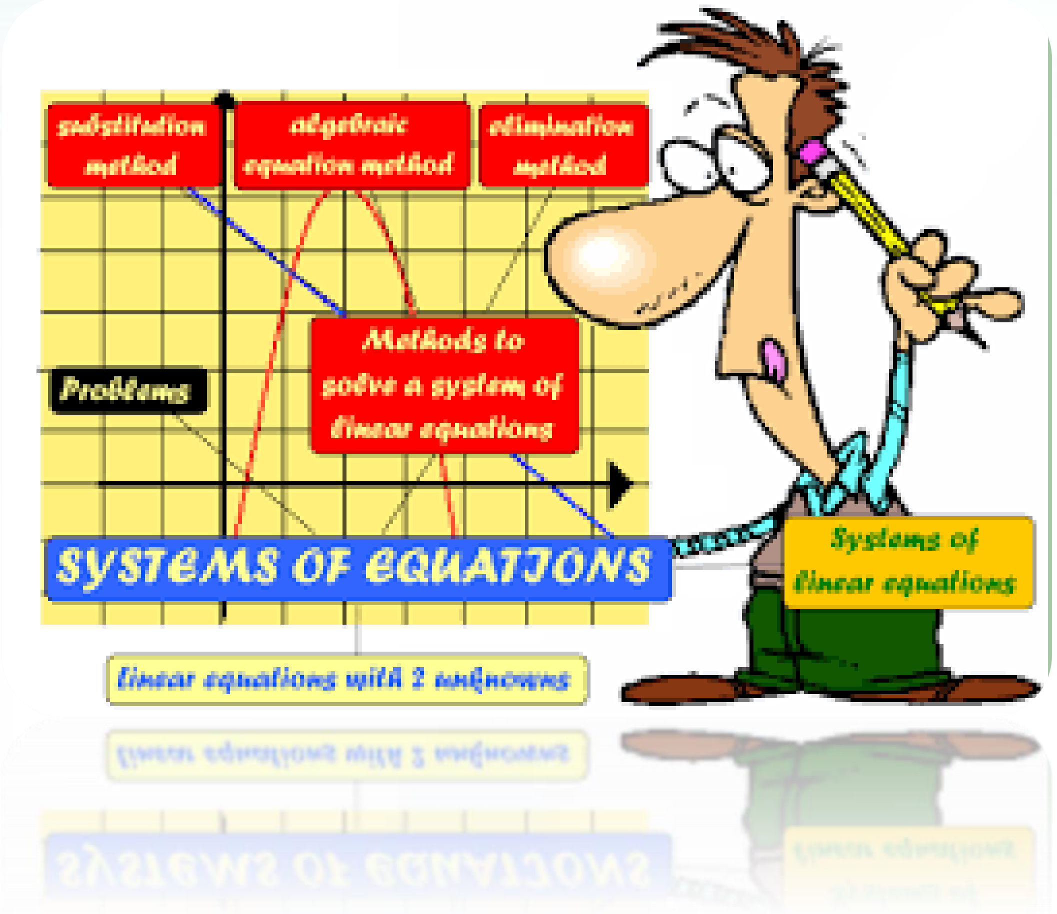
Lecture on
Numerical Methods Unit 5

SOLUTIONS TO LINEAR ALGEBRIC EQUATIONS (Part I)

Prepared By: Er. Sarbesh Chaudhary
H.O.D (Electronics Department)
MMP

CONTENTS

- Basic Fundamentals
- Systems of Linear equations
- Solutions of Linear Equations
- Direct or EliminationMethods
    - Basic Gauss Elimination
    - Gauss Elimination with partial pivoting
    - Gauss Jordon method
    - LU decomposition methods
        - Do Little Algorithm
        - Crout Algorithm
    - Matrix Inversion Method
    - Cholesky’s Method



BASIC FUNDAMENTALS

- **VECTORS**

Vector : a one dimensional array of numbers

Examples :

row vector $\begin{bmatrix} 1 & 4 & 2 \end{bmatrix}$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\textit{Identity vectors } e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

BASIC FUNDAMENTALS

- **MATRICES**

Matrix : a two dimensional array of numbers

Examples :

$$\text{zero matrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{identity matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{diagonal} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \quad \text{Tridiagonal} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

BASIC FUNDAMENTALS

- **Upper and Lower Triangular Matrix**

- **Upper Triangular Matrix**
- It is a square matrix in which all the items under the main diagonal are zero.

Example:

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

- **Lower Triangular Matrix**
- It is a square matrix in which all the elements above the main diagonal are zero.

