Numerical Differentiation

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Differentiating continuous function :-

1) Two point forward difference formula

$$f'(x_i) = \frac{f(x_i+h) - f(x_i)}{h}$$

2) Two point backward difference formula

$$f'(x_i) = \frac{f(x_i) - f(x_i-h)}{h}$$

3) Three point formula

$$f'(x_i) = \frac{f(x_i+h) - f(x_i-h)}{2h}$$

Estimate approximate derivative of $f(x) = x^2$ at $x=1$ for $h=0.2$ & 0.05 using a) two point forward difference formula b) two point backward difference formula c) Three point formula. Compare with true value.

Soln:-

Two point forward difference formula

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

For $h=0.2$

$$f'(1) = \frac{f(1+0.2) - f(1)}{0.2} = \frac{1.2^2 - 1^2}{0.2} = 2.2$$

True value, $f'(x) = 2x$, $f'(1) = 2$

$$\text{Error} = \left|\frac{2-2.2}{2}\right| = 10\%$$

For $h=0.05$

$$f'(1) = \frac{f(1+0.05) - f(1)}{0.05} = \frac{1.05^2 - 1^2}{0.05} = 2.05$$

$$\text{Error} = \left|\frac{2-2.05}{2}\right| = 2.5\%$$

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Two point Backward Difference formula

$$f'(x) = \frac{f(x) - f(x-h)}{h}$$

For $h=0.2$

$$f'(1) = \frac{f(1) - f(0.8)}{0.2} = \frac{1^2 - 0.8^2}{0.2} = 1.8$$

$$\text{Error} = \left|\frac{2-1.8}{2}\right| = 10\%$$

For $h = 0.05$,

$$f'(1) = \frac{f(1) - f(0.95)}{0.05} = \frac{1^2 - 0.95^2}{0.05} = 1.95$$

$$\text{Error} = \left|\frac{2-1.95}{2}\right| = 2.5\%$$

Three point formula,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

For $h=0.2$,

$$f'(1) = \frac{f(1.2) - f(0.8)}{2\times 0.2} = \frac{1.2^2 - 0.8^2}{0.4} = 2$$

$$\text{Error} = \left|\frac{2-2}{2}\right| = 0\%$$

For $h=0.05$,

$$f'(1) = \frac{f(1.05) - f(0.95)}{2\times 0.05} = \frac{1.05^2 - 0.95^2}{2\times 0.05} = 2$$

$$\text{Error} = \left|\frac{2-2}{2}\right| = 0\%$$

\# Find value of derivative at $x=45°$ for $f(x) = \sin x + 1$ by using $h=0.1$ & 0.001.

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Differentiating discrete (tabulated) functions

1) **Derivative using Newton's Divided Difference formula:-**

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cancel{a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3)} + \ldots$$

$$+ a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \ldots$$

Now,

$$f'(x) = a_1 + a_2\{(x-x_0)+(x-x_1)\} + a_3\{(x-x_0)(x-x_2) + (x-x_1)(x-x_2) + (x-x_0)(x-x_1)\} + a_4\{(x-x_0)(x-x_1)(x-x_2) +$$

Again,

$$f''(x) = 2a_2 + 2a_3\{(x-x_0)+(x-x_1)+(x-x_2)\}$$

$$f'(x) = a_1 + a_2\{(x-x_0)+(x-x_1)\} + a_3\{(x-x_0)(x-x_2) + (x-x_1)(x-x_2) + (x-x_0)(x-x_1)\} + a_4[(x-x_3).\{(x-x_0)(x-x_2) + (x-x_1)(x-x_2) + (x-x_0)(x-x_1)\} + (x-x_0)(x-x_1)(x-x_2)] +$$

max & min

Use Newton's divided difference to find 1st & 2nd derivative of function tabulated below at $x = 1.8$

x	1	2	3	4
$f(x)$	0	7	26	63

Soln:-

x	$f(x)$ (a_0)	(a_1)	(a_2)	(a_3)
1	0			
		7		
2	7		6	
		19		1
3	26		9	
		37		
4	63			

We've,

$$f'(x) = a_1 + a_2\{(x-x_0)+(x-x_1)\} + a_3\{(x-x_0)(x-x_2) + (x-x_1)(x-x_2) + (x-x_0)(x-x_1)\} + \cdots$$

$$f'(1.8) = 7 + 6\{(1.8-1)+(1.8-2)\} + 1\{(1.8-1)(1.8-3) + (1.8-2)(1.8-3) + (1.8-1)(1.8-2)\}$$

$$= 9.72$$

$$f''(x) = 2a_2 + 2a_3\{(x-x_0)+(x-x_1)+(x-x_2)\}$$

$$f''(1.8) = 2\times 6 + 2\times 1\{(1.8-1)+(1.8-2)+(1.8-3)\}$$

$$= 10.8$$

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Derivative Using Newton's Forward Difference Formula:-

$$f(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}$$

$$\Delta^3 f(x_0) + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f(x_0) + \cdots$$

where $s = \frac{x-x_0}{h}$ $\quad \frac{ds}{dx} = \frac{1}{h}$

$$f'(x) = \frac{d f(x)}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{1}{h}\left[\Delta f(x_0) + \frac{(2s-1)}{2!}\Delta^2 f(x_0) + \frac{(3s^2-6s+2)}{3!}\right.$$

$$\left.\Delta^3 f(x_0) + \frac{(4s^3-18s^2+22s-6)}{4!}\Delta^4 f(x_0) + \cdots\right]$$

