Interpolation & extrapolation :-

Given n+1 data points $(x_0, y_0), (x_1, y_1) \cdots (x_n, y_n)$, interpolation is the process of finding an eqn $y = f(x)$ that passes through above n+1 data points & using this eqn to find value of y at x, $x_0 < x < x_n$.

Extrapolation is the process of finding an eqn $y = f(x)$ that passes through above n+1 data points & using this eqn to find value of y at x, $x < x_0$ or $x > x_n$.

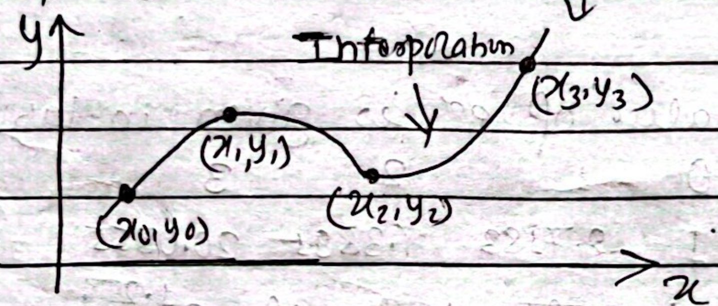

fig: Interpolation & extrapolation

Lagrange interpolation :-

A second order polynomial can be written in the form

$$P_2(x) = b_1(x-x_0)(x-x_1) + b_2(x-x_1)(x-x_2) + b_3(x-x_2)(x-x \quad \text{---(1)}$$

Let (x_0, f_0), (x_1, f_1) & (x_2, f_2) are three interpolating points. Substituting these points in eqn(1), we get

$$P_2(x_0) = f_0 = b_2(x_0-x_1)(x_0-x_2)$$

$$b_2 = \frac{f_0}{(x_0-x_1)(x_0-x_2)}$$

DATE ______ PAGE ______

$$P_2(x_1) = f_1 = b_3(x_1-x_2)(x_1-x_0)$$

$$b_3 = \frac{f_1}{(x_1-x_2)(x_1-x_0)}$$

$$P_2(x_2) = f_2 = b_1(x_2-x_0)(x_2-x_1)$$

$$b_1 = \frac{f_2}{(x_2-x_0)(x_2-x_1)}$$

Substituting these values of b_1, b_2 & b_3 in eq^n(1), we get,

$$P_2(x) = \frac{f_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f_1(x-x_2)(x-x_0)}{(x_1-x_2)(x_1-x_0)} + \frac{f_2(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

The value of e^x is given in table below:

x	0	1	2	3
e^x	1	2.7183	7.3891	20.0855

Determine the value of $e^{1.2}$ by using 2nd order Lagrange interpolation.

Soln:-

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2$

$f_0 = 1 \quad f_1 = 2.7183 \quad f_2 = 7.3891$

$x = 1.2$

$$P_2(x) = f_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{1(1.2-1)(1.2-2)}{(0-1)(0-2)} + \frac{2.7183(1.2-0)(1.2-2)}{(1-0)(1-2)} +$$

$$\frac{7.3891(1.2-0)(1.2-1)}{(2-0)(2-1)}$$

DATE ______ PAGE ______

$$= 1\times(-0.08) + 2.7183\times 0.96 + 7.3891\times 0.12$$

$$P_2(1.2) = 3.41626$$

Newton's Divided Difference Interpolation :-

Let us consider a polynomial of degree n of the form given below:

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots a_n(x-x_0)\cdots(x-x_{n-1}) \quad \text{---}(1)$$

Let us suppose $(x_0, f(x_0)), (x_1, f(x_1)) \cdots (x_n, f(x_n))$ are given interpolating points.

At $x = x_0$, eqn (1) becomes,

$$P_n(x_0) = a_0 = f(x_0)$$

Similarly at $x = x_1$, eqn (1) becomes,

$$p(x_1) = a_0 + a_1(x_1 - x_0) = f(x_1)$$

$$\therefore a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$, eqn (1) becomes,

$$p(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

$$a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) - a_0 - a_1(x_2 - x_0)$$

$$a_2 = \frac{f(x_2) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{(f(x_1) - f(x_0))\cancel{(x_2 - x_0)}}{(x_1 - x_0)\cancel{(x_2 - x_0)}(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1) + f(x_1) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1) - f(x_0)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

$$= \frac{f(x_2) - f(x_1)}{(x_2 - x_0)(x_2 - x_1)} + \frac{f(x_1) - f(x_0)}{(x_2 - x_1)}\left[\frac{1}{(x_2 - x_0)} - \frac{1}{(x_1 - x_0)}\right]$$

