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NUMERICAL METHODS

BACHELOR OF ENGINEERING

Er. Arbind Kumar Gupta

POKHARA UNIVERSITY

PURBANCHAL UNIVERSITY

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NUMERICAL METHODS

Bachelor of Engineering

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Publisher

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Syllabus of Pokhara University
Numerical Methods (3-1-3)

Evaluation

	Theory	Practical	Total
Sessional	30	20	50
Final	50	–	50
Total	80	20	100

Course Objectives

1. To introduce numerical methods for interpolation, regressions, and root finding to the solution of problems.
2. To solve elementary matrix arithmetic problems analytically and numerically.
3. To find the solution of ordinary and partial differential equations.
4. To proved knowledge of relevant high level programming language for computing, implementing, solving, and testing of algorithms.

Course Contents

1. **Solution of Nonlinear Equations** **[10 hrs]**
   - 1.1 Review of calculus and Taylor's theorem
   - 1.2 Errors in numerical calculations
   - 1.3 Bracketing methods for locating a root, initial approximation and convergence criteria
   - 1.4 False position method, secant method and their convergence, Newton's method and fixed point iteration and their convergence.
2. **Interpolation and Approximation** **[7 hrs]**
   - 2.1 Lagrangian's polynomials
   - 2.2 Newton's interpolation using difference and divided differences
   - 2.3 Cubic spline interpolation
   - 2.4 Curve fitting: Least square lines for linear and nonlinear data
3. **Numerical Differentiation and Integration** **[5 hrs]**
   - 3.1 Newton's differentiation formulas
   - 3.2 Newton-Cote's, Quadrature formulas
   - 3.3 Trapezoidal and Simpson's Rules
   - 3.4 Gaussian integration algorithm
   - 3.5 Romberg integration formulas
4. **Solution of Linear Algebraic Equations** **[10 hrs]**
   - 4.1 Matrices and their properties
   - 4.2 Elimination methods, Gauss Jordan method, pivoting
   - 4.3 Method of factorization: Dolittle, Crout's and Cholesky's methods
   - 4.4 The inverse of matrix
   - 4.5 Ill-conditional system
   - 4.6 Iterative methods: Gauss Jacobi, Gauss Seidel, Relaxation methods
   - 4.7 Power method

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5. **Solution of Ordinary Differential Equations** [8 hrs]
   - 5.1 Overview of initial and boundary value problems
   - 5.2 The Taylor's series method
   - 5.3 The Euler method and its modifications
   - 5.4 Huen's method
   - 5.5 Runge-Kutta methods
   - 5.6 Solution of higher order equations
   - 5.7 Boundary value problems: Shooting method

6. **Solution of Partial Differential Equations** [5 hrs]
   - 6.1 Review of partial differential equations
   - 6.2 Elliptical equations, parabolic equations, hyperbolic equations and their relevant examples

Laboratory

Use of Matlab/Math-CAD/C/C++ or any other relevant high level programming language for applied numerical analysis. The laboratory experiments will consists of program development and testing of:

1. Solution of nonlinear equations
2. Interpolation, extrapolation, and regression
3. Differentiation and integration
4. Linear systems of equations
5. Ordinary differential equations (ODEs)
6. Partial differential equations (PDEs)

Text Books

1. Gerald, C.F. and Wheatly, P.O., '*Applied Numerical Analysis*', (7th Edition), New York: Addison Wesley Publishing Company.
2. Guha, S. and Srivastava, R., '*Numerical Methods: For Engineers and Scientists*', Oxford University Press.
3. Grewal, B. S. and Grewal, J. S. '*Numerical Methods in Engineering and Science*', (8th Edition), New Delhi: Khanna Publishers, 2010.
4. Balagurusamy, E., '*Numerical Methods*', New Delhi: Tata McGraw Hill, 2010.

References

1. Moin, Parviz. *Fundamentals of Engineering Numerical Analysis.* Cambridge University Press, 2001.
2. Lindfield, G.R. & Penny, J.E.T. *Numerical Methods: Using MATLAB.* Academic Press, 1012.
3. Schilling, J. & Harris, S.L. *Applied Numerical Methods for Engineers using MATLAB and C.* Thomson Publishers, 2005.
4. Sastry, S.S. Introductory Methods of Numerical Analysis (3rd Edition). New Dehli: Prentice Hall of India, 2002.
5. Rao, S.B. and Shantha, C.K. *Numerical Methods with Programs in Basic, Fortran and Pascal.* Hyderabad: Universities Press 2000.
6. Pratap, Rudra. Getting Started with MATLAB. Oxford University Press, 2010.
7. Vedamurthy, V.N. & Lyengar, N. *Numerical Methods.* Noida: Vikash Publication House, 2009

Syllabus of Purbanchal University
Numerical Methods (BRG370CO)

Year: III **Semester: I**

Teaching Schedule Hours/Week			Examination Scheme				Total
			Internal Assessment		Final		
Theory	Tutorial	Practical	Theory	Practical	Theory	Practical	
3	2	2/2	20	50	80	0	150

Course Objectives

To solve the engineering problems by using the theory of numerical computational procedures.

1. **Introduction** [4 hrs]
   - 1.1 Numerical computing process
   - 1.2 New trends in numerical computing
   - 1.3 Application in numerical computing
   - 1.4 Taxonomy of errors in numerical method
   - 1.5 Absolute, relative and percentage errors

2. **Solution of Non-Linear Equation** [7 hrs]
   - 2.1 Iterative methods and stopping criteria
   - 2.3 Bisection method and its convergence
   - 2.3 Horner's method
   - 2.4 Newton-Raphson method and its convergence
   - 2.5 Secant method and its convergence
   - 2.6 Evaluation of polynomials using Horner's rule

3. **Curve Fitting** [8 hrs]
   - 3.1 Interpolation
     - 3.1.1 Linear interpolation
     - 3.1.2 Lagrange interpolation
     - 3.1.3 Newton interpolation
     - 3.1.4 Newton divided different interpolation
     - 3.1.5 Spine interpolation: Cubic spines
     - 3.1.6 Control interpolation (Gauss forward/backward formulae)
   - 3.2 Regression
     - 3.2.1 Least square regression
     - 3.2.2 Fitting transcendental equations
     - 3.2.3 Fitting a polynomial function

4. **Numerical Differentiation and Integration** [7 hrs]
   - 4.1 Differentiating continuous function
     - 4.1.1 Forward difference quotient
     - 4.1.2 Backward difference quotient
     - 4.1.3 Central difference quotient
   - 4.2 Newton cotes methods of integration
     - 4.2.1 Trapezoidal rule and composite trapezoidal rule

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4.2.2 Simpson's $\frac{1}{3}$ rule and its composite

4.2.3 Simpson's $\frac{3}{8}$ rule

4.2.4 Bode's rule

4.3 Romberg integration

4.4 Gaussian integration

5. Linear Algebraic Equations [10 hrs]

5.1 Elimination approach

5.1.1 Basic gauss elimination

5.1.2 Gauss elimination with partial pivoting

5.1.3 Gauss Jordon method

5.1.4 LU decomposition methods

5.1.4.1 Do little algorithm

5.1.4.2 Crout algorithm

5.1.5 Matrix inversion method

5.1.6 Cholesky method

5.2 Interative method

5.2.1 Iconic method

5.2.2 Gauss-Seidal method

5.2.3 Eigen values and eigen vectors using power method and inverse power method

6. Solution of Ordinary Differential Equations [6 hrs]

6.1 Euler's method

6.2 Heun's method (predictor-corrector method)

6.3 Fourth order Runge-Kutta method

6.4 Systems of differential equations using Heun's method

6.5 Higher order differential equations using Heun's method

7. Solutions of Partial Differential Equations [3 hrs]

7.1 Elliptic equations

7.1.1 Poisson's equations

7.1.2 Laplace's equations

7.2 Parabolic equations

7.3 Hyperbolic equations

Laboratories

1. Review of properties of programming language
2. Bisection method
3. Newton-Raphson method
4. Secant method and Horner's rule
5. Lagrange interpolation
6. Linear regression
7. Basic gauss elimination method
8. Gauss Seidal method
9. Matrix inversion method
10. Trapezoidal rule
11. Simpson's $\frac{1}{3}$ rule
12. Simpson's $\frac{3}{8}$ rule
13. Solution of differential equation using Euler's method
14. Solution of differential equation using Runge-Kutta method

References

1. E. Balagurusamy, "*Numerical Methods*" Tata McGraw Hill.
2. S. Yakwitz and F. Azidarousky, "*An Introduction to Numerical Computations*", 2nd Edition, Macmillan Publishing Co., New York.
3. W. Cdhency and D Kaxaid, "*Numerical Mathematics 4 Computing*", 2nd Editor, Books/Cole Publishing.
4. C.F. Gerald and P.O. Wheatley, "*Applied Numerical Analysis*", 4th Edition, Addipon Wesley Publishing Co., New York.
5. W. It press, B.P. Flannery *et.al*, "*Numerical Recises Inc*", 1st Edition, Cambridge Press 1988.

Marking Scheme

Group A (Attempt six) **[6×10=60]**

Chapter no.	Chapter Topic	Credit	Final Marks
1	Introduction	4	10
2	Solution of non-linear equation	7	10
3	Curve fitting	8	10
4	Numerical differentiation and integration	7	10
5	Linear algebraic equation	10	10
6	Solution of ordinary differential equation	6	10
7	Solution of partial differential equation	3	10

Group B (Attempt any two) **[2×10=20]**

There may be three or four question related to program or algorithm.

NOTE

There may be minor variation in marks distribution according to credit hour allocated.

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Chapter 1

INTRODUCTION TO NUMERICAL METHOD

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	1.1.1 Characteristics	1
	1.1.2 Approximation	2
1.2	New Trends in Numerical Computing	2
1.3	Applications in Numerical Computing	2
1.4	Errors	3
	1.4.1 Taxonomy of Errors in Numerical Methods	3
1.5	Absolute and Relative Errors	4

1.1 NUMERICAL COMPUTING PROCESS

The process of numerical computing can be divided into following four phases.

i) Formulation of a mathematical model.
ii) Construction of an appropriate numerical method.
iii) Implementation of method to obtain solution.
iv) Validation of the solution.

1.1.1 Characteristics

i) Accuracy

It usually depends on number of iteration being carried out.

ii) Rate of convergence

This rate depends on how fast a desired result can be achieved.

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iii) Numerical stability

A numerical Method is usually unstable if solution is unreachable despi of infinite number of iterations.

iv) Efficiency

Efficiency depends on the effort of human or computer to get desire solution.

1.1.2 Approximation

An approach designed or used to solve complex mathematical problem by using basic arithmetic operations. This can be achieved by using:

- Computer based algorithmic mathematics.
- Most of numerical methods. [Usually iterative methods]

1.2 NEW TRENDS IN NUMERICAL COMPUTING

Numerical computation plays a very important role in solving real life mathematical, physical and engineering problems. The development of digital computers has enhanced the speed and accuracy of numerical computations.

In the field of science and engineering complex mathematical problems are solved using the methods and algorithms of numerical methods. This has helped in both the field to tackle the complex numericals in a more relevant and accurate way to obtain the desired solution.

The numerical method involves a large number of arithmetic calculations, which includes the iterative methods. Hence, use of such algorithm in science and engineering provide greater accuracy.

Before the invention of computer, the calculation of such complex numerical was very difficult, but the numerical method provided us with methods and their algorithms to solve such problems with great accuracy and efficiency. The computer also used their algorithm to compute such numerical, so on the whole the numerical method has provided a great help to the field of science and Engineering.

1.3 APPLICATIONS IN NUMERICAL COMPUTING

i) It helps engineers to calculate area of irregular fields using various rules like $\left(\frac{1}{3}\right)$ Simpson's rule, $\left(\frac{3}{8}\right)$ Simpson's rule, etc.

ii) It helps engineers to solve the complex differential equation problems consisting of large number of variables (motion of air, nature of soil settlement, afflux of water during flood, water hammer, etc. phenomenon) using techniques such as Runge-Kutta 4 method, shooting method, etc.

iii) It helps engineers to approximate the missing (lost) data by various interpolating techniques like Newton's forward and backward interpolation, cubic spline interpolation, etc.

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iv) It helps engineers to best fit the data obtained from field and to determine the nature of phenomenon. *For example;* rating curve, infiltration coefficient are determined by curve fitting.

1.4 ERRORS

Errors are integral part of any numerical calculations. Different types of errors arise during the process of numerical computing. All these errors contribute to the total error in the final result. The figure below shows the "Taxonomy of Errors", which includes contribution of different errors to the total errors obtained in any numerical computation.

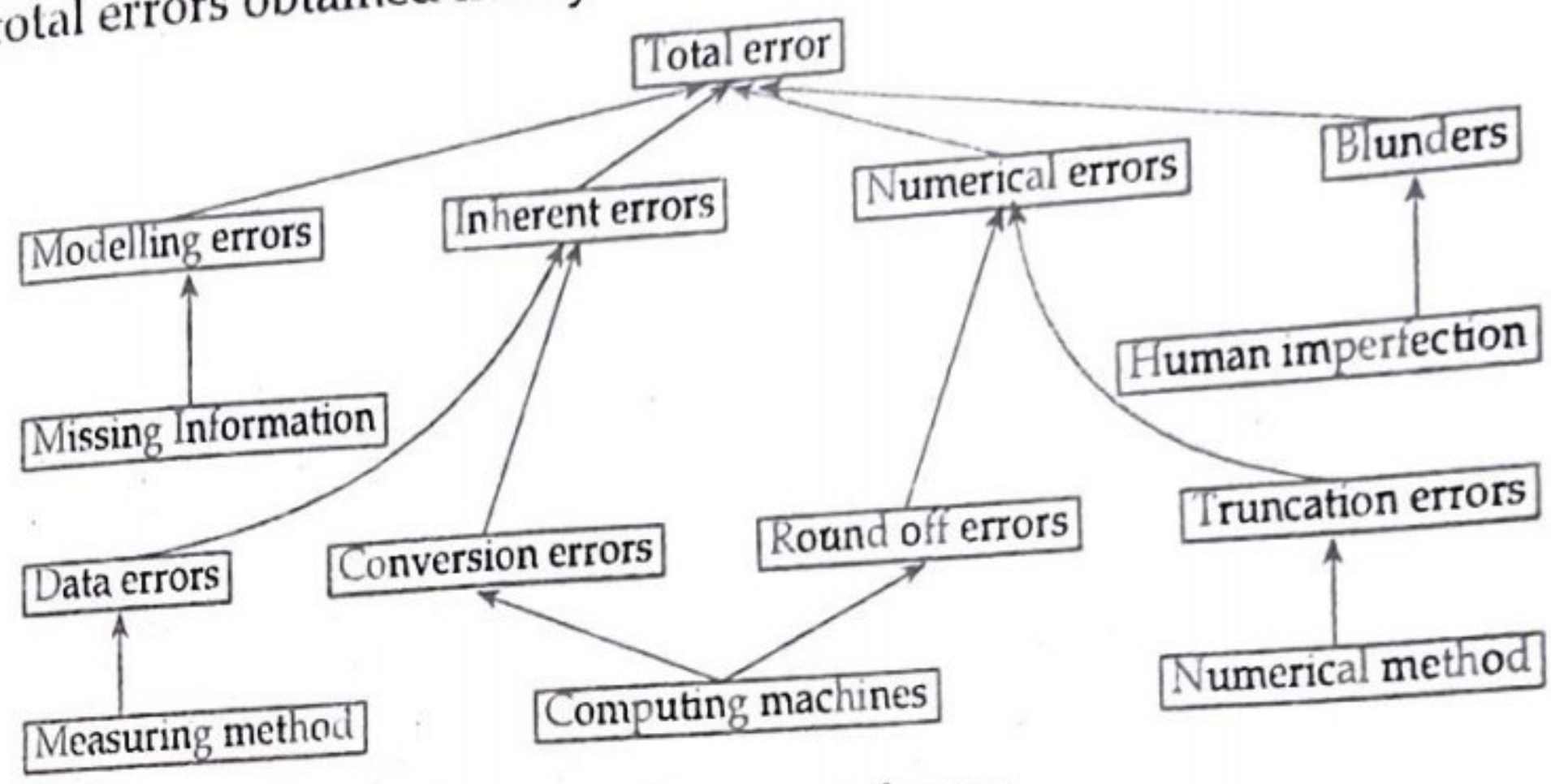

Figure: Taxanomy of errors

1.4.1 Taxonomy of Errors in Numerical Methods

In any numerical computation, we come across the following types of errors.

i) Inherent errors

Inherent errors are those errors which are present in the data supplied to the model. Inherent errors (also known as input errors) contain two components normally, Data errors and Conversion errors.

ii) Data errors

Data errors (also known as **empirical error**) arise when data for a problem are obtained by some experimental means and are therefore of limited accuracy and precision.

iii) Conversion errors

Conversion errors (also known as representation errors) arise due to the limitation of computer to store the data exactly; we know that, the floating point representation retains only a specified number of digits.

iv) Round off errors

Round off errors occurs when a fixed number of digits are used to represent exact number. Since, the numbers are stored at every stage of computation; round off errors is introduced at the end of every arithmetic operation.

v) Truncation errors

Truncation errors arise from using an approximation in place of exact mathematical procedure. It is the error resulting from the truncation of the numerical process. We often use some finite number of terms to estimate the sum of an infinite series. *For example;*

$S = \sum_{i=0}^{\infty} a_i x^i$ is replaced by some finite sum which give rise to truncation errors.

1.5 ABSOLUTE AND RELATIVE ERRORS

There are many ways, which can be used to describe error. The simplest is the absolute errors; this is the difference between the measured or calculated value and the true value.

$$\varepsilon = |\text{True value} - \text{Approximation}|$$

The absolute error does not consider the order of magnitude of the value under consideration. For this, we usually consider the relative error.

$$\varepsilon = \frac{|\text{True value} - \text{Approximation}|}{|\text{True value}|}$$

and, the percentage error is;

$$\varepsilon_p = \frac{|\text{True value} - \text{Approximation}|}{|\text{True value}|} \times 100\%$$

SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. Discuss the application of numerical method in the field of science and engineering. [2017 Spring]

Solution:

The application of numerical method in the field of science and engineering are given below:

i) Usually used in computer algorithm
ii) Used for multidimensional root finding
ii) Network simulation
iv) Train and traffic signal
v) Weather prediction
vi) Build up algorithm

2. Write short notes on errors in numerical computations. [2019 Fall]

Solution: See the definition part.

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SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. **Explain the importance of numerical method for an engineer. [2016]**

Solution:

The importance of numerical method for an engineers are;

i) It help engineers to solve various non-linear equations.

ii) Generally, differential equations consisting of more than 2 variables are very complicated to solve which can be solved using numerical methods like: Runge Kutta 4 method, etc.

iii) It help engineers to calculate area of irregular fields using various rules like $\left(\frac{1}{3}\right)$ Simpsons rule, $\left(\frac{3}{8}\right)$ Simpsons rule, etc.

iv) It help engineers in finding solutions to complex differentiation problems.

2. **What is error? Explain different types of error. [2017]**

Solution: See the definition part.

3. **Explain the importance and scope of numerical method for civil engineers. [2017]**

Solution: See the definition part.

4. **Define error with an example. Find the truncation error if the number 752.6835 is truncated to four significant digits. [2018]**

Solution:

Error with an example

See the definition part.

Numerical

Number (N) = 752.6835, it contains 7 significant digits.

When it is truncated to 4-significant figures

$\bar{N} = 752.6$

$\therefore$ Error = $N - \bar{N} = 752.6835 - 752.6 = 0.0835$

NOTE: In truncation we do not perform round off.

5. **What is application of numerical method in civil engineering? [2018]**

Solution: See the definition part.

6. **Explain the concept of significant digit. Differentiate absolute and relative error. [2019]**

Solution: See the definition part.

7. **Explain the application of numerical computing in civil engineering? [2019]**

Solution: See the definition part.

Chapter 2

SOLUTION OF NONLINEAR EQUATIONS

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2.6	Newton–Raphson Method	19
2.7	Fixed Point Iteration Method	23

2.1 ITERATIVE METHODS AND STOPPING CRITERIA

Iterative method is the method in which we evaluate the solution of problem by selecting a initial guess value. Iterative method is divided into two types: (i) intermediate value theorem and (ii) open interval value theorem.

"If a function f(x) = 0 is continuous in the interval [a, b] such that f(a) and f(b) are of opposite sign then there exits at least one or an odd number of roots between a and b" is the statement of intermediate value theorem.

2.1.1 Stopping Criteria

On solving, the problem we find its approximate solution 'c'; such that $f(c) \approx 0$; then we stop the iteration if $|f(c)| <$ Error.

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2.2 BISECTION METHOD

Bisection method is an iterative method used for the solution of non-linear equations, also known as binary chopping or half-interval method.

It is based on intermediate value property. It means if f(x) is continuous in the interval [a, b] and f(a) and f(b) have different sign then the equation f(x) = 0 has at least one root between x = a and x = b.

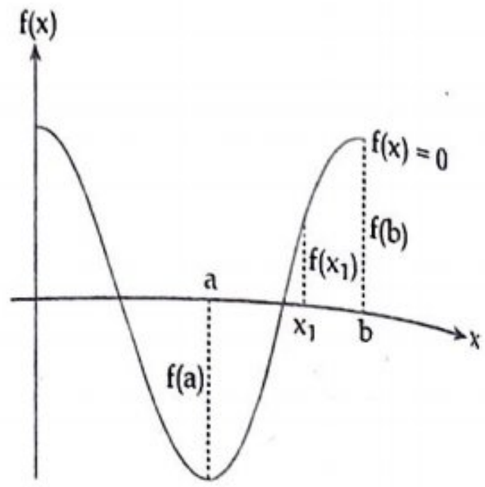


Figure: Illustration of bisection method

This method is most reliable and simplest iterative method for solution of non linear equation.

$$a < x < b$$

where, x is not of non linear equation.

Let f(x) = 0 be continuous between 'a' and 'b'. By definition let f(a) be negative and f(b) be positive. The first approximation to the root is

$$x_1 = \frac{a+b}{2};$$

where, a and b is selected such that,

$$f(a) \times f(b) < 0.$$

Now, if $f(x_1) = 0$ then x_1 is the root of f(x) otherwise the root lies between 'a' and x_1 or x_1 and 'b' according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

Algorithm of Bisection Method

1. Define function f(x) and Error (E).
2. Take two initial value for root as x_1 and x_2.
3. Compute $f(x_1)$ and $f(x_2)$.
4. If $f(x_1) \times f(x_2) > 0$, then x_1 and x_2 do not converge and root does not lies in between x_1 and x_2 and go to step (9); otherwise
5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$
   Else,
   set $a = x_2$ and $b = x_1$
6. Calculate root, $x_n = \frac{a+b}{2}$ and also calculate $f(x_n)$.
7. If $f(x_n) > 0$ (i.e. positive) then,
   set $b = x_n$
   Else,
   set $a = x_n$

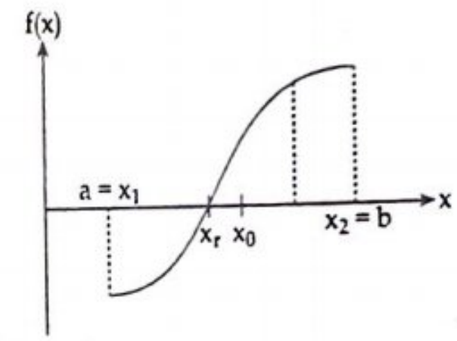
8. Repeat till step (7), until absolute value of $\frac{b-a}{a}$ is less than E, then print root.
   Root = $\left(\frac{a+x_n}{2}\right)$, go to step (9), else to step (6).
9. Stop.

Convergence of Bisection Method

Here, x_0 is the midpoint of x_1 and x_2 so depending upon the sign of $f(x_0)$, $f(x_1)$ and $f(x_2)$, x_1 or x_2 is set equal to x_0.

i.e., $f(x_0) \times f(x_1) < 0 \Rightarrow x_2 = x_0$

$f(x_0) \times f(x_2) < 0 \Rightarrow x_1 = x_0$



If n is the number of iteration then,

$$\frac{x_2 - x_1}{2^n} = \frac{\Delta x}{2^n} = \varepsilon_n;$$

which is the error at n^{th} iteration.

$$\therefore \quad \varepsilon_{n+1} = \left|\frac{x_2 - x_1}{2^n + 1}\right|$$

$$= \frac{1}{2}\left|\frac{\Delta x}{2^n}\right|$$

$$\varepsilon_{n+1} = \frac{1}{2}\varepsilon_n \qquad (1)$$

The error decreases linearly with each state by factor $\frac{1}{2}$. Therefore, the bisection method is linearly convergent, the convergence is slow to achieve high degree of accuracy and large number of iteration may be required.

Example 1

Find the root of the equation $e^x - 3x = 0$ correct up to three decimal place using bisection method.

Solution:

Here,

$$f(x) = e^x - 3x$$

Now, let us follow following steps to find initial guess x_1 and x_2 using calculator.

Here, the initial guess be 1 and 2

$$f(1) = -0.281$$

$$f(2) = 1.389$$

$$\because \quad f(1) = -0.281 < 0$$

and, $f(2) = 1.389 > 0$

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Therefore the root lies in between 1 and 2.

so, $a = 1$ and $b = 2$ and $E = 0.0001$

Now, let us calculate root by tabulation method.

n	X	Y	$A = \frac{X+Y}{2}$	$f(x_n) = C$
1	1	2	1.5	– 0.0183
2	1.5	2	1.75	0.5046
3	1.5	175	1.625	0.2034
4	1.5	1.625	1.5625	0.0832
5	1.5	1.5625	1.5313	0.0303
6	1.5	1.5313	1.5157	0.0055
7	1.5	1.5157	1.5079	– 0.0065
8	1.5079	1.5157	1.5118	– 0.0005
9	1.5118	1.5157	1.5138	0.0026
10	1.5118	1.5138	1.5128	0.0010
11	1.5118	1.5128	1.5123	0.0003
12	1.5118	1.5123	1.5121	– 0.0001
13	1.5121	1.5123	1.5122	0.0001 END

Since, 'a' and 'b' values are equation three decimal place so the root of equation $e^x - 3x$ is 1.5122.

Example 2

Find at least one root of $x^3 - 2x - 5 = 0$ with the accuracy of 0.08% using bisection method.

Solution:

Given that;

$$x^3 - 2x - 5 = 0$$

Let the initial guess be 2 and 3

$f(2) = -1$

and, $f(3) = 16$

$f(2) \times f(3) < 0$; so, the root lies between 2 and 3.

Now, for $\varepsilon = 0.08\%$;

$$\text{Number of iterations} = \frac{\ln(b-a) - \ln \varepsilon}{\ln 2}$$

$$= \frac{\ln(3-2) - \ln\left(\frac{0.08}{100}\right)}{\ln 2}$$

$$= 10.2877$$

$$\approx 11$$

Now let us calculate the root using tabulation method.

n	a	b	x_n	$f(x_n)$
1	2	3	2.5	5.625
2	2	2.5	2.25	1.891
3	2	2.25	2.215	0.346
4	2	2.125	2.0625	– 0.351
5	2.0625	2.125	2.0938	– 0.0083
6	2.0938	2.125	2.1094	0.1671
7	2.0938	2.1094	2.1016	0.0789
8	2.0938	2.1016	2.0977	0.0352
9	2.0938	2.0977	2.0958	0.0139
10	2.0938	2.0958	2.0948	0.0027
11	2.0938	2.0948	2.0943	END

After 11 iterations, the solution is 2.0943. Thus, the solution of the equation $f(x) = x^3 - 2x - 5$ is 2.0943.

Example 3

Find at least one root of the equation $x^2 + \tan x + e^x = 0$ correct up to 3 significant digits using bisection method.

Solution:

Given that;

$$f(x) = x^2 + \tan x + e^x$$

NOTE: Set the calculator in radian for trigonometric function equation.

Let, the initial guess be (–1) and (–0.75)

For $f(-1) = -0.189$

$f(-0.75) = 0.103$

$\because$ For $f(-1) = -0.189 < 0$ and $f(-0.75) = 0.103 > 0$

Therefore, the root lies in between –1 and –0.75 with error 0.0001.

Now, let us calculate root by tabulation method.

n	a	b	x_n	$f(x_n)$
1	–1	–0.75	–0.875	–0.0149
2	–0.75	–0.875	–0.8125	0.0482
3	–0.875	–0.8125	–0.84375	0.0179
4	–0.875	–0.84375	–0.85938	0.0019
5	–0.875	–0.85938	–0.86719	-0.0064
6	–0.86719	–0.85938	–0.86328	-0.0023
7	–0.86328	–0.85938	–0.86133	-0.002
8	–0.86133	–0.85938	–0.860355	0.0008
9	–0.86133	–0.860355	–0.86084	0.0003
10	–0.86133	–0.86084	–0.861085	0.00006
11	–0.86133	–0.861085	–0.8612	END

Thus, the required solution of equation $f(x) = x^2 + \tan x + e^x$ is – 0.8612.

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Example 4

Find out all the real roots of $x^3 - 6x^2 + 11x - 6 = 0$ using bisection method to two decimal places.

Solution:

Given that;

$$f(x) = x^3 - 6x^2 + 11x - 6$$

Let the initial guess be 2.5 and 4 for first root

$$f(2.5) = -0.375$$

and, $f(4) = 6$

$\because$ $f(2.5) = -0.370 < 0$

and, $f(4) = 6 > 0$,

Therefore the root lies in between 2.5 and 4 with given E = 0.001.

Now, let us calculate root by tabulation method.

n	a	b	x_n	$f(x_n)$
1	2.5	4	3.25	0.70313
2	2.5	3.25	2.875	−0.2051
3	2.875	3.25	3.0625	0.1370
4	2.875	3.0625	2.9688	−0.0595
5	2.9688	3.0625	3.0157	0.03214
6	2.9688	3.0157	2.9923	−0.01522
7	2.9923	3.0157	3.004	0.00805
8	2.9923	3.004	2.9982	−0.00360
9	2.9982	3.004	3.0011	0.00220
10	2.9982	3.0011	2.9996	−0.00080
11	2.9996	3.001	3.0003	0.00060
12	2.9996	3.0003	2.9999	- 0.0002
13	2.9999	3.0003	3.0001	+ 0.0002
14	2.9999	3.0001	3	0
15	3	3.0001	3.00	END

Since, 'a' and 'b' are same up to three decimal place, so the root of equation $f(x) = x^3 - 6x^2 + 11x - 6$ is 3.00.

For 2nd root let the initial guess be

$a = 0$ and $b = 1.5$

$f(a) = -6$ and $f(b) = 0.375$

so, the root lies in between 0 and 1.5.

n	a	b	x_n	$f(x_n)$
1	0	1.5	0.75	−0.703
2	0.75	1.5	1.125	0.205
3	0.75	1.125	0.937	−0.138
4	0.937	1.125	1.031	0.059
5	0.937	1.031	0.984	−0.033
6	0.984	1.031	1.007	0.014
7	0.984	1.007	0.995	−0.009
8	0.995	1.007	1.001	0.002
9	0.995	1.001	0.998	−0.004
10	0.998	1.001	0.999	−0.001
11	0.999	1.001	1	0 (END)

Hence, the root is 1.

For 3rd root

Let initial guess be a = 1.8 and b = 2.1

$$f(1.8) = 0.192$$

$$f(2.1) = -0.099$$

$$f(1.8) \times f(2.1) < 0$$

n	a	b	x_n	$f(x_n)$
1	1.8	2.1	1.95	0.0499
2	1.95	2.1	2.025	- 0.025
3	1.95	2.025	1.9875	0.0125
4	1.9875	2.025	2.00625	- 0.00625
5	1.9875	2.00625	2.0016	- 0.0016
6	1.9969	2.00625	2.0016	- 0.0016
7	1.9969	2.0016	1.999	0.00075
8	1.999	2.0016	2.0003	- 0.0003
9	1.999	2.0003	1.9996	0.00035
10	1.9996	2.0003	1.9999	0.00005
11	1.9999	2.0003	2.0001	- 0.00009
12	1.9999	2.0001	2.00	0 (END)

so, the 3rd root is 2.

2.3 HORNER'S METHOD

Any polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots\ldots\ldots$ is evaluated using Horner's rule at $x = x_0$ as;

$$f(x_0) = a_0 + x_0\left(a_1 + x_0\left(a_2 + x_0\left(a_3 + x_0(a_4 + x_0(a_5 + + \ldots\ldots\ldots))\right)\right)\right)$$

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For example;

$f(x) = x^4 + 4x^2 - 5$ at $x = 1$

Comparing this equation with $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$; we get,

$x_0 = 1$
$a_0 = -5$
$a_1 = 0$
$a_2 = 4$
$a_3 = 0$
$a_4 = 1$

$$\therefore \quad f(x=1) = a_0 + x_0\left(a_1 + x_0\left(a_2 + x_0(a_3 + x_0(a_4))\right)\right)$$
$$= -5 + 1\left(0 + 1\left(4 + 1(0 + 1(1))\right)\right)$$
$$= -5 + 1\left(0 + 1(4 + 1(1))\right)$$
$$= -5 + 1(0 + 1(4 + 1))$$
$$= -5 + 1(0 + 5)$$
$$= -5 + 5$$
$$= 0$$

2.4 FALSE POSITION METHOD

In bisection method, the interval between x_1 and x_2 is divided into two equal halves, irrespective of their location of root. The root may be closer to one end or the other. Now let us join point x_1 and x_2 by a straight line. The point of intersection of this line with x-axis gives an improved estimate of the root and is called false position of the root. This point then replaces one of the initial guess that has a function value of the same sign as $f(x_0)$. This process is repeated with the new value of x_1 and x_2. Since, this method uses the false position of the root repeatedly so it is called **false position method**. It is also called **linear interpolation**.

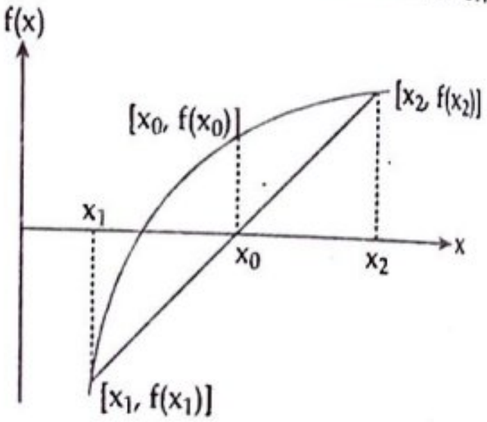

Figure: Illustration of false position method

False Position Method Algorithm (Regular False Method)

1. Define function f(x) and Error (E).
2. Take two initial point of interval for root as x_1 and x_2.
3. Calculate $f(x_1)$ and $f(x_2)$.
4. If $f(x_1) \times f(x_2) > 0$ then, root does not lie in between x_1 and x_2, go to step 9.
5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$
   Else,
   Set $a = x_2$ and $b = x_1$

6. Calculate root $x_n = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ and $f(x_n)$.
7. If $f(x_n) > 0$ (positive) then set $b = x_n$
   Else,
   Set $a = x_n$
8. Repeat step 7 till $\left(\text{abs}\dfrac{b-a}{b}\right) < E$ and print root.
9. Stop

NOTE:

The bisection and regular false method has similar algorithm except the formula for calculation of x_n.

Example 5

Find a real root of the following equation correct to 4 decimal place using False position method. $e^{\cos x} - \sin x - 1 = 0$

Solution:

Given that;

$$f(x) = e^{\cos x} - \sin x - 1$$

Let the initial guess be 3 and 4

$\because \quad f(3) = -0.769 < 0$

and, $\quad f(4) = 0.2769 > 0$,

So, root lies between 3 and 4.

Then according to false position method;

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{3 \times 0.2769 - 4 \times (-0.769)}{0.2769 + 0.769} = 3.73525$$

Let us calculate root from tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	3	4	−0.7695	0.2769	3.73537	−0.003945
2	3.73537	4	−0.003945	0.2769	3.73908	0.00003489
3	3.73537	3.73908	−0.003945	0.0000349	3.73905	0.000002707
4	3.73537	3.73905	−0.003945	0.000002707	3.73904	−0.00000802
						END

Since, the value of two consecutive x_n is similar up to 4 decimal place, so the root of equation $f(x) = e^{\cos x} - \sin x - 1$ is **3.73904**.

Example 6

Find root of the equation $x \log_{10} x = 1.2$ by false position method correct up to 4 decimal places.

Solution:

Given that;

$$f(x) = x \log_{10} x - 1.2$$

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Let the initial guess be 2 and 3.

$\because \quad f(2) = -0.597 < 0$

and, $\quad f(3) = 0.2313 > 0$

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

so, root lies between a = 2 and b = 3, with Error = 0.00001

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	2	3	−0.597	0.2313	2.72075	−0.01732
2	2.72075	3	−0.01732	0.2313	2.74020	−0.000389
3	2.74020	3	−0.000389	0.2313	2.74063	−0.000014
4	2.74063	3	−0.000014	0.2313	2.74064	−0.0000053

Since, the value of x_n in 3rd and 4th iteration are same up to 4 decimal place so the root of equation $f(x) = x\log_{10}x - 1.2$ is 2.74064.

Example 7

Use the method of false position to find the fourth root of 32 correct up to three decimal place.

Solution:

Given that;

$$x = (32)^{\frac{1}{4}}$$

$$f(x) = x^4 - 32 = 0$$

Let the initial guess be 2 and 3

$\because \quad f(2) = -16 < 0$

and, $\quad f(3) = 49 > 0$

Therefore, the root lies between 2 and 3.

Hence, a = 2 and b = 3

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Let us calculate the root using tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	2	3	−16	49	2.2461	−6.54833
2	2.2461	3	−6.54833	49	2.2349	−2.27833
3	2.3349	3	−2.27833	49	2.3644	−0.74757
4	2.3644	3	−0.74757	49	2.3739	−0.24225
5	2.3739	3	−0.24225	49	2.3769	−0.08141
6	2.3769	3	−0.08141	49	2.3779	−0.02766
7	2.3779	3	−0.02766	49	2.3782	−0.01153
8	2.3782	3	−0.01153	49	2.3783	−0.00614
						END

Since, the value of x_n in 7th and 8th iteration are similar up to 3 decimal place so the root of equation $f(x) = x^4 - 32$ is 2.3783.

2.5 SECANT METHOD

Secant method is like false position and bisection method; it uses two initial estimations but does not require that they must bracket the root. Let us consider the following figure.

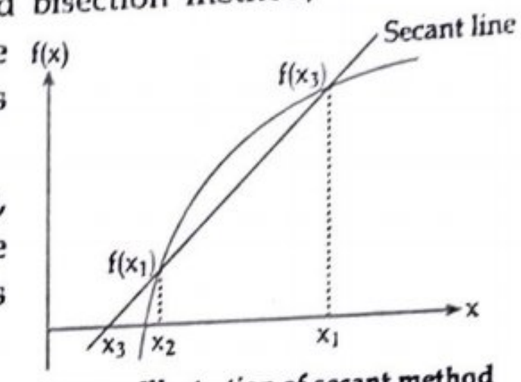

Figure: Illustration of secant method

The point x_1 and x_2 are starting point, they do not bracket the root. Slope of line (secant line) passing through x_1 and x_2 is given by;

$$\frac{f(x_1)}{x_1 - x_3} = \frac{f(x_2)}{x_2 - x_3}$$

$$f(x_1)(x_2 - x_3) = f(x_2)(x_1 - x_3)$$

$$x_3[f(x_2) - f(x_1)] = f(x_2)\cdot x_1 - f(x_1)\cdot x_2$$

$$x_3 = \frac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$$

Now, by adding and subtracting $f(x_2)x_2$ to the numerator and rearranging the terms; we get,

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \quad (1)$$

This equation is called secant formula.

The secant formula in general form is;

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Secant Method Algorithm

1. Define function f(x) and Error (E)
2. Take two initial guess of interval for root as x_1 and x_2
3. Calculate $f(x_1)$ and $f(x_2)$
4. If $f(x_1) \times f(x_2) > 0$ then root does not lie in between x_1 and x_2, so go to step 8.
5. If $f(x_1) < 0$ then set $a = x_1$ and $b = x_2$
   Else,
   Set $a = x_2$ and $b = x_1$
6. Calculate root by $x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$ and x_n
7. If $\left[bs\frac{(b-a)}{b}\right] < E$ then print root = x_n
   Else,
   Go to step 5
8. Stop.

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Example 8

Find an approximate root of $x\log_{10}x - 1.2 = 0$ using secant method up to three decimal place of accuracy.

Solution:

Given that;

$$f(x) = x\log_{10}x - 1.2$$

Let the initial guess be 2 and 3

$\because \quad f(2) = -0.597 < 0$

and, $\quad f(3) = 0.2313 > 0$

Therefore, the root lies in between 2 and 3, where $a = 2$ and $b = 3$,

$$\text{Error} = 0.0001$$

By secant formula; we have,

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now let us use tabular form for the calculation of root.

n	a	b	f(a)	f(b)	x_n
1	2	3	−0.597	0.2313	$x_1 = 2.7207$
2	3	$x_1 = 2.7207$	0.2313	−0.01736	$x_2 = 2.7401$
3	2.7207	$x_2 = 2.7401$	−0.01736	−0.000476	$x_3 = 2.7406$

The 2nd and 3rd iteration has similar value of x_n up to 3 decimal place, so the root of equation $f(x) = x\log_{10}x - 1.2$ is 2.7406.

Example 9

Find at least one root of the equation $x^2 + \tan x + e^x = 0$ correct up to 3 decimal place with secant method.

Solution:

Given that;

$$f(x) = x^2 + \tan x + e^x$$

Let the initial guess be −1 and −0.75

$\because \quad f(-1) = -0.189 < 0$

and, $\quad f(-0.75) = 0.1032 > 0$

Therefore, the root lies in between −1 and −0.75, where $a = -1$ and $b = -0.75$

Let us use secant method and find the root, by secant method formula

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now let us use tabular form for the calculation of root.

n	a	b	f(a)	f(b)	x_n
1	−1	−0.75	−0.189	0.1032	−0.83829
2	−0.75	−0.83829	0.1032	0.023378	−0.868522
3	−0.83892	−0.868522	0.023378	−0.0078748	−0.86090
4	−0.868522	−0.86090	−0.0078748	-2.578×10^{-5}	−0.86114
5	−0.86090	−0.86114	2.57×10^{-5}	4.54×10^{-6}	−0.86114

Since, the 5th and 6th iteration has similar value up to three decimal place so the root of equation $f(x) = x^2 + \tan x + e^x$ is −0.8611.

Example 10

Calculate a root of non-linear equation $f(x) = 3x + \sin x - e^x$ using secant method. The absolute error of function value of your calculated root should be less than 10^{-5}.

Solution:

Given that;

$$f(x) = 3x + \sin x - e^x$$

Let the initial guess be 0 and 1

$\because \quad f(0) = -1 < 0$

and, $\quad f(1) = 1.1231 > 0$

Therefore, the root lies in between 0 and 1

Now let us calculate the root of the given equation using tabular form.

n	a	b	f(a)	f(b)	x_n
1	0	1	−1	1.1231	0.4710
2	1	0.4710	1.1231	0.26518	0.30748
3	0.4710	0.30748	0.26518	−0.13489	0.3626
4	0.30748	0.3626	−0.13489	0.00544	0.36046
5	0.3626	0.36046	0.00544	0.0000958	0.36042

Hence, the value of x_n is 4th and 5th iteration is similar up to 4 decimal place, the root of equation $f(x) = 3x + \sin x - e^x$ is 0.36042.

2.6 NEWTON–RAPHSON METHOD

Consider the graph below of f(x). Let us assume that x_1 is an approximate root of $f(x) = 0$. Draw a tangent at the curve f(x) at $x = x_1$ as shown in figure. The point of intersection of this tangent with the x-axis gives the second approximation to the root. Let the point of intersection be x_2. The slope of the tangent is given by;

$$\tan\alpha = \frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

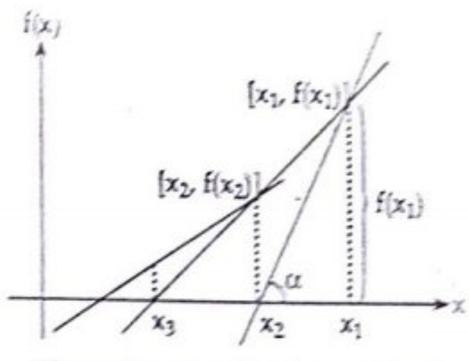


Figure: Newton-Raphson method

where, $f(x_1)$ is the slope of $x = x_1$.

Solving for x_2; we obtain,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This is called Newton Raphson method.

In general $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

This method of successive approximation is called Newton-Raphson method. The process is terminated when the difference between two successive values is within a prescribed limit.

Example 11

Derive iterative formula for Newton Raphson method using Taylor series.

Solution:

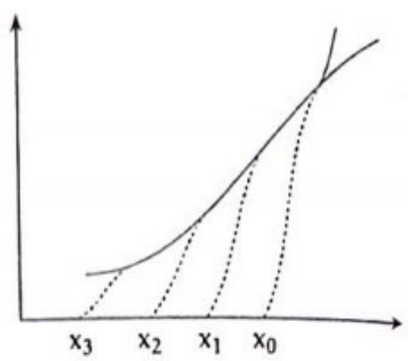
Let, x_0 be the approximate root of $f(x) = 0$

and, $x_1 = x_0 + h$ be correct root so that;

$f(x_1) = 0_1$;

where, h is a small interval.

i.e., $h = (x_1 - x_0)$



We know that;

Taylor series is expanded as;

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \ldots\ldots\ldots + \frac{f^n(x)}{n!}h^n \quad (1)$$

since, h is so small, higher power of h is also small.

Thus, neglecting higher power of x; we get;

$$f(x+h) = f(x) + f'(x)h = 0$$

or, $h = \frac{-f(x)}{f'(x)}$

or, $(x_1 - x_0) = \frac{-f(x_0)}{f'(x_0)}$ at $x = x_0$

$\therefore$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly;

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Thus,

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}};$$

which is the required iterative formula for Newton Raphson method.

Limitation of Newton Raphson Method

The Newton-Raphson method has certain limitation and pit falls. The method will fail in the following situations.

i) Division by zero may occur if $f'(x_1)$ is zero or very close to zero.

ii) If the initial guess is too far away from the required root, the process may converge to some other root.

iii) A particular value in the iteration sequence may repeat, resulting in an infinite loop. This occurs when the tangent to the curve f(x) at $x = x_{i+1}$ cuts the x-axis again at $x = x_i$.

Convergence of Newton-Raphson Method

Let x_n be an estimate of the root of the function f(x). If x_n and x_{n+1} are close to each other, then using Taylor's series expansion, we can state,

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(R)}{2}(x_{n+1} - x_n)^2 \quad (1)$$

where 'R' lie somewhere in the interval x_n to x_{n+1} and third and higher order have been dropped.

Let us assume that the exact root of f(x) is x_r. Then, $x_{n+1} = x_r$.

Therefore, $f(x_{n+1}) = 0$ and substituting then value in (1); we get,

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad (2)$$

We know that the Newton's iterative formula is given by;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Rearranging the terms; we get,

$$f(x_n) = f'(x_n)(x_n - x_{n+1})$$

Substituting this for $f(x_n)$ in equation (2) yields;

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(R)}{2}(x_r - x_n)^2 \quad (3)$$

We know the error in the estimate x_{n+1} is given by;

$$e_{n+1} = x_r - x_{n+1}$$

Similarly,

$$e_{n+1} = x_r - x_n$$

Now equation (3) can be expressed in terms of these errors as;

$$0 = f'(x_n)e_{n+1} + \frac{f''(R)}{2}e_n^2$$

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Rearranging the terms; we get,

$$e_{n+1} = -\frac{f''(R)}{2f'(x_n)} e_n^2 \quad (4)$$

This equation (4) shows that the error is roughly propogational to the square of the error in the previous iteration. Therefore, the Newton-Raphson method is said to have quadratic convergence.

Algorithm of Newton Raphson Method

1. Assign an initial value to 'x' say x_0.
2. Evaluate $f(x_0)$ and $f'(x_0)$.
3. Find the improved estimate of x_0

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
4. Check for accuracy of the latest estimate.
5. Compare relative error to a predefined value E.

   If $\left|\frac{x_1 - x_0}{x_1}\right| \le E$ stop;

   Otherwise continue.
6. Replace x_0 by x_1 and repeat 3, 4 and 5.

Example 12

Find the roots of the following equation correct to three decimal places by Newton-Raphson Method. $x^3 + x^2 + x + 7 = 0$

Solution:

$$f(x) = x^3 + x^2 + x + 7 = 0$$
$$f'(x) = 3x^2 + 2x + 1$$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	−2.5	−4.875	14.75	−2.1694
2	−2.1694	−0.67294	10.78	−2.10697
3	−2.10697	−0.021166	10.10402	−2.10487
4	−2.10487	2.8084×10^{-5}	10.0816	−2.10487

∴ Root = −2.10487

Example 13

Find out at least one root of the equation $x \cdot e^x - 2 = 0$ correct up to 3 decimal place using Newton-Raphson method.

Solution:

$$f(x) = x \times e^x - 2$$
$$f'(x) = x \times e^x + e^x$$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	−1.17563	2.473081	0.97537
2	0.97537	0.58682	5.238974	0.863358
3	0.863358	0.047116	4.41822	0.852693
4	0.852693	3.812700×10^{-4}	4.34633	0.85260

∴ Root = 0.85260

Example 14

Find the reciprocal of 3 using Newton Raphson method.

Solution:

$$x = \frac{1}{3}$$
$$f(x) = x - \frac{1}{3}$$
$$f'(x) = 1$$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	0.1666	1	0.33333
2	0.3333	-3.333×10^{-5}	1	0.3333
3	0.3333	-3.333×10^{-5}	1	0.3333

∴ Root = 0.3333

2.7 FIXED POINT ITERATION METHOD

Any function in the form of

$$f(x) = 0; \quad (1)$$

can be manipulated such that 'x' is on the left hand side of the equation

$$x = g(x) \quad (2)$$

Equation (1) and (2) are equivalent and therefore a root of equation (2) is root of equation (1). The root of equation (2) is given the point of interaction of the curve $y = x$ and $y = g(x)$. This intersection point is known as the fixed point of g(x).

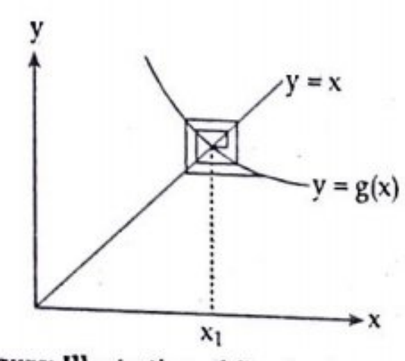

Figure: Illustration of fixed point method

The equation $x = g(x)$ is known as the fixed point equation. It provides a convenient from for predicting the value of 'x' as a function of 'x'. If x_0 is the initial guess to a root, the next approximation is given by;

$$x_1 = g(x_0)$$

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Further approximation is given by;

$$x_2 = g(x_1)$$

Thus iteration process can be expressed in general form as;

$$x_{i+1} = g(x_i)$$

where, i = 0, 1, 2, 3, ...; which is fixed point iteration formula.

This method of solution is known as the method of successive approximations or method of direct substitution. The iteration terminates if two successive approximations aggress within some specified error.

Example 15

Find a real root of the following equation, correct to three decimal place using the fixed point iteration method $\sin x + 3x - 2 = 0$.

Solution:

Given that;

$$f(x) = \sin x + 3x - 2 \quad (1)$$

Now for fixed point iteration method, arranging equation (1) in terms of $\emptyset$

or, $3x = -\sin x + 2$

or, $x = \dfrac{-\sin x + 2}{3}$

$$\emptyset(x) = \frac{-\sin x + 2}{3}$$

By iteration formula;

$$x_{n+1} = \emptyset(x) = \frac{-\sin x + 2}{3}$$

Calculating the root of equation in tabular form

n	x_n	x_{n+1}
1	0	0.6667
2	0.6667	0.4605
3	0.4605	0.5185
4	0.5185	0.50147
5	0.50147	0.50643
6	0.50643	0.50498
7	0.50498	0.50540
8	0.50540	0.50528
9	0.50528	0.50531
10	0. 50531	0.50530

Since, in 10th and 11th iteration x_{n+1} value is same up to 3 decimal place so the root is 0.5053 for $f(x) = \sin x + 3x - 2$.

Example 16

Find a real root of the equation $x^3 + x^2 - 1 = 0$ by the fixed point iteration method, correct up to six decimal place.