Example:

$$D = \begin{bmatrix} -22 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

BASIC FUNDAMENTALS

- **Augmented Matrix**
- It is called extended or augmented matrix that is formed by the Coefficient matrix and the vector of independent terms, which are usually separated with a dotted line Example:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 5 & 2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$$

- The augmented matrix A|B is represented as follows:

$$A|B = \left(\begin{array}{ccc:c} 1 & 3 & 2 & 4 \\ 2 & 0 & 1 & 3 \\ 5 & 2 & 2 & 1 \end{array}\right)$$

BASIC FUNDAMENTALS

- **Transpose Matrix**

Let any matrix $A=(a_{ij})$ of mxn order, then A^T is its transpose if the A rows are the A^T columns

Example:

$$A_{2\times 3} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -4 & 3 \end{pmatrix} \Rightarrow A^t{}_{3\times 2} = \begin{pmatrix} 1 & 0 \\ 2 & -4 \\ -2 & 3 \end{pmatrix}$$

$$B_{1\times 2} = 2 \quad -9 \quad \Rightarrow \quad B^t{}_{2\times 1} = \begin{pmatrix} 2 \\ -9 \end{pmatrix}$$

Properties:

a. $(A^t)^t = A$

b. $(A+B)^t = A^t + B^t$

c. $(A \cdot B)^t = B^t \cdot A^t$

d. $(k \cdot A)^t = k \cdot A^t,\ k \in \square$

BASIC FUNDAMENTALS

- **Symmetric Matrix**

It is a square matrix in which the elements are symmetric about the main diagonal

Example:

$$A = \begin{bmatrix} 1 & \boxed{4} & \boxed{5} \\ \boxed{4} & -9 & \boxed{6} \\ \boxed{5} & \boxed{6} & 3 \end{bmatrix}$$

If A is a symmetric matrix, then:

a. The product A x A^T is defined and is a symmetric matrix.

b. The sum of symmetric matrices is a symmetric matrix.

c. The product of two symmetric matrices is a symmetric matrix if the matrices commute

BASIC FUNDAMENTALS

- **Determinant of a MATRICES**
- If a matrix has a line (row or column) of zeros, the determinant is zero.
- If a matrix has two equal rows or proportional, the determinant is null
- If we permute two parallels lines of a square matrix, its determinant changes sign.
- If we multiply all elements of a determinant line by a number, the determinant is multiplied by that number.
- If a matrix line is added another line multiplied by a number, the determinant does not change.
- The determinant of a matrix is equal to its transpose,
- If A has inverse matrix, A^{-1}, it is verified that: $\det(A^{-1}) = 1/\det(A)$

BASIC FUNDAMENTALS

Example :

$$\det\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2\begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} -1\begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12 + 5) - 1(15 - 0) = -82$$

BASIC FUNDAMENTALS

- **Addition of Matrices**

The addition of two matrices A and B is possible only if they have the same size

$$C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$$

- **Multiplication of Matrices**

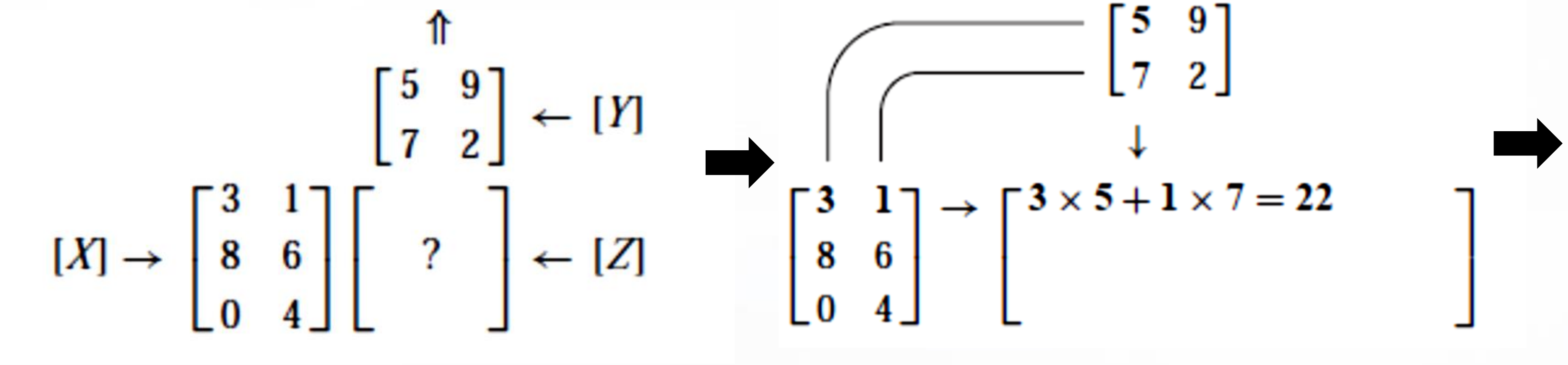
Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