Again,

$$f''(x) = \frac{d f'(x)}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 f(x_0) + \frac{(6s-6)}{3!}\Delta^3 f(x_0) + \right.$$

$$\left.\frac{(12s^2-36s+22)}{4!}\Delta^4 f(x_0) + \cdots\right]$$

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Find 1st & 2nd derivative of functions tabulated below at 1.1 using Newton's Forward Difference formula.

x	1.0	1.2	1.4	1.6	1.8
$f(x)$	0	0.1	0.5	1.25	2.4

Soln:-

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0	0.1	0.3	0.05	0
1.2	0.1	0.4	0.35	0.05	
1.4	0.5	0.75	0.4		
1.6	1.25	1.15			
1.8	2.4				

$$s = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.2} = 0.5$$

$$h = 1.2 - 1.0 = 0.2$$

$$f'(x) = \frac{1}{h}\left[\Delta f(x_0) + \frac{(2s-1)}{2!}\Delta^2 f(x_0) + \frac{(3s^2 - 6s + 2)}{3!}\Delta^3 f(x_0) + \dots\right]$$

$$f''(1.1) = \frac{1}{0.2}\left[0.1 + \left(\frac{3 \times 0.5^2 - 6 \times 0.5 + 2}{6}\right) 0.05\right]$$

$$= 0.4895$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 f(x_0) + \frac{(6s-6)}{3!}\Delta^3 f(x_0) + \dots\right]$$

$$= \frac{1}{0.2^2}\left[0.3 + (0.5-1) \times 0.05\right]$$

$$= 6.875$$

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Derivative using Newton's Backward Difference Formula :-

$$f(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \frac{s(s+1)(s+2)}{3!}\nabla^3 f(x_n)$$

$$+ \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f(x_n) + \cdots$$

Where $s = \frac{x - x_n}{h}$ $\quad \frac{ds}{dx} = \frac{1}{h}$

$$f'(x) = \frac{df(x)}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{1}{h}\left[\nabla f(x_n) + \frac{(2s+1)}{2!}\nabla^2 f(x_n) + \frac{(3s^2+6s+2)}{3!}\nabla^3 f(x_n)\right.$$

$$\left. + \frac{(4s^3+18s^2+22s+6)}{4!}\nabla^4 f(x_n) + \cdots\right]$$

$$f''(x) = \frac{df'(x)}{ds} \cdot \frac{ds}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 f(x_n) + \frac{(6s+6)}{3!}\nabla^3 f(x_n) + \frac{(12s^2+36s+22)}{4!}\nabla^4 f(x_n) + \cdots\right]$$

$$s = \frac{x - x_n}{h} = \frac{1.1 - 1.8}{0.2} = -3.5$$

$$f'(1.1) = \frac{1}{0.2}\left[1.15 + \left(\frac{2\times(-3.5)+1}{2}\right)0.4 + \left\{\frac{3(-3.5)^2 + 6\times(-3.5)+2}{6}\right\}0.05\right] = 0.4895$$

$$f''(1.1) = \frac{1}{0.2^2}\left[0.4 + (-3.5+1)\times 0.05\right] = 6.875$$

Maxima & Minima of tabulated functions:-

Where,

$$f'(x) = \frac{1}{h}\left\{\Delta f(x_0) + \frac{1}{2!}(2s-1)\Delta^2 f(x_0) + \frac{1}{3!}(3s^2-6s+2)\Delta^3 f(x_0)\right\}$$

For maxima & minima,

$$f'(x) = 0$$

$$\Delta f(x_0) + \frac{1}{2!}(2s-1)\Delta^2 f(x_0) + \frac{1}{3!}(3s^2-6s+2)\Delta^3 f(x_0) = 0$$

$$\frac{1}{2}\Delta^3 f(x_0)\, s^2 + \{\Delta^2 f(x_0) - \Delta^3 f(x_0)\}s + \{\Delta f(x_0) - 1/2\,\Delta^2 f(x_0) + 1/3\,\Delta^3 f(x_0)\} = 0$$

$$as^2 + bs + c = 0$$

where

$a = 1/2\,\Delta^3 f(x_0)$

$b = \Delta^2 f(x_0) - \Delta^3 f(x_0)$

$c = \Delta f(x_0) - 1/2\,\Delta^2 f(x_0) + 1/3\,\Delta^3 f(x_0)$

Now,

$x = x_0 + sh$

Find maxm & minm values of function tabulated below:

x	0	1	2	3
$f(x)$	-5	-7	-3	13

Solns:-

x	$f(x)$	Δf
0	-5	
		-2
1	-7	
		4
2	-3	
		16
3	13	

$a = 1/2\,\Delta^3 f(x_0)$

$b = \Delta^2 f(x_0) -$

$c = \Delta f(x_0) -$

$= -2 - 1/2 \times$

$as^2 + bs + c =$

$3s^2 - 3 = 0$

$x = x_0 + sh$

$x = s$

Now,

$f(x) = f(x_0)$

$= -5 +$

$f(x) = x^3 - 3$

$f'(x) = 3x^2 -$

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x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-5			
		-2		
1	-7		6	
		4		6
2	-3		12	
		16		
3	13			