Solution:

Given that;

$$f(x) = x^3 + x^2 - 1 \quad (1)$$

For fixed point iteration method, arranging equation (1) in terms of $\emptyset(x)$

$$x^3 + x^2 - 1 = 0$$

or, $x^3 + x^2 = 1$

or, $x^2(x+1) = 1$

or, $x^2 = \dfrac{1}{x+1}$

or, $x = \sqrt{\dfrac{1}{x+1}}$

$$\emptyset(x) = \sqrt{\frac{1}{x+1}}$$

Now, calculating the root of equation in tabular form;

n	x_n	x_{n+1}
1	0	1
2	1	0.707106
3	0.707106	0.765365
4	0.765365	0.7526316
5	0.7526316	0.755361
6	0.755361	0.754773
7	0.754773	0.754900
8	0.754900	0.754873
9	0.754873	0.754878
10	0.754878	0.754878

Since, the value of x_{n+1} in 9th and 10th iteration has similar value up to 6 decimal place so the root of equation $f(x) = x^3 + x^2 - 1$ is 0.754878.

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SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. Find the square root of 17. Using fixed point iteration method. **[2009 Fall]**

Solution:

Given that;

$$x^2 = 17$$

$$f(x) = x^2 - 17$$

or, $x = \frac{17}{x}$

or, $x + x = \frac{17}{x} + x$

or $2x = \frac{17 + x^2}{x}$

or, $x = \frac{17 + x^2}{2x}$

$$\emptyset(x_n) = \frac{17 + x_n^2}{2x_n}$$

Let calculate root by tabular form

n	x_n	x_{n+1}
1	1	9
2	9	5.444
3	5.444	4.28335
4	4.28335	4.1261
5	4.1261	4.1231
6	4.1231	4.1231

Since, the 5th and 6th iteration has similar value of x_{n+1} up to 4 decimal place. So, $f(x) = x^2 - 17$ has root 4.1231.

2. Find the root of the equation $f(x) = x^3 + 4x - 1$ with accuracy 0.0001 using False position method. **[2011 Spring]**

Solution:

Given that;

$$f(x) = x^3 + 4x - 1$$

Let the initial guess be 0 and 1

$\because$ $f(0) = -1 < 0$

and, $f(1) = 4 > 0$

Therefore, the root lies between 0 and 1.

Here, a = 0 and b = 1, Error = 0.0001

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Let us calculate root in tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	0	1	−1	4	0.20	−0.192
2	0.2	1	− 0.192	4	0.2366	- 0.0401
3	0.2366	1	- 0.0403	4	0.2442	- 0.0085
4	0.2442	1	- 0.0085	4	0.2458	- 0.00183
5	0.2458	1	- 0.0019	4	0.2461	- 0.0004
6	0.2461	1	- 0.00070	4	0.2462	- 0.0001
7	0.2462	1	- 0.0002	4	0.2462	END

Since, the value of x_n in 6th and 7th iteration is same up to 4 decimal place, so the root of equation $f(x) = x^3 + 4x - 1$ is 0.2462.

3. Calculate the root of the given equation using bisection method correct up to 3 decimal place $f(x) = \sin x - \frac{1}{x}$. **[2012 Fall]**

Solution:

Given that;

$$f(x) = \sin x - \frac{1}{x}$$

$$f(1) = -0.1585$$

and, $f(2) = 0.4092$

$$f(1) \times f(2) < 0$$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	2	1.5	0.3308
2	1	1.5	1.25	0.1489
3	1	1.25	1.125	0.0134
4	1	1.125	1.0625	−0.0676
5	1.0625	1.125	1.09375	−0.0260
6	1.09375	1.125	1.1094	−0.00595
7	1.1094	1.125	1.1172	0.00378
8	1.1094	1.1172	1.1133	−0.001069
9	1.1133	1.1172	1.11525	0.00136
10	1.1133	1.11525	1.114275	0.000149
11	1.1133	1.114275	1.11379	−0.000458
12	1.11379	1.114275	1.11403	−0.000158
13	1.11403	1.114275	1.1142	−0.000053
				END

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Since, 'a' and 'b' are same up to three decimal place, so the root of equation $f(x) = \sin x - \frac{1}{x}$ is 1.1142.

4. Find the root of the equation $\cos x = xe^x$ using the regula-false method correct up to four decimal place. **[2012 Fall]**

Solution:

Given that;

$$f(x) = \cos x - xe^x$$

Let the initial guess be 0 and 1

$\because$ $f(0) = 1 > 0$

and, $f(1) = -2.177 < 0$

So, root lies between 0 and 1, but according to algorithm step 5.

$a = 1$

and $b = 0$

Error = 0.00001

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Now, let us calculate using tabular form

n	a	b	f(a)	f(b)	x_n	$f(x_n)$
1	1	0	−2.177	1	0.31476	0.51967
2	1	0.31476	−2.177	0.51967	0.44681	0.20332
3	1	0.44681	−2.177	0.20332	0.49408	0.07061
4	1	0.49408	−2.177	0.07061	0.50997	0.02354
5	1	0.50997	−2.177	0.02354	0.51521	0.007735
6	1	0.51521	−2.177	0.00773	0.51692	0.002545
7	1	0.51692	−2.177	0.002545	0.51748	0.0008436
8	1	0.51748	−2.177	0.0008436	0.51766	0.0002962
9	1	0.51766	−2.177	0.0002962	0.51772	0.00011366
10	1	0.51772	−2.177	0.000113366	0.51774	0.000053
						END

Since, the value of x_n in 9th and 10th iteration are similar up to 4 decimal place so the root of equation $f(x) = \cos x - xe^x$ is 0.51774.

5. Find the root of the equation $f(x) = 3x - \cos x - 1$ using Newton Raphson method, correct up to three decimal place. **[2012 Fall]**

Solution:

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

Number of iteration	x_i	$f(x_i)$	$f'(x_i)$	$x_i - \frac{f(x_i)}{f'(x_i)}$
1	0.5	−0.377758	3.47942	0.60851
2	0.60851	5.029×10^{-3}	3.571645	0.60710
3	0.60710	-5.884×10^{-6}	3.57048	0.60710

$\therefore$ Root = 0.60710

6. Find a positive root of the equation $f(x) = \cos x - 3x + 1$ correct up to 3 decimal place by using bisection method. **[2013 Fall]**

Solution:

Given that;

$$f(x) = \cos x - 3x + 1$$

Let the initial guess be 0 and 1.

$f(0) = 2 > 0$

and, $f(1) = -1.459 < 0$

Now, let us calculate root using tabular form.

n	a	b	x_n	$f(x_n)$
1	1	0	0.5	0.37758
2	1	0.5	0.75	−0.5183
3	0.75	0.5	0.625	−0.06404
4	0.625	0.5	0.5625	0.1584
5	0.625	0.5625	0.59375	0.04759
6	0.625	0.59375	0.6094	−0.00821
7	0.6094	0.59375	0.6016	0.01963
8	0.6094	0.6016	0.6055	0.00572
9	0.6055	0.6094	0.60745	- 0.00124
10	0.6055	0.60745	0.6065	0.00224
11	0.6065	0.60745	0.60698	0.00045
12	0.60698	0.60745	0.6072	- 0.0004
13	0.60698	0.6072	0.60709	0.00004
14	0.60709	0.6072	0.6071	END

Since, a, b and x_n have same value up to 3 decimal place so the root of equation $(x) = \cos x - 3x + 1$ is 0. 6071.

7. Find the root of the equation $3x = \cos x + 1$ using NR method with the tolerance is 10E - 5. **[2017 Spring]**

Solution:

Given that;

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$3x = \cos x + 1$

or, $f(x) = 3x - \cos x - 1 = 0$

Now, assume initial value of $x_0 = 1$.

and, $f'(x) = 3 + \sin x$

Now, according to NR method, next assumption is given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$= x_{n-1} - \frac{3x_{n-1} - \cos x_{n-1} - 1}{3 + \sin x_{n-1}}$$

Now, in table

n	x_{n-1}	x_n	$\in = \lvert x_{n-1} - x_n \rvert$
1	1	0.66854	-0.33145
2	0.66854	0.66665	-0.00188
3	0.66665	0.66664	-0.000005

Hence, root of equation is 0.66664.

NOTE:

i) $10E - 5 = 10^{-5}$

ii) Calculator $\Rightarrow A : B = A - \dfrac{3A - \cos A - 1}{3 + \sin A} : C = B - A : A = B$

8. Find the roots of $e^x \tan x = 1$ by creating iterative formula of Newton-Rapshon method. **[2018 Spring]**

Solution:

Given equation,

$e^x \tan x = 1$

or, $e^x \tan x - 1 = 0$

Let, $f(x) = e^x \tan x - 1$

Now, first derivative of $f(x)$,

$$f'(x) = \frac{d}{dx}(e^x \tan x - 1)$$

$$= e^x \sec^2 x + e^x \tan x$$

$$= e^x (\sec^2 x + \tan x)$$

Then, next approximate value of root of equation $f(x) = 0$, according Newton Raphson's method is;

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{e^x \tan x - 1}{e^x (\sec^2 x + \tan x)}$$

So, assume initial value of x. *i.e.*, $x_0 = 3$.

And, in tabulated form, to find out the root correct up to decimals.

i	x_i	x_{i+1}	$\in = \lvert x_{i+1} - x_i \rvert$	Remark
0	3	3.21912	0.21912	
1	3.21912	3.18434	0.03478	
2	3.18434	3.18303	0.00131	
3	3.18303	3.18303	0.00000012	$\in < 0.00005$

So, root of equation $e^x \tan x = 1$ is 3.1830.

NOTE:

To calculate values in table; we use calculator in following way:

i) Type $\Rightarrow Y = X - \dfrac{e^x \tan x - 1}{e^x (\sec^2 x + \tan x)}$

ii) Press "calc".

9. Solve $f(x) = xe^x - 1$ by secant method for tolerance value 0.0001. **[2018 Spring]**

Solution:

Given equation;

$f(x) = xe^x - 1 = 0$

Now, iterative formula by using secant method

$$x_n = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

Then, this process is repeated to get root for tolerance value 0.00001 and illustrated in table form, let assume initial value a = 0 and b = 1.

n	a	b	$f(a)$	$f(b)$	x_n
1	0	1	-1	1.7182	0.3678
2	1	0.3678	1.7182	-0.4686	0.5632
3	0.3678	0.5032	-0.4686	-0.1677	0.5786
4	0.5032	0.5786	-0.1677	0.0319	0.5665
5	0.5786	0.5665	0.0319	-0.0017	0.5671
6	0.5665	0.5671	-0.0017	-0.0011	0.5671

In step 5 and 6, $\in < 0.0001$. So, root of $f(x) = xe^x - 1$ is 0.5671.

NOTE:

We can calculate the values in tables using

$A : B : C = Ae^A - 1 : D = Be^B - 1:$

$$E = \frac{A \cdot D - B \cdot C}{D - C} : A = B : B = E$$

10. Derive an iterative formula for NR-method. Find a positive real root of the equation $x \log_{10}(x) = 1.2$ by using this formula correct to four significant digits. **[2018 Fall]**

Solution:

Derivation of NR-method

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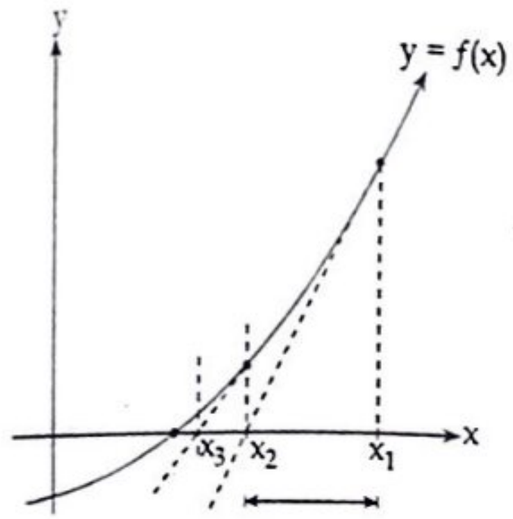
Suppose we have a equation $f(x) = 0$. Let $y = f(x)$. And, this curve shown in figure.



The principle of NR-method is that slope on the curve always coverage toward real root which we can also see in above curve.

Hence, iterative formula for NR-method is derived in this way:

$$\text{Slope} = \frac{P}{b} = \frac{f(x_1)}{x_1 - x_2} = f'(x_1)$$

or, $$\frac{f(x_1)}{f'(x_1)} = x_1 - x_2$$

or, $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\therefore$ $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Solution:

Given equation;

$$x \times \log_{10}(x) = 1.2$$

or, $$f(x) = x \times \log_{10}(x) - 1.2 \quad (1)$$

Now, first differentiation of $f(x)$ is

$$f'(x) = x \times \frac{1}{x} + \log_{10}(x) - 0$$

$$f'(x) = \log_{10}(x) + 1 \quad (2)$$

Then, we know that,

Next assumption is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x \times \log_{10}(x) - 1.2}{\log_{10}(x) + 1}$$

Now, for correct up to 4 significant digits, iterative calculation is done in table below:

n	x_n	x_{n+1}	$\epsilon = \lvert x_{n+1} - x_n \rvert$	Remark
1	1	2.2	1.2	
2	2.2	2.5327	0.3327	
3	2.5327	2.6594	0.1267	
4	2.6594	2.7087	0.0493	
5	2.7087	2.7280	0.0193	
6	2.7280	2.7356	0.0076	
7	2.7356	2.7386	0.0030	
8	2.7386	2.7398	0.0012	
9	2.7398	2.7403	0.0005	
10	1.7403	2.7403	0.002	
11	2.7405	2.7403	0.0000	error is less than 0.0001

Hence, for significance up to 4 digits, root of given equation is 2.7405.

11. Using secant method find a root of the equation $e^x \sin x - x^2 = 0$ correct up to three decimal places. [2018 Fall]

Solution:

Given equation,

$$e^x \sin x - x^2 = 0$$

We know that, iterative formula for root using secant method,

$$x_n = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$$

Let assume initiate assumption a = 1 and b = 2

Now, iterative process is done in table below for significance up to 3 digits.

n	a	$f(a)$	b	$f(b)$	x_n	$x_n - b$
1	1	-0.952	2	-3.742	0.658	1.341
2	2	-3.742	0.658	-0.410	0.492	0.165
3	0.658	-0.410	0.492	-0.228	0.284	0.207
4	0.492	-0.228	0.207	-0.038	0.149	0.037
5	0.207	-0.038	0.149	-0.019	0.091	0.037
6	0.149	-0.019	0.091	-0.006	0.060	0.030
7	0.091	-0.006	0.060	-0.002	0.040	0.019
8	0.060	-0.002	0.040	-0.0008	0.029	0.010
9	0.040	-0.0008	0.029	-0.0003	0.022	0.006
10	0.029	-0.0003	0.022	-0.00009	0.019	0.002
11	0.022	-0.00009	0.019	-0.00002	0.017	0.001
12	0.019	-0.0002	0.017	-0.00001	0.017	0.000

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NOTE:

We calculate values in above using calculation by using following steps:

i) Type : $A : B : C = e^A \cdot \sin A - A^2 : D = e^B \cdot \sin B - B^2$:

$$E = \frac{AD - BC}{D - C} : F = |E - D| : A = B : B = E$$

ii) Press ⇒ calc

Hence, root of equation $e^x \sin x - x^2 = 0$ is 0.017.

12. Find where the graph of $y = x - 3$ and $y = \ln(x)$ intersect using bisection method. Get the intersection value correct to four decimal places. [2019 Fall]

Solution:

The graph of $y = x - 3$ and $y = \ln x$ intersect when $x - 3 = \ln x$.

or, $x - 3 - \ln(x) = 0$

Let, $f(x) = x - 3 - \ln(x)$

x	1	2	3	4	5
f (x)	- 2	- 1.693	- 1.098	- 0.386	0.3905

Then,

$f(4) = -0.386$

$f(5) = 0.3905$

$f(4) \cdot f(5) < 0$

The initial guess be 4 and 5.

Hence, we have the following iteration.

a	b	$x_n = \frac{a+b}{2}$	$f(x_n)$
4	5	4.5	- 0.00407
4.5	5	4.75	0.191855
4.5	4.75	4.625	0.0935236
4.5	4.625	4.5625	0.044629
4.5	4.5625	4.53125	0.020252
4.5	4.53125	4.515625	0.00808
4.5	4.515625	4.5078125	0.002000
4.5	4.5078125	4.50390625	- 0.00103
4.50390625	4.5078125	4.505859	0.000480
4.50390625	4.505859	4.504882	- 0.000279
4.504882	4.505859	4.505370	0.0000999
4.504882	4.505370	4.505126	- 0.0000898

∴ The value of $x = 4.505126$.

∴ $y = x - 3 = 4.505126 - 3 = 1.505126$

Hence, the point of interaction is $(x, y) = (4.505126, 1.505126)$.

13. Find the value of $\sqrt{18}$ using Newton Raphson method. [2019 Fall]

Solution:

Let, $x = \sqrt{18}$

or, $x^2 = 18$

or, $x^2 - 18 = 0$

i.e., $f(x) = x^2 - 18$

∴ $f'(x) = 2x$

Let, $x = 4.5$ be the initial guess.

Hence, we have the following iteration by Newton Raphson method.

x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
$x_0 = 4.5$	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.5 - \frac{(4.5)^2 - 18}{2 \times 4.5} = 4.25$
$x_1 = 4.25$	$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.24264$
$x_2 = 4.24264$	$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.24264$

Hence, the value of $\sqrt{18}$ is 4.24264.

14. Using secant method, find the zero of function $f(x) = 2x - \log_{10} x - 7$ correct up to three decimal places. [2019 Spring]

Solution:

Here;

$f(x) = 2x - \log_{10} x - 7$

Let us assume that root lies between 1 to 5 *i.e.*, [1, 5]

x	1	2	3	4	5
f(x)	-5	-3.301	-1.477	0.3979	2.301

From table, we have,

$f(3) = -1.477$

$f(4) = 0.3979$

i.e., $f(3) \times F(4) < 1$

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so, root lies between (3, 4).

Let a = 3 and b = 4.

a	b	f(a)	f(b)	$x_n = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$
3	4	-1.477	0.39794	3.78777
4	3.78777	0.39794	-2.8393×10^{-3}	3.78927
3.78777	3.78927	-2.8393×10^{-3}	-4.622×10^{-6}	3.78927

$\therefore$ f(x) = 0 at x = 3.78927

Understanding the problem:

Using calculator, we solved the problem *i.e.*, $2x - \log_{10}x - 7 = 0$ and we get,

x = 3.789

So, just for formality, we assumed root lies between 1 and 5.

For table generation

Mode + (9) ↲ (CASIO FX - 991 EX)

$f(x) = 2x - \log_{10}x - 7$ ↲

g(x) = ↲

start = 1 ↲

end = 5 ↲

step = 1 ↲

We get the table

Setting formula in calculator

$$A : B : C = 2A - \log_{10}A - 7 : D = 2B - \log_{10}B - 7 : E = \frac{A.D - B.C}{D - C} :$$

A = B : B = E + (calc)

Now, just press enter and fill the boxes.

15. Find the root of the equation log x − cos x = 0 correct to three decimal places by using N-R method. [2019 Spring]

Solution:

Here;

$$f(x) = \ln x - \cos x$$

$$f'(x) = \frac{1}{x} - (-\sin x) = \frac{1}{x} + \sin x$$

Let $x_0 = 2$ be the initial guess.

Using formula given by Newton Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i) $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\ln 2 - \cos 2}{\frac{1}{2} + \sin 2} = 1.2128$$

ii) $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.2128 - \frac{\ln 1.2128 - \cos 1.2128}{\frac{1}{1.2128} + \sin 1.2128} = 1.3022$$

Following the same pattern, we fill the table.

S.N.	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	1.2128
2	1.2128	1.30221
3	1.30221	1.30296
4	1.30296	1.302964

$\therefore$ f(x) = 0 at x = 1.30296

NOTE:

Keep calculator in radian (R) mode, because we have trigonometic expression in this question.

Calculator code

$$A : B = A - \frac{\ln(A) - \cos A}{\frac{1}{A} + \sin A} : A = B + calc$$

16. Write short note on "Algorithm of Bisection Method". [2019 Spring]

Solution: See the theory section 2.2.

17. Find the positive real root of the equation $\cos x + e^x + x^2 = 3$ using false position method, correct to 3 decimal places. [2020 Fall]

Solution:

Here;

$$\cos x + ex + x^2 = 3$$

$\therefore$ $f(x) = \cos x + e^x + x^2 - 3 = 0$

Let root lies between −2 to 2 *i.e.* [−2, 2]

x	−2	−1	0	1	2
f(x)	0.7191	−1.091	−1	1.2585	7.9729

From table, we have,

f(−2) = 0.7191, f(−1) = −1.091

i.e., $f(-2) \times f(-1) < 1$ so, root lies between (−2, −1).

Again, from table we have,

f(0) = −1, f(1) = 1.2585

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i.e., $f(0) \times f(1) < 0$. So, root lies between (0, 1)

Now, we see that root lies between (–2, –1) and (0, 1). Let's find the root that lies between (0, 1).

Let a = 0 and b = 1.

a	b	f(a)	f(b)	$x_n = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$	$f(x_n)$
0	1	–1	1.2585	0.44275	–0.34341
0.44275	1	–0.34341	1.2585	0.562205	–0.083306
0.562205	1	–0.083306	1.2585	0.589384	–0.01846
0.589384	1	–0.018465	1.2585	0.595321	-4.01×10^{-3}
0.595321	1	-4.011×10^{-3}	1.2585	0.5966	-8.67×10^{-4}

$\therefore$ $f(x) = 0$ at $x = 0.5966$ with error of (-8.67×10^{-4}).

NOTE:

In this question, triangonometric expression is present. So, keep the calculator in radian mode.

Calculator code

$$A : B : C = \cos A + e^A + A^2 - 3 : D = \cos B + e^B + B^2 - 3 :$$

$$E = \frac{A.D - B.C}{D - C} : F = \cos E + e^E + E^2 - 3 : A = E + (\text{calc})$$

18. Discuss the drawbacks of Newton-Raphson method. Find the real root of the equation x sin x – cos x = 0 using Newton-Raphson method, correct to 3 decomal places. [2020 Fall]

Solution:

For the first part

See the theory section 2.6.

For the second part

Here;

$$x \sin x - \cos x = 0$$

$\therefore$ $f(x) = x \sin x - \cos x$

$\therefore$ $f'(x) = x \cos x + \sin x - (-\sin x)$

$= x \cos x + \sin x + \sin x$

$= x \cos x + 2\sin x$

Let $x_0 = 1$ be the initial guess.

Using formula given by Newton Raphson method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{1 \times \sin 1 - \cos 1}{1 \times \cos 1 + 2 \times \sin 1}$

$= 0.8645$

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.8645 - \frac{0.8645 \times \sin 0.8645 - \cos 0.8645}{0.8645 \times \cos 0.8645 + 2 \sin 0.8645}$

$= 0.8605$

x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$f(x_{n+1})$ [Error]
1	0.8645	0.00866
0.8645	0.8605	0.00034
0.8605	0.8603	–0.000069
0.8603	0.86033	–0.000007

$\therefore$ $f(x) = 0$ at $x = 0.86033$ with error of 7×10^{-6}

Calculator code

$$A : B = A - \frac{A \sin A - \cos A}{A \cos A + 2\sin A} : C = B \sin B - \cos B : A = B + \text{calc}$$

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SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. Prove that the Newton-Raphson method is quadrically convergent. [2015]

Solution: See the definition part.

2. Evaluate the following polynomial using Horner's rule

$f(x) = x^4 + 4x^2 - 5$ at $x = 2$ [2015]

Solution:

Here;

$f(x) = x^4 + 4x^2 - 5$

Any polynomial $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ is evaluated using Horner's rule at $x = x_0$ as;

$f(x_0) = a_0 + x_0(a_1 + x_0(a_2 + x_0(a_3 + x_0 \cdot a_4)))$

We have;

$x_0 = 2, a_0 = -5, a_1 = 0, a_2 = 4, a_3 = 0, a_4 = 1$

$\therefore \quad f(2) = -5 + 2(0 + 2(4 + 2(0 + 2 \times 1)))$

$= -5 + 2(0 + 2(4 + 2 \times 2))$

$= -5 + 2(0 + 2 \times (8))$

$= -5 + 2(16)$

$= -5 + 32$

$= 27$

$\therefore \quad f(2) = 27$

3. Find the root of the non-linear equation $xe^x - 1 = 0$ using bisection method correct up to 3 decimal places. [2016]

Solution:

Here;

$f(x) = xe^x - 1$

Let's assume that $f(x)$ has solution in [-3, 3]

x	-3	-2	-1	0	1	2	3
$f(X)$	-1.149	-1.27	-1.367	-1	1.718	13.778	59.256

Trick: To generate table. [Casio FX - 991 EX]

Mode + 9

$f(x) = xe^x - 1$ ↵

g(x) = ↵

start: -3 ↵

end: +3 ↵

Step: 1 Table is generated

Are you confused how we choose interval [-3, 3]
Solve, $xe^x - 1 = 0$ using calculator we get 0.567. So, just to show formality we use [-3, 3] interval.

From above table, we can clearly see that value of function changes from (-) to (+) in 0 to 1

i.e., $f(0) = -1$

$f(1) = 1.718$

$\therefore$ Root lies between 0 and 1

Let $a = 0$

$b = 1$

(-) a	(+) b	$x_n = \frac{a+b}{2}$	$f(x_n)$
0	1	0.5	-0.1756
0.5	1	0.75	0.587
0.5	0.75	0.625	0.1676
0.5	0.625	0.5625	-0.01278
0.5625	0.625	0.5937	0.07514
0.5625	0.5937	0.5781	0.03054
0.5625	0.5781	0.5703	8.74×10^{-3}
0.5625	0.5703	0.5664	-2.052×10^{-3}
0.5664	0.5703	0.5683	3.33×10^{-3}
0.5664	0.5683	0.56735	5.71×10^{-4}

$\therefore \quad f(x) = xe^x - 1, f(x) = 0$ at $x = 0.56735$.

Using calculator to get table

$A : B : C = \frac{A+B}{2} : D = C.e^C - 1$ calc

A = 0 ↵

B = 1 ↵

Now, other values C and D are calculated automatically.

The value of D = $f(x_n)$ = Negative so replace A (*i.e.,* '-)) by C and continue similarly.

NOTE:

Correct up to 3 decimal places means up to 0.0001 or 10^{-4}

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4. **Prove that the Newton Raphson method is quadratically convergent.** [2017]

Solution: See the solution of Q. no. 1

5. **Use secant method to find an approximate root of the equation $f(x) = 3x + \sin x + e^x$. Correct up to 3 decimal places.** [2017]

Solution:

Here,

$f(x) = 3x + \sin x + e^x$

Let us assume that root of $f(x)$ lies between -2 to 2 *i.e.*, [-2, 2]

x	-2	-1	0	1	2
$f(x)$	-6.773	-3.473	1	6.559	14.298

From table, we have

$f(-1) = -3.473$

$f(0) = 1$

i.e., $f(-1) \times f(0) < 1$. So, root lies between (-3.473, 1)

Let a = -1 and b = 0

a	b	$f(a)$	$f(b)$	$x_n = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)}$
-1	0	-3.4736	1	-0.2235
0	-0.2235	1	-0.09259	-0.020459
-0.2235	-0.20459	-0.0926	-1.957×10^{-3}	-0.20418
-0.20459	-0.20418	-1.957×10^{-3}	4.0349×10^{-6}	-0.20418

$\therefore$ $f(x) = 0$ at $x = -0.20418$

Understanding the problem

Using calculator, we solved the problem

i.e., $3x + \sin x + e^x = 0$ and we get

$x = -0.2041$ **(Note: Put calculator in radian (R) mode)**

So, just for formality we assumed root lies between -2 to 2.

For table generation;

Mode + (9) (CASIO FX - 991 EX)

$f(x) = 3x + \sin x + e^x$ ↲

$g(x) =$ ↲

start = -2 ↲

end = +2 ↲

step = 1 ↲

We get the table.

$A : B : C = 3A + \sin A + e^A : D = 3B + \sin B + e^B :$

$E = \frac{A \cdot D - B \cdot C}{D - C} : A = B : B = E$

calc

Now, just press enter and fill the box.

6. **Show that Newton Raphson method is said to have quadratic convergence.** [2018]

Solution: See the solution of Q. no. 1

7. **Find the real root of equation $x^2 - 2x - 3 = 0$ using Newton Raphson method correct up to three decimal places** [2018]

Solution:

Here;

$f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2$

Let $x_0 = 2$ be the initial guess.

Using formula given by Newton Raphson method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = +2 - \frac{2^2 - 2 \times 2 - 3}{2 \times 2 - 2} = 3.5$

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 3.5 - (3.5)^2 - \frac{2 \times 3.5 - 3}{2 \times 3.5 - 2} = 3.05$$

Following the same pattern, we fill the table

	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
i)	2	3.5
ii)	3.5	3.05
iii)	3.05	3.0006
iv)	3.0006	3.0000000093

$\therefore$ $x = 3$ is one root of $f(x) = x^2 - 2x - 3$.

Calculator $\Rightarrow A : B = A - \frac{A^2 - 2A - 3}{2A - 2} : A = B$

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8. **Use secant method to find root of $f(x) = x^2 - 13x - 12$. Take error tolerance = 0.0001** [2019]

Solution:

Here,

$$f(x) = x^2 - 13x - 12$$

x	-2	-1	0	1	2
f(x)	18	2	-12	-24	-34

We see,

$$f(-1) = 2$$

and, $f(0) = -12$

i.e., $f(-1) \times f(0) < 0$

so, root lies in between 0 and −1.

Let, $a = -1, b = 0$

a	b	f(a)	f(b)	$x_n = \frac{a\,f(b) - b\,f(a)}{f(b) - b\,f(a)}$	Error
-1	0	2	-12	-0.85714	
0	-0.85714	-12	-0.12244	-0.865979	0.008839
-0.85714	-0.865979	-0.1224	7.65×10^{-3}	-0.865459	5.2×10^{-4}
-0.865979	-0.865459	7.65×10^{-3}	-4.32×10^{-6}	-0.865459	< 0.0001

$\therefore$ $x = -0.865459$

NOTE:

Calculator code

$$A : B : C = A^2 - 13A - 12 : D = B^2 - 13B - 12 : E = \frac{AD - BC}{D - C} :$$

$$A = B : B = E$$

9. **Prove that convergence of bisection method is linear.** [2019]

Solution: See the definition part.

Chapter 3

INTERPOLATION AND APPROXIMATION

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3.1 INTERPOLATION

The computing of value for a tabulated function at a point is called interpolation. Suppose we are given the following value of $y = f(x)$ for a set of values of 'x'.

x	x_0	x_1	x_2	x_n
$y = f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_n)$

Then the process of finding the value of 'y' corresponding to any value of 'x' is called interpolation. Thus, interpolation is the technique to estimate the value of a function for any intermediate value of the independent variable.

3.1.1 Linear Interpolation

Using equation of straight line (co-linearity condition) to find the missing value is linear interpolation. *For example;*

x	1	2	3
y	5	?	15

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Using linear interpolation; we have,

$$\frac{x_1 - x_2}{x_1 - x_3} = \frac{y_1 - y_2}{y_1 - y_3}$$

or, $\frac{1-2}{1-3} = \frac{5-y_2}{5-15}$

or, $\frac{-1}{-2} = \frac{5-y_2}{-10}$

or, $\frac{10}{-2} = 5 - y_2$

or, $y_2 = 5 + 5$

$\therefore$ $y_2 = 10$

3.1.2 Interpolation with Unequal Intervals

There are basically two methods to find the interpolation with unequal intervals.

a) Lagrange's interpolation method

b) Newton's divided difference interpolation method

i) Lagrange's Interpolation Method

Suppose $y = f(x)$ be a function with $f(x_0)$, $f(x_1)$, $f(x_2)$..... $f(x_n)$ corresponding to the values x_0, x_1, x_2..... x_n, then;

$$y_0 = f(x_0), y_1 = f(x_1) \ldots\ldots y_n = f(x_n)$$

The Lagrange's interpolation formula is given by;

$$y = f(x)$$

$$= \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} \times y_1 \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} \times y_n$$

Example 1

Use Lagrange method to find f(2.5) from following data.

x	1	2	4	5	7
f(x)	1	1.414	1.732	2.00	2.6

Solution:

Here,

$x_0 = 1$	$y_0 = 1$
$x_1 = 2$	$y_1 = 1.414$
$x_2 = 4$	$y_2 = 1.732$
$x_3 = 5$	$y_3 = 2.00$
$x_2 = 7$	$y_2 = 2.6$

By Lagrange's formula; we have,

$$y = f(x)$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times y_3 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

$$y = f(x)$$

$$= \frac{(x-2)(x-4)(x-5)(x-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1 + \frac{(x-1)(x-4)(x-5)(x-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.414 + \frac{(x-1)(x-2)(x-5)(x-7)}{(4-1)(4-2)(4-5)(4-7)} \times 1.732 + \frac{(x-1)(x-2)(x-4)(x-7)}{(5-1)(5-2)(5-4)(5-7)} \times 2 + \frac{(x-1)(x-2)(x-4)(x-5)}{(7-1)(7-2)(7-4)(7-5)} \times 2.6$$

$$y = f(2.5)$$

$$= \frac{(2.5-2)(2.5-4)(2.5-5)(2.5-7)}{(1-2)(1-4)(1-5)(1-7)} \times 1 + \frac{(2.5-1)(2.5-4)(2.5-5)(2.5-7)}{(2-1)(2-4)(2-5)(2-7)} \times 1.414 + \frac{(2.5-1)(2.5-2)(2.5-5)(2.5-7)}{(4-1)(4-2)(4-5)(4-7)} \times 1.732 + \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-7)}{(5-1)(5-2)(5-4)(5-7)} \times 2 + \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-5)}{(7-1)(7-2)(7-4)(7-5)} \times 2.6$$

$$y = f(2.5) = -0.117188 + 1.193063 + 0.811875 - 0.421875 + 0.040625$$

$$y = f(2.5) = 1.5065$$

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Example 2

Use Lagrange interpolation and estimate the square of 3.25 if:

x	1	2	3	4	5
x^2	1	4	9	16	25

Solution:

Here,

$x_0 = 1$	$y_0 = 1$
$x_1 = 2$	$y_1 = 4$
$x_2 = 3$	$y_2 = 9$
$x_3 = 4$	$y_3 = 16$
$x_2 = 5$	$y_2 = 25$

$$y = f(3.25)$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times y_3$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} \times y_4$$

$$y = f(3.25)$$

$$= \frac{(3.25-2)(3.25-3)(3.25-4)(3.25-5)}{(1-2)(1-3)(1-4)(1-5)} \times 1$$

$$+ \frac{(3.25-1)(3.25-3)(3.25-4)(3.25-5)}{(2-1)(2-3)(2-4)(2-5)} \times 4$$

$$+ \frac{(3.25-1)(3.25-2)(3.25-4)(3.25-5)}{(3-1)(3-2)(3-4)(3-5)} \times 9$$

$$+ \frac{(3.25-1)(3.25-2)(3.25-3)(3.25-5)}{(4-1)(4-2)(4-3)(4-5)} \times 16$$

$$+ \frac{(3.25-1)(3.25-2)(3.25-3)(3.25-4)}{(5-1)(5-2)(5-3)(5-4)} \times 25$$

$$= 0.017089 - 0.492188 + 8.30566 + 3.28125 - 0.54932$$

$$= 10.5625$$

ii) Newton's Divided Difference Interpolation Method

If $f(x_1)$, $f(x_2), f(x_3)$ …… be the value of $y = f(x)$ corresponding to x_1, x_2, x_3 …… Then, Newton divided difference interpolation is given by;

$$y = y_0 + (x-x_0)f[x_0, x_1] + (x-x_0)(x-x_1)f[x_0, x_1, x_2] + \ldots\ldots$$

$$+(x-x_0)(x-x_1)\ldots\ldots(x-x_n)f[x_0, x_1, \ldots\ldots x_n]$$

Now, we can construct the divided difference table as follows;

x	f	1st divided difference Δf	2nd divided difference $\Delta^2 f$	3rd divided difference $\Delta^3 f$	4th divided difference $\Delta^4 f$
x_0	f_0				
		$\frac{f_1 - f_0}{x_1 - x_0} = \Delta f_0$			
x_1	f_1		$\frac{\Delta f_1 - \Delta f_0}{x_2 - x_0} = \Delta^2 f_0$		
		$\frac{f_2 - f_1}{x_2 - x_1} = \Delta f_1$		$\frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0} = \Delta^3 f_0$	
x_2	f_2		$\frac{\Delta f_2 - \Delta f_1}{x_3 - x_1} = \Delta^2 f_1$		$\frac{\Delta^3 f_1 - \Delta^3 f_0}{x_4 - x_0}$
		$\frac{f_3 - f_2}{x_3 - x_2} = \Delta f_2$		$\frac{\Delta^2 f_2 - \Delta^2 f_1}{x_4 - x_1} = \Delta^3 f_1$	
x_3	f_3		$\frac{\Delta f_3 - \Delta f_2}{x_4 - x_2} = \Delta^2 f_2$		
		$\frac{f_4 - f_3}{x_4 - x_3} = \Delta f_3$			
x_4	f_4				

Example 3

Use Newton's Divided Difference method interpolate value of x = 1.175

Temperature	1.1	2.0	3.5	5	7.1
Viscosity	0.6981	1.4715	2.1287	2.0521	1.4480

Solution:

Let us first construct Newton divided difference table:

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x	y = f(x)	1st difference	2nd difference	3rd difference	4th difference
1.1	0.6981				
		$\frac{1.4715-0.6981}{2.0-1.1} = 0.8593$			
2.0	1.4715		$\frac{0.4381-0.8593}{3.5-1.1} = -0.1755$		
		$\frac{2.1287-1.4715}{3.5-2.0} = 0.4381$		$\frac{-0.1630+0.1755}{5-1.1} = 0.0031$	
3.5	2.1287		$\frac{-0.0510-0.4381}{5-2.0} = -0.1630$		$\frac{0.019-0.0031}{7.1-1.1} = 0.00265$
		$\frac{2.0521-2.1287}{5-3.5} = -0.0510$		$\frac{-0.0657+0.1630}{7.1-2.0} = 0.0190$	
5	2.0521		$\frac{-0.2876+0.0510}{7.1-3.5} = -0.0657$		
		$\frac{1.4480-2.0521}{7.1-5} = 0.2875$			
7.1	1.4480				

For x = 1.175; which falls in between 1.1 and 2.0 interval

$$y = f(1.175)$$

$$= y_0 + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$

$$+(x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$+(x - x_0)(x - x_1)(x - x_2)(x - x_3)f(x_0, x_1, x_2, x_3, x_4)$$

$$= 0.6981 + (1.175 - 1.1)0.85933 + (1.175 - 1.1)(1.175 - 2.0)$$

$$\times -0.1755 + (1.175 - 1.1)(1.175 - 2)(1.175 - 3.5) \times 0.0031$$

$$+(1.175 - 1.1)(1.175 - 2)(1.175 - 3.5)(1.175 - 5) \times 0.0026495$$

$$y = f(1.175) = 0.77239$$

Example 4

The following are the measurement of 't' (time) made on a curve recorded by an oscillograph a change in the condition of an electric current (I).

t (time)	1.2	2.0	2.5	3.0
I	1.36	0.58	0.34	0.20

Find the value of 'I' when t = 1.6 with appropriate Newton's Gregory Interpolation method.

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	1.36			
		$\frac{0.58 - 1.36}{2 - 1.2} = -0.975$		
2.0	0.58		$\frac{-0.48 + 0.975}{2.5 - 1.2} = 0.38077$	
		$\frac{0.34 - 0.58}{2.5 - 2.0} = -0.48$		$\frac{0.2 - 0.38077}{3 - 1.2} = -0.100428$
2.5	0.34		$\frac{-0.28 + 0.48}{3 - 2} = 0.2$	
		$\frac{0.20 - 0.34}{3 - 2.5} = -0.28$		
3.0	0.20			

$$y = y_0 + (x - x_0)\Delta y + (x - x_0)(x - x_1)\Delta^2 y + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y$$

$$y = 1.36 + (x - 1.2) \times -0.975 + (x - 1.2)(x - 2) \times 0.38077$$

$$+(x - 1.2)(x - 2)(x - 2.5) \times -0.100428$$

$$y = 1.36 + (1.6 - 1.2) \times -0.975 + (1.6 - 1.2)(1.6 - 2) \times 0.38077$$

$$+(1.6 - 1.2)(1.6 - 2)(1.6 - 2.5) \times 0.100428$$

$$y = f(1.6) = 0.894615$$

3.1.3 Interpolation with Equal Intervals

There are two methods to find the interpolation with equal intervals.

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a) Newton forward interpolation method/Newton-Gregory forward interpolation

b) Newton backward interpolation method/Newton-Gregory backward interpolation

a) Newton Forward Interpolation Method

We know that, Newton's forward interpolation formula as;

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0 + \ldots$$

where, $p = \frac{x_p - x_0}{h}$

x_p = Value at which interpolation is to be found.

x_0 = Initial value relative to x_p.

h = Interval of 'x'.

Example 5

Apply Newton's forward difference formula to find y(3.5)

X	1	2	3	4	5	6	7	8
Y	1	8	27	64	125	216	343	512

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$	$\Delta^7 y$
1	1							
		8 – 1 = 7						
2	8		12					
		27 – 8 = 19		6				
3	27		18		0			
		64 – 27 = 37		6		0		
4	64		24		0		0	
		125 – 64 = 61		6		0		0
5	125		30		0		0	
		216 – 125 = 91		6		0		
6	216		36		0			
		343 – 216 = 127		6				
7	343		42					
		512 – 343 = 169						
8	512							

x_p = 3.5 lie in between 3 and 4.

Therefore, $x_0 = 3$ and $y = 27$

$h = 1$

$$p = \frac{x_p - x_0}{h} = \frac{3.5-3}{1} = 0.5$$

Let y_p be the value of 'x' when x = 3.5.

Using Newton's forward difference formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 27 + 0.5 \times 37 + \frac{0.5(0.5-1)}{2!} \times 24 + \frac{(0.5-2)(0.5-1)0.5}{3!} \times 6 + 0$$

$$= 27 + 18.5 - 3 + 0.375$$

$$y_p = 42.875$$

Example 6

Generate forward difference table from the following data:

X	1	2	3	4	5	6
f(x)	2	9	28	65	126	217

Solution:

X	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	2					
		9 – 2 = 7				
2	9		19 – 7 = 12			
		28 – 9 = 19		18 – 12 = 6		
3	28		37 – 19 = 18		6 – 6 = 0	
		65 – 28 = 37		24 – 18 = 6		0
4	65		61 – 37 = 24		6 – 6 = 0	
		126 – 65 = 61		30 – 24 = 6		
5	126		91 – 61 = 30			
		217 – 126 = 91				
6	217					

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b) Newton Backward Interpolation Method

We know that, Newton's backward interpolation formula as;

$$y_p = y_n$$

$$= p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \ldots$$

where, $p = \frac{x_p - x_n}{h}$

x_p = Value at which interpolation is to be found.

x_n = Final value relative to x_p.

h = Interval 'x'.

Example 7

Use appropriate method of interpolation to get f(0.675) from the given table.

X	0.125	0.25	0.375	0.5	0.625	0.75
f(x)	0.7916	0.7733	0.7437	0.7041	0.6532	0.6022

Solution:

The forward difference table;

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.125	0.7916					
		−0.0018				
0.25	0.7733		- 0.0113			
		- 0.0296		0.0013		
0.375	0.7437		−0.01		- 0.0026	
		−0.0396		- 0.0013		0.0151
0.5	0.7041		−0.0113		0.0125	
		−0.0509		0.0112		
0.625	0.6532		- 0.0001			
		−0.051				
0.75	0.6002					

Now,

$x_p = 0.675$

$x_n = 0.75$

$h = 0.125$

Here,

$$p = \frac{x_p - x_n}{h} = \frac{0.675 - 0.75}{0.125} = -0.6$$

Using Newton's backward interpolation formula; we get,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \ldots\ldots + y_n$$

$$= -0.6 \times (-0.051) + \frac{-0.6(-0.6+1)}{2!} \times (-0.0001) + \frac{-0.6(-0.6+1)(-0.6+2)}{3!} \times (0.0112) + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times (0.0125) + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!} \times (0.0151) + 0.6022$$

$$= 0.0306 + 1.2 \times 10^{-5} - 6.272 \times 10^{-4} - 4.2 \times 10^{-4} - 3.45 \times 10^{-4} + 0.6022$$

$\therefore \quad y_p = 0.6314198$

The difference table (Forward)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0		forward	
		$\Delta y_0 = y_1 - y_0$		
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y = \Delta^2 y_1 - \Delta^2 y_0$
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		
x_3	y_3			

The difference table (Backward)

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		$\nabla y_0 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_0 = \nabla y_1 - \nabla y_0$	
		$\nabla y_1 = y_2 - y_1$		$\nabla^3 y = \nabla^2 y_1 - \nabla^2 y_0$
x_2	y_2		$\nabla^2 y_1 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_3 - y_2$		
x_3	y_3		backward	

3.1.4 Cubic Spline Interpolation

The concept behind cubic spline interpolation is from the mechanical drafting tool called 'spline' used for drawing smooth curves. These curve resemble cubic curve and hence the name cubic spline has been given to

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the piecewise cubic interpolating polynomial. We consider the construction of the cubic spline function which would interpolate the point $(x_0, f_0), (x_1, f_1), \ldots\ldots (x_n, f_n)$. The cubic spline s(x) consists of (n – 1) cubic corresponding to (n – 1) sub interval.

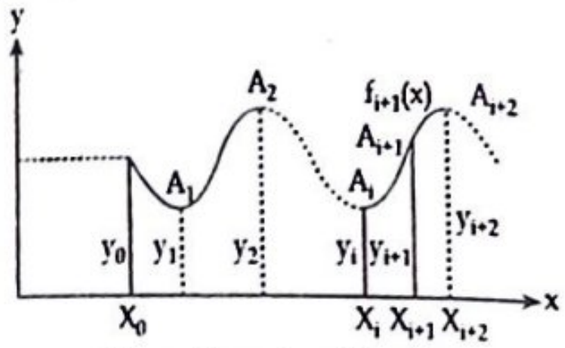

Figure: Curve for cubic spline

Formula

1. $h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$

   where, a_0 and $a_n = 0$

2. $S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i)$

Important Points

1. This method is applicable for uniformly and non-uniformly distributed for any value of 'x'.
2. $h_i = x_i = x_{i-1}$ and $v_i = x - x_i$
3. Number of coefficient is equal to number of points.
4. Evaluate equation (1) by starting i = 1 and onward a numbers of time equal to number of unknown coefficients.
5. Now evaluate equation (2) at i = a value given by position of interval.

Example 8

Given the data points

x =	4	9	16
f =	2	3	4

Estimate the functional value of 'f' at x = 7 using cubic splines.

Solution:

Here,

$$h_i = x_i - x_{i-1}$$

and, $v_i = x - x_i$

In this question there are 3 points

$$a_0 = a_2 = 0$$

For formula 1

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} = 6\left[\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right]$$

For i = 1

$$\cancel{h_1 a_0}^{\,0} + 2a_1(h_1 + h_2) + \cancel{h_2 a_2}^{\,0} = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$h_1 = x_1 - h_0 = 9 - 4 = 5$$

$$2a_1 \times (5 + 7) = 6\left[\frac{4-3}{7} - \frac{3-2}{5}\right]$$

$$24a_1 = 6\left[\frac{1}{7} - \frac{1}{5}\right]$$

$$a_1 = \frac{6}{24}\left[\frac{1}{7} - \frac{1}{5}\right]$$

$$\therefore \quad a_1 = -0.0143$$

From second formula; we have,

$$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 U_i - U_i^3) + \frac{a_i}{6h_i}(U_{i-1}^3 - h_i^2 U_{i-1}) + \frac{1}{h_i}(f_i U_{i-1} - f_{i-1} U_i)$$

$$S_1(x) = \frac{a_0}{6h_1}(h_1^2 U_1 - U_1^3) + \frac{a_1}{6h_1}(U_0^2 - h_1^2 - U_0) + \frac{1}{h_1}(f_1 U_0 - f_0 U_1)$$

$$U_0 = x - x_0 = x - 4$$

$$U_1 = x - x_1 = x - 9$$

$$S_1(7) = \frac{-0.0143}{6 \times 5}[(x-4)^3 - 5^2(x-4)] + \frac{1}{5}[3(x-4) - 2(x-9)]$$

$$= \frac{-0.0143}{30}[(7-4)^3 - 5^2(7-4)] + \frac{1}{5}[3(7-4) - 2(7-9)]$$

$$= 0.0228 + 2.6$$

$$S_1(7) = 2.6228$$

Example 9

Find 'y' at x = 8 from following data using Natural cubic spline interpolation

X	3	5	7	9
Y	3	2	3	1

Solution:

In this question, there are 4 points

a_0, a_1, a_2, and a_3

$a_0 = a_3 = 0$

We have three interval and three cubics and therefore, only a_1 and a_2 are to be determined.

When i = 1

$$\cancel{h_1 a_0}^{\,0} + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left[\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right]$$

$$h_1 = x_1 - x_0 = 5 - 3 = 2$$

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$h_2 = x_2 - x_1 = 7 - 5 = 2$

$$\begin{bmatrix} 2(2+2) & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$d_1 = \frac{6}{h^2}[f_2 - 2f_1 + f_0] = \frac{6}{4}[3 - 2 \times 2 + 3] = 3$$

$$d_2 = \frac{6}{h^2}[f_3 - 2f_2 + f_1] = \frac{6}{4}[1 - 6 + 3] = -3$$

$$a_1 = \frac{d_1 \times 8 - d_2 \times 1}{60} = \frac{3 \times 8 - (-3 \times 1)}{60} = 0.45$$

$$a_2 = \frac{d_2 \times 8 - d_1 \times 1}{60} = \frac{-3 \times 8 - 3 \times 1}{60} = -0.45$$

The target point $x = 8$

$$S_2(x) = \frac{a_1}{6}(U_2 - U_2^3) + \frac{a_2}{6}(U_1^3 - U_1) + (f_2 U_1 - f_1 U_2)$$

$$S_2(x) = \frac{a_1}{6}[(x - x_2) - (x - x_2)^3] + \frac{a_2}{6}[(x - x_1)^3 - (x - x_1)] + [f_2(x - x_1) - f_1(x - x_2)]$$

$$S_2(8) = \frac{0.45}{6}[(8 - 7) - (8 - 7)^3] + \frac{(-0.45)}{6}[(8 - 5)^3 - (8 - 5)] + [3(8 - 5) - 2(8 - 7)]$$

$$S_2(8) = 5.2$$

3.2 CURVE FITTING

3.2.1 Least Square Method/Fitting Linear Equation (Straight Line)

The equation $y = a + b$; (1)

which is an equation of first degree in 'x' and 'y'. It represents a straight line. Usually fitting a straight line means fitting the values of the parameters 'a' and 'b' of the straight line given by equation (1).

Method

Let, $y = a + b$; (1)

be the equation of straight line having

$$\sum Y = nA + b\sum X \quad (2)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (3)$$

Equation (2) and (3) are called normal equation.

Example 10

Obtain a relation of the form $y = ae^{bx}$ for the following data by the method of least square.

X	0.0	0.5	1.0	1.5	2.0	2.5
Y	0.10	0.45	2.15	9.15	40.35	180.75

Solution:

The given curve is $y = ae^{bx}$ (1)

Taking log on both sides; we have,

$\log y = \log a + bx \log e$; which is of the formal.

$Y = A + BX$ (2)

$Y = \log y$

$A = \log a$

$B = b \log e$

The normal equation of (2) is;

$$\sum Y = nA + b\sum X \quad (3)$$

$$\sum XY = A\sum X + b\sum X^2 \quad (4)$$

To solve using calculator to find value of $\sum X, \sum Y, \sum XY, \sum X^2$ etc.
→ Press ⇒ (Mode) and 3
→ Select 3, put the value of 'x' and 'y' and press [AC]
→ Press ⇒ (Shift) and (Stat)
→ Now select (4) : SUM and take the required submation vale.

X	Y	Y = log y	X^2	XY
0.0	0.10	−1	0.0	0
0.5	0.45	−0.34679	0.25	−0.1734
1.0	2.15	0.33244	1	0.33244
1.5	9.15	0.96142	2.25	1.44213
2.0	40.35	1.60584	4	3.21168
2.5	180.75	2.25708	6.25	5.6427
$\sum X = 7.5$		$\sum Y = 3.80999$	$\sum X^2 = 13.75$	$\sum XY = 10.4556$

$3.80999 = 6A + B(7.5)$

$10.4556 = A(7.5) + B(13.75)$

On solving; we get,

$A = -0.99161$

$B = 1.3013$

$\therefore \quad a = \text{antilog}(-0.99161) = 0.101951$

$$b = \frac{B}{\log e} = \frac{1.3013}{\log e} = 2.99635$$

Hence, the required curve is $y = 0.101951e^{2.99635x}$.

