Sets, Functions, and Predicates

CSC165 Week 1 - Part 2

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From last class:

$x \in S$	"x is an element of A"
$A \subset S$	"A is a subset of S"
$A \subseteq S$	"A is a subset of S or A is equal to S"

$A \subseteq A$	is always true
$\varnothing \subseteq A$	always true (but not always an element)

$\varnothing$ = the empty set = { }

Let A = {1,x,$\varnothing$}

Let B = $\varnothing$ = { }

Let C = { $\varnothing$ } = { { } }

P($\varnothing$) = P({ }) = $\varnothing$

P({$\varnothing$}) = P({{}}) = { {$\varnothing$}, $\varnothing$}

$$\boxed{A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}}$$

Example of Cartesian Product:

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$

$\mathbb{Z} \times \mathbb{R} = \{ (n,y) \mid n \in \mathbb{Z} \text{ and } y \in \mathbb{R} \}$

$\mathbb{Z} \times \mathbb{Z} = \{ (n,m) \mid n \in \mathbb{Z} \text{ and } m \in \mathbb{Z} \}$

$R \times \varnothing = \varnothing$

Functions

A function is a mapping from its Domain to its Codomain.

$f: A \longrightarrow B$ means $f(a) = b$ where $a \in A$ and $b \in B$

Example: $f(x) = \frac{1}{|x|}$ $f : \mathbb{R} \setminus \{0\} \longrightarrow (0,\infty)$

For $k \in \mathbb{Z}^+$, we say f is a k-ary function if

$$f: A_1 \times A_2 \times \dots \times A_k \longrightarrow B$$

$$f(a_1,a_2,\dots,a_k) = b$$

Summation Notation

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =$

$1 + 2 + 3 + \ldots + 99 + 100 =$

$1 + 4 + 9 + \ldots + 81 + 100 =$

$3 + 9 + 19 + \ldots + 163 + 201 =$

Summation Notation

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \sum_{i = 1}^{10} i$$

$$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{i = 1}^{10} i$$

$$4 + 9 + \ldots + 81 + 100 + 121 = \sum_{i = 2}^{11} i^2$$

$$3 + 9 + 19 + \ldots + 163 + 201 = \sum_{i = 1}^{10} 2i^2 + 1$$

$$\sum_{i=1}^{3} 3\sin(2i) \quad =$$

$$\sum_{i=1}^{5} 3i + i^2 \quad =$$

$$\sum_{i=8}^{2} (i+3)^{1/2} - 6 \quad =$$

Product Notation

$$\prod_{k=-1}^{2} 3k + 2 =$$

$$\prod_{i=2}^{5} (x - i)^2 =$$

$$\prod_{i=10}^{9} i - 5 =$$

Propositional Logic

A proposition is a statement that can be true or false.

Examples:

$S = 3 < 4$

$T = 3 \geq 4$

U = "It is raining right now."

V = "This loop will terminate in a finite number of iterations."

NOT (negation)

Ex: Let $q = \neg (x < 3)$

p	¬ p
TRUE	
FALSE	

AND (conjunction)

Ex: $(x < 2) \wedge (x > 3)$

Ex: It is January and it is sunny.

p	q	$p \wedge q$

OR inclusive (disjuction)

Ex: $(x < 2) \vee (x > 3)$

Ex: It is January or it is sunny.

p	q	$p \vee q$

Exclusive OR is used here:

"I want a red car or a blue car"

We understand here that the person is not looking for two cars: one of each.

Implies (implication) (conditional) $\sim p \vee q$

$p \Longrightarrow q$ "if p is true, then q must be true"

p is the hypothesis

q is the conclusion

Ex: If it rains, then I bring an umbrella

Ex: $(n \in \mathbb{N}) \Longrightarrow (n \in \mathbb{Z})$

p	q	$p \Rightarrow q$

iff (if and only if) (biconditional) $p \implies q \wedge q \implies p$

p is true if and only if q is true

Ex: $(n \in \mathbb{N}) \iff (n \in \mathbb{Z})$

Ex: It is Monday if and only if it is the first CSC165 lecture of the week

Ex: A number is a natural number if and only if it is a positive integer.