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DATE ______ PAGE ______

$$= \frac{f(x_2)-f(x_0)}{(x_2-x_0)(x_2-x_1)} + \frac{f(x_1)-f(x_0)}{(x_2-x_0)}\left[\frac{x_1-\cancel{x_0}-x_2+\cancel{x_0}}{(x_2-x_0)(x_1-x_0)}\right]$$

$$= \frac{f(x_2)-f(x_0)}{(x_2-x_0)(x_2-x_1)} - \frac{f(x_1)-f(x_0)\cancel{(x_2-x_1)}}{\cancel{(x_2-x_1)}(x_2-x_0)(x_1-x_0)}$$

$$a_2 = \frac{\dfrac{f(x_2)-f(x_0)}{(x_2-x_0)} - \dfrac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)}$$

1st DD,

$$a_0 = f[x_0] = f(x_0)$$

2nd DD,

$$a_1 = f[x_1, x_0] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$$

3rd DD,

$$a_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{(x_2-x_0)}$$

$$a_2 = \frac{\dfrac{f(x_2)-f(x_1)}{(x_2-x_1)} - \dfrac{f(x_1)-f(x_0)}{(x_1-x_0)}}{(x_2-x_0)}$$

The upward velocity of a rocket is given a a function of time in table below:

Time(t)	0	10	15	20	22.5	30
Velocity(V)	0	227.04	362.78	517.35	602.97	901.

Determine the value of velocity at $t = 16$ sec with using 3rd order Newton's Divide difference polynomial.

Soln:-

3rd order polynomial,

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

Four points,

$x_0 = 10$, $x_1 = 15$, $x_2 = 20$, $x_3 = 22.5$

$f(x_0) = 227.04$ $f(x_1) = 362.78$ $f(x_2) = 517.35$ $f(x_3) = 602$

$$a_0 = f[x_0] = f(x_0) = 227.04$$

$$a_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$a_1 = 27.148$$

$$a_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{517.35 - 362.78}{20 - 15}$$

$$= 30.914$$

$$a_2 = \frac{30.914 - 27.148}{20 - 10} = 0.3766 = f[x_2, x_1, x_0]$$

$$a_3 = f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$$

$$f[x_3, x_2, x_1] = \frac{f[x_3, x_2] - f[x_2, x_1]}{x_3 - x_1}$$

$$f[x_3, x_2] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{602.97 - 517.35}{22.5 - 20} = 34.248$$

$$f[x_3, x_2, x_1] = \frac{34.248 - 30.914}{22.5 - 15} = 0.44453$$

$$a_3 = \frac{0.44453 - 0.3766}{22.5 - 10} = 0.0054344$$

$$p(16) = 227.04 + 27.148(16-10) + 0.3766(16-10)(16-15) + 0.0054344(16-10)(16-15)(16-20)$$

$$= 392.057$$

Divided Difference Table :-

	$\leftarrow a_0$			
x_0	$f(x_0)$	$\leftarrow a_1$		
		$f[x_1, x_0]$	$\leftarrow a_2$	
x_1	$f(x_1)$		$f[x_2, x_1, x_0]$	$\leftarrow a_3$
		$f[x_2, x_1]$		$f[x_3, x_2, x_1, x_0]$
x_2	$f(x_2)$		$f[x_3, x_2, x_1]$	
		$f[x_3, x_2]$		
x_3	$f(x_3)$			

Given the following data points, create the table of divided difference. Use the table to estimate the value of $f(1.8)$ by using second & third order polynomial.

x	1	2	3	4
$f(x)$	0	7	26	63

Soln:-

x	$f(x)$ a_0	a_1	a_2	a_3
1	0			
		$\frac{7-0}{2-1} = 7$		
2	7		6	
		19		1
3	26		9	
		37		
4	63			