Example 11

Fit the following set of data to a curve of from $y = ae^{bx}$ from the following observation by least square method.

X	1	2	3	4	5	6
Y	5.5	6.5	9.4	15.2	30.6	49.8

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Solution:

The given curve is $y = ae^{bx}$

Taking log on both sides; we have,

$$\log y = \log a + bx \log x$$

Equation (2) is form of equation given below.

$$Y = A + BX$$

$$Y = \log y$$

$$A = \log a$$

$$B = b \log e$$

Now, the normal equation (3) is;

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

X	Y	Y = log y	X^2	XY
1	5.5	0.740363	1	0.740363
2	6.5	0.81291	4	1.62582
3	9.4	0.97313	9	2.91939
4	15.2	1.18184	16	4.72736
5	30.6	1.48000	25	7.40000
6	49.8	1.69723	36	10.18338
$\sum X = 21$		$\sum Y = 6.885473$	$\sum X^2 = 91$	$\sum XY = 27.596313$

Put these values in equation (4) and (5); we have,

$$6.885473 = 6A + 21B$$

$$27.596313 = 21A + 91B$$

$$A = 0.44815$$

$$B = 0.19964$$

$$a = \text{antilog}(A) = \text{antilog}(0.44815) = 2.806403$$

$$b = \frac{B}{\log e} = \frac{0.19984}{\log e} = 0.46015$$

The equation (1) becomes;

$$y = 2.806403e^{0.46015x}$$

Example 12

Fit the following data to the curve $y = \log_e(ax + b)$

X	0	1	2	3	4	5	6
Y	0.9	1	1.5	1.9	2.1	2.4	2.5

Solution:

Given that;

$$y = \log_e(ax + b)$$

or, $e^y = ax + b$

Comparing with $Y = A + BX$

$$Y = e^y$$

$$A = b$$

$$B = a$$

x	y	e^y
0	0.9	2.4596
1	1	2.7183
2	1.5	4.4817
3	1.9	6.6859
4	2.1	8.1661
5	2.4	11.0231
6	2.5	12.1825

$$\sum X = 21$$

$$\sum Y = 47.7173$$

$$\sum XY = 192.61487$$

$$\sum X^2 = 91$$

The normal equations are;

$$47.7173 = 7A + 21B$$

$$192.61487 = 21A + 91B$$

Solving; we get,

$$A = 1.51716 = b$$

$$B = 1.76653 = a$$

$$\therefore \quad y = \log_e(1.76653x + 1.51716)$$

Example 13

Use suitable method to fit quadratic curve $y = ax^2 + bx + c$ for the following data.

X	−3	−2	−1	0	1	2	3
Y	4.63	2.11	0.67	0.09	0.63	2.15	4.56

Solution:

The quadratic equation $y = ax^2 + bx + c$; then the normal equation is;

$$nc + b\sum X + a\sum X^2 = \sum Y \quad (1)$$

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$$c\sum X + b\sum X^2 + a\sum X^3 = \sum XY$$
$$c\sum X^2 + b\sum X^3 + a\sum X^4 = \sum X^2Y$$

X	Y	X^2	X^3	X^4	XY	X^2Y
−3	4.63	9	−27	81	−13.89	41.67
−2	2.11	4	−8	16	−4.22	8.44
−1	0.67	1	−1	1	−0.67	0.67
0	0.09	0	0	0	0	0
1	0.63	1	1	1	0.63	0.63
2	2.15	4	8	16	4.3	8.6
3	4.56	9	27	81	13.68	41.04
$\sum X = 0$	$\sum Y = 14.84$	$\sum X^2 = 28$	$\sum X^3 = 0$	$\sum X^4 = 196$	$\sum XY = -0.17$	$\sum X^2Y = 101.05$

so, the equation (1), (2) and (3); we get,

$7c + 28a = 14.84$

$28b = -0.17$

$28c + 196a = 101.05$

$b = -\frac{0.17}{28} = -0.006071 = -0.0061$

$7c + 28a = 14.84$

$28c + 196a = 109.05$

$a = 0.4963$

$c = 0.13476$

$y = 0.4963x^2 - 0.0061x + 0.13476$

SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. **Find the missing value in the following table of a chemical dissolved in water using least square approximation to straight-line fitting.** **[2005 Fall]**

Temperature	10°	15°	20°	25°	30°	35°
Solubility	20	22	23	?	26	27

Solution:

The given equation of straight line is;

$y = mx + C$ (1)

The normal equation of (1) is;

$\sum Y = nC + m\sum X$ (2)

$\sum XY = C\sum X + m\sum X^2$ (3)

X	Y	X^2	XY
10	20	100	200
15	22	225	330
20	23	400	460
30	26	900	780
35	27	1225	945
$\sum X = 110$	$\sum Y = 118$	$\sum X^2 = 2850$	$\sum XY = 2715$

The equation (2) and (3) now becomes;

$118 = 110m + 5C$ (4)

$2715 = 285m + 110C$ (5)

On solving (4) and (5); we get,

$M = 0.2767$

$C = 17.5116$

The required equation curve is $y = 0.2767x + 17.5116$ at $x = 25$

$y = 0.2767 \times 25 + 17.5116 = 24.429$

2. **Use the data linearization method and determine the exponential fit of $y = Ce^{Ax}$ for the following data** **[2006 Spring]**

X	0	1	2	3	4
Y	1.5	2.5	3.5	5.0	7.5

Solution:

The given curve is $y = Ce^{Ax}$ (1)

Taking log on both sides; we get,

$\log y = \log C + Ax\log e$ (2)

$Y = \log y$

$V = \log C$

$a = \log e$

The normal equation of (2) are;

$$\sum Y = nV + a\sum X \quad (3)$$

$$\sum XY = V\sum X + a\sum X^2 \quad (4)$$

X	Y	Y = logy	X^2	XY
0	1.5	0.1760	0	0.2640
1	2.5	0.3979	1	0.9947
2	3.5	0.5440	4	1.9040
3	5.0	0.6989	9	3.4945
4	7.5	0.8750	16	3.5629
$\sum X = 10$		$\sum Y = 2.6918$	$\sum X^2 = 30$	$\sum XY = 7.0828$

Now, equation (3) and (4) becomes;

$$2.6918 = 5V + 10a$$

$$7.0824 = 10V + 30a$$

$$V = 0.1985$$

$$a = -0.1699$$

$$\therefore \quad A = \frac{a}{\log e} = 0.3911$$

$$\therefore \quad C = \text{antilog}(0.1985) = 1.5794$$

Hence, the required curve is $y = 1.5794e^{0.3911x}$.

3. **The following set of data represents the position of a car in a road at specified time, calculate the position of the car at T = 1.75 hours.** **[2008 Spring]**

Time (hr)	0	0.5	1.0	1.5	2.0
Position (km)	0	0.25	1.0	2.25	4.0

Solution:

x = t	y = p	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	0				
		0.25			
0.5	0.25		0.5		
		0.75		0	
1	1		0.5		0
		1.25		0	
1.5	2.25		0.5		
		1.75			
2	4				

$x_p = 1.75$ lie in between 1.5 and 2 therefore, $x_n = 2$ and $y_n = 4$, $h = 0.5$

$$p = \frac{x_p - x_n}{h} = -0.5$$

By Newton's backward interpolation formula; we have,

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{(p+1)(p+2)p}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 4 + (-0.5 \times 1.75) + \frac{-0.5(-0.5-1)}{2!} \times 0.5 + 0 + 0$$

$$\therefore \quad y_{1.75} = 3.0625 \text{ km}$$

4. **Find the polynomial using Lagrange's method also calculate absolute and relative error at x = 2.7.** **[2009 Fall]**

x	3.2	2.7	1.0	4.8
f(x)	22.0	17.8	14.2	38.3

Solution:

Here

$$x_0 = 3.2$$

$$x_1 = 1.0$$

$$x_2 = 4.8$$

$$y_0 = 22.0$$

$$y_1 = 14.8$$

$$y_2 = 38.3$$

The required polynomial according to Lagrange's formula is;

$$y = 22 \times \frac{(x-2.7)(x-1)(x-4.8)}{(3.2-2.7)(3.2-1)(3.2-4.8)} + 17.8 \times \frac{(x-3.2)(x-1)(x-4.8)}{(2.7-3.2)(2.7-1)(2.7-4.8)} + 14.2 \times \frac{(x-3.2)(x-2.7)(x-4.8)}{(1-3.2)(1-2.7)(1-4.8)} + 38.3 \times \frac{(x-3.2)(x-2.7)(x-1)}{(4.8-3.2)(4.8-2.7)(4.8-1)}$$

$$= -12.5(x-2.7)(x-1)(x-4.8) + 9.972(x-3.2)(x-1)(x-4.8) + 3(x-3.2)(x-2.7)(x-1) - (x-3.2)(x-2.7)(x-4.8)$$

Here, the value of 'x' at which corresponding 'y' is to be calculated is taken according to Lagrange's interpolation.

$$y = f(x)$$

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$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times y_2$$

Now, at $x = 2.7$

$$y = f(2.7)$$

$$= \frac{(2.7-1)(2.7-4.8)}{(3.2-1)(3.2-4.8)} \times 22 + \frac{(2.7-3.2)(2.7-4.8)}{(1-3.2)(1-4.8)} \times 14.2 + \frac{(2.7-3.2)(2.7-1)}{(4.8-3.2)(4.8-1)} \times 38.3$$

or, $y = f(2.7) = 22.3125 + 1.7835 + 5.3455$

or, $y = f(2.7) = 18.7416$

But from table; we have,

$$y = f(x) = 17.8$$

so Absolute error $= 18.7416 - 17.8 = 0.9416$

Relative error $= \frac{18.7416 - 17.8}{17.8} = 0.0529$

5. **The heat of water Hf(t) and the quantity of water Q (t^3) flowing per second are related by the law $Q = CH^{\alpha}$. Find out the best fit value for coefficient C and α for the following data** [2009 Fall]

H	5	10	15	20	25	30
Q	20	150	360	800	1500	2200

Solution:

The given equation is $Q = CH^{\alpha}$ (1)

Let, $Q = y$ and $C = a$, $H = x$ and $\alpha = b$

We get,

$$y = ax^b$$

Taking log on both sides; we get,

$$\log y = \log a + bx \log x \quad (2)$$

Comparing (2) with

$$Y = A + BX \quad (3)$$

We get,

$$Y = \log y$$
$$A = \log a$$
$$X = \log x$$

The normal equation of (3) is;

$$\sum Y = nA + b\sum X \quad (4)$$
$$\sum XY = A\sum X + b\sum X^2 \quad (5)$$

X	y	x = log H	Y = log y	X^2	XY
5	20	0.699	1.301	0.489	0.909
10	150	1.0	2.176	1	2.176
15	360	1.176	2.5563	1.383	3.006
20	800	1.301	2.903	1.693	3.777
25	1500	1.398	3.176	1.954	9.44
30	2200	1.477	3.342	2.182	4.936
		$\sum X = 7.051$	$\sum Y = 15.454$	$\sum X^2 = 8.7$	$\sum XY = 19.244$

Now equation (4) and (5) can be written as;

$$15.454 = 6A + 7.051x$$
$$19.244 = 7.051A + 8.7x$$

On solving; we get,

$$B = 2.617 = \alpha$$
$$A = -0.499$$
$$a = \text{antilog}(A) = 0.317 = C$$

Hence,

$$Q = 0.317H^{2.617}$$

6. **Estimate the value of $\sin\theta$ at $\theta = 25$ using Newton-Gregory forward difference formula with the help of the following table.** [2010 Spring]

θ	10	20	30	40	50
sin θ	0.1736	0.3420	0.5000	0.6428	0.7660

Solution:

The difference table is tabulated below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736				
		0.1684			
20	0.3420		−0.0104		
		0.158		−0.0048	
30	0.5		−0.0152		0.0004
		0.1428		−0.0044	
40	0.6428		−0.0196		
		0.1232			
50	0.7660				

$$x_p = 25$$

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$x_0 = 20$

and, $y_0 = 0.3420$

$$p = \frac{x_p - x_0}{h} = \frac{25-20}{10} = 0.5$$

Let y_p be the value at $x = 25$

From Newton's forward interpolation formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{p(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 0.3420 + 0.5 \times 0.158 + \frac{0.5(0.5-1)}{2!}(-0.0152) + \frac{(0.5-2)(0.5-1)0.5}{3!}(-0.0044)$$

$\therefore \quad y_{25} = 0.4226$

7. **Find Newton's forward difference interpolation polynomial for the following data. Use the obtained formula to estimate the value of f(0.24)** **[2010 Fall]**

X	0.1	0.2	0.3	0.4	0.5
f(x)	1.40	1.56	1.76	2.00	2.28

Solution:

The difference table is tabulated below.

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.2		0	
0.3	1.76		0.04		0
		0.24		0	
0.4	2.0		0.04		
		0.28			
0.5	2.28				

Here, $x_p = 0.24$ lie in between 0.3 and 0.2 therefore, $x_0 = 0.2$ and $y_0 = 1.56$,

$h = 0.3 - 0.2 = 0.1$

$$p = \frac{x_p - x_0}{h} = \frac{0.24-0.2}{0.1} = 0.4$$

Let y_p be the value of 'x' when $x = 0.24$

Using Newton's forward interpolation formula; we have,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{(p-2)(p-1)p}{3!}\Delta^3 y_0 + \frac{p(p-3)(p-2)(p-1)}{4!}\Delta^4 y_0$$

$$= 1.56 + 0.4 \times 0.2 + \frac{0.4(0.4-1)}{2!} \times 0.04 + 0 + 0$$

$\therefore \quad y_p = 1.6352$

8. **The growth of bacteria (N) in a culture after t hours is given by the following table.**

Time t (hrs)	0	1	2	3	4	4
Bacteria (N)	32	47	65	92	132	124

If the relation between bacteria N and time if of the form $N = ab^t$. Using least square method, estimate the N at t = 5 hrs. **[2017 Spring]**

Solution:

Here;

$N = ab^t$

Taking *ln* on both side,

or, $\ln N = \ln a + t \ln b$

Let, $\ln N = Y$, $\ln a = A$, $\ln b = B$

Then,

$$Y = A + Bt \quad (1)$$

Then, according to L.S.M.,

$$\Sigma Y = nA + \Sigma t \cdot B \quad (2)$$

$$\Sigma tY = \Sigma t \cdot A + \Sigma t^2 \cdot B \quad (3)$$

t	N	Y = *ln* N	tY	t^2
0	32	3.465	0	0
1	47	3.850	3.850	1
2	65	4.174	8.348	4
3	92	4.522	13.565	9
4	132	4.883	19.531	16
4	124	4.820	19.281	16
Sum 14		**25.714**	**64.575**	**46**

so, equation (2) and equation (3) becomes,

$25.714 = 6A + 14B$

$64.575 = 14A + 46B$

and, solving above equation,

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$A = 3.485$ $\quad$ $B = 0.343$

or, $\ln a = 3.485$ $\quad$ or, $\ln b = 0.343$

or, $a = 32.622$ $\quad$ or, $b = 1.409$

so, equation becomes;

$$N = (32.622) \times (1.409)^t$$

$\therefore$ N at t = 5 hrs $= 32.622 \times (1.409)^5 = 181.16 \approx 182$.

9. The following table given the percentage of criminals for different age groups. Using interpolation for find the percentage of criminals under the age of 35.

Under age	25	30	40	50
% of criminals	52	67.3	84.1	94.4

[2017 Spring]

Solution:

Here; 35 is nearer to 50. Hence, we use backward interpolation. [35 is located at nearby mid-point. So, central difference interpolation should be used. But for simplicity we are using backward interpolation]

Constructing difference table

Age (x)	% of criminal (y)	∇y	$\nabla^2 y$	$\nabla^3 y$
25	52			
		15.3		
30	67.3		1.5	
		16.8		−8
40	84.1		−6.5	
		10.3		
50	94.4			

Taking

$$x_n = 50, x = 35, h = 10$$

$$\therefore \quad P = \frac{x - x_n}{h} = \frac{35 - 50}{10} = -1.5$$

$$\therefore \quad y_{35} = y_n + P.\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$= 94.4 + (-1.5) \times 10.3 + \frac{(-1.5) \times (-0.5)}{2} \times (-6.5) + \frac{(-1.5) \times (-0.5) \times (0.5)}{6} \times (-8)$$

$$= 76.0125$$

Therefore, % of criminals under the age of 35 = 76.0125%.

10. Determine the constants a and b by the method of least squares such that $y = ae^{bx}$.

[2018 Spring]

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Solution:

Here;

$$y = ae^{bx}$$

Taking $\ln$ on both side,

or, $\ln y = \ln (ae^{bx})$

or, $\ln y = \ln a + bx$

Let, $\ln y = Y$ and $\ln a = A$, then

or, $Y = A + bx$

This is linear function. And, we know that in least squares method; we can obtain value of constant by solving these two equations,

$$\Sigma Y = nA + b\Sigma x \quad (1)$$

$$\Sigma XY = A.\Sigma x + b\Sigma x^2 \quad (2)$$

And

x	y	$Y = \ln y$	xY	x^2
2	4.077	1.405	2.811	4
4	11.084	2.406	9.622	16
6	30.128	3.406	20.432	36
8	81.897	4.406	35.244	64
10	222.62	5.406	54.055	100
Sum = 30		**17.030**	**122.164**	**220**

NOTE 1:

We calculate value in table using calculator by:

i) Type "A = $\ln$y : xA : x^2"

ii) Then press "calc".

So, $\Sigma x = 30$, $\Sigma Y = 17.030$, $\Sigma xY = 122.164$, $\Sigma x^2 = 220$

Then, equation (1) and equation (2) becomes

$$17.030 = 5A + b \times 30 \quad (3)$$

$$122.164 = 30A + b \times 220 \quad (4)$$

By solving equation (3) and equation (4);

$$A = 0.4084 \text{ and } b = 0.4996$$

NOTE 2:

To solve equation, we uses the calculator.

And, we know

$$\ln a = A$$

or, $a = e^A = e^{0.4084} = 1.504$

Hence, constants are calculated.

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11. From the following table, find the number of student who obtained less than 45 marks. [2018 Spring]

Marks	30-40	40-50	50-60	60-70	70-80
No. of student	31	42	51	35	31

Solution:

First we prepare cumulative frequency table as follows:

Mark less than (x)	40	50	60	70	80
Number of students (y)	31	73	124	159	190

Now, constructing difference table

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

We have to find number of students who obtained less than 45 marks.

Taking $x_0 = 40$ and $x = 45$, we have

$$P = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{5}{10} = 0.5 \qquad [\because h = 10]$$

Therefore, using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + P\Delta y_{40} + \frac{P(P-1)}{2!}\Delta^2 y_{40} + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_{40} + \frac{P(P-1)(P-2)(P-3)}{4!}\cdot\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5 \times (-0.5)}{2} \times 9 + \frac{(0.5)(-0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$
$$= 47.87$$

The number of students with marks less than 45 is 47.87 *i.e.*, 48.

NOTE:

If we were asked to find number of students who obtained marks between 40 and 45.

The number of students with marks less than 40 is 31.

Hence, the number of students getting marks between 40 and 45

$= 48 - 31 = 17.$

12. Generate a Lagrange's interpolation polynomial for the function: $y = \cos \pi x$, taking the pivotal point $0, \frac{1}{4}$ and $\frac{1}{2}$. [2018 Fall]

Solution:

Here;

k	x_k	y_k
1	0	1
2	$\frac{1}{4}$	0.707
3	$\frac{1}{2}$	0

Hence, for these pivotal point.

Now, we know that; Lagrange's interpolation polynomial is given by

$$y(x) = \sum_{j=1}^{n} y_j(x)$$

where, $y_j(x) = y \prod_{k=1}^{n} \frac{x - x_k}{x_j - x_k}$

i.e., $$y(x) = \sum_{j=0}^{n} \left[y_j \times \prod_{k=0}^{n} \left(\frac{x - x_k}{x_j - x_k} \right) \right] \qquad [\text{if } j \neq k]$$

Expanding

$$y = \frac{(x - x_1)(x - x_2) \cdots\cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots\cdots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \cdots\cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots\cdots (x_1 - x_n)} y_1 + \cdots\cdots + \frac{(x - x_0)(x - x_1) \cdots\cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots\cdots (x_n - x_{n-1})} y_n$$

Hence,

$$y_1(x) = y_1 \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

$$= 1\left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$= 1\left(\frac{x - \frac{1}{4}}{0 - \frac{1}{4}}\right)\left(\frac{x - \frac{1}{2}}{0 - \frac{1}{2}}\right) = 8\left(x - \frac{1}{4}\right)\left(x - \frac{1}{2}\right)$$

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Similarly,

$$y_2(x) = 0.707\left(\frac{x-0}{\frac{1}{4}-0}\right)\left(\frac{x-\frac{1}{2}}{\frac{1}{4}-\frac{1}{2}}\right)$$

$$= -11.312x\left(x-\frac{1}{2}\right)$$

$$y_3(x) = 0\left(\frac{x-0}{\frac{1}{2}-0}\right)\left(\frac{x-\frac{1}{2}}{\frac{1}{2}-\frac{1}{2}}\right)$$

$$= 0$$

so, $y(x) = \sum_{r=0}^{n} y_r(x)$

$$\boxed{y(x) = 8\left(x-\frac{1}{4}\right)\left(x-\frac{1}{2}\right) - 11.312x\left(x-\frac{1}{2}\right)}$$

13. Fit a curve of the form: $y = \frac{1}{a+bx}$ by using the method of least square with the following data points: [2018 Fall]

x	1	2	3	4	5
f(x)	3.33	2.20	1.52	1.00	0.91

Solution:

Given that;

$$y = \frac{1}{a+bx}$$

or, $y(a+bx) = 1$ if $\frac{1}{a+bx} \neq 0$

or, $ay + bxy = 1$

or, $ay = 1 - bxy$

or, $y = \frac{1}{a} - \frac{b}{a}xy$

Let, $\frac{1}{a} = A$, $-\frac{b}{a} = B$ and $xy = X$

So, $y = A + BX$

This is linear equation.

And, for linear equation, we can use; according to L.S.M.,

$$\Sigma y = nA + B\Sigma X \quad (1)$$

$$\Sigma Xy = \Sigma X \cdot A + B\Sigma X^2 \quad (2)$$

x	y	X = xy	Xy	X^2
1	3.33	3.33	11.0889	11.0889
2	2.20	4.40	9.6800	19.3600
3	1.52	4.56	6.9312	20.7936
4	1.00	4.00	4.0000	16.0000
5	0.91	4.55	4.1405	20.7025
	8.96	20.84	35.8406	87.945

so, equitation becomes,

$$8.96 = 5A + 20.84\,B \quad (1)$$

$$35.8406 = 20.84A + 87.945\,B \quad (2)$$

Now, by solving about two equations.

$A = 7.378$, $B = -1.388$

And,

$\frac{1}{a} = A = 7.578$ $\frac{-b}{a} = B = -1.388$

or, $a = 0.132$ or, $\frac{b}{0.132} = 1.388$

or, $b = 0.183$

So, fitting curve will be

$$y = \frac{1}{0.132 + 0.183x}$$

14. The function y = f(x) is given at the points (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for x = 9.5 using Lagrange interpolation formula. [2019 Fall]

Solution:

We have,

x	$x_0 = 7$	$x_1 = 8$	$x_2 = 9$	$x_3 = 10$
y	$y_0 = 3$	$y_1 = 1$	$y_2 = 1$	$y_3 = 9$

Putting x = 9.5 and substituting the above values in Lagrange's formula; we get,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} \times 3$$

$$+ \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} \times 1$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} \times 1$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-8)} \times 9$$

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$$= 0.1875 + (-0.3125) + 0.9375 + 2.8125$$
$$= 3.625$$

15. The following table shows pressure and specific volume of dry saturated stream. [2019 Fall]

V	38.4	20	8.51	4.44	3.03
P	10	20	50	100	150

Fit a curve of the form: $PV^{\alpha} = \beta$ by using least square method.

Solution:

Given that;

$$PV^{\alpha} = \beta$$
or, $\log PV^{\alpha} = \log\beta$

or, $\log P + \log V^{\alpha} = \log\beta$

or, $\log P + \alpha\log V = \log\beta$

or, $\log V = \frac{1}{\alpha}\log\beta - \frac{1}{\alpha}\log P$

or, $Y = A + bX$ (1)

where, $Y = \log V$

$X = \log P$

$A = \frac{1}{\alpha}\log\beta$

$b = -\frac{1}{\alpha}$

The normal equations are:

$$\sum Y = nA + b\sum X$$ (2)

$$\sum XY = A\sum X + b\sum X^2$$ (3)

Calculating the value from the table; we have,

$X = \log P$	$Y = \log V$	X^2	XY
1	1.58	1	1.58
1.30	1.30	1.69	1.69
1.6989	0.9299	2.89	1.58
2	0.67473	4	1.35
2.176	0.4814	4.735	1.047
8.1749	4.9386	14.311	7.1919

Substituting the values in the equation (2) and (3); w get,

$$4.9386 = 5A + 8.1749b$$ (4)

$$7.1919 = 8.1749A + 14.311b$$ (5)

Solving equation (4) and (5); we get,

$A = 2.51$

$b = -0.9337$

$\therefore \quad b = -\frac{1}{\alpha}$

or, $\alpha = -\frac{1}{b} = -\frac{1}{-0.9337} = 1.07$

Also,

$$A = \frac{1}{\alpha}\log_{10}(\beta)$$

or, $2.51 = \frac{1}{1.07}\log_{10}(\beta)$

$\therefore \quad \beta = 484.95$

Hence, the required equation becomes;

$$PV^{1.07} = 484.95$$

16. Obtain divided difference table for the given data set. [2019 Fall]

x	-1	2	5	7
y	-8	3	1	12

Solution:

x	y	First divided differences	Second divided differences	Third divided differences
-1	-8			
		$\frac{3-(-8)}{2-(-1)} = \frac{11}{3}$		
2	3		$\frac{-\frac{2}{3}-\frac{11}{3}}{5+1} = -\frac{13}{18}$	
		$\frac{1-3}{5-2} = -\frac{2}{3}$		$\frac{\frac{37}{30}+\frac{13}{18}}{7+1} = \frac{\frac{88}{45}}{8} = \frac{11}{45}$
5	1		$\frac{\frac{11}{2}+\frac{2}{3}}{7-2} = \frac{\frac{37}{6}}{5} = \frac{37}{30}$	
		$\frac{12-1}{7-5} = \frac{11}{2}$		
7	12			

17. The voltage V across a capacitor at time t seconds is given in following table:

Time t (sec)	0	2	4	6	8
Voltage (V)	150	63	28	12	5.6

If the relationship between voltage V and time t is of the form $V = ae^{kt}$. Using least-square approximation estimate the temperature at t = 2.6 minutes. [2019 Spring]

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Solution:

Here,

$V = ae^{kt}$

Taking ln on both sides,

$ln\, V = ln\,(ae^{kt})$

or, $ln\, V = ln\, a + ln\, e^{kt}$

or, $ln\, V = ln\, a + kt\, ln\, e$ $[\because ln\, e = 1]$

or, $ln\, V = ln\, a + kt$

or, $y = A + bx$ (1)

where, $y = ln\, V$

$A = ln\, a$

$b = k$

$x = t$

From equation (1), using method of least squares, we get,

$\Sigma y = nA + b\Sigma x$ (2)

$\Sigma xy = A\Sigma x + b\Sigma x^2$ (3)

V	x = t	y = ln V	xy	x^2
150	0	5.01	0	0
63	2	4.143	8.286	4
28	4	3.332	13.328	16
12	6	2.485	14.91	36
5.6	8	1.7227	13.7816	64
Total	20	16.6927	50.3056	120

Putting valves in equation (2) and (3), we get,

$16.6927 = 5A + 20\,b$ (4)

$50.3056 = 20\,A + 120\,b$ (5)

Solving equation (4) and (5), we get,

$A = 4.98506$

or, $ln\, a = 4.98506$

$\therefore$ $a = e^{4.98506} = 146.2123$

$b = -0.41163 = k$

Now,

$V = ae^{kt}$

or, $V = 146.2123\, e^{-0.41163\, t}$

$\therefore$ At $t = 2.6$ minutes $= 2.6 \times 60$ sec $= 156$ sec

$\therefore$ $V = 146.2123 \times e^{-0.41163 \times 156}$

$= 1.89 \times 10^{-26}$ $V \approx 0$

18. From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Solution: See the Q. no. (11) 2018 Spring]

19. Write short notes on: Cubic splines. **[2019 Spring]**

Solution: See the theory section 3.1.4.

20. From the following experimental data, it is known that the relation connects V and t as $V = at^b$. Find the possible values of a and b.

V	300	400	500	600
T	61	26	7	2.6

[2020 Fall]

Solution:

Here;

$V = at^b$

Taking ln on both sides

$ln\, V = ln\,(at^b)$

or, $ln\, V = ln\, a + ln\, t^b$

or, $ln\, V = ln\, a + b\, ln\, t$

or, $y = A + bx$ (1)

where, $y = ln\, V$

$A = ln\, a$

$x = ln\, t$

From equation (1), using method of least squares, we get,

$\Sigma y = nA + b\Sigma x$ (2)

$\Sigma xy = A\Sigma x + b\Sigma x^2$ (3)

V	T	x = ln T	y = ln V	xy	x^2
350	61	4.1108	5.878	24.163	16.898
400	26	3.258	5.99	19.515	10.614
500	7	1.946	6.214	12.092	3.787
600	2.6	0.956	6.397	6.115	0.913
Total		10.2708	24.479	61.885	32.212

Putting values in equation (2) and (3), we get,

$24.479 = 4A + 10.2708\,b$ (4)

$61.885 = 10.2708A + 32.212b$ (5)

Solving (4) and (5), we get,

$A = 6.546$

$b = -0.166$

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Now,

$A = 6.456$

or, $\ln a = 6.456$

$\therefore$ $a = e^{6.456} = 635.51$

Hence, the relation between V and t becomes

$V = 636.51\, t^{-0.166}$

21. The following table gives the viscosity of oil as the function of temperature. Use Lagrange's interpolation formula to find the viscosity of oil at a temperature of 140 degree celsius.

T(degree celsius)	110	130	160	190
Viscosity	10.8	8.1	5.5	4.8

[2020 Fall]

Solution:

Here;

Let's take T (degree celsius) as x and viscosity as y.

$x_0 = 110$	$y_0 = 10.8$
$x_1 = 130$	$y_1 = 8.1$
$x_2 = 160$	$y_2 = 5.5$
$x_3 = 190$	$y_3 = 4.8$

$y = ?$ When $x = 140$

By Lagrange's formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8$$

$$+ \frac{(140-110)(140-160)(140-190)}{(130-110)(130-160)(130-190)} \times 8.1$$

$$+ \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5$$

$$+ \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$= -1.35 + 6.75 + 1.834 - 0.2$

$= 7.034$

$\therefore$ Viscousity $= 7.034$ at $t = 140°$ C.

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. Fit a quadratic curve to the data in the table below; [2015]

x	1.0	2.1	3.2	4.0
t	2	2.5	3	4

Solution:

Here, we have to fit quadratic curve $y = ax^2 + bx + c$ in the data given in table below;

Now,

$y = ax^2 + bx + c$

$\therefore$ $\Sigma y = a\Sigma x^2 + b\Sigma x + nc$ (1)

$\therefore$ $\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$ (2)

$\therefore$ $\Sigma x^2y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$ (3)

x	y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2.1	2.5	4.41	9.261	19.448	5.25	11.025
3.2	3	10.24	32.768	104.857	9.6	30.72
4	4	16	64	256	16	64
10.3	11.5	31.65	107.029	381.305	32.85	107.745

Putting the values of table in equation (1), (2) and (3); we get,

$11.5 = 31.65a + 10.3b + 4c$ (4)

$32.85 = 107.029a + 31.65b + 10.3c$ (5)

$107.745 = 381.305a + 107.029b + 31.65c$ (6)

Solving equation (4), (5) and (6) using calculator

$a = 0.168$

$b = -0.2057$

$c = 2.074$

$\therefore$ The equation of fitted quadratic curve is;

$y = 0.168x^2 - 0.2054x + 2.074$

NOTE:

You can also use Jacobi iteration method or Gauss-Siedel iteration method to solve above equation (4), (5) and (6).

2. Estimate the value of $f(1.35)$ using Newton's interpolation polynomial.

x	1.2	1.3	1.4	1.5
$f(x)$	1.063	1.091	1.119	1.145

Also, find the polynomial. [2015]

Solution:

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Here;

$x = 1.35$ lies exactly at midpoint.

So, let us determine whether it will be appropriate to use Newton's forward or backward interpolation formula.

Let's say functional value of $x = 1.35$ lies exactly at midpoint of value of $x = 1.3$ and $x = 1.4$.

i.e., $f(x = 1.35) = \dfrac{f(x = 1.3) + f(x = 1.4)}{2} = \dfrac{1.019 + 1.119}{2} = 1.105$

Now, checking difference of functional value from beginning and end.

$\Delta_1 = |f(1.2) - f(1.35)| = |1.063 - 1.105| = 0.042$

$\Delta_2 = |f(1.5) - f(1.35)| = |1.145 - 1.105| = 0.04$

Here, difference is less from backward. Hence, it is more precise to use Newton's backward interpolation.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1.2	1.063			
		0.028		
1.3	1.091		0	
		0.028		-2×10^{-3}
1.4	1.119		-2×10^{-3}	
		0.026		
1.5	1.145			

Using Newton's backward interpolation formula:

$$y_P = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

Let $x_P = x$ then

$$P = \frac{x_P - x_n}{h} = \frac{x - 1.5}{0.1} = (10x - 15)$$

Then,

$$y = 1.145 + (10x - 15)(0.026) + \frac{(10x - 15)\cdot(10x - 14)}{2}(-2 \times 10^{-3})$$
$$+ \frac{(10x - 15)(10x - 14)(10x - 13)}{6} \times (-2 \times 10^{-3})$$

or, $y = 1.145 + 0.26x - 0.39 + (100x^2 - 290x + 210) \times (-1 \times 10^{-3})$
$+ (100x^2 - 290x + 210)(10x - 13) \times (-3.34 \times 10^{-4})$

or, $y = 1.145 + 0.26x - 0.39 - 0.1x^2 + 0.29x - 0.21$
$+ (1000x^3 - 4200x^2 + 5870x - 2730) \times (-3.34 \times 10^{-4})$

$= 1.145 + 0.26x - 0.39 - 0.1x^2 + 0.29x - 0.21 - 0.334x^3$

$+ 1.4028x^2 - 1.9606x + 0.9118$

or, $y = -0.334x^3 + 1.3028x^2 - 1.4106x + 1.4568$

Therefore, the required polynomial is

$y = -0.334x^3 + 1.3028x^2 - 1.4106x + 1.4568$

Now,

$f(x) = ?$ when $x = 1.35$

$f(x = 1.35) = -0.334 \times (1.35)^3 + 1.3028 \times (1.35)^2 - 1.4106 \times (1.35) + 1.4568$
$= 1.10508$

NOTE:

We can also interpolate $f(x = 1.35)$ with respect to $x = 1.4$ to get quick result but it is less accurate.

$$P = \frac{x_P - x_n}{h} = \frac{1.35 - 1.4}{0.1} = -0.5$$

$$\therefore \quad y = y_n + P.\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{2!}\nabla^3 y_n$$

$$= 1.119 + (-0.5) \times (0.028) + \frac{(-0.5) \times (0.5)}{2!} \times 0$$

$= 1.119 - 0.014$

$= 1.105$

When we put $x_P = x$, we can also get the polynomial in very easy way.

3. **Fit the power curve $y = ax^b$ from the data given below. Also, find the value of y at $x = 5$.** [2016]

x	1	2	3	4
y	0.5	2	4.5	8

Solution:

Here;

$y = ax^b$

Taking $\log_e$ on both side; we get,

$\ln y = \ln (ax^b)$

or, $\ln y = \ln a + b.\ln x$

or, $Y = A + b.X$ (1)

where, $Y = \ln y$

$A = \ln a$

$X = \ln x$

From equation (1); using method of least squares; we get,

$\Sigma Y = nA + b\,\Sigma X$ (2)

$\Sigma XY = A\,\Sigma X + b\,\Sigma X^2$ (3)

x	y	X = ln x	Y = ln y	XY	X^2
1	0.5	0	-0.6931	0	0
2	2	0.6931	0.6931	0.4804	0.4804
3	4.5	1.0986	1.5040	1.6523	1.2069
4	8	1.3863	2.0794	2.8826	1.9218
		3.178	3.5834	5.0153	3.6091

Putting values in equation (2) and (3); we get,

$3.5834 = 4A + 3.178b$ (4)

$5.0153 = A(3.178) + 3.6091b$ (5)

Solving equation (4) and (5); we get,

$A = -0.693$

$b = 2$

Now,

$A = -0.693$

or, $\ln a = -0.693$

$\therefore$ $a = e^{-0.693} = 0.5$

Therefore, the equation of power curve is;

$\boxed{y = 0.5 \cdot x^2}$

$f(x = 5) = ?$

$\therefore$ $f(x = 5) = 0.5 \times (5)^2 = 12.5$

4. **From the given below estimate the value of $e^{2.5}$ using Lagrange's interpolation formula.** [2016]

x	1	2	3
$e^x - 1$	1.718	6.389	19.086

Solution:

Here, using Lagrange's interpolation formula;

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \cdot y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \cdot y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \cdot y_2$$

where,

$x_0 = 1$ $\quad y_0 = 1.718$

$x_1 = 2$ $\quad y_1 = 6.389$

$x_2 = 3$ $\quad y_2 = 19.086$

$x = 2.5$

$$\therefore \quad y = \frac{(2.5 - 2)(2.5 - 3)}{(1 - 2)(1 - 3)} \cdot (1.718) + \frac{(2.5 - 1)(2.5 - 3)}{(2 - 1)(2 - 3)} \cdot (6.389) + \frac{(2.5 - 1)(2.5 - 2)}{(3 - 1)(3 - 2)} \cdot (19.086)$$

$= 11.734$

Now, the function is $e^{2.5} - 1$

so, $e^{2.5} - 1 = 11.734$

$\therefore$ $e^{2.5} = 12.734$

5. **Use the method of least squares to fit a straight line to the following data.** [2017]

x	0	5	10	15	20
y	7	11	16	20	26

Solution:

Here, we know the equation of straight line is;

$y = mx + c$

i.e., $y = ax + b$ (1)

Using method of least squares in equation; we get,

$\Sigma y = a\,\Sigma x + b \cdot n$ (2)

$\Sigma xy = a\,\Sigma x^2 + b\,\Sigma x$ (3)

x	y	xy	x^2
0	7	0	0
5	11	55	25
10	16	160	100
15	20	300	225
20	26	520	400
50	80	1035	750

Putting values in (2) and (3); we get,

$80 = 50a + 5b$ (4)

$1035 = 750a + 50b$ (5)

Solving equation (4) and (5); we get,

$a = 0.94$ and $b = 6.6$

Therefore, the equation of straight line that can be fitted to above data is

$y = 0.94x + 6.6$

6. **Use Newton's forward method to find the value of y at x = 21 and x = 28 from the following data.**

x	20	23	26	29
y	0.342	0.3907	0.4384	0.4848

Estimate value of y when x = 25 [2017]

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Solution:

Here, constructing Newton's interpolation table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	0.342			
		0.0487		
23	0.3907		-0.001	
		0.0477		-0.0003
26	0.4384		-0.0013	
		0.0464		
29	0.4848			
		∇y	$\nabla^2 y$	$\nabla^3 y$

a) y = ? when x = 21

x = 21 is near to 20. So, we use Newton's forward interpolation

$$P = \frac{x - x_0}{h} = \frac{21 - 20}{3} = 0.334$$

$$\therefore \quad y_P = y_0 + P\cdot\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$= 0.342 + (0.334)(0.0487) + \frac{(0.334)(-0.667)}{2!} \times (-0.001) + \frac{(0.334)(-0.667)(-1.667)}{6!} \times (-0.0003)$$

$\therefore \quad y_P = 0.3584$

i.e., y = 0.3584 when x = 21

b) y = ? when x = 28

x = 28 is near to 29 (last term).

So, we use Newton's backward interpolation

$$P = \frac{x - x_n}{h} = \frac{28 - 29}{3} = -0.334$$

$$\therefore \quad y_P = y_n + P\cdot\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$= 0.4848 + (-0.334)(0.0464) + \frac{(-0.334)(0.667)}{2!} \times (-0.0013) + \frac{(-0.334)(0.667)(1.667)}{6} \times (-0.0003)$$

$\therefore \quad y_P = 0.4694$

i.e., y = 0.4694 when x = 28

c) y = ? when x = 25

x = 25 is near to 20 (last term) as compared to 20 (first term).

So, we use Newton's backward interpolation formula

$$P = \frac{x - x_n}{h} = \frac{25 - 29}{3} = -1.334$$

$$\therefore \quad y_P = y_n + P\cdot\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$= 0.4848 + (-1.334)(0.0464) + \frac{(-1.334)(-0.334)}{2!} \times (-0.0013) + \frac{(-1.334)(-0.334)(0.667)}{6!} \times (-0.0003)$$

$\therefore \quad y_P = 0.4226$

i.e., y = 0.4226 when x = 25.

7. **Fit a curve $y = ae^{bx}$ to the following data:** [2018]

x	5.01	10.00	15.05	25.10
y	40.1	45.2	60.3	70.3

Solution:

Here, the given curve is

$y = ae^{bx}$ (1)

Taking ln on both the side, we get

$\ln y = \ln a + bx \cdot \ln e$

or, $\ln y = \ln a + bx$ [∵ ln e = 1]

$\therefore \quad Y = A + bx$ (2)

where, $Y = \ln y$

$A = \ln a$

x	y	Y = ln y	xY	x^2
5.01	40.1	3.691	18.492	25.1
10	45.2	3.811	38.11	100
15.05	60.3	4.099	61.690	226.502
25.10	70.3	4.252	106.725	630.01
55.16		15.853	225.017	981.612

Using method of least squares in equation (2); we get,

$\Sigma Y = nA + b\Sigma x$ (3)

$\Sigma xY = A\Sigma x + b\Sigma x^2$ (4)

Putting values in (3) and (4) from table; we get,

$15.853 = 4A + 55.16x$ (5)

$225.017 = 55.16A + 981.612x$ (6)

Solving equation (5) and (6); we get,

A = 3.5636 and b = 0.02898

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We have,

$A = \ln a$

or, $3.5636 = \ln a$

$\therefore$ $a = e^{3.5636} = 35.29$

Therefore, the required equation of curve is;

$y = 35.29 \cdot e^{0.02898x}$

8. **Determine the constants a and b by the method of least squares such that $y = ae^{bx}$. Fits the following data:** [2019]

x	2	4	6	8	10
y	4.077	11.084	30.128	81.892	222.62

Solution:

Here,

$y = ae^{bx}$

or, $\ln y = \ln(ae^{bx}) = \ln a + \ln e^{bx}$ $[\because \ln e = 1]$

or, $\ln y = \ln a + bx$

i.e., $Y = A + bx$

By using method of least squares; we get,

$\sum Y = nA + b\sum x$ (1)

$\sum xY = A\sum x + b\sum x^2$ (2)

x	y	$Y = \ln y$	xY	x^2
2	4.077	1.4053	2.8107	4
4	11.084	2.4055	9.622	16
6	30.128	3.4054	20.4327	36
8	81.892	4.4054	35.2432	64
10	222.62	5.4054	54.054	100
30		17.02724	122.16380	220

Now,

$\sum Y = nA + b\sum x$

or, $17.02724 = 5A + b \times 30$ (3)

$\sum xY = A\sum x + b\sum x^2$

or, $122.16380 = A \times 30 + b \times 220$ (4)

Solving the equation (3) and (4); we get,

$A = 0.40539$

or, $\ln a = 0.40539$

$\therefore$ $a = e^{0.40539} = 1.49988$

$\therefore$ $b = 0.5$

$\therefore$ $y = 1.49988 \cdot e^{0.5x}$

Chapter 4

NUMERICAL DIFFERENTIATION AND INTEGRATION

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4.1 DIFFERENTIATING CONTINUOUS FUNCTIONS

We have,

Taylor series is;

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots\ldots \quad (1)$$

Neglecting higher order of derivative,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

Again,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots\ldots\ldots$$

Neglecting higher order derivative,

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Adding equation (1) and (3) by neglecting higher derivative; we get,

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + 0$$

$$\therefore \quad f''(x) = \frac{f(x+h) - 2f(x) - f(x-h)}{h^2}$$

4.1.1 Newton Forward Difference Quotient

Newton's forward difference formula is,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

Differentiating with respect to p; we get,

$$\frac{dy}{dP} = y_0 + \frac{2P-1}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

$$\because \quad P = \left(\frac{x - x_0}{h}\right) \Rightarrow \frac{dP}{dx} = \frac{1}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dP} \times \frac{dP}{dx} = \frac{1}{h}\left[y_0 + \frac{2P-1}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \frac{4P^3 - 18P^2 - 22P - 6}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\right] \quad (1)$$

At $x = x_0$, $P = 0$. Hence, putting $P = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \ldots\right] \quad (2)$$

Differentiating equation (1) with respect to x; we get,

$$\frac{d^2y}{dx^2} = \frac{d}{dP}\left(\frac{dy}{dP}\right)\frac{dP}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6P-6}{3!}\Delta^3 y_0 + \frac{12P^2 - 36P - 22}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\right]\frac{1}{h}$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \ldots\ldots\ldots\right]$$

Similarly;

$$\left(\frac{d^3y}{dx^3}\right) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

4.1.2 Backward Difference Quotient

Similarly, using Newton's Backward difference;

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2P+1}{2!}\nabla^2 y_n + \frac{3P^3 + 6P + 2}{3!}\nabla^3 y_n + \ldots\ldots\ldots\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6P+6}{3!}\nabla^3 y_n + \frac{6P^2 + 18P + 11}{12!}\nabla^4 y_n + \ldots\ldots\ldots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\ldots\ldots\right]$$

4.1.3 Central Difference Quotient

$$y_p = y_0 + \frac{P}{1!}\left(\frac{\Delta y_0 - \Delta y_{-1}}{2}\right) + \frac{P^2}{2!}\Delta^2 y_{-1} + \frac{P(P^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\ldots \quad (1)$$

$$P = \frac{x - x_0}{h}$$

$$\frac{dP}{dx} = \frac{1}{h}$$

Differentiating equation (1) at $x = x_0$ *i.e.*, $P = 0$; we get,

$$\left(\frac{dP}{dx}\right)_{x_0} = \frac{1}{h}\left[-\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \ldots\right]$$

Similarly;

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\right]$$

Example 1

A slider in a machine moves along a fixed straight rod. Its displacement x cm, along the rod is given below at different instant of time t seconds. Find the velocity of slider and its acceleration at t = 0.2 sec.

t	0.0	0.1	0.2	0.3	0.4
x	30.13	31.62	32.87	33.64	33.95

Solution:

The difference table is;

t	x	Δ	Δ²	Δ³	Δ⁴
0	30.13				
		1.49			
0.1	31.62		− 0.24		
		1.25		− 0.24	

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0.2	32.87		− 0.48		0.26
		0.77		0.02	
0.3	33.64		− 0.46		
		0.31			
0.4	33.95				

As derivative is required near the middle of table we use Sterling's formula;

According to Sterling's formula; (For P = 0)

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left[\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots\ldots\right]$$

$h = 0.1$

$t_0 = 0.2$

$\Delta x_0 = 0.77$

$\Delta x_{-1} = 1.25$

$\Delta^2 x_{-1} = -0.48$

$\Delta^3 x_{-1} = 0.02$

$\Delta^3 x_{-2} = -0.24$

$$\left(\frac{dx}{dt}\right)_{0.2} = \frac{1}{0.1}\left[\left(\frac{0.77 + 1.25}{2}\right) - \frac{1}{6}\left(\frac{0.02 - 0.24}{2}\right)\right] = 10(1.01 + 0.01833)$$

$$= 10.2833 \text{ m/sec.}$$

and, $$\frac{d^2x}{dt^2} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} + \ldots\ldots\ldots\right]$$

or, $$\left(\frac{d^2x}{dt^2}\right)_{0.2} = \frac{1}{(0.1)^2}\left[-0.48 - \frac{1}{12} \times 0.26\right] = -50.167 \text{ m/s}^2$$

Example 2

The distance y covered in time t by an object moving in a straight line is given below; approximate velocity at t = 1 sec.

t (sec)	0	1	2	3	4	5
y (meter)	0	15	71	143	245	367

Solution:

The difference table is;

t	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	0					
		15				
1	15		41			
		56		− 25		
2	71		16		39	.
		72		14		− 63
3	143		30		− 24	
		102		− 10		
4	245		20			
		122				
5	367					

$$\left(\frac{dy}{dt}\right)_{y_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$h = 1$

$x_0 = 1$

$y_0 = 15$

$\Delta y_0 = 56$

$\Delta^2 y_0 = 16$

$\Delta^3 y_0 = 14$

$\Delta^4 y_0 = -24$

$$\therefore \quad \left(\frac{dy}{dt}\right) = \frac{1}{1}\left[56 - \left(\frac{1}{2} \times 16\right) + \frac{1}{3} \times (14) - \frac{1}{4} \times (-24)\right]$$

$$= 56 - 8 + \frac{14}{3} + 6$$

$$= 58.667 \text{ m/sec.}$$

Example 3

Find y′(0.2) and y″(0.2) from the following data;

x	0.1	0.2	0.3	0.4	0.5
y	2.6	8.2	15.4	25.6	37.8

Solution:

The difference table is;

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	2.6				
		5.6			
0.2	8.2		1.6		
		7.2		1.4	

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0.3	15.4		3		−2.4
		10.2		−1	
0.4	25.6		2		
		12.2			
0.5	37.8				

$h = 0.1$

$$y'(0.2) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right]$$

$$= \frac{1}{0.1}\left[7.2 - \frac{1}{2}\times 3 - \frac{1}{3}\times 1\right]$$

$$= 53.667$$

$$y''(0.2) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{(0.1)^2}[3+1]$$

$$= 400$$

$\therefore$ $y'(0.2) = 53.667$

$\therefore$ $y''(0.2) = 400$

Example 4

A rod is rotating in a plane. The following table gives the angle θ (radian) through which the rod is turned for various values of the time t seconds, calculate the angular velocity and angular acceleration of the rod when t = 0.2 seconds.

t	0.0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Solution:

The forward table is;

t	θ	Δθ	$\Delta^2\theta$	$\Delta^3\theta$	$\Delta^4\theta$	$\Delta^5\theta$	$\Delta^6\theta$
0	0						
		0.12					
0.2	0.12		0.25				
		0.37		0.01			
0.4	0.49		0.26		0		
		0.63		0.01		0	
0.6	1.12		0.27		0		0
		0.9		0.01		0	
0.8	2.02		0.28		0		

		1.18		0.01			
1.0	3.20		0.29				
		1.47					
1.2	4.67						

$$\text{Angular velocity (W)} = \left(\frac{d\theta}{dt}\right)_{0.2}$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{0.2}\left[0.37 - \frac{1}{2}\times 0.26 + \frac{1}{3}\times 0.01\right] = \frac{0.2433}{0.2}$$

$$= 1.2167 \text{ rad/sec.}$$

$$\text{Angular acceleration } (\alpha) = \frac{d^2\theta}{dt^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\ldots\ldots\right]$$

$$= \frac{1}{(0.2)^2}[0.26 - 0.01]$$

$$= \frac{0.25}{(0.2)^2} = 6.25 \text{ rad/sec}^2$$

$\therefore$ Angular velocity = 1.2167 rad/sec.

$\therefore$ Angular acceleration = 6.25 rad/sec²

4.2 NUMERICAL INTEGRATION

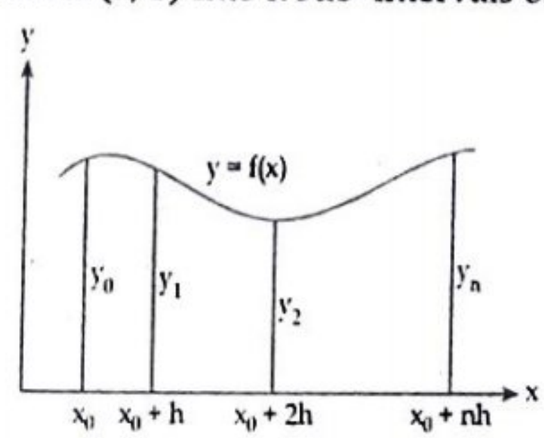
The process of evaluating a definite integral from a set of tabulated value of the integral f(x) is called **numerical integration.**

$$I = \int_a^b f(x)dx$$

Newton–Cotes Formula

Let $I = \int_a^b f(x)\,dx$, where f(x) takes values $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots\ldots, x_n$.

