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Errors in Numerical calculation :-

1) True error :-

It is the difference between the true value (exact value) & the approximate value.

True Error (E_t) = True value − Approximate value

\# The derivative of a function f(x) at a particular value of x can be approximately calculated by

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ & $h = 0.3$, find the approximate value of $f'(2)$, true value of $f'(2)$ & true error.

Soln:-

Approximate value of derivative is calculated by using formula,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $x = 2$ & $h = 0.3$,

$$f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3}$$

$$= \frac{7e^{0.5\times2.3} - 7e^{0.5\times2}}{0.3}$$

$$= 10.265$$

For exact value,

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$f(x) = 7e^{0.5x}$

$f'(x) = 7 \times 0.5 e^{0.5x} = 3.5e^{0.5x}$

True value of

$f'(2) = 3.5e^{0.5\times2} = 9.514$

Now,

True error (E_t) = True value − Approximate value

$= 9.514 - 10.265 = -0.751$

2) Relative true error:-

It is defined as the ratio between true error & true value.

Relative true error $(E_r) = \dfrac{\text{True Error}}{\text{True value}}$

The derivative of a function f(x) at a particular value of x can be approximately calculated by

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ & $h = 0.3$, find relative true error at $x = 2$.

Soln:-

True error (E_t) = True value − Approximate value

$= 9.514 - 10.265 = -0.751$

Now,

Relative True error,

$E_r = \dfrac{\text{True error}}{\text{True value}} = \dfrac{-0.751}{9.514} = -0.078$

Absolute relative true error $= |E_r|$

$= 0.07893$

$= 7.893\%$

3) Approximate Error:-

It is defined as the difference between the present approximation & previous approximation

Ea = Present approximation − Previous approximation

The derivative of a function f(x) at a particular value of x can be approximately calculated by

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$ & at $x = 2$, find the following

a) $f'(2)$ using $h = 0.3$

b) $f'(2)$ using $h = 0.15$

c) approximate error for the value of $f'(2)$ for part (b).

Soln:-

We've,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

(a) For $x = 2$ & $h = 0.3$,

$$f'(2) = \frac{f(2+0.3) - f(2)}{0.3}$$

$$= \frac{f(2.3) - f(2)}{0.3} = 10.265$$

b) For $x = 2$ & $h = 0.15$,

$$f'(2) = \frac{f(2+0.15) - f(2)}{0.15}$$

$$= \frac{f(2.15) - f(2)}{0.15}$$

$$= \frac{7e^{0.5\times 2.15} - 7e^{0.5\times 2}}{0.15}$$

$$= 9.879$$

c) Approximate error,

E_a = Present approximation − Previous approximation

$= 9.879 - 10.265 = -0.386$

4) **Relative approximate error:-**

It is defined as the ratio between approximate error & present approximation.

$$E_{ra} = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

The derivative of a function $f(x)$ at a particular value of x can be approximately calculated by

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $f(x) = 7e^{0.5x}$, find the relative approximate error in calculating $f'(2)$ using values from $h = 0.3$ & $h = 0.15$.

Soln:-

E_a = Present approximation − Previous approximation

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$E_a = -0.386$

Relative approximate error,

$$E_{ra} = \frac{\text{Approximate Error}}{\text{Present Approximation}}$$

$$= \frac{-0.386}{9.879} = -0.0390$$

Absolute relative approximate error,

$|E_{ra}| = 0.0390 = 3.9\%$

Sources of errors:-

1) Round off error:-

A computer can represent a number approximately. For example, number like 1/3 may be represented as 0.333333 on PC. Then round off error in this case is $1/3 - 0.333333 = 0.0000003\bar{3}$. Other number such as π & $\sqrt{2}$ need to be approximated in computer calculations.

2) Truncation error:-

It is defined as the error caused by truncating a mathematical procedure. For example, Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

If we use three terms to calculate e^x, then

$$e^x = 1 + x + \frac{x^2}{2!}$$

Truncation error $= e^x - \left(1 + x + \frac{x^2}{2!}\right)$

$$= \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Propagation of error:-

If a calculation is made with numbers that are not exact, then the calculation itself will have an error. Propagation of error is the effect of variables uncertainties on the uncertainty of a function based on them.

