Lecture on
Numerical Methods Unit 5

# SOLUTIONS TO LINEAR ALGEBRIC EQUATIONS (Part II)

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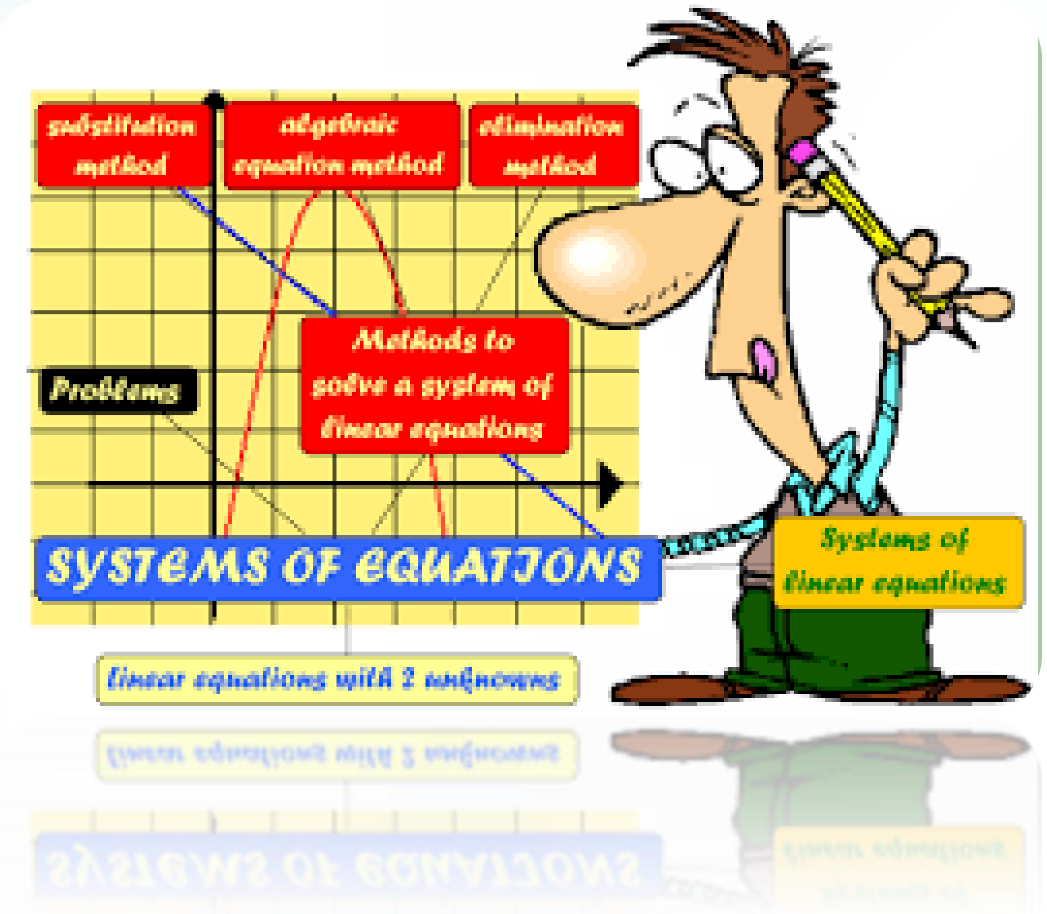
# CONTENTS

- Iterative method
- Gauss- Jacobi method
- Gauss- seidal method
- Eigen values and  eigen vectors inverse power method



# ITERATIVE METHOD

# GAUSS– JACOBI ITERATION METHOD

- In this method, a fixed set of values is used to calculate all the variables and then repeated for the next iteration with the values obtained previously.