Let us divide the interval (a, b) into n sub-intervals of width h so that,



$x_0 = a, x_1 = x_0 + h_1, x_2 = x_0 + 2h_1, \ldots\ldots\ldots, x_0 + 2h = b.$

$$I = \int_{x_0}^{x_0+2h} f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr \quad [\text{Put } x = x_0 + rh,\ dx = h\,dr]$$

$$= h\int_0^n \left[y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] dr$$

[Using Newton's forward interpolation]

On integrating term by term we get,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2(r-2)}{24}\Delta^3 y_0 + \ldots\ldots\right] \quad \ldots\ldots(1)$$

This is called Newton's-Cotes formula.

This gives the general quadration formula for equidistant ordinate and is also known as Newton's cote formula.

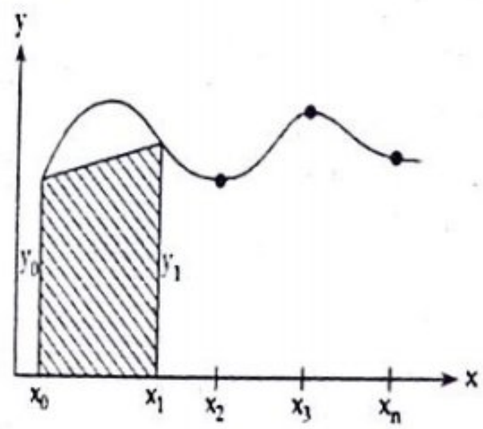
There are following to solve integration.

i) Trapezoidal rule

ii) Simpson's $\frac{1}{3}$ rule

iii) Simpson's $\frac{3}{8}$ rule

4.2.1 Trapezoidal Rule



Take n = 1 and neglecting the higher order differences more than one in Newton's cote quadrature formula; we get,

$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}[2y_0 + \Delta y_0] = \frac{h}{2}[y_0 + y_0 + \Delta y_0]$$

$$= \frac{h}{2}[y_0 + y_1] \qquad [\because y_0 + \Delta y_0 = y_1]$$

In the interval $(x_0 + h, x_0 + 2h)$; we get,

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right]$$

$$= \frac{h}{2}[y_1 + y_2]$$

Similarly;

$$\int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}[y_2 + y_3]$$

In general;

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}[y_{n-1} + y_n]$$

Adding all; we get,

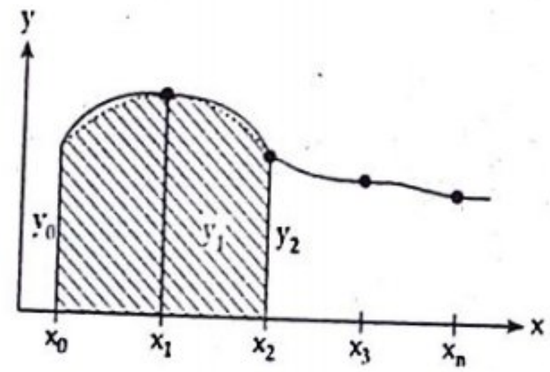
$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[y_0 + y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1} + y_n]$$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$$

This is called Trapezoidal formula.

4.2.2 Simpson's $\frac{1}{3}$ Rule

Putting $n = 2$ in Newton's-cotes formula and taking curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as parabola as in the figure, a polynomial of second order so that difference of order higher than second vanish; we get,



$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly;

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

.....

.....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n);$$

where, n being even.

Adding all these integrals,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots + y_{n-2})]$$

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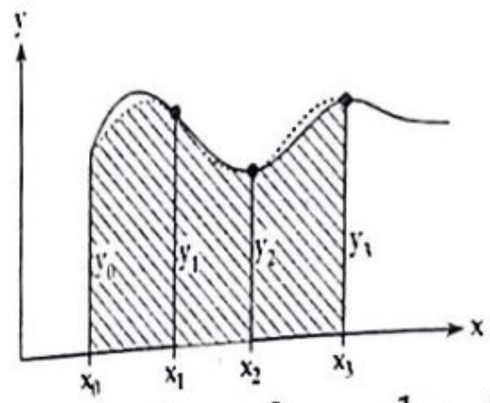
4.2.3 Simpson's $\frac{3}{8}$ Rule

Putting $n = 3$ in Newton's-cotes formula and taking the curve through (x_i, y_i): $i = 0, 1, 2, 3$ as polynomial of third order, the difference above third order vanishes.



$$\therefore \quad \int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly;

$$\int_{x_0+3h}^{x_0+5h} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all integrals;

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots\dots\dots + y_{n-1}) + 2(y_3 + y_6 + \dots\dots\dots + y_{n-3})\right]$$

4.2.4 Bode's Rule

Bode's rule is also known as Boole's rule. Consider $y = f(x)$ over $[x_0, x_4]$; where, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$ and $x_4 = x_0 + 4h$. Boole's rule is;

$$BR(f, h) = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)]$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_4]$ and we have the expression;

$$\int_{x_0}^{x_4} f(x)dx = BR(f, h)$$

The remainder term for Boole's rule is;

$$R_{BR}(f, h) = -\frac{8}{945} f^{(6)}(C)h^{(7)};$$

where, C lies somewhere between x_0 and x_4 and finally we have the equality;

$$\int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8}{945} f^{(6)}(C)h^{(7)}$$

Let's take an example;

$$I = \int_0^4 x\,dx \qquad \left[h = \frac{4-0}{4} = 1, x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4\right]$$

$$= \frac{2 \times 1}{45} \times [7 \times (0) + 32 \times 1 + 12 \times 2 + 32 \times 3 + 7 \times 4]$$

$$= \frac{2}{45} \times [32 + 24 + 96 + 28]$$

$$= \frac{2}{45} \times 180$$

$$= 8$$

Alternatively;

$$I = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \left[\frac{(4)^2}{2} - 0\right] = 8$$

Example 5

The velocity 'V' of particle at a distance 'S' from a point on its path is given in the table below.

S (f(t))	0	10	20	30	40	50	60
V (f(t)/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel a distance of 60 feet by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule.

Solution:

If 't' second be the time taken to travel a distance s(f(t)); then,

$$\frac{ds}{dt} = v$$

or, $$\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)}$$

S (ft)	0	10	20	30	40	50	60
$\frac{1}{v}$	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$

Then,

$$|t|_{s=0}^{s=60} = \int_0^{60} y\,ds$$

where, $h = 10$

By Simpson's $\frac{1}{3}$ rule; we know,

$$\int_0^{60} y\,ds = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$$

$$= \frac{10}{3}\left[\frac{1}{47} + 4\left(\frac{1}{58} + \frac{1}{65} + \frac{1}{52}\right) + 2\left(\frac{1}{64} + \frac{1}{61}\right)\right] + \frac{1}{38}$$

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$= 1.0635$ seconds

By Simpson's $\frac{3}{8}$ rule; we know,

$$\int_0^{60} y\,ds = \frac{3h}{8}\left[y_0 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 + y_6\right]$$

$$= \frac{30}{8}\left[\frac{1}{47} + 3\left(\frac{1}{58} + \frac{1}{64} + \frac{1}{61} + \frac{1}{52}\right) + 2 \times \frac{1}{65} + \frac{1}{38}\right]$$

$$= 1.0644 \text{ seconds}$$

Example 6

Evaluate the following using Simpson's $\frac{1}{3}$ rule (take h = 0.2)

$$\int_0^2 \frac{4e^x}{1+x^3}dx$$

Solution:

Given that;

$$\int_0^2 \frac{4e^x}{1+x^3}dx$$

$h = 0.2$

We have,

$$f(x) = y = \frac{4e^x}{1+x^3}$$

	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y = f(x)	4	4.84684	5.6084	5.9938	5.88767	5.43656	4.86820	4.3325	3.88778	3.54195	3.28402
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_n

By Simpson's $\frac{1}{3}$ rule; we have,

$$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \ldots) + 4(y_1 + y_3 + y_5 + y_7 + \ldots)\right]$$

$$\int_0^2 \frac{4e^x}{1+x^3}dx = \frac{0.2}{3}\left[(4 + 3.28402) + 2(5.6084 + 5.88767 + 4.86820 + 3.88778) + 4(4.84684 + 5.9938 + 5.43656 + 4.3325 + 3.54195)\right]$$

$$\int_0^2 \frac{4e^x}{1+x^3}dx = 9.62630$$

4.3 GAUSS INTEGRATION/GAUSS LEGENDRE INTEGRATION

Gauss Integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point rather than on the basis of equal spacing.

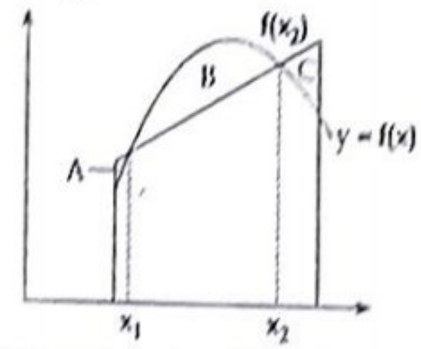

Figure: Illustration of Gauss integration

Let us consider a curve, a straight line has been moved up such that the area B = A + C, the sampling points are moved away from the end point. The problem is to compute the value of x_1 and x_2 and to choose appropriate weights w_1 and w_2.

The method of implementing the strategy of obtaining the value of x_i and w_i and that of the integral of f(x) is called Gauss Legendre integration.

Gauss integration assumes an approximation of the form.

$$I_g = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

Formula

i) **For two point**

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

where, $w_1 = w_2 = 1$

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

ii) **For three point**

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

Here,

$$w_1 = \frac{5}{9} = 0.555$$

$$w_2 = \frac{8}{9} = 0.8889$$

$$w_3 = \frac{5}{9} = 0.555$$

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$$x_1 = -\sqrt{\frac{3}{5}} = -0.77459$$

$$x_2 = 0$$

$$x_1 = \sqrt{\frac{3}{5}} = 0.77459$$

$$\int_{-1}^{1} f(x)dx = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

Comparison between Simpson's Method and Gaussian Integration

Simpson 's method for calculation of integration is one of Newton's cotes formula. The Newton-cotes formula was derived by integrating the Newton-Gregory forward difference interpolating polynomial. Consequently, all the rules were based on evenly faced sampling points within the range of integral.

Gauss integration is based on the concept that the accuracy of numerical integration can be improved by choosing the sampling point wisely rather than on the basis of equal sampling.

Example 7

Evaluate the integral $I = \int_0^1 e^{-x^2}dx$ and compare the result in both the condition of Simpson's $\frac{1}{3}$ rule and 3 point gauss Legendre method.

Solution:

For Simpson's method find, the integral for n = 3 as,

Given that;

$$I = \int_0^1 e^{-x^2}dx$$

We have,

$$nh = b - a$$

or, $$h = \frac{b-a}{n} = \frac{1-0}{3} = \frac{1}{3} = 0.33$$

Here,

$$f(x) = e^{-x^2}$$

X	0	$\frac{1}{3}$	$\frac{2}{3}$	1
Y	1	0.89682	0.646876	0.36788
	y_0	y_1	y_2	y_3

By Simpson's $\frac{1}{3}$ rule; we have,

$$I = \frac{0.33}{3}[1 + 0.36788 + 2(0.646876) + 4(0.89682)] = 0.694323$$

According to Gauss Legendre; we have,

We need to change the limit of integral;

$$I = \int_0^1 e^{-x^2}dx; \text{ from } (0, 1) \text{ to } (-1, 1) \text{ by transformation}$$

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0)$$

$$x = \frac{u+1}{2}$$

$$dx = \frac{1}{2}du$$

Now,

$$I = \int_0^1 e^{-x^2}dx = \int_{-1}^{1} e^{-\left(\frac{u+1}{2}\right)^2}\frac{1}{2}du$$

$$\therefore \quad f(u) = \frac{1}{2}e^{-\left(\frac{u+1}{2}\right)^2}$$

$$f(0) = \frac{1}{2}e^{-\left(\frac{u+1}{2}\right)^2} = \frac{1}{2} \times e^{-\frac{1}{4}} = 0.3894$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.49368$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.2275$$

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] = 0.7467$$

Example 8

Evaluate the integral $I = \int_0^{1.5} \sin x\, dx$, compare the absolute error in both condition for Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule.

Solution:

Given that;

$$I = \int_0^{1.5} \sin x\, dx$$

$$nh = b - a$$

or, $$h = \frac{b-a}{n}$$

For n = 6,

$$h = \frac{1.5-0}{6} = 0.25$$

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Here,

$f(x) = \sin x$

X	0	0.25	0.5	0.75	1	1.25	1.5
f(x)	0	0.24740	0.47943	0.68164	0.84147	0.94898	0.997495
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Note: Make sure calculator is in radian mode.

By Simpson's $\frac{1}{3}$ rule; we have,

$$I = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.25}{3}[(0 + 0.997495) + 2(0.47943 + 0.84147) + 4(0.24740 + 0.681647) + 0.94898)]$$

$$I = 0.929763$$

By Simpson's $\frac{3}{8}$ rule; we have,

$$I = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3 \times 0.25}{8}[(0 + 0.997495) + (2 \times 0.68164) + 3(0.24740 + 0.47943 + 0.84147 + 0.94898)]$$

$$I = 0.9293077$$

Using calculator;

$$\int_0^{1.5} \sin x \, dx = 0.929263$$

so, absolute errors are calculated as;

$$E_1 = |0.929263 - 0.929763| = 0.0005$$

$$E_2 = |0.929263 - 0.9293077| = 0.00045$$

4.4 ROMBERG INTEGRATION

The accuracy of numerical integration can be improved in two ways;

i) By changing the number of subinterval (*i.e.*, by decreasing 'h') this decreases the magnitude of error terms.

ii) By using higher order methods, this eliminates the lower order errors term, here the order of the method is varied and therefore, this method is known as variable order approach.

The Romberg integration method uses the trapezoidal rule. It use trapezoidal rule in iterative way.

Trapezoidal Rule

$$I = \int_a^b f(x)dx = \frac{h}{2}[f(a) + f(b)]$$

Now,

Find 'I' with 'h', which will give I_1 and E_1 similarly, 'I' is found with h_2 which will give I_2.

$$E_1 = -\frac{(b-a)h_1^2}{12}y''(x)$$

$$E_2 = -\frac{(b-a)h_2^2}{12}y''(x)$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2}$$

$$\text{or,} \quad \frac{E_1}{E_2 - E_1} = \frac{h_2^2}{h_1^2 - h_2^2}$$

$$I = I_1 + E_1 = I_2 + E_2$$

$$E_2 - E_1 = I_1 - I_2$$

Now,

$$\frac{E_1}{E_2 - E_1} = \frac{h_2^2}{h_1^2 - h_2^2}$$

$$E_1 = \frac{h_2^2}{h_1^2 - h_2^2}(I_1 - I_2)$$

$$I = I_1 + \frac{h_2^2}{h_1^2 - h_2^2}(I_1 - I_2)$$

$$= \frac{I_1 h_2^2 - I_2 h_1^2}{h_1^2 - h_2^2}$$

If $h_1 = h$

$$h_2 = \frac{h}{2}$$

$$\therefore \quad I = \frac{I_1\left(\frac{h}{2}\right)^2 - I_2(h)^2}{(h)^2 - \left(\frac{h}{2}\right)^2} = \frac{4I_2 - I_1}{3}$$

$$\therefore \quad I\left(h, \frac{h}{2}\right) = \frac{4I\left(\frac{h}{2}\right) - I(h)}{3} \qquad (1)$$

We use trapezoidal rule several times successively having 'h' and apply in equation (1) to each pair of values as per the following scheme.

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$I(h)$			
	$I\left(h,\frac{h}{2}\right)$		
$I\left(\frac{h}{2}\right)$		$I\left(h,\frac{h}{2},\frac{h}{4}\right)$	
	$I\left(\frac{h}{2},\frac{h}{4}\right)$		$I\left(h,\frac{h}{2},\frac{h}{4},\frac{h}{8}\right)$
$I\left(\frac{h}{4}\right)$		$I\left(\frac{h}{2},\frac{h}{4},\frac{h}{8}\right)$	
	$I\left(\frac{h}{4},\frac{h}{8}\right)$		
$I\left(\frac{h}{8}\right)$			

Procedure to Find Romberg Integration

Use trapezoidal value to find

I_1: Divide the interval into two parts

I_2: Divide interval into four parts

I_3: Divide interval into eight parts

Then, calculate

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$

and, $I_5 = I_3 + \frac{I_3 - I_2}{3}$

Now, calculate

$$I = I_5 + \frac{1}{3}(I_5 - I_4)$$

Example 9

Use Romberg integration method to evaluate the integral $\int_1^2 \frac{dx}{x}$ correct up to 3 place taking size as $h = \frac{b-a}{2}$

Solution:

We have,

$$h = \frac{b-a}{2} = \frac{2-1}{2} = 0.5$$

Therefore, taking $h = 0.5$, $\frac{0.5}{2} = 0.25$ and $\frac{0.25}{2} = 0.125$

Let us calculate the given integral using trapezoidal rule

$$f(x) = y_i = \frac{1}{x}$$

i) When $h = 0.5$

X_i	1	1.5	2
Y_i	1	0.6667	0.5

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.5}{2}[1 + 0.5 + 2 \times 0.6667]$$
$$= 0.70835$$

ii) When $h = 0.25$

X_i	1	1.25	1.5	1.75	2
Y_i	1	0.8	0.6667	0.57143	0.5

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{0.25}{2}[1 + 0.5 + 2(0.8 + 0.6667 + 0.57143)]$$
$$I_2 = 0.697033$$

iii) When $h = 0.125$

X_i	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
Y_i	1	0.8889	0.8	0.72727	0.6667	0.6154	0.57143	0.533	0.5

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.125}{2}[1 + 0.5 + 2(0.8889 + 0.8 + 0.72727 + 0.6667]$$
$$+ 0.6154 + 0.57143 + 0.533)]$$

$\therefore$ $I_3 = 0.69408$

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$
$$= 0.697033 + \frac{0.697033 - 0.70835}{3}$$

$\therefore$ $I_4 = 0.69326$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$
$$= 0.69408 + \frac{0.69408 - 0.697033}{3}$$

$\therefore$ $I_5 = 0.693095$

$$I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.693095 + \frac{0.693095 - 0.69326}{3}$$

$\therefore$ $I = 0.69304$

$\therefore$ $\int_1^2 \frac{dx}{x} = 0.69304$

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Example 10

Evaluate $\int_{0.2}^{1.5} e^{-x^2} dx$ using Romberg method correct up to 3 decimal place.

Solution:

We have,

$$h = \frac{b-a}{n}$$

Taking n = 2

$$h = \frac{1.5-0.2}{2} = 0.65$$

Taking h = 0.65, 0.325, 0.162 respectively

$$f(x) = y_i = e^{-x^2}$$

i) When h = 0.65

X	0.2	0.85	1.5
Y	0.9607	0.4855	0.1053

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{0.65}{2}[0.960 + 0.1053 + 2 \times 0.4855]$$

$\therefore \quad I_1 = 0.6617$

ii) When h = 0.325

X	0.2	0.525	0.850	1.175	1.5
Y	0.9607	0.759	0.4855	0.251	0.1053

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.325}{2}[0.9607 + 0.1053 + 2(0.759 + 0.4855 + 0.251)]$$

$\therefore \quad I_2 = 0.6589$

iii) When h = 0.1625

X	0.2	0.363	0.525	0.688	0.850	1.013	1.175	1.338	1.5
Y	0.9607	0.876	0.759	0.6233	0.4855	0.35691	0.251	0.167	1.1053

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.1625}{2}[0.9607 + 1.1053 + 2(0.876 + 0.759 + 0.6233 + 0.4855 + 0.35691 + 0.251 + 0.167)]$$

$\therefore \quad I_3 = 0.6565$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.6589 + \frac{1}{3}(0.6589 - 0.6617) = 0.657$$

Similarly,

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.6565 + \frac{0.6565 - 0.6589}{3} = 0.6557$$

$$\therefore \quad I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.6557 + \frac{0.6557 - 0.657}{3}$$

$$\therefore \quad \int_{0.2}^{1.5} e^{-x^2} dx = 0.6553$$

Example 11

Evaluate the following integral using Romberg method.

$$\int_0^2 \frac{e^x + \sin x}{1 + x^2} dx$$

Solution:

Here,

$$h = \frac{b-a}{n}$$

Taking n = 2

$$\therefore \quad h = \frac{b-a}{n} = \frac{2-0}{2} = 1$$

Taking h = 1, 0.5, 0.25

Here,

$$f(x) = y = \frac{e^x + \sin x}{1 + x^2}$$

i) When h = 1

X	0	1	2
Y	1	1.77988	1.659671

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[1 + 1.659671 + 2 \times 1.77988]$$

$$= 3.10972$$

ii) When h = 0.5

X_i	0	0.5	1	1.5	2
Y_i	1	1.70252	1.77988	1.68590	1.659671

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 1.659671 + 2(1.70252 + 1.77988 + 1.68590)]$$

$\therefore \quad I_2 = 3.24906$

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iii) When h = 0.25

X_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Y_i	1	1.44134	1.70252	1.79112	1.77988	1.73242	1.68590	1.6587	1.659671

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 1.659671 + 2(1.44134 + 1.70252 + 1.79112 + 1.77988 + 1.73242 + 1.68590 + 1.6587)]$$

$\therefore \quad I_3 = 3.28043$

Now,

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 3.24906 + \frac{3.24906 - 3.10792}{3} = 3.2955$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 3.28043 + \frac{3.28043 - 3.24906}{3} = 3.29088$$

$$I = I_5 + \frac{1}{3}(I_5 - I_4) = 3.29088 + \frac{3.29088 - 3.2955}{3} = 3.28934$$

$$\therefore \quad \int_0^2 \frac{e^x + \sin x}{1 + x^2} dx = 3.28934$$

Example 12

Use Romberg integration method $I = \int_0^2 \frac{e^x + e^{-x}}{2} dx$ correct up to three decimal place.

Solution:

Here,

$$h = \frac{b - a}{n}$$

Taking n = 2

$$h = \frac{2 - 0}{2}$$

Taking h = 1, 0.5, 0.25

$$y(i) = f(x_i) = \frac{e^x + e^{-x}}{2}$$

i) When h = 1

X_i	0	1	2
Y_i	1	1.54308	3.76219

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{1}{2}[1 + 3.76219 + 2 \times 1.54308] = 3.9242$$

ii) When h = 0.5

X_i	0	0.5	1	1.5	2
Y_i	1	1.12763	1.54308	2.35241	3.76219

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}[1 + 3.76219 + 2(1.12763 + 1.54308 + 2.35241)]$$

$\therefore \quad I_2 = 3.70211$

iii) When h = 0.25

X_i	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
Y_i	1	1.03141	1.12763	1.29468	1.54308	1.88842	2.35241	2.964188	3.76219

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.25}{2}[1 + 3.76219 + 2(1.03141 + 1.12763 + 1.59468 + 1.56308 + 1.878842 + 2.35241 + 2.964188)]$$

$\therefore \quad I_3 = 3.64573$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 3.70211 + \frac{3.70211 - 3.9242}{3} = 3.62808$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 3.64573 + \frac{3.64573 - 3.70211}{3} = 3.62694$$

so, $I = I_5 + \frac{1}{3}(I_5 - I_4)$

$$= 3.62694 + \frac{3.62694 - 3.62808}{3} = 3.62656$$

Example 13

Use Romberg Integration find the integral of $e^x \sin x$ between the limit −1 and 1.

Solution:

Here,

$$\int_{-1}^{1} e^x \sin x \, dx$$

$$h = \frac{b - a}{n}$$

Taking n = 2

$$h = \frac{1 + 1}{2} = 1$$

Taking h = 1, 0.5, 0.25

$$f(x) = e^x \sin x$$

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i) When h = 1

X_i	−1	0	1
Y_i	−0.30959	0	2.2874

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[-0.30959 + 2.2874 + 2 \times 0]$$
$$= 0.98892$$

ii) When h = 0.5

X_i	−1	−0.5	0	0.5	1
Y_i	−0.30956	−0.29079	0	0.79044	2.2874

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$
$$= \frac{0.5}{2}[-0.30956 + 2.2874 + 2(-0.29079 + 0 + 0.79044)]$$
$$\therefore \quad I_2 = 0.744285$$

iii) When h = 0.25

X_i	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1
Y_i	−0.30956	−0.32198	−0.29079	−0.192978	0	0.317673	0.79044	1.44303	2.2874

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$
$$= \frac{0.25}{2}[-0.30956 + 2.2874 + 2(-0.32198 - 0.2979 - 0.192978 + 0$$
$$+0.317673 + 0.79044 + 1.44303)]$$
$$\therefore \quad I_3 = 0.683654$$
$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.744285 + \frac{0.744285 - 0.98892}{3} = 0.66274$$
$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.683654 + \frac{0.683654 - 0.744285}{3} = 0.66344$$
$$\therefore \quad I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.66344 + \frac{0.66344 - 0.66274}{3} = 0.66367$$

SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. Evaluate $\int_3^5 \frac{1}{x^6+5}dx$ by using Gauss Legendre two and three point formula. **[2006 Fall]**

Solution:

By changing the limit (3, 5) to (−1, 1) by transformation,

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(5-3)u + \frac{1}{2}(5+3) = u + 4$$
$$dx = du$$
$$I = \int_3^5 \frac{1}{x^6+5}dx = \int_{-1}^1 \frac{1}{(u+4)^6+5}du$$
$$f(u) = \frac{1}{(u+4)^6+5}$$
$$f(0) = \frac{1}{(4)^6+5} = \frac{1}{4101}$$
$$f\left(-\frac{1}{\sqrt{3}}\right) = 6.2012 \times 10^{-4}$$
$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{1}{\left(-\sqrt{\frac{3}{5}}+4\right)^6+5} = 8.8424 \times 10^{-4}$$
$$f\left(\frac{1}{\sqrt{3}}\right) = 1.0866 \times 10^{-4}$$
$$f\left(\sqrt{\frac{3}{5}}\right) = 8.4371 \times 10^{-5}$$

By Gauss three points formula;

$$\int_{-1}^1 f(x)\,du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$
$$= \left(\frac{8}{9} \times \frac{1}{4101}\right) + \frac{5}{9}(8.8424 \times 10^{-4} + 8.4371 \times 10^{-5})$$
$$= 7.5486 \times 10^{-4}$$

By Gauss-2 point formula is;

$$\int_{-1}^1 f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$
$$= 6.2012 \times 10^{-4} + 1.0866 \times 10^{-4}$$
$$= 7.2878 \times 10^{-4}$$

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2. Apply Romberg method to evaluate $\int_4^{5.2} \log x\, dx$. **[2006 Fall]**

Solution:

Here,

$$h = \frac{b-a}{n}$$

For n = 2

$$h = \frac{5.2-4}{2} = 0.6$$

Taking h = 0.6, 0.3, 0.15 respectively

Here,

$f(x) = \log x_i$

i) When h = 0.6

X_i	4	4.6	5.2
Y_i	0.60	0.66	0.71

$$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1]$$

$$= \frac{0.6}{2}[0.60206 + 0.716003 + 2 \times 0.662758]$$

$$= 0.793074$$

ii) When h = 0.3

X_i	4	4.3	4.6	4.9	5.2
Y_i	0.60	0.63	0.66	0.69	0.71

$$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.3}{2}[0.60 + 0.71 + 2(0.63 + 0.66 + 0.69)]$$

$\therefore$ $I_2 = 0.7905$

iii) When h = 0.15

X_i	4	4.15	4.3	4.45	4.6	4.75	4.9	5.05	5.2
Y_i	0.60	0.61	0.63	0.64	0.66	0.67	0.69	0.70	0.71

$$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.15}{2}[0.60 + 0.71 + 2(0.61 + 0.63 + 0.64 + 0.66 + 0.67 + 0.69 + 0.7)]$$

$\therefore$ $I_3 = 0.78825$

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.7905 + \frac{0.7905 - 0.793}{3} = 0.789642$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.78825 + \frac{0.78825 - 0.7905}{3} = 0.7875$$

$$\therefore \quad I = I_5 + \frac{1}{3}(I_5 - I_4) = 0.7875 + \frac{0.7875 - 0.789642}{3} = 0.786786$$

3. Using Gauss-quadrature formula for n = 3, evaluate

$$\int_0^1 \frac{dx}{1+x^2}$$ **[2007 Spring]**

Solution:

By changing limit (0, 1) to (−1, 1) by transformation

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{u}{2} + \frac{1}{2} = \frac{u+1}{2}$$

$$dx = \frac{1}{2}du$$

Now,

$$I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^1 \frac{1}{1+\left(\frac{u+1}{2}\right)^2}\frac{1}{2}du$$

$$\therefore \quad f(u) = \frac{2}{u^2 + 2u + 5}$$

From Gauss Legendre three point formula; we have,

$$f(0) = 0.4$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.49373$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.27975$$

$$\int_{-1}^1 f(x)\, du = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9} \times 0.4\right) + \frac{5}{9} \times (0.49373 + 0.27975) = 0.785267$$

4. Using two and three point Gauss Legendre formula. Evaluate

$$\int_{0.2}^{1.5} e^{-x^2} dx$$ **[2009 Fall]**

Solution:

Changing limit from (0.2, 1.5) to (−1, 1)

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{1}{2}(1.5-0.2)u + \frac{1}{2}(1.5+0.2)$$

$$= (0.65u + 0.85)$$

Now,

$$dx = 0.65\, du$$

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$$I = \int_{0.2}^{1.5} e^{-x^2} dx = \int_{-1}^{1} e^{-(0.65u+0.85)^2} \times 0.65\, du$$

$$\therefore \quad f(u) = e^{-(0.65u+0.85)^2} \times 0.65$$

$$f(0) = 0.3156$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = 0.57646$$

$$f\left(\sqrt{\frac{3}{5}}\right) = 0.10407$$

$$f\left(-\frac{1}{\sqrt{3}}\right) = 0.518848$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 0.144845$$

Using Gauss 2 point formula; we have,

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.518848 + 0.144845 = 0.663693$$

Using Gauss 3 point formula; we have,

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \left(\frac{8}{9} \times 0.3156\right) + \frac{5}{9}(0.57646 + 0.10407) = 0.6586055$$

5. **A slider in a machine moves along a fixed straight rod. Its distance X (cm) along the rod is given below for various values of time t seconds. Find the velocity and the acceleration of the slider when t = 0.2.**

t	0	0.1	0.2	0.3
x	30.13	31.62	32.87	33.95

[2017 Spring]

Solution:

Here, difference table:

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$
0	30.13			
		1.49		
0.1	31.62		-0.24	
		1.25		0.07
0.2	32.87		-0.17	
		1.08		
0.3	33.95			

Now, using backward difference formula.

$$\frac{dx}{dt} = \frac{1}{h}\left[\nabla y_n + \frac{2P+1}{2!}\nabla^2 y_n + \frac{3P^2+6P+2}{3!}\nabla^3 y_n + \dots\dots\right]$$

and, $$\frac{d^2x}{dt^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6P+6}{3!}\nabla^3 y_n + \dots\dots\right]$$

Here, $h = 0.3 - 0.2 = 0.1$

$\nabla y_n = 1.08$

$\nabla^2 y_n = -0.17$

$\nabla^3 y_n = 0.07$

$$P = \frac{x - x_n}{h} = \frac{0.2 - 0.3}{0.1} = -1$$

so, velocity V at 0.2

$$\left(\frac{dx}{dt}\right)_{0.2} = \frac{1}{0.1}\left[1.08 + \frac{2 \times (-1) + 1}{2!} \times (-0.17) + \frac{3(-1)^2 + 6(-1) + 2}{3!} \times 0.07\right]$$

$$= 11.533 \text{ cm/sec}$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.2} = \frac{1}{0.1^2}\left[-0.17 + \frac{6 \times (-1) + 6}{3!} \times 0.07\right]$$

$$= 17 \text{ cm/s}^2$$

So, at time 0.2 sec, velocity V = 11.533 cm/sec. acceleration a = 17 cm/s².

6. **The velocity 'v' of a particle at a distance 'S' from a point on its path is given by the following table:**

S (meter)	0	10	20	30	40	50
V(meter/sec)	47	58	64	65	61	52

Estimate time taken to travel 60 meters by using Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule. **[2017 Spring]**

Solution:

Here, we know that;

$$V = \frac{ds}{dt}$$

or, $$dt = \frac{ds}{v}$$

or, $$T = \int_a^b dt = \int_a^b \frac{1}{V} ds = \int_a^b f(V)\, ds$$

so,

n	S	V	$f(V) = \frac{1}{V}$
0	0	47	0.02128
1	10	58	0.01724

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2	20	64	0.01563
3	30	65	0.01539
4	40	61	0.01639
5	50	52	0.01923
6	60	38	0.02632

Now, using Simpsons $\frac{1}{3}$ rule

$$T = \int_0^{60} f(V)\, ds = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

Here, h = 10 sec.

$$T = \frac{10}{3}[(0.02128 + 0.02632) + 4(0.01724 + 0.01539 + 0.01923) + 2(0.01563 + 0.01639)]$$

$$= 1.0636 \text{ sec.}$$

Then, using Simpson $\frac{3}{8}$ rule,

$$T = \int_0^{60} f(V)\, ds = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times 10}{8}[(0.02128 + 0.02632) + 3(0.01724 + 0.01563 + 0.01639 + 0.01923]$$

$$= 1.0644 \text{ sec.}$$

7. From the following table x and y obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 1.2.

x	1.0	1.2	1.4	1.6	1.8
y	2.7183	3.3201	4.0552	4.9530	6.0496

[2018 Spring]

Solution:

Using forward difference formula,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2P-1}{2!}\Delta^2 y_0 + \frac{3P^2 - 6P + 2}{3!}\Delta^3 y_0 + \frac{4P^3 - 18P^2 + 22P - 6}{4!}\Delta^4 y_0 + \dots\dots\right] \quad (1)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6P-6}{3!}\Delta^3 y_0 + \frac{12P^2 - 36P + 22}{4!}\Delta^4 y_0 + \dots\dots\right] \quad (2)$$

where, $P = \left(\frac{x - x_0}{h}\right)$

and, the forward difference table,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	2.7183				
		0.6018			
1.2	3.3201		0.1333		
		0.7351			
1.4	4.0552		0.1627	0.0294	
		0.8978			0.0067
1.6	4.9530		1.1988	0.0361	
		1.0966			
1.8	6.0496				

so, $\Delta y_0 = 0.6018$, $x_0 = 1.0$

$\Delta^2 y_0 = 0.1333$, $h = 1.2 - 1.0 = 0.2$

$\Delta^3 y_0 = 0.0294$, $\Delta^4 y_0 = 0.0067$

at x = 1.2,

$$P = \frac{1.2 - 1.0}{0.2} = 1$$

so, by putting values in equation (1) and equation (2);

$$\left(\frac{dy}{dx}\right)_{1.2} = \frac{1}{0.2}\left[0.6018 + \frac{2 \times 1 - 1}{2!} \times 0.1333 + \frac{3 \times 1^2 - 6 \times 1 + 2}{3!} \times 0.0294 + \frac{4 \times 1^3 - 18 \times 1^2 + 22 \times 1 - 6}{4!} \times 0.0067 + \dots\dots\right]$$

$$\left(\frac{dy}{dx}\right)_{1.2} = 3.321$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.2} = \frac{1}{0.2^2}\left[0.1333 + \frac{6-6}{3!} \times 0.0294 + \frac{12 - 36 + 22}{4!} \times 0.0067\right]$$

or, $$\left(\frac{d^2y}{dx^2}\right)_{1.2} = 3.319$$

8. Find the integral value $I = \int_0^1 \frac{dx}{1 + x^2}$ correct to three decimal place using Romberg integration.

[2018 Spring]

Solution:

Here, taking step size h = 0.5, 0.25 and 0.125 successively.

i) When h = 0.5, the value of $y = (1 + x^2)^{-1}$ are;

x	0	0.5	1
y	1	0.8000	0.5

$$I(h) = \frac{0.5}{2}(1 + 0.5 + 2 \times 0.8) = 0.775$$

ii) When h = 0.25, the value $y = (1 + x^2)^{-1}$ are:

x	0	0.25	0.5	0.75	1
y	1	0.9411	0.8000	0.6400	0.500

$$I\left(\frac{h}{2}\right) = \frac{0.25}{2}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)] = 0.7828$$

iii) When h = 0.125, the value of $(1 + x^2)^{-1}$ are:

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9411	0.8767	0.8000	0.7191	0.6400	0.5663	0.500

$$I\left(\frac{h}{4}\right) = \frac{0.125}{2}[1 + 0.5 + 2(0.9846 + 0.9411 + 0.8767 + 0.8 + 0.7191 + 0.64 + 0.3663)]$$

$$= 0.7847$$

So, using Romberg's formula

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right] = \frac{1}{3}[4 \times 0.775 - 0.7828] = 0.7854$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = 0.7853$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right] = \frac{1}{3}(4 \times 0.7853 - 0.7854)$$

$$= 0.7853$$

so, $\int_0^1 \frac{1}{1 + x^2}\,dx = 0.785$

9. Evaluate the integral $I = \int_0^{\pi/2} (1 + 3\cos 2x)\,dx$, compare the result in both condition for Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule. **[2018 Fall]**

Solution:

Here, take step-size $h = \frac{\pi}{12}$

Then, if $y_n = (1 + 3\cos 2x_n)$

n	0	1	2	3	4	5	6
x_n	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y_n	4	3.598	2.5	1	−0.5	−1.598	−2

i) Using Simpson's $\frac{1}{3}$ rule

$$I = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$= \frac{\pi}{36}[4 + (-2) + 4(3.598 + 1 - 1.598) + 2(2.5 - 0.5)] = \frac{\pi}{2}$$

$$= 1.571$$

ii) Using Simpson's $\frac{3}{8}$ rule,

$$I = \frac{3h}{8}[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3 \times \pi}{8 \times 12}[4 + (-2) + 3(3.598 + 2.5 - 0.5 - 1.598) + 2(1)]$$

$$= \frac{\pi}{2} = 1.571$$

In this problem, in both ways we get exactly same value which is $\frac{\pi}{2} = 1.571$.

$$I = \int_0^{\pi/2} (1 + 3\cos 2x)\,dx = \frac{\pi}{2} = 1.571$$

10. The following data gives corresponding value of pressure 'P' and specific volume 'V' of stem. **[2018 Fall]**

P	105	42.7	25.3	16.7	13
V	2	4	6	8	10

Find the rate of change of volume when pressure is 105 and 13.

Solution:

Here, we know that rate of change of volume at a certain pressure is given by;

$$\left(\frac{dV}{dP}\right) \text{ at } P = 105 \text{ and } P = 13$$

But, we see that interval in 'P' is not uniform, but it is uniform in 'V'. So, we find $\frac{dP}{dV}$ and reciprocal it finally.

V	P	ΔP	Δ^2P	Δ^3P	Δ^4P
2	105				
		−62.3			
4	42.7		44.9		
		−17.4		−36.1	
6	25.3		8.8		32.2
		−8.6		−3.9	
8	16.7		4.9		
		−3.7			
10	13				

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i) $\left(\frac{dV}{dP}\right)_{at\ P=105}$ [Using forward interpolation]

$$\frac{dP}{dV} = \frac{1}{h}\left[\Delta P_0 + \frac{2k-1}{2!}\Delta^2 P_0 + \frac{3k^2-6k+2}{3!}\Delta^3 P_0 + \frac{4k^3-18k^2+22k-6}{4!}\Delta^4 P_0 + \dots\dots\right]$$

Here,

$$h = V_n - V_{n-1} = 2$$

$$k = \frac{V - V_0}{h} = 0 \qquad [\because V = V_0 = 2]$$

so, solve equation becomes;

$$\frac{dP}{dV} = \frac{1}{h}\left[\Delta P_0 + \frac{1}{2}\Delta^2 P_0 + \frac{1}{3}\Delta^3 P_0 - \frac{1}{4}\Delta^4 P_0 + \dots\dots\right]$$

$$= \frac{1}{2}\left[-62.3 - \frac{1}{2}\times(44.9) + \frac{1}{3}\times(-36.1) - \frac{1}{4}\times(+32.2)\right]$$

$$= -52.416$$

$$\therefore \left(\frac{dV}{dP}\right)_{at\ P=105} = -\frac{1}{52.416} = -0.019$$

ii) $\left(\frac{dV}{dP}\right)_{at\ P=13}$ [Using backward interpolation]

$$\frac{dP}{dV} = \frac{1}{h}\left[\nabla P_n + \frac{2k+1}{2!}\nabla^2 P_n + \frac{3k^2+6k+2}{3!}\nabla^3 P_n + \dots\dots\right]$$

Here,

$$h = 2$$

$$k = \frac{P - P_n}{h} = 0 \qquad [\because P = P_n]$$

Hence, above equation becomes;

$$\frac{dP}{dV} = \frac{1}{h}\left[\nabla P_n + \frac{1}{2}\nabla^2 P_n + \frac{1}{3}\Delta^3 P_n + \frac{1}{4}\nabla^4 P_n + \dots\dots\right]$$

$$= \frac{1}{2}\left[-3.7 + \frac{1}{2}\times 4.9 + \frac{1}{3}\times(-3.9) + \frac{1}{4}\times(32.2)\right]$$

$$= 2.75$$

$$\therefore \left(\frac{dV}{dP}\right)_{at\ P=105} = \frac{1}{2.75} = 0.3636$$

11. Evaluate $\int_{-2}^{2}\frac{x}{x+2e^x}dx$ by using trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019 Fall]**

Solution:

We have,

$$nh = b - a$$

or, $6h = 2 - (-2)$

or, $h = \frac{4}{6} = \frac{2}{3}$

$$f(x) = \frac{x}{x+2e^x}dx$$

Now,

x_i	-2	$-\frac{4}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{4}{3}$	2
y_i	1.16	1.65	-1.85	0	0.146	0.213	0.12
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By trapezoidal rule; we have,

$$L = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{2}{2\times 3}\times[(1.16 + 0.12) + 2(1.65 - 1.85 + 0 + 0.146 + 0.213)]$$

$$= 0.5327$$

By Simpson's $\frac{1}{3}$ rule; we have,

$$i = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{2}{3\times 3}\times[(1.16 + 0.12) + 2(-1.85 + 0.146) + 4(1.65 + 0 + 0.213)]$$

$$= \frac{2}{9}\times[1.28 + (-3.4) + 7.452]$$

$$= 1.1849$$

By Simpson's $\frac{3}{8}$ rule; we have,

$$i = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3\times 2}{8\times 3}\times[(1.16 + 0.12) + 2\times 0 + 3(1.65 - 1.85 + 0.146 + 0.213)]$$

$$= 0.43925$$

12. Using three-point Gaussian quadrature formula, evaluate

$$\int_0^1 \frac{dx}{1+x}$$ **[2019 Fall]**

Solution:

We first change the limits (0, 1) to – 1 to so that,

$$x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{1}{2}(u+1)$$

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$$I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}du}{1+\frac{1}{2}(u+1)} = \int_{-1}^{1} \frac{du}{u+3}$$

Gauss-formula for n = 3 is;

$$I = \frac{8}{9}f(0) + \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)$$

where, $f(u) = \frac{1}{u+3}$

Thus,

$$I = \frac{8}{9}\left(\frac{1}{3}\right) + \frac{5}{9}\left(-\frac{1}{-\sqrt{\frac{3}{5}}+3} + \frac{1}{\sqrt{\frac{3}{5}}+3}\right) = \frac{8}{27} + \frac{25}{63} = 0.29629 + 0.3968$$

$$= 0.6931$$

Otherwise, using the table;

$$I = w_1 f(u_1) + w_2 f(u_2) + w_3 f(u_3)$$

where, $f(u_i) = \frac{1}{u_i + 3}$

Using the abscissa and weight corresponding to n = 3 in the above table we obtain,

$$I = \frac{1}{3-0.7746}(0.555) + \frac{1}{3-0}(0.889) + \frac{1}{3+0.7746}(0.555)$$

$$= 0.4497 \times 0.55555 + \frac{1}{3}(0.889) + 0.2649 \times 0.5555$$

$$= 0.6931$$

13. Write an algorithm for Romberg's integration method. [2019 Fall]

Solution:

Step 1 : Start

Step 2 : Enter the function, lower and upper limit of integration and interval size h.

Step 3 : Evaluate

$I(h)$

$I\left(\frac{h}{2}\right)$

$I\left(\frac{h}{3}\right)$

Step 4 :

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left(4I\left(\frac{h}{2}\right) - I(h)\right)$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left(4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right)$$

$$I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left(4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right)$$

Step 5 : Print $I\left(h, \frac{h}{2}, \frac{h}{4}\right)$

Step 6 : Stop

14. The following table gives the velocity of a vehicle at various points of time.

Time, t (seconds)	1	2	4	5
Velocity, v (m/sec)	0.25	1	2.2	4

Find the acceleration of the vehicle at t = 1.1 seconds and t = 2.5 seconds using any suitable differential formula. [2019 Spring]

Solution:

Here;

Let time (t) as x and velocity (v) as y. As the values of x are not equi-spaced, we shall use Newton's divided difference formula.

Newton's divided difference table.

x	f(x)	1st div diff Δ	2nd div diff Δ	3rd div diff Δ
1	0.25			
		$\frac{1-0.25}{2-1} = 0.75$		
2	1		$\frac{0.6-0.75}{4-1} = -0.05$	
		$\frac{2.2-1}{4-2} = 0.6$		$\frac{0.4-(-0.05)}{5-1} = 0.1125$
4	2.2		$\frac{1.8-0.6}{5-2} = 0.4$	
		$\frac{4-2.2}{5-4} = 1.8$		
5	4			

Newton's divided formula is given by;

$$f(x) = y = y_0 + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$= y_0 + (x - x_0) f(x_0, x_1) + (x^2 - xx_1 - xx_0 + x_0x_1) f(x_0, x_1, x_2) + (x^3 - x^2x_1 - x^2x_0 + xx_0x_1 - x^2x_2 + xx_1x_2 + xx_0x_2 - x_0x_1x_2) f(x_0, x_1, x_2, x_3)$$

$$\therefore \quad f'(x) = f(x_0, x_1) + (2x - x_1 - x_0) f(x_0, x_1, x_2) + (3x^2 - 2xx_1 - 2xx_0 + x_0x_1 - 2xx_2 + x_1x_2 + x_0x_2) f(x_0, x_1, x_2, x_3)$$

$$\therefore \quad f''(x) = 2f(x_0, x_1, x_2) + (6x^2 - 2x_1 - 2x_0 - 2x_2) f(x_0, x_1, x_2, x_3)$$

Here,

$x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 5$

i) Acceleration $\left(a = \frac{dv}{dt}\right)$ at t = 1.1 sec

i.e., f(x) = ? at x = 1.1 sec

$\therefore$ $f(1.1) = 0.75 + (2 \times 1.1 - 2 - 1)(-0.05) + (3 \times 1.1^2 - 2 \times 1.1 \times 1 - 2 \times 1.1 \times 2 + 1 \times 2 - 2 \times 1.1 \times 4 + 2 \times 4 + 1 \times 4)(0.1125)$

$= 0.75 + (+0.04) + (0.2508)$

$= 1.0408$

ii) Acceleration $\left(a = \frac{dv}{dt}\right)$ at t = 2.5 sec

i.e., f(x) = ? at x = 2.5 sec

$\therefore$ $f(2.5) = 0.75 + (2 \times 2.5 - 2 - 1)(-0.05) + (3 \times 2.5^2 - 2 \times 2.5 \times 1 - 2 \times 2.5 \times 2 + 1 \times 2 - 2 \times 2.5 \times 4 + 2 \times 4 + 1 \times 4)(0.1125)$

$= 0.75 - 0.1 - 0.2531$

$= 0.3969$

15. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin u}{u} du$ by using Trapezoidal, Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule with n = 6. **[2019 Spring]**

Solution:

Here,

$a = 0, b = \frac{\pi}{2}, n = 6$

$nh = b - a$

$\therefore$ $h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$

$f(u) = y = \frac{\sin u}{u}$

u	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y_i	1	0.9886	0.9549	0.9003	0.8269	0.7379	0.6366
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

NOTE:

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

Make sure that calculator is in radian mode.

i) By Trapezoidal rule; we have,

$I = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$

$= \frac{\pi}{24}[(1 + 0.6366) + 2(0.9886 + 0.9549 + 0.9003 + 0.8269 + 0.7379)]$

$= 1.3684$

ii) By Simpson's $\frac{1}{3}$ rule; we have,

$I = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$

$= \frac{\pi}{36}[(1 + 0.6366) + 2(0.9549 + 0.8269) + 4(0.9886 + 0.9003 + 0.7379)]$

$= \frac{\pi}{36}[1.6366 + 3.5636 + 10.507]$

$= 1.3707$

iii) By Simpson's $\frac{3}{8}$ rule, we have,

$I = \frac{3h}{8}[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$

$= \frac{3}{8} \times \frac{\pi}{12}[(1 + 0.6366) + 2 \times 0.9003 + 3(0.9886 + 0.9549 + 0.8269 + 0.7379)]$

$= \frac{3\pi}{96}[1.6366 + 1.8006 + 10.5249]$

$= 1.3707$

16. Integrate the given integral using Romberg integration.

$\int_1^2 \frac{1}{1 + x^3} dx$ **[2020 Fall]**

Solution:

Here,

Let, $h = 0.5, \frac{h}{2} = 0.25, \frac{h}{4} = 0.125$

i) When h = 0.5, the value of $y = \frac{1}{1 + x^3}$ is shown in the table

x	1	1.5	2
y	0.5	0.2285	0.1112
	y_0	y_1	y_2

$$I(h) = I(0.5) = \frac{h}{2}[(y_0 + y_2) + 2y_1]$$

$$= \frac{0.5}{2}[0.5 + 0.1112 + 2 \times 0.2285]$$

$$= 0.26705$$

ii) When h = 0.25

x	1	1.25	1.50	1.75	2
y	0.5	0.3386	0.2285	0.1572	0.1112
	y_0	y_1	y_2	y_3	y_4

$$\therefore \quad I\left(\frac{h}{2}\right) = I(0.25) = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.25}{2}[(0.5 + 0.1112) + 2(0.3386 + 0.2285 + 0.1572)]$$

$$= \frac{0.25}{2}[0.6112 + 1.4486]$$

$$= 0.25747$$

iii) When h = 0.125

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y	0.5	0.4125	0.3386	0.2778	0.2285	0.1889	0.1572	0.1317	0.1112
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\therefore \quad I\left(\frac{h}{2}\right) = I(0.125)$$

$$= \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.125}{2}[(0.5 + 0.1112) + 2(0.4125 + 0.3386 + 0.2778 + 0.2285 + 0.1889 + 0.1572 + 0.1317)]$$

$$= \frac{0.125}{2}[0.6112 + 3.4704]$$

$$= 0.2551$$

Now,

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}\right) - I(h)\right]$$

$$= \frac{1}{3}[4 \times 0.2551 - 0.26705]$$

$$= 0.25427$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.2551 - 0.25747]$$

$$= 0.25431$$

$$\therefore \quad I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{2}\right) - I\left(h, \frac{h}{2}\right)\right]$$

$$= \frac{1}{3}[4 \times 0.25431 - 0.25427]$$

$$= 0.25432$$

17. Use Gauss-Legendre 2-point and 3-point formula to evaluate.

$$\int_{0.5}^{1.5} e^{x^2}\, dx$$

[2019 Spring]

Solution:

First we change the limit of integration from (0.5, 1.5) to (0, 1) using transformation

$$x = \frac{b-a}{2}z + \frac{a+b}{2}$$

or, $$x = \frac{1.5 - 0.5}{2}z + \frac{1.5 - 0.5}{2}$$

$$\therefore \quad x = \frac{z}{2} + 1$$

$$\therefore \quad dx = \frac{dz}{2}$$

$$g(z) = e^{\left(\frac{z}{2}+1\right)^2}$$

$$\therefore \quad I = \int_{0.5}^{1.5} e^{x^2}\, dx = \frac{1}{2} \cdot \int_{-1}^{1} e^{\left(\frac{z}{2}+1\right)^2} dz = \frac{1}{2} \cdot \int_{-1}^{1} g(z)\, dz$$

Using 2-point formula; we have,

$$\int_{-1}^{1} g(z)\, dz = \sum_{i=1}^{n} \omega_i\, g(z_i)$$

$$= \omega_1\, g(z_1) + \omega_2\, g(z_2)$$

$$= 1 \cdot e^{\left(\left(-\frac{1}{2} \times -\frac{1}{\sqrt{3}}\right)+1\right)^2} + 1 \cdot e^{\left(\left(-\frac{1}{2} \times \frac{1}{\sqrt{3}}\right)+1\right)^2}$$

$$= 5.2629 + 1.6586$$

$$= 6.9215$$

18. Compute the integral using Gaussian 3-point formula.

$$\int_{2}^{5} \frac{e^x + \sin x}{1 + x^2}\, dx$$

[2020 Fall]

Solution:

Here; first we change the limit of integration from (2, 5) to (–1, 1) using transformation.

so, $x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right)$

$= \left(\frac{5-2}{2}\right)z + \left(\frac{5+2}{2}\right)$

$= 1.5z + 3.5$

$\therefore \quad dx = 1.5\,dz$

$\therefore \quad g(z) = \dfrac{e^{1.5z+3.5} + \sin(1.5z+3.5)}{1+(1.5z+3.5)^2}$

$\therefore \quad I = \int_2^5 \dfrac{e^x + \sin x}{1+x^2}\,dx$

$= 1.5\int_{-1}^{1} \dfrac{e^{1.5z+3.5} + \sin(1.5z+3.5)}{1+(1.5z+3.5)^2}\,dz$

$= 1.5\int_{-1}^{1} f(z)\,dz$

$\therefore \quad \int_{-1}^{1} g(z)\,dz = \dfrac{b-a}{2}\sum_{i=1}^{n}\omega_i\, g(z_i)\,dx \left[\dfrac{1-(-1)}{2} = \dfrac{b-a}{2}\right]$

$= \omega_1 g(z_1) + \omega_2 g(z_2) + \omega_3 g(z_3)$

$$= \frac{5}{9} \times \frac{e^{1.5\times\left(-\sqrt{\frac{3}{5}}\right)+3.5} + \sin\left(1.5\left(-\sqrt{\frac{3}{5}}\right)+3.5\right)}{1+\left(1.5\times\left(-\sqrt{\frac{3}{5}}\right)+3.5\right)^2}$$

$$+ \frac{8}{9}\cdot\frac{e^{0+3.5} + \sin(0+3.5)}{1+(0+3.5)^2}$$

$$+ \frac{5}{9}\cdot\frac{e^{\left(1.5\times\left(\sqrt{\frac{3}{5}}\right)+3.5\right)} + \sin\left(1.5\left(\sqrt{\frac{3}{5}}\right)+3.5\right)}{1+\left(1.5\times\left(\sqrt{\frac{3}{5}}\right)+3.5\right)^2}$$

$= 0.837 + 2.198 + 2.6107$

$= 5.6457$

$\therefore \quad I = 1.5\int_{-1}^{1} g(z)\,dz = 1.5 \times 5.6457 = 8.4685$

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. **Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule and hence evaluate the value of π. Take h = 0.125. [2015]**

Solution:

Here;

$h = 0.125$

Let us construct table for various values of x and its corresponding value.