If f is a function of several variables $x_1, x_2, \ldots x_n$, then the maximum possible value of the error in f is

$$\Delta f = \left|\frac{\partial f}{\partial x_1}\Delta x_1\right| + \left|\frac{\partial f}{\partial x_2}\Delta x_2\right| + \cdots + \left|\frac{\partial f}{\partial x_n}\Delta x_n\right|$$

\# Find the bounds for the propagation error in adding two numbers. For example if one is calculating $X+Y$ where $X = 1.5 \pm 0.05$ & $Y = 3.4 \pm 0.04$.

Soln:-

The maximum possible value of X & Y are $X = 1.55$ & $Y = 3.44$.

Hence,

$X + Y = 1.55 + 3.44 = 4.99$ is the maximum value of $X+Y$.

The minimum possible value of X & Y are

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$X = 1.45$ & $Y = 3.36$

Hence,

$X+Y = 1.45+3.36 = 4.81$ is the minimum value of $X+Y$.

Hence,

$$4.81 \leq X+Y \leq 4.99$$

Subtraction of numbers that are nearly equal can create unwanted inaccuracies. Using the formula for error propagation, show that this is true.

Soln:-

Let, $z = x-y$

Then,

$$|\Delta z| = \left|\frac{\partial z}{\partial x}\Delta x\right| + \left|\frac{\partial z}{\partial y}\Delta y\right|$$

$$= |(1)\Delta x| + |(-1)\Delta y|$$

$$= |\Delta x| + |\Delta y|$$

So, absolute relative change is

$$\left|\frac{\Delta z}{z}\right| = \frac{|\Delta x| + |\Delta y|}{|x-y|}$$

As x & y become close to each other, the denominator becomes small & hence create large relative errors.

For example, if

$x = 2 \pm 0.001$ $y = 2.003 \pm 0.001$

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$$\left|\frac{\Delta z}{z}\right| = \frac{|0.001|+|0.001|}{|2-2.003|} = 0.6667 = 66.67\%$$

Review of Taylor theorem:-

Taylor's theorem is given by

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \ldots \quad (1)$$

$$f(h+x) = f(h) + f'(h)x + \frac{f''(h)}{2!}x^2 + \frac{f'''(h)}{3!}x^3 + \ldots \quad (2)$$

If we put $h=0$ in eqn (2), we get,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \quad (3)$$

(Maclaurin series)

Derive Maclaurin series of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Soln:-

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

Now,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

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Solution of non-linear eqns:-

Methods of solving non-linear eqns can be categorized into four types:-

1) Direct Analytical Method
2) Graphical Method
3) Trial & Error Method
4) Iterative Method

Trial & Error Method :-

In this method, we make a series of guess for x & then evaluate $f(x)$ at that x if it is close to zero, it is one of the approximate root of the given non-linear equation.

Otherwise make another guess for x & repeat the same process again until x for which $f(x) = 0$ is found.

Solve the equation $f(x) = 2x^2+x-6=0$.

Soln:-

Step1: Guess $x=0 \Rightarrow f(x) = -6$

Step2: Guess $x=1 \Rightarrow f(x) = -3$

Step3: Guess $x=2 \Rightarrow f(x) = +4$

Step4: Guess $x=1.5 \Rightarrow f(x) = 0$

Step5: Guess $x=-1 \Rightarrow f(x) = -5$

Step6: Guess $x=-2 \Rightarrow f(x) = 0$

Hence roots are $x=1.5$ & $x=-2$

Iterative Methods:-

An iterative method starts with an approximate value of root & then finds more & more correct approximate solution in successive iterations.

A specific implementation of an iterative method including termination criteria, is an algorithm of the iterative method.

An iterative method is called convergent if the corresponding sequence converges for given initial approximations.