Evaluating $f(1.8)$ by using second order polynomial

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$

At $x = 1.8$

$$f(1.8) = 0 + 7(1.8-1) + 6(1.8-1)(1.8-2) = 4.64$$

Evaluating $f(1.8)$ by using third order polynomial,

$$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

At $x = 1.8$,

$$f(1.8) = 0 + 7(1.8-1) + 6(1.8-1)(1.8-2) + 1(1.8-1)(1.8-2)(1.8-3)$$

$$= 4.832$$

Newton's Forward Difference interpolation :-

$$P_4(x) = f[x_0] + S\Delta f(x_0) + \frac{S(S-1)}{2!}\Delta^2 f(x_0) + \frac{S(S-1)(S-2)}{3!}\Delta^3 f(x_0)$$

$$+ \frac{S(S-1)(S-2)(S-3)}{4!}\Delta^4 f(x_0)$$

where $S = \frac{x-x_0}{h}$ $h = x_{i+1} - x_i$

\# Construct Newton's forward difference table for data points in table below & then approximate the value of f(1.1) by using Newton's forward difference formula.

x	1.0	1.3	1.6	1.9	2.2
f(x)	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

Soln:-

$h = x_1 - x_0 = 1.3 - 1.0 = 0.3$

$S = \frac{x-x_0}{h} = \frac{1.1-1.0}{0.3} = 0.3333$

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1.0	0.7651977			
		-0.1451117		
1.3	0.6200860		-0.0195721	
		-0.1646838		0.0106723
1.6	0.4554022		-0.0088998	
		-0.1735836		0.0110271
1.9	0.2818186		+0.0021273	
		-0.1714563		
2.2	0.1103623			

$\Delta^4 f(x)$

0.0003548

$P_4(x) = f[x_0] + S\Delta f(x_0) + \ldots$

$\frac{S(S-1)(S-2)(S-3)}{4!}\Delta\ldots$

$f(1.1) = 0.7651977 + 0\ldots$

$\frac{0.3333(0.3333-1)(-0.0195721)}{2} + \ldots$

$+ \frac{0.3333(0.3333-1)(0.3333\ldots}{24}$

$= 0.71965$

Newton's Backward\ldots

$P_4(x) = f(x_n) + S\nabla\ldots$

$\nabla^3 f(x_n) + \frac{S(S+1)(S+2)(\ldots}{4!}$

\# Estimate the value\ldots

a) Newton Gregory fo\ldots

b) Newton Gregory ba\ldots

with the help of\ldots

θ	10	20
Sin θ	0.1736	0.3420

Soln:-

DATE ______ PAGE ______

$$P_4(x) = f[x_0] + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) +$$

$$\frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0)$$

$$f(1.1) = 0.7651977 + 0.3333 \times (-0.1451117) +$$

$$\frac{0.3333(0.3333-1)}{2}(-0.0195721) + \frac{0.3333(0.3333-1)(0.3333-2)}{6}(0.010672$$

$$+ \frac{0.3333(0.3333-1)(0.3333-2)(0.3333-3)}{24}(0.0003548)$$

$$= 0.71965$$

Newton's Backward Difference interpolation :-

$$P_4(x) = f[x_n] + s\nabla f(x_n) + \frac{s(s+1)}{2}\nabla^2 f(x_n) + \frac{s(s+1)(s+2)}{3!}$$

$$\nabla^3 f(x_n) + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f(x_n)$$

Estimate the value of $\sin\theta$ at $\theta = 25°$ using
a) Newton Gregory forward difference formula
b) Newton Gregory backward difference formula

with the help of following table.

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5	0.6428	0.7660

Soln:-

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
θ	$\sin\theta$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.1580		-0.0048	
30	0.5		-0.0152		+0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

a) Forward difference formula,

$$x_0 = 10 \quad h = 20-10 = 10 \quad x = 25$$

$$s = \frac{x - x_0}{h} = \frac{25-10}{10} = 1.5$$

$$P_4(x) = f[x_0] + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$

$$+ \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0)$$

$$\sin 25^\circ = 0.1736 + 1.5 \times \overset{0.1684}{\cancel{0.1736}} + \frac{1.5(1.5-1)}{2} \times (-0.0104) +$$

$$\frac{1.5(1.5-1)(1.5-2)}{6} \times (-0.0048) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24} \times (+0.0004)$$

$$= \cancel{0.4220}\ 0.42261$$

DATE ______ PAGE ______

b) <u>Newton's Backward difference formula,</u>

$$x_n = 50 \quad x_0 = 10 \quad h = 10 \quad x = 25$$

$$S = \frac{x - x_n}{h} = \frac{25 - 50}{10} = -2.5$$

$$P_4(x) = f(x_n) + S\nabla f(x_n) + \frac{S(S+1)}{2!}\nabla^2 f(x_n) + \frac{S(S+1)(S+2)}{3!}$$