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5663	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Simpson's $\frac{1}{3}$ rule to calculate integration; we have,

$I = \frac{h}{3}[y_0 + y_8 + 2(y_2 + y_4 + y_6) + 4(y_1 + y_3 + y_5 + y_7)$

$= \frac{0.125}{3}[1 + 0.5 + 2(0.9411 + 0.8 + .064) + 4(0.9846 + 0.8767 + 0.7191 + 0.5663)]$

$= 0.785375$

Using mathematical method to find integration using formula, we have,

$I = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$

Now; we have,

$\frac{\pi}{4} = 0.785375$

$\therefore \quad \pi = 3.1415$

NOTE:

We can generate table directly using calculator (CASIO FX - 991 EX)

Menu + (9) or menu + table ↵

$f(x) = \frac{1}{1+x^2}$ ↵

g(x) = ↵

start = 0 ↵

end = 1 ↵

step = 0.125 ↵

Table is automatically generated.

2. How can we improve the accuracy of a numerical integration process? Use Romberg integration to evaluate $\int_0^1 \frac{dx}{1+x^2}dx$? [2016]

Solution:

The process of evaluating a definite integral from a set of tabulated value of the integral $f(x)$ is called numerical integration.

$$I = \int_a^b f(x)\,dx$$

We can improve the accuracy of a numerical integration process by choosing lower value of interval size or making large number of intervals.

$$h = \frac{b-a}{n}$$

where, h = Interval size

n = Number of intervals

Numerical

Let $h = 1$, $\frac{h}{2} = 0.5$ and $\frac{h}{4} = 0.25$ successively. Let's evaluate the integral using trapezoidal rule.

i) When h = 1, the value of $y = \frac{1}{1+x^2}$ is shown in the table

x	0	1	2	3
y	1	0.5	0.2	0.1
	y_0	y_1	y_2	y_3

$$\therefore \quad I(h) = I(1) = \frac{h}{2}[y_0 + y_3 + 2(y_1 + y_2)]$$

$$= \frac{1}{2}[1 + 0.1 + 2(0.5 + 0.2)] = 1.25$$

ii) When h = 0.5

x	0	0.5	1	1.5	2	2.5	3
y	1	0.8	0.5	0.3076	0.2	0.1379	0.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$\therefore \quad I = \left(\frac{h}{2}\right) = I(0.5) = \frac{h}{2}[y_0 + y_6 + 2(y_1 + \dots\dots + y_5)]$$

$$= \frac{0.5}{2}[1 + 0.1 + 2(0.8 + 0.5 + 0.3076 + 0.2 + 0.1379]$$

$$= 1.24775$$

iii) When h = 0.25

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
y	1	0.9411	0.8	0.64	0.5	0.3902	0.3076	0.2461	0.2	0.1649	0.1379	0.1167	0.1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}

$$I = \left(\frac{h}{4}\right) = I(0.25) = \frac{h}{2}[y_0 + y_{12} + 2(y_1 + y_2 + \dots\dots + y_{10} + y_{11})]$$

$$= \frac{0.25}{2}[1 + 0.1 + 2(0.9411 + 0.8 + 0.64 + .05 + .03902 + 0.3076 + 0.2461 + 0.2 + 0.1649 + .01379 + 0.1167)]$$

$$= 1.248625$$

$$I\left(h, \frac{h}{2}\right) = \frac{1}{3}\left[4 \cdot I\left(\frac{h}{2}\right) - I(h)\right] = \frac{1}{3}[4 \times 1.24775 - 1.25] = 1.247$$

$$I\left(\frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4 \cdot I\left(\frac{h}{4}\right) - I\left(\frac{h}{2}\right)\right] = \frac{1}{3}[4 \times 1.248625 - 1.24775] = 1.248917$$

$$\therefore \quad I\left(h, \frac{h}{2}, \frac{h}{4}\right) = \frac{1}{3}\left[4I\left(\frac{h}{2}, \frac{h}{4}\right) - I\left(h, \frac{h}{2}\right)\right]$$

$$= \frac{1}{3}[4 \times 1.248917 - 1.247]$$

$$= 1.249556$$

3. Use Simpson's $\frac{3}{8}$ rule to evaluate $\int_0^6 \frac{dx}{1+x^2}$. Also find the error. Take n = 6 [2017]

Solution:

Here, let us assume interval size 1(*i.e.*, h = 1)

$$\therefore \quad n = \frac{b-a}{h} = \frac{6-0}{1} = 6$$

[*i.e.*, n is a multiple of 3 (OK)]

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.05888	0.0384	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule, we have

$$I = \frac{3h}{8}[y_0 + y_6 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3 \times 1}{8}[1 + 0.027 + 2 \times 0.1 + 3(0.5 + 0.2 + 0.05888 + 0.0384)]$$

$$= 1.35707$$

Using mathematical formula of integration;

$$T = \int_0^6 \frac{dx}{1+x^2}[\tan^{-1} x]_0^6 = \tan^{-1} 6 - \tan^{-1} 0 = 1.405647$$

$$\therefore \quad \text{Error} = |T - I| = |1.405647 - 1.35707| = 0.04858$$

4. Compute integral $\int_{-2}^{2} e^{\frac{-x}{2}} dx$ using Gaussian two point formula. [2017]

Solution:

Here, first we change the limit of integration from (-2, 2) to (-1, 1) using transformation formula.

$$\left[\int_a^b e^{\frac{-x}{2}} dx = \int_{-2}^{2} e^{\frac{-x}{2}} dx\right]$$

$$x = \frac{b-a}{2} \cdot z + \frac{a+b}{2}$$

$$= \frac{2-(-2)}{2} z + \frac{2+2}{2}$$

$$= 2z$$

$$\therefore \quad dx = 2\,dz$$

$$\left[\int_a^b f(x)\,dx = \left(\frac{b-a}{2}\right)\int_{-1}^{1} g(z)\,dz\right]$$

$$\therefore \quad g(z) = e^{-\frac{2z}{2}} = e^{-z}$$

Now,

$$\int_{-2}^{2} e^{\frac{-x}{2}} dx = \frac{2-(-2)}{2}\int_{-1}^{1} e^{-z}\,dz$$

$$= 2\sum_{i=1}^{2} \omega_i\, g(z_i)$$

$$= 2[\omega_1 \cdot g(z_1) + \omega_2 \cdot g(z_2)]$$

$$= 2\left[1 \times e^{-(-1/\sqrt{3})} + 1 \times e^{-(-1/\sqrt{3})}\right]$$

$$= 4.6854$$

5. Using Simpson's $\frac{3}{8}$ rule, evaluate $\int_0^1 \frac{dx}{1+x^2}$. [2018]

Solution:

Here, let us assume n = 6 (*i.e.*, n must be multiple of 3).

Then,

$$h = \frac{b-a}{n} \qquad [i.e., a = 0, b = 1]$$

$$= \frac{1-0}{6} = 0.16667$$

Let us now generate table,

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	0.9729	0.9	0.8	0.6923	0.5901	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ rule, we get

$$I = \frac{3h}{8}[y_0 + y_6 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3 \times 0.16667}{8}[1 + 0.5 + 2 \times 0.8 + 3(0.9729 + 0.9 + 0.6923 + 0.5901)]$$

$$= 0.785384$$

Using mathematical formula

$$I = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} = 0.785398$$

6. Estimate the following intervals by Simpson's $\frac{1}{3}$ rule.

$$\int_1^2 \frac{e^x}{x} dx,\ n = 4$$ [2019]

Solution:

Here,

a = 1 (Lower limit)

b = 2 (Upper limit)

n = 4

$$y = f(x) = \frac{e^x}{x}$$

$$h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

x	1	1.25	1.5	1.75	2
y	2.7182	2.7922	2.9877	3.2883	3.6945
	y_0	y_1	y_2	y_3	y_4

By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule; we have,

$$\int_1^2 \frac{e^x}{x} dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{\frac{1}{4}}{3}[(2.7182 + 3.6945) + 4(2.7922 + 3.2883) + 2 \times 2.9877]$$

$$= \frac{1}{12} \times 36.7101$$

$$= 3.059175$$

7. Estimate the interval using Gauss quadrature with n = 2

$$I = \int_0^1 \frac{1}{1+x^2} dx$$ [2019]

Solution:

Here, we have to evaluate the integral using 2 point formula.

First we need to change the limit of integration from (0, 1) to (−1, 1).

i.e., $\int_a^b f(x)dx = \int_{-1}^{1} g(z)dz$

$a = 0$

$b = 1$

$f(x) = \frac{1}{1+x^2}$

$\therefore \quad x = \left(\frac{b-a}{2}\right)z + \left(\frac{a+b}{2}\right) = \left(\frac{1-0}{2}\right)z + \left(\frac{0+1}{2}\right) = \frac{z+1}{2}$

$\therefore \quad dx = \frac{dz}{2}$

$\therefore \quad g(z) = \frac{1}{1+\left(\frac{z+1}{2}\right)^2} = \frac{4}{4+(z+1)^2}$

$$\therefore \quad \int_0^1 \frac{1}{1+x^2}dx = \frac{b-a}{2}\sum_{i=1}^{2} w_i g(z_i)$$

$$= \frac{1}{2}[w_1 g(z_1) + w_2 g(z_2)]$$

$$= \frac{1}{2}\left[1 \times \frac{4}{4+\left(-\frac{1}{\sqrt{3}}+1\right)^2} + 1 \times \frac{4}{4+\left(\frac{1}{\sqrt{3}}+1\right)^2}\right]$$

$$= 0.7868$$

Chapter 5

SYSTEM OF LINEAR ALGEBRAIC EQUATIONS

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Introduction

The techniques for solving system of linear algebraic equation are;

- Eliminative technique
- Iterative technique

5.1 ELIMINATIVE TECHNIQUE

In this technique, the system is reduces to a form, from which the solution can be obtained by direct and simple substitution. This method do not contains any truncation error. It is also called direct method. Some eliminative techniques used are as follows;

- Method of determinant
- Matrix inversion method
- Gauss elimination method

- Gauss Jordan method
- LU decomposition/Factorization method

I) Iterative Technique

Iterative technique requires some assumption on initial values. These initial assumption are refined until they reach to least error *i.e.*, negligible error. Some iterative techniques are as follows;

- Jacobi iteration method
- Gauss seidal iteration method

ii) Ill Conditioned System

The solution of the given system of equation depends upon the condition under which the system of equation is given. That is the system of equations will have unique solution infinitely many solution and no solution depending upon the nature of the system of equations. But, small change in the coefficient of variable in the system of equation shows large deviation in the solution then the system is known as ILL conditioned system.

Mathematically; let the two given equations be;

$$a_{11}x_1 + a_{12}x_2 = b_1$$

and, $$a_{21}x_1 + a_{22}x_2 = b_2$$

If two straight lines represented by the above two equations be almost parallel then the slopes must be nearly equal.

i.e., $$\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}}$$

or, $$a_{11}a_{22} = a_{12}a_{21}$$

or, $$a_{11} - a_{22} - a_{12}a_{21} = 0$$

This expression is the determinant of the coefficient of matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Thus, the condition for ILL defined system of the equation is that, the determinant of the coefficient matrix is small enough or approximately equal to zero.

iii) Method of Determinant

Let us consider the system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then, the augmented matrix of the given system is;

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Now, determinant of the coefficient are;

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_1 = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_2 = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}$$

and, $$\Delta_3 = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

Now, by Cramer's rule, solutions of these equations are;

$$x = \frac{\Delta_1}{\Delta}$$

$$y = \frac{\Delta_2}{\Delta}$$

and, $$z = \frac{\Delta_3}{\Delta}$$

iv) Matrix Inversion Method

Let us consider the system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then, this system can be written as;

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

where, $$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Hence,

$$AX = B$$

or, $X = A^{-1}B$

where, A^{-1} is the inverse matrix of A.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

and, 'A', 'B' and 'C' are cofactor of 'a', 'b' and 'c' respectively and such matrix is called matrix of co-factor.

5.1.1 Gauss Elimination Method

In this method, the unknown values are eliminated successively and system is reduced to a triangular system from which these unknown values are determined using substitution. Gauss elimination can be done using two methods.

i) Forward elimination

Given system is reduces to upper triangular matrix.

i.e., $$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_3'' \end{pmatrix}$$

ii) Backward elimination

Given system is reduces to lower triangular matrix.

i.e., $$\begin{pmatrix} a_1'' & 0 & 0 & : & d_1'' \\ a_2' & b_2' & 0 & : & d_2' \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Consider a system of linear equation;

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then,

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - \frac{a_2}{a_1}R_1$, $R_3 \to R_3 - \frac{a_3}{a_1}R_1$

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & b_3' & c_3' & : & d_3' \end{pmatrix}$$

where, $b_2' = b_2 - \frac{a_2}{a_1}b_1$, $c_2' = c_2 - \frac{a_2}{a_1}c_1$ and $d_2' = d_2 - \frac{a_2}{a_1}d_1$.

Step 3: Elimination of 'y' from third row

i.e., $R_3 \to R_3 - \frac{b_3'}{b_2'}R_2$

$$\begin{pmatrix} a_1 & b_1 & c_1 & : & d_1 \\ 0 & b_2' & c_2' & : & d_2' \\ 0 & 0 & c_3'' & : & d_2'' \end{pmatrix}$$

where, $c_3'' = c_3 - \frac{b_3'}{b_2'}c_2$ and $d_3'' = d_3 - \frac{b_3'}{b_2'}d_2$.

Step 4: Evaluate the unknown values using backward substitution

i.e., $c_3''z = d_3''$

or, $z = \frac{d_3''}{c_3''}$

$$b_2'y + c_2'z = d_2'$$

or, $y = \frac{d_2' - c_2'z}{b_2'}$

$$a_1x + b_1y + c_1z = d_1$$

or, $x = \frac{d_1 - b_1y - c_1z}{a_1}$

NOTE:

Similarly; in backward elimination technique, the given augmented matrix is reduced into lower triangular matrix and values are determined using forward substitution.

Example 1

Solve the system of linear equations by using Gauss elimination method.

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 4 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 0 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - \frac{1}{2}R_1$

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 0 & -2 & 1 & : & -8 \\ 0 & -2 & 0.5 & : & -3 \end{pmatrix}$$

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Step 3: Elimination of 'y' from third row

Applying $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 0 & -2 & 1 & : & -8 \\ 0 & 0 & -\frac{1}{2} & : & 5 \end{pmatrix}$$

Step 4: Using backward substitution

From R_3;

$$-\frac{1}{2}x_3 = 5;$$

or, $x_3 = -10$

From R_2;

$$-2x_2 + x_3 = -8$$

or, $x_2 = \frac{8-10}{2} = -1$

From R_1;

$$2x_1 + 2x_2 + x_3 = 6$$

or, $2x_1 = 6 + 2 + 10$

or, $x_1 = 9$

Hence,

$x_1 = 9$
$x_2 = -1$; which is the required solution of the system.
$x_3 = -10$

Example 2

Solve the system of linear equations by using Gauss elimination method.

$$x + 2y + 3z = 6$$
$$2x + 3y + 5z = 10$$
$$2x - y + 3z = 4$$

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 2 & 3 & 5 & : & 10 \\ 2 & -1 & 3 & : & 4 \end{pmatrix}$$

Step 2: Elimination of 'x' from second and third row;

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -1 & : & -2 \\ 0 & -5 & -3 & : & -8 \end{pmatrix}$$

Step 3: Elimination of 'y' from third row

Applying $R_3 \to R_3 - 5R_2$

$$\begin{pmatrix} 1 & 2 & 3 & : & 6 \\ 0 & -1 & -1 & : & -2 \\ 0 & 0 & 2 & : & 2 \end{pmatrix}$$

Step 4: Using backward substitution

From R_3;

$$2z = 2$$

or, $z = 1$

From R_2;

$$-y - z = -2$$

or, $y = 1$

From R_1;

$$x + 2y + 3z = 6$$

or, $x = 1$

Hence,

$x = 1$
$y = 1$; which is the required solution of the system.
$z = 1$

5.1.2 Gauss Elimination Using Pivoting

Pivoting is the process of recording the equation in order to get better accuracy. Pivoting can be done only if the pivot element is non-zero. The main diagonal element of augmented matrix is called pivot element. The pivoting is necessary if the main diagonal element of augmented matrix is zero. Pivoting can be applied by two ways;

i) Partial pivoting
ii) Complete pivoting

The partial pivoting involves the following steps.

- Search and locate largest absolute value among coefficient in first column.
- Exchange the first row with row containing that element.
- Eliminate the first variable in the second equation.
- When second row becomes pivot row, search for the coefficient in the second column from second row to n^{th} row and locate the largest coefficient.
- Exchange the second row with the row containing largest coefficient.
- Continue the process until $(n-1)^{th}$ unknown are eliminated.

Process for Complete Pivoting

- Check for the largest coefficient of unknown.
- Interchange the row and column, so that the largest coefficient of unknown appears as first element of augmented matrix.

Example 3

Solve the given system of linear equations using partial pivoting.

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Solution:

The augmented matrix of given system is;

$$\begin{pmatrix} 2 & 2 & 1 & : & 6 \\ 4 & 2 & 3 & : & 4 \\ 1 & -1 & 1 & : & 0 \end{pmatrix}$$

Check the first column for largest coefficient. Here, R_2 is pivot equation. [Since, 4 is the largest coefficient.]

Interchanging R_1 and R_2; we have,

$$\begin{pmatrix} \boxed{4 \quad 2 \quad 3 \quad : \quad 4} \\ 2 \quad 2 \quad 1 \quad : \quad 6 \\ 1 \quad -1 \quad 1 \quad : \quad 0 \end{pmatrix} \text{ Pivot}$$

Applying $R_2 \to R_2 - \frac{1}{2}R_1$, $R_3 \to R_3 - \frac{1}{4}R_1$

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ 0 & 1 & -0.5 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \end{pmatrix}$$

Now, check the largest value for second column, except first row. (Leave first row)

-1.5 is the largest coefficient.

$\therefore$ R_3 is pivot equation.

Interchanging R_2 and R_3; we have,

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \\ 0 & 1 & -0.5 & : & 4 \end{pmatrix} \text{ Pivot}$$

Applying $R_3 \to R_3 + \frac{2}{3}R_2$

$$\begin{pmatrix} 4 & 2 & 3 & : & 4 \\ 0 & -1.5 & 0.25 & : & -1 \\ 0 & 0 & -0.33 & : & 3.33 \end{pmatrix}$$

Using backward substitution;

From R_3;

$$x_3 = -10$$

From R_2;

$$-1.5x_2 + 0.25x_3 = -1$$

or, $x_2 = -1$

From R_1;

$$4x_1 + 2x_2 + 3x_3 = 4$$

or, $x_1 = 9$

Hence,

$$x_1 = 9$$
$$x_2 = -1$$
$$x_3 = -10$$

Example 4

Solve the following system using complete pivoting.

$$2x + 7y + 3z = 5$$
$$3x + 4y + 9z = 15$$
$$2x + 3y + z = 9$$

Solution:

The augmented matrix of given system is;

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \text{const.} \end{array} \\ \begin{pmatrix} 2 & 7 & 3 & : & 5 \\ 3 & 4 & \boxed{9} & : & 15 \\ 2 & 3 & 1 & : & 9 \end{pmatrix} \end{array}$$

Here, 9 is the largest coefficient in R_2.

Interchanging R_2 and R_1; we have,

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \text{const.} \end{array} \\ \begin{pmatrix} 3 & 4 & 9 & : & 15 \\ 2 & 7 & 3 & : & 5 \\ 2 & 3 & 1 & : & 9 \end{pmatrix} \end{array}$$

Again, interchanging C_1 and C_3; we have,

$$\begin{array}{c} \begin{array}{cccc} z & y & x & \text{const.} \end{array} \\ \begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 3 & 7 & 2 & : & 5 \\ 1 & 3 & 2 & : & 9 \end{pmatrix} \end{array}$$

Applying $R_2 \to R_2 - \frac{1}{3}R_1$, $R_3 \to R_3 - \frac{1}{9}R_1$

$$\begin{array}{c} \begin{array}{cccc} z & y & x & \text{const.} \end{array} \\ \begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 0 & \boxed{\frac{17}{3}} & 1 & : & 0 \\ 0 & \frac{23}{9} & \frac{5}{3} & : & \frac{22}{3} \end{pmatrix} \end{array}$$

Here, $\frac{17}{3}$ is the largest term among R_2 and R_3

Applying $R_3 \rightarrow R_3 - \frac{23}{51}R_2$

$$\begin{array}{c} \begin{array}{cccc} z & y & x & \text{const.} \end{array} \\ \begin{pmatrix} 9 & 4 & 3 & : & 15 \\ 0 & \frac{17}{3} & 1 & : & 0 \\ 0 & 0 & \frac{62}{51} & : & \frac{22}{3} \end{pmatrix} \end{array}$$

Using backward substitution

From R_3;

$$x = \frac{187}{31}$$

From R_2;

$$y = -\frac{33}{31}$$

From R_1;

$$z = \frac{4}{31}$$

Hence,

$$x = \frac{187}{31}$$

$$y = -\frac{33}{31}$$

$$z = \frac{4}{31}$$

5.1.3 Gauss Jordan Method

Gauss Jordan method is the modified form of Gauss elimination method. In this method, the augmented matrix is reduced to unit diagonal matrix. Here, we get the solution without using back substitution.

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_3 + a_{32}x_2 + a_{33}x_3 = b_3$$

The augmented matrix is the form of;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & : & b_1 \\ a_{21} & a_{22} & a_{23} & : & b_2 \\ a_{31} & a_{32} & a_{33} & : & b_3 \end{pmatrix}$$

After the completion of Gauss Jordan method the equation becomes;

$$\begin{pmatrix} 1 & 0 & 0 & : & B_1 \\ 0 & 1 & 0 & : & B_2 \\ 0 & 0 & 1 & : & B_3 \end{pmatrix}$$

Hence, the required solution is;

$$x_1 = B_1$$
$$x_2 = B_2$$
$$x_3 = B_3$$

Example 5

Solve the following system of linear equation using Gauss Jordan elimination method.

$$x + 2y + z = 8$$
$$2x + 3y + 4z = 20$$
$$4x + 3y + 2z = 16$$

Solution:

Arranging

$$4x + 3y + 2z = 16$$
$$2x + 3y + 4z = 20$$
$$x + 2y + z = 8$$

Step 1

The augmented matrix of the given system is;

$$\begin{pmatrix} 4 & 3 & 2 & : & 16 \\ 2 & 3 & 4 & : & 20 \\ 1 & 2 & 1 & : & 8 \end{pmatrix}$$

Step 2

Applying $R_2 \rightarrow 2R_2 - R_1$, $R_3 \rightarrow 4R_3 - R_1$

$$\begin{pmatrix} 4 & 3 & 2 & : & 16 \\ 0 & 3 & 6 & : & 24 \\ 0 & 5 & 2 & : & 16 \end{pmatrix}$$

Step 3

Applying $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow 3R_3 - 5R_2$

$$\begin{pmatrix} 4 & 0 & -4 & : & -8 \\ 0 & 3 & 6 & : & 24 \\ 0 & 0 & -24 & : & -72 \end{pmatrix}$$

Applying $R_1 \rightarrow 6R_1 - 3R_3$, $R_2 \rightarrow 4R_2 + R_3$

$$\begin{pmatrix} 24 & 0 & 0 & : & 24 \\ 0 & 12 & 0 & : & 24 \\ 0 & 0 & -24 & : & -72 \end{pmatrix}$$

Hence,

From R_3;

$$z = 3$$

From R_2;

$$y = 2$$

From R_1;

$$x = 1$$

Hence,

$$x = 1$$
$$y = 2$$
$$z = 3$$

5.1.4 Inverse of Matrix

We know that 'X' will be the inverse of matrix A, if

$$AX = I$$

where, 'I' is the unit matrix of same order as 'A'.

It is required to determine the element of 'X' such that $AX = I$ is satisfied.

Consider the system of inverse in matrix form $AX = I$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This equation is equivalent to the following three equations which are;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (1)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2)$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (3)$$

We can therefore apply the Gauss elimination method of each of these systems and the resulting in each case will be corresponding column of A^{-1}.

Example 6

Find the inverse of matrix.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Solution:

Let, $x = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ be the inverse of A.

Then,

$$AX = I$$

The augmented matrix is;

$$\left(\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right)$$

Applying $R_1 \to \frac{R_1}{2}$;

$$\left(\begin{array}{ccc:ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right)$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc:ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1.5 & -1.5 & 1 & 0 \\ 0 & 3.5 & 8.5 & -0.5 & 0 & 1 \end{array}\right)$$

Applying $R_2 \to \frac{R_2}{0.5}$

$$\left(\begin{array}{ccc:ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 3.5 & 8.5 & -0.5 & 0 & 1 \end{array}\right)$$

Applying $R_3 \to R_3 - 3.5R_2$

$$\left(\begin{array}{ccc:ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -2 & 10 & -7 & 1 \end{array}\right)$$

This is equivalent to three systems.

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix} = \begin{pmatrix} 0.5 \\ -3 \\ 10 \end{pmatrix}$$

By using backward substitution; we get,

From R_3;

$$x_{31} = -5$$

From R_2;

$$x_{21} = 12$$

From R_1;

$$x_{11} = -3$$

Second system

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -7 \end{pmatrix}$$

From R_3;

$$x_{32} = 3.5$$

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From R_2;

$$x_{22} = -8.5$$

From R_1;

$$x_{12} = 2.5$$

Third system

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

From R_3;

$$x_{33} = -0.5$$

From R_2;

$$x_{23} = 1.5$$

From R_1;

$$x_{13} = -0.5$$

Hence,

$$A^{-1} = \begin{pmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{pmatrix}$$

5.1.5 LU Decomposition Methods

This method is based on the fact that every square matrix 'A' can be expressed as the product of a lower triangular matrix and upper triangular matrix.

Let us consider the system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_3 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of equation can be written in the matrix form as;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

or, $AX = B$

where, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and, $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Let, $A = LU$ (1)

where, $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $L = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$.

Now, equation (1) becomes;

$$LUX = B \quad (2)$$

or, $$LZ = B \quad (3)$$

where, $Z = UX$.

Now, we can solve equation in two stages.

i) Solve the equation $LZ = B$ for 'z' by forward substitution.

ii) Solve equation $UX = Z$ for 'x' using 'z' by backward substitution.

Doolittle and Cholesky Factorization

The factorization is done by assuming diagonal element of 'L' and 'U' to be unity. The factorization with 'L' having unit diagonal values is called Doolittle factorization while other one with 'U' having uni-diagonal element called Cholesky's factorization.

5.1.5.1 Doolittle Factorization

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = A$$

5.1.5.2 Cholesky's Factorization

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{pmatrix} = A$$

Example 7

Solve the following linear equation by using Doolittle LU method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Solution:

The given system of linear equation can be written in matrix form as;

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

or, $AX = B$

where, $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and, $B = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$.

Let, $LU = A$

where, $L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$ and, $U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$.

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$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

or, $$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

Equating corresponding elements; we get,

$U_{11} = 3$

$U_{12} = 2$

$U_{13} = 1$

$l_{21}U_{11} = 2$

or, $l_{21} = \frac{2}{3}$

$l_{21}U_{12} + U_{22} = 3$

or, $U_{22} = \frac{5}{3}$

$l_{21}U_{13} + U_{23} = 2$

or, $U_{23} = \frac{4}{3}$

$l_{31}U_{11} = 1$

or, $l_{31} = \frac{1}{3}$

$l_{31}U_{12} + l_{32}U_{22} = 2$

or, $l_{32} = \frac{4}{5}$

$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 3$

or, $U_{33} = \frac{8}{5}$

Hence,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix}$$

and, $$U = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{pmatrix}$$

We know that;

$LZ = B$

i.e., $$\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 14 \end{pmatrix}$$

Using forward substitution; we have,

$z_1 = 10$

$z_2 = \frac{22}{3}$

$z_3 = \frac{24}{5}$

Again, we have,

$UX = Z$

or, $$\begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{pmatrix}$$

Using backward substitution; we have,

From R_3;

$x_3 = 3$

From R_2;

$x_2 = 2$

From R_1;

$x_1 = 1$

5.2 ITERATIVE SOLUTION OF LINEAR EQUATION

5.2.1 Gauss Jacobi Method

Consider the system of the linear equations;

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad (1)$$

Then,

$$\left.\begin{aligned} x &= \frac{d_1 - b_1y - c_1z}{a_1} \\ y &= \frac{d_2 - b_2y - c_2z}{a_2} \\ z &= \frac{d_3 - b_3y - c_3z}{a_3} \end{aligned}\right\} \quad (2)$$

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Here, a_1, b_2 and c_3 must be larger than other coefficient in each equation.
Suppose the initial approximation be;

$$x = x_0$$
$$y = y_0$$
and, $z = z_0$

Substituting these values in equation (2); we have,

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_1x_0 - c_2z_0]$$

$$z_1 = \frac{1}{c_2}[d_3 - a_3x_0 - b_3y_0]$$

Substituting these values of x_1, y_1 and z_1 in equation (2); we get the second approximation x_2, y_2 and z_2. This process is continued till the difference between two consecutive approximations is negligible.

Example 8

Solve the following system using Gauss Jacobi iteration method.

$$3x + 4y + 15z = 54.8$$
$$x + 12y + 3z = 39.66$$
$$10x + y - 2z = 7.74$$

Solution:

Arranging the given system in larger in order.

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now, the given system can be written as;

$$x = \frac{1}{10}[7.74 - y + 2z]$$

$$y = \frac{1}{12}[39.66 - x - 3z]$$

and, $z = \frac{1}{15}[54.8 - 3x - 4y]$

Let initial approximation be;

$$x = 0$$
$$y = 0$$
and, $z = 0$

Then, successive iteration is given as;

n	x	y	z
Initial	0	0	0
1	0.7740	3.3050	3.6533
2	1.1742	2.3272	2.6172
3	1.0647	2.5529	2.7979
4	1.0783	2.5168	2.7596
5	1.0748	2.5252	2.7665
6	1.0748	2.5239	2.7651
7	1.0746	2.5242	2.7653
8	1.0746	2.5241	2.7653

The iteration at 7th and 8th are similar up to 3 decimal points.

i.e., $x = 1.074$
$y = 2.524$
and, $z = 2.765$

Gauss Seidal Iteration Method

It is the modified version of Gauss Jordan method. In Gauss seidal, the latest approximation on Gauss Jordan method is used. *i.e.*, the process is repeated till the values of 'x', 'y' and 'z' obtained to desired accuracy.

From Gauss Jordan approximation; we have,

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_2x_0 - c_2z_0]$$

$$z_1 = \frac{1}{c_3}[d_3 - a_3x_0 - b_3y_0]$$

and, Gauss seidal approximation;

$$x_1 = \frac{1}{a_1}[d_1 - b_1y_0 - c_1z_0]$$

$$y_1 = \frac{1}{b_2}[d_2 - a_2x_1 - c_2z_0]$$

and, $z_1 = \frac{1}{c_3}[d_3 - a_3x_1 - b_3y_1]$

Example 9

Solve the following equation by an iterative method.

$$10x - 2y - z - w = 3$$
$$-2x + 10y - z - w = 15$$
$$-x - y + 10z - 2w = 27$$
$$-x - y - 2z + 10w = -9$$

Solution:

Here, the given system is diagonally dominant. It can be written as;

$$x = \frac{1}{10}[3 + 2y + z + w]$$

$$y = \frac{1}{10}[15 + 2x + z + w]$$

$$z = \frac{1}{10}[27 + 2w + x + y]$$

and, $w = \frac{1}{10}[x + y + 2z - 9]$

Let initial approximation be $y = z = w = 0$.

Now, tabulating the successive iterations; we have,

n	x	y	z	w
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9984	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	2	3	0
7	1	2	3	0
8	1	2	3	0

The iteration at seventh and eighth positions is similar (or exact).

Hence,

$$x = 1$$

$$y = 2$$

and, $z = 3$

5.2.2 Eigen Value and Eigen Vector Using Power Method

If a 'A' is any square matrix of order $n \times n$ with elements a_{ij} we can find a column matrix 'X' and a constant λ.

Such that;

$$AX = \lambda X$$

where, λ is called eigen value and 'X' is called corresponding eigen vector.

We start with the column vector $X^{(0)}$ which is as near the solution as possible and evaluate $AX^{(0)}$ which is written as $\lambda^{(1)}X^{(1)}$ after normalization. This gives the first approximation $\lambda^{(1)}$ to the engine value and $X^{(1)}$ to the eigen vector.

Similarly $X^{(1)}$ to the eigen vector, and we evaluate;

$$AX^{(1)} = A^{(2)}X^{(2)};$$

which gives us second approximation.

Repeat the process till $[X^{(n)} - X^{(n-n)}]$ becomes negligible, then $\lambda^{(n)}$ will be the largest eigen value and $X^{(n)}$ be the corresponding eigen vector.

NOTE:

For initial approximation of eigen vector take 1 for the rows not having zero and zero for other.

For example;

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 5 & 3 \\ 8 & 0 & 2 \end{bmatrix}, X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In first row, there is no zero element.

$\therefore$ First element of $X^{(0)}$ is 1.

In second row, there is zero element.

$\therefore$ Second element of $X^{(0)}$ is 0.

Similarly; third element of $X^{(0)}$ is 0.

$$\text{For } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 6 & 9 \end{bmatrix}, X^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

If all row contains '0'

$$X^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and for all non zero element;

$$X^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Take largest value as common and divide the whole element of the vector with largest value.

Example 10

Find the largest eigen value and corresponding vector of the following matrix using power method.

$$A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Solution:

Here, row 2 and row 3 contains zero element.

$$\text{Initial eigen vector} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

First Iteration

$$\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0.429 \\ 0 \end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.429 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.571 \\ 1.858 \\ 0 \end{pmatrix} = 3.571\begin{pmatrix} 1 \\ 0.520 \\ 0 \end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.520 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.120 \\ 2.040 \\ 0 \end{pmatrix} = 4.120\begin{pmatrix} 1 \\ 0.495 \\ 0 \end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.495 \\ 0 \end{pmatrix} = \begin{pmatrix} 3.971 \\ 1.990 \\ 0 \end{pmatrix} = 3.971\begin{pmatrix} 1 \\ 0.501 \\ 0 \end{pmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth Iteration

$$AX^{(5)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.501 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.006 \\ 2.002 \\ 0 \end{pmatrix} = 4.006\begin{pmatrix} 1 \\ 0.500 \\ 0 \end{pmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh Iteration

$$AX^{(6)} = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.500 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \lambda^{(7)}X^{(7)}$$

$\therefore$ Largest eigen value = 4

and, Eigen vector $= \begin{pmatrix} 1 \\ 0.500 \\ 0 \end{pmatrix}$

SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. **Find the largest eigen value and corresponding vector of the following matrix.**

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ **[2009 Spring]**

Solution:

Here, all row contains zero element,

$\therefore$ Initial Eigen vector of the matrix $X^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

First Iteration

$$AX^{(0)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \lambda^{(1)}X^{(1)}$$

Second Iteration

$$AX^{(1)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.5 \\ 0 \end{pmatrix} = 2.5\begin{pmatrix} 0.8 \\ 1 \\ 0 \end{pmatrix} = \lambda^{(2)}X^{(2)}$$

Third Iteration

$$AX^{(2)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0.8 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 2.6 \\ 0 \end{pmatrix} = 2.8\begin{pmatrix} 1 \\ 0.929 \\ 0 \end{pmatrix} = \lambda^{(3)}X^{(3)}$$

Fourth Iteration

$$AX^{(3)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.929 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.857 \\ 2.929 \\ 0 \end{pmatrix} = 2.929\begin{pmatrix} 0.975 \\ 1 \\ 0 \end{pmatrix} = \lambda^{(4)}X^{(4)}$$

Fifth Iteration

$$AX^{(4)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0.975 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.975 \\ 2.95 \\ 0 \end{pmatrix} = 2.975\begin{pmatrix} 1 \\ 0.992 \\ 0 \end{pmatrix} = \lambda^{(5)}X^{(5)}$$

Sixth Iteration

$$AX^{(5)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0.992 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.983 \\ 2.992 \\ 0 \end{pmatrix} = 2.992\begin{pmatrix} 0.997 \\ 1 \\ 0 \end{pmatrix} = \lambda^{(6)}X^{(6)}$$

Seventh Iteration

$$AX^{(6)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0.997 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.997 \\ 2.994 \\ 0 \end{pmatrix} = 2.997\begin{pmatrix} 1 \\ 0.999 \\ 0 \end{pmatrix}$$

$$= \lambda^{(7)}X^{(7)}$$

Largest eigen value = 2.997

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and, Eigen vector $= \begin{pmatrix} 1 \\ 0.999 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

2. Solve the system of linear equations by using Gauss elimination method.

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7$$

[2009 Fall]

Solution:

Step 1: The augmented matrix of given system is;

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ -6 & 8 & -1 & -4 & : & 5 \\ 3 & 1 & 4 & 11 & : & 2 \\ 5 & -9 & -2 & 4 & : & 7 \end{pmatrix}$$

Step 2: Elimination of 'x'

Applying $R_2 \to R_2 + \frac{6}{10}R_1$, $R_3 \to R_3 - \frac{3}{10}R_1$, $R_4 \to R_4 - \frac{5}{10}R_1$

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & \frac{19}{5} & \frac{4}{5} & -1 & : & \frac{43}{5} \\ 0 & \frac{31}{10} & \frac{31}{10} & \frac{19}{2} & : & \frac{1}{5} \\ 0 & -\frac{11}{2} & -\frac{7}{2} & \frac{3}{2} & : & 4 \end{pmatrix}$$

or, $$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 31 & 31 & 95 & : & 2 \\ 0 & -11 & -7 & 3 & : & 8 \end{pmatrix}$$

Step 3: Elimination of 'y'

Applying $R_3 \to R_3 - \frac{31}{19}R_2$, $R_4 \to R_4 + \frac{11}{19}R_2$

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 0 & 465 & 1960 & : & -1295 \\ 0 & 0 & -89 & 2 & : & 625 \end{pmatrix}$$

Step 4: Elimination of 'z'

Applying $R_4 \to R_4 + \frac{89}{465}R_3$

$$\begin{pmatrix} 10 & -7 & 3 & 5 & : & 6 \\ 0 & 19 & 4 & -5 & : & 43 \\ 0 & 0 & 465 & 1960 & : & -1295 \\ 0 & 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Step 5: Determination of unknowns using backward substitution

From R_4;

$$u = 1$$

From R_3;

$$465z + 1960u = -1295$$

or, $z = -7$

From R_2;

$$19y + 4z - 5u = 43$$

or, $y = 4$

From R_1;

$$10x - 7y + 3z + 5u = 6$$

or, $x = 5$

Hence,

$x = 5$
$y = 4$
$z = -7$; which is the required solution of the system.
$u = 1$

3. Solve the following system of linear equation by Gauss-seidal iteration method.

$$9x_1 + 2x_2 - 3x_3 = 10$$
$$5x_1 + 11x_3 - 24x_4 = 30$$
$$2x_1 + 8x_2 - 2x_4 = 15$$
$$x_2 + x_3 + 7x_4 = 25$$

[2010 Fall]

Solution:

Writing them in order of highest coefficients;

$$9x_1 + 2x_2 - 3x_3 = 10$$
$$2x_1 + 8x_2 - 2x_4 = 15$$
$$5x_1 + 11x_3 - 24x_4 = 30$$
$$x_2 + x_3 + 7x_4 = 25$$

$$x_1 = \frac{1}{9}(10 - 2x_2 + 3x_3) \quad (1)$$

$$x_2 = \frac{1}{8}(15 - 2x_1 + 2x_4) \quad (2)$$

$$x_3 = \frac{1}{11}(30 - 5x_1 + 24x_4) \quad (3)$$

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$$x_4 = \frac{1}{7}(25 - x_2 - x_3) \quad (4)$$

Iteration 1

Let $x_2 = x_3 = x_4 = 0$ (initial guess)

$$x_1^{(1)} = 1.111$$

$$x_2^{(1)} = \frac{1}{8}[15 - (2 \times 1.111) + 0] = 1.59725$$

$$x_3^{(1)} = \frac{1}{11}[30 - (5 \times 1.111) + 0] = 2.2222$$

$$x_4^{(1)} = \frac{1}{7}(25 - 1.59725 - 2.2222) = 3.0258$$

Iteration 2

$$x_1^{(2)} = \frac{1}{9}[10 - (2 \times 1.59725) + (3 \times 2.2222)] = 1.4969$$

$$x_2^{(2)} = \frac{1}{8}[15 - (2 \times 1.4969) + (2 \times 3.0258)] = 2.2572$$

$$x_3^{(2)} = \frac{1}{11}[30 - (5 \times 1.4969) + (24 \times 3.0258)] = 8.6486$$

$$x_4^{(2)} = \frac{1}{7}(25 - 2.2572 - 8.6486) = 2.0135$$

Iteration 3

$$x_1^{(3)} = \frac{1}{9}[10 - (2 \times 2.2572) + 3 \times (8.6486)] = 3.4924$$

$$x_2^{(3)} = \frac{1}{8}[15 - (2 \times 3.4924) + (2 \times 2.0135)] = 1.5053$$

$$x_3^{(3)} = \frac{1}{11}[30 - (5 \times 3.4924) + (24 \times 2.0135)] = 5.533$$

$$x_4^{(3)} = \frac{1}{7}(25 - 1.5053 - 5.533) = 2.566$$

n	x_1	x_2	x_3	x_4
3	3.4924	1.5053	5.533	2.566
4	2.6209	1.8613	7.1345	2.2863
5	3.0756	1.6777	6.3175	2.4293
6	2.8441	1.7713	6.7348	2.3563
7	2.9624	1.7235	6.5217	2.3935
8	2.9020	1.7479	6.6303	2.3745
9	2.9328	1.7354	6.5749	2.3842
10	2.9171	1.7418	6.6032	2.3793
11	2.9251	1.7385	6.5888	2.3818
12	2.9210	1.7402	6.5962	2.3805

13	2.9231	1.7398	6.5924	2.3812
14	2.922	1.7398	6.5944	2.3808
15	2.9226	1.7395	6.5933	2.3810

Thus, from 14th and 15th iterations;

$$x_1 = 2.9226$$
$$x_2 = 1.7395$$
$$x_3 = 6.5933$$
$$x_4 = 2.3810$$

4. **Solve the following set of equation using LU factorization method.**

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10$$

[2017 Fall]

Solution:

Here, in matrix form;

$$AX = B$$

where, $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

And, decompose A into LU. *i.e.*, A = LU

Let, find out U using row operation method,

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

Operation: $R_2 \to R_2 - \frac{7R_1}{5}, R_3 \to R_3 - \frac{3R_1}{5}$

$$\sim \begin{bmatrix} 5 & 2 & 1 \\ 0 & +\frac{19}{5} & -\frac{32}{5} \\ 0 & \frac{41}{5} & \frac{17}{5} \end{bmatrix}$$

Operation: $R_3 \to R_3 - \frac{41}{5} \times \frac{5}{19} R_2$

$$\sim \begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix}$$

so, $L = \begin{bmatrix} 1 & 0 & 0 \\ +\frac{7}{5} & 1 & 0 \\ +\frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix}$

Let, $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

and, $LY = B$

or, $\begin{bmatrix} 1 & 0 & 0 \\ \frac{7}{5} & 1 & 0 \\ \frac{3}{5} & \frac{41}{19} & 1 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$

Equating

i) $Y_1 = 4$

ii) $\frac{7}{5}Y_1 + Y_2 = 8 \Rightarrow \frac{7}{5} \times 4 + Y_2 = 8 \Rightarrow Y_2 = \frac{12}{5}$

iii) $\frac{3}{5}Y_1 + \frac{41}{19}Y_2 + Y_3 = 10 \Rightarrow \frac{3}{5} \times 4 + \frac{41}{19} \times \frac{12}{5} + Y_3 = 10$

or, $Y_3 = \frac{46}{19}$

Now,

$UX = Y$

or, $\begin{bmatrix} 5 & -2 & 1 \\ 0 & \frac{19}{5} & -\frac{32}{5} \\ 0 & 0 & \frac{327}{19} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{12}{5} \\ \frac{46}{19} \end{bmatrix}$

Equating, we get

i) $\frac{327}{19} z = \frac{46}{19}$

or, $z = \frac{46}{327}$

ii) $\frac{19}{5} y - \frac{32}{5} z = \frac{12}{5}$

or, $\frac{19}{5} y - \frac{32}{5} \times \frac{46}{327} = \frac{12}{5}$

or, $y = \frac{284}{327}$

iii) $5x - 2y + z = 4$

or, $5x - 2 \times \frac{284}{327} + \frac{46}{327} = 4$

or, $x = \frac{122}{109}$

5. **Solve the equation by Gauss-Jacobi method.**

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

[2017 Fall]

Solution:

Here, re-write equation in other form,

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 + 3y - 2x)$$

Now, $a_{ii} > \sum_{i \neq j} a_{ij}$ [Diagonally Dominant]

So, it is convergence.

n	x	y	z
1	0	0	0
2	0.85	-0.9	1.25
3	1.02	-0.965	1.03
4	1.0012	-1.0015	1.0032
5	1.0003	-1.0000	0.9996
6	0.9999	-1.0000	0.9999
7	1.0000	-1.0000	1.0000
8	1.0000	-1.0000	1.0000

Note: To calculate these value; in calculator

i) Type $\frac{1}{20}(17 - y + 2A) : \frac{1}{20}(-18 - 3x + A) : \frac{1}{20}(25 + 3y - 2x)$

ii) Press calc and input value.

Hence, solution of equation:

$x = 1, y = -1, z = 1$

6. **Determine the largest Eigen value and the corresponding Eigen vector of the matrix:**

$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$ **using the power method.** **[2017 Fall]**

Solution:

Let, eigen vector be $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and, according to power method,

$$AX^n = \lambda X^{n+1}$$

where, X_{n+1} is next assumption, given by equation.

Hence,

$$1 \Rightarrow AX^0 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^1 X^1$$

$$2 \Rightarrow AX^1 = \begin{bmatrix} 16.25 \\ -7.5 \\ -15 \end{bmatrix} = 16.25\begin{bmatrix} 1 \\ -0.461 \\ -0.923 \end{bmatrix} = \lambda^2 X^2$$

$$3 \Rightarrow AX^2 = \begin{bmatrix} 19.615 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.98 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^3 X^3$$

$$4 \Rightarrow AX^3 = \begin{bmatrix} 19.711 \\ -9.807 \\ -19.610 \end{bmatrix} = 19.711\begin{bmatrix} 1 \\ -0.497 \\ -0.995 \end{bmatrix} = \lambda^4 X^4$$

$$5 \Rightarrow AX^4 = \begin{bmatrix} 19.976 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.999 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^5 X^5$$

$$6 \Rightarrow AX^5 = \begin{bmatrix} 19.982 \\ -9.988 \\ -19.97 \end{bmatrix} = 19.982\begin{bmatrix} 1 \\ -0.499 \\ -0.999 \end{bmatrix} = \lambda^6 X^6$$

$$7 \Rightarrow AX^6 = \begin{bmatrix} 19.998 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^7 X^7$$

Hence, in 6th and 7th integration, eigen vector are correct up to 2 decimal.

so, Eigen value = 20

and, its corresponding eigen vector $= \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$

7. Solve the following system of equation using factorization method

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

[2018 Spring]

Solution:

Here;

$$A = LU$$

where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

Let, find out L and U using row operation method,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Operation: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{bmatrix}$$

Operation: $R_3 \to R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}$$

so, $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}$

and, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

and, let $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

Then,

$$LY = B$$

or, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 8 \end{bmatrix}$

Now; equating above matrix

i) $Y_1 = 6$

ii) $2Y_1 + Y_2 = 9$ or, $Y_2 = -3$

iii) $3Y_1 + 5Y_2 + Y_3 = 8$ or, $Y_3 = 5$

and, $UX = Y$

where, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

or, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$

Equating above matrix,

i) $18z = 5$

or, $z = \frac{5}{18}$

ii) $-y - 5z = -3$

or, $y = \frac{29}{18}$

iii) $x + 2y + 3z = 6$

or, $x + 2 \times \frac{29}{18} + 3 \times \frac{5}{18} = 6$

or, $x = \frac{35}{18}$

8. Determine the higher eigen value and its corresponding eigen vector for the following matrix using power method.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$ **[2018 Spring]**

Solution:

Here, let eigen vector be $X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now, using power method, we have,

$$\Rightarrow \quad AX^1 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix} = 10\begin{bmatrix} -0.1 \\ 0.4 \\ 1 \end{bmatrix} = \lambda^2 X^2$$

$$\Rightarrow \quad AX^2 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} -0.1 \\ 0.4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 4.5 \\ 11.7 \end{bmatrix} = 11.7\begin{bmatrix} 0.009 \\ 0.385 \\ 1 \end{bmatrix} = \lambda^3 X^3$$

$$\Rightarrow \quad AX^3 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.009 \\ 0.385 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.162 \\ 0.795 \\ 11.329 \end{bmatrix} = 11.529\begin{bmatrix} 0.014 \\ 0.416 \\ 1 \end{bmatrix} = \lambda^4 X^4$$

$$\Rightarrow \quad AX^4 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.014 \\ 0.416 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.262 \\ 4.874 \\ 11.65 \end{bmatrix} = 11.65\begin{bmatrix} 0.022 \\ 0.419 \\ 1 \end{bmatrix} = \lambda^5 X^5$$

$$\Rightarrow \quad AX^5 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.022 \\ 0.419 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.278 \\ 4.904 \\ 11.651 \end{bmatrix} = 11.651\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix} = \lambda^6 X^6$$

$$\Rightarrow \quad AX^6 = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.024 \\ 0.421 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.287 \\ 4.913 \\ 11.66 \end{bmatrix} = 11.66\begin{bmatrix} 0.025 \\ 0.421 \\ 1 \end{bmatrix} = \lambda^7 X^7$$

Hence, eigen value is 11.66 and its corresponding eigen vector is $\begin{bmatrix} 0.025 \\ 0.421 \\ 1 \end{bmatrix}$.