There are many ways of solving non-linear eqn using iterative approach:

1) Bisection Method
2) Newton-Raphson Method
3) Secant Method
4) Fixed-Point Method

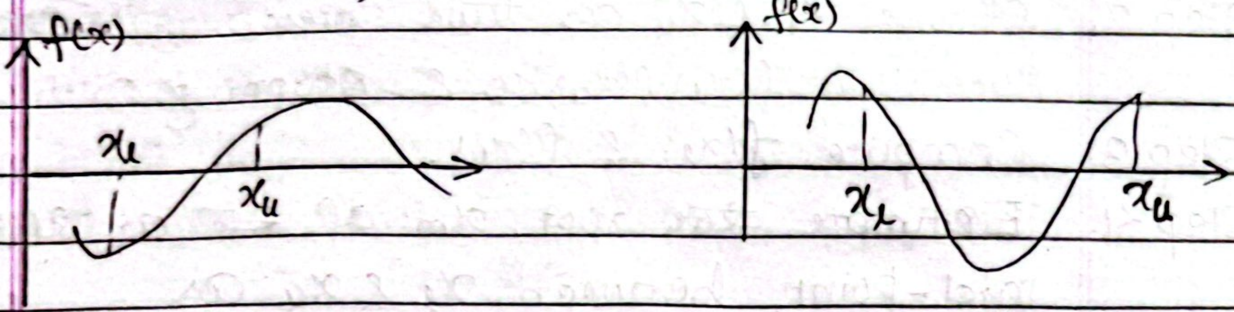
Bisection Method:-

It is also called Binary-Search method or half interval method.

An equation $f(x)=0$ where $f(x)$ is real continuous function, has at least one root between x_l & x_u if $f(x_l)f(x_u)<0$.

Note that if $f(x_l)f(x_u)>0$, there may or may not be any root between x_l & x_u.

Since the method is based on finding the root between two points, the method falls under the category of bracketing methods.



(a) At least one root exists if the function changes sign.

(b) Even number of roots may exists if the function do not change sign.

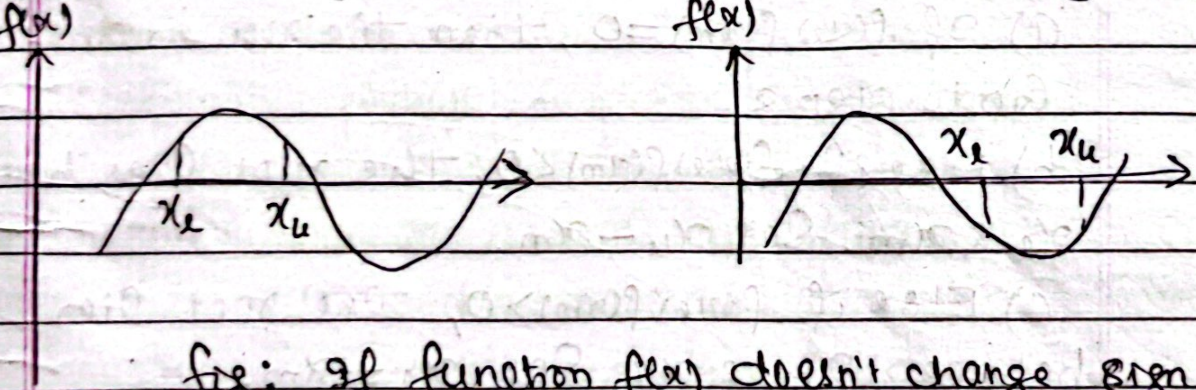


fig: If function f(x) doesn't change sign between two points, there may not be any roots of f(x)=0 between two points.

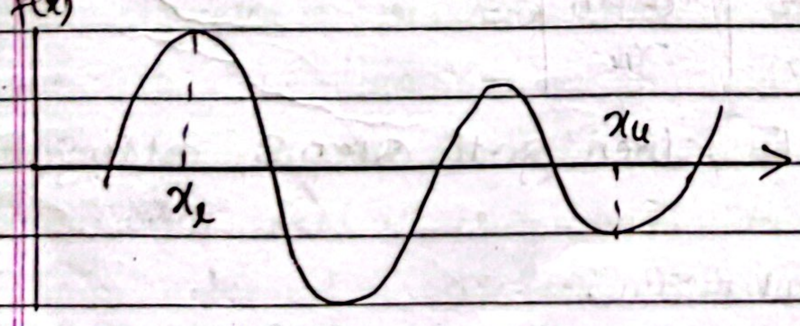


fig: If function f(x) changes sign between two points, more than one root of f(x) = 0 exists between the two points.