$$\nabla^3 f(x_n) + \frac{S(S+1)(S+2)(S+3)}{4!}\nabla^4 f(x_n)$$

$$\sin 25^\circ = 0.7660 + (-2.5)\times(0.1232) + \frac{(-2.5)(-2.5+1)}{2}\times$$

$$(-0.0196) + \frac{(-2.5)(-2.5+1)(-2.5+2)}{6}(-0.0044) +$$

$$\frac{(-2.5)(-2.5+1)(-2.5+2)(-2.5+3)}{24}(+0.0004)$$

$$= \cancel{0.4200}\ 0.42213$$

<u>Spline interpolation:-</u>

It is a form of interpolation where interpolant is a special type of piecewise polynomial called spline. It avoids the problem of Runge's phenomenon. Runge's phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree.

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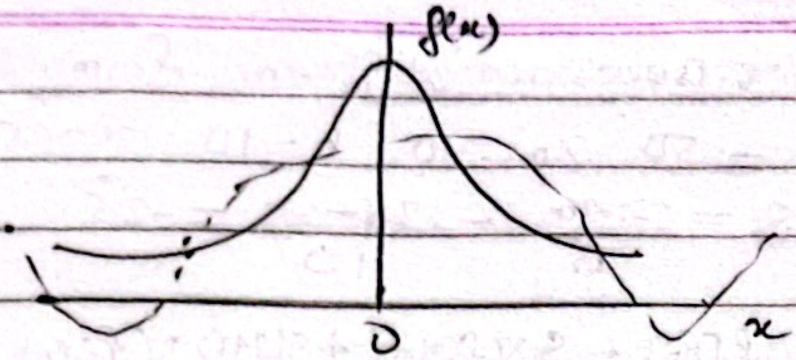

fig: Runge's phenomenon

Cubic Spline interpolation :-

$$h_i = x_{i+1} - x_i \qquad \text{for } i = 0 \text{ to } n-1$$

$$b_i = \frac{f(x_{i+1}) - f(x_i)}{h_i} \qquad "$$

$$u_i = 2(h_{i-1} + h_i) \qquad \text{for } i = 1 \text{ to } n-1$$

$$v_i = 6(b_i - b_{i-1}) \qquad "$$

$$e_0 = 0$$

$$h_{i-1}e_{i-1} + u_i e_i + h_i e_{i+1} = v_i \qquad i = 1 \text{ to } n-1$$

$$e_n = 0$$

$$S_i(x) = \frac{e_{i+1}}{6h_i}(x_p - x_i)^3 + \frac{e_i}{6h_i}(x_{i+1} - x_p)^3 + \left(\frac{f(x_{i+1})}{h_i} - \frac{e_{i+1}h_i}{6}\right)(x_p - x_i) + \left(\frac{f(x_i)}{h_i} - \frac{e_i h_i}{6}\right)(x_{i+1} - x_p)$$

Find $f(6)$ ~~using~~ using cubic spline interpolation for the data given below:

