Due before 17:00 (EST) on Tuesday 16 March 2021

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions that are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Handwritten submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set3.pdf**.
- Each problem set may be completed in groups of up to three—**except for Problem Set 0**. If you are working in a group for this problem set, please consult https://github.com/MarkUsProject/Markus/wiki/Student-Guide for a brief explanation of how to create a group on MarkUs.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with one or more partner(s), you must form a group on MarkUs, and make one submission per group.
- Your submitted file(s) should not be larger than **5MB**. You might exceed this limit if you use a word processor like Microsoft Word to create a PDF; in that case, you should look into PDF compression tools to make your PDF smaller, but please make sure that your PDF is still legible!
- Submissions must be made *before* the due date on MarkUs. Please see the Assessment section on the course website for details on how late submissions will be handled.
- MarkUs is known to be slow when many students try to submit right before a deadline. **Aim to submit your work at least one hour before the deadline. It is your responsibility to meet the deadline.** You can submit your work more than once; the most recent version submitted within the deadline (or within the late submission period) is the version that will be marked.
- The work you submit must be that of your group; you may not use or copy from the work of other groups, or external sources like websites or textbooks. Please see the section on Academic Integrity in the course syllabus for further details.

Additional instructions

- When doing a proof by induction, always label the step(s) that use the induction hypothesis.
- You may not use forms of induction we have not covered in lecture, except where indicated in the question.
- Please follow the same guidelines as Problem Set 2 for all proofs.

1. [14 marks] Proofs by induction.

(a) [**4 marks**] Prove each of the following identities by induction.

i. $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

ii. $\displaystyle\sum_{k=1}^{n} k! \cdot k = (n+1)! - 1$

(b) [**4 marks**] Prove that any integer $n \geq 10$ is expressible as a sum of 3's and/or 5's. For example, 21 is expressible as a sum of 3's and/or 5's because $21 = 5 + 5 + 5 + 3 + 3$.

Hint: Let $P(n)$ denote the predicate that n is expressible as a sum of 3's and/or 5's. Start by showing that $P(10)$, $P(11)$, and $P(12)$ are all true (so your proof will have more than one base case). Then, show that $P(k) \Rightarrow P(k+3)$. It is also possible to do this with a different argument and a "standard" inductive step ($P(k) \Rightarrow P(k+1)$).

(c) [**6 marks**] Sometimes it's possible to use induction "backwards", proving things from k to $k-1$ instead of vice versa! Consider the statement

$$P(n) : \forall x_1, x_2, \ldots, x_n \in \mathbb{R}^{\geq 0}, x_1 \cdots x_n \leq \left(\frac{x_1 + \cdots + x_n}{n}\right)^n$$

where $n \in \mathbb{N}$ and $n \geq 1$.

i. Prove that $P(2)$ is true.
Hint: Think about the quantity $(x_1 + x_2)^2 - (x_1 - x_2)^2$.

ii. Prove that, for each $n \geq 2$, if $P(2)$ and $P(n)$ are true, then $P(2n)$ is also true.
Use this to prove that $P(2^m)$ is true for all $m \in \mathbb{N}$, where $m \geq 1$.

iii. Prove that $P(k) \Rightarrow P(k-1)$ for all $k \geq 2$. (*Hint*: Set $x_k = (x_1 + \cdots + x_{k-1})/(k-1)$.)

iv. Why is $P(n)$ is true for all $n \in \mathbb{N}$, where $n \geq 1$? An informal argument here is fine (you do NOT have to provide a rigorous proof).

2. [7 marks] Number representations. On Worksheet #10, we looked at representing rational numbers in binary notation. Here, we'll also consider representations in two other bases that appear often in computer science. Recall that a **binary representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_2,$$

where $a_i, b_i \in \{0, 1\}$ and

$$x = \sum_{i=0}^{k-1} a_i 2^i + \sum_{i=1}^{m} b_i 2^{-i}.$$

Likewise, an **octal (base 8) representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_8,$$

where $a_i, b_i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ and

$$x = \sum_{i=0}^{k-1} a_i 8^i + \sum_{i=1}^{m} b_i 8^{-i}.$$

Finally, a **hexadecimal (base 16) representation of the rational number** x is

$$x = (a_{k-1}a_{k-2}\cdots a_1a_0 \,.\, b_1b_2\cdots b_m)_{16},$$

where $a_i, b_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$ (with $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, and $F = 15$) and

$$x = \sum_{i=0}^{k-1} a_i 16^i + \sum_{i=1}^{m} b_i 16^{-i}.$$

(a) [**3 marks**] In each of the following equalities, find the missing representation. To get full credit, you MUST show your work.

i. $(EA)_{16} = (x)_8$

ii. $(755)_8 = (x)_2$

iii. $(9009)_{10} = (x)_{16}$

(b) [**4 marks**] On Worksheet #10, we encountered representations of fractional numbers for which the representations have repeating digits after the decimal point. For example, $1/3$ has the representation $(0.\overline{3})_{10}$, where the overline indicates that the 3 repeats. Likewise, $1/3$ has the representation $(0.\overline{01})_2$ in binary notation. Prove that every fraction p/q (where $p, q \in \mathbb{N}$, $q \neq 0$, and $\gcd(p, q) = 1$) has a base-b representation *without* repeating digits if and only if there exists an $m \in \mathbb{N}$ such that $q \mid b^m$. **You can use the fact that every natural number has a base-b representation.**

3. [9 marks] Asymptotic notation.

For each part of this question, you *may* (but are not required to) use any of the following facts.

- Fact 1: $\forall n \in \mathbb{Z}^+, n \leq 2^n$
- Fact 2: $\forall x, y \in \mathbb{R}^{\geq 0}, x \leq y \Leftrightarrow \log_2(x) \leq \log_2(y)$
- Fact 3: $\forall x, y \in \mathbb{R}^{\geq 0}, x \leq y \Leftrightarrow 2^x \leq 2^y$

(a) [**3 marks**] Prove each of the following statements. You do NOT have to use induction on n. (Keep it simple!)

i. $n \in \mathcal{O}(n^{1+\epsilon})$, for any real number $\epsilon > 0$.

ii. $\log_2(n) \in \mathcal{O}(n)$

iii. $2^n \in \mathcal{O}(n!)$

(b) [**3 marks**] Suppose that $f \colon \mathbb{N} \to \mathbb{R}^{\geq 0}$, and that

$$f(n) \in \mathcal{O}(\log_2(n)).$$

Prove that there exists a constant $c > 0$ such that

$$2^{f(n)} \in \mathcal{O}(n^c).$$

(c) [**3 marks**] The mathematical function $e^x \colon \mathbb{R} \to \mathbb{R}$ can be represented as series by the formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Suppose that $f \colon \mathbb{N} \to \mathbb{R}^{\geq 0}$ and that $f(n)$ is eventually dominated by 1.
Prove that

$$e^{f(n)} - 1 \in \mathcal{O}(f(n)).$$