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Algorithm of Bisection Method:-

Step 1. Start

Step 2. Choose x_l & x_u as two guesses for the root such that $f(x_l) f(x_u) < 0$ & stopping criterion E.

Step 3. Compute $f(x_l)$ & $f(x_u)$.

Step 4. Estimate the root, x_m of eqn. as the mid-point between x_l & x_u as

$$x_m = \frac{x_l + x_u}{2}$$

Step 5. Now check the following

a) If $f(x_l) f(x_m) = 0$, then the root is x_m. Goto step 8.

b) Else if $f(x_l) f(x_m) < 0$, the root lies between x_l & x_m. Set $x_u = x_m$.

c) Else if $f(x_l) f(x_m) > 0$, the root lies between x_m & x_u. Set $x_l = x_m$.

Step 6 Find the absolute relative approximate error as

$$|E_{ra}| = \left| \frac{x_l - x_u}{x_u} \right|$$

Step 7 If $|E_{ra}| > E$, then goto step 3, else goto step 8

Step 8 Print Root $= x_m$.

Step 9 Stop

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Solve the equation $f(x) = 3x^2 - 6x + 2 = 0$ using Bisection Method.

Soln:-

Assume $x_l = 1$, $x_u = 2$, $E = 0.05$

Iteration 1

$x_l = 1 \quad f(x_l) = -1$

$x_u = 2 \quad f(x_u) = 2$

$x_m = \frac{x_l + x_u}{2} = 1.5 \quad f(x_m) = -0.25$

Since $f(x_m) f(x_u) < 0$,

$x_l = x_m = 1.5 \quad x_u = 2$

$\text{Error} = \left|\frac{x_l - x_u}{x_u}\right| = \left|\frac{1.5-2}{2}\right| = 0.25$

Iteration 2

$x_l = 1.5 \quad f(x_l) = -0.25$

$x_u = 2 \quad f(x_u) = 2$

$x_m = 1.75 \quad f(x_m) = 0.6875$

Since $f(x_m) f(x_l) < 0$,

$x_l = 1.5 \quad x_u = x_m = 1.75$

$\text{Error} = \left|\frac{x_l - x_u}{x_u}\right| = \left|\frac{1.5 - 1.75}{1.75}\right| = 0.1428$

Iteration 3

$x_l = 1.5 \quad f(x_l) = -0.25$

$x_u = 1.75 \quad f(x_u) = 0.6875$

$x_m = 1.625 \quad f(x_m) = 0.1718$

Since $f(x_m) f(x_l) < 0$,

$x_l = 1.5 \quad x_u = x_m = 1.625$

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$$\text{Error} = \left|\frac{x_l - x_u}{x_u}\right| = \left|\frac{1.5-1.625}{1.625}\right| = 0.0769$$

Iteration 4

$x_l = 1.5 \quad f(x_l) = -0.25$

$x_u = 1.625 \quad f(x_u) = 0.6875$

$x_m = 1.5625 \quad f(x_m) = -0.0507$

Since $f(x_m) f(x_u) < 0$,

$x_l = x_m = 1.5625 \quad x_u = 1.625$

$$\text{Error} = \left|\frac{x_l - x_u}{x_u}\right| = \left|\frac{1.5625-1.625}{1.625}\right| = 0.0384$$

Since Error < 0.05, the root of eqn is

$x_m = 1.5625$

Advantages of Bisection method :-

1) Bisection method is always convergent.
2) As iterations are conducted, the interval gets halved. So one can guarantee the decrease in the error in the solution of eqn.

Disadvantages of Bisection method:-

1) The convergence of Bisection method is slow.
2) If one of the initial guess is closer to the root, it will take larger number of iterations to reach the root.
3) If function f(x) is such that it just touches x-axis, it will be unable to find lower & upper guess.

Convergence of Bisection method :-

In Bisection method, interval is halved in every iteration.