$a = \frac{1}{2}\Delta^3 f(x_0) = \frac{1}{2} \times 6 = 3$

$b = \Delta^2 f(x_0) - \Delta^3 f(x_0) = 6 - 6 = 0$

$c = \Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0)$

$= -2 - \frac{1}{2} \times 6 + \frac{1}{3} \times 6 = -3$

$as^2 + bs + c = 0$

$3s^2 - 3 = 0 \qquad s = \pm 1$

$x = x_0 + sh \qquad h = 1$

$x = s$

Now,

$$f(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$

$$= -5 + x(-2) + \frac{x(x-1)}{\cancel{2}} \times \overset{3}{\cancel{6}} + \frac{x(x-1)(x-2)}{\cancel{6}} \times \cancel{6}$$

$f(x) = x^3 - 3x - 5$

$f'(x) = 3x^2 - 3 \qquad f''(x) = 6x$

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At $x=1$ $f''(1) = 6 > 0$, we've ~~maxima~~ minima at $x=1$ &

$f(1) = -7$

At $x=-1$ $f''(-1) = -6 < 0$, we've maxima at $x=-1$ &

$f(-1) = -3$

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Numerical Integration

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Newton cotes integration formula:-

$$I = \int_{x_0}^{x_n} f(x)\,dx$$

Let us divide the interval (x_0, x_n) into n sub-interval of equal width i.e. $h = \frac{x_n - x_0}{n}$.

Therefore,

$$f(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) + \dots$$

where $s = \frac{x - x_0}{h}$ $\quad h\,ds = dx$

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} \left\{ f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0) - \right\} dx$$

$$= h\left[s f(x_0) + \frac{s^2}{2}\Delta f(x_0) + \frac{1}{2}\left(\frac{s^3}{3} - \frac{s^2}{2}\right)\Delta^2 f(x_0) + \frac{1}{3!}\left(\frac{s^4}{4} - s^3 + s^2\right)\Delta^3 f(x_0) + \dots \right]_0^n$$

$$= h\left[n f(x_0) + \frac{n^2}{2}\Delta f(x_0) + \frac{1}{12}(2n^3 - 3n^2)\Delta^2 f(x_0) + \frac{1}{24}(n^4 - 4n^3 + 4n^2)\Delta^3 f(x_0) + \dots \right]$$

$$= nh\left[f(x_0) + \frac{n}{2}\Delta f(x_0) + \frac{1}{12}(2n^2 - 3n)\Delta^2 f(x_0) + \frac{1}{24}(n^3 - 4n^2 + 4n)\Delta^3 f(x_0) + \dots \right]$$ (Quadrature formula)

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Trapezoidal rule:-

It assumes $n=1$ & neglect higher order forward differences.

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[f(x_0) + \tfrac{1}{2}\Delta f(x_0)\right]$$

$$= \left(\frac{x_1 - x_0}{1}\right)\left[f(x_0) + \tfrac{1}{2}\{f(x_1) - f(x_0)\}\right]$$

$$= \frac{(x_1 - x_0)}{2}\left[f(x_0) + f(x_1)\right]$$

\# Find $\int_0^1 e^{-x^2}\,dx$ by using Trapezoidal rule.

Soln:-

$x_0 = 0 \quad x_1 = 1 \quad f(x) = e^{-x^2}$

$$\int_0^1 e^{-x^2}\,dx = \left(\frac{1-0}{2}\right)\left(e^0 + e^{-1}\right) = 0.6839$$

$$\boxed{\int_2^8 (x^3+2)\,dx = 1572}$$

Composite trapezoidal rule:-

In order to improve the accuracy of Trapezoidal rule, integration interval can be divided into K segments of equal width,

$$h = \frac{x_n - x_0}{K}$$

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Now,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \cdots$$

$$= \frac{h}{2}\{f(x_0) + f(x_0+h)\} + \frac{h}{2}\{f(x_0+h) + f(x_0+2h)\}$$

$$+ \cdots$$

$$= \frac{h}{2}\left[f(x_0) + 2\{f(x_0+h) + f(x_0+2h) + \cdots\} + f(x_n)\right]$$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{K-1} f(x_0+ih) + f(x_n)\right]$$

(Multiple Segment Trapezoidal rule)

Use Trapezoidal rule to approximate $\int_0^2 e^{x^2}dx$ taking ten number of intervals.

Soln:-

$x_0 = 0 \quad x_n = 2 \quad K = 10$

$h = \frac{x_n - x_0}{K} = \frac{2-0}{10} = 0.2$

$f(x) = e^{x^2}$

$f(x_0) = f(0) = 1$

$f(x_1) = f(x_0+h) = f(0.2) = 1.0408$

$f(x_2) = f(x_0+2h) = f(0.4) = 1.1735$

$f(x_3) = f(x_0+3h) = f(0.6) = 1.4333$

$f(x_4) = f(x_0+4h) = f(0.8) = 1.8964$

$f(x_5) = f(x_0+5h) = f(1) = 2.7182$

$f(x_6) = f(x_0+6h) = f(1.2) = 4.2206$

$f(x_7) = f(x_0+7h) = f(1.4) = 7.0993$

$f(x_8) = f(1.6) = 12.93$

$f(x_9) = f(1.8) = 25.53$

$f(x_{10}) = f(2) = 54.59$

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$$\int_0^2 e^{x^2}dx = \frac{0.2}{2}\Big[1 + 2(1.0408 + 1.1735 + 1.4333 + 1.8964 + 2.7182 + 4.2206 + 7.0993 + 12.935 + 23.533) + 54.598\Big] = 17.16862$$

$\int_1^{?} \sqrt{1+x^2}\,dx$ for $k=4$ & $k=8$ (12.78 & 12.78) (Simpson's 3/8)

Simpson's 1/3 rule :-

It assumes $n=2$ & neglect higher order forward diff

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[f(x_0) + \Delta f(x_0) + \tfrac{1}{6}\Delta^2 f(x_0)\right]$$

$$= 2h\left[f(x_0) + \{f(x_1) - f(x_0)\} + \tfrac{1}{6}\{f(x_2) - 2f(x_1) + f(x_0)\}\right]$$

$$= \frac{2h}{6}\left[6\cancel{f(x_0)} + 6f(x_1) - 6\cancel{f(x_0)} + f(x_2) - 2f(x_1) + f(x_0)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

where $h = \dfrac{x_2 - x_0}{2}$

Evaluate $\int_0^{\pi/2} \sin x\,dx$ using Simpson's 1/3 rule. Compare with true value.