The product $C = AB$ is defined only if $m = p$

$$C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \forall i, j$$

BASIC FUNDAMENTALS

- Suppose that we want to multiply [X] by [Y] to yield [Z],
- [Z] = [X][Y] = $\begin{bmatrix} 3 & 1 \\ 8 & 6 \\ 0 & 4 \end{bmatrix}$ x $\begin{bmatrix} 5 & 9 \\ 7 & 2 \end{bmatrix}$
- A simple way to visualize the computation of [Z] is to raise [Y],as in



$$[Z] = \begin{bmatrix} 22 & 29 \\ 82 & 84 \\ 28 & 8 \end{bmatrix}$$

SYSTEMS OF LINEAR EQUATIONS

A system of linear equations can be presented in different forms

$$\left.\begin{array}{l} 2x_1 + 4x_2 - 3x_3 = 3 \\ 2.5x_1 - x_2 + 3x_3 = 5 \\ x_1 \qquad\qquad - 6x_3 = 7 \end{array}\right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form — Matrix form

SOLUTIONS OF LINEAR EQUATIONS

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is a solution to the following equations :}$$

$$x_1 + x_2 = 3$$
$$x_1 + 2x_2 = 5$$

- A set of equations is **inconsistent** if there exists no solution to the system of equations:

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 5$$

These equations are inconsistent

SOLUTIONS OF LINEAR EQUATIONS

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 6$$

have infinite number of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \infty \\ 0.5(3-\infty) \end{bmatrix} \text{ is a solution for all } \infty$$

SOLUTIONS OF LINEAR EQUATIONS

- There are two classes of methods for solving system of linear, algebraic equations:
  1.Direct methods.
  2. Indirect or Iterative methods
- **Direct methods**
- They transform the original equation into equivalent equations (equations that have the same solution) that can be solved more easily.
- The transformation is carried out by applying certain operations.
- The solution does not contain any *truncation errors* but the <u>*round off errors*</u> is introduced due to floating point operations.

SOLUTIONS OF LINEAR EQUATIONS

- **Indirect methods**
- *Iterative* or *indirect methods*, start with a guess of the solution **x**, and then repeatedly refine the solution until a certain convergence criterion is reached.

- Generally less efficient than direct methods since, large number of operations or iterations required.

- Iterative procedures are self-correcting, meaning that round off errors (or even arithmetic mistakes) in one iteration cycle are corrected in subsequent cycles.

- The solution contains *truncation error*.
- A serious ***drawback*** of iterative methods is that they do not always converge to the solution.

SOLUTIONS OF LINEAR EQUATIONS

- Some of the Direct & Indirect Methods are Listed below:
- **Direct Methods:**
  1. Gauss Elimination with partial pivoting
  2. Gauss Jordon method
  3. LU decomposition methods
  4. Do Little Algorithm
  5. Crout Algorithm
  6. Matrix Inversion Method
  7. Cholesky's Method
- **Indirect or Iterative Methods:**
  1. Jacobi's Iteration Method
  2. Gauss-Seidal Iteration Method

DIRECT METHODS

GAUSS ELIMINATION METHOD

- **Gaussian Elimination Method procedure:**

1. Write the augmented matrix for the system of linear equations.

2. Use elementary row operations on the augmented matrix [A|B] to transform A into upper triangular form. If a zero is located on the diagonal, switch the rows until a nonzero is in that place. If you are unable to do so, stop; the system has either infinite or no solutions.

3. Use back substitution to find the solution of the problem.

Karl Friedrich Gauss
Great mathematician 19th century

GAUSS ELIMINATION METHOD

- We illustrate the method using the 3×3 system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad \text{.............} (1)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step 1: The ***augmented matrix*** of the system is $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$ ....(2)

Step 2: Transform A into ***upper triangular*** form

a)First stage of elimination

We assume $a_{11} \neq 0$. This element a_{11} in the 1×1 position is called the first pivot. Performing the elementary row operations $R_2 - (a_{21}/a_{11})R_1$ and $R_3 - (a_{31}/a_{11})R_1$ respectively.

We obtain the new augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & b_3^{(1)} \end{array}\right] \qquad \text{...............}(3)$$

GAUSS ELIMINATION METHOD

b) Second stage of elimination

- We assume $a_{22}^{(1)} \neq 0$. This element $a_{22}^{(1)}$ in the 2 × 2 position is called the *second pivot.*
- performing the elementary row operation $R_3 - (a_{32}^{(1)} / a_{22}^{(1)})R_2$. *We obtain the new* augmented matrix as

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ 0 & 0 & a_{33}^{(2)} & b_3^{(2)} \end{array}\right] \quad \text{..........(4)}$$

- The element $a_{33}^{(2)} \neq 0$ is called the *third pivot*.