9. Write a program to solve a system of linear equations by Gauss sideal method. **[2018 Spring]**

Solution:

Code in C program

```
# include <stdio.h>
int main ()
{
    float a[50] [50], b[50], x[50], s
    int i, j, n, m, k;
    printf ("enter no. of equation =");
    scanf ("%d", &n);
    for (i = 0; i <= n; i++)
    {
        printf ("enter the equation :\n");
        for (j = 0; k < n; j++);
        {
            scanf ("%f", &a[i][j]);
        }
        scanf ("%f", &b[i]);
    }
    printf ("enter no. of iteration =");
    scanf ("%d", &m);
    for (i = 0; i < m; i++)
    {
        s = 0;
        for (j = 0; j < n; j++)
        {
            for (k = 0; k < n; k++)
            {
                if (k==j) continue;
                s = + a[j][k];
            }
            x[j] = s/a[j][k];
        }
    }
    printf("\n");
    for (i = 0; i < n; i++)
    {
        print("x[%d] = %f", i, x[i]);
```

```
    }
    return 0;
}.
```

10. Find the inverse of matrix using Gauss Jordan Method.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ [2018 Fall]

Solution:

Here, using Gauss Jordan method,

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Operation 1: $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -12 & -3 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Operation 2: $R_2 \to R_2 - \frac{R_3}{2}$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -13 & -4 & 1 & -\frac{1}{2} \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Operation 3: $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 16 & 5 & -1 & \frac{1}{2} \\ 0 & 1 & -13 & -4 & 1 & -\frac{1}{2} \\ 0 & -2 & -24 & -6 & 2 & 1 \end{array}\right]$$

Operation 4: $R_3 \to \frac{R_3}{24}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 16 & 5 & -1 & \frac{1}{2} \\ 0 & 1 & -13 & -4 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{2} & 0 \end{array}\right]$$

Operation 5: $R_1 \to R_1 - 16R_3$ and $R_2 \to R_2 + 13R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{4} & -\frac{1}{12} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{12} & 0 \end{array}\right]$$

Hence, inverse of given matrix is;

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{12} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{12} & 0 \end{bmatrix}$$

11. Find the largest eigen value and the corresponding eigen vector of the following square matrix using power method.

$$\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$ [2018 Fall]

Solution:

Here;

$$\text{Let } A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Let, assume eigen vector $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now,

$$AX^1 = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.75 \\ 0.5 \\ -1 \end{bmatrix} = \lambda^2 X^2$$

Here, $X^1 \neq X^2$ for the significance of 3 digits.

So, according to power method, next assumption will be X^2.

and, similarly

$$AX^2 = \begin{bmatrix} 16.25 \\ -7.5 \\ -15 \end{bmatrix} = 16.25\begin{bmatrix} 1 \\ -0.461 \\ -0.923 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = \begin{bmatrix} 19.615 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.98 \\ -0.5 \\ -1 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 19.711 \\ -9.807 \\ -19.61 \end{bmatrix} = 19.711\begin{bmatrix} 1 \\ -0.497 \\ -0.955 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = \begin{bmatrix} 19.976 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.99 \\ -0.5 \\ -1 \end{bmatrix}$$

$$= \lambda^6 X^6$$

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$$AX^6 = \begin{bmatrix} 19.982 \\ -9.88 \\ -19.97 \end{bmatrix} = 19.982 \begin{bmatrix} 1 \\ -0.499 \\ -0.999 \end{bmatrix} = \lambda^7 X^7$$

$$AX^7 = \begin{bmatrix} 19.998 \\ -10 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^8 X^8$$

$$AX^8 = \begin{bmatrix} 20 \\ -10 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^9 X^9$$

Here, X^8 and X^9 are correct up to precision of 3 decimal digits. So, eigen value = 20 and it corresponding eigen vector = $\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$.

12. Solve the following equation:

$$6x_1 - 2x_2 + x_3 = 4$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

By Gauss factorization method. [2018 Fall]

Solution:

Here, equation

$$A = \begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

Now, according to Gauss factorization method,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

and, $A = LU$

or, $LUX = B$ [∵ Let $V = UX$]

or, $LV = B$

Now, find out L and U by row operation method,

$$\begin{bmatrix} 6 & -2 & 1 \\ -2 & 7 & 2 \\ 1 & 2 & -5 \end{bmatrix}$$

Operation: $R_2 \to R_2 + \frac{R_1}{3}$ and $R_3 \to R_3 - \frac{R_1}{6}$

$$\sim \begin{bmatrix} 6 & -2 & 1 \\ 0 & \frac{19}{3} & \frac{7}{3} \\ 0 & \frac{7}{3} & -\frac{31}{6} \end{bmatrix}$$

Operation: $R_3 \to R_3 - \frac{7}{19} R_2$

so,
$$U = \begin{bmatrix} 6 & -2 & 1 \\ 0 & \frac{19}{3} & \frac{7}{3} \\ 0 & 0 & -\frac{229}{38} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ +\frac{1}{6} & +\frac{7}{19} & 1 \end{bmatrix}$$

Then,

$AX = B$

or, $LUX = B$

or, $LY = B$ [∵ Let $UX = Y$]

And,

$LY = B$

or,
$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{6} & \frac{7}{19} & 1 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$$

Equating the above matrix

i) $Y_1 = 4$

ii) $-\frac{Y_1}{3} + Y_2 = 5$ or, $Y_2 = \frac{19}{3}$

iii) $\frac{Y_1}{6} + \frac{7}{19} Y_2 + Y_3 = -1$ or, $Y_3 = -4$

and, $UX = Y$

or,
$$\begin{bmatrix} 6 & -2 & 1 \\ 0 & \frac{19}{3} & \frac{7}{3} \\ 0 & 0 & -\frac{229}{38} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \frac{19}{3} \\ -4 \end{bmatrix}$$

Solving above matrix; we have,

i) $-\frac{229}{38} z = -4$

or, $z = \frac{152}{229}$

ii) $\frac{19}{3} y + \frac{7}{3} z = \frac{19}{3}$

or, $\frac{19}{3}y + \frac{7}{3} \times \frac{152}{229} = \frac{19}{3}$

or, $y = \frac{173}{229}$

iii) $6x - 2y + z = 4$

or, $6x - 2 \times \frac{173}{229} + \frac{152}{229} = 4$

or, $x = \frac{185}{229}$

13. Using Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$ [2019 Fall]

Solution:

We have,

$$= \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

(Operate $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$)

$$= \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$\left(\text{Operate } R_2 \to \frac{1}{2}R_2 \text{ and } R_3 \to \frac{1}{2}R_3\right)$

$$= \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

(Operate $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 1R_2$)

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$\left(\text{Operate } R_1 + 3R_3,\ R_2 - \frac{3}{2}R_3 \text{ and } \frac{1}{2}R_2\right)$

$$= \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

Hence, the inverse of the matrix is $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$.

14. Determine the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ using power method.

[2019 Fall]

Solution:

Let initial approximation to required eigen vector be $X = [1, 0, 0]$. Then,

$$AX = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} = \lambda' X'$$

so, the first approximation to the eigen value is $\lambda' = 6$ and the corresponding eigen vector $X' = \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ 1 \end{bmatrix}$.

Now,

$$AX' = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{7}{3} \\ 8 \end{bmatrix} = 8\begin{bmatrix} \frac{1}{48} \\ \frac{7}{24} \\ 1 \end{bmatrix}$$

$\lambda^2 = 8$

$$X^2 = \begin{bmatrix} \frac{1}{48} \\ \frac{7}{24} \\ 1 \end{bmatrix}$$

On repeating, we have,

$$AX^2 = \begin{bmatrix} 1.14 \\ 0.25 \\ 6 \end{bmatrix} = 6\begin{bmatrix} \frac{1.14}{6} \\ \frac{0.25}{6} \\ 1 \end{bmatrix} = \lambda^3 X^3$$

$$AX^3 = \begin{bmatrix} 2.065 \\ -0.073 \\ 6.265 \end{bmatrix} = 6.265 \begin{bmatrix} \frac{2.065}{6.265} \\ -\frac{0.073}{6.265} \\ 1 \end{bmatrix} = \lambda^4 X^4$$

$$AX^4 = \begin{bmatrix} 2.36 \\ 0.27 \\ 6.9426 \end{bmatrix} = 6.9426 \begin{bmatrix} \frac{2.36}{6.9426} \\ \frac{0.27}{6.9426} \\ 1 \end{bmatrix} = \lambda^5 X^5$$

$$AX^5 = \begin{bmatrix} 2.22 \\ 0.51 \\ 7.1562 \end{bmatrix} = 7.1562 \begin{bmatrix} \frac{2.22}{7.1562} \\ \frac{0.51}{7.1562} \\ 1 \end{bmatrix} = \lambda^6 X^6$$

$$AX^6 = \begin{bmatrix} 2.0964 \\ 0.5259 \\ 7.0751 \end{bmatrix} = 7.0751 \begin{bmatrix} \frac{2.0964}{7.0751} \\ \frac{0.5259}{7.0751} \\ 1 \end{bmatrix} = \lambda^7 X^7$$

$$AX^7 = \begin{bmatrix} 2.07 \\ 0.48 \\ 7.000 \end{bmatrix} = 7 \begin{bmatrix} \frac{2.17}{7} \\ \frac{0.48}{7} \\ 1 \end{bmatrix} = \lambda^8 X^8$$

Hence, the largest eigen value is 7.000 and corresponding eigen vector is $\begin{bmatrix} 0.2957 \\ 0.0685 \\ 1 \end{bmatrix}$.

15. Write short notes on: "ill conditioned method". **[2019 Spring]**

Solution:

A linear system is said to be ill-conditioned if small changes in the coefficients of the equations result in large changes in the values of the unknowns. On the contrary, a system is well conditioned if small changes in the coefficients of the system also produce small changes in the solution. Ill-conditioning of a system is usually expected when the determinant of the coefficient matrix is small. The coefficient matrix of an ill-conditioned system is called an ill-conditioned matrix.

Method to solve "Ill-conditioned system"

Let us consider the system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \quad (1)$$

Let $x', y'\ z'$ be an approximate solution. Substituting these values on the left-hand sides, we get new values of d_1, d_2, d_3 as d_1', d_2', d_3'. So that new system is;

$$\left.\begin{aligned} a_1x' + b_1y' + c_1z' &= d_1' \\ a_2x' + b_2y' + c_2z' &= d_2' \\ a_3x' + b_3y' + c_3z' &= d_3' \end{aligned}\right\} \quad (2)$$

Subtracting each equation in (2) from the corresponding equations in (1), we obtain;

$$\left.\begin{aligned} a_1x_e + b_1y_e + c_1z_e &= k_1 \\ a_2x_e + b_2y_e + c_2z_e &= k_2 \\ a_3x_e + b_3y_e + c_3z_e &= k_3 \end{aligned}\right\} \quad (3)$$

where, $x_e = x - x'$, $y_e = y - y'$, $z_e = z - z'$

and, $k_1 = d_1 - d_1'$, $k_2 = d_2 - d_2'$, $k_3 = d_3 - d_3'$

We solve the system (3) for x_e, y_e, z_e giving $x = x' + x_e$, $y = y' + y_e$, $z = z' + z_e$ which will be better approximations for x, y, z. We can repeat the procedure for improving the accuracy.

16. Obtain the solution of the following system using the polittle LU decomposition method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

[2019 Spring]

Solution: See the solution of example no. 7

17. Determine the largest eigen value and corresponding eigen vector for the matrix using power method correct up to 3-decimal places.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

[2019 Spring]

Solution:

Here;

Let $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial guess

i) $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \lambda^{(1)} \cdot X^{(1)}$$

ii) $AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$= \lambda^{(2)} \cdot X^{(2)}$

iii) $AX^{(2)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix} = -4\begin{bmatrix} -0.75 \\ 1 \\ -0.75 \end{bmatrix}$

$= \lambda^{(3)} \cdot X^{(3)}$

iv) $AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.75 \\ 1 \\ -0.75 \end{bmatrix} = \begin{bmatrix} -2.5 \\ 3.5 \\ -2.5 \end{bmatrix} = 3.5\begin{bmatrix} -0.714 \\ 1 \\ -0.714 \end{bmatrix}$

$= \lambda^{(4)} \cdot X^{(4)}$

v) $AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.714 \\ 1 \\ -0.714 \end{bmatrix} = \begin{bmatrix} -2.428 \\ 3.428 \\ -2.428 \end{bmatrix}$

$= 3.428\begin{bmatrix} -0.7083 \\ 1 \\ -0.7083 \end{bmatrix} = \lambda^{(5)} \cdot X^{(5)}$

vi) $AX^{(5)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.7083 \\ 1 \\ -0.7083 \end{bmatrix} = \begin{bmatrix} -2.4166 \\ 3.4166 \\ -2.4166 \end{bmatrix}$

$= 3.4166\begin{bmatrix} -0.7073 \\ 1 \\ -0.7073 \end{bmatrix} = \lambda^{(6)} \cdot X^{(6)}$

vii) $AX^{(6)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.7073 \\ 1 \\ -0.7073 \end{bmatrix} = \begin{bmatrix} -2.4146 \\ 3.4146 \\ -2.4146 \end{bmatrix}$

$= 3.4146\begin{bmatrix} -0.7071 \\ 1 \\ -0.7071 \end{bmatrix} = \lambda^{(7)} \cdot X^{(7)}$

$\therefore$ $X^{(7)} - X^{(6)}$ is negligible.

$\lambda^{(7)} = 3.4146$ is largest eigen value.

$X^{(8)} = \begin{bmatrix} -0.7071 \\ 1 \\ -0.7071 \end{bmatrix}$ is largest eigen vector.

NOTE:

If we were to find smallest eigen value for

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Let initial eigen vector is $x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

i) $AX = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$= \lambda^{(1)} \cdot X^{(1)}$

ii) $AX^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -4 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1.5 \\ -1 \end{pmatrix}$

$= \lambda^{(2)} \cdot X^{(2)}$

iii) $AX^{(2)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.5 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.5 \\ 5 \\ -3.5 \end{pmatrix} = 3.5\begin{pmatrix} -1 \\ 1.43 \\ -1 \end{pmatrix}$

$= \lambda^{(3)} \cdot X^{(3)}$

iv) $AX^{(3)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.43 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.43 \\ 4.86 \\ -3.43 \end{pmatrix} = 3.43\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix}$

$= \lambda^{(4)} \cdot X^{(4)}$

v) $AX^{(4)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix} = \begin{pmatrix} -3.42 \\ 4.84 \\ -3.42 \end{pmatrix}$

$= 3.42\begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix} = \lambda^{(5)} \cdot X^{(5)}$

Here,

$X^{(5)} - X^{(4)}$ is nearly negligible.

$\therefore$ Smallest eigen value = 3.42

Smallest eigen vector $= \begin{pmatrix} -1 \\ 1.42 \\ -1 \end{pmatrix}$.

18. **Solve the following set of equations by using LU-decomposition method.**

$$3x + 2y + 7z = 32$$
$$2x + 3y + z = 40$$
$$3x + 4y + z = 56$$

[2020 Fall]

Solution:

Here, writing the given equations in matrix form, we get,

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

The above equation is represented in the form AX = B.

where, $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$

Let, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now,

$$LU = A$$

or, $$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

or, $$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

On equating corresponding elements; we get,

a) $u_{11} = 3,\ u_{12} = 2,\ u_{13} = 7$

b) $u_{11}l_{21} = 2 \therefore l_{21} = \frac{2}{3}$

c) $l_{21}u_{12} + u_{22} = 3 \Rightarrow u_{22} = 3 - \left(\frac{2}{3} \times 2\right) = \frac{5}{3}$

d) $l_{21}u_{13} + u_{23} = 1 \Rightarrow u_{23} = 1 - \left(\frac{2}{3} \times 7\right) = \frac{-11}{3}$

e) $l_{31}u_{11} = 3 \Rightarrow l_{31} = 1$

f) $l_{31}u_{12} + l_{32}u_{22} = 2 \Rightarrow l_{32} = \frac{4 - (1 \times 2)}{\left(\frac{5}{3}\right)} = \frac{6}{5}$

g) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$$\Rightarrow \quad u_{33} = 1 - \left[1 \times 7 + \frac{6}{5} \times \left(-\frac{11}{3}\right)\right] = -\frac{8}{5}$$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix}$

Using, LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ 1 & \frac{6}{5} & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ \left(\frac{2}{3}\right)v_1 + v_2 \\ v_1 + \left(\frac{6}{5}\right)v_2 + v_3 \end{bmatrix} = \begin{bmatrix} 32 \\ 40 \\ 56 \end{bmatrix}$$

a) $v_1 = 32$

b) $\frac{2}{3} \times 32 + v_2 = 40 \quad \therefore \quad v_2 = \frac{56}{3}$

c) $32 + \frac{6}{5} \times \left(\frac{56}{3}\right) + v_3 = 56 \quad \therefore \quad v_3 = \frac{8}{5}$

Again,

$$UX = V$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & \frac{5}{3} & -\frac{11}{3} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 32 \\ \frac{56}{3} \\ \frac{8}{5} \end{bmatrix}$$

or, $$\begin{bmatrix} 3x_1 + 2x_2 + 7x_3 \\ \left(\frac{5}{3}\right)x_2 - \left(\frac{11}{3}\right)x_3 \\ -\left(\frac{8}{5}\right)x_3 \end{bmatrix} = \begin{bmatrix} 32 \\ \frac{56}{3} \\ \frac{8}{5} \end{bmatrix}$$

Using backward substitution

a) $-\frac{8}{5}x_3 = \frac{8}{5}$

$\therefore \quad x_3 = -1$

b) $\frac{5}{3}x_2 - \frac{11}{3}x_3 = \frac{56}{3}$

or, $\frac{5}{3}x_2 - \frac{11}{3}(-1) = \frac{56}{3}$

$\therefore \quad x_2 = 9$

c) $3x_1 + 2x_2 + 7x_3 = 32$

or, $3x_1 + 2 \times 9 + 7 \times (-1) = 32$

$\therefore \quad x_1 = 7$

$\therefore \quad x_1 = 7, x_2 = 9$ and $x_3 = -1$

19. Determine the largest eigen value and the corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}$$ **using the power method.** **[2020 Fall]**

Solution:

Here,

Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial guess.

i) $$Ax = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 13 \end{bmatrix} = 13\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix}$$
$$= \lambda^{(1)} \cdot x^{(1)}$$

ii) $$Ax^{(1)} = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.692 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.692 \\ 11.768 \\ 11.768 \end{bmatrix} = 11.768\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix}$$
$$= \lambda^{(2)} \cdot x^{(2)}$$

iii) $$Ax^{(2)} = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.738 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.738 \\ 11.952 \\ 11.952 \end{bmatrix} = 11.952\begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix}$$
$$= \lambda^{(3)} \cdot x^{(3)}$$

iv) $$Ax^{(3)} = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.731 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.731 \\ 11.924 \\ 11.924 \end{bmatrix} = 11.924\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$
$$= \lambda^{(4)} \cdot x^{(4)}$$

v) $$Ax^{(4)} = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix}\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8.732 \\ 11.928 \\ 11.928 \end{bmatrix} = 11.928\begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$$
$$= \lambda^{(5)} \cdot x^{(5)}$$

$x^{(5)} - x^{(4)}$ is negligible.

$\therefore$ $\lambda^{(5)} = 11.928$ is largest eigen value.

$x^{(5)} = \begin{bmatrix} 0.732 \\ 1 \\ 1 \end{bmatrix}$ is largest eigen vector.

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. **Solve the following set of equations using Gauss Siedel method.**

$$3x_1 + 6x_2 + x_3 = 16$$
$$2x_1 + 4x_2 + 3x_3 = 13$$
$$x_1 + 3x_2 + 2x_3 = 9$$

[2015]

Solution:

Here, the given system of equations is not diagonally dominant. So, solving them as they are:

$$x_1 = \frac{13 - 4x_2 - 3x_3}{2}$$

$$x_2 = \frac{16 - 3x_1 - x_3}{6}$$

$$x_3 = \frac{9 - x_1 - 3x_2}{2}$$

Let $((x_1)_0, (x_2)_0, (x_3)_0) = (0, 0, 0)$ be the initial guess.

$(x_2)_n$	$(x_3)_n$	$(x_1)_{n+1} = \frac{[13 - 4(x_2)_n - 3(x_3)_n]}{2}$	$(x_2)_{n+1} = \frac{[16 - 3(x_1)_n - (x_3)_n]}{6}$	$(x_3)_{n+1} = \frac{[9 - (x_1)_n - 3(x_2)_n]}{2}$
0	0	6.5	-0.58334	2.125
-0.58334	2.125	4.47916	0.07291	2.15104
0.07291	2.15104	3.12760	0.74436	1.81966
0.74436	1.81966	2.28179	1.2225	1.52536
1.2225	1.52536	1.76696	1.52895	1.32308
1.52895	1.32308	1.45746	1.71742	1.19513
1.71742	1.19513	1.27245	1.83125	1.11689
1.83125	1.11689	1.16215	1.89944	1.0697
1.89944	1.0697	1.0964	1.94013	1.0416
1.94013	1.0416	1.0574	1.9643	1.0247
1.9643	1.0247	1.0341	1.9788	1.0147
1.9788	1.0147	1.0203	1.98739	1.0087
1.98739	1.0087	1.0120	1.9925	1.0062
A	B	$C = \frac{13 - 4A - 3B}{2}$	$D = \frac{16 - 3C - B}{6}$	$E = \frac{9 - C - 3D}{2}$

$x_1 = 1,$

$x_2 = 2$

and, $x_3 = 1.$

Calculator formula

$$A : B : C = \frac{13 - 4A - 3B}{2} : D = \frac{16 - 3C - B}{6} : E = \frac{9 - C - 3D}{2}$$

$$A = D : B = E$$

2. Find the largest eigen value and corresponding eigen vector of the following matrix, using power method.

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}$$ [2015]

Solution:

Here;

Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be the initial guess and given matrix be denoted as;

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}$$

i) $$Ax = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 8 \end{bmatrix} = 12\begin{bmatrix} 0.5 \\ 1 \\ 0.667 \end{bmatrix} = \lambda^1 X^1$$

ii) $$AX^1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \\ 0.667 \end{bmatrix} = \begin{bmatrix} 4.334 \\ 9.335 \\ 6.667 \end{bmatrix} = 9.335\begin{bmatrix} 0.464 \\ 1 \\ 0.7141 \end{bmatrix} = \lambda^2 X^2$$

iii) $$AX^2 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0.464 \\ 1 \\ 0.7141 \end{bmatrix} = \begin{bmatrix} 4.3562 \\ 9.4985 \\ 6.6421 \end{bmatrix} = 9.4985\begin{bmatrix} 0.4586 \\ 1 \\ 0.6992 \end{bmatrix} = \lambda^3 X^3$$

iv) $$AX^3 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0.4586 \\ 1 \\ 0.6992 \end{bmatrix} = \begin{bmatrix} 4.3156 \\ 9.4132 \\ 6.6164 \end{bmatrix} = 9.4132\begin{bmatrix} 0.4543 \\ 1 \\ 0.6965 \end{bmatrix} = \lambda^4 X^4$$

v) $$AX^4 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0.4543 \\ 1 \\ 0.6965 \end{bmatrix} = \begin{bmatrix} 4.3016 \\ 9.3911 \\ 6.6051 \end{bmatrix} = 9.3911\begin{bmatrix} 0.4528 \\ 1 \\ 0.6953 \end{bmatrix} = \lambda^5 X^5$$

v) $$AX^5 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 5 \\ 2 & 5 & 1 \end{bmatrix}\begin{bmatrix} 0.4528 \\ 1 \\ 0.6953 \end{bmatrix} = \begin{bmatrix} 4.2961 \\ 9.3821 \\ 6.6009 \end{bmatrix} = 9.3821\begin{bmatrix} 0.4523 \\ 1 \\ 0.6949 \end{bmatrix} = \lambda^6 X^6$$

$X^6 - X^5$ is negligible

$\therefore$ $\lambda^6 = 9.3821$ is largest eigen value

$X^8 = \begin{bmatrix} 0.4528 \\ 1 \\ 0.6949 \end{bmatrix}$ is largest eigen vector

3. Solve the following equations by Gauss-Siedel method.

$$2x - 7y - 10z = -17$$
$$5x + y + 3z = 14$$
$$x + 10y + 9z = 7$$ [2016]

Solution:

Here, writing the equations in diagonally dominant from.

$5x + y + 3z = 14$ (1) $|5| > |1| + |3|$

$x + 10y + 9z = 7$ (2) $|10| \geq |1| + |9|$

$2x - 7y - 10z = -17$ (3) $|-10| > |2| + |-7|$

The equations can also be written in the following form:

$$x = \frac{14 - y - 3z}{5}$$

$$y = \frac{7 - x - 9z}{10}$$

$$z = \frac{-17 - 2x + 7y}{-10}$$

y_n	z_n	$x_{n+1} = \frac{(14 - y_n - 3z_n)}{5}$	$y_{n+1} = \frac{(7 - x_n - 9z_n)}{10}$	$(x_3)_{n+1} = \frac{(-17 - 2x_n + 7y_n)}{-10}$
0	0	2.8	0.42	1.966
0.42	1.966	1.5364	-1.22304	2.8634
-1.2230	2.8634	1.3265	-2.009	3.3721
-2.009	3.3721	1.1786	-2.4527	3.6526
-2.4527	3.6526	1.0989	-2.69730	3.8079
-2.6973	3.8079	1.0547	-2.8326	3.8937
-2.8326	3.8937	1.0302	-2.9074	3.9412
-2.9074	3.9412	1.0167	-2.9487	3.9674
-2.9487	3.9674	1.0092	-2.9716	3.9820
-2.9716	3.9820	1.0051	-2.9843	3.990
-2.9843	3.990	1.0028	-2.991	3.994
A	B	$C = \frac{14 - A - 3B}{5}$	$D = \frac{7 - C - 9B}{10}$	$E = \frac{-17 - 2C + 7D}{-10}$

$\therefore$ $x = 1$, $y = -3$ and $z = 4$.

Calculator formula

$$A : B : C = \frac{14 - A - 3B}{5} : D = \frac{7 - C - 9B}{10} : E = \frac{-17 - 2C + 7D}{-10} :$$

$$A = D : B = E$$

4. Solve the following system of equations by using Dolittle LU decomposition method.

$$x + y - 2z = 3$$
$$4x + 2y + z = 5$$
$$3x - y + 3z = 8$$ [2016]

Solution:

Here, writing the given equations in matrix form; we get,

$$\begin{bmatrix} 1 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

The above equation is represented in the from AX = B

where, $A = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$

Let, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

and, $u = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Now,

$$LU = A$$

OR, $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 3 \end{bmatrix}$

or, $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 2 & 1 \\ 3 & -1 & 3 \end{bmatrix}$

Equating corresponding elements, we get

a) $u_{11} = 1, u_{12} = 1, u_{13} = -2$

b) $u_{11}\, l_{21} = 4 \Rightarrow l_{21} = \frac{4}{1} = 4$

c) $l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = 2 - (4 \times 1) = -2$

d) $l_{21}u_{13} + u_{23} = 1 \Rightarrow u_{23} = 1 - (4 \times (-2)) = 9$

e) $l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{1} = 3$

f) $l_{31}u_{12} + l_{32}u_{22} = -1 \Rightarrow l_{32} = \frac{-1 - (3 \times 1)}{-2} = 2$

g) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow u_{33} = 3 - [3 \times (-2) + 2 \times 9] = -9$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

and, $U = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 9 \\ 0 & 0 & -9 \end{bmatrix}$

Using LV = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

a) $V_1 = 3$

b) $4V_1 + V_2 = 5$

$\therefore V_2 = -7$

c) $3V_1 + 2V_2 + V_3 = 8$

$\therefore V_3 = 8 - 3 \times 3 - 2 \times (-7) = 13$

Again,

$$UX = V$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 9 \\ 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 13 \end{bmatrix}$$

Using backward substitution

a) $-9x_3 = 13$

$\therefore x_3 = -1.445$

b) $-2x_2 + 9 \times (-1.445) = -7$

$\therefore x_2 = -3$

c) $x_1 + (-3) - 2 \times (-1.445) = 3 \Rightarrow x_1 = 3.11$

$\therefore x_1 = 3.11, x_2 = -3, x_3 = -1.445$

5. **Solve the following system of equations using Gauss-Jordan method correct up to 3 decimal places.**

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

[2017]

Solution:

Here, the given equations are:

$x + y + 54z = 110$ (1)

$6x + 15y + 2z = 72$ (2)

$27x + 6y - z = 85$ (3)

Writing the equations in augmented form;

$$\begin{bmatrix} 1 & 1 & 54 & : & 110 \\ 6 & 15 & 2 & : & 72 \\ 27 & 6 & -1 & : & 85 \end{bmatrix}$$

Applying $R_2 \to R_2 - 6R_1$ and $R_3 \to R_3 - 27R_1$

$$\sim \begin{bmatrix} 1 & 1 & 54 & : & 110 \\ 0 & 9 & -322 & : & -588 \\ 0 & -21 & -1459 & : & -2885 \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{9}$

$$\sim \begin{bmatrix} 1 & 1 & 54 & : & 110 \\ 0 & 1 & -35.77 & : & -65.33 \\ 0 & -21 & -1459 & : & -2885 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_2 + 21R_1$

$$\sim \begin{bmatrix} 1 & 0 & 89.77 & : & 175.33 \\ 0 & 1 & -35.77 & : & -65.33 \\ 0 & 0 & -2210.17 & : & -4256.93 \end{bmatrix}$$

Applying $R_3 \to \dfrac{R_3}{-2210.17}$

$$\sim \begin{bmatrix} 1 & 0 & 89.77 & : & 175.33 \\ 0 & 1 & -35.77 & : & -65.33 \\ 0 & 0 & 1 & : & 1.926 \end{bmatrix}$$

Applying $R_1 \to R_1 - 89.77R_3$ and $R_2 \to R_2 + 35.77R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2.433 \\ 0 & 1 & 0 & : & 3.563 \\ 0 & 0 & 1 & : & 1.926 \end{bmatrix}$$

$\therefore$ $x = 2.433$

$y = 3.563$

$z = 1.926$

6. Solve the following system:

$$8x + 3y + 4z = 5$$
$$3x + 10y + 5z = 6$$
$$4x + 5y + 16z = 7$$

Using Gauss Jacobi iteration method. **[2018]**

Solution:

Here, writing the equations in diagonally dominant order:

$8x + 3y + 4z = 5$ (1) $|8| > |3| + |4|$

$3x + 10y + 5z = 6$ (2) $|10| > |3| + |5|$

$4x + 5y + 16z = 7$ (3) $|16| > |4| + |5|$

The equations can also be written in following form:

$$x = \frac{5 - 3y - 4z}{8}$$

$$y = \frac{6 - 3x - 5z}{10}$$

$$z = \frac{7 - 4x - 5y}{16}$$

Let initial guess $(x_0, y_0, z_0) = (0, 0, 0)$

x_n	y_n	z_n	$x_{n+1} = \frac{5 - 3y_n - 4z_n}{8}$	$y_{n+1} = \frac{6 - 3x_n - 5z_n}{10}$	$z_{n+1} = \frac{7 - 4x_n - 5y_n}{16}$
0	0	0	0.625	0.6	0.4375
0.625	0.6	0.4375	0.1812	0.1937	0.0937
0.1812	0.1937	0.0937	0.5054	0.4987	0.3316
0.5054	0.4987	0.3316	0.2721	0.2825	0.1552
0.2721	0.2825	0.1552	0.4414	0.4407	0.2871
0.4414	0.4407	0.2811	0.3191	0.3269	0.1894
0.3191	0.3269	0.1894	0.4076	0.4095	0.2555

0.4076	0.4095	0.2555	0.3436	0.3499	0.2076
0.3436	0.3499	0.2076	0.3899	0.3931	0.2422
0.3899	0.3931	0.2422	0.3564	0.3619	0.2171
0.3564	0.3619	0.2171	0.3807	0.3844	0.2323
0.3807	0.3844	0.2323	0.3631	0.3681	0.222
0.3631	0.3681	0.222	0.3758	0.3799	0.2316
0.3758	0.3799	0.2316	0.3666	0.3714	0.2249
0.3666	0.3714	0.2249	0.3733	0.3776	0.2297
0.3733	0.3776	0.2257	0.3685	0.3731	0.2261
0.3685	0.3731	0.2261	0.372	0.376	0.228

Calculator formula

$$A : B : C : D = \frac{5 - 3B - 4C}{8} : E = \frac{6 - 3A - 5C}{10} : F = \frac{7 - 4A - 5B}{16} :$$

$$A = D : B = E : C = F$$

$\therefore$ $x = 0.372$

$y = 0.376$

$z = 0.228$

7. Solve the system using Gauss-Jordan method.

$$2x + 4y - 6z = -8$$
$$x + 3y + z = 10$$
$$2x - 4y - 2z = -12$$

[2019]

Solution:

Here, the given equations are;

$2x + 4y - 6z = -8$ (1)

$x + 3y + z = 10$ (2)

$2x - 4y - 2z = -12$ (3)

Step 1: To eliminate 'x' from the equation (2) and (3)

i) Operate equation (2) × equation (2) – equation (1)

$2y + 8z = 28$ (4)

ii) Operate equation (1) – equation (3)

$8y - 4z = 4$ (5)

Step 2: To eliminate 'z' from equation (4) and (5)

Operate equation (4) + 2 × equation (5)

$18y = 36$

$\therefore$ $y = 2$

Putting y in the equation (5) to get 'z'; we have,

$8 \times 2 - 4 \times z = 4$

$\therefore$ $z = 3$

Putting 'y' and 'z' in the equation (1) to get 'x'

$$2x + 4 \times 2 - 6 \times 3 = -8$$

or, $2x + 8 - 18 = -8$

$\therefore$ $x = 1$

$\therefore$ $x = 1, y = 2$ and $z = 3$

NOTE

Gauss Jordan method is a modification of Gauss-elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately reducing the system to a diagonal matrix form *i.e.*, each equations involving only one unknowns. From these equations x, y, z can be evaluated easily.

Chapter 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION

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Solution of ordinary differential equation means finding an explicit expression for 'y' in terms of a finite numbers of elementary functions of 'x'.

Suppose the first order differential;

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

All numerical technique for selling differential equation involves a series of estimate of y(x) starting from the given conditions. There are two basic approaches to estimate the values of y(x). They are known as one-step method and multistep methods.

In step by step method, we start from $y(x_0) = y_0$ and proceed with the approximate value of 'y' of the solution of 'y' of equation (1) at;

$$x = x_1 = x_0 + h$$

In second step, we compute an approximate value y_2 of the solution at $x = x_2 = x_0 + 2h$ etc.

Hence, there are 5 methods for solving the differential equation, numerically.

i) Taylor series method
ii) Picard's method
iii) Euler's Method
iv) Runga-kutta method

v) Heun's methods

i) Taylor Series Method

Suppose $y = f(x)$ be a solution of the equation $\frac{dy}{dx} = f(x, y)$; (1)

with an initial approximation $y(x_0) = y_0$. Expanding by Taylor's series about the point x_0; we get,

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} \times f'(x) + \frac{(x-x_0)^2}{2!} \times f''(x_0) + \frac{(x-x_0)^3}{3!} \times f''(x) + \ldots\ldots$$

This equation can be written as;

$$y(x) = y_0 + \frac{(x-x_0)}{1!} \times y_0' + \frac{(x-x_0)^2}{2!} \times y_0'' + \frac{(x-x_0)^3}{3!} \times y_0''' + \ldots\ldots \quad (2)$$

When x_0 and y_0 denote the initial values of 'x' and 'y' put $x = x_1 = x_0 + h$

Then equation (2) becomes;

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \ldots\ldots$$

Similarly;

$$y(x_0 + 2h) = y_1 + \frac{h}{1!} \times y_1' + \frac{h^2}{2!} \times y_1'' + \frac{h^3}{3!} \times y_1''' + \frac{h^4}{4!} \times y_1^{iv} + \ldots\ldots$$

This formula is known as Taylor's series formula.

Example 1

Given that $\frac{dy}{dx} = x + y, y(0) = 1$ using Taylor series method compute y(0.1)

Solution:

Here,

$$y' = \frac{dy}{dx} = x + y$$

$$y(0) = 1$$

i.e., $x_0 = 0$

and, $y_0 = 1$

Taking $h = 0.1$ then,

$$x_1 = x_0 + h$$

We know, Taylor's series expansion,

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \ldots\ldots \quad (1)$$

Now,

$$y' = x + y$$

or, $y_0' = x_0 + y_0 = 0 + 1 = 1$

$$y'' = 1 + y'$$

or, $y_0'' = 1 + y_0' = 1 + 1 = 2$

$$y''' = y''$$

or, $y_0''' = y_0'' = 2$

$$y^{iv} = y'''$$

or, $y_0^{iiv} = y_0''' = 2$

Substituting these values in equation (1); we get,

$$y(0 + 0.1) = 1 + \frac{0.1}{1!} \times 1 + \frac{(0.1)^2}{2!} \times 2 + \frac{(0.1)^3}{3!} \times 2 + \frac{(0.1)^4}{4!} \times 2 + \ldots\ldots$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12}$$

$$y(0.1) = 1.11034$$

Example 2

Given that $y^i = 3x + \frac{y}{2}$ and $y(0) = 1$. Find y(0.1) and y(0.2) taking h = 0.1 using Taylor series method.

Solution:

Here,

$$y' = 3x + \frac{y}{2} \quad (1)$$

$$y(0) = 1$$

$$x_0 = 0$$

$$y_0 = 1$$

Taking $h = 0.1$; then,

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

We know that; Taylor's series expansion,

$$y(x_0 + h) = y_0 + \frac{h}{1!} \times y_0' + \frac{h^2}{2!} \times y_0'' + \frac{h^3}{3!} \times y_0''' + \frac{h^4}{4!} \times y_0^{iv} + \ldots\ldots \quad (2)$$

$$y' = 3x + \frac{y}{2}$$

or, $y_0' = 3x_0 + \frac{y_0}{2} = 3 \times 0 + \frac{1}{2} = 0.5$

$$y'' = 3 + \frac{y'}{2}$$

or, $y_0'' = 3 + \frac{y_0'}{2} = 3 + \frac{0.5}{2} = 3.25$

$$y''' = \frac{y''}{2}$$

or, $y_0''' = \frac{3.25}{2} = 1.625$

$$y^{iv} = \frac{y'''}{2}$$

or, $y_0^{iiv} = \frac{y_0'''}{2} = \frac{1.625}{2} = 0.8125$

Substituting there values in (2); we get,

$$y(0+0.1) = 1 + \frac{0.1}{1} \times 0.5 + \frac{(0.1)^2}{2} \times 3.25 + \frac{(0.1)^3}{6} \times 1.625 + \frac{(0.1)^4}{24} \times 0.8125$$

$$y(0.1) = 1.0665$$

Again for y(0.2); we have,

$x_2 = 0.2$

$x_2 = x_1 + h$

$0.2 = 0.1 + h$

$h = 0.1$

$y^i = 3x + \frac{y}{2}$

or, $y_1' = 3x_1 + \frac{y_1}{2} = 3 \times 0.1 + \frac{1.0665}{2} = 0.833$

$y'' = 3 + \frac{y'}{2}$

or, $y_1'' = 3 + \frac{y_1'}{2} = 3 + \frac{0.833}{2} = 3.4165$

$y''' = \frac{y''}{2}$

or, $y_1''' = \frac{y_1''}{2} = \frac{3.4165}{2} = 1.70825$

$y^{iv} = \frac{y'''}{2}$

or, $y_1^{iiv} = \frac{y_1'''}{2} = \frac{1.7083}{2} = 0.8542$

Now,

$$y(0.1+0.1) = y_1 + \frac{h}{1!} \times y_1' + \frac{h^2}{2!} \times y_1'' + \frac{h^3}{3!} \times y_1''' + \frac{h^4}{4!} \times y_1^{iv} + \ldots\ldots$$

$$y(0.2) = 1.0665 + 0.1 \times 0.8333 + \frac{(0.1)^2}{2} \times 3.4166 + \frac{(0.1)^3}{6} \times 1.7083 + \frac{(0.1)^4}{24} \times 0.8542$$

$$y(0.2) = 1.16720$$

ii) Picard's Method

Suppose the given differential equation is of the form;

$$\frac{dy}{dx} = f(x, y) \quad (1)$$

Let us solve equation (1) with the given initial condition $y(x_0) = y_0$

The given differential equation can be written as

$$dy = f(x, y)\, dx \quad (2)$$

Integrating equation (2) form $x_0 + x$ with respect to 'x'

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\, dx$$

$$y - y_0 = \int_{x_0}^{x} f(x, y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \quad (3)$$

For first approximation we replace 'y' by y_0 in equation (3); we get,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For second approximation, we replace 'y' by y_1; we get,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Similarly, for other approximation we make a general form

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

We continue this process until we get two successive approximation value equal.

Example 3

Use Picard's Method, find 'y' when x = 0.2. Given that y(0) = 1 and $\frac{dx}{dy} = x - y$

Solution:

Here,

$$\frac{dy}{dx} = x - y = f(x, y)$$

$y(0) = 1$

i.e., $x_0 = 0$

and, $y_0 = 1$

Using Picard's formula; we have,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

First approximation;

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_1 = y_0 + \int_{0}^{x} (x - y_0)\, dx$$

$$y_1 = y_0 + \int_0^x (x-1)\,dx$$

$$y_1 = y_0 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$y_1 = y_0 + \frac{x^2}{2} - x$$

$$y_1 = 1 + \frac{x^2}{2} - x$$

Second approximation;

$$y_2 = y_0 + \int_0^x f(x, y_1)dx$$

$$y_2 = 1 + \int_0^x f(x, y_1)dx$$

$$y_2 = 1 + \int_0^x \left(x - 1 - \frac{x^2}{2} + x\right)dx$$

$$y_2 = 1 + \frac{x^2}{2} - x - \frac{x^3}{6} + \frac{x^2}{2}$$

$$y_2 = 1 - x + x^2 - \frac{x^3}{6}$$

At $x = 0.2$;

$$y_2 = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{6} = 0.83867$$

Third approximation;

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2)dx$$

$$y_3 = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6}\right)dx$$

[We use the function y_2, not the numerical value of y_2]

$$y_3 = 1 + \frac{x^2}{2} - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_3 = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24}$$

at $x = 0.2$

$$y_3 = 1 - 0.2 + (0.2)^2 - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24} = 0.8374$$

Fourth approximation;

$$y_4 = y_0 + \int_{x_0}^x f(x, y_3)dx$$

$$y_4 = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{24}\right)dx$$

$$y_4 = 1 + \frac{x^2}{2} - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

At $x = 0.2$;

$$y_4 = 1 + \frac{(0.2)^2}{2} - 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} - \frac{(0.2)^5}{120}$$

$$y_4 = 0.8375$$

$\therefore$ $y_3 = y_4$ is equal up to 3 decimal place

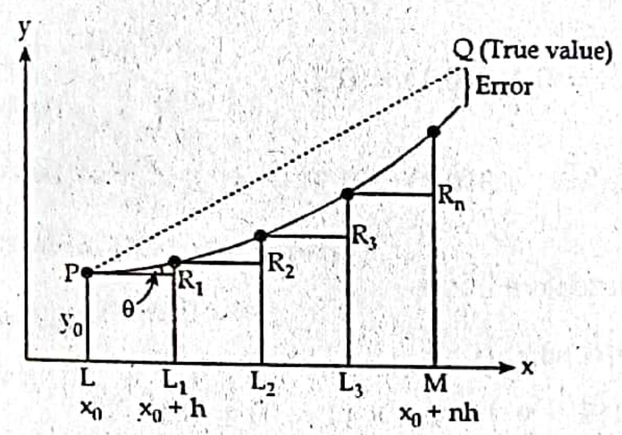
$\therefore$ $y(0.2) = 0.837$

6.1 EULER'S METHOD: RK–1

Consider the equation,

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in the figure.



We have to find the ordinate of any other point Q on this curve.

Let us divide LM into n–sub–intervals each of width h at $L_1, L_2, \ldots\ldots\ldots$ so that h is quite small.

Let the interval LL_1, we approximate the curve by tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$ then,

$$y_1 = L_1P_1 + R_1P_1$$

$$= y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P$$

$$= y_0 + h\,f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then,

$$y_2 = y_1 + h\,f(x_0 + 2h, y_2)$$

Repeating n–times;

$$y_n = y_{n-1} + h\,f(x_0 + \overline{n-1}h, y_{n-1})$$

Example 4

Solve the equation $\frac{dy}{dx} = 1 - y$ with initial condition $y(0) = 0$ using Euler's method and tabulate the solution at $x = 0.1, 0.2$

Solution:

Given that;

$$\frac{dy}{dx} = 1 - y = f(x, y)$$

$$y(0) = 0$$

i.e., $x_0 = 0$

and, $y_0 = 0$

Taking $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = ?$$

$$x_2 = x_0 + 2h = 0 + 2(0.1) = 0.2$$

$$y_2 = ?$$

$$x_3 = x_0 + 3h = 0 + 3(0.1) = 0.3$$

$$y_3 = ?$$

From Euler's method; we have,

$$y_1 = y_0 + h[f(x_0, y_0)]$$

$$y_1 = 0 + 0.1(1 - y_0) = 0.1 \times (1 - 0) = 0.1$$

$$y_2 = y_1 + h[f(x_1, y_1)] = 0.1 + 0.1 \times (1 - 0.1) = 0.19$$

$$y_3 = y_2 + h[f(x_2, y_2)] = 0.19 + 0.1 \times (1 - 0.19) = 0.271$$

Tabulating result

x	0	0.1	0.2	0.3
y	0	0.1	0.19	0.271

Example 5

Solve $\sin x + \cos y$, $y(0) = \pi$ in the range $0 \le x \le 2$ by dividing the interval into 5 sub intervals using Euler's method.

Solution:

x	y	$\frac{dy}{dx} = \sin x + \cos y$	Old $y_0 + 0.4\left(\frac{dy}{dx}\right)$ = New y
0	π	$(\sin 0 + \cos \pi) = -1$	$\pi + 0.4 \times (-1) = 2.7415$
0.4	2.7415	$\sin(0.4) + \cos(2.7415) = -0.5316$	$2.7415 + 0.4(-0.5316) = 2.5288$
0.8	2.5288	$\sin(0.8) + \cos(2.5288) = -0.10068$	$2.5288 + 0.4(-0.10068) = 2.4885$
1.2	2.4885	$\sin(1.2) + \cos(2.4885) = 0.137815$	$2.4885 + 0.4(0.137815) = 2.5436$
1.6	2.5436	$\sin(1.6) + \cos(2.5436) = 0.173091$	$2.5436 + 0.4(0.173091) = 2.61283$
2.0	2.61283	$\sin(2.0) + \cos(2.612863) = 0.04586$	$2.61283 + 0.4(0.04586) = 2.6313$

$\therefore$ Required approximate value of $y = 2.6313$.

6.2 RUNGE–KUTTA METHOD

This method refers to a family of one-step method used for numerical solution of initial value problems. They all based on the general form of extrapolation equation.

a) **First order R–K method**

$$y_1 = y_0 + h\,f(x_0, y_0);$$ which is Euler's method.

b) **Second order R–K2 method**

This method is a **Heun's method** or **modified Euler's method**

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where, $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

c) **Third order R–K method (R–K3)**

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where, $k_1 = h\,f(x_0, y_0)$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k}{2}\right)$$

$$k_3 = h\,f(x_0 + h, y_0 + 2k_2 - k_1)$$

d) **Fourth order R–K method**

This method is commonly used and is often referred to as **Runga-Kutta method** classically forth order **R–K method.**

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $k_1 = h\,f(x_0, y_0)$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Similarly, for second interval

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where, $k_1 = h\,f(x_1, y_1)$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

Example 6

Apply R-K forth method to find an approximate value of 'y' when x = 0.2.

Solution:

Given that;

$$\frac{dy}{dx} = x + y$$

$$y(0) = 1$$

i.e., At $x_0 = 0$;

$$y_0 = 1$$

$$h = 0.2$$

$\therefore$ $x_1 = x_0 + h = 0 + 0.2 = 0.2$

$$y_1 = ?$$

From R-K forth order method; we have,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_0, y_0) = 0.2 \times (0 + 1) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= h f(0.1, 1.1)$$

$$= 0.2 \times (0.1 + 1.1) = 0.24$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$$= h f(0.1, 1.12)$$

$$= 0.2 \times (0.1 + 1.12)$$

$$= 0.244$$

and, $k_4 = h f(x_0 + h, y_0 + k_3)$

$$= h f(0 + 0.2, 1 + 0.244)$$

$$= h f(0.2, 1.244)$$

$$= 0.2 \times (0.2 + 1.244)$$

$$= 0.2888$$

Now,

$$y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.2 + 2 \times 0.24 + 2 \times 0.244 + 0.2888)$$

$$= 1.2428$$

6.3 SHOOTING METHOD

Solving Boundary Value Problem Using Shooting Method

Shooting method is used for solving the boundary value problem. In this method, the given boundary value problem is first converted into an equivalent initial value problem and then solved.

Consider the equation

$$y''' = f(x, y, y')$$

$$y(a) = A$$

and, $y(b) = B$

$$y' = z$$

Then,

$$y' = z$$

$$z' = f(x, y, z)$$

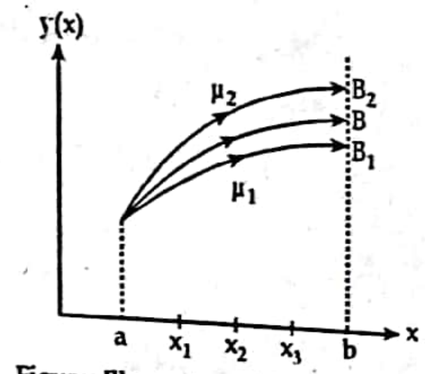


Figure: Illustration of shooting method

In order to solve this set as an initial value problem. We need two test conditions at x = a and therefore, required another condition for 'z' at x = a

Let us assume that $z(a) = \mu_1$ where, μ_1 is a queue.

Thus,

$$y^i = z$$

$$y(a) = A$$

$$z^i = f(x, y, z)$$

$$z(a) = \mu_1 \qquad (1)$$

Equation (1) can be solved by Heun's or R-K fourth method.

Example 7

Using shooting method solve the following equation $\frac{d^2y}{dx^2} = 6x$, y(1) = 2 and y(2) = 9 in the interval 1 and 2.

Solution:

Here,

$$\frac{dy}{dx} = z$$

or, $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{dz}{dx}$$

so, the given equation becomes;

$$\frac{dy}{dx} = 6x = f(x, y, z) \qquad (1)$$

$$\frac{dy}{dx} = z = g(x, y, z) \quad (2)$$

Now, let us convert the given problem into an initial value problem.

i) **Assumption 1**

Let, $z_0 = 0$

Then,

$x_0 = 1$

$y_0 = 2$

$z_0 = 0$

Using Euler's method;

Take h = 0.5

$$\begin{aligned} y_1 = y(1.5) &= y_0 + hg(x_0, y_0, z_0) \\ &= 2 + 0.5 \times g(1, 2, 0) \\ &= 2 + 0.5 \times 0 \\ &= 2 \end{aligned}$$

$$\begin{aligned} z_1 = z(1.5) &= z_0 + hf(x_0, y_0, z_0) \\ &= 0 + 0.5 \times f(1, 2, 0) \\ &= 0 + 0.5 \times (6 \times 1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} y_2 = y(2) &= y_1 + hg(x_1, y_1, z_1) \\ &= 2 + 0.5 \times g(1.5, 2, 3) \\ &= 2 + 0.5 \times 3 \\ &= 3.5 \end{aligned}$$

$$\begin{aligned} z_2 = z(2) &= z_1 + hf(x_1, y_1, z_1) \\ &= 3 + 0.5 \times f(1.5, 2, 3) \\ &= 3 + 0.5 \times (6 \times 1.5) \\ &= 7.5 \end{aligned}$$

so, when $x_0 = 1, y_0 = 2, z_0 = 0$;

$x_2 = 2$

$y_2 = 3.5$

$z_2 = 7.5$

ii) **Assumption 2**

Let, $z_0 = 1$

Then,

$x_0 = 1$

$y_0 = 2$

$z_0 = 1$

Using Euler's method;

Take h = 0.5

$$\begin{aligned} y_1 = y(1.5) &= y_0 + hg(x_0, y_0, z_0) \\ &= 2 + 0.5 \times g(1, 2, 1) \\ &= 2 + 0.5 \times 1 \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} z_1 = z(1.5) &= z_0 + hf(x_0, y_0, z_0) \\ &= 1 + 0.5 \times f(1, 2, 1) \\ &= 1 + 0.5 \times (6 \times 1) \\ &= 4 \end{aligned}$$

so, $x_1 = 1.5$

$y_1 = 2.5$

$z_1 = 4$

Also,

$$\begin{aligned} y_2 = y(2) &= y_1 + hg(x_1, y_1, z_1) \\ &= 2.5 + 0.5 \times g(1.5, 2.5, 4) \\ &= 2.5 + 0.5 \times 4 \\ &= 4.5 \end{aligned}$$

$$\begin{aligned} z_2 = z(2) &= z_1 + hf(x_1, y_1, z_1) \\ &= 4 + 0.5 \times f(1.5, 2.5, 4) \\ &= 4 + 0.5 \times (6 \times 1.5) \\ &= 8.5 \end{aligned}$$

From assumption 1;

When $z_0 = 0$;

$y_2 = 3.5$

From assumption 2;

When $z_0 = 1$;

$y_2 = 4.5$

But, we know that;

$y_2 = y(2) = 9$ [From question]

By interpolation; we have,

$$(m - 0) = \frac{1 - 0}{4.5 - 3.5} \times (9 - 3.5)$$

$\therefore \quad m = 5.5$

Thus, the real value of z_0 is 5.5.

