Learning Objectives

By the end of this worksheet, you will:

- Know the definition of bipartite graphs.

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1. **Bipartite graphs.** Let $G = (V, E)$ be a graph. We say that G is **bipartite** when it satisfies the following properties:

- There exist subsets $V_1, V_2 \subset V$ such that $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, and V_1 and V_2 form a *partition* of V.[1]
- Every edge in E has exactly one endpoint in V_1 and one in V_2. (Equivalently, no two vertices in V_1 are adjacent, and no two vertices in V_2 are adjacent.)

When G is bipartite, we call the partitions V_1 and V_2 a **bipartition of** G. TIP: bipartite graphs are typically drawn such that V_1 and V_2 are clearly separated (e.g., with all the vertices of V_1 on the left, and all the vertices of V_2 on the right).

(a) Prove that the following graph $G = (V, E)$ is bipartite.

$$V = \{1, 2, 3, 4, 5, 6\} \quad \text{and} \quad E = \{(1, 2), (1, 6), (2, 3), (3, 4), (4, 5), (5, 6)\}$$

(b) Let m and n be positive integers. A **complete bipartite graph on** (m, n) **vertices** is a graph $G = (V, E)$ that satisfies the following properties:

i. G is bipartite, with bipartition V_1, V_2 (as defined above).
ii. (new) $|V_1| = m$ and $|V_2| = n$.
iii. (new) For all vertices $u \in V_1$ and $w \in V_2$, u and w are adjacent.

How many edges are in a complete bipartite graph on (m, n) vertices? Your answer will depend on m and n. Explain your answer.

[1] That is, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$.

(c) Recall that a *cycle* in a graph $G = (V, E)$ is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $k \geq 3$, $v_k = v_0$, and G contains every edge between consecutive vertices: $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$.

In this question, we will be concerned with the *parity* of the lengths of cycles in bipartite graphs—the parity of an integer is either 0 (when the number is even) or 1 (when the number is odd).

Explore. Draw a few different bipartite graphs and make sure they contain some cycles. What do you notice about the parity of the lengths of these cycles (are they even or odd)? Can you draw a bipartite graph with cycles whose lengths have either parity?

Prove. Make a conjecture about the parity of every cycle length in a bipartite graph, and prove it.