Estimating the Double WEll

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Abstract

Supposed we have a double –well potential SDE, how do we estimate its parameters

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1 The Double Well

Consider the generic, parametrized, Double-Well System

$$dX = U(X;\theta)\,\mathrm{d}t + \sqrt{2D(X;\theta)}dW \tag{1}$$

where U is a potential field gradient, $U = -\nabla V$ for example something like

$$dX = \underbrace{-\left(aX^3 - bX - A\frac{X}{c}e^{-(X/c)^2/2}\right)}_{U(X;\theta)}\mathrm{d}t + \underbrace{\sigma}_{\sqrt{2D}}dW \tag{2}$$

Whose parameter set is:

$$\theta = \{a, b, A, c, \sigma\}$$

or a subset thereof.

We want to estimate θ from observations $\{X_t\}$. Now assume that we have very high frequency (cts.!) observations, $\{X_{k\Delta t}\}_0^K$ such that $\Delta t \to 0, K\Delta t \to T$, i.e. we know the full realization $\{X_t\}_0^T$ over some interval $[0, T]$.

We want to estimate the values of θ.

1.1 Estimating σ for cts. observations

We now discuss why the estimation of σ can be fairly accurate in the high-frequency (cts.) context.

Suppose, for illustration sake that σ is constant.

Then recall that the quadratic variation formula for an Ito Diffusion (such as X_t) states that:

$$\lim_{\Delta \to 0} \sum_{k \geq 1} (X_k - X_{k-1})^2 = \int_0^T \sigma(X_t)\,\mathrm{d}t \tag{3}$$

i.e for Δ very small and σ constant, we will have:

$$\sigma \approx \frac{1}{T} \sum_{k \geq 1} (X_k - X_{k-1})^2 \tag{4}$$

and the probability of estimating σ incorrectly due to X fluctuations goes to zero.

Note that this formula does not depend on the other possibly unknown parameters in the drift.

Equation (3) can readily be used for inference when σ is NOT constant.

I.e if we think that σ is a nonlinear function of X such as for example $\sigma(X) = \sigma_0 + \tanh(X_t/d)$, then one needs to employ a nonlinear root-finder for the equation:

$$\lim_{\Delta \to 0} \sum_{k \geq 1} (X_k - X_{k-1})^2 = \int_0^T \sigma_0 + \tanh(X_t/d)\,\mathrm{d}t$$

Finally, one can try a non-parametric approach to estimate $\sigma(X)$ over different segments of the realized X_t (e.g. in the high state vs. low state for the double-well potential).

Finally, one can devise all kind of SDE-based estimations to estimate all the parameters, θ simultaneously ...

References