Due before 17:00 on Tuesday 23 February 2021

Note: **solutions may be incomplete, and meant to be used as guidelines only**. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[12 marks] Number theory.** For this question, you may use any of the following definitions or facts *without proof*. In other words, you may refer to any of these definitions, and any of these facts may appear as a justification for some deduction in your proofs, but *do NOT prove* these facts as part of your justification.

- All definitions, facts, and statements proven in worksheets 1–8.
- Theorem 2.1 (Quotient-Remainder Theorem) from the Course Notes.
- *Fact 1:* $\forall x \in \mathbb{R}, 0 \leq x - \lfloor x \rfloor < 1$.
- *Definition 1:* An *odd function* is any function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
- *Definition 2:* An *even function* is any function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

Now, for each statement below:

(i) Write down whether the statement is true or false.

(ii) If the statement is true, write it in predicate notation and then write a detailed proof of the statement.

(iii) If the statement is false, write its *negation* in predicate notation, and then write a detailed proof of its negation.

State all definitions you use in your proofs (other than the ones above).

(a) **[4 marks]** There exist 4 consecutive integers whose product is **not** divisible by 12. (Integers are *consecutive* when their difference is 1, e.g., $2, 3, 4, 5$ are consecutive.)

Solution

(i) False

(iii) $\forall n \in \mathbb{Z}, 12 \mid n(n+1)(n+2)(n+3)$

Let $n \in \mathbb{Z}$. First we will show that $n(n+1)(n+2)(n+3)$ is divisible by 4.
By Theorem 2.1, we know that $\exists r \in \mathbb{Z}, n = 2k + r \land 0 \leq r < 2$.

Case 1: Assume $n = 2k + 0$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &= (2k)(2k+1)(2k+2)(2k+3) \\ &= 4(k)(2k+1)(k+1)(2k+3) \end{aligned}$$

Case 2: Assume $n = 2k + 1$. Then

$$\begin{aligned} n(n+1)(n+2)(n+3) &= (2k+1)(2k+2)(2k+3)(2k+4) \\ &= 4(2k+1)(k+1)(2k+3)(k+2) \end{aligned}$$

Next we will show that the product is divisible by 3.
By Theorem 2.1, we know that $\exists r \in \mathbb{Z}, n = 3k + r \land 0 \leq r < 3$.

Case 1: Assume $n = 3k+0$. Then $n = 3k$ is divisible by 3 so $n(n+1)(n+2)(n+3)$ is also divisible by 3.

Case 2: Assume $n = 3k+1$. Then $n+2 = 3k+3 = 3(k+1)$ is divisible by 3 so $n(n+1)(n+2)(n+3)$ is also divisible by 3.

Case 3: Assume $n = 3k+2$. Then $n+1 = 3k+3 = 3(k+1)$ is divisible by 3 so $n(n+1)(n+2)(n+3)$ is also divisible by 3.

We now know that $4 \mid n(n+1)(n+2)(n+3)$, so

$$\exists r \in \mathbb{Z}, n(n+1)(n+2)(n+3) = 4r$$

We also know that $3 \mid n(n+1)(n+2)(n+3)$, so

$$\exists s \in \mathbb{Z}, n(n+1)(n+2)(n+3) = 3s$$

This means $4r = 3s$, which means $4r$ is a multiple of 3.
From lecture we know that $\forall p \in \mathbb{N}, Prime(p) \Rightarrow (p \nmid a \land p \nmid b \Rightarrow p \nmid ab)$. This is logically equivalent to:

$$\forall p \in \mathbb{N}, Prime(p) \land p \mid ab \Rightarrow p \mid a \lor p \mid b$$

The number 3 is prime and $3 \mid 4r$. Since 3 is not a factor of 4, $3 \mid r$ which implies that $\exists t \in \mathbb{Z}, r = 3t$.
So $n(n+1)(n+2)(n++3) = 4r = 4(3t) = 12t$. Therefore $\forall n \in \mathbb{Z}, \exists t \in \mathbb{Z}, n(n+1)(n+2)(n+3) = 12t$.

(b) [**4 marks**] For all real numbers x greater than or equal to 6, $4x^2 - 3\lfloor x \rfloor^2 \geq 9$.
(Hint: Try to find a lower bound for $(x-3)^2$ first.)

Solution

(i) True

(ii) $\forall x \in \mathbb{R}, x \geq 6 \Rightarrow 4x^2 - 3\lfloor x \rfloor^2 \geq 9$

Let $x \in \mathbb{R}$ and assume $x \geq 6$.

$$
\begin{aligned}
& \lfloor x \rfloor \leq x && \text{(by Fact 1)} \\
\Longrightarrow\ & \lfloor x \rfloor^2 \leq x^2 && \text{(square both sides, since } x \geq 6 \Rightarrow \lfloor x \rfloor \geq 6) \\
\Longrightarrow\ & -3\lfloor x \rfloor^2 \geq -3x^2 && \\
\Longrightarrow\ & 4x^2 - 3\lfloor x \rfloor^2 \geq 4x^2 - 3x^2 && \\
\Longrightarrow\ & 4x^2 - 3\lfloor x \rfloor^2 \geq x^2 && \\
\Longrightarrow\ & 4x^2 - 3\lfloor x \rfloor^2 \geq 6^2 && \text{(since } x \geq 6) \\
\Longrightarrow\ & 4x^2 - 3\lfloor x \rfloor^2 \geq 9 &&
\end{aligned}
$$

(c) [**4 marks**] There exist odd functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x) - h(x)$ is a non-constant even function.
(A function $f : \mathbb{R} \to \mathbb{R}$ is *constant* when $\exists k \in \mathbb{R}, \forall x \in \mathbb{R},\ f(x) = k$.)

Solution

(i) False

(iii) $\forall g, h : \mathbb{R} \to \mathbb{R}, (\forall x \in \mathbb{R}, g(-x) = -g(x)) \land (\forall x \in \mathbb{R}, h(-x) = -h(x)) \Rightarrow \big(\exists k \in \mathbb{R}, \forall x \in \mathbb{R}, g(x) - h(x) = k\big) \lor \big(\exists x \in \mathbb{R}, g(-x) - h(-x) \neq g(x) - h(x)\big)$

Let $g, h : \mathbb{R} \to \mathbb{R}$. Assume $(\forall x \in \mathbb{R}, g(-x) = -g(x)) \land (\forall x \in \mathbb{R}, h(-x) = -h(x))$.
Note that either $g - h$ is constant, or it is non-constant.

Case 1: Assume $g - h$ is constant.
Then by definition, $\exists k \in \mathbb{R}, \forall x \in \mathbb{R}, g(x) - h(x) = k$ so the conclusion holds.

Case 2: Assume $g - h$ is non-constant $(\forall k \in \mathbb{R}, \exists x \in \mathbb{R}, g(x) - h(x) \neq k)$.
Then $\exists x_0 \in \mathbb{R}, g(x_0) - h(x_0) \neq 0$, and

$$\begin{aligned} g(-x_0) - h(-x_0) &= (-g(x_0)) - (-h(x_0)) && \text{(by assumption that } g \text{ and } h \text{ are odd)} \\ &= -(g(x_0) - h(x_0)) \\ &\neq g(x_0) - h(x_0) && \text{(since } g(x_0) - h(x_0) \neq 0\text{).} \end{aligned}$$

So $\exists x \in \mathbb{R}, g(-x) - h(-x) \neq g(x) - h(x)$ (pick $x = x_0$), and the conclusion holds.

2. [8 marks] Sets of real numbers. Consider the following predicates, defined for arbitrary sets $S, T \subseteq \mathbb{R}$:

$$G_0(S,T) : \forall x \in S, \forall y \in T, x > y$$
$$G_1(S,T) : \forall x \in S, \exists y \in T, x > y$$
$$G_2(S,T) : \exists x \in S, \forall y \in T, x > y$$
$$G_3(S,T) : \exists x \in S, \exists y \in T, x > y$$

(a) [**2 marks**] For fixed (but arbitrary) **non-empty** sets S and T, do any of these predicates imply each other?

To answer this question, for each predicate, write any true implications for which it can be the hypothesis. For example, you would write "$G_0(S,T) \Rightarrow G(S,T)$" if $G(S,T)$ was on the list and was true whenever $G_0(S,T)$ is true.

For each implication you write, **explain** in 1–3 sentences why the implication is true.

(Note: this will require you to consider all 16 possible implications between the four predicates.)

Solution

Let $S, T \subseteq \mathbb{R}$ such that $S \neq \varnothing$, $T \neq \varnothing$.

Lemma: For all predicates $P : \mathbb{R} \to \{\texttt{True}, \texttt{False}\}$, $\forall x \in S, P(x) \Rightarrow \exists x \in S, P(x)$.

Assume $\forall x \in S, P(x)$. Let $x_0 =$ some element in S—any element will do, and since $S \neq \varnothing$ there is always at least one value we can choose. Then $P(x_0)$ by our assumption. Hence, $\exists x \in S, P(x)$.

- $G_i(S,T) \Rightarrow G_i(S,T)$ for $i = 0, 1, 2, 3$, because $p \Rightarrow p$ is a tautology.
- $G_0(S,T) \Rightarrow G_1(S,T)$ by the lemma above, applied to variable y and the predicate $x > y$ (for any $x \in \mathbb{R}$)
- $G_0(S,T) \Rightarrow G_2(S,T)$ by the lemma above, applied to variable x and the predicate $\forall y \in T, x > y$
- $G_0(S,T) \Rightarrow G_3(S,T)$ by the lemma above, applied twice: first with variable y and the predicate $x > y$ (for any $x \in \mathbb{R}$), next with variable x and the predicate $\exists y \in T, x > y$
- $G_1(S,T) \Rightarrow G_3(S,T)$ by the lemma above, applied to variable x and the predicate $\exists y \in T, x > y$
- $G_2(S,T) \Rightarrow G_3(S,T)$ by the lemma above, applied to variable y and the predicate $x > y$ (for the same $x \in \mathbb{R}$ that makes $G_2(S,T)$ true)

(b) [**6 marks**] Prove or disprove each of the following statements. State clearly whether you are attempting a proof or a disproof.

- $G_0([3,5],[0,2])$

Solution

We **prove** $\forall x \in [3,5], \forall y \in [0,2], x > y$.
Let $x \in [3,5]$. Let $y \in [0,2]$. Then, $x \geq 3 > 2 \geq y$.

- $G_0(\mathbb{R}, \mathbb{R})$

Solution

We **disprove** $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x > y$.
Let $x = 1$ and $y = 1.5$. Then, $x \leq y$.

- $G_1((0,1],(0,1))$

Solution

We **prove** $\forall x \in (0,1], \exists y \in (0,1), x > y$.
Let $x \in (0,1]$. Let $y = x/2$. Then $y \in (0,1)$ and $x > y$ (because $x > 0$).

- $G_2(\mathbb{Z},(-\infty,0))$

Solution

We **prove** $\exists x \in \mathbb{Z}, \forall y \in (-\infty,0), x > y$.
Let $x = 0$. Let $y \in (-\infty,0)$. Then $y < 0 = x$.

- $G_2(\{10\},\varnothing)$, where $\varnothing$ is the empty set.

Solution

We **prove** $\exists x \in \{10\}, \forall y \in \varnothing x > y$.
Let $x = 10$. Then, $\forall y \in \varnothing, x > y$ is vacuously true. (Equivalently, there is no $y \in \varnothing$ that can make $x > y$ false.)

- $G_3([0,1]\cap\mathbb{Q},\mathbb{R}\setminus\mathbb{Q})$

Solution

We **prove** $\exists x \in [0,1]\cap\mathbb{Q}, \exists y \in \mathbb{R}\setminus\mathbb{Q}, x > y$.
Let $x = 1$. Let $y = \pi/4$. Then $y \in \mathbb{R}\setminus\mathbb{Q}$ and $y < 1 = x$.

3. [10 marks] Pascal's Triangle. Pascal's Triangle is the following arrangement of numbers:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\iddots & & & & & \vdots & & & & & \ddots
\end{array}
\quad = \quad
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\iddots & & & & & \vdots & & & & & \ddots
\end{array}
$$

where $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \leq n \Rightarrow \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$, and $\forall n \in \mathbb{N}, n! = \prod_{i=1}^{n} i = (n)(n-1)\cdots(2)(1)$.
(In particular, note that $0! = 1$.)

<u>Binomial Theorem:</u> $\forall a, b \in \mathbb{R}, \forall n \in \mathbb{N}, (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

Note that the n^{th} row of Pascal's Triangle contains the coefficients in the simplified expansion of any binomial of the form $(a+b)^n$. (Where we number rows starting at the top, and counting from 0, so the top row is the 0^{th} row, the second row from the top is the 1^{st} row, and so on.)

The three statements below are true and can be proven by at least one of these methods:

- indirect proof (proof by contrapositive);
- direct proof.

Prove each statement so that you use *each* type of proof listed above *at least once*.

(a) [**4 marks**] Consider a function $f(x) = \left(x + \dfrac{c}{x}\right)^b$, where $b, c \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.
Then $\forall b, c \in \mathbb{Z}^+, \forall x \in \mathbb{R}^*$, if $f(x)$ has no constant term, then b is an odd number.
(A *constant term* is one whose value does not depend on x.)

<u>Solution</u>

Indirect proof. Let $b, c \in \mathbb{Z}^+$ and $x \in \mathbb{R}^+$. Assume b is even, i.e., $\exists k \in \mathbb{Z}, b = 2k$. Then,

$$
\begin{aligned}
f(x) &= \left(x + \frac{c}{x}\right)^b \\
&= \left(x + \frac{c}{x}\right)^{2k} \\
&= \left(x^2 + 2c + \frac{c^2}{x^2}\right)^k
\end{aligned}
$$

The full expansion of $f(x)$ will contain the term $(2c)^k$, where x does not appear. So $f(x)$ has a non-zero constant term.

(b) [**3 marks**] Let $f(n, k) = \dbinom{n}{k}$, where $n, k \in \mathbb{N}$. Then $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}^+$ is an onto function.

Solution

By definition of "onto", we prove that $\forall y \in \mathbb{Z}^+, \exists n, k \in \mathbb{N}, f(n,k) = y$. Let $y \in \mathbb{Z}^+$. Let $n = y$, and $k = 1$. Then

$$\begin{aligned} f(n,k) &= \frac{n!}{k!(n-k)!} \\ &= \frac{y!}{1!(y-1)!} \\ &= \frac{y(y-1)!}{(y-1)!} \\ &= y \end{aligned}$$

Where in the second-last line, we rely on the fact that for all $y \in \mathbb{Z}^+$, $y! = \prod_{i=1}^{y} i = y \cdot \prod_{i=1}^{y-1} i = y(y-1)!$.

(c) **[3 marks]** $\forall n, k \in \mathbb{Z}^+, k < n \Rightarrow \binom{n}{k} \div \binom{n-1}{k} > 1$

Solution

Let $n, k \in \mathbb{Q}^+$. Assume $k < n$. Then,

$$\begin{aligned} \binom{n}{k} \div \binom{n-1}{k} &= \frac{n!}{k!(n-k)!} \div \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)(n-k-1)!} \cdot \frac{k!(n-k-1)!}{(n-1)!} \\ &= \frac{n}{n-k} \end{aligned}$$

And this allows us to conclude:

$$\begin{aligned} & k > 0 \\ \Longrightarrow\ & n > n - k \\ \Longrightarrow\ & \frac{n}{n-k} > 1 && (n - k > 0 \text{ since } k < n) \\ \Longrightarrow\ & \binom{n}{k} \div \binom{n-1}{k} > 1 \end{aligned}$$