Step 3: *Back Substitution*

From Equation 4 we have

- From the third row, we get $\boldsymbol{x_3 = b_3^{(2)} / a_{33}^{(2)}}$
- From the second row, we get $\boldsymbol{x_2 = (b_2^{(1)} - a_{23}^{(1)} x_3)/a_{22}^{(1)}}$
- From the first row, we get $\boldsymbol{x_1 = (b1 - a_{12}x_2 - a_{13} x_3)/a_{11}}$

GAUSS ELIMINATION METHOD

Example

1. Write the augmented matrix for the system of linear equations.

$$\begin{aligned} 2y + z &= 4 \\ x + y + 2z &= 6 \\ 2x + y + z &= 7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

2. Use elementary row operations on the augmented matrix [A|b] to transform A into upper triangular form.

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right]\begin{array}{l} (r_2) \\ (r_1) \\ \end{array} \quad \begin{array}{c}\text{Change row 1 to row 2} \\ \text{and vice versa}\end{array} \quad \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right]\begin{array}{l} \\ \\ (r_3) + (-2 * r_1) \end{array}$$

GAUSS ELIMINATION METHOD

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right]_{(r_3)+(\frac{1}{2}*r_2)} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -5/2 & -3 \end{array}\right]$$

3. Use back substitution to find the solution of the problem

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -5/2 & -3 \end{array}\right]$$

The last row in the augmented matrix represents the equation: $-\frac{5}{2}z = -3 \Rightarrow z = \frac{6}{5}$

The second row of the augmented matrix represents the equation:

$$2y + z - 4 \Rightarrow y - \frac{4-z}{2} - \frac{4-6/5}{2} \Rightarrow y - \frac{7}{5}$$

Finally, the first row of the augmented matrix represents the equation

$$x + y + 2z = 6 \Rightarrow x = 6 - y - 2z = 6 - 7/5 - 2\ 6/5 \Rightarrow x = \frac{11}{5}$$

GAUSS ELIMINATION METHOD

- **Gaussian Elimination Method with Partial Pivoting**
- Gauss elimination method fails if any one of the pivots in the equations becomes zero.
- To overcome this difficulty, the equations are to be rewritten in a slightly different order such that the pivots are not zero.

Partial pivoting Procedure:

Step 1: The numerically largest coefficient of x1 is selected from all the equations as pivot and the corresponding equation becomes the first equation.

Step 2: The numerically largest coefficient of x2 is selected from all the remaining equations as pivot and the corresponding equation becomes the second equation.

- This process is repeated till an equation into a simple variable is obtained.
- Remaining 1^{st} and last method is same as GEM.

GAUSS ELIMINATION METHOD

- **Example :** Solve the system of equations using the Gauss elimination with partial pivoting.

$$x1 + 10x2 - x3 = 3$$

$$2x1 + 3x2 + 20x3 = 7$$

$$10x1 - x2 + 2x3 = 4$$

- ***Solution:***

Step 1: We have the augmented matrix as $\left[\begin{array}{ccc|c} 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \\ 10 & -1 & 2 & 4 \end{array}\right]$

Step 2: We perform the following elementary row transformations and do the eliminations.

$$R_1 \leftrightarrow R_3 : \left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 2 & 3 & 20 & 7 \\ 1 & 10 & -1 & 3 \end{array}\right]. \ R_2 - (R_1/5),\ R_3 - (R_1/10) :$$

$$\left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & 3.2 & 19.6 & 6.2 \\ 0 & 10.1 & -1.2 & 2.6 \end{array}\right]. \quad R_2 \leftrightarrow R_3 : \left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & 10.1 & -1.2 & 2.6 \\ 0 & 3.2 & 19.6 & 6.2 \end{array}\right].$$

$$R_3 - (3.2/10.1)R_2 : \left[\begin{array}{ccc|c} 10 & -1 & 2 & 4 \\ 0 & 10.1 & -1.2 & 2.6 \\ 0 & 0 & 19.98020 & 5.37624 \end{array}\right].$$

GAUSS ELIMINATION METHOD

Step 3:Back substitution gives the solution.

Third equation gives **x3** = 5.37624/ 19.98020 = 0.26908.

Second equation gives **x2** = (1/10.1)(2.6 + 1.2x3) =(1/10.1)(2.6 + 1.2(0.26908)) = 0.28940.

First equation gives **x1** = (1/10)(4 + x2 – 2x3) = (1/10)(4 + 0.2894 – 2(0.26908)) = 0.37512.