After n^{th} iteration, size of interval is reduced to

$$\Delta n = \frac{x_u - x_l}{2^n}$$

Now, we can say maximum error after n^{th} iteration is

$$E_n = \pm \Delta n$$

$$|E_n| = \frac{x_u - x_l}{2^n}$$

Similarly, after $(n+1)^{th}$ iteration, maximum error is given by

$$|E_{n+1}| = \frac{x_u - x_l}{2^{n+1}} = \frac{x_u - x_l}{2^n \cdot 2} = \frac{|E_n|}{2}$$

This equation shows that error is halved after each iteration of Bisection method, therefore we can say that Bisection method converges linearly.

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Newton Raphson method :-

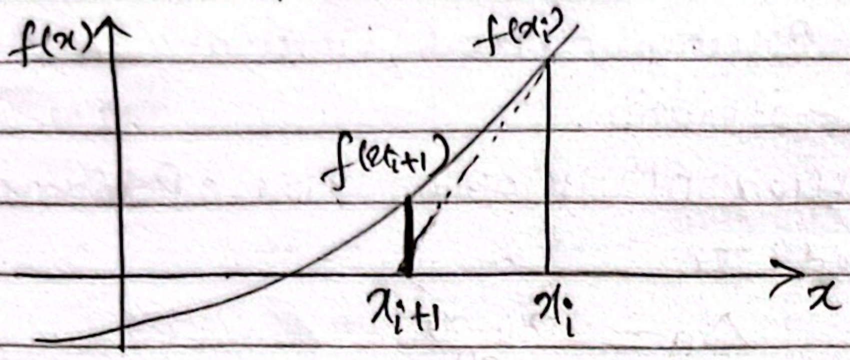

fig: Geometrical interpretation of Newton Raphson method

Newton Raphson method is based on the principle that if the initial guess of root $f(x) = 0$ is at x_i & if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root.

Using defn of slope of a function at $x = x_i$,

$$f'(x_i) = \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}} \quad \text{---} (1)$$

If $f(x)$ has root at x_{i+1}, then $f(x_{i+1}) = 0$

Putting this in above eqn, we get,

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

This gives, $x_i - x_{i+1} = \frac{f(x_i)}{f'(x_i)}$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}} \quad \text{....} (2)$$

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This eq^n is called Newton Raphson formula for solving non-linear eq^n. So, starting with initial guess x_i, we can find the next guess x_{i+1} by using above eq^n. We can repeat this process until the root within desirable tolerance is found.

Algorithm of Newton Raphson method :-

Step1 : Start

Step2 : Input initial guess x_0 & precision E.

Step3 : Evaluate $f'(x)$ symbolically.

Step4 : Use x_0 to estimate the new value of root x_1 as

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step5 : Find the absolute relative approximate error $|E_{ra}|$ as

$$|E_{ra}| = \left|\frac{x_1 - x_0}{x_1}\right|$$

Step6 : If $|E_{ra}| > E$,

$x_0 = x_1$ go to step 4.

else go to step 7.

Step7 : Print root $= x_1$

Step8 : Stop

Solve the eqn $e^{-x}-3x=0$ by using Newton Raphson method. Assume error precision is 0.01.

Soln:-

$$f(x) = e^{-x} - 3x$$

$$f'(x) = -e^{-x} - 3$$

Let $x_0 = 1$ & $E = 0.01$

Iteration 1

$x_0 = 1$ $f(x_0) = -2.632$ $f'(x_0) = -3.367$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-2.632)}{(-3.367)} = 0.218$$

$$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{0.218 - 1}{0.218}\right| = 3.587$$

Iteration 2

$x_0 = 0.218$ $f(x_0) = 0.1501$ $f'(x_0) = -3.8041$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.218 - \frac{0.1501}{(-3.8041)} = 0.2574$$

$$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{0.2574 - 0.218}{0.2574}\right| = 0.1517$$

Iteration 3

$x_0 = 0.2574$ $f(x_0) = 0.0008592$ $f'(x_0) = -3.773$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.2576$$

$$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = 0.00078$$

Since Error < 0.01, Root = 0.2576

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Convergence of Newton Raphson method :-

By Taylor Series,

$$f(x_r) = f(x_n) + f'(x_n)(x_r - x_n) + \frac{1}{2!} f''(\epsilon)(x_r - x_n)^2 \quad --(1)$$

Where ϵ lies between x_n & x_r.