Soln:-

$f(x) = \sin x$

$x_0 = 0 \quad x_2 = \pi/2 \quad x_1 = \ldots$

$h = \dfrac{x_2 - x_0}{2} = \pi/4$

$f(x_0) = f(0) = 0$

$f(x_1) = f(x_0 + h) = f(\pi/4) \ldots$

$f(x_2) = f(\pi/2) = 1$

We've,

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) \ldots\right.$$

$$\int_0^{\pi/2} \sin x\,dx = \frac{\pi}{12}\left(\ldots\right.$$

True value,

$$\int_0^{\pi/2} \sin x\,dx = \ldots$$

$$\text{Error} = \left|\frac{1 - 1.002\ldots}{1}\right.$$

$$\boxed{\int_0^2 (3x^2 + 2x - 5)\,dx = \ldots}$$

Composite Simpson's 1/3 ...

Divide the ... segments & apply ... over every two ... need to be even ... width of se...

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$x_0 = 0 \quad x_2 = \pi/2 \quad x_1 = x_0 + h$

$h = \frac{x_2 - x_0}{2} = \pi/4$

$f(x_0) = f(0) = 0$

$f(x_1) = f(x_0+h) = f(\pi/4) = 0.7071$

$f(x_2) = f(\pi/2) = 1$

Now,

$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$

$\int_0^{\pi/2} \sin x\,dx = \pi/12\,(0 + 4 \times 0.7071 + 1) = 1.0022$

True value,

$\int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = 1$

$\text{Error} = \left|\frac{1 - 1.0022}{1}\right| = 0.22\%$

$$\boxed{\int_0^2 (3x^2+2x-5)\,dx = 2}$$

Composite Simpson's 1/3 rule:-

Divide the interval (x_0, x_n) into K segments & apply Simpson's 1/3 rule repeatedly over every two segments. Note that K need to be even.

Width of segment, $h = \frac{x_n - x_0}{K}$

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Now,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] +$$

$$\frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right] + \cdots$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1) + f(x_3) + \cdots\} + 2\{f(x_2) + f(x_4) + \cdots\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{K-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{K-2} f(x_i) + f(x_n)\right]$$

Use Simpson's 1/3 rule to evaluate $\int_0^{\pi} \sqrt{5+\sin(x)}\,dx$ considering six sub-intervals.

Soln:-

$f(x) = \sqrt{5+\sin(x)}$

$x_0 = 0 \quad x_n = \pi \quad K = 6$

$h = \dfrac{x_n - x_0}{K} = \pi/6$

$f(x_0) = f(0) = 2.236$

$f(x_1) = f(x_0+h) = f(\pi/6) = 2.3452$

$f(x_2) = f(x_0+2h) = f(\pi/3) = 2.4219$

$f(x_3) = f(x_0+3h) = f(\pi/2) = 2.4494$

$f(x_4) = f(x_0+4h) = f(2\pi/3) = 2.4219$

$f(x_5) = f(x_0+5h) = f(5\pi/6) = 2.3452$

$f(x_6) = f(x_0+6h) = f(\pi) = 2.2360$

$\int_0^{\pi} \sqrt{5+\sin(x)}$

2.34

Simpson's 3

Put

difference

$\int_{x_0}^{x_3} f(x$

=

$2f(x_3) +$

=

ane

Integra

Simpson

Soln:-

$$\int_0^{\pi} \sqrt{5+\sin(x)}\, dx = \frac{\pi}{18}\{2.236 + 4(2.3452 + 2.4494 + 2.3452) + 2(2.4219 + 2.4219) + 2.2360\}$$

$$= 7.4558$$

Simpson's 3/8 rule:-

Put $n=3$ & neglect higher order forward difference in quadrature formula,

$$\int_{x_0}^{x_3} f(x)\, dx = 3h\left[f(x_0) + \frac{3}{2}\Delta f(x_0) + \frac{3}{4}\Delta^2 f(x_0) - \frac{1}{8}\Delta^3 f(x_0)\right]$$

$$= 3h\left[f(x_0) + \frac{3}{2}(f(x_1) - f(x_0)) + \frac{3}{4}\{f(x_2) - 2f(x_1) + f(x_0)\} + \frac{1}{8}\{-f(x_0) + 3f(x_1) - 3f(x_2) + f(x_3)\}\right]$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

where $h = \frac{x_3 - x_0}{3}$

Integrate the following function by using Simpson's 3/8 rule:

$$\int_0^{\pi/2} \frac{\sin x}{x}\, dx$$

Soln:-

$$f(x) = \frac{\sin x}{x}$$

$x_0 = 0 \quad x_3 = \pi/2 \qquad h = \frac{x_3 - x_0}{3} = \pi/6$

$f(x_0) = 1$

$f(x_1) = f(x_0 + h) = f(\pi/6) = 0.9549$

$f(x_2) = f(x_0 + 2h) = f(\pi/3) = 0.8269$

$f(x_3) = f(\pi/2) = 0.6366$

We've,

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

$$\int_0^{\pi/2} \frac{\sin x}{x}\,dx = \frac{3}{8} \times \frac{\pi}{6}\left[1 + 3 \times 0.9549 + 3 \times 0.8269 + 0.6366\right]$$

$$= 1.3709 \qquad (0.02467)$$

\# $\int_0^2 (2 + \cos(2\sqrt{x}))\,dx = 3.46$

Composite Simpson's 3/8 rule:-

It divides the interval (x_0, x_n) into K segments & apply Simpson's 3/8 rule repeatedly over every three segments. Therefore K needs to be multiple of 3.

Width of segment,

$$h = \frac{x_n - x_0}{K}$$

Now,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \dots$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] + \frac{3h}{8}\left[f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6)\right] + \dots$$

$$= \frac{3h}{8}\left[f(x_0) + 3\{f(x_1) + f(x_2) + f(x_4) + f(x_5)\} + 2\{f(x_3) + f(x_6) + \cdots\} + f(x_n)\right]$$

$$= \frac{3h}{8}\left[f(x_0) + 3\sum_{\substack{i=1 \\ i \bmod 3 \neq 0}}^{n-1} f(x_i) + 2\sum_{\substack{i=1 \\ i \bmod 3 = 0}}^{n-1} f(x_i) + f(x_n)\right]$$

Calculate the integral value of following tabulated function from $x=0$ to $x=1.6$ using Simpson's 3/8 rule:

x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$f(x)$	0	0.24	0.55	0.92	1.63	1.84	2.37	2.95	3.56

Soln:-

$h = 0.2 \quad x_0 = 0 \quad x_n = 1.6$

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\{f(x_0) + 3\{f(x_1) + f(x_2) + f(x_4) + f(x_5) + f(x_7)\} + 2\{f(x_3) + f(x_6)\} + f(x_8)\}$$

$$= \frac{3\times0.2}{8}\{0 + 3(0.24 + 0.55 + 1.63 + 1.84 + 2.95) + 2(0.92 + 2.37) + 3.56\}$$

$= 2.38275$

$$\boxed{\int_0^3 \frac{1}{x+4}\,dx \quad K=9}$$

Romberg Integration :-

$$\begin{tikzcd}
T(0,0) \arrow[d] \arrow[dr] & & \\
T(1,0) \arrow[r] \arrow[d] \arrow[dr] & T(1,1) \arrow[dr] & \\
T(2,0) \arrow[r] & T(2,1) \arrow[r] & T(2,2)
\end{tikzcd}$$

$$T(0,0) = \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

$$T(i,0) = \frac{T(i-1,0)}{2} + \frac{h}{2^i}\sum_{K=1}^{2^{i-1}} f(x_0 + (2K-1)h/2^i)$$ (recursive Trapezoidal rule)

where $h = x_1 - x_0$

$$T(m+K,K) = \frac{4^K T(m+K,K-1) - T(m+K-1,K-1)}{4^K - 1}$$ (recursive Romberg integration formula)

Compute Romberg estimate T(2,2) for $\int_0^1 \frac{1}{(1+x)}\,dx$.

Soln:-

$$T(0,0) = \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

$x_0 = 0 \quad x_1 = 1 \quad f(x) = 1/(1+x) \quad h = x_1 - x_0 = 1$

$$T(0,0) = 1/2\,(1 + 0.5) = 0.75$$

$$T(1,0) = \frac{T(0,0)}{2} + \frac{h}{2}\cdot f(x_0 + h/2)$$

$$T(1,0) = \frac{0.75}{2} + \frac{1}{2}$$

$= 0.7083$

$$T(2,0) = \frac{T(1,0)}{2} +$$

$$= \frac{0.7083}{2} +$$

$$= \frac{0.7083}{2}$$

$$T(1,1) = \frac{4T(1,0}{4}$$

$= 0.694$

$$T(2,1) = 4T(2$$

$= 0.693$

$$T(2,2) = 4T(2$$

$$= \frac{0.69}{2}$$

T(2,2) $\int_0 1/x$

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$$T(1,0) = \frac{0.75}{2} + \frac{1}{2} f(1/2) = \frac{0.75}{2} + \frac{1}{2}(0.6666)$$

$$= 0.7083$$

$$T(2,0) = \frac{T(1,0)}{2} + \frac{h}{4}\left\{f(x_0 + h/4) + f(x_0 + 3h/4)\right\}$$

$$= \frac{0.7083}{2} + \frac{1}{4}\left\{f(1/4) + f(3/4)\right\}$$

$$= \frac{0.7083}{2} + 1/4(0.8 + 0.5714) = 0.697$$

$$T(1,1) = \frac{4T(1,0) - T(0,0)}{4-1} = \frac{4 \times 0.7083 - 0.75}{3}$$

$$= 0.6944$$

$$T(2,1) = \frac{4T(2,0) - T(1,0)}{4-1} = \frac{4 \times 0.697 - 0.7083}{3}$$

$$= 0.6932$$

$$T(2,2) = \frac{4^2 T(2,1) - T(1,1)}{4^2-1} = \frac{16 \times 0.6932 - 0.6944}{15}$$