All definitions are biconditional statements.

p	q	$p \iff q$

Example of Evaluating a Truth Table: $(p \vee q) \implies (r \implies (p \wedge q))$

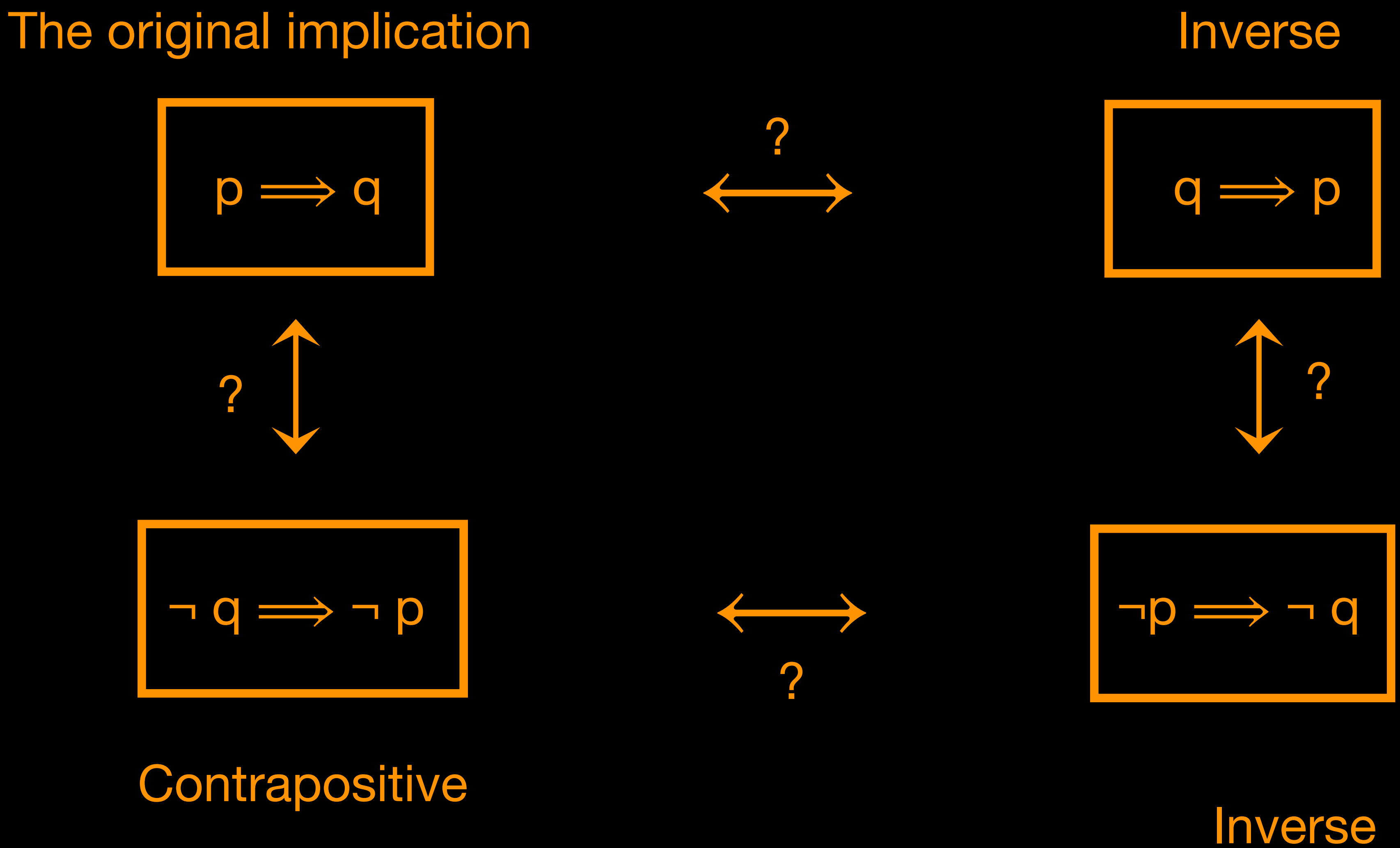
p	q	r	$(p \vee q)$	$(p \wedge q)$	$r \implies (p \wedge q)$	$(p \vee q) \Rightarrow (r \Rightarrow (p \wedge q))$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Two statements are said to be logically equivalent if they are true for the same input and false for the same input:

$p \iff q$ means that p and q are logically equivalent to each other.

Example: $(x = 2) \iff ((x \in \mathbb{Z}) \wedge (x \geq 0))$

$(x = 2)$	$((x \in \mathbb{Z}) \wedge (x \geq 0))$	$(x = 2) \iff ((x \in \mathbb{Z}) \wedge (x \geq 0))$
T	T	
T	F	
F	T	
F	F	


The original implication
Inverse
$p \implies q$
?
$q \implies p$
?
?
$\neg q \implies \neg p$
$\neg p \implies \neg q$
?
Contrapositive
Inverse

The original statement is “All cats are animals”.

This is logically equivalent to the implication:

“If it is a cat, then it is an animal”

Let A = “it is a cat” and B = “it is an animal”

$A \Rightarrow B$ = the original statement

(contrapositive) $\neg B \Rightarrow \neg A$ =

(converse) $B \Rightarrow A$ =

(inverse) $\neg A \Rightarrow \neg B$ =

The original implication

$$p \implies q$$

$$\Updownarrow$$

$$\neg q \implies \neg p$$

Contrapositive

Inverse

$$q \implies p$$

$$\Updownarrow$$

$$\neg p \implies \neg q$$

Inverse