x	0	5	7	8	10
$f(x)$	0	2	-1	-2	20

DATE ______ PAGE ______

Sol^n:-

$h_0 = x_1 - x_0 = 5 - 0 = 5$

$h_1 = x_2 - x_1 = 7 - 5 = 2$

$h_2 = x_3 - x_2 = 8 - 7 = 1$

$h_3 = x_4 - x_3 = 10 - 8 = 2$

$b_0 = \frac{f(x_1) - f(x_0)}{h_0} = \frac{2-0}{5} = 0.4$

$b_1 = \frac{f(x_2) - f(x_1)}{h_1} = \frac{-1-2}{2} = -1.5$

$b_2 = \frac{f(x_3) - f(x_2)}{h_2} = \frac{-2 \cancel{8} - (-1)}{1} = -1$

$b_3 = \frac{f(x_4) - f(x_3)}{h_3} = \frac{20 - (-2)}{2} = 11$

$u_1 = 2(h_0 + h_1) = 2(5+2) = 14$

$u_2 = 2(h_1 + h_2) = 2(2+1) = 6$

$u_3 = 2(h_2 + h_3) = 2(1+2) = 6$

$v_1 = 6(b_1 - b_0) = 6(-1.5 - 0.4) = -11.4$

$v_2 = 6(b_2 - b_1) = 6(-1 + 1.5) = 3$

$v_3 = 6(b_3 - b_2) = 6(11 + 1) = 72$

$e_0 = 0$

$h_0 e_0 + u_1 e_1 + h_1 e_2 = v_1 \quad 14e_1 + 2e_2 = -11.4$

$h_1 e_1 + u_2 e_2 + h_2 e_3 = v_2 \quad 2e_1 + 6e_2 + e_3 = 3$

$h_2 e_2 + u_3 e_3 + h_3 e_4 = v_3 \quad e_2 + 6e_3 = 72$

$e_4 = 0$

$e_1 = -0.6245 \quad e_2 = -1.3288 \quad e_3 = 12.2214$

$S_1(x) = \frac{e_2}{6h_1}(x_p - x_1)^3 + \frac{e_1}{6h_1}(x_2 - x_p)^3 + \left(\frac{f(x_2)}{h_1} - \frac{e_2 h_1}{6}\right)$

$(x_p - x_1) + \left(\frac{f(x_1)}{h_1} - \frac{e_1 h_1}{6}\right)(x_2 - x_p)$

$$= \frac{-1.3288}{6\times 2}(6-5)^3 + \frac{(-0.6245)}{6\times 2}(7-6)^3 + \left(\frac{-1}{2} - \frac{(-1.3288)}{6}\right)(6-5) + \left(\frac{2}{2} - \frac{(-0.6245)\times 2}{6}\right)(7-6)$$

$$= 0.9883$$

Regression:-

It is a form of predictive modeling technique which investigates the relationship between dependent & independent variables.

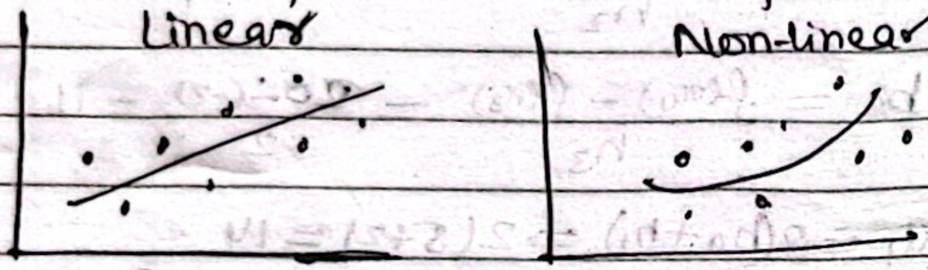

fig: Linear & Non-linear regression

linear regression :-

Fitting a straight line is simplest approach of regression analysis, which is called linear regression.

St. line can be represented by using mathematical eqⁿ,

$y = f(x) = a + bx$, where a and b are regression coefficients to be determined.

The sum of square of individual errors can be expressed as

DATE ______ PAGE ______

$$E = \sum e_i^2 = \sum_{i=1}^{n} (y_i - f(x_i))^2$$

$$= \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

We choose a & b such that E is minimum

$$\frac{\partial E}{\partial a} = 0 \quad \& \quad \frac{\partial E}{\partial b} = 0$$

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0$$

$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - \sum_{i=1}^{n} bx_i = 0$$

$$na + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{--- (1)} \qquad \times \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} ax_i - \sum_{i=1}^{n} bx_i^2 = 0$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i \quad \text{--- (2)} \qquad \times n$$

From (1) & (2),

$$b\left(\sum_{i=1}^{n} x_i\right)^2 - bn\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i - n\sum_{i=1}^{n} x_i y_i$$

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{--- (3)}$$

DATE ______ PAGE ______

From ①,

$$a = \frac{\sum_{i=1}^{n} y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n} \quad \text{—— (4)}$$

\# The value of x & their corresponding value of y are shown in table below:

x	0	1	2	3	4
$y = f(x)$	2	3	5	4	6

a) Find least square regression line $y = ax + b$

b) Estimate value of y when $x = 10$.