$$= 0.69312$$

$$\# \; T(2,2) \quad \int_1^2 1/x \, dx \quad (0.693175)$$

Gaussian Integration :-

$$\int_{-1}^{1} f(x)\,dx = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

Now,

$$\int_{a}^{b} f(x)\,dx = c\int_{-1}^{1} g(z)\,dz$$

Let, $x = Az + B$

$a = A(-1) + B$

$B - A = a$

$b = A(1) + B$

$B + A = b$

$B = \frac{a+b}{2}$ $\quad A = \frac{b-a}{2}$

Thus, $x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right)$

$dx = \left(\frac{b-a}{2}\right)dz$

$$\int_{a}^{b} f(x)\,dx = \left(\frac{b-a}{2}\right)\int_{-1}^{1} g(z)\,dz$$

Compute the integral $\int_{-2}^{2} e^{-x/2}\,dx$ using Gaussian two-point formula.

Soln:-

$a = -2 \quad b = 2 \qquad x = 2z$

$$\int_{-2}^{2} e^{-x/2}\,dx = \left(\frac{2+2}{2}\right)\int_{-1}^{1} e^{-z}\,dz$$

$$= 2\left(e^{-1/\sqrt{3}} + e^{1/\sqrt{3}}\right) = 4.6854$$

$\int_{2}^{4} (x^3+1)\,dx = 61.99$

Chapter(4)

System of eqn:-

$a_{11}x_1 + a_{12}x_2 + \ldots$

$a_{21}x_1 + a_{22}x_2 + \ldots$

$a_{31}x_1 + a_{32}x_2 + \ldots$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$AX = \ldots$

Consistent sys...

Unique Soln

Inf...

Ill-conditioned...

It ...

difficult to ...

lines interse...

E.g. x ...

0.4...

Chapter(4) :

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System of eqn :-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$AX = C$$

Consistent system — Inconsistent System

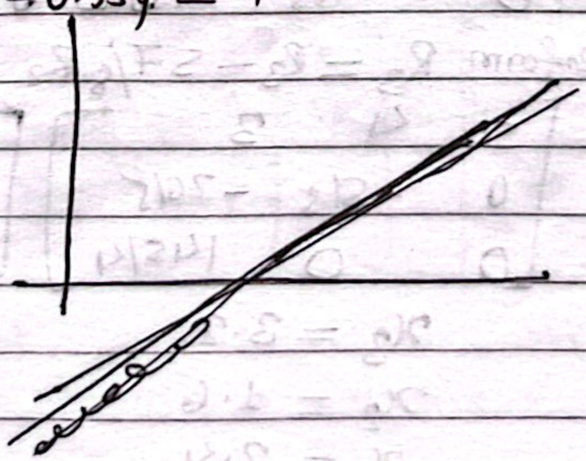
Consistent system: Unique soln — Infinite soln

Ill-conditioned system :-

It has many solutions & it is difficult to find the exact point at which lines intersect.

E.g. $x - 2y = -2$

$0.45x - 0.95y = -1$



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Solving System of linear eq^{n}s:-

1) Direct method

a) Naive Gauss elimination method

Two steps:

i) Forward elimination of unknown

ii) Back substitution

\# Solve the following system of linear eq^{n}s using Gauss elimination method:

$$5x_1 + 4x_2 + 3x_3 = 28$$
$$7x_1 + 4x_2 - x_3 = 20$$
$$8x_1 - 5x_2 + 4x_3 = 24$$

Sol^n:-

$$\begin{bmatrix} 5 & 4 & 3 \\ 7 & 4 & -1 \\ 8 & -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 20 \\ 24 \end{bmatrix}$$

Perform $R_2 = R_2 - (7/5)R_1$ & $R_3 = R_3 - (8/5)R_1$

$$\begin{bmatrix} 5 & 4 & 3 \\ 0 & -8/5 & -26/5 \\ 0 & -57/5 & -4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ -96/5 \\ -\frac{104}{5} \end{bmatrix}$$

Perform $R_3 = R_3 - 57/8 R_2$

$$\begin{bmatrix} 5 & 4 & 3 \\ 0 & -8/5 & -26/5 \\ 0 & 0 & 145/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ -96/5 \\ 116 \end{bmatrix}$$

$$x_3 = 3.2$$
$$x_2 = 1.6$$
$$x_1 = 2.4$$

b) Gauss elimination

\# Solve using Ga[...] partial pivoting:

$$0.02x + 0.01y = \ldots$$
$$x + 2y + z = 1 \ldots$$
$$y + 2z + w = \ldots$$
$$100z + 200w = \ldots$$

Sol^n:-

$$\begin{bmatrix} 0.02 & 0.01 & \ldots \\ 1 & 2 & \ldots \\ 0 & 1 & \ldots \\ 0 & 0 & \ldots \end{bmatrix}$$

Since largest [...] a_{21} & make it [...]

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & \ldots \\ 0.02 & 0.01 & \ldots \\ 0 & 1 & \ldots \\ 0 & 0 & \ldots \end{bmatrix}$$

Perform $R_2 = \ldots$

$$\begin{bmatrix} 1 & 2 & \ldots \\ 0 & -0.03 & \ldots \\ 0 & 1 & \ldots \\ 0 & 0 & \ldots \end{bmatrix}$$

Since lar[...] a_{32} & m[...]