Now, the initial value problem becomes;

$x_0 = 1$

$y_0 = 2$

$z_0 = 5.5$

Solving it by Euler's method; we have,

$$y_1 = y(1.5) = y_0 + hg(x_0, y_0, z_0)$$
$$= 2 + 0.5 \times g(1, 2, 5.5)$$
$$= 2 + 0.5 \times 5.5$$
$$= 4.75$$
$$z_1 = z(1.5) = z_0 + hf(x_0, y_0, z_0)$$
$$= 5.5 + 0.5 \times f(1, 2, 5.5)$$
$$= 5.5 + 0.5 \times (6 \times 1)$$
$$= 8.5$$
$$y_2 = y(2) = y_1 + hg(x_1, y_1, z_1)$$
$$= 4.75 + 0.5 \times g(1.5, 4.75, 8.5)$$
$$= 4.75 + 0.5 \times 8.5$$
$$= 9$$
$$z_2 = z(2) = z_1 + hf(x_1, y_1, z_1)$$
$$= 8.5 + 0.5 \times f(1.5, 4.75, 8.5)$$
$$= 8.5 + 0.5 \times (6 \times 1.5)$$
$$= 13$$

Thus, the solution of the boundary value problem is;

x	y	z
1	2	5.5
1.5	4.75	8.5
2	9	13

SOLVED NUMERICAL (POKHARA UNIVERSITY)

1. **Solve the differential equation for y(0. 5) using R-K fourth order method. $10\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 6x = 0$, with $y(0) = 1$ and $y'(0) = 0$ take $h = 0.5$.** **[2011 Spring, 2011 Fall, 2010 Fall]**

Solution:

Here, the given equation is of second order.

For this form;

$$\frac{d^2y}{dx^2} = \emptyset\left(x, y, \frac{dy}{dx}\right) \quad (1)$$

We put

$$\frac{dy}{dx} = z$$
$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From (1), (2) and (3); we get,

$$\frac{dz}{dx} = \emptyset(x, y, z) \quad (4)$$

and, $$\frac{dy}{dx} = f(x, y, z) \quad (5)$$

Equation (4) and (5) gives a system of simultaneous equation which can be solved easily.

Here,

$$10y'' + (y')^2 + 6x = 0$$
$$y(0) = 1$$
$$y'(0) = 0$$
$$h = 0.5$$
$$y' = z = f(x, y, z)$$
$$z_0 = 0$$

Then,

$$10z' + z^2 + 6x = 0$$
$$z' = -\left(\frac{6x + z^2}{10}\right) = \emptyset(x, y, z)$$

Now, using Rungee-Kutta method;

$$k_1 = hf(x_0, y_0, z_0) = 0.5 \times f(0, 1, 0) = 0$$
$$l_1 = h\emptyset(x_0, y_0, z_0) = 0.5 \times \emptyset(0, 1, 0) = 0$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.5 \times f(0.25, 1, 0) = 0$$

$$l_2 = h\emptyset\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= 0.5 \times \emptyset(0.25, 1, 0) = -0.075$$
$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.5 \times f(0.25, 1, -0.0375) = -0.01875$$
$$l_3 = h\emptyset\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$
$$= 0.5 \times \emptyset(0.25, 1, -0.0375) = -0.075$$
$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.5 \times f(0.5, 0.98125, -0.075) = -0.0375$$
$$l_4 = h\emptyset(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= 0.5 \times \emptyset(0.5, 0.98125, -0.075) = -0.0125$$

Then,

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0125$$
$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = -0.075$$
$$y(0.5) = y_0 + k = 1 - 0.0125 = 0.9875$$
$$z(0.5) = z_0 + l = 0 + (-0.075) = -0.075$$

2. Solve the following initial value problem using the modified method for $0 < x < 0.6$ with an interval of 0.2.

$$\frac{dy}{dx} = \sin x + \cos y;\ y(0) = 3$$

[2012 Spring]

Solution:

x	$y' = \sin x + \cos y$	Mean slope	Old y + 0.2 (mean slope) = New y
0	$\sin(0) + \cos(3) = -0.9899$	-0.9899	$3 + 0.2(-0.9899) = 2.80202$
0.2	$\sin(0.2) + \cos(2.802) = -0.7442$	$\frac{1}{2}(-0.7442 - 0.9899) = -0.867$	$3 + 0.2(-0.867) = 2.8266$
	$\sin(0.2) + \cos(2.8266) = -0.7521$	$\frac{1}{2}(-0.7521 - 0.9899) = -0.871$	$3 + 0.2(-0.871) = 2.8258$
	$\sin(0.2) + \cos(2.8258) = -0.7519$	$\frac{1}{2}(-0.7519 - 0.9899) = -0.8709$	$3 + 0.2(-0.8709) = 2.82582$ Since, 2 values are almost equal; hence, $y(0.2) = 2.8258$
0.2	$\sin(0.2) + \cos(2.8258) = -0.7519$	-0.7519	$2.8258 + 0.2(-0.7519) = 2.6754$

0.4	$\sin(0.4) + \cos(2.6754) = -0.5039$	$\frac{1}{2}(-0.5039 - 0.7519) = -0.6279$	$2.8258 + 0.2(-0.6279) = 2.7$
0.4	$\sin(0.4) + \cos(2.7) = -0.515$	$\frac{1}{2}(-0.515 - 0.7519) = -0.63345$	$2.8258 + 0.2(-0.63345) = 2.7$ ∴ $y(0.4) = 2.7$
0.4	$\sin(0.4) + \cos(2.7) = -0.515$	-0.515	$2.7 + 0.2(-0.515) = 2.597$
0.6	$\sin(0.6) + \cos(2.597) = -0.2907$	$\frac{1}{2}(-0.2907 - 0.515) = -0.40285$	$2.7 + 0.2(-0.40285) = 2.6194$
0.6	$\sin(0.6) + \cos(2.6194) = -0.302$	$\frac{1}{2}(-0.302 - 0.515) = -0.4085$	$2.7 + 0.2(-0.4085) = 2.61825$
0.6	$\sin(0.6) + \cos(2.6183) = -0.3015$	$\frac{1}{2}(-0.3015 - 0.515) = -0.40825$	$2.7 + 0.2(-0.40825) = 2.61825$ ∴ $y(0.6) = 2.61825$

3. Use RK-4 method to find y(0.2) for $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$, given that y = 1 and $\frac{dy}{dx} = 0$ when x = 0.

[2017 Fall]

Solution:

Here;

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$$

or, $y'' = xy' - y$

Let, $y' = z$ with initial value $y_0 = 1$, $z_0 = 0$ and $x_0 = 0$

so, $z' = xz - y$

or, $z' = f(x, y, z)$

Now, using RK-4 method; we have,

$$x_0 = 0,\ y_0 = 1,\ z_0 = 0,\ h = 0.1$$
$$k_1 = hf(x_0, y_0, z_0) = 0.1\,f(0, 1, 0) = -0.1$$

and, $l_1 = hz_0 = 0.1 \times 0 = 0$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{l_1}{2}, z_0 + \frac{k_1}{2}\right)$$
$$= 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}, 0 + \frac{-0.1}{2}\right)$$
$$= -0.1$$

and, $l_2 = h\left(z_0 + \frac{k_1}{2}\right) = 0.1\left(0 + \frac{-0.1}{2}\right) = -0.005$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{l_2}{2}, z_0 + \frac{k_2}{2}\right) = -0.1005$$

and, $l_3 = h\left(z_0 + \frac{k_2}{2}\right) = -0.005$

$$k_4 = hf(x_0 + h, y_0 + l_3 + z_0 + k_3) = -0.1005$$

and, $l_4 = h(z_0 + k_3) = -0.010$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = -0.1003$$

$$l = \frac{l_1 + 2l_2 + 2l_3 + l_4}{6} = -0.005$$

$\Rightarrow$ $x_1 = 0 + 0.1 = 0.1$

$y_1 = 1 - 0.005 = 0.995$

$z_1 = 0 - 0.1003 = -0.1003$

Using RK-4 method,

$$k_1 = hf(x_1, y_1, z_1) = 0.1\, f(0.1, 0.995, -0.1003) = -0.1005$$

and, $l_1 = hz_1 = 0.1 \times -0.1003 = -0.010$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{l_1}{2}, z_1 + \frac{k_1}{2}\right)$$

$$= 0.1\, f\left(0.1 + \frac{0.1}{2}, 0.995 + \frac{-0.010}{2}, -0.1003 + \frac{-0.1003}{2}\right)$$

$$= -0.1013$$

and, $l_2 = h\left(z_1 + \frac{k_1}{2}\right) = -0.015$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{l_2}{2}, z_1 + \frac{k_2}{2}\right) = -0.1010$$

and, $l_3 = h\left(z_1 + \frac{k_2}{2}\right) = -0.015$

$$k_4 = hf(x_1 + h, y_1 + l_3 + z_1 + k_3) = -0.1020$$

and, $l_4 = h(z_1 + k_3) = -0.020$

so, $l = \frac{l_1 + 2l_2 + 2l_3 + l_4}{6} = -0.015$

and $y_2 = y_1 + l = 0.98$

so, $y(0.2) = 0.98$

4. Write = a program in any higher level language for solution of ordinary differential equation using Euler's method. [2017 Fall]

Solution:

In C language

```
# include <stdio.h>
float fun (float x, float y)
```



```
{
    return (x² + y²);
}
int main ()
{
    float x[100], y[100], h, k;
    int i, n;
    printf ("enter the initial value of x and y":\n");
    scanf("%f%f", &x[0], &y[0]);
    printf ("enter the step size h =");
    scanf ("%f", &h);
    printf ("enter the no. of sub-division n =");
    sacnf ("%d", &n);
    for (i = 0; i <= n; i++)
    {
        k = n * fun (x[0], y[0]);
        x[i + 1] = x[i] + h;
        y[i + 1] = y[i] + k;
    }
    printf ("display the solution of ODE:\n");
    printf ("\tx\ty");
    for (i = 0; i <=n; i++)
    {
        printf ("\n\t%d\t%d", x[i], y[i]);
    }
    return 0;
}
```

5. Use the Rung-Kutta method to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$ for the interval $0 \le x \le 0.4$ with $h = 0.1$. [2018 Spring]

Solution:

Here, using RK-4 method in;

$10y' = x^2 + y^2$ with $y_0 = 1$ and $x_0 = 0$

or, $y' = \frac{x^2 + y^2}{10} = f'(x)$

Now, taking step size $h = 0.1$

and, we know that,

$$k_1 = hf(x_i, y_i) \quad (1)$$

$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right) \quad (2)$$

$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right) \quad (3)$$

$$k_4 = hf(x_i + h, y_i + k_3) \quad (4)$$

and, $$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \quad (5)$$

$$x_i = x_{i-1} + h \text{ and } y_i = y_{i-1} + k \quad (6)$$

Hence, using above equations; we solve for $0 \le x \le 0.4$ with $h = 0.1$ in below table.

i	x_{i-1}	y_{i-1}	k_1	k_2	k_3	k_4	k	x_i	y_i
1	0	1	0.0100	0.0101	0.0101	0.0103	0.0101	0.1	1.0101
2	0.1	1.011	0.0104	0.0105	0.0105	0.0108	0.0105	0.2	1.0206
3	0.2	1.0206	0.0110	0.0112	0.0112	0.0116	0.0112	0.3	1.0318
4	0.3	1.0318	0.0119	0.0120	0.0120	0.0124	0.01203	0.4	1.0439

Hence, this is the solution of given first order differential equation.

6. Solve the boundary value problem.

$y''(x) = y(x)$

$y(0) = 0$ and $y(1) = 1.1752$ by shooting method taking $m_0 = 0.8$ and $m_1 = 0.9$ **[2018 Spring]**

Solution:

Here;

Let, $y'(x) = z(x)$ (1)

Then,

$y''(x) = y(x)$

or, $z'(x) = y(x)$ $[y''(x) = z'(x) \because z(x) = y'(x)]$ (2)

We know $y(0) = 0$ and assume $z(0) = 0.8$ as given in question.

So using Euler method, take step size $h = 0.2$

We know that,

$k = hy_i$ and $l = hz_i$

$x_{i+1} = x_i + h$, $y_{i+1} = y_i + l$, $z_{i+1} = z_i + k$

Using above formula;

For $m_0 = y_0' = z_0 = 0.8$

i	x_i	y_i	z_i	k	l	x_{i+1}	y_{i+1}	z_{i+1}
0	0	0	0.8	0	0.160	0.2	0.160	0.8
1	0.2	0.160	0.8	0.032	0.16	0.4	0.320	0.832
2	0.4	0.320	0.832	0.064	0.166	0.6	0.486	0.896
3	0.6	0.486	0.896	0.097	0.179	0.8	0.665	0.993
4	0.8	0.665	0.993	0.133	0.199	1.0	0.864	1.126

For $y_0' = z_0 = 0.9$

i	x_i	y_i	z_i	k	l	x_{i+1}	y_{i+1}	z_{i+1}
0	0	0	0.9	0	0.180	0.2	0.18	0.9
1	0.2	0.180	0.9	0.036	0.180	0.4	0.360	0.936
2	0.4	0.360	0.936	0.072	0.187	0.6	0.547	1.008
3	0.6	0.547	1.008	0.109	0.202	0.8	0.749	1.117
4	0.8	0.749	1.117	0.150	0.223	1.0	0.972	1.167

Now, interpolating

x_4	y_0'
0.864	0.8
0.972	0.9
1.1752	1.088

For $y_0' = 1.088$

i	x_i	y_i	z_i	k	l	x_{i+1}	y_{i+1}	z_{i+1}
0	0	0	1.088	0	0.218	0.2	0.218	1.088
1	0.2	0.218	1.088	0.044	0.218	0.4	0.436	1.132
2	0.4	0.436	1.132	0.087	0.226	0.6	0.662	1.219
3	0.6			0.133	0.244	0.8	0.906	1.452
4	0.8	0.906	1.452	0.181	0.290	1.0	1.196	1.633

and, this is the solution of boundary value problem of given differential equation.

7. Solve the given differential equation by RK-4 order method $y'' - x^2y' - 2xy = 0$ with initial condition $y(0) = 1$, $y'(0) = 0$, for $y(0.1)$ taking $h = 0.1$. **[2018 Fall]**

Solution:

Here,

$$y'' - x^2y' - 2xy = 0 \quad (1)$$

Let $y' = z$ and $y'' = z'$

Then, equation (1) becomes;

$z' - x^2z - 2xy = 0$

or, $z' = x(xz + 2y)$ **(1) with initial condition $x = 0, y = 1, z = 0$**

$= f(x, y, z)$

and, $y' = z$ **(2) with initial condition $y = 1$ and $z = 0$**

and, using RK-4 method;

$k_1 = hf(x_0, y_0, z_0) = 0.1 \times f(0, 1, 0) = 0.000$

and, $l_1 = hz_0 = 0.1 \times 0 = 0.000$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{l_1}{2}, z_0 + \frac{k_1}{2}\right) = 0.010$$

and, $l_2 = h\left(z_0 + \frac{k_1}{2}\right) = 0.000$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{l_2}{2}, z_0 + \frac{k_2}{2}\right) = 0.010$$

and, $l_3 = h\left(z_0 + \frac{k_2}{2}\right) = 0.000$

$$k_4 = hf(x_0 + h, y_0 + l_3 + z_0 + k_3) = 0.020$$

and, $l_4 = h\,(z_0 + k_3) = 0.001$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.01$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) = 0.000$$

so, $x_1 = x_0 + h = 0.1$

$y_1 = y_0 + l = 1$

$z_1 = z_0 + k = 0.01$

and, $y(0.1) = 1$

8. **Solve differential equation $y' = y - \frac{2x}{y}$ using approximate method within $0 \le x \le 0.2$ with initial condition $y(0) = 1$ and step size $h = 0.1$.** **[2018 Fall]**

Solution:

Here;

$$y' = y - \frac{2x}{y} \text{ with initial condition } y(0) = 1.$$

$$y' = f(x, y) = y - \frac{2x}{y}$$

Now, using RK-4 method due to more accuracy than other method.

Now, step size $h = 0.1$.

$x_0 = 0, y_0 = 1, h = 0.1$

so, $k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.095$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.095}{2}\right) = 0.095$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0 + 0.1, 1 + 0.095) = 0.091$$

so, $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 0.380$

Hence,

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + k = 1 + 0.380 = 1.38$

and, $x_1 = 0.1, y_1 = 1.38, h = 0.1$

Again,

$k_1 = kf(x_1, y_1) = 0.1\,f(0.1, 1.38) = 0.124$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.38 + \frac{0.124}{2}\right) = 0.124$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.124$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.124$$

So, $k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = 0.124$

Hence,

$x_2 = x_1 + h = 0.2$

$y_2 = y_1 + k = 1.38 + 0.124 = 1.504$

so,

x	0	0.1	0.2
y	1	1.380	1.504

9. **Use Picard's method to approximate the value of y when $x = 0.1$, $x = 0.2$ and $x = 0.4$ given that $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 1 + xy$ correct to three decimal places. (Use up to second approximation).** **[2019 Fall]**

Solution:

i) Here, initially $y = 1$ at $x = 0$;

$$\frac{dy}{dx} = 1 + xy$$

Using formula; we have,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

First approximation; $y = y_0 = 1, x_0 = 1$

$$y_1 = 1 + \int_0^x (1 + xy)\,dx = 1 + \int_0^x (1 + x)\,dx = 1 + x + \frac{x^2}{2}$$

Second approximation; $(y = y_1)$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx = 1 + \int_0^x (1 + xy)\,dx$$

$$= 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

At $x = 0.1$;

$$y = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} = 1.1053$$

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ii) At $x_0 = 0.1, y_0 = 1.153, x = 0.2, y = ?$

First approximation;

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx$$

$$= 1.1053 + \int_{0.1}^{x} (1 + xy)\,dx$$

$$= 1.1053 + \int_{0.1}^{x} (1 + 1.1053x)\,dx$$

$$= 1.1053 + \left[x + \frac{1.1053x^2}{2}\right]_{0.1}^{x}$$

$$= 1.1053 + x + \frac{1.1053x^2}{2} - 0.1 - \frac{1.1053 \times (0.1)^2}{2}$$

$$= 0.999 + x + \frac{1.1053x^2}{2}$$

Second approximation;

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1.1053 + \int_{0.1}^{x} 1 + x\left(0.999 + x + \frac{1.1053x^2}{2}\right) dx$$

$$= 1.1053 + \int_{0.1}^{x} 1 + 0.999x + x^2 + \frac{1.1053x^3}{2}\,dx$$

$$= 1.1053 + \left[x + \frac{0.999x^2}{2} + \frac{x^3}{3} + \frac{1.1053x^4}{8}\right]_{0.1}^{x}$$

$$= x + \frac{0.999x^2}{2} + \frac{x^3}{3} + \frac{1.1053x^4}{8} + 0.999$$

At $x = 0.2$;

$$y = 0.2 + \frac{0.999 \times (0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{1.1053 \times (0.2)^4}{8} + 0.999$$

or, $y = 1.22186$

Similarly, we can evaluate y at $x = 0.4$ using $y_0 = 1.22186$ and $x_0 = 0.2$.

On solving; we get,

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{1.22186x^4}{4} + 1$$

At $x = 0.4, y = 1.5038$;

Hence,

x	y
0.1	1.1053
0.2	1.2216
0.4	1.5038

NOTE:
We can also approximate y at $x = 0.1, 0.2$ and 0.4, using first expression; but it gives less accurate result.

i.e., $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{2}$

x	y
0.1	1.1053
0.2	1.223
0.4	1.5077

10. Using Runge–Kutta method of second order (RK–2) obtain the solution of equation $y'' = y + xy'$ with initial condition $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$ taking $h = 0.1$. **[2019 Fall]**

Solution:

Let, $z = \frac{dy}{dx}$

i.e., $z = y'$ then above equation can be written as;

$z' = y + xz$

$$z' = \frac{dz}{dx} = d\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Initial condition;

$x_0 = 0$

$y_0 = 1$

$z_0 = 0$

$h = 0.1$

The given equations can be expressed as simultaneous differential equation as;

$$y' = z = f(x, y, z)$$

$$z' = y + xz = g(x, y, z)$$

Step 1:

$(x_0 = 0, y_0 = 1, z_0 = 0, h = 0.1)$

$$m_1 = f(x_0, y_0, z_0) = f(0, 1, 0) = 0$$

$$l_1 = g(x_0, y_0, z_0) = g(0, 1, 0) = 1 + 0 \times 0 = 1$$

$$m_2 = f(x_0 + h, y_0 + m_1h, z_0 + l_1h)$$

$$= f(0 + 0.1, 1 + 0 \times 0.1, 0 + 1 \times 0.1)$$

$$= f(0.1, 1.1, 0.1)$$

$$= 0.1$$

$$l_2 = g(x_0 + h, y_0 + m_1h, z_0 + l_1h)$$

$$= g(0.1, 1.1, 0.1)$$

$= 1.1 + 0.1 \times 0.1$

$= 1.11$

$m = \frac{m_1 + m_2}{2} = \frac{0 + 0.1}{2} = 0.05$

$l = \frac{l_1 + l_2}{2} = \frac{1 + 1.11}{2} = 1.055$

$y_1 = y_0 + mh = 1 + 0.05 \times 0.1 = 1.005$

$z_1 = z_0 + lh = 0 + 1.055 \times 0.1 = 0.1055$

At $x_0 = 0.1, y = 1.005, z = 0.1055$

Step 2:

$(x_0 = 0.1, y_0 = 1.005, z = 0.1055, h = 0.1)$

$m_1 = f(x_0, y_0, z_0) = f(0.1, 1.005, 0.1055) = 0.1055$

$l_1 = g(x_0, y_0, z_0) = g(0.1, 1.005, 0.1055)$

$= 1.005 + 0.1 \times 0.1055 = 1.01555$

$m_2 = f(x_0 + h, y_0 + m_1h, z_0 + l_1h)$

$= f(0.1 + 0.1, 1.005 + 0.1055 \times 0.1, 0.1055 + 1.01555)$

$= f(0.2, 1.01555, 0.20705)$

$= 0.20705$

$l_2 = g(x_0 + h, y_0 + m_1h, z_0 + l_1h) = g(0.2, 1.01555, 0.20705)$

$= 1.01555 + 0.2 \times 0.20705$

$= 1.05696$

$m = \frac{m_1 + m_2}{2} = \frac{0.1055 + 0.20705}{2} = 0.15627$

$l = \frac{l_1 + l_2}{2} = \frac{1.01555 + 1.05696}{2} = 1.03625$

At $x = 0.2$;

$y = y_0 + mh = 1.005 + 0.15627 \times 0.1 = 1.020627$

$z = z_0 + lh = 0.1055 + 1.03625 \times 0.1 = 0.209125$

x	0	0.1	0.2
y	1	1.005	1.020627
z	0	0.1055	0.209125

11. Solve the given differential equation by Heun's method $y'' - y' - 2y = 3e^x$ with initial condition $y(0) = 0$, $y'(0) = -2$, for $y(0.2)$ taking $h = 0.1$.

[2019 Spring]

Solution:

Here;

$y'' - y' - 2y = 3e^x$

Let $y' = \frac{dy}{dx} = z$

Then, $y'' = \frac{d^2y}{dx^2} = z'$

So, given equation can be expressed as simultaneous differential equation.

$y' = z = f(x, y, z)$

$y'' = y' + 2y + 3e^x$

$z' = z + 2y + 3e^x = g(x, y, z)$

x	0	0.1	0.2
y	0	y_1 or $y(0.1)$	y_2 or $y(0.2)$
y'	-2		

NOTE:

Heun's method is RK–2 method.

Now, end conditions:

$y(0) = 0$ and $z(0) = -2$

Step 1:

$x_0 = 0, y_0 = 0, z_0 = -2$ and $h = 0.1$

$m_1 = f(x_0, y_0, z_0)$

$= f(0, 0, -2)$

$= -2$

$l_1 = g(x_0, y_0, z_0)$

$= g(0, 0, -2)$

$= -2 + 2 \times 0 + 3e^0$

$= 1$

$m_2 = f(x_0 + h, y_0 + m_1h, z_0 + l_1h)$

$= f(0 + 0.1, 0 + (-2) \times 0.1, -2 + 1 \times 0.1)$

$= f(0.1, -0.2, -1.9)$

$= -1.9$

$l_2 = g(x_0 + h, y_0 + m_1h, z_0 + l_1h)$

$= g(0.1, -0.2, -1.9)$

$= -1.9 + 2 \times (-0.2) + 3e^{0.1}$

$= 1.0155$

$m = \frac{m_1 + m_2}{2} = \frac{-2 - 1.9}{2} = -1.95$

$l = \frac{l_1 + l_2}{2} = \frac{1 + 1.0155}{2} = 1.0078$

$y_1 = y_0 + mh = 0 + (-1.95) \times 0.1 = -0.195$

$z_1 = z_0 + lh = -2 + 1.0078 \times 0.1 = -1.8992$

Step 2:

$x_1 = 0.1, y_1 = -0.195, z_1 = -1.8992$ and $h = 0.1$

$m_1 = f(x_1, y_1, z_1)$
$= f(0.1, -0195, -1.8992)$
$= -1.8992$

$l_1 = f(x_1, y_1, z_1)$
$= f(0.1, -0195, -1.8992)$
$= -1.8992 + 2 \times (-0.195) + 3e^{0.1}$
$= 1.026$

$m_2 = f(x_1 + h, y_1 + m_1h, z_1 + l_1h)$
$= f(0.1 + 0.1, -0.195 + (-1.8992 \times 0.1), -1.8992 + (1.026 \times 0.1)$
$= f(0.2, -0.3849, -1.7966)$
$= -1.7966$

$l_1 = g(x_1 + h, y_1 + m_1h, z_1 + l_1h)$
$= g(0.2, -0.3849, -1.7966)$
$= -1.7966 + 2 \times (-0.3849) + 3e^{0.2}$
$= 1.0978$

$$m = \frac{m_1 + m_2}{2} = \frac{-1.8992 - 1.7966}{2} = -1.8479$$

$$l = \frac{l_1 + l_2}{2} = \frac{1.026 + 1.0978}{2} = 1.0619$$

$y_2 = y_1 + mh = -0.195 + (-1.8479) \times 0.1 = -0.3798$

$\therefore$ $y(0.2) = 0.3798$

12. **Solve**

$y' = y + e^x, y(0) = 0$ for $y(0.2)$

and, $y(0.4)$ by RK-4th order method. **[2019 Spring]**

Solution:

Here;

$y' = y + e^x = f(x, y)$

Take $h = 0.2$

x	0	0.2	0.4
y	0	y_1 or y (0.2)	y_2 or y (0.4)

Step 1:

$x_0 = 0, y_0 = 0, h = 0.2$

$m_1 = f(x_0, y_0) = f(0, 0) = 0 + e^0 = 1$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right)$$
$$= f\left(0 + \frac{0.2}{2}, 0 + 1 \cdot \frac{0.2}{2}\right)$$
$= f(0.1, 0.1)$
$= 0.1 + e^{0.1}$
$= 1.205$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + m_2 \cdot \frac{h}{2}\right)$$
$$= f\left(0.1, 0 + 1.205 \times \frac{0.2}{2}\right)$$
$= f(0.1, 0.1205)$
$= 0.1205 + e^{0.1}$
$= 1.2256$

$m_4 = f(x_0 + h, y_0 + m_3h)$
$= f(0 + 0.2, 0 + 1.2256 \times 0.2)$
$= f(0.2, 0.24512)$
$= 0.24512 + e^{0.2}$
$= 1.4665$

$$m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$
$$= \frac{1 + 2 \times 1.205 + 2 \times 1.2256 + 1.4665}{6}$$
$= 1.22128$

$y_1 = y_0 + mh = 0 + 1.22128 \times 0.2 = 0.24425$

$\therefore$ $y(0.2) = 0.24425$

Step 2:

$x_1 = 0.2, y_1 = 024425, h = 0.2$

$m_1 = f(x_1, y_1)$
$= f(0.2, 0.24425)$
$= 0.24425 + e^{0.2}$
$= 1.4656$

$$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + m_1 \cdot \frac{h}{2}\right)$$
$$= f\left(0.2 + \frac{0.2}{2}, 0.24425 + 1.4656 \cdot \frac{0.2}{2}\right)$$
$= f(0.3, 0.39081)$
$= 0.39081 + e^{0.3}$
$= 1.7407$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + m_2 \cdot \frac{h}{2}\right)$$
$$= f\left(0.2 + \frac{0.2}{2}, 0.24425 + 1.7407 \cdot \frac{0.2}{2}\right)$$
$= f(0.3, 0.41832)$
$= 0.41832 + e^{0.3}$
$= 1.76817$

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$$m_4 = f(x_1 + h, y_1 + m_3 h)$$
$$= f(0.2 + 0.2, 0.24425 + 1.76817 \times 0.2)$$
$$= f(0.4, 0.59788)$$
$$= 0.59788 + e^{0.4}$$
$$= 2.0897$$
$$m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$
$$= \frac{1.4656 + 2 \times 1.7407 + 2 \times 1.76817 + 2.0897}{6}$$
$$= 1.76217$$
$$y_2 = y_1 + mh = 0.24425 + 1.76217 \times 0.2 = 0.59668$$
$$\therefore \quad y(0.4) = 0.59668$$

13. Write short notes on: "Finite differences" **[2020 Fall]**

Solution:

Finite difference method is a method of solving Boundary value problems. In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and resulting linear system of equations are solved by any standard procedures.

From Taylor's series

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} \cdot y''(x) + \frac{h^3}{3!} y'''(x) + \dots\dots\dots \quad \text{(a)}$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!} \cdot y''(x) - \frac{h^3}{3!} y'''(x) + \dots\dots\dots \quad \text{(b)}$$

Subtracting equation (b) from equation (a); we have,

$$y(x+h) - y(x-h) = 2hy'(x) + 2\frac{h^3}{3!} y'''(x)$$

or, $$2h\left[y'(x) + \frac{h^3}{3!} y'''(x)\right] = y(x+h) - y(x-h)$$

or, $$y'(x) + \frac{h^2}{3!} y'''(x) = \frac{y(x+h) - y(x-h)}{2h}$$

h is small, $h^2 \rightarrow$ neglect

$$\therefore \quad y'(x) = \frac{y(x+h) - y(x-h)}{2h}$$

Similarly, adding equation (a) and (b); we have,

$$y(x+h) + y(x-h) = 2y(x) + \frac{2h^2}{2!} y''(x)$$

or, $$y(x+h) + y(x-h) - 2y(x) = h^2 y''(x)$$

$$\therefore \quad y''(x) = \frac{y(x+h) + y(x-h) - 2y(x)}{h^2}$$

In terms of i, it is written as;

$$y_i' = \frac{(y_{i+1} - y_{i-1})}{2h}$$

$$y_i'' = \frac{y_{i+1} + y_{i-1} - 2y_i}{h^2}$$

14. Write short notes on "Picard's iterative formula". **[2020 Fall]**

Solution: See the theory section (ii).

15. Write short notes on: "Algorithm for second order Runge-Kutta (RK-2) method". **[2020 Fall]**

Solution:

Runge Kutta (RK–2) method is also known as modified Euler's method or Heun's method.

Algorithm of RK–2 method for the solution of first order differential equation.

Let's consider the equation $y' = f(x, y)$.

$$y(x_0) = y_0, h \text{ (given)}$$
$$m_1 = f(x_0, y_0)$$
$$m_2 = f(x_0 + h, y_0 + m_1 h)$$
$$m = \frac{m_1 + m_2}{2}$$
$$y_1 = y_0 + mh$$

We continue above procedure in loop until we get the required value of y for the given value of x.

Algorithm of RK–2 method for the solution of second order differential equation.

Let, $y' = z = f(x, y, z)$

$$z' = g(x, y, z)$$
$$y(x_0) = y_0, z(x_0) = z_0, h \text{ (given)}$$
$$m_1 = f(x_0, y_0, z_0)$$
$$l_1 = g(x_0, y_0, z_0)$$
$$m_2 = f(x_0 + h, y_0 + m_1 h, z_0 + l_1 h)$$
$$l_2 = g(x_0 + h, y_0 + m_1 h, z_0 + l_1 h)$$
$$m = \frac{m_1 + m_2}{2}$$
$$l = \frac{l_1 + l_2}{2}$$

$y_1 = y_0 + mh$

$z_1 = z_0 + lh$

We continue above procedure in loop until we get the required value of y and z for the given value of x.

16. Apply Runge-Kutta fourth order method to find an approximate value of y when x = 0.3 given that $y' = 2.5y + e^{0.3x}$; with an initial y(0) = 1, taking h = 0.3. **[2020 Fall]**

Solution:

Here;

$y' = 2.5y + e^{0.3x} = f(x, y)$

$y(0) = 1$

$y(0.3) = ?, h = 0.3$

$x_0 = 0, y_0 = 1, h = 0.3$

$m_1 = f(x_0, y_0) = f(0, 1) = 2.5 \times 1 + e^0 = 5.218$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right)$

$= f\left(0 + \frac{0.3}{2}, 1 + 5.218 \cdot \frac{0.3}{2}\right)$

$= f(0.15, 1.7827)$

$= 2.5 \times 1.7827 + e^{0.3\times0.15}$

$= 5.5028$

$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + m_2 \cdot \frac{h}{2}\right)$

$= f\left(0 + \frac{0.3}{2}, 1 + 5.5028 \cdot \frac{0.3}{2}\right)$

$= f(0.15, 1.8254)$

$= 2.5 \times 1.8254 + e^{0.3\times0.5}$

$= 5.6095$

$m_4 = f(x_0 + h, y_0 + m_3h)$

$= f(0 + 0.3, 1 + 5.6095 \times 0.3)$

$= f(0.3, 2.6828)$

$= 2.5 \times 2.6828 + e^{0.3\times0.3}$

$= 7.8012$

$m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$

$= \frac{5.218 + 2 \times 5.5028 + 2 \times 5.6095 + 7.8012}{6}$

$= 5.8739$

$y_1 = y_0 + mh = 1 + 5.8739 \times 0.3 = 2.7622$

$\therefore$ $y(0.3) = 2.7622$

17. Solve the Boundary value problem (BVP) using shooting method by dividing into four sub-intervals implying Euler's method.

$$y'' + 2y' - y = x$$

Subject to boundary condition y(1) = 2 and y(2) = 4. **[2020 Fall]**

Solution:

Let, $y' = z$

Then, given equation will become;

$z' + 2z - y = x$ and $y' = z$

with conditions;

$y(1) = 2$ and $y(2) = 4$

x	1	1.25	1.5	1.75	2
y	2	y_1	y_2	y_3	4

Step 1:

$x_0 = 1, y_0 = 2, h = 0.25, z_0 = 2$ (let)

$y_{n+1} = y_n + hy_n'$

or, $y_{n+1} = y_n + hz_n$

$z_{n+1} = z_n + hz'$

or, $z_{n+1} = z_n + h(x + y - 2z)$

$y_1 = y_0 + hz_0$

$= 2 + 0.25 \times 2$

$= 2.5$

$z_1 = z_0 + h(x_0 + y_0 - 2z_0)$

$= 2 + 0.25(1 + 2 - 2 \times 2)$

$= 1.75$

$y_2 = y_1 + hz_1$

$= 2.5 + 0.25 \times 1.75$

$= 2.9375$

$z_2 = z_1 + h(x_1 + y_1 - 2z_1)$

$= 1.75 + 0.25(1.25 + 2.5 - 2 \times 1.75)$

$= 1.8125$

$y_3 = y_2 + hz_2$

$= 2.9375 + 0.25 \times 1.8125$

$= 3.3906$

$z_3 = z_2 + h(x_2 + y_2 - 2z_2)$

$= 1.8125 + 0.25(1.5 + 2.9375 - 2 \times 1.8125)$

$= 2.0156$

$y_4 = y_3 + hz_3$

$= 3.3906 + 0.25 \times 2.0156$

$= 3.8945$

$\therefore \quad y_4 = 3.8945 < 4$

Step 2:

$x_0 = 1, y_0 = 2, h = 0.25, z_0 = 4$ (let)

$y_1 = y_0 + h \times z_0$
$= 2 + 0.25 \times 4$
$= 3$

$z_1 = z_0 + h(x_0 + y_0 - 2z_0)$
$= 4 + 0.25\,(1 + 2 - 2 \times 4)$
$= 2.75$

$y_2 = y_1 + h \times z_1$
$= 3 + 0.25 \times 2.75$
$= 3.6875$

$z_2 = z_1 + h\,(x_1 + y_1 - 2z_1)$
$= 2.75 + 0.25\,(1.25 + 3 - 2 \times 2.75)$
$= 2.4375$

$y_3 = y_2 + h \times z_2$
$= 3.6875 + 0.25 \times 2.4375$
$= 4.297$

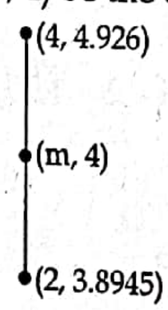
$z_3 = z_2 + h\,(x_2 + y_2 - 2z_2)$
$= 2.4375 + 0.25\,(1.5 + 3.6875 - 2 \times 2.4375)$
$= 2.5156$

$y_4 = y_3 + h \times z_3$
$= 4.297 + 0.25 \times 2.5156$
$= 4.926$

$\therefore \quad y_4 = 4.926 > 4$

Step 3:

Let (2, 3.8945), (4, 4.926) and (m, 4) be the collinear points.



$$\frac{4 - m}{4 - 2} = \frac{4.926 - 4}{4.926 - 3.8945}$$

$\therefore \quad m = 2.2045$

Step 4:

$x_0 = 1, y_0 = 2, h = 0.25, z_0 = 2.2045$

$y_1 = y_0 + h \times z_0$
$= 2 + 0.25 \times 2.2045$

$= 2.551$

$z_1 = z_0 + h(x_0 + y_0 - 2z_0)$
$= 2.2045 + 0.25\,(1 + 2 - 2 \times 2.2045)$
$= 1.8522$

$y_2 = y_1 + h \times z_1$
$= 2.551 + 0.25 \times 1.8522$
$= 3.014$

$z_2 = z_1 + h\,(x_1 + y_1 - 2z_1)$
$= 1.8522 + 0.25\,(1.5 + 2.551 - 2 \times 1.8522)$
$= 1.9389$

$y_3 = y_2 + h \times z_2$
$= 3.014 + 0.25 \times 1.9389$
$= 3.4987$

$z_3 = z_2 + h\,(x_2 + y_2 - 2z_2)$
$= 1.9389 + 0.25\,(1.5 + 3.014 - 2 \times 1.9389)$
$= 2.0979$

$y_4 = y_3 + h \times z_3$
$= 3.4987 + 0.25 \times 2.0979$
$= 4.02$

i.e., $y_4 = 4.02 \approx$ (OK)

x	1	1.25	1.5	1.75	2
y	2	2.551	3.014	3.4987	4.02

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. Solve the differential equation for y (2) $\cdot \frac{dy}{dx} = x + y$, y(0) = 1 using fourth order RK-method taking h = 1. [2015]

Solution:

Here;

$$m = \frac{dy}{dx} = x + y$$

i) At initial condition, x = 0, y = 1

$$m_1 = f(x_0, y_0) = f(0, 1) = 0 + 1 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right) = f\left(0 + \frac{1}{2}, 1 + 1 \times \frac{1}{2}\right)$$

$$= f(0.5, 1.5) = 0.5 + 1.5 = 2$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + m_2 \cdot \frac{h}{2}\right) = f\left(0 + \frac{1}{2}, 1 + 2 \times \frac{1}{2}\right)$$

$$= f\left(\frac{1}{2}, 2\right) = 2.5$$

$$m_4 = (x_0 + h, y_0 + m_3 h) = f(0 + 1, 1 + 2.5 \times 1)$$

$$= f(1, 3.5) = 1 + 3.5 = 4.5$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6} = \frac{1 + 2 \times 2 + 2 \times 2.5 + 4.5}{6}$$

$$= 2.4167$$

$$\therefore \quad y_1 = y_0 + m \times h = 1 + 2.4167 \times 1 = 3.4167$$

i.e., y (1) = 3.4167

or, y = 3.4167 when x = 1

ii) $x_1 = x_0 + h = 0 + 1 = 1$

$y_1 = 3.4167$

$$m_1 = f(x_1, y_1) = f(1, 3.4167)$$

$$= 1 + 3.4167 = 4.4167$$

$$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + m_1 \cdot \frac{h}{2}\right)$$

$$= f\left(1 + \frac{1}{2}, 3.4167 + 4.4167 \times \frac{1}{2}\right)$$

$$= f(1.5, 5.6250)$$

$$= 1.5 + 5.6250 = 7.1250$$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + m_2 \cdot \frac{h}{2}\right) = f\left(1 + \frac{1}{2}, 3.4167 + 7.1250 \times \frac{1}{2}\right)$$

$$= f(1.5, 6.9792)$$

$$= 1.5 + 6.9792 = 8.4792$$

$$m_4 = (x_1 + h, y_1 + m_3 h) = f(1 + 1, 8.4792 + 1 \times 3.4167)$$

$$= f(2, 11.8959) = 13.8959$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

$$= \frac{4.4167 + 2 \times 7.1250 + 2 \times 8.4792 + 13.8959}{6}$$

$$= 8.2535$$

$$\therefore \quad y_2 = y_1 + m \times h = 3.4167 + 8.2535 \times 1 = 11.6702$$

i.e., y (2) = 11.6702

or, y = 11.6702 when x = 2

NOTE

RK-4 method can be done using alternative formula also see the Q. no. 3.

2. Applying RK-4 method to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x^2 + y$, y (0) = 1. Take h = 0.1. [2015]

Solution:

Here;

a) $\frac{dy}{dx} = x^2 + y$ and y (0) = 1 (*i.e.,* $x_0 = 0, y_0 = 1$)

$$m_1 = f(x_0, y_0) = f(0, 1) = 0^2 + 1 = 1$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + 1 \times \frac{0.1}{2}\right)$$

$$= f(0.05, 1.05) = (0.05)^2 + 1.05 = 1.0525$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + m_2 \cdot \frac{h}{2}\right) = f\left(0 + \frac{0.1}{2}, 1 + 1.0525 \times \frac{0.1}{2}\right)$$

$$= f(0.05, 1.0526) = (0.05)^2 + 1.0526 = 1.0551$$

$$m_4 = (x_0 + h, y_0 + m_3 h) = f(0 + 0.1, 1 + 1.0551 \times 0.1)$$

$$= f(0.1, 1.1055) = (0.1)^2 + 1.1055 = 1.1155$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

$$= \frac{1 + 2 \times 1.0525 + 2 \times 1.0551 + 1.1155}{6}$$

$$= 1.0551$$

$$\therefore \quad y_1 = y_0 + m \times h$$

$$= 1 + 1.0551 \times 0.1 = 1.1055$$

b) $x_1 = 0.1, y_1 = 1.1055 \Rightarrow y(1) = 1.1055$

$$m_1 = f(x_1, y_1) = f(0.1, 1.1055)$$

$$= (0.1)^2 + 1.1055 = 1.1155$$

$$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + m_1 \cdot \frac{h}{2}\right)$$

$$= f\left(0.1 + \frac{0.1}{2}, 1.1055 + 1.1155 \times \frac{0.1}{2}\right)$$

$$= f(0.15, 1.1613)$$

$$= 0.15^2 + 1.1613 = 1.1838$$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + m_2 \cdot \frac{h}{2}\right)$$

$$= f\left(0.1 + \frac{0.1}{2}, 1.1055 + 1.1828 \times \frac{0.1}{2}\right)$$

$$= f(0.15, 1.1647)$$

$$= 0.15^2 + 1.1647 = 1.1872$$

$$m_4 = (x_1 + h, y_1 + m_3 h)$$

$$= f(0.1 + 0.1, 1.1055 + 1.1872 \times 0.1)$$

$$= f(0.2, 1.2242) = (0.2)^2 + 1.2242 = 1.2642$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

$$= \frac{1.1155 + 2 \times 1.1838 + 2 \times 1.1872 + 1.2642}{6}$$

$$= 1.18695$$

$$\therefore \quad y_2 = y_1 + m \times h$$

$$= 1.1055 + 1.18695 \times 0.1$$

$$= 1.2242$$

Hence,

$$y(0.2) = y_2 = 1.2242$$

i.e., $y = 1.2242$ when $x = 0.2$

NOTE

RK–4 method can be done using alternative formula, which has been used in Q. No.: (3) below.

3. Apply Runge-Kutta Fourth order method to find an approximate value of y when x = 0.2 given by $\frac{dy}{dx} = x + y^2$ and y = 1 when x = 0. Take h = 0.1. [2017]

Solution:

Here;

$$\frac{dy}{dx} = x + y^2, y(0) = 1 \text{ (i.e., } x_0 = 0, y_0 = 1)$$

$$h = 0.1 \text{ (step size)}$$

a) $x_0 = 0, y_0 = 1, h = 0.1$

$$k_1 = h \cdot f(x_0, y_0) = 0.1 \cdot f(0, 1) = 0.1 \times (0 + 1^2) = 0.1$$

$$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1 \cdot f(0.05, 1.05) = 0.1152$$

$$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \cdot f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1152}{2}\right)$$

$$= 0.1 \cdot f(0.05, 1.057) = 0.1168$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3)$$

$$= 0.1 \cdot f(0 + 0.1, 1 + 0.1168)$$

$$= 0.1 \cdot f(0.1, 1.1168) = 0.1347$$

$$\therefore \quad k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} = \frac{0.1 + 0.1152 \times 2 + 0.1168 \times 2 + 0.1347}{6}$$

$$= 0.1165$$

$$\therefore \quad y_1 = y_0 + k = 1 + 0.1165 = 1.1165$$

b) $y_1 = 1.1165, x_1 = 0.1, h = 0.1$

$$k_1 = h \cdot f(x_1, y_1) = 0.1 \cdot f(0.1, 1.1165)$$

$$= 0.1 \times [0.1 + (1.1165)^2] = 0.1347$$

$$k_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1 \cdot f\left(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1347}{2}\right)$$

$$= 0.1 \cdot f(0.15, 1.1838) = 0.1551$$

$$k_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 \cdot f\left(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1551}{2}\right)$$

$$= 0.1 \cdot f(0.15, 1.19405) = 0.1576$$

$$k_4 = h \cdot f(x_1 + h, y_1 + k_3)$$

$$= 0.1 \cdot f(0.1 + 0.1, 1.1165 + 0.1576)$$

$$= 0.1 \cdot f(0.2, 1.2741) = 0.1823$$

$$\therefore \quad k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$= \frac{0.1347 + 2 \times 0.1551 + 2 \times 0.1576 + 0.1823}{6}$$

$$= 0.1571$$

$$\therefore \quad y_2 = y_1 + k = 1.1165 + 0.1571 = 1.2736$$

Hence,

$y = 1.2736$ when $x = 0.2$

i.e., $y(0.2) = 1.2736$

4. Solve the initial value problem $y' = 2x + 3y$, $y(0) = 1$ at $x = 0.2$ using Runge-Kutta Fourth order method. Take $h = 0.1$. **[2018]**

Solution:

Here;

$$y' = \frac{dy}{dx} = 2x + 3y, \; y(0) = 1$$

a) $y_0 = 1, x_0 = 0, h = 0.1$

$$m_1 = f(x_0, y_0) = f(0, 1) = 2 \times 0 + 3 \times 1 = 3$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right)$$

$$= f\left(0 + \frac{0.1}{2}, 1 + 3 \times \frac{0.1}{2}\right)$$

$$= f(0.05, 1.15)$$

$$= 2 \times 0.05 + 3 \times 1.15 = 3.55$$

$$m_3 = f(x_0 + h, y_0 + m_3 h)$$

$$= f\left(0 + \frac{0.1}{2}, 1 + 3.55 \times \frac{0.1}{2}\right)$$

$$= f(0.05, 1.1775)$$

$$= 2 \times 0.05 + 3 \times 1.1775 = 3.6325$$

$$m_4 = (x_0 + h, y_0 + m_3 h)$$

$$= f\left(0 + 0.1, 1 + 3.6325 \times \frac{0.1}{2}\right)$$

$$= f(0.1, 1.1816) = 2 \times 0.1 + 3 \times 1.1816 = 3.7449$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

$$= \frac{3 + 2 \times 3.55 + 2 \times 3.6325 + 3.7449}{6}$$

$$= 3.5183$$

$$\therefore \quad y_1 = y_0 + m \times h$$

$$= 1 + 3.5183 \times 0.1 = 1.3518$$

b) $x_1 = x_0 + h = 0 + 0.1 = 0.1, y_1 = 1.3518$

$$m_1 = f(x_1, y_1) = f(0.1, 1.3518)$$

$$= 2 \times 0.1 + 3 \times 1.3518 = 4.2554$$

$$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + m_1 \cdot \frac{h}{2}\right)$$

$$= f\left(0.1 + \frac{0.1}{2}, 1.3518 + 4.2554 \times \frac{0.1}{2}\right)$$

$$= f(0.15, 1.5646)$$

$$= 2 \times 0.15 + 3 \times 1.5646 = 4.9937$$

$$m_3 = f\left(x_1 + \frac{h}{2}, y_1 + m_2 \cdot \frac{h}{2}\right)$$

$$= f\left(0.1 + \frac{0.1}{2}, 1.3518 + 4.9937 \times \frac{0.1}{2}\right)$$

$$= f(0.15, 1.6014)$$

$$= 2 \times 0.15 + 3 \times 1.6014 = 5.1044$$

$$m_4 = f(x_1 + h, y_1 + m_3 h)$$

$$= f\left(0.1 + 0.1, 1.3518 + 5.1044 \times \frac{0.1}{2}\right)$$

$$= f(0.2, 1.6070) = 2 \times 0.2 + 3 \times 1.6070 = 5.2210$$

$$\therefore \quad m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

$$= \frac{4.2554 + 2 \times 4.9937 + 2 \times 5.1044 + 5.2210}{6}$$

$$= 1.9454$$

$$\therefore \quad y_2 = y_1 + m \times h = 1.3518 + 4.9454 \times 0.1 = 1.8463$$

Hence,

$$y(0.2) = 1.8463$$

i.e., $y = 1.8463$ when $x = 0.2$

5. Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.5$, given that $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$ and $y = 1$ when $x = 0$, take $h = 0.25$. **[2019]**

Solution:

We have,

$$\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$$

x	0	0.25	0.5
y	1	y_1	y_2

Step 1: To find y(0.25)

Here,

$$x_0 = 0, y_0 = 1, h = 0.25$$

$$k_1 = hf(x_0, y_0) = 0.25 f(0, 1) = 0.25 \times \frac{(0)^2 - (1)^2}{(0)^2 + (1)^2} = -0.25$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.25 \times f\left(0 + \frac{0.25}{2}, 1 + \frac{-0.25}{2}\right)$$

$$= 0.25 \times f(0.125, 0.875)$$

$$= 0.25 \times \frac{(0.125)^2 - (0.875)^2}{(0.125)^2 + (0.875)^2}$$

$$= -0.24$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.25 \times f(0.125, 0.88) = -0.2401$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.25 \times f(0.25, 1 - 0.2401)$$

$$= 0.25 \times f(0.25, 0.7599)$$

$$= -0.27244$$

$$\therefore \quad k_4 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{-0.25 - 2 \times 0.24 - 2 \times 0.2401 - 0.27244}{6}$$

$$= -0.247106$$

$$\therefore \quad y(0.25) = y_0 + k = 1 - 0.247106 = 0.7529$$

Step 2: To find y(0.5)

$$x_1 = 0.25, y_1 = 0.7529, h = 0.25$$

$$k_1 = hf(x_1, y_1) = 0.25\, f(0.25, 0.729) = -0.2$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.25 \times f(0.375, 0.6529) = -0.1259$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.25 \times f(0.375, 0.68995) = -0.1359$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.25 \times f(0.5, 0.617) = -0.0518$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{-(0.2 + 2 \times 0.1259 + 2 \times 0.1359 + 0.0518)}{6}$$

$$= -0.1292$$

$$\therefore \quad y(0.5) = y_1 + k = 0.7529 - 0.1292 = 0.62366$$

Chapter 7

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATION

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We classify partial differential equation of second order. The general lines partial equation of second order in two independent variable in the form;

$$A\frac{d^2U}{dx^2} + B\frac{d^2U}{dx\,dy} + C\frac{d^2U}{dy^2} + D\frac{dU}{dx} + \frac{dU}{dy} + FU = 0$$

or, $$AU_{xx} + BU_{yy} + CU_{yy} + DU_x + EU_y + FU = 0 \qquad (1)$$

where, A, B, C, D and E are function of 'x' and 'y'.