Since x_r is root of eqn, $f(x_r) = 0$

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{1}{2!} f''(\epsilon)(x_r - x_n)^2 - (2)$$

dividing eqn(2) by $f'(x_n)$ & rearranging gives

$$\frac{f(x_n)}{f'(x_n)} + (x_r - x_n) = -\frac{f''(\epsilon)}{2f'(x_n)}(x_r - x_n)^2 \quad --(3)$$

We know that according to Newton Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}$$

Eqn(3) can be written as

$$x_n - x_{n+1} + x_r - x_n = -\frac{f''(\epsilon)}{2f'(x_n)}(x_r - x_n)^2$$

$$x_r - x_{n+1} = -\frac{f''(\epsilon)}{2f'(x_n)}(x_r - x_n)^2 \quad --(4)$$

Let $E_n = x_r - x_n$ $E_{n+1} = x_r - x_{n+1}$, eqn(4) can be written as

$$E_{n+1} = \frac{-f''(\epsilon)}{2f'(x_n)} E_n^2$$

Taking absolute value of both sides,

$$|E_{n+1}| = \frac{|f''(\epsilon)|}{2|f'(x_n)|} \cdot E_n^2 \quad ---(5)$$

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Eqn(5) shows that Newton Raphson method has quadratic rate of convergence.

Fixed point method:-

In fixed point method, we rearrange the function $f(x)=0$ such that x is on left hand side of eqn.

This can be expressed as below:

$$f(x)=0 \quad ---(1)$$

is written as

$$x=g(x) \quad --(2)$$

Eqn(2) is called fixed point eqn.

This can be achieved by algebraic manipulation or by simply adding x to both sides of original eqn.

For example, $x^2+3x-1=0$

$$x=\frac{1-x^2}{3}$$

$$\sin x=0$$

$$x=x+\sin x$$

Algorithm of fixed point method:-

Step1: Start

Step2: Input initial guess x_0 & precision E.

Step3: Convert $f(x)=0$ to form $x=g(x)$.

Step4: Estimate new value of the root x_1 as

$$x_1=g(x_0)$$

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Step5: Find the absolute relative approximate error, $|E_{ra}|$ as

$$|E_{ra}| = \left|\frac{x_1 - x_0}{x_1}\right|$$

Step6: If $|E_{ra}| > E$,

$x_0 = x_1$

goto Step4

else

go to Step7

Step7: Print root $= x_1$

Step8: Stop

Solve the eqn $1 + 1/2 \sin x - x = 0$ using fixed point method.

Soln:-

Above eqn can be written as

$$x = x + 1/2 \sin x$$

Assume $x_0 = 0$ & $E = 0.05$

Iteration 1

$x_0 = 0 \quad x_1 = 1$

$$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{1-0}{1}\right| = 1$$

Iteration 2

$x_0 = 1 \quad x_1 = 1 + 1/2 \sin(1) = 1.42$

$$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{1.42 - 1}{1.42}\right| = 0.2957$$

Iteration 3

$x_0 = 1.42 \quad x_1 = 1 + \frac{1}{2}\sin(1.42) = 1.4943$

$\text{Error} = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{1.4943 - 1.42}{1.4943}\right| = 0.0497$

Since Error < 0.05,

Root $= 1.4943$

Convergence of Fixed point method

Fixed point method is given as

$$x_{n+1} = g(x_n)$$

Let x_r be the root of eq^n,

$$x_r = g(x_r)$$

Subtracting above two eq^{n}s, we get,

$$x_r - x_{n+1} = g(x_r) - g(x_n) \quad \text{---(1)}$$

From mean value theorem, there exist at least one point $x = t$ in the interval (x_r, x_n) such that

$$g'(t) = \frac{g(x_r) - g(x_n)}{x_r - x_n}$$

$$g(x_r) - g(x_n) = g'(t)(x_r - x_n)$$

Substituting this value in eq^n(1), we get,

$$x_r - x_{n+1} = g'(t)(x_r - x_n) \quad \text{---(2)}$$

Since $e_{n+1} = x_r - x_{n+1}$

$e_n = x_r - x_n$

Eq^n(2) becomes,

$$e_{n+1} = g'(t)\, e_n \quad \text{--(3)}$$

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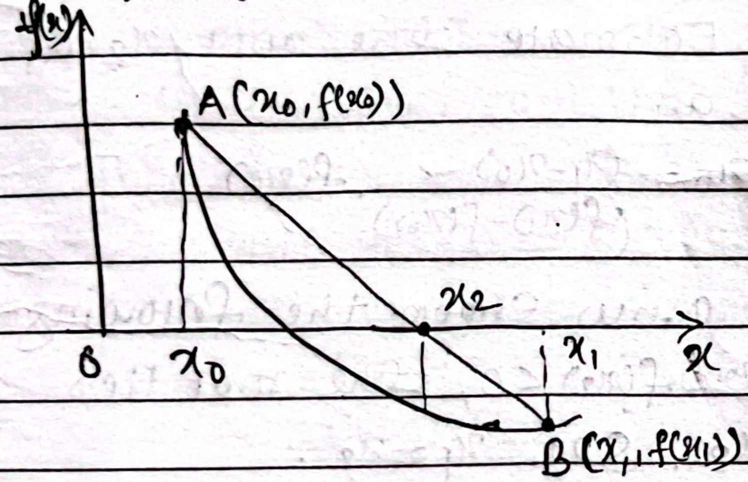
Eq^n(3) says that error decreases with each iteration only if $g'(t) < 1$.
Hence we say that fixed point method converges only under the condition $g'(t) < 1$.

Method of False position or Regula-Falsi method

This is the oldest method of finding root of eq^n & closely resembles the bisection method. We choose two points x_0 & x_1 such that $f(x_0) f(x_1) < 0$.



Eq^n of the chord joining the points $A(x_0, f(x_0))$ & $B(x_1, f(x_1))$ is

$$y - f(x_0) = \frac{(f(x_1) - f(x_0))}{(x_1 - x_0)} (x - x_0)$$

Taking point of intersection of the chord with x-axis as approximation of the root.

$$y = 0 \ \& \ x = x_2$$

$$0 - f(x_0) = \frac{(f(x_1) - f(x_0))}{(x_1 - x_0)} (x_2 - x_0)$$

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$$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} f(x_0)$$

Which is approximation of the root.

Algorithm of method of False position :-

Step1 : Start

Step2 : Choose x_0 & x_1 as two guesses for the root such that $f(x_0) f(x_1) < 0$ & stopping criterion E.

Step3 : Compute $f(x_0)$ & $f(x_1)$.

Step4 : Estimate the root, x_2 of the eqn $f(x) = 0$ as

$$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} \cdot f(x_0)$$

Step5 : Now check the following

a) If $f(x_0) f(x_2) < 0$, the root lies between x_0 & x_2. Set $x_1 = x_2$.

b) Else if $f(x_0) f(x_2) > 0$, the root lies between x_2 & x_1. Set $x_0 = x_2$.

Step6 : Find the absolute relative approximate error as

$$|E_{ra}| = \left| \frac{x_0 - x_1}{x_1} \right|$$

Step7 : If $|E_{ra}| > E$, then go to step3, else go to step 8.

Step8 : Print root $= x_2$

Step9 : Stop

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Find the root of eqn $\cos x = xe^x$ using Regula-falsi method correct upto three decimal places.

Soln:-

$$f(x) = \cos x - xe^x$$

Iteration 1

$x_0 = 0 \quad f(x_0) = 1$

$x_1 = 1 \quad f(x_1) = -2.17798$

$$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} \cdot f(x_0) = 0.31467$$

$f(x_2) = 0.51987$

$f(x_2) f(x_1) < 0$, root lies betn 0.31467 & 1.

Iteration 2

$x_0 = 0.31467 \quad f(x_0) = 0.51987$

$x_1 = 1 \quad f(x_1) = -2.17798$

$$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} \cdot f(x_0) = 0.44673$$

$f(x_2) = 0.20356$

Since $f(x_2) f(x_1) < 0$, root lies betn 0.44673 & 1

Iteration 3

$x_0 = 0.44673 \quad f(x_0) = 0.20356$

$x_1 = 1 \quad f(x_1) = -2.17798$

$$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} \cdot f(x_0) = 0.49402$$

$f(x_2) = 0.07078$

Since $f(x_2) \cdot f(x_1) < 0$, root lies betn 0.49402 & 1.

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Iteration 4

$x_0 = 0.49402 \qquad f(x_0) = 0.07078$

$x_1 = 1 \qquad f(x_1) = -2.17798$

$x_2 = x_0 - \frac{(x_1 - x_0)}{(f(x_1) - f(x_0))} f(x_0) = 0.50995$

$f(x_2) = 0.0235965$

Since $f(x_2) f(x_1) < 0$, root lies betn 0.50995 & 1

Iteration 5 $x_2 = 0.51520$

Iteration 6 $x_2 = 0.51692$

Iteration 7 $x_2 = 0.51748$

Iteration 8 $x_2 = 0.51767$

Convergence of False position method:

The formula for false position method is given by

$$x_{n+1} = x_{n-1} - \frac{(x_n - x_{n-1})}{(f(x_n) - f(x_{n-1}))} f(x_{n-1}) \quad (1)$$

Let x_r be actual root of eqn & e_n be error estimate in nth iteration.

$$x_{n+1} = e_{n+1} + x_r$$

$$x_n = e_n + x_r$$

$$x_{n-1} = e_{n-1} + x_r$$

Substituting these values in eqn(1), we get,

$$e_{n+1} = e_{n-1} - \frac{(e_n - e_{n-1})}{(f(x_n) - f(x_{n-1}))} f(x_{n-1})$$

$$= \frac{e_{n-1} f(x_n) - e_{n-1} \cancel{f(x_{n-1})} - e_n f(x_{n-1}) + e_{n-1} \cancel{f(x_{n-1})}}{(f(x_n) - f(x_{n-1}))}$$

$$e_{n+1} = \frac{e_{n-1} f(x_n) - e_n f(x_{n-1})}{(f(x_n) - f(x_{n-1}))} \quad \text{--- (2)}$$

According to mean value theorem, there exists at least one point say $x = t_n$ in the interval x_n & x_r such that

$$f'(t_n) = \frac{f(x_n) - f(x_r)}{x_n - x_r} \quad \text{--- (3)}$$

Since $f(x_r) = 0$ & $e_n = x_n - x_r$, eq^n(3) becomes,

$$f'(t_n) = \frac{f(x_n)}{e_n}$$

$$f(x_n) = e_n f'(t_n)$$

Similarly, $f(x_{n-1}) = e_{n-1} \cancel{f'(t_{n-1})} f'(t_{n-1})$

Substituting these value in eq^n(2),

$$e_{n+1} = \frac{e_{n-1} e_n f'(t_n) - e_n e_{n-1} f'(t_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$= e_{n-1} e_n \frac{(f'(t_n) - f'(t_{n-1}))}{f(x_n) - f(x_{n-1})}$$

$$e_{n+1} \propto e_{n-1} e_n \quad \text{--- (4)}$$

By def^n, rate of convergence, the method is of order p if

$$e_n \propto e_{n-1}^p \quad \text{--- (5)}$$

or $$e_{n+1} \propto e_n^p \quad \text{--- (6)}$$

From (4), (5) & (6),

$$e_n^p \propto e_{n-1} e_{n-1}^p$$

$$e_n \propto e_{n-1}^{(p+1)/p} \quad \text{--- (7)}$$

Comparing (5) & (7), we get,

$$p = (p+1)/p$$

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$$p^2 - p - 1 = 0$$

$$p = \frac{1 \pm \sqrt{5}}{2}$$

$$p = 1.62 \text{ (golden ratio)}$$

Thus,

$$e_n \propto e_{n-1}^{1.62}$$

This eqn shows that convergence of false position method is superlinear.