$R_2 \leftrightarrow R_3$

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Gauss elimination with partial pivoting

Solve using Gauss elimination method with partial pivoting:

$0.02x + 0.01y = 0.02$

$x + 2y + z = 1$

$y + 2z + w = 4$

$100z + 200w = 800$

Soln:-

$$\begin{bmatrix} 0.02 & 0.01 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 100 & 200 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0.02 \\ 1 \\ 4 \\ 800 \end{bmatrix}$$

Since largest value among a_{11}, a_{21}, a_{31} & a_{41} is a_{21} & make it pivot element by performing $R_1 \Leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0.02 & 0.01 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 100 & 200 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0.02 \\ 4 \\ 800 \end{bmatrix}$$

Perform $R_2 = R_2 - 0.02R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -0.03 & -0.02 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 100 & 200 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 800 \end{bmatrix}$$

Since largest value among a_{22}, a_{32} & a_{42} is a_{32} & make it pivot element by performing $R_2 \Leftrightarrow R_3$

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$$\begin{bmatrix}1 & 2 & 1 & 0\\0 & 1 & 2 & 1\\0 & -0.03 & -0.02 & 0\\0 & 0 & 100 & 200\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}1\\4\\0\\800\end{bmatrix}$$

Perform $R_3 = R_3 + 0.03R_2$

$$\begin{bmatrix}1 & 2 & 1 & 0\\0 & 1 & 2 & 1\\0 & 0 & 0.04 & 0.03\\0 & 0 & 100 & 200\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}1\\4\\0.12\\800\end{bmatrix}$$

Since larger value betn a_{33} & a_{43} is a_{43} & make it pivot element & by ~~perfo~~ performing $R_3 \leftrightarrow R_4$

$$\begin{bmatrix}1 & 2 & 1 & 0\\0 & 1 & 2 & 1\\0 & 0 & 100 & 200\\0 & 0 & 0.04 & 0.03\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}1\\4\\800\\0.12\end{bmatrix}$$

Perform $R_4 = R_4 - \frac{0.04}{100} R_3$

$$\begin{bmatrix}1 & 2 & 1 & 0\\0 & 1 & 2 & 1\\0 & 0 & 100 & 200\\0 & 0 & 0 & -0.05\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}1\\4\\800\\-0.2\end{bmatrix}$$

$w = 4,\ z = 0,\ y = 0\ \ x = 1$

Gauss Jordan method :-

$$\left[\begin{array}{ccc|c}a_{11} & a_{12} & a_{13} & c_1\\a_{21} & a_{22} & a_{23} & c_2\\a_{31} & a_{32} & a_{32} & c_3\end{array}\right]$$

Apply row operations
matrix into the follo

$$\left[\begin{array}{ccc|c}1 & 0 & 0 & b_1\\0 & 1 & 0 & b_2\\0 & 0 & 1 & b_3\end{array}\right]$$

$$X = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} = \text{solution}$$

\# Use Gauss Jordan
System of linear eqn

$a+b+2c = 1$

$2a-b+d = -2$

$a-b-c-2d =$

$2a-b+2c-d$

Soln:-

Augmented matri

$$\begin{bmatrix}1 & 1 & 2 & \\2 & -1 & 0 & \\1 & -1 & -1 & \\2 & -1 & 2 & \end{bmatrix}$$

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Gauss Jordan method :-

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$$

Apply row operations to reduce the augmented matrix into the following form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

$$X = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \text{Solution vector}$$

Use Gauss Jordan method to solve following system of linear eqns

$$a+b+2c = 1$$
$$2a-b+d = -2$$
$$a-b-c-2d = 4$$
$$2a-b+2c-d = 0$$

Soln:-

Augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 2 & -1 & 0 & 1 & -2 \\ 1 & -1 & -1 & -2 & 4 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right]$$

Perform $R_2 = R_2 - 2R_1$, $R_3 = R_3 - R_1$ & $R_4 = R_4 - 2R_1$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -3 & -4 & 1 & -4 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right]$$

Perform $R_2 = -\frac{1}{3}R_2$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & -2 & -3 & -2 & 3 \\ 0 & -3 & -2 & -1 & -2 \end{array}\right]$$

Perform $R_1 = R_1 - R_2$ $R_3 = R_3 + 2R_2$ & $R_4 = R_4 + 3R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & -1/3 & -8/3 & 17/3 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

Perform $R_3 = -3R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2/3 & 1/3 & -1/3 \\ 0 & 1 & 4/3 & -1/3 & 4/3 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

Perform $R_1 = R_1 - \frac{2}{3}R_3$, $R_2 = R_2 - \frac{4}{3}R_3$ & $R_4 = R_4 - 2R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & -18 & 36 \end{array}\right]$$

Perform $R_4 = -\frac{1}{18}R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Perform $R_1 = R_1 + 5R_4$, R_2 …

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Hence $a = 1$, $b = 2$, $c =$ …

Matrix Inversion:-

Step 1: Augment … identity matrix as be…

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right.$$

Step 2: Apply Gauss… augmented matrix … matrix to identity

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right.$$

The right hand … matrix is inverse

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$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & 11 \\ 0 & 1 & 0 & -11 & 24 \\ 0 & 0 & 1 & 8 & -17 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Perform $R_1 = R_1 + 5R_4$, $R_2 = R_2 + 11R_4$ & $R_3 = R_3 - 8R_4$,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