**Philipp Ludwig von Seidel**
23 October 1821, Zweibrücken, Germany – 13 August 1896, Munich

**PROCEDURE:**

1. Make sure the diagonal term in the matrix Ax=B has no any zero entries i.e a11, a22, a33…..amn should not be equal to zero. If there are then interchange row and column to get non zero diagonal entries.
2. Make sure that Matrix Ax=B must be diagonally dominant.
3. Calculate first iteration for $x^1_1$, $x^1_2 \ldots x^1_n$ using the generalized formula below with initial approximation of $x^0_1 = x^0_2 \ldots x^0_n = 0$

   If $x^k_1$, $x^k_2 \ldots x^k_n$ are the $k^{th}$ iterates then,

**Johann Carl Friedrich Gauss**
30 April 1777 Brunswick – 23 February 1855 Brunswick

# GAUSS– JACOBI ITERATION METHOD

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \cdots\cdots a_{1n}x_n^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{k} \cdots\cdots a_{2n}x_n^{(k)}\right)$$

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(k)} - a_{n3}x_3^{(k)} \cdots\cdots a_{n,n-1}x_{n-1}^{(k)}\right)$$

Where k= 0,1,2…n

Example

Solve the following equations by Jacobi’s method correct up to 2 decimal places.

a) 15x + 3y – 2z = 85

2x + 10y + z = 51

x – 2y + 8z = 5

# GAUSS– JACOBI ITERATION METHOD

***Solution:***

In the above equations:

$$|a11|=|15| > |3| + |-2|$$

$$|a22|=|10| > |2| + |1|$$

$$|a33|=|8| > |1| + |-2|$$

Hence, the equations are diagonally dominant thus Jacobi's method is applicable. We rewrite the given equations as follows:

$$x = 1/15(85-3y+2z)........eq1$$

$$y = 1/10(51-2x-z)..........eq2$$

$$Z =1/8(5+2y-x)...............eq3$$

Let the initial approximations be: $x^0 = y^0 = z^0 = 0$, using above 3 eqations and Gauss-Jacobi formula we have after 1st iteration,

$$x= 85/15= 17/3= 5.667$$

$$y= 51/10= 5.1$$

$$z= 5/8= 0.625$$

# GAUSS– JACOBI ITERATION METHOD

- Tabulating the values of x, y & z after each iterations we get

| Iteration (k) | x | y | z |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 5.667 | 5.1 | .665 |
| 2 | 4.73 | 3.904 | 1.192 |
| 3 | 5.045 | 4.035 | 1.010 |
| 4 | 4.994 | 3.99 | 1.003 |
| 5 | 5.002 | 4.001 | 0.998 |
| 6 | 5.0 | 4.0 | 1.0 |
| 7 | 5.0 | 4.0 | 1.0 |

Thus, x=5.0, y= 4.0, z=1.0 is the answer.

# GAUSS– JACOBI ITERATION METHOD

***PRACTICE QSN***

a) Use the Jacobi iterative scheme to obtain the solutions of the system of equations correct to three decimal places. (***Ans:*** x= 0.333, y= -0.444, z= 0.555)

$$x + 2y + z = 0$$

$$3x + y - z = 0$$

$$x - y + 4z = 3$$

b) Use Jacobi iterative scheme to obtain the solution of the system of equations correct to 4 decimal places. (***Ans:*** $x_1$= 2.4255 , $x_2$= 3.5730 & $x_3$= 1.9260

$$\begin{aligned} 27x_1 + 6x_2 - x_3 \quad &= 85 \\ 6x_1 + 15\,x_2 + 2x_3 &= 72 \\ x_1 + \quad x_2 + 54x_3 &= 110 \end{aligned}$$

# GAUSS– SEIDAL ITERATION METHOD

- Modification of Gauss- Jacobi method, named after Carl Friedrich Gauss and Philipp Ludwig Von Seidal.

- This method requires fewer iteration to produce the same degree of accuracy.

- This method is almost identical with Gauss –Jacobi method except in considering the iteration equations.

- The sufficient condition for convergence in the Gauss – Seidal method is that the system of equation must be strictly diagonally dominant

Philipp Ludwig Von Seidal

# GAUSS– SEIDAL ITERATION METHOD

- Consider a system of strictly diagonally dominant equation as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \text{Eq1}$$

- If the absolute value of the largest coefficient in each equation is greater than the sum of the absolute values of all the remaining coefficients, then the Gauss-Seidal iteration method will converge.
- If this condition is not satisfied, then Gauss-Seidal method is not applicable.
- Here, in Eq.(1), we assume the coefficient *a*11, *a*22 and *a*33 are the largest coefficients.