Soln:-

x_i	y_i	x_i^2	x_iy_i
0	2	0	0
1	3	1	3
2	5	4	10
3	4	9	12
4	6	16	24
$\sum x_i = 10$	$\sum y_i = 20$	$\sum x_i^2 = 30$	$\sum x_iy_i = 49$

$$a = \frac{n\sum_{i=1}^{n} x_iy_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum x_i\right)^2} = \frac{5\times49 - 10\times20}{5\times30 - 10^2}$$

$$= 0.9$$

$$b = \frac{\sum_{i=1}^{n} y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n} = \frac{20}{5} - 0.9\times\frac{10}{5} = 2.2$$

$$y = 0.9x + 2.2$$

At $x = 10$, $y = 0.9\times10 + 2.2 = 11.2$

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Non-linear regression:-

1) Fitting exponential model

Exponential model is given by

$$y = ae^{bx} \quad \text{---(1)}$$

Taking natural log on both sides, we get,

$$\log y = \log(ae^{bx})$$

$$= \log a + bx \quad \text{---(2)}$$

This eq^n is similar to $y = a + bx$

$$b = \frac{n\sum_{i=1}^{n} x_i \log y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \log y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \quad \text{---(3)}$$

$$\log a = \frac{\sum_{i=1}^{n} \log y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n} = K \quad \text{---(4)}$$

$$a = e^K \quad \text{---(5)}$$

Fit the curve $y = ae^{bx}$ through the data given below:

x	-4	-2	0	1	2	4
$y = f(x)$	0.57	1.32	4.12	6.65	11	30.3

Soln:-

x_i	y_i	$\log y_i$	$x_i \log y_i$	x_i^2
-4	0.57	-0.562	2.248	16
-2	1.32	0.277	-0.555	4
0	4.12	1.415	0	0
1	6.65	1.894	1.894	1
2	11	2.398	4.796	4
4	30.3	3.411	13.645	16

$\sum x_i = 1$ $\quad \sum \log y_i = 8.835$ $\quad \sum x_i \log y_i = 22.03$ $\quad \sum x_i^2 = 41$

DATE ______ PAGE ______

$$b = \frac{n\sum_{i=1}^{n} x_i \log y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \log y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$= \frac{6 \times 22.03 - 1 \times 8.835}{6 \times 41 - (1)^2} = 0.503$$

$$\log a = \frac{\sum_{i=1}^{n} \log y_i}{n} - b\frac{\sum_{i=1}^{n} x_i}{n}$$

$$= \frac{8.835}{6} - 0.503 \times \frac{1}{6} = 1.388$$

$$a = e^{1.388} = 4.006$$

$$y = 4.006\, e^{0.503x}$$

2) Fitting polynomial model:-

$$y = a_0 + a_1 x + a_2 x^2$$

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

$$\frac{\partial E}{\partial a_0} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - a_2 x_i^2)(-1) = 0$$

$$n a_0 + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i \quad \text{--- (1)}$$

$$\frac{\partial E}{\partial a_1} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - a_2 x_i^2)(-x_i) = 0$$

$$a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i y_i \quad \text{--- (2)}$$

$$\frac{\partial E}{\partial a_2} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - a_2 x_i^2)(-x_i^2) = 0$$

$$a_0 \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + a_2 \sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} x_i^2 y_i \quad \text{--- (3)}$$

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$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i^2 y_i \end{bmatrix}$$

Fit the quadratic curve through the following data

x	1	2	3	4
$f(x)$	6	11	18	27

Soln:-

	x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
	1	6	1	1	1	6	6
	2	11	4	8	16	22	44
	3	18	9	27	81	54	162
	4	27	16	64	256	108	432
Σ	10	62	30	100	354	190	644

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 62 \\ 190 \\ 644 \end{bmatrix}$$

$$y = 3 + 2x + x^2$$

Secant method:-

It uses two initial estimates but doesn't require that they must bracket the root.

~~Abstre~~ Newton Raphson method of solving non-linear eqⁿ is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{--- (1)}$$

The derivative $f'(x_i)$ is approximated as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \text{--- (2)}$$

From (1) & (2),

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{--- (3)}$$

This eqⁿ is called secant formula.

Algorithm of Secant method:-

Step1 : Start

Step2 : Input ~~choose~~ x_0 & x_1 as two initial guesses & precision E.

Step3 : Evaluate $f(x_0)$ & $f(x_1)$.

Step4 : Estimate the root x_2 as

$$x_2 = x_1 - \frac{f(x_1)\,(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Step5 : Find absolute relative approximate error, $|E_{ra}|$ as

$$|E_{ra}| = \left|\frac{x_2 - x_1}{x_2}\right|$$

Step6 :

Step 3.