<u>PRACTICE QSN:</u>

1) Solve the following equations by Gauss elimination method:

a) $2x + 4y - 6z = -4$

$x + 5y + 3z = 10$

$x + 3y + 2z = 5$ ***Ans:*** x=-3, y=-2, z=1

b)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 4x_1 + 4x_2 + x_3 + x_4 &= 11 \\ x_1 - x_2 - x_3 + 2x_4 &= 0 \\ 2x_1 + x_2 + 2x_3 - 2x_4 &= 2 \end{aligned}$$

Ans: *x1=1, x2=2, x3=-1, x4=0*

2) Solve the following equations using the Gauss elimination with partial pivoting.

a)
$$\begin{aligned} 2x_1 + x_2 + x_3 - 2x_4 &= -10 \\ 4x_1 + 2x_3 + x_4 &= 8 \\ 3x_1 + 2x_2 + 2x_3 &= 7 \\ x_1 + 3x_2 + 2x_3 - x_4 &= -5 \end{aligned}$$

Ans: x1=5, x2=6, x3=-10,x4=8

b) $5x + 12y + 9z = 5$

$8x + 11y + 20z = 35$

$16x + 5y + 7z = 29$

Ans: x = 1.426, y = -1.808, z = 2.174

GAUSS JORDAN METHOD

- Gauss-Jordan method is a modification of Gauss elimination method.
- The series of operations performed are quite similar to the Gauss elimination method.
- In the Gauss elimination method, an upper triangular matrix is derived while in the Gauss-Jordan method an identity matrix is derived. Hence, back substitutions are not required.

Wilhelm Jordan

Famous German Geodesist

GAUSS JORDAN METHOD

- **PROCEDURE:**

1. Write the augmented matrix for the system of linear equations.

2. Use elementary row operations on the augmented matrix [A|b] to transform A into diagonal form. If a zero is located on the diagonal, switch the rows until a nonzero is in that place. If you are unable to do so, stop; the system has either infinite or no solutions.

3. By dividing the diagonal element and the right-hand-side element in each row by the diagonal element in that row, make each diagonal element equal to one.

GAUSS JORDAN METHOD

- **Example:**
- **Step 1:** Write the augmented matrix for the system of linear equations.

$$\begin{aligned} 2y + z &= 4 \\ x + y + 2z &= 6 \\ 2x + y + z &= 7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right]$$

- **Step 2:** Use elementary row operations on the augmented matrix [A|b] to transform A into diagonal form.

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 7 \end{array}\right] \begin{array}{l} (r_2) \\ (r_1) \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 2 & 1 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \\ (r_3)+(-2*r_1) \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 4 \\ 0 & -1 & -3 & -5 \end{array}\right] \begin{array}{l} (r_1)+(-\frac{1}{2}*r_2) \\ \\ (r_3)+(\frac{1}{2}*r_2) \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 4 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & -\frac{5}{2} & -3 \end{array}\right] \begin{array}{l} (r_1)+(\frac{3}{5}*r_3) \\ (r_2)+(\frac{2}{5}*r_3) \\ \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{5} \\ 0 & 2 & 0 & -\frac{14}{5} \\ 0 & 0 & -\frac{5}{2} & -3 \end{array}\right]$$

GAUSS JORDAN METHOD

- **Step 3:** By dividing the diagonal element and the right-hand-side element in each row by the diagonal element in that row, make each diagonal element equal to one.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/5 \\ 0 & 2 & 0 & -14/5 \\ 0 & 0 & -5/2 & -3 \end{array}\right] \begin{array}{l} \\ (r_2)*(-1/2) \\ (r_3)*(-2/5) \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/5 \\ 0 & 1 & 0 & 7/5 \\ 0 & 0 & 1 & 6/5 \end{array}\right]$$

- Hence, $x=\frac{11}{5}$, $y=\frac{7}{5}$, and $z=\frac{6}{5}$

PRACTICE QSN: Use to Gauss-Jordan method solve the system

- $$\begin{aligned} 10x+y+z&=12 \\ 2x+10y+z&=13 \\ x+4y+3z&=5 \end{aligned}$$ ***Ans***: x=y=z=1

b) $$\begin{aligned} x + 3y + 2z &= 17 \\ x + 2y + 3z &= 16 \\ 2x - y + 4z &= 13 \end{aligned}$$ ***Ans:*** *x=4,y=3,z=2*

LU DECOMPOSITION METHOD

- Linear algebraic notation can be rearranged to give

$$A\,X - B = 0 \qquad eq1$$

- Suppose that this equation could be expressed as an upper triangular system:

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \qquad eq2$$

- Elimination is used to reduce the system to upper triangular form. The above equation can also be expressed in matrix notation and rearranged to give

$$U\,X - D = 0 \qquad eq3$$

LU DECOMPOSITION METHOD

- Now, assume that there is a lower diagonal matrix with 1's on the diagonal,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

- That has the property that when Eq. 3 is pre-multiplied by it, Eq. 1 is the result. That is,

$$L\,U\,X - D = A\,X - B \qquad eq5$$

- If this equation holds, it follows from the rules for matrix multiplication that

$$L\,U = A \qquad eq6$$

$$L\,D = B \qquad eq7$$

LU DECOMPOSITION METHOD

- A two-step strategy for obtaining solutions can be based on Eqs. 3, 6 & 7.
- **LU decomposition step. [A] is factored or "decomposed"** into lower [L] and upper [U] triangular matrices.
- **Substitution step. [L] and [U] are used to determine a** solution {X} for a right-hand side {B}. This step itself consists of two steps. First, Eq. 7 is used to generate an intermediate vector {D} by forward substitution.
- Then, the result is substituted into Eq. 3, which can solved by back substitution for [X].In the other hand, Gauss Elimination can be implemented in this way.

LU DECOMPOSITION METHOD

- LU-decomposition of a non-singular matrix (when it exists) is not unique. In practice we use one of the following alternatives to get unique factorization:

  (a) **Crout's formalism: All the diagonal elements of U is chosen 1.**

  (b) **Dolittle's formalism: All the diagonal elements of L is chosen 1.**

Crout's Method

- Consider the matrix equation of the system of 3 equations in 3 unknowns

$$AX = B$$

- We write matrix A as a product of an Upper and Lower Triangular matrices[1]

$$A = LU$$

Prescott Durand Crout

American Mathematician

$$\text{Where, } L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \; and \; U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

LU DECOMPOSITION METHOD

- Since $\boldsymbol{A} = \boldsymbol{LU}$ $\therefore \boldsymbol{AX} = \boldsymbol{B}$ eq(1) Gives $\boldsymbol{LUX} = \boldsymbol{B}$ eq(2)
- Let us take $\boldsymbol{UX} = \boldsymbol{Y}$ eq(3)
- Y is some unknown matrix which is to be evaluated
- Then $\mathbf{LY} = \mathbf{B}$ eq(4)
- Therefore to find the solution of the system (1) we will have to solve (4) and then (3), but before that we will have to evaluate the values of L and U

PROCEDURE:

- Use the following steps to solve the System of Linear algebraic equations.

  - **Step 1:** Write $A = LU = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

LU DECOMPOSITION METHOD

- **Step 2:** Calculate the Product of L and U

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

- • **Step 3:** write L and U
- • **Step 4:** Solve LY = B by forward substitution
- • **Step 5:** Solve UX = Y by backward substitution

LU DECOMPOSITION METHOD

Example: Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 11 \\ 5 & 14 & 12 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 5 \\ 21 \\ 15 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- (i) Determine a lower triangular matrix **L and an upper triangular matrix U such that** L U = A using Crout's algorithm.
- (ii) Use the above factorization to solve the equation **A X = B.**

Solution: i)

$$\text{Let} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 11 \\ 5 & 14 & 12 \end{pmatrix} = \mathbf{L}\,\mathbf{U} = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix} \begin{pmatrix} 1 & l & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & al & am \\ b & bl+d & bm+dn \\ c & cl+e & cm+en+f \end{pmatrix}$$

LU DECOMPOSITION METHOD

- Equating the corresponding elements of the two matrices, we have

$$
\begin{aligned}
&a = 1 \\
&b = 3 \\
&c = 5 \\
&a\,l = 2 &&\text{or} && l = \frac{2}{1} = 2 \\
&a\,m = 3 &&\text{or} && m = 3 \\
&b\,l + d = 4 &&\text{or} && d = 4 - 3(2) = -2 \\
&c\,l + e = 14 &&\text{or} && e = 14 - 5(2) = 4 \\
&b\,m + d\,n = 11 &&\text{or} && n = \frac{1}{-2}\left(11 - 3(3)\right) = -1 \\
&c\,m + e\,n + f = 12 &&\text{or} && f = 12 - 5(3) - 4(-1) = 1
\end{aligned}
$$

- Thus,

$$
\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 5 & 4 & 1 \end{pmatrix} \qquad \mathbf{U} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}
$$

LU DECOMPOSITION METHOD

- (ii) The equation can be written as **AX = LUX = LY = B**

Where, **UX = Y** and $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

- Consider the solution of **LY = B** $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \\ 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 21 \\ 15 \end{pmatrix}$