# GAUSS– SEIDAL ITERATION METHOD

- We can rewrite Eq.(1) as

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3) \qquad \text{Eq2}$$

$$x_3 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2)$$

Let the initial approximations be $x^0{}_1$ , $x^0{}_2$ and $x^0{}_3$ respectively. The following iterations are then carried out.

Iteration 1: The first improvements of $x_1$, $x_2$ and $x_3$ are obtained as

$$x_{11} = \frac{1}{a_{11}}\left(b_1 - a_{12}\dot{x}_2 - a_{13}x_3^0\right)$$

$$x_{21} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_{11} - a_{23}x_3^0\right) \qquad \text{Eq3}$$

$$x_{31} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_{11} - a_{32}x_{21}\right)$$

# GAUSS– SEIDAL ITERATION METHOD

- Iteration 2: The second improvements of $x_1$, $x_2$ and $x_3$ are obtained as

$$x_{12} = \frac{1}{a_{11}}(b_1 - a_{12}x_{11} - a_{13}x_{31})$$

$$x_{22} = \frac{1}{a_{22}}(b_2 - a_{21}x_{12} - a_{23}x_{31}) \qquad \text{Eq4}$$

$$x_{32} = \frac{1}{a_{33}}(b_3 - a_{31}x_{12} - a_{32}x_{22})$$

- The above iteration process is continued until the values of $x_1$, $x_2$ and $x_3$ are obtained to a pre-assigned or desired degree of accuracy. In general, the initial approximations are assumed as $x^0_1 = x^0_2 = x^0_3 = 0$ .
- Gauss- Seidal method generally converges for any initial values of $x^0_1$ , $x^0_2$ , $x^0_3$ .
- The convergence rate of Gauss-Seidal method is found to be twice to that of Jacobi's method.

# GAUSS– SEIDAL ITERATION METHOD

**Example**

- Solve the following equations by Gauss-Seidal method.

$$8x + 2y - 2z = 8$$

$$x - 8y + 3z = -4$$

$$2x + y + 9z = 12$$

***Solution:***

In the above equations:

$$|8| > |2| + |-2|$$

$$|-8| > |1| + |3|$$

$$|9| > |2| + |1|$$

So, the conditions of convergence are satisfied and we can apply Gauss-Seidal method. Then we rewrite the given equations as follows:

# GAUSS– SEIDAL ITERATION METHOD

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y^0 - c_1 z^0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z^0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

- Let the initial approximations be: $x^0 = y^0 = z^0 = 0$

Iteration 1: $\boxed{x_1} = \frac{d_1}{a_1} = \frac{8}{8} = 1.0$

$$\boxed{y_1} = \frac{1}{b_2}(d_2 - a_2 x_1) = \frac{1}{-8}(-4 - 1 \times 1.0) = 0.625$$

$$\boxed{z_1} = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1) = \frac{1}{9}(12 - 2) = 2 \times 1.0 - 1 \times 0.625) = 1.042$$

# GAUSS– SEIDAL ITERATION METHOD

- Iteration 2:

$$\boxed{x_2} = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1) = \frac{1}{8}(8 - 2\times 0.625 - (-2)\times 1.042) = 1.104$$

$$\boxed{y_2} = \frac{1}{b_2}(d_2 - a_2 x_2 - c_2 z_1) = \frac{1}{-8}(-4 - 1\times 1.104 - 3\times 1.042) = 1.029$$

$$\boxed{z_2} = \frac{1}{c_3}(d_3 - a_3 x_2 - b_3 y_2) = \frac{1}{9}(12 - 2\times 1.104 - 1\times 1.029) = 0.974$$

Similarly, tabulating the result below we have,

| iteration | (k) | x | y | z |
|---|---|---|---|---|
| 1 | 0 | 1.0 | 0.625 | 1.042 |
| 2 | 1 | 1.104 | 1.029 | 0.974 |
| 3 | 2 | 0.986 | 0.989 | 1.004 |
| 4 | 3 | 1.004 | 1.002 | 0.999 |
| 5 | 4 | 0.999 | 1.0 | 1.0 |
| 6 | 5 | 1.0 | 1.0 | 1.0 |

Hence, x=1.0, y=1.0 & z=1.0

# GAUSS– SEIDAL ITERATION METHOD

- ***PRACTICE QSN***
- a) Use the Gauss-Seidal iterative scheme to obtain the solutions of the system of equations correct to three decimal places. (***Ans:*** x= 0.333, y= -0.444, z= 0.555)

$$x + 2y + z = 0$$

$$3x + y - z = 0$$

$$x - y + 4z = 3$$

- b) Use Gauss-Seidal iterative scheme to obtain the solution of the system of equations. (***Ans:*** x1= 1.0 , x2= 2.0, x3= 3.0 & x4= 0.0

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9 \end{aligned}$$

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

## POWER METHOD

- Power method is particularly useful for estimating numerically
- largest or smallest eigenvalue and its corresponding eigenvector.
- The intermediate (remaining) eigenvalues can also be found.
- The power method, which is an iterative method, can be used
- when

  (i) The matrix A of order n has n linearly independent eigenvectors.

  (ii) The eigenvalues can be ordered in magnitude as

  $|\lambda 1| \geq |\lambda 2| \geq |\lambda 3| \geq : : : \geq |\lambda n|$
- When this ordering is adopted, the eigenvalue $\lambda 1$ with the greatest magnitude is called the ***dominant eigenvalue*** of the matrix A
- And the remaining eigenvalues $\lambda 2; \lambda 3; : : : ; \lambda n$ are called the subdominant eigenvalues of A.

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

***Procedure:***

- Let $A = [a_{ij}]$ be a matrix of order nx n.
- We start from any vector $x_0 \neq 0$ with n components such that

$$Ax_0 = x \text{ ...eq1}$$

- In order to get a convergent sequence of eigenvectors simultaneously scaling method is adopted.
- In which at each stage each components of the resultant approximate vector is to be divided by its absolutely largest component.
- Then use the scaled vector in the next step.
- This absolutely largest component is known as numerically largest eigenvalue.

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

- Accordingly x in eq -(1) can be scaled by dividing each of its components by absolutely largest component of it.
- Thus

$$Ax_0 = x = \lambda_1 x_1 \qquad ; x_1 \text{ is the scaled vector of } x$$

- Now scaled vector $x_1$ is to be used in the next iteration to obtain

$$Ax_1 = x = \lambda_2 x_2$$

- Proceeding in this way, finally we get

$$Ax_n = \lambda_{n+1} x_{n+1}; \text{ where } n = 0; 1; 2; 3; :::$$

- Where $\lambda_{n+1}$ is the numerically largest eigenvalue up to desired accuracy and $x_{n+1}$ is the corresponding eigenvector.

**NOTE :** The initial vector $x_0$ is usually (not necessarily) taken as a vector with all components equal to 1.

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

**Characteristics:**

- The main advantage of this method is its simplicity. And it can handle sparse matrices too large to store as a full square array.
- Its disadvantage is its possibly slow convergence

**INVERSE POWER METHOD**

- If $\lambda$ is the eigenvalue of A, then the reciprocal $1/\lambda$ is the eigenvalue of $A^{-1}$.
- The reciprocal of the largest eigenvalue of $A^{-1}$ will be the smallest eigenvalue of A.

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

***Example 1:*** Use power method to estimate the largest eigen value and the corresponding eigen vector of

$$A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$$

***Solution:***

We begin with an initial nonzero approximation of $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We then obtain the following approximations.

**Iterations** **Approximations**

$$Ax_0 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2x_1$$

$$Ax_1 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.75 \end{bmatrix} = 8x_2$$

$$Ax_2 = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -0.75 \end{bmatrix} = \begin{bmatrix} 6.75 \\ -5 \end{bmatrix} = 6.75 \begin{bmatrix} 1 \\ -0.7407 \end{bmatrix} = 6.75x_3$$

Hence, largest eigen value is 6.75 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ -0.7407 \end{bmatrix}$

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

**Example 2:** Use power method to approximate a dominant eigenvalue and the corresponding eigenvector of matrix A below, correct to 3-significant figures, after 11 iterations.

$$A=\begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

***Solution:***

We begin with an initial nonzero approximation of $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

We then obtain the following approximations**.**

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

**Iterations** **Approximations**

$Ax_0 = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix} = 12.00\begin{pmatrix} 0.5000 \\ 0.6667 \\ 1.000 \end{pmatrix}$ $= 12.00\, x_1$

$Ax_1 = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}\begin{pmatrix} 0.5000 \\ 0.6667 \\ 1.000 \end{pmatrix} = \begin{pmatrix} 2.3337 \\ 3.3339 \\ 5.3342 \end{pmatrix} = 5.3342\begin{pmatrix} 0.4375 \\ 0.6250 \\ 1.000 \end{pmatrix}$ $= 5.3342\, x_2$

similarly, $Ax_3 = 5.3342\begin{pmatrix} 0.4375 \\ 0.6250 \\ 1.000 \end{pmatrix}$ $Ax_4 = 4.500\begin{pmatrix} 0.4167 \\ 0.6111 \\ 1.000 \end{pmatrix}$, $Ax_5 = 4.222\begin{pmatrix} 0.4079 \\ 0.6053 \\ 1.000 \end{pmatrix}$, $Ax_6 = 4.105\begin{pmatrix} 0.4038 \\ 0.6026 \\ 1.000 \end{pmatrix}$,

$Ax_7 = 4.051\begin{pmatrix} 0.4019 \\ 0.6013 \\ 1.000 \end{pmatrix}$

$Ax_8 = 4.025\begin{pmatrix} 0.4009 \\ 0.6006 \\ 1.000 \end{pmatrix}$

Therefore, the dominant eigenvalue, correct to 3-significant figure is 4.01 and the corresponding eigenvector is $\begin{bmatrix} 0.400 \\ 0.600 \\ 1.00 \end{bmatrix}$

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

***Example 3:*** Use inverse power method to approximate the least eigenvalue and the corresponding eigenvector of given matrix, correct to 3-significant figures, after 6 iterations.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$$

***Solution:*** We first obtain the inverse of the given matrix A, which can be easily calculated as

$$B = A^{-1} = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$$

We now calculate the dominant eigenvalue and its corresponding eigenvector of matrix B= $A^{-1}$ by power method (as described in earlier examples) by taking an initial non-zero approximation of $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

The series of approximations are obtained as follows:

# EIGEN VALUES AND EIGEN VECTORS USING POWER METHOD & INVERSE POWER METHOD

$$x_1 = 1.000\begin{pmatrix}-1.000\\1.000\\0.000\end{pmatrix}, x_2 = 42.00\begin{pmatrix}1.000\\-0.8333\\0.2143\end{pmatrix}, x_3 = -37.93\begin{pmatrix}1.000\\-0.8343\\0.2141\end{pmatrix}$$

$$x_4 = -37.95\begin{pmatrix}1.000\\-0.8343\\0.2141\end{pmatrix}, x_5 = -37.95\begin{pmatrix}1.000\\-0.8343\\0.2141\end{pmatrix}.$$

- Thus, the dominant eigenvalue of B= $A^{-1}$ is -37.95 and the corresponding eigenvector is $\begin{pmatrix}1.000\\-0.8343\\0.2141\end{pmatrix}$
- Therefore, by inverse power method, the least eigenvalue of A is

  A = 1/-3.795 = -0.02635
- and its corresponding eigenvector is $\begin{pmatrix}1.000\\-0.8343\\0.2141\end{pmatrix}$ (same as that of $A^{-1}$ )


Thank You !