Else

Step7 :

Step8 :

Use Sec

eqⁿ

With

Solⁿ:-

Iteration

$x_0 =$

$x_1 =$

$x_2 =$

Error

Iteration

$x_0 =$

$x_1 =$

$x_2 =$

Iteration

$x_0 =$

$x_1 =$

$x_2 =$

Step 6: If $|E_{ra}| > E$, $x_0 = x_1$ & $x_1 = x_2$ go to Step 3.

Else go to Step 7.

Step 7: Print root $= x_2$

Step 8: Stop

Use Secant method to estimate the root of eqn $x^2 - 4x - 10 = 0$ with initial estimates of $x_0 = 4$ & $x_1 = 2$.

Soln:-

$$f(x) = x^2 - 4x - 10, \quad E = 0.05$$

Iteration 1

$x_0 = 4 \quad f(x_0) = -10$

$x_1 = 2 \quad f(x_1) = -14$

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 9$$

$$\text{Error} = \left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{9-2}{9}\right| = 0.7777$$

Iteration 2

$x_0 = 2 \quad f(x_0) = -14$

$x_1 = 9 \quad f(x_1) = 35$

$x_2 = 4 \quad \text{Error} = 1.25$

Iteration 3

$x_0 = 9 \quad f(x_0) = 35$

$x_1 = 4 \quad f(x_1) = -10$

$x_2 = 5.1111 \quad \text{Error} = 0.2173$

Iteration 4

$x_0 = 4$ $\quad f(x_0) = -10$

$x_1 = 5.1111$ $\quad f(x_1) = -4.3207$

$x_2 = 5.9563$ $\quad$ Error $= 0.1418$

Iteration 5

$x_0 = 5.1111$ $\quad f(x_0) = -4.3207$

$x_1 = 5.9563$ $\quad f(x_1) = \cancel{5.0331}\ 1.6507$

$x_2 = \cancel{5.5014}$ 5.72 $\quad$ Error $= 0.041$

Iteration 6

$x_0 = 5.9563$ $\quad f(x_0) = 5.0331$

$x_1 = 5.5014$ $\quad f(x_1) = -1.7392$

$x_2 = 5.6182$ $\quad$ Error $=$

Convergence of Secant method:-

Secant formula is

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---} (1)$$

Let x_r be root of eqn $f(x) = 0$ & e_n be error estimate of x_n.

$$x_{n+1} = e_{n+1} + x_r$$

$$x_n = e_n + x_r$$

$$x_{n-1} = e_{n-1} + x_r$$

Substituting these eqn in (1),

$$e_{n+1} = e_n - \frac{f(x_n)(e_n - e_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$e_{n+1} = \frac{e_{n-1} f(x_n) - e_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{---} (2)$$

Horner's method for polynomial evaluation:-

It is an algorithm for evaluating polynomials efficiently.

We've given,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$= a_0 + x(a_1 + a_2 x + \cdots + a_n x^{n-1})$$

$$= a_0 + x(a_1 + x(a_2 + a_3 x + \cdots + a_n x^{n-2})$$

$$\vdots$$

$$p(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + x(a_4 + \cdots (a_{n-1} + x a_n)$$

Now we can define a new sequence of constants as follows:

$$b_n = a_n$$

$$b_{n-1} = a_{n-1} + b_n x_0$$

Then b_0 is the value of $p(x_0)$.

\# Evaluate the polynomial $p(x) = 2x^4 - x^3 + 3x^2 - 5x + 4$ at $x = 2$ using Horner's method.

Soln:-

$a_4 = 2 \quad a_3 = -1 \quad a_2 = 3 \quad a_1 = -5 \quad a_0 = 4, \; x_0 = 2$

Now,

$$b_4 = a_4 = 2$$

$$b_3 = a_3 + b_4 x_0 = -1 + 2 \times 2 = 3$$

$$b_2 = a_2 + b_3 x_0 = 3 + 3 \times 2 = 9$$

$$b_1 = a_1 + b_2 x_0 = -5 + 9 \times 2 = 13$$

$$b_0 = a_0 + b_1 x_0 = 4 + 13 \times 2 = 30$$

\# $p(x) = 3x^3 - 4x^2 + 5x - 6$ at $x = 2$ (12)

Synthetic division :-

Let $p(x)$ be a polynomial of degree n. If we divide $p(x)$ by $(x-x_0)$, we get another polynomial $q(x)$, which is quotient of degree $n-1$.