- Using forward elimination, we have *y1 = 5, y2 = -3, y3 = 2*
- Now consider the solution of **UX = Y**

LU DECOMPOSITION METHOD

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$$

- Using backward elimination, we have *x =1, y = -1, z = 2*

and hence

$$X = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

PRACTICE QSN: Solve the following system of equations by Crout's Method

$2x1 - 4x2 + x3 = 4,$

$6x1 + 2x2 - x3 = 10,$

$-2x1 + 6x2 - 2x3 = -6$

MATRIX INVERSION METHOD

- Consider a set of three simultaneous linear algebraic equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad eq(1)$$

- Equation (1) can be expressed in the matrix form

$$\mathbf{Ax = b} \qquad \text{eq(2)}$$

- Pre-multiplying by the inverse $\mathbf{A^{-1}}$, we obtain the solution of x as

$$\mathbf{x = A^{-1}b} \qquad \text{eq(3)}$$

- If the matrix A is non-singular, that is, if det (A) is not equal to zero, then Eq. (3) has a unique solution.

MATRIX INVERSION METHOD

- The solution for *x1 is*

$$x_1 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = \frac{1}{|\mathbf{A}|}\left\{ b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_2 \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_3 \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \right\}$$

$$= \frac{1}{|\mathbf{A}|}\{ b_1 C_{11} + b_2 C_{21} + b_3 C_{31} \}$$

- where A is the determinant of the coefficient matrix A, and *C11, C21 and C31 are the cofactors of A* corresponding to element 11, 21 and 31.
- We can also write similar expressions for *x2 and x3 by replacing the* second and third columns by the *y column respectively. Hence, the complete solution can be written in* matrix form as follows:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

MATRIX INVERSION METHOD

or $$\{x\} = \frac{1}{|\mathbf{A}|}[C_{ji}]\{b\} = \frac{1}{|\mathbf{A}|}[adj\ \mathbf{A}]\{b\}$$

Hence $$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} adj\ \mathbf{A} \text{ and Adj } \mathbf{A} = \mathbf{A}^{-1} \text{ abs } [\mathbf{A}]$$

- Although this method is quite general but it is not quite suitable for large systems because evaluation of A^{-1} *by co-factors becomes very cumbersome.*

Example: Obtain the solution of the following linear simultaneous equations by the matrix inversion method.

(*a*) $$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

MATRIX INVERSION METHOD

Solution:

(*a*)
$$\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1}|-1| = -1$$
$$C_{12} = (-1)^{1+2}|4| = -4$$
$$C_{21} = (-1)^{1+}|3| = -3$$
$$C_{22} = (-1)^{2+2}|1| = 1$$

- Hence $\mathbf{C} = \begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}$ $\quad \mathbf{C}^{\mathbf{T}} = \begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$ $\quad \mathbf{A}^{-1} = \dfrac{\mathbf{C}^{\mathbf{T}}}{|\mathbf{A}|} = \dfrac{-1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}$

Hence
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -1 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 12 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -5 & -36 \\ -20 & +12 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} -41 \\ -8 \end{bmatrix}$$

Therefore,
$$x_1 = \frac{-41}{-13} = 3.15$$

and
$$x_2 = \frac{-8}{-13} = 0.62$$

MATRIX INVERSION METHOD

PRACTICE QSN: Obtain the solution of the following linear simultaneous equations by the matrix inversion method.

$$\begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

Ans: *x1 = 0, x2 = 1, x3 = 2.*

Cholesky's Method

- Cholesky's decomposition method is faster than the *LU decomposition. There is no need for pivoting. If the* decomposition fails, the matrix is not positive definite.
- Consider the system of linear equations:

$$a11x1 + a12x2 + a13x3 = b1$$

$$a21x1 + a22x2 + a23x3 = b2$$

$$a31x1 + a32x2 + a33x3 = b3$$

- The above system can be written as

$$\mathbf{Ax = b} \qquad eq(1)$$

Andre Louis Cholesky

French Military & Mathematician

Where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{eq(2)}$$

Cholesky's Method

- Let **A = *LU***

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Equation (1) can be written as $LUX = b$ *eq(3)*
- If we write $UX = V$
- Equation (3) becomes $LV = b$ *eq(4)*
- Equation (4) is equivalent to the system

$v1 = b1$

$l21v1 + v2 = b2$

$l31v1 + l32v2 + v3 = b3$ *eq(5)*

Cholesky's Method

- The above system can be solved to find the values of *v1, v2 and v3 which give us the matrix V.*

$$UX = V$$

- then becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1$$

$$u_{22}x_2 + u_{23}x_3 = v_2$$

$$u_{33}x_3 = v_3 \qquad eq(6)$$

- which can be solved for *x3, x2 and x1 by the backward substitution process.*
- In order to compute the matrices *L and U, we write Eq. (1) as*