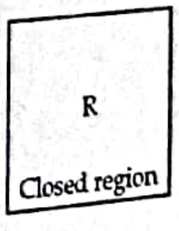
i) Elliptical at point (x, y) if $B^2 - 4AC < 0$

ii) Parabolic if $B^2 - 4AC = 0$

iii) Hyperbolic if $B^2 - 4AC > 0$

7.1 ELLIPTICAL EQUATION

A differential equation is elliptical in a region 'R' if $b^2 - 4AC < 0$ at all points of the region. The boundary condition of this type of equation specify the function 'u' at every point of the closed boundary of the region 'R' within which the solution f(x, y) is to be determined as shown in the figure. Boundary conditions are given at every or some point of the boundary.



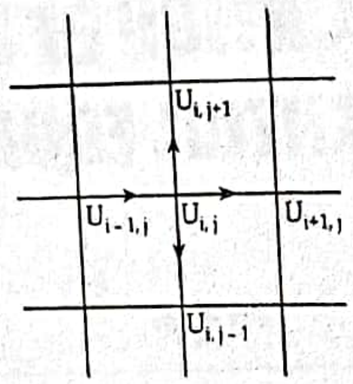
7.1.1 Laplace's Equation

An important equation of the elliptic type is;

$$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}=0$$

i.e., $U_{xx}+U_{yy}=0$ (1)

This is called Laplace's equation.

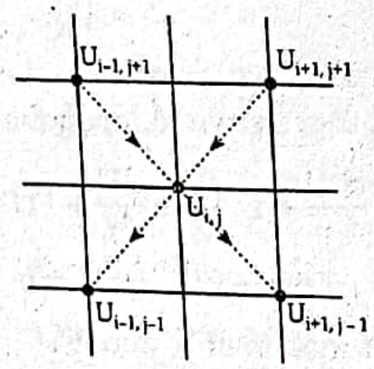


Taking square mesh and putting (1); we get,

$$U_{ij}=\frac{1}{4}[U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}] \quad (2)$$

That the value of 'u' at any interior meshes point is the arithmetic mean of its values at the four neighboring mesh point. This is called standard five point formula.

Instead of equation (2) we can also write the formula as;



$$U_{i,j}=\frac{1}{4}[U_{i-1,j-1}+U_{i+1,j-1}+U_{i-1,j-1}+U_{i+1,j+1}] \quad (3)$$

This is called diagonal five point formula (DFPE)

7.1.2 Poisson Equation

The partial differential equation is;

$$\nabla^2U=f(x,y)$$

or, $$\frac{d^2U}{dx^2}+\frac{d^2U}{dy^2}=f(x,y)$$

or, $U_{xx}+U_{yy}=f(x,y)$ (1)

where, f(x, y) is given function of 'x' and 'y' is called poisons equation.

$$U_{i-1,j}+U_{i+1,j}+U_{i,j-1}+U_{i,j+1}-4U_{ij}=h^2f(ih,jh) \quad (2)$$

Example 1

Solve the Poisson's partial differential equation $u_{xx}+u_{yy}=-10(x^2+y^2+10)$ over the region $0\le x\le 3$ and $0\le y\le 3$ with boundary condition $u(0, y)=0$, $u(3, y)=0$, $u(x, 0)=0$ and $u(x, 3)=0$. Assuming mess length 1.

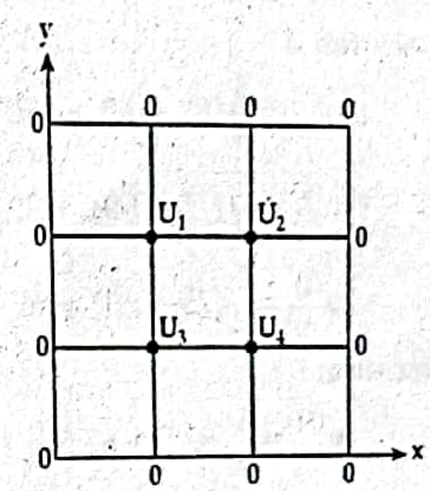
Solution:

Here,

$h=1$

The standard 5-point formula is,

$$u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4u_{i,j}=-10(i^2+j^2+10) \quad (1)$$



For $u_1(i=1, j=2)$,

$$0+u_2+u_3-4u_1=-10(1+4+10)$$

or, $$u_1=\frac{1}{4}(u_2+u_3+150) \quad (2)$$

For $u_2(i=2, j=2)$,

$$u_2=\frac{1}{4}(u_1+u_4+180) \quad (3)$$

For $u_3(i=1, j=1)$,

$$u_3=\frac{1}{4}(u_1+u_4+120) \quad (4)$$

For $u_4(i=2, j=1)$,

$$u_4=\frac{1}{4}(u_2+u_3+150) \quad (5)$$

From equation (2) and (5); we get,

$u_4=u_1$

$\therefore$ $u_1=\frac{1}{4}(u_2+u_3+150)$

$\therefore$ $u_2=\frac{1}{2}(u_1+90)$

$\therefore$ $u_3=\frac{1}{2}(u_1+60)$

Solving equations by G-S method,

Iteration 1

Let $u_2=u_3=0$,

$u_1^{(1)}=37.5$

$u_2^{(1)}=\frac{1}{2}(37.5+90)\approx 64$

$u_3^{(1)}=\frac{1}{2}(37.5+60)\approx 49$

Iteration 2

$$u_1^{(2)} = \frac{1}{4}(64 + 49 + 150) \approx 66$$

$$u_2^{(2)} = \frac{1}{2}(66 + 90) \approx 78$$

$$u_3^{(2)} = \frac{1}{2}(66 + 60) \approx 63$$

Iteration 3

$$u_1^{(3)} = \frac{1}{4}(78 + 63 + 150) \approx 73$$

$$u_2^{(3)} = \frac{1}{2}(73 + 90) \approx 82$$

$$u_3^{(3)} = \frac{1}{2}(73 + 60) \approx 67$$

Iteration 4

$$u_1^{(4)} = \frac{1}{4}(82 + 67 + 150) \approx 75$$

$$u_2^{(4)} = \frac{1}{2}(75 + 90) \approx 82.5$$

$$u_3^{(4)} = \frac{1}{2}(75 + 60) \approx 67.5$$

Iteration 5

$$u_1^{(5)} = \frac{1}{4}(82.5 + 67.5 + 150) \approx 75$$

$$u_2^{(5)} = \frac{1}{2}(75 + 90) \approx 82.5$$

$$u_3^{(5)} = \frac{1}{2}(75 + 60) \approx 67.5$$

Hence, the values of iteration 4 and 5 are same.

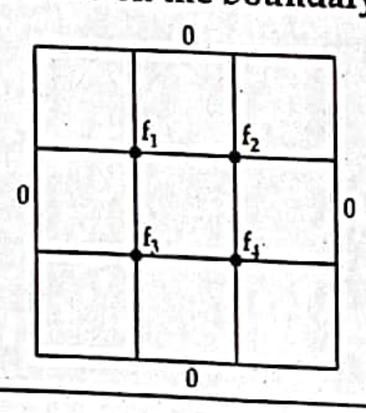
$\therefore \quad u_1 = 75 = u_4$

$\therefore \quad u_2 = 82.5$

$\therefore \quad u_3 = 67.5$

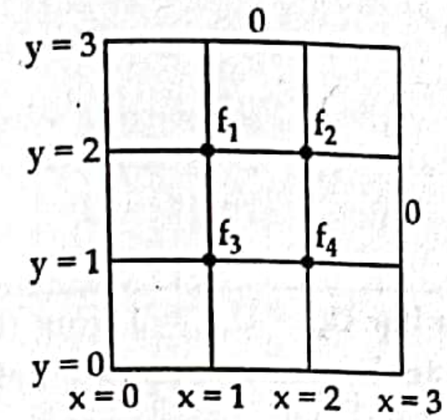
Example 2

Solve the Poisson equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3,\ 0 \le y \le 3$ with $f = 0$ on the boundary. Take $h = 1$



Solution:

The domain is divided into square of 1 unit size as shown.



Now, applying standard 5 point formula at each grid point; we get,

At $f_1(x = 1, y = 2)$;

$$0 + 0 + f_2 + f_3 - 4f_1 = 2 \times 1^2 \times 2^2 = 8$$

or, $$f_1 = \frac{1}{4}(f_2 + f_3 - 8) \quad (1)$$

At $f_2(x = 2, y = 2)$;

$$f_1 + f_4 - 4f_2 = 2 \times 2^2 \times 2^2 = 32$$

or, $$f_2 = \frac{1}{4}(f_1 + f_4 - 32) \quad (2)$$

At $f_3(x = 1, y = 1)$;

$$f_1 + f_4 - 4f_3 = 2 \times 1^2 \times 1^2 = 2$$

or, $$f_3 = \frac{1}{4}(f_1 + f_4 - 2) \quad (3)$$

At $f_4(x = 2, y = 1)$;

$$f_2 + f_3 - 4f_4 = 2 \times 2^2 \times 1^2 = 8$$

or, $$f_4 = \frac{1}{4}(f_2 + f_3 - 8) \quad (4)$$

Now, using successive iteration method; we have,

Let's use Gauss-Seidal method; assume $f_2 = f_3 = f_4 = 0$ as initial guess

Iteration	f_1	f_2	f_3	f_4
1	– 2	– 8.5	– 1	– 4.375
2	– 4.375	– 10.185	– 2.6875	– 5.21875
3	– 5.21875	– 10.6094	– 3.1094	– 5.43
4	– 5.43	– 10.715	– 3.215	– 5.482
5	– 5.483	– 10.741	– 3.241	– 5.496
6	– 5.496	– 10.748	– 3.248	– 5.499
7	– 5.499	– 10.749	– 3.249	– 5.499
8	– 5.499	– 10.749	– 3.249	– 5.499

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Since, the 7th and 8th iteration are same;

$\therefore \quad f_1 = -5.5$

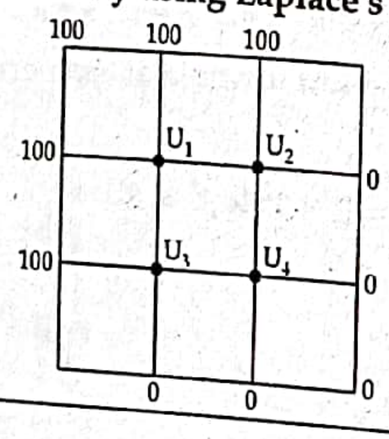
$f_2 = -10.75$

$f_3 = -3.25$

and, $f_4 = -5.5$

Example 3

Solve the elliptical equation $U_{xx} + U_{yy} = 0$ from the following square mesh with boundary values by using Laplace's method.



Solution:

Here, using standard 5-point formula;

$$U_1 = \frac{1}{4}(100 + 100 + U_2 + U_3)$$

or, $$U_1 = \frac{1}{4}(200 + U_2 + U_3) \quad (1)$$

Similarly;

$$U_2 = \frac{1}{4}(100 + U_1 + U_4 + 0)$$

$$\therefore \quad U_2 = \frac{1}{4}(U_1 + U_4 + 100)$$

$$U_3 = \frac{1}{4}(U_1 + 100 + 0 + U_4)] \quad (2)$$

$$\therefore \quad U_3 = \frac{1}{4}(U_1 + U_4 + 100)$$

and, $$U_4 = \frac{1}{4}(0 + 0 + U_2 + U_3) \quad (3)$$

$$U_4 = \frac{1}{4}(U_2 + U_3)$$

Again, using 5-point formula; we have, (4)

$$U_1 = \frac{1}{4}(100 + 100 + 100 + U_4)$$

For initial condition;

Let $U_4 = 0$

$$\therefore \quad U_1 = \frac{1}{4} \times 300 = 75$$

Then,

$U_2 = 43.75$

$U_3 = 43.75$

$U_4 = 21.88$

Tabulating the successive approximation

Iteration	U_1	U_2	U_3	U_4
Initial	75	43.75	43.75	21.88
1	71.88	48.44	48.44	24.22
2	74.22	49.61	49.61	24.81
3	74.81	49.91	49.91	24.99
4	74.96	49.98	49.98	24.99
5	74.99	50.00	50.00	25.00
6	75.00	50.00	50.00	25.00
7	75.00	50.00	50.00	25.00

From sixth and seventh iteration; we have,

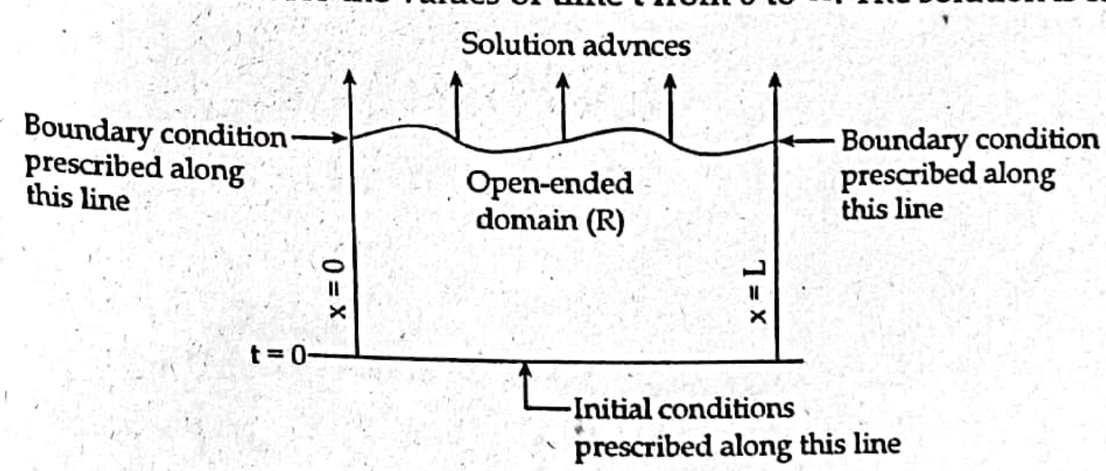
$U_1 = 75$

$U_2 = U_3 = 50$

and, $U_4 = 25$

7.2 PARABOLIC EQUATIONS

The one-dimensional heat conduction equation $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$ is a well known example of parabolic partial differential equations. The solution of this equation is a temperature function u(x, t) which is defined for values of x from 0 to l and for the values of time t from 0 to ∞. The solution is not



defined in a closed domain but advances in an open-ended region from initial values, satisfying the prescribed boundary conditions.

In general, the study of pressure waves in a fluid, propagation of heat and unsteady state problems lead to parabolic type of problems.

7.3 HYPERBOLIC EQUATIONS

The wave equation $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest example of hyperbolic partial differential equations. Its solution is the displacement function u(x, t) defined for values of x from 0 to l and for t from 0 to ∞, satisfying the initial and boundary conditions. The solution as for parabolic equations, advances in an open-ended region (refer 7.2 for figure). In the case of hyperbolic equations however, we have two initial conditions and two boundary conditions.

Such equations arise from convective type of problems in vibrations, wave mechanics and gas dynamics.

SOLVED NUMERICAL (POKHARA UNIVERSITY)

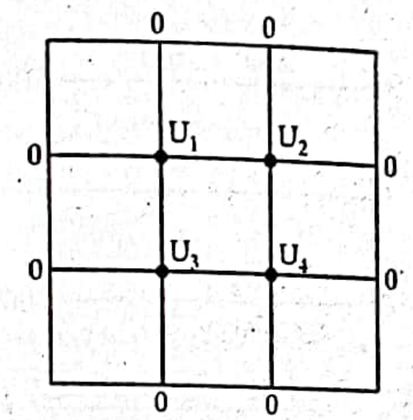
1. **Solve the Poisson equation $\nabla^2 f = 2x^2 + y$ over the square domain $1 \le x \le 4$, $1 \le y \le 4$ with $f = 0$ on the boundary Take $h = k = 1$**
[2005 Spring, 2007 Spring, 2010 Fall]

Solution:

The given equation is;

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} = 2x^2 + y^2 \qquad (1)$$

Note that the boundary limits are 1–4 not, 0–3.



Here,

$$U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j} = (2x^2 + y)h^2 \qquad (2)$$

For U_3;

$$x = 2, y = 2$$

$$U_3 = \frac{1}{4}(U_1 + U_4 - 10) \qquad (a)$$

For U_1;

$$x = 2, y = 3$$

$$U_3 = \frac{1}{4}(U_2 + U_3 - 11) \qquad (b)$$

For U_2;

$$x = 3, y = 3$$

$$U_4 + U_1 - 4U_2 = [2 \times (3)^2 + 3] \times (1)^2$$

or, $$U_2 = \frac{1}{4}(U_1 + U_4 - 21) \qquad (c)$$

For U_4;

$$x = 3, y = 2$$

$$U_2 + U_3 - 4U_4 = [2 \times (3)^2 + 2] \times (1)^2$$

or, $$U_4 = \frac{1}{4}(U_2 + U_3 - 20) \qquad (d)$$

On solving (a), (b), (c) and (d) by Gauss seidel method; we have,

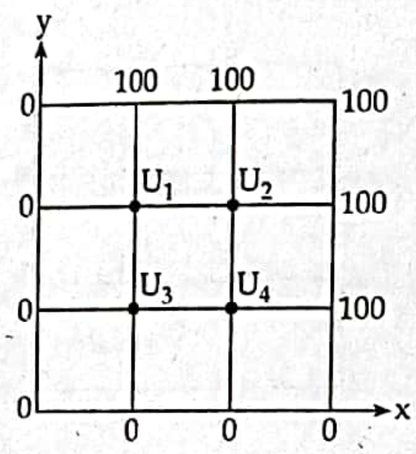
$U_1 = -6.625$

$U_3 = -9.125$

$U_3 = -6.375$

$U_3 = -8.875$

2. Solve the Poisson's equation $u_{xx} + u_{yy} = -81\,xy$, $0 < x < 1$, $0 < y < 1$ with boundary condition $u(0, y) = u(x, 0) = 0$ and $u(1, y) = u(x, 1) = 100$ and taking $h = \frac{1}{3}$. [2012 Spring]



Solution:

Here,

$$h = \frac{1}{3}$$

The standard 5-point formula for given equation is;

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2(ih, jh) = h^2[-81(ih \cdot jh)]$$

$$= h^4(-81)ij = -ij \quad (1)$$

For $u_1(i = 1, j = 2)$,

$$0 + u_2 + u_3 + 100 - 4u_1 = -2 \quad (2)$$

For $u_2(i = 2, j = 2)$,

$$u_1 + 100 + u_4 + 100 - 4u_2 = -4 \quad (3)$$

For $u_3(i = 1, j = 1)$,

$$0 + u_4 + 0 + u_1 - 4u_3 = -1 \quad (4)$$

For $u_4(i = 2, j = 1)$,

$$u_3 + u_2 - 4u_4 = -102 \quad (5)$$

Subtracting (5) from (2); we get,

$$-4u_1 + 4u_4 = 0$$

or, $u_1 = u_4$

Equation (3) becomes,

$$2u_1 - 4u_2 = -204 \quad (6)$$

Equation (4) becomes,

$$2u_1 - 4u_3 = -1 \quad (7)$$

Now, multiply equation (2) by 4 and add equation (6); we get,

$$-14u_1 + 4u_3 = -612 \quad (8)$$

Add equation (7) and (8) gives,

$$-12u_1 = -613$$

Thus,

$$u_1 = \frac{613}{12} = 51.0833 = u_4$$

$$u_2 = \frac{1}{2}(u_1 + 102) = 76.5477$$

$$u_3 = \frac{1}{2}\left(u_1 + \frac{1}{2}\right) = 25.7916$$

3. Solve the Poisson equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = k = 1$. [2018 Spring]

Solution:

Here; we know that;

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{1}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{1}{k^2}(u_{i,j-1} - 2u_{i,j} + u_{i,j+1})$$

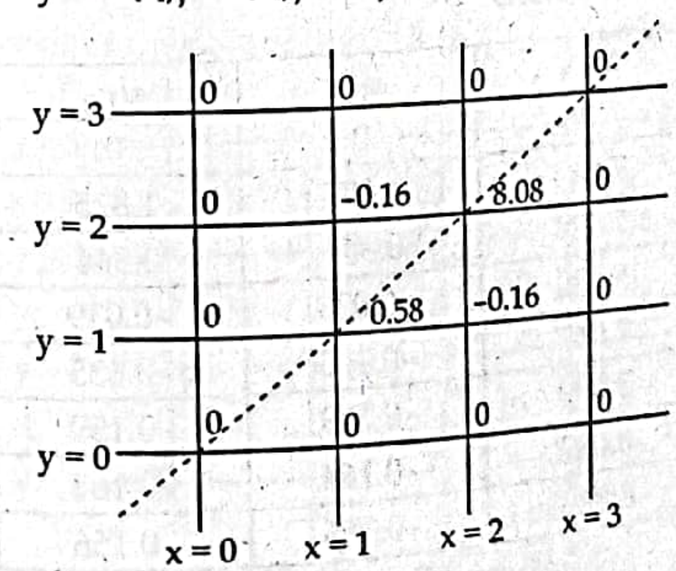
[$\because$ Given $h = k = 1$]

$$\nabla^2 f = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}$$

so, $\nabla^2 f = 2x^2 y^2$

or, $u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 2x^2 y^2$

or, $u_{i,j} = \frac{1}{4}(2x^2 y^2 - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} + u_{i,j+1})$



Given, boundary value is 0. So,

$f_{0,i} = 0$ for $i \in [0, 3]$

$f_{j,0} = 0$ for $j \in [0, 3]$

$f_{3,i} = 0$ for $i \in [0, 3)$

$f_{j,3} = 0$ for $j \in (0, 3)$

Now,

For i = 1 and j = 1

$$u_{1,1} = \frac{1}{4}(2 \times 1^2 + 1^2 - u_{0,1} - u_{2,1} - u_{1,0} - u_{1,2})$$

or, $$u_{1,1} = \frac{1}{4}(2 - u_{2,1} - u_{1,2}) \quad (1)$$

For i = 1 and j = 2

$$u_{1,2} = \frac{1}{4}(2 \times 1^2 + 2^2 - u_{0,2} - u_{2,2} - u_{1,1} - u_{1,3})$$

or, $$u_{1,2} = \frac{1}{4}(8 - 0 - u_{2,2} - u_{1,1} - 0)$$

or, $$u_{1,2} = \frac{1}{4}(8 - u_{2,2} + u_{1,1}) \quad (2)$$

For i = 2 and j = 1

$$u_{2,1} = \frac{1}{4}(2 \times 2^2 + 1^2 - u_{1,1} - u_{3,1} - u_{2,0} - u_{2,2})$$

or, $$u_{2,1} = \frac{1}{4}(8 - u_{1,1} - u_{2,2}) \quad (3)$$

For i = 2 and j = 2

$$u_{2,2} = \frac{1}{4}(2 \times 2^2 \times 2^2 - u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3})$$

or, $$u_{2,2} = \frac{1}{4}(32 - u_{1,2} - u_{2,1}) \quad (4)$$

Now, using above equations

n	$u_{2,2}$	$u_{2,1}$	$u_{1,2}$	$u_{1,1}$
0	0	0	0	0.5
1	7.063	1.875	1.875	-0.4389
2	7.828	0.344	0.344	0.328
3	8.020	-0.039	-0.039	0.520
4	8.067	-0.135	-0.135	0.567
5	8.079	-0.159	-0.159	0.579
6	8.082	-0.164	-0.164	0.582
7	8.083	-0.166	-0.166	0.583

so, $u_{1,1} = 0.58$

$u_{1,2} = -0.16$

$u_{2,1} = -0.16$

$u_{2,2} = 8.08$

NOTE:
We can also use the property of symmetry about the line; and find that $u_{1,2} = u_{2,1}$. Hence, we just need to 3 equations only. Which further simplify the problem?

4. **Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh $0 \le x \le 3, 0 \le y \le 3$ with $u = 0$ on the boundary and mesh length $h = k = 1$.** **[2018 Fall]**

Solution:

Here;

$$\nabla^2 u = -10(x^2 + y^2 + 10)$$

or, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10) \quad (1)$$

and, we know that;

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}]$$

so, equation (1) becomes,

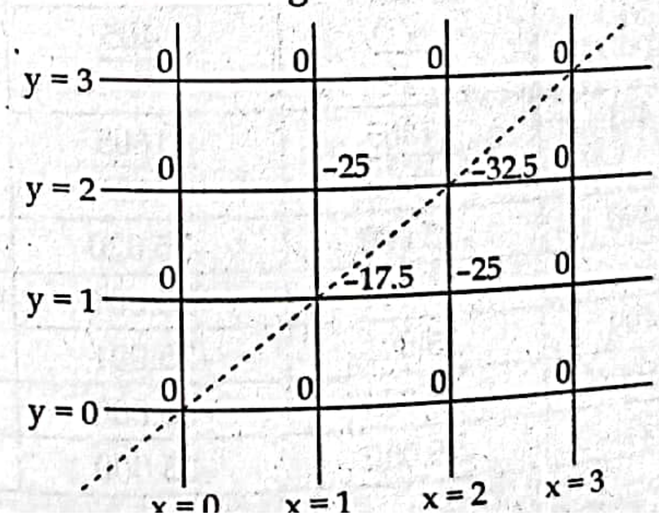
$$\frac{1}{h^2}[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}] + \frac{1}{k^2}[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] = -10(x^2 + y^2 + 10)$$

or, $$[u_{i,j-1} - 2u_{i,j} + u_{i,j+1} + u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] = -10(x^2 + y^2 + 10)$$

$[\because h = k = 1]$

or, $$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(x^2 + y^2 + 10) \quad (2)$$

Now, in square mesh of mesh length h = k = 1.



or, $u_{i,j} = \frac{1}{4}[-10(x^2 + y^2 + 10) - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}]$

Now,

For $i = 1, j = 1$

so, $u_{1,1} = \frac{1}{4}[-10(1^2 + 1^2 + 10) - u_{0,1} - u_{2,1} - u_{1,0} - u_{1,2}]$

$u_{1,1} = \frac{1}{4}(-120 - u_{2,1} - u_{1,2})$ (1)

For $i = 1, j = 2$

so, $u_{1,2} = \frac{1}{4}[-10(1^2 + 2^2 + 10) - u_{0,2} - u_{1,3} - u_{2,2} - u_{1,1}]$

$= \frac{1}{4}(-150 - 0 - 0 - u_{2,2} - u_{2,2})$

$= \frac{1}{4}(-150 - u_{2,2} - u_{1,1})$ (2)

For $i = 2, j = 1$

so, $u_{2,1} = \frac{1}{4}[-10(2^2 + 1^2 + 10) - u_{2,0} - u_{2,2} - u_{1,1} - u_{3,1}]$

$= \frac{1}{4}(-150 - u_{1,1} - u_{2,2})$ (3)

For $i = 2, j = 2$

so, $u_{2,2} = \frac{1}{4}[-10(2^2 + 2^2 + 10) - u_{1,2} - u_{3,2} - u_{2,1} - u_{2,3}]$

$= \frac{1}{4}(-180 - u_{1,2} - 0 - u_{2,1} - 0)$

$= \frac{1}{4}(-180 - u_{1,2} - u_{2,1})$ (4)

i	$u_{1,1}$	$u_{1,2}$	$u_{2,1}$	$u_{2,2}$
0	-30	0	0	
1	-15	-30	-30	-30
2	$-\frac{135}{8}$	$-\frac{105}{4}$	$-\frac{105}{4}$	$-\frac{255}{8}$
3	$-\frac{555}{32}$	$-\frac{405}{16}$	$-\frac{405}{16}$	$-\frac{1035}{32}$
4	-17.461	$-\frac{1605}{64}$	$-\frac{1605}{64}$	$-\frac{4155}{128}$
5	-17.490	-25.020	-25.020	-32.490
6	-17.500	-25.004	-25.004	-32.498
7	-17.499	-25.001	-25.001	-32.499
8	-17.500	-25.000	-25.000	-32.500
9	-17.500	-25.000	-25.000	-32.500

Hence,

$u_{1,1} = -17.500$

$u_{1,2} = -25.000$

$u_{2,1} = -25.000$

$u_{2,2} = -32.500$

NOTE:

We can also use symmetry property from figure along diagonal.

i.e., $u_{1,2} = u_{2,1}$

5. **Given the Poisson's equation: $\nabla^2 u = -10(x^2 + y^2 + z^2)$ over the square domain such that $0 \le x \le 3$ and $0 \le y \le 3$ with Dirichlet boundary condition of $u(x, y) = 0$. Calculate steady state temperatures at interior points by using successive over relaxation method up to 5th iteration. Assume $h = k = 1$. [2017 Fall]**

Solution:

Here;

At $z = 0$

so, $\nabla^2 u = u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}$

or, $u_{i,j} = \frac{1}{4}[10(x^2 + y^2) + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$

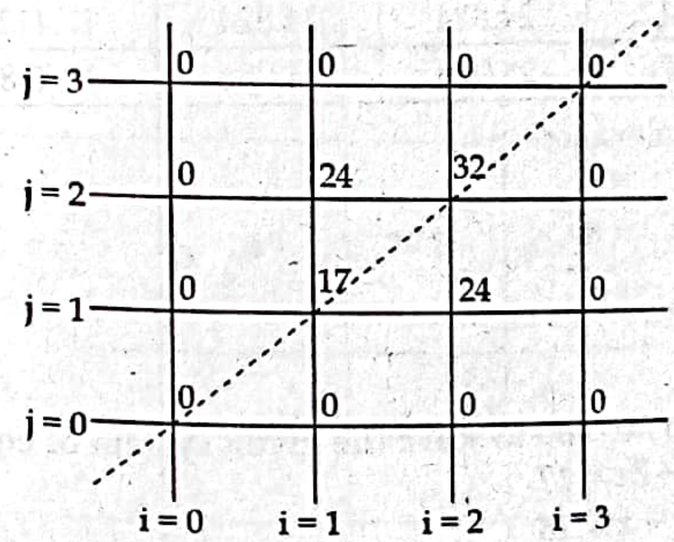
Now,

$x_n = x_0 + nh$

or, $x_n = x_0 + n$

$y_n = y_0 + nk$

or, $y_n = y_0 + n$ [$\because h = k = 1$]



At boundary $u = 0$

so, $u_{0,i} = u_{i,0} = u_{3,i} = u_{i,3} = 0$

For $i \in [0, 3]$

Now,

For $i = 1$ and $j = 1$

$$u_{1,1} = \frac{1}{4}(10 \times 2 + u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2})$$

$$= \frac{1}{4}(20 + u_{1,2} + u_{2,1})$$

For i = 1 and j = 2

$$u_{1,2} = \frac{1}{4}[10(1^2 + 2^2) + u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3}]$$

$$= \frac{1}{4}(50 + u_{1,1} + u_{2,2})$$

For i = 2 and j = 1

$$u_{2,1} = \frac{1}{4}[10(2^2 + 1^2) + u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2}]$$

$$= \frac{1}{4}(50 + u_{1,1} + u_{2,2})$$

For i = 2 and j = 2

$$u_{2,2} = \frac{1}{4}[10(2^2 + 2^2) + u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3}]$$

$$= \frac{1}{4}(80 + u_{1,2} + u_{2,1})$$

Now, from above equation

n	$u_{2,2}$	$u_{2,1}$	$u_{1,2}$	$u_{1,1}$
0	0	0	0	5
1	26.875	13.75	13.75	11.875
2	31.094	22.188	22.188	16.094
3	32.148	24.297	24.297	17.148
4	32.412	24.824	24.824	17.412
5	32.478	24.956	24.956	17.478

Now, in 5th success relaxation

$u_{1,1} = 17$

$u_{1,2} = 24$

$u_{2,1} = 24$

$u_{2,2} = 32$

6. Use relaxation method to solve the given system of equations.

$$20x + y - 2z = 17$$
$$3x + 20y - z = 18$$
$$2x - 3y + 20z = 25$$

[2019 Fall]

Solution:

Here,

$$20x + y - 2z = 17 \quad (1)$$

$$3x + 20y - z = 18 \quad (2)$$

$$2x - 3y + 20z = 25 \quad (3)$$

Condition of convergence check;

$|20| > |1| + |-2| \Rightarrow 20 > 3$

$|20| > |3| + |-1| \Rightarrow 20 > 4$

$|20| > |2| + |-3| \Rightarrow 20 > 5$ (Ok)

Let, R_1, R_2, R_3 represent residual values in the equation (1), (2) and (3); we have,

$$R_1 = 17 - 20x - y + 2z \quad (4)$$

$$R_2 = 18 - 3x - 20y + z \quad (5)$$

$$R_3 = 25 - 2x + 3y - 20z \quad (6)$$

Writing in the table form;

	R_1	R_2	R_3
δ_x	−20	−3	−2
δ_y	−1	−20	3
δ_z	2	1	−20

NOTE:

To make $R_1 = 0$, we make the change in x, $R_2 = 0$ change in y and $R_3 = 0$ change in z.

Relaxation table

	R_1	R_2	R_3	Remarks
$\delta_x = \delta_y = \delta_z = 0$	17	18	25	Put x = 0, y = 0 and z = 0 in equation (4), (5) and (6)
$\delta_x = 0, \delta_y = 0, \delta_z = \frac{25}{20}$	19.5	19.25	0	Minimize maximum value of R *i.e.*, 25 is minimized to O.
$\delta_x = \frac{19.5}{20}, \delta_y = 0, \delta_z = \frac{25}{20}$	0	16.325	− 1.95	
$\delta_x = \frac{19.5}{20}, \delta_y = \frac{16.325}{20}, \delta_z = \frac{25}{20}$	− 0.81625	0	0.4987	
$\delta_x = \frac{19.5}{20}, \delta_y = \frac{16.325}{20},$	− 0.7664	0.0249	7.5×10^{-4}	

$\delta_z = \frac{25}{20} + \frac{0.4987}{20}$ $= 1.2749$				
$\delta_x = \frac{19.5}{20}$, $\delta_y = 1.2749$, $\delta_z = \frac{16.325}{20} + \frac{0.0249}{20}$ $= 0.81749$	-0.76769	10^{-4}	4.47×10^{-3}	
$\delta_x = \frac{19.5}{20}$, $\delta_y = 0.81749$, $\delta_z = 1.2749 + \frac{4.47 \times 10^{-3}}{20}$ $= 1.2751$	-0.76729	3×10^{-4}	4.7×10^{-4}	
$\delta_x = \frac{19.5}{20} - \frac{0.76729}{20}$ $= 0.9366$ $\delta_y = 0.81749$, $\delta_z = 1.2751$	7.1×10^{-4}	0.1155	0.7727	
$\delta_x = 0.9366$, $\delta_y = 1.2751$, $\delta_z = 0.81749 + \frac{0.1155}{20}$ $= 0.8232$	-5×10^{-3}	1.3×10^{-3}	0.0944	
$\delta_x = 0.9366$, $\delta_y = 0.8232$, $\delta_z = 1.2751 + \frac{0.0944}{20}$ $= 1.2798$	4.4×10^{-3}	6×10^{-3}	4×10^{-4}	

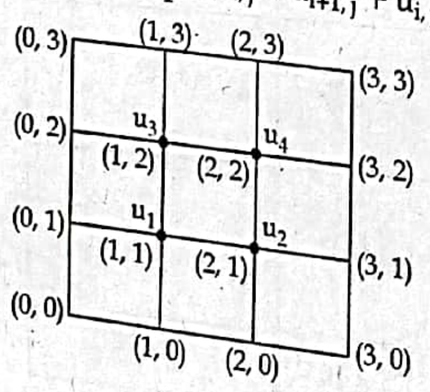
$\therefore$ $x = 0.9366$, $y = 0.8232$ and $z = 1.2798$

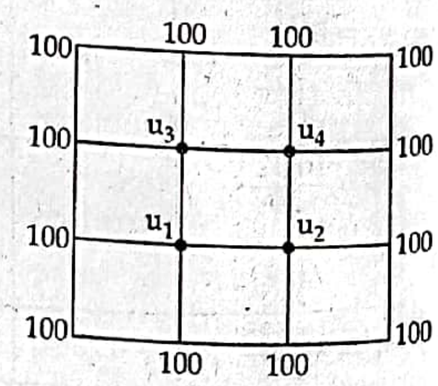
7. Given the Poisson's equation $\Delta^2 F = 4x^2y^2$ over the square domain $0 \le x \le 3$, $0 \le y \le 3$ with Dirichlet boundary condition $f(x, y) = 100$ and $h = k = 1$. Calculate steady state temperature at interior node by using Gauss–Siedel method. Iterate until the successive values at any point is correct to two decimal places. [2019 Fall]

Solution:

Here, finite difference equation for Poisson's equation is;

$$U_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - h^2 f(ih, jk)]$$





We have,

$$h = k = 1$$

$$f(x, y) = 4x^2y^2$$

or, $$f(ih, jk) = 4(ih)^2(jk)^2 = 4i^2j^2 \times h^2k^2 = 4i^2j^2 \qquad [\because h = k = 1]$$

Hence, the above equation becomes;

$$U_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - (1)^2 \times 4i^2j^2]$$

or, $$U_{i,j} = \frac{1}{4}(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4i^2j^2)$$

$\therefore$ $$U_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4i^2j^2) \qquad (1)$$

i) For $i, j = 1, 1$;

$$U_{1,1} = U_1 = \frac{1}{4}[u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 4 \times (1)^2 \times (1)^2]$$

$$= \frac{1}{4}(u_2 + 100 + \mu_3 + 100 - 4)$$

$$= \frac{1}{4}(u_2 + u_3 + 196) \qquad (2)$$

ii) For $i, j = 1, 2$;

$$U_{1,2} = U_3 = \frac{1}{4}[u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 4 \times (1)^2 \times (1)^2]$$

$$= \frac{1}{4}(u_4 + 100 + 100 + u_1 - 16)$$

$$= \frac{1}{4}(u_1 + u_4 + 184) \qquad (3)$$

iii) For $i, j = 2, 1$;

$$U_{2,1} = U_2 = \frac{1}{4}[u_1 + u_4 + 100 + 100 - 4 \times (2)^2 \times 1]$$

$$= \frac{1}{4}(u_1 + u_4 + 184) \qquad (4)$$

iv) For $i, j = 2, 2$;

$$U_{2,2} = U_4 = \frac{1}{4}[u_3 + u_2 + 100 + 100 - 4 \times (2)^2 \times (2)^2]$$

$$= \frac{1}{4}(u_2 + u_3 + 136) \qquad (5)$$

From equation (3) and (4); we find,

$$u_2 = u_3$$

Hence, equations can be modified as;

$$u_1 = \frac{1}{4}(u_2 + u_3 + 196) = \frac{1}{4}(2u_2 + 196) \qquad (6)$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 184) \qquad (7)$$

$$u_4 = \frac{1}{4}(u_2 + u_2 + 136) = \frac{1}{4}(2u_2 + 136) \quad (8)$$

Solving equation (6), (7) and (8) by Gauss Siedel method; we have,

Initial guess is $u_1 = u_2 = u_4 = 0$.

n	u_1	u_2	u_4
0	0	0	0
1	49	58.25	63.125
2	78.125	81.3125	74.6562
3	89.6562	87.0781	77.539
4	92.539	88.5195	78.2597
5	93.2597	88.87988	78.43994
6	93.4399	88.9699	78.4849
7	93.4849	88.9924	78.4962
8	93.4962	88.998	78.499
9	93.4990	88.999	78.4997
10	93.5	89	78.5

$\therefore$ $u_1 = 93.5$

$u_2 = 89$

$u_3 = 89$

$u_4 = 78.5$

NOTE:

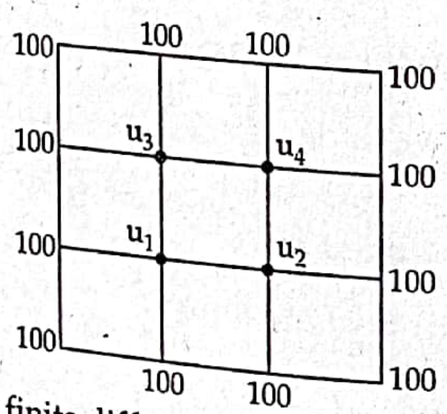
Calculator code

$$A : B : C : A = \frac{2B + 196}{4} : B = \frac{A + C + 184}{4} : C = \frac{2B + 136}{4}$$

8. **Solve the Poison's equation $u_{xx} + u_{yy} = 243\,(x^2 + y^2)$ over a square domain $0 \le x \le 1, 0 \le y \le 1$ with step size $h = \frac{1}{3}$ with $u = 100$ on the boundary.** [2019 Spring]

Solution:

Here;



Poisson's equation for finite difference equation is;

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(ih, jh)]$$

We have,

$$h = \frac{1}{3}$$

$$f(x, y) = 243\,(x^2 + y^2)$$

$$\therefore \quad f(ih, jh) = 243\,[(ih)^2 + (ij)^2] = 243 \times h^2\,[i^2 + j^2]$$

$$= 243 \times \left(\frac{1}{3}\right)^2 [i^2 + j^2]$$

$$= 27\,(i^2 + j^2)$$

Hence, above equation becomes;

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - \left(\frac{1}{3}\right)^2 \times 27(i^2 + j^2)\right]$$

$$= \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 3\,(i^2 + j^2)] \quad (1)$$

a) For $i = j = 1$

$$u_{1,1} = u_1 = \frac{1}{4}[u_2 + u_3 + 100 + 100 - 3\,(1^2 + 1^2)]$$

$$= \frac{1}{4}(u_2 + u_3 + 194) \quad (2)$$

b) For $(i, j) = (1, 2)$

$$u_{1,2} = u_3 = \frac{1}{4}[100 + 100 + u_4 + u_1 - 3\,(1^2 + 2^2)]$$

$$= \frac{1}{4}(u_1 + u_4 + 185) \quad (3)$$

c) For $(i, j) = (2, 1)$

$$u_{2,1} = u_2 = \frac{1}{4}[u_4 + u_1 + 100 + 100 - 3\,(1^2 + 2^2)]$$

$$= \frac{1}{4}(u_1 + u_4 + 185) \quad (4)$$

d) For $(i, j) = (2, 2)$

$$u_{2,2} = u_4 = \frac{1}{4}[u_2 + u_3 + 100 + 100 - 3\,(2^2 + 2^2)]$$

$$= \frac{1}{4}(u_2 + u_3 + 176) \quad (5)$$

From equation (3) and (4), we get,

$$u_2 = u_3$$

Hence other equations are modified as

$$u_1 = \frac{1}{4}(2u_2 + 194) \quad (6)$$

$$u_2 = \frac{1}{4}(u_1 + u_4 + 185)$$

$$u_4 = \frac{1}{4}(2u_2 + 176) \qquad (7)$$

Solving equation (6), (7), (8) by Gauss siedel method. (8)
Initial guess $u_1 = u_2 = u_4 = 0$.

n	u_1	u_2	u_4
0	0	0	0
1	48.5	58.375	73.1875
2	77.6875	83.9687	85.9843
3	90.4843	90.3671	89.1835
4	93.6836	91.9668	89.9834
5	94.4834	92.3667	90.1833
6	94.6833	92.4666	90.2333
7	94.7333	92.4916	90.2458
8	94.7458	92.4979	90.2489
9	94.7489	92.4994	90.2497
10	94.7497	92.4998	90.2499
11	94.75	92.5	90.25

$\therefore \quad u_1 = 94.75$

$\therefore \quad u_2 = u_3 = 92.5$

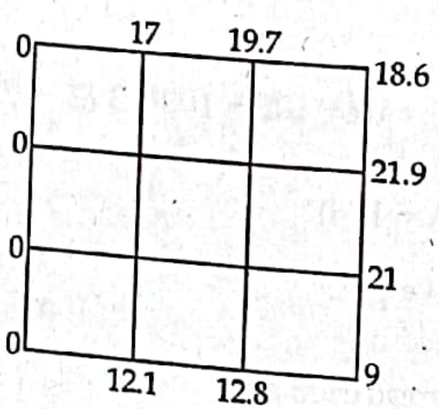
$\therefore \quad u_4 = 90.25$

NOTE:

Calculator code

$$A : B : C : A = \frac{2B + 194}{4} : B = \frac{A + C + 185}{4} : C = \frac{2B + 176}{4} + \text{calc}$$

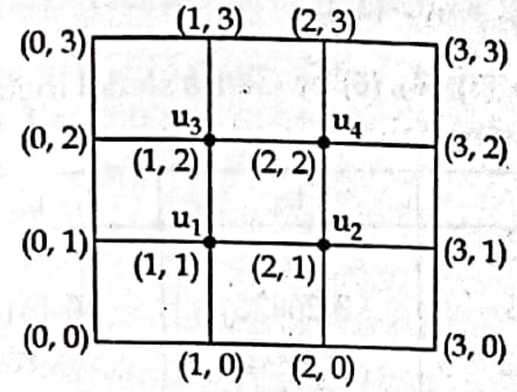
9. **Solve the Poisson equation $\nabla^2 f = 4x^2y + 3xy^2$ over the square domain $1 \le x \le 3$, $1 \le y \le 3$, with the boundary is given in figure below. Take $h = k = 1$.** **[2020 Fall]**



Solution:

Here;

$h = k = 1$



Poisson's equation for finite difference is;

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(ih, jk)]$$

We have,

$h = k = 1$

$f(x, y) = 4x^2y + 3xy^2$

$\therefore \quad f(ih, jk) = 4(ih)^2 (jk) + 3(ih)(jk)^2$

$= 4i^2j + 2ij^2 \qquad [\because h = k = 1]$

Hence, above equation becomes

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 1^2 \times (4i^2j + 3ij^2)]$$

$$= \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4i^2j - 3ij^2] \qquad (1)$$

a) For $i = j = 1$

$$u_{1,1} = u_1 = \frac{1}{4}[u_2 + u_3 + 0 + 12.1 - 4 \times 1^2 \times 1 - 3 \times 1 \times 1^2]$$

$$\therefore \quad u_1 = \frac{1}{4}(u_2 + u_3 + 5.1) \qquad (2)$$

b) For $i, j = 1, 2$

$$u_{1,2} = u_3 = \frac{1}{4}[u_4 + u_1 + 0 + 17 - 4 \times 1^2 \times 2 - 3 \times 1 \times 2^2]$$

$$\therefore \quad u_3 = \frac{u_1 + u_4 - 3}{4} \qquad (3)$$

c) For $i, j = 2, 1$

$$u_{2,1} = u_2 = \frac{1}{4}[u_1 + u_4 + 12.8 + 21 - 4 \times 2^2 \times 1 - 3 \times 2 \times 1^2]$$

$$\therefore \quad u_2 = \frac{u_1 + u_4 + 11.8}{4} \qquad (4)$$

d) For $(i, j) = (2, 2)$

$$u_{2,2} = u_4 = \frac{1}{4}[u_2 + u_3 + 19.7 + 21.9 - 4 \times 2^2 \times 2 - 3 \times 2 \times 2^2]$$

CamScanner

$$\therefore \quad u_4 = \frac{1}{4}(u_2 + u_3 - 14.4)$$

Solving equation (2), (3), (4), (5) by Gauss siedel method. (5)

Initial guess $u_1 = u_2 = u_3 = u_4 = 0$

n	u_1	u_2	u_3	u_4
0	0	0	0	0
1	1.275	3.26875	-0.43125	-2.8906
2	1.9943	2.7234	-0.9765	-3.1632
3	1.7117	2.5871	-1.1129	-3.2314
4	1.6435	2.5530	-1.1469	-3.2484
5	1.6265	2.5445	-1.1555	-3.2527
6	1.6222	2.5423	-1.1576	-3.2538
7	1.6212	2.5418	-1.1582	-3.2540
8	1.6209	2.5417	-1.1582	-3.2541

$\therefore \quad u_1 = 1.6209$

$u_2 = 2.5417$

$u_3 = -1.1582$

$u_4 = -3.2541$

Calculator code

$$A : B : C : D : A = \frac{B + C + 5.1}{4} : B = \frac{A + D + 11.8}{4} : C = \frac{A + D - 3}{4} :$$

$$D = \frac{B + C - 14.4}{4} + \text{calc}$$

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

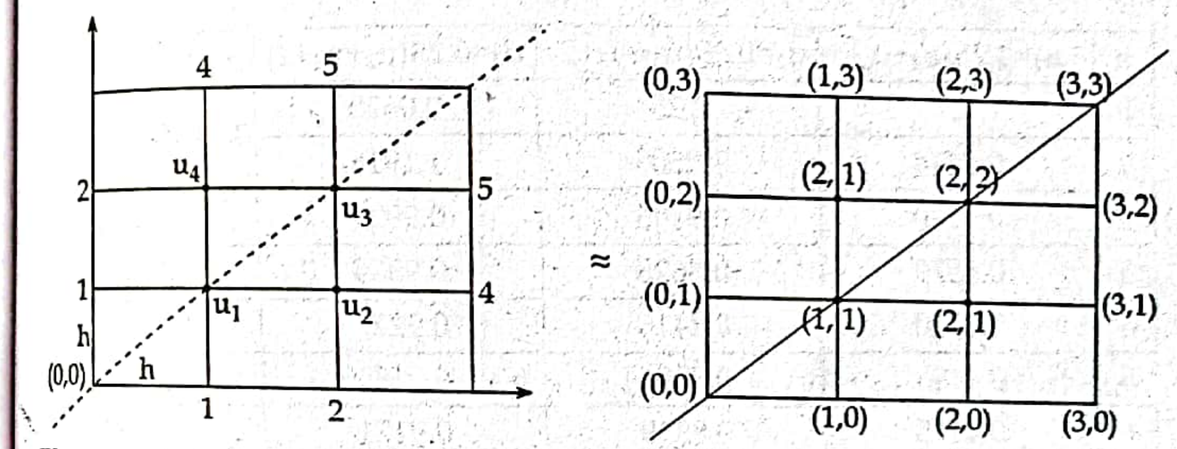
1. **Solve the Poisson's equation $\nabla^2 f - 2xy = 0$ over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = x + y$ on the boundary. Take h = 1.** **[2016]**

Solution:

Here;

$f = x + y; \quad 0 \le x \le 3, \quad 0 \le y \le 3, \quad h = 1$

Constructing Square Mesh



The co-ordinate (x, y) = (ih, jh)

Now,

$$\nabla^2 f - 2xy = 0 \qquad (1)\ [\nabla^2 f - f(x, y) = 0]$$

Using standary five point formula for given Poisson's equation; we have,

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 \cdot f(x, y)]$$

$$= \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 1^2(2xy)]$$

$$= \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 2(ih)(jh)]$$

$$= \frac{1}{2}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 2ij) \qquad (2)$$

a) For i = 1, j = 1

$$u_1 = \frac{1}{4}(u_4 + u_2 + 1 + 1 - 2 \times 1 \times 1)$$

$$= 0.25\,(u_4 + u_2) \qquad (3)$$

b) When i = 2, j = 1

$$u_2 = 0.25\,(u_1 + u_3 + 2 + 4 - 2 \times 2 \times 1)$$

$$= 0.25\,(u_1 + u_3 + 2) \qquad (4)$$

c) When i = 2, j = 2

$$u_3 = 0.25\,(u_2 + u_4 + 5 + 5 - 2 \times 2 \times 2)$$

$$= 0.25\,(u_2 + u_4 + 2) \qquad (5)$$

d) When $i = 1, j = 2$

$$u_4 = 0.25\,(u_1 + u_3 + 2 + 4 - 2 \times 1 \times 2)$$
$$= 0.25\,(u_1 + u_3 + 2)$$

Now, guess values;

$$u_1 = 0.25\,(4 + 4) = 2 \qquad (6)$$
$$u_2 = 0.25\,(2 + 5 + 2) = 2.25$$
$$u_3 = 0.25\,(4 + 2.25 + 2) = 2.0625$$
$$u_4 = 0.25\,(2 + 2.0625 + 2) