This document shows **all versions** of each question (or part of a question) on the test, along with their sample solution. *Remember that sample solutions may be partial, showing only key elements without all the details. Also, that there could be many other correct solutions besides the one shown here.* Each individual test paper contained only one version of each question (or each part).

1. [8 marks] Short answer questions. No justification is required for any part of this question.

(a) **[2 marks]** Let $S = \{2, 3, 5, 7\}$. Find a set $S_1 \subseteq \mathbb{N}$ such that:

- $6 > |S_1| > |S|$, and
- $\forall x \in S,\ \exists y \in S_1,\ x \cdot |S_1| = y$

Solution

$S_1 = \{10, 15, 25, 35, 36\}$ (36 could be replaced with any other natural number)

(a) **[2 marks]** Let $T = \{1, 2, 3, 4, 5\}$. Find sets $T_1 \subseteq T$ and $T_2 \subseteq T$ such that:

- $|T_1 \times T_2| < |T \times T|$, and
- $T_1 \times T_2 = T_2 \times T_1$

Solution

$T_1 = \{1, 2\}$, $T_2 = \{1, 2\}$.

(b) **[2 marks]** Recall that $\mathcal{P}(S)$ is the set of all subsets of S. Express the following set **without** using $\mathcal{P}$:

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))$$

Solution

$\Big\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\Big\}$

(b) **[2 marks]** Recall that $\mathcal{P}(S)$ is the set of all subsets of S. Express the following set **without** using $\mathcal{P}$:

$$\mathcal{P}(\mathcal{P}(\{0\}))$$

Solution

$\Big\{\varnothing, \{\varnothing\}, \{\{0\}\}, \{\varnothing, \{0\}\}\Big\}$

(c) **[2 marks]**

p	q	Student Number	Truth Value
`False`	`False`	6	`True`
`False`	`True`	7	`False`
`True`	`False`	8	`True`
`True`	`True`	9	`False`

Create a table like the one on the left, where you will write the last 4 digits of your student number, in the order they appear, from top to bottom in the column labelled "Student Number". In the column labelled "Truth Value" write `True` if the digit from your student number is even, and write `False` if the digit from your student number is odd. For example, if your student number were 123456789, your truth table would look like the one on the left.

Then, write a propositional formula using only the symbols p, q, $\wedge$, $\vee$, and $\neg$ (you may use each symbol any number of times) that is logically equivalent to your truth table (in other words, the truth table for your formula is the same as the truth table you generated from your student number).

Solution

Solutions vary.

(d) **[2 marks]** Suppose we want to **prove** the following statement:

$$\forall n \in \mathbb{N},\ n > 2 \Rightarrow \exists x, y \in \mathbb{N},\ \mathit{Prime}(x) \land \mathit{Prime}(y) \land x + y = n$$

Write the complete *proof header* for a proof, introducing all variables and assumptions. You may write statements like "Let $d =$ _____" without filling in the blank. The last statement of your proof header should be "We will prove that ..." where you clearly state what remains to be proved. **Careful: we are NOT asking you to write a proof!** Only the proof *headers*.

Solution

Let $n \in \mathbb{N}$.

Assume that $n > 2$. Let $x =$ _____ and let $y =$ _____.

We will prove that $\mathit{Prime}(x) \land \mathit{Prime}(y) \land x + y = n$ is true.

(d) **[2 marks]** Suppose we want to **disprove** the following statement:

$$\forall n \in \mathbb{N},\ \exists n_0 \in \mathbb{N},\ n_0 > n \land \mathit{Prime}(n_0) \land \mathit{Prime}(n_0 + 2)$$

Write the complete *proof header* for a proof, introducing all variables and assumptions. You may write statements like "Let $d =$ _____" without filling in the blank. The last statement of your proof header should be "We will prove that ..." where you clearly state what remains to be proved. **Careful: we are NOT asking you to write a proof!** Only the proof *headers*.

Solution

Let $n =$ _____.

Let $n_0 \in \mathbb{N}$.

We will prove that $n_0 \le n \lor \neg \mathit{Prime}(n_0) \lor \neg \mathit{Prime}(n_0 + 2)$ is true.

Equivalently:

Assume $n_0 > n$ and $\mathit{Prime}(n_0)$.

We will prove that $\neg \mathit{Prime}(n_0 + 2)$ is true.

(Or any other valid combination of assumptions obtained by transforming some of the disjunctions into implications.)

2. **[10 marks] Translations.** Let P be the set of all people and C be the set of all courses, and suppose we define the following predicates:

- $\mathit{Enrolled}(s,c)$: "s is enrolled in course c", where $s \in P$ and $c \in C$.
- $\mathit{Teaches}(p,s)$: "p teaches s", where $p \in P$ and $s \in P$ ($\mathit{Teaches}(x,y)$ is *not* the same as $\mathit{Teaches}(y,x)$).

Translate each of the following statements into predicate logic. No explanation is necessary. *Do not define any of your own predicates or sets, and use only the quantifiers and propositional operators from class.* You may use $=$ and $\neq$ to compare whether two people or courses are the same.

(a) **[2 marks]** There is a class in which everyone is enrolled.

Solution

$$\exists c \in C, \forall p \in P,\ \mathit{Enrolled}(p,c)$$

(b) **[2 marks]** There is exactly one person who teaches everyone.

Solution

$$\exists p \in P,\ (\forall s \in P, \mathit{Teaches}(p,s)) \land \forall p_0 \in P,\ (\forall s_0 \in S, \mathit{Teaches}(p_0,s_0)) \Rightarrow p_0 = p$$

(c) **[2 marks]** Nobody is enrolled in the same course as a person whom they teach.

Solution

$$\forall p \in P, \forall s \in P, \mathit{Teaches}(p,s) \Rightarrow \forall c \in C, \mathit{Enrolled}(s,c) \Rightarrow \neg \mathit{Enrolled}(p,c)$$

(d) **[2 marks]** Everyone who doesn't teach anybody is enrolled in a course.

Solution

$$\forall p \in P,\ (\forall p_0 \in P,\ \neg \mathit{Teaches}(p,p_0)) \Rightarrow (\exists c \in C,\ \mathit{Enrolled}(p,c))$$

(e) **[2 marks]** Some student takes every course.

Solution

$$\exists p \in P, \forall c \in C,\ \mathit{Enrolled}(p,c)$$

(f) **[2 marks]** Nobody teaches every student.

Solution

$$\forall p \in P, \exists s \in P,\ \neg \mathit{Teaches}(p,s)$$

(g) **[2 marks]** Every person who teaches at least one student does not take any courses.

Solution

$$\forall p \in P, (\exists s \in P,\ \mathit{Teaches}(p, s)) \Rightarrow \forall c \in C, \neg \mathit{Enrolled}(p, c)$$

(h) **[2 marks]** Every person who teaches at least one person also takes at least one course.

Solution

$$\forall p \in P, (\exists s \in P,\ \mathit{Teaches}(p, s)) \Rightarrow \exists c \in C,\ \mathit{Enrolled}(p, c)$$

(i) **[2 marks]** There are two people that take at least two courses in common.

Solution

$$\exists a, b \in P, \exists c, d \in C,\ a \neq b \land c \neq d \land \mathit{Enrolled}(a, c) \land \mathit{Enrolled}(a, d) \land \mathit{Enrolled}(b, c) \land \mathit{Enrolled}(b, d)$$

(j) **[2 marks]** If two people take the same course, then they are taught by the same person.

Solution

$$\forall a, b \in P,\ (\exists c \in C, a \neq b \land \mathit{Enrolled}(a, c) \land \mathit{Enrolled}(b, c)) \Rightarrow (\exists d \in P,\ \mathit{Teaches}(d, a) \land \mathit{Teaches}(d, b))$$

(k) **[2 marks]** Nobody teaches themselves.

Solution

$$\forall p \in P, \neg \mathit{Teaches}(p, p)$$

(l) **[2 marks]** Exactly one person teaches themselves.

Solution

$$\exists p \in P,\ \mathit{Teaches}(p, p) \land \forall p_0 \in P,\ \mathit{Teaches}(p_0, p_0) \Rightarrow p = p_0$$

(m) **[2 marks]** Everyone teaches themselves and at least one other person.

Solution

$$\forall p \in P, \exists q \in P,\ p \neq q \land \mathit{Teaches}(p, p) \land \mathit{Teaches}(p, q)$$

(n) **[2 marks]** There is a course with exactly two people enrolled.

Solution

$$\exists c \in C, \exists a, b \in P, a \neq b \land \mathit{Enrolled}(a, c) \land \mathit{Enrolled}(b, c) \land \forall p \in P,\ \mathit{Enrolled}(p, c) \Rightarrow p = a \lor p = b$$

(o) **[2 marks]** Every course has at least one person enrolled.