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Cholesky's Method

- Multiplying the matrices on the left and equating the corresponding elements of both sides, we obtain

$$u_{11} = a_{11},\ u_{12} = a_{12},\ u_{13} = a_{13}$$

$$\left.\begin{aligned} l_{21}u_{11} &= a_{21} \Rightarrow l_{21} = \frac{a_{21}}{a_{11}} \\ l_{31}u_{11} &= a_{31} \Rightarrow l_{31} = \frac{a_{31}}{a_{11}} \end{aligned}\right\}$$

$$\left.\begin{aligned} l_{21}u_{12} + u_{22} &= a_{22} \Rightarrow u_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12} \\ l_{21}u_{13} + u_{23} &= a_{23} \Rightarrow u_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13} \end{aligned}\right\}$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{1}{u_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}}a_{12}\right]$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$$

- The value of *u33 can be computed from last eq above*

Cholesky's Method

- To obtain the elements of *L and U, we first find the first row of U and the first column of L.*
- *Then, we* determine the second row of *U and the second column of L. Finally, we compute the third row of U.*

EXAMPLE:

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$

Solution:

We have

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

Cholesky's Method

Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}$

Multiplying and equating we get:

$$l \times u_{11} = 2 \Rightarrow u_{11} = 2$$

$$l \times u_{12} = 1 \Rightarrow u_{12} = 1$$

$$l \times u_{13} = 4 \Rightarrow u_{13} = 4$$

$$l_{21} \times u_{11} = 8 \Rightarrow l_{21} = 8/u_{11} = 8/2 = 4$$

$$l_{21} \times u_{12} + u_{22} = -3 \Rightarrow \boxed{u_{22}} = -3 - l_{21} \times u_{12} = -3 - 4 \times 1 = -7$$

$$l_{21} \times u_{13} + u_{23} = 2 \Rightarrow \boxed{u_{23}} = 2 - l_{21} \times u_{13} = 2 - 4 \times 4 = -14$$

$$l_{31} \times u_{11} = 4 \Rightarrow \boxed{l_{31}} = \frac{4}{u_{11}} = \frac{4}{2} = 2$$

Cholesky's Method

$$l_{31}\times u_{12} + l_{32}\times u_{22} = 11 \Rightarrow \boxed{l_{32}} = \frac{11 - l_{31}\times u_{12}}{u_{22}} = \frac{11-2\times1}{-7} = -\frac{9}{7}$$

$$l_{31}\times u_{13} + l_{32}\times u_{23} + l\times u_{33} = -1 \Rightarrow \boxed{u_{33}} = -1 - l_{31}\times u_{13} - l_{32}\times u_{23} = -1 - 2\times 4 - \left(-\frac{9}{7}(-14)\right) = -27$$

We get,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}$$

and the given system can be written as,

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

Cholesky's Method

- Writing: $LV = B$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & -\frac{9}{7} & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

- Which gives,

$$\boxed{V_1} = 12 \Rightarrow \boxed{V_2}$$

$$4V_1 + V_2 = 20 \Rightarrow \boxed{V_2} = 20 - 4 \times 12 = -28$$

$$2V_1 - \frac{9}{7}V_2 + V_3 = 33 \Rightarrow \boxed{V_3} = 33 + \frac{9}{7}(-28) - 2 \times 12 = -27$$

- The solution to the original system is given by:

UX = V

$$\begin{bmatrix} 2 & 1 & 4 \\ 0 & -7 & -14 \\ 0 & 0 & -27 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ -27 \end{bmatrix}$$

From backward substitution we have

2x + y + 4z = 12
−7y − 14z = −28
−27z = −27 ⇒ z = 1

Substituting z in eq b we get,

7y = 28 – 14 × 1 ⇒ y =14/7 ⇒ y = 2

Now substituting y & z value in eq a we get,

2x = 12 – y – 4z = 12 – 2 – 4 × 1

or, x= 6/2 ⇒ x = 2

Cholesky's Method

PRACTICE QSN:

a)

$$x_1 + x_2 + x_3 - x_4 = 2$$
$$x_1 - x_2 - x_3 + 2x_4 = 0$$
$$4x_1 + 4x_2 + x_3 + x_4 = 11$$
$$2x_1 + x_2 + 2x_3 - 2x_4 = 2$$

Ans: $x_4 = 0,\ x_3 = -1,\ x_2 = 2,\ x_1 = 1.$

b)

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16$$

Ans: $x=1,\ y=3,\ z=5$


Thank You !