Filtering in 1-D SDEs

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Abstract

Infer an unknown, but assumed smooth signal driving an observed SDE.

Contents

1 SDE Filtering

Having observed the realization of an SDE we want to know what kind of stimulation 'drove' it.

Consider the generic parametrized and perturbed SDE system

$$dX = U(x, u(t); \theta)\,\mathrm{d}t + \sqrt{2D}dW \tag{1}$$

and, for illustration case, specialize eq. (1) to the OU system:

$$dX = \underbrace{(-\beta(X) + u(t))}_{U(x,\alpha;\theta)}\,\mathrm{d}t + \underbrace{\sigma}_{\sqrt{2D}}dW \tag{2}$$

where we know β, σ, but do not know the perturbation: $u(t)$. We want to get an estimate of $u(t)$ given only observations of X.

1.1 Some basic concepts/notation

If the process X is observed continuously, its log 'likelihood' l can be obtained directly via the Radon-Nikodym theorem ([1] eq. 15 e.g.) as

$$l(X|u) = \int_0^T \frac{U(X, u; \theta)}{\sigma^2}dX - \frac{1}{2}\int_0^T \frac{U^2(X, u; \theta)}{\sigma^2}dt \tag{3}$$

2 Specifying the problem via the log-likelihood of the cts. trajectory

In general the problem of determining $u(t)$ given knowledge of X_t can be tackled in several ways. One, following [2],[1] but also work of Susanne++ (Anders' dissertation) is to determine u as the solution to the cts. log-likelihood.

$$u(t) = \arg\max_u l(X|u)$$

In fact, if we assume that $u(t) = \int \dot{u}dt$ i.e that u is some C^1 function for example, we have a classical Calculus of Variations problem (I follow Rogers here, b/c I'm a little uncertain on some of the derivations, I need to recheck them, but it seems right)

Basically, Rogers re-writes the log-likelihood as:

$$l(X|u) = -\frac{1}{2}\int_0^T \frac{\big(\dot{x} - U(x,u;\theta)\big)^2}{\sigma} dt \tag{4}$$

where $\dot{x}$ is obtained by smoothing a fine, but discrete skeleton of X_t with a C^2 spline, such that the time-derivative exists (of course the idealized stochastic process X is not cts. differentiable)

Note that in this case, $\dot{x}$ is actually given data to the problem and the goal is to minimize eq. (4) wrt. u.

if we drop irrelevant constants and let

$$\psi(u,\dot{u},t) = \frac{\big(\dot{x} - U(x,u;\theta)\big)^2}{\sigma}$$

be the 'action', then we have the standard Calculus of Variations problem:

$$u = \arg\min_{u\in C^1}\left\{\int_0^T \psi(u,\dot{u},t)\,\mathrm{d}t\right\}$$

In principle this will result in a pair of BVPs (ODEs) arising from the Euler-Lagrange equations those can be solved using any Optimal Control software (there are many). One can also avoid the Calculus of Variations (indirect methods) and go for direct (numerical Optimal Control) methods. The important thing is that one is solving ODEs, not PDEs!!!

For my reminder, the EL equations look like ($p = \dot{u}$)

$$\partial_u\psi = \partial_t\partial_p\psi = 0!!! \tag{5}$$

Ok, here we have a problem, as in since the derivative $\dot{u}$ does not enter the equation, we have that it does not matter what $\dot{u}$ is, ok, this obvious, we just set

$$\partial_u\psi = 0 \implies \big(\dot{x} - U(x,u;\theta) = 0$$

and we are done... Well, that's the Maximum Likelihood Approach to finding u ...

It would be more interesting if u was another Stochastic Process, then we are really looking into Anders' dissertation. (for the Fitzhugh-Nagumo system)

References

[1] PCB Phillips and Jun Yu. Maximum likelihood and Gaussian estimation of continuous time models in finance. Handbook of financial time series, pages 1–33, 2009.

[2] L C G Rogers. Least-action filtering. (September 2008):1–25, 2013.