Hence $a = 1$, $b = 2$, $c = -1$ & $d = -2$

Matrix Inversion :-

Step1: Augment the coefficient matrix with identity matrix as below:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Step2: Apply Gauss Jordan method to augmented matrix to reduce coefficient matrix to identity matrix as below:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_{11}' & a_{12}' & a_{13}' \\ 0 & 1 & 0 & a_{21}' & a_{22}' & a_{23}' \\ 0 & 0 & 1 & a_{31}' & a_{32}' & a_{33}' \end{array}\right]$$

The right hand side of above augmented matrix is inverse of original coefficient matrix

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\# Find inverse of the matrix
$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Soln:-

Augmented matrix is
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Perform $R_2 = R_2 - 2R_1$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Perform $R_2 = 1/5 R_2$
$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Perform $R_1 = R_1 + R_2$ & $R_3 = R_3 + 2R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & 3/5 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & -4/5 & 2/5 & 1 \end{array}\right]$$

Perform $R_3 = 5R_3$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3/5 & 3/5 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Perform $R_1 =$ ...
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, inve...
$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \\ -4 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

Matrix factori...

A =

low...
trian...
m...

Now, Ax

LUx

$L^{-1}LU$

Ux

Lz

Dolittle L

l_{11}

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Perform $R_1 = R_1 - 3/5 R_3$ & $R_2 = R_2 + 2/5 R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

Therefore, inverse of matrix A is

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{bmatrix}$$

Matrix factorization :-

$$A = LU$$

L → lower triangular matrix

U → upper triangular matrix

Now, $AX = C$

$LUX = C$

$L^{-1}LUX = L^{-1}C = Z$ (let)

$UX = Z \quad \cdots (2)$

$LZ = C \quad \cdots (1)$

Dolittle LU decomposition,

$l_{11} = l_{22} = l_{33} = 1$

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Factorize the following matrix by using Dolittle LU decomposition

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Soln:-

$$A = LU$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$u_{11} = 25$ $\quad u_{12} = 5$ $\quad u_{13} = 1$

$l_{21} u_{11} = 64$ $\quad l_{21} u_{12} + u_{22} = 8$ $\quad l_{21} u_{13} + u_{23} = 1$

$l_{21} = 2.56$ $\quad u_{22} = -4.8$ $\quad u_{23} = -1.56$

$l_{31} u_{11} = 144$ $\quad l_{31} u_{12} + l_{32} u_{22} = 12$ $\quad l_{31} u_{13} + l_{32} u_{23} + u_{33} = 1$

$l_{31} = 5.76$ $\quad l_{32} = 3.5$ $\quad u_{33} = 0.7$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -4 & 6 & 1 \\ -4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

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Solve following system of eqns by using Dolittle LU decomposition method

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Soln:-

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

Solve $LZ = C$ by using forward substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$z_1 = 10$ $\quad 2/3\, z_1 + z_2 = 14$ $\quad 1/3\, z_1 + 4/5\, z_2 + z_3 = 14$

$z_2 = 22/3$ $\quad z_3 = 24/5$

Again, solve $UX = Z$ by using backward substitution

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

$8/5\, x_3 = 24/5$ $\quad 5/3\, x_2 + 4/3\, x_3 = 22/3$

$x_3 = 3$ $\quad x_2 = 2$

$3x_1 + 2x_2 + x_3 = 10$

$x_1 = 1$

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Cholesky method:-

If A is symmetrical, then

$$A = LL^T = U^TU$$

Factorize ~~the~~ given matrix by using Cholesky decomposition

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 80 & 44 \\ 7 & 44 & 89 \end{bmatrix}$$

Soln:-

$$A = U^TU$$

$$\begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 80 & 44 \\ 7 & 44 & 89 \end{bmatrix}$$

$u_{11}^2 = 1$ $\quad u_{11}u_{12} = 4$ $\quad u_{11}u_{13} = 7$

$u_{11} = 1$ $\quad u_{12} = 4$ $\quad u_{13} = 7$

$u_{12}^2 + u_{22}^2 = 80$ $\quad u_{12}u_{13} + u_{22}u_{23} = 44$

$u_{22} = \sqrt{80 - 4^2} = 8$ $\quad u_{23} = 2$

$u_{13}^2 + u_{23}^2 + u_{33}^2 = 89$

$u_{33} = 6$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 8 & 0 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

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Solve following system of eqⁿs by using cholesky decomposition technique

$$4x_1 + 10x_2 + 8x_3 = 44$$
$$10x_1 + 26x_2 + 26x_3 = 128$$
$$8x_1 + 26x_2 + 61x_3 = 214$$

Solⁿ:-

$$A = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Solve the eqⁿ $LZ = C$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$z_1 = 22$ $\quad 5z_1 + z_2 = 128$ $\quad 4z_1 + 6z_2 + 3z_3 = 214$

$z_2 = 18$ $\quad z_3 = 6$

Solve the eqⁿ $UX = Z$

$$\begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$3x_3 = 6$ $\quad x_2 + 6x_3 = 18$ $\quad 2x_1 + 5x_2 + x_3 = 22$

$x_3 = 2$ $\quad x_2 = 6$ $\quad x_1 = -8$

$x + y + 3z = 6$

$x + 5y + 5z = 20$

$3x + 5y + 19z = 106$

Iterative methods for solving System of linear eq^ns

1) Jacobi Iteration method

Use Jacobi iteration method to obtain the soln of following eq^ns:

$$6x_1 - 2x_2 + x_3 = 11$$
$$x_1 + 2x_2 - 5x_3 = -1$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$

Soln:-

Rewrite the eq^ns such that each eq^n has unknown with largest coefficient on LHS.