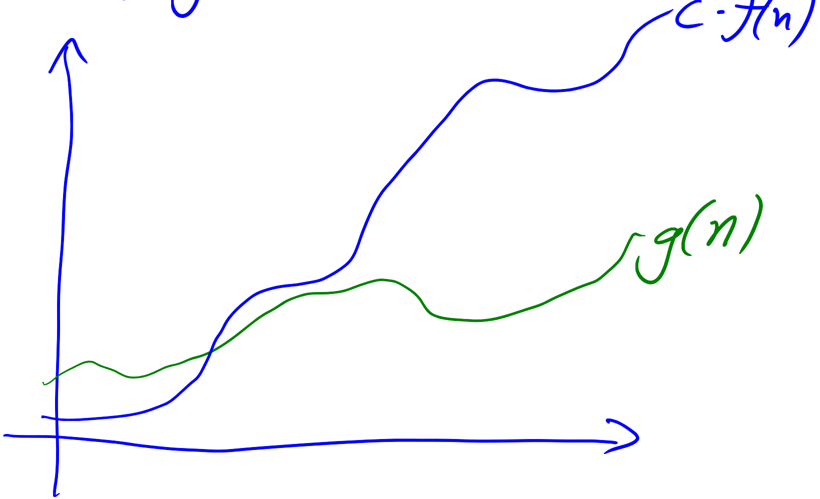
• PS2: marking <u>almost</u> done but not quite...

Recall...  For functions $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$

$g \in O(f)$

"$f$ is an upper bound on $g$"

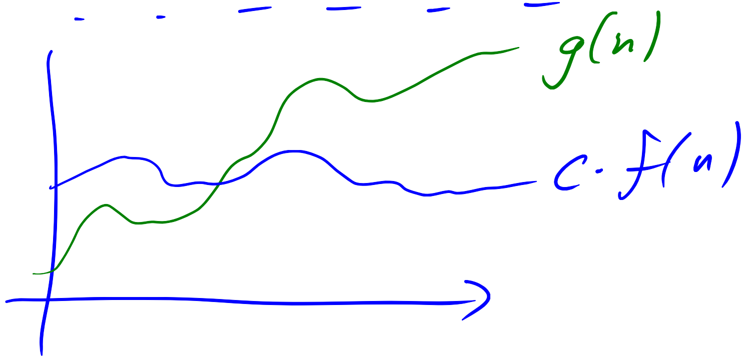
$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},$
$n \geq n_0 \Rightarrow g(n) \leq c \cdot f(n)$




$g \in \Omega(f)$:

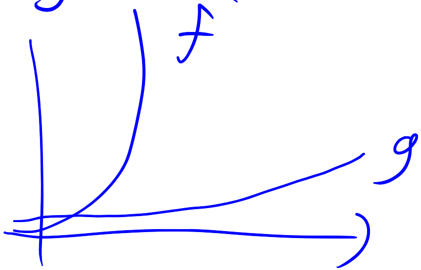
$\exists c, n_0 \in \mathbb{R}^+, \forall n \in \mathbb{N},$
$n \geq n_0 \Rightarrow g(n) \geq c \cdot f(n)$

"$f$ is a lower bound on $g$"



Note: $g \in \Omega(f) \iff f \in O(g)$

Note:




$3n \in O(n^n)$

$n \le 10^{10^{10^{10}}}$

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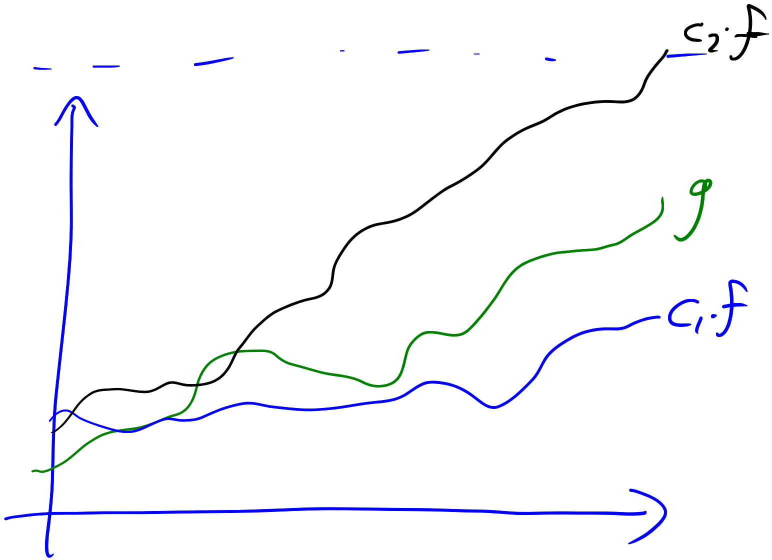
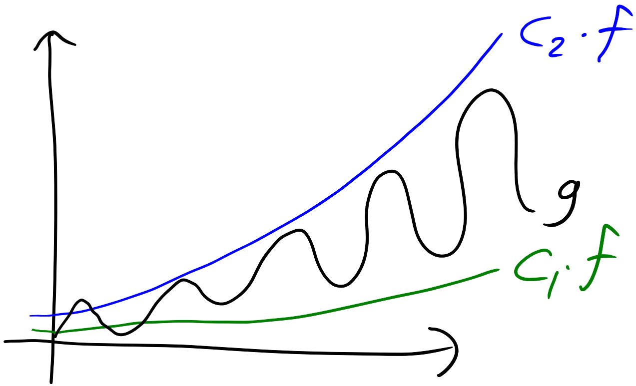
$g \in \Theta(f)$ :

$g \in O(f) \wedge g \in \Omega(f)$

"f is a tight bound on g"



Note: $g \in \Theta(f)$ does **not** imply $g = f$




EX: Prove $\forall a,b \in \mathbb{R}^+,\ an+b \notin \Omega(n^2)$.

$\forall c, n_0 \in \mathbb{R}^+, \exists n \in \mathbb{N},\ n \geq n_0 \wedge \cancel{g(n)} < c \cdot \cancel{f(n)}$

$an+b < c \cdot n^2$

Let $a, b \in \mathbb{R}^+$.

Let $c, n_0 \in \mathbb{R}^+$

Let $n =$ ______ ?

WTS: $n \geq n_0 \wedge an+b < c \cdot n^2$

ROUGH WORK

- to ensure $n \geq n_0$, pick $n = \max(\lceil n_0 \rceil, ?)$
- to ensure $an + b < c \cdot n^2$

  $\iff cn^2 - an - b > 0$ ... algebra: solve and

Let $n = \max(\lceil n_0 \rceil, \bigcirc)$

Then, $n \geq \lceil n_0 \rceil \geq n_0$ and

$cn^2 - an - b > 0 \iff an + b < cn^2$

Also, $n \in \mathbb{N}$ because $\lceil n_0 \rceil \in \mathbb{N}, \forall n_0 \in \mathbb{R}^+$

and $\bigcirc \in \mathbb{N}$

Properties of $O$, $\Omega$, $\Theta$
(Theorems 5.1–5.8 in the notes)

Example: Prove $\forall f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$,
$$g \in O(f) \implies f+g \in \Theta(f)$$
where $f+g : \mathbb{N} \to \mathbb{R}^{\geq 0}$ is defined as $f(n)+g(n)$ $\forall n \in \mathbb{N}$.

Proof: Let $f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}$. Assume $g \in O(f)$,
i.e., $\exists c_0, n_0 \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n \geq n_0 \Rightarrow g(n) \leq c_0 f(n)$

Let $c_1 =$ $1$, $c_2 =$ $c_0 + 1$, $n_1 =$ $n_0$

WTS: $\forall n \in \mathbb{N},\ n \geq n_1 \Rightarrow c_1 f(n) \leq f(n)+g(n) \leq c_2 f(n)$

## ROUGH WORK:

- $f(n)+g(n) \geq f(n) \Rightarrow$ pick $c_1 = 1$
- $f(n)+g(n) \leq f(n)+c_0 f(n)$ as long as $n \geq n_0$

$= (c_0+1) f(n)$

$c_2 = c_0 + 1$

Then $c_1, c_2, n_1 \in \mathbb{R}^+$ and

$\forall n \in \mathbb{N},\ n \geq n_1 \Rightarrow$

$c_1 f(n) = f(n) \leq f(n)+g(n)$ (b.c. $g(n) \geq 0$)

$\wedge$ $f(n)+g(n) \leq f(n)+c_0 f(n)$ (b.c. $n \geq n_1 = n_0$ and by assumption)

$= c_2 f(n)$