Solution

$$\forall c \in C, \exists p \in P, \mathit{Enrolled}(p, c)$$

(p) **[2 marks]** If someone is enrolled in a course, then someone else teaches that person.

Solution

$$\forall p \in P, (\exists c \in C,\ \ \mathit{Enrolled}(p, c)) \Rightarrow \exists q \in P,\ \ p \neq q \wedge \mathit{Teaches}(q, p)$$

3. [6 marks] Disproofs.

Consider the following statement:

$$\forall x, y \in \mathbb{Z},\ x \mid y \land y \mid x \Rightarrow x = y$$

(a) **[1 mark]** Write the *negation* of this statement in predicate logic.

Solution

$$\exists x, y \in \mathbb{Z},\ x \mid y \land y \mid x \land x \neq y$$

(b) **[5 marks]** *Disprove* the original statement by proving its negation. Take the time to write your proof carefully: it will be marked on its *structure* as well as its *content*,

Solution

Proof. Let $x = 5$. Let $y = -5$. Thus we have $x, y \in \mathbb{Z}$. We'll prove that $x \mid y \land y \mid x \land x \neq y$.

- First $x \neq y$ since $5 \neq -5$.
- Second $x \mid y$: take $k = -1$ then we have $5 = (-1)(-5)$ so $x \mid y$
- Third $y \mid x$: take $k = -1$ then we have $-5 = (-1)5$ so $y \mid x$

□

3. [6 marks] Disproofs.

Consider the following statement:

$$\forall x, y \in \mathbb{R}, x > 0 \land y > 0 \Rightarrow \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

(a) **[1 mark]** Write the *negation* of this statement in predicate logic.

Solution

$$\exists x, y \in \mathbb{R},\ x > 0 \land y > 0 \land \lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$$

(b) **[5 marks]** *Disprove* the original statement by proving its negation. Take the time to write your proof carefully: it will be marked on its *structure* as well as its *content*,

Solution

Proof. Let $x = 0.5$. Let $y = 0.5$. Thus we have $x, y \in \mathbb{R}$. We'll prove that $x > 0 \land y > 0 \land \lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$.

- First $x > 0$ since $0.5 > 0$.
- Second $y > 0$ since $0.5 > 0$.
- Third

$$\begin{aligned} \lfloor x + y \rfloor &= \lfloor 0.5 + 0.5 \rfloor \\ &= \lfloor 1 \rfloor \\ &= 1 \end{aligned}$$

whereas

$$\begin{aligned}\lfloor x \rfloor + \lfloor y \rfloor &= \lfloor 0.5 \rfloor + \lfloor 0.5 \rfloor \\ &= 0 + 0 \\ &= 0 \neq 1\end{aligned}$$

□

3. [6 marks] Disproofs.

Consider the following statement:

$$\forall x, y \in \mathbb{R},\ x > 0 \land y > 0 \Rightarrow \sqrt{4x} + \sqrt{2y} = \sqrt{4x + 2y}$$

(a) **[1 mark]** Write the *negation* of this statement in predicate logic.

Solution

$$\exists x, y \in \mathbb{R},\ x > 0 \land y > 0 \land \sqrt{4x} + \sqrt{2y} \neq \sqrt{4x + 2y}$$

(b) **[5 marks]** *Disprove* the original statement by proving its negation. Take the time to write your proof carefully: it will be marked on its *structure* as well as its *content*,

Solution

Proof. Let $x = 1$. Let $y = 2$. Thus we have $x, y \in \mathbb{R}$. We'll prove that $x > 0 \land y > 0 \land \sqrt{4x} + \sqrt{2y} \neq \sqrt{4x + 2y}$.

- First $x > 0$ since $1 > 0$.
- Second $y > 0$ since $2 > 0$.
- Third

$$\begin{aligned}\sqrt{4x} + \sqrt{2y} &= \sqrt{4 \cdot 1} + \sqrt{2 \cdot 2} \\ &= \sqrt{4} + \sqrt{4} \\ &= 2 + 2 \\ &= 4\end{aligned}$$

whereas

$$\begin{aligned}\sqrt{4x + 2y} &= \sqrt{4 \cdot 1 + 2 \cdot 2} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \neq 4\end{aligned}$$

□