Assume $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

& $q(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0$

$p(x) = (x-x_0)\,q(x)$

$a_n x^n + a_{n-1}x^{n-1} + \cdots a_0 = (x-x_0)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots b_0)$

Comparing the coefficients on both sides, we get,

$b_{n-1} = a_n$

$b_{n-2} = a_{n-1} + x_0 b_{n-1}$

$\vdots$

$b_0 = a_1 + x_0 b_1$

Thus,

$b_n = 0$

$b_{i-1} = a_i + x_0 b_i \quad i = n, n-1, \cdots 1$

Remainder $= a_0 + x_0 b_0$

Divide $(x^4 - 3x + 5)$ by $(x-4)$ using synthetic division & find quotient & remainder.

Soln:-

$p(x) = x^4 - 3x + 5$

$a_4 = 1 \quad a_3 = 0 \quad a_2 = 0 \quad a_1 = -3 \quad a_0 = 5$

$x_0 = 4$

$b_3 = a_4 = 1$

$b_2 = a_3 + x_0 b_3 = 4$

$b_1 = a_2 + x_0 b_2 = 16$

$b_0 = a_1 + x_0 b_1 = 3 + 4 \times 16 = 61$

Quotient,

$$q(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0$$
$$= x^3 + 4x^2 + 16x + 61$$

Remainder,

$$R = a_0 + x_0 b_0$$
$$= 5 + 4 \times 61 = 249$$

Find root of eqn $x^3 - x^2 - 14x + 24 = 0$ using Newton's method & synthetic division.

Soln:-

$f(x) = x^3 - x^2 - 14x + 24$, $x_0 = 1$, $E = 0.05$

$f'(x) = 3x^2 - 2x - 14$

Iteration 1

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{10}{(-13)} = 1.7692$$

$$\text{Error} = \left| \frac{x_1 - x_0}{x_1} \right| = 0.4346$$

Iteration 2

$$x_1 = 1.7692 - \frac{1.6388}{(-8.1481)} = 1.9703$$

Error $= 0.102$

DATE ________ PAGE ______

Iteration 3

$$x_1 = 1.9703 - \frac{0.1825}{-6.2943} = 1.9992$$

Error = 0.014

Hence third root of eq^n = 1.9992

Use synthetic division to deflate the polynomial by $(x-1.9992)$, which gives the polynomial

$$g(x) = b_2x^2 + b_1x + b_0$$

where $f(x) = x^3 - x^2 - 14x + 24$

$$a_3 = 1 \quad a_2 = -1 \quad a_1 = -14 \quad a_0 = 24$$

$$b_2 = a_3 = 1$$

$$b_1 = a_2 + x_0 b_2 = -1 + 1.9992 \times 1 = 0.9992$$

$$b_0 = a_1 + x_0 b_1 = -14 + 1.9992 \times 0.9992 = -12.00$$

$$g(x) = x^2 + 0.9992x - 12.002$$

$$x_0 = 1.9992$$

$$g'(x) = 2x + 0.9992$$

Iteration 1

$$x_1 = 1.9992 - \frac{(-6.00759)}{4.9976} = 3.2012$$

Error = 0.37541

Iteration 2

$$x_1 = 3.2012 - \frac{1.4443}{7.4016} = 3.0060$$

Error = 0.0649

Iteration 3

$$x_1 = 3.0060 - \frac{0.03763}{7.0112} = 3.0006$$

Error = 0.00178

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DATE ________ PAGE ________

Hence 2nd root of eqn $= 3.0006$

Again use synthetic division to deflate the polynomial by $(x-3.0006)$, which gives the polynomial,

$h(x) = b_1 x + b_0$

$g(x) = x^2 + 0.9992x - 12.002$

$a_2 = 1 \quad a_1 = 0.9992 \quad a_0 = -12.002$

$b_1 = a_2 = 1$

$b_0 = a_1 + x_0 b_1 = 0.9992 + 3.0006 \times 1 = 3.999$

$h(x) = x + 3.9998$

1st root is given by

$x + 3.9998 = 0$

$\therefore x = -3.9998$

$x^3 + 3x - 1 = 0$