Idea: look at

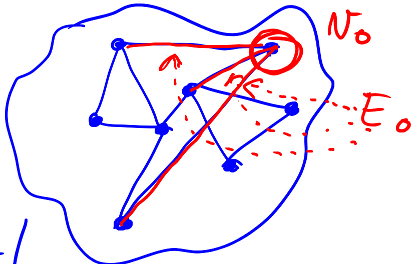
$G'=(V',E')$ where

$V' = V_1 - \{v_0\}$

$E' = E_1 - \{$every edge in E_1 that contains $v_0\}$,

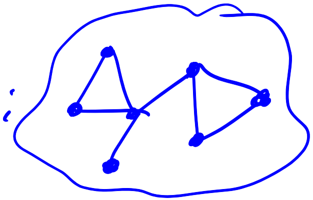
where v_0 is some vertex in V_1

G_1 :



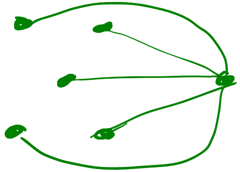
$|V_1| = n+1$

G' :



$|V'| = n$

$|V'| = n \Rightarrow$ IH applies to G'



v_0

Q: how many edges in G'?

- look at $E_0 = \{$all edges in E_1 that contain $v_0\}$
- $|E'| = |E_1| - |E_0|$

• <u>claim 1</u>: $|E_0| \leq n$ because there are at most n other vertices to form edges with v_0

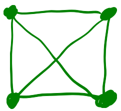
• <u>claim 2</u>: $|E_0| \geq 1$

<u>Proof</u>: In G_1, there are at most $\frac{n(n-1)}{2}$ many edges that do <u>not</u> contain v_0

but $|E_1| \geq \frac{n(n-1)}{2} + 1$ (by assumption).

So E_1 contains at least one edge with v_0.

$n = 4$ $\quad \frac{n(n-1)}{2} + 1 = 7$



v_0

From Claim 1 and assumption about $|E_1|$

$$|E'| = |E_1| - |E_0|$$

$$\geq \left(\frac{n(n-1)}{2} + 1\right) - n = \frac{(n-1)(n-2)}{2}$$

<u>PROBLEM</u>: IH requires $|E'| \geq \frac{(n-1)(n-2)}{2} + 1$

Let's introduce cases.

- <u>Case 1</u>: If G_1 contains a vertex v_0 with <u>fewer</u> than n edges adjacent to v_0, then G' contains $\geq |E_1| - |E_0|$

$$\geq \left(\frac{n(n-1)}{2} + 1\right) - (n-1)$$

$$\geq \frac{(n-1)(n-2)}{2} + 1 \quad \text{edges}$$

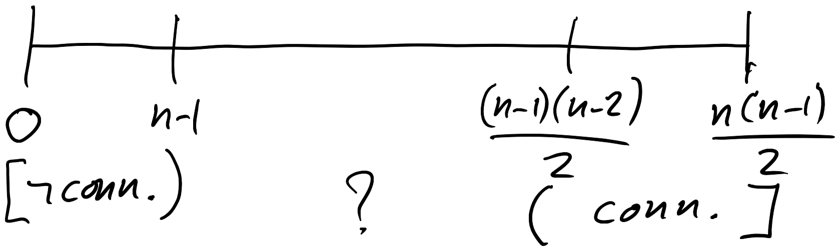
so by $P(n)$, G' is connected.

So G_1 is connected, since N_0 has at least one edge to a vertex in G!

- Case 2: If all vertices in G_1 have n adjacent edges, then $|E_1| = \frac{(n+1)(n)}{2}$ → not necessary for conclusion

Then, G_1 is connected: for all $u, v \in V$ with $u \neq v$, $(u,v) \in E_1$.

So far...

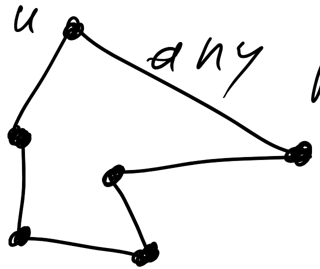


$|E|$ as a function of $n = |V|$

Q: Are there graphs with $\frac{(n-1)(n-2)}{2}$ edges that are <u>not</u> connected? (Yes: Ex. 6.7)

Q: Are there graphs with $n-1$ edges that are connected? (Yes: exercise...)

<u>Def 1</u>: A <u>cycle</u> in a graph $G=(V,E)$ is any path from u to itself, for some $u \in V$ (requires at least 3 edges)



<u>Def 2</u>: A <u>tree</u> is any graph that is connected and acyclic (does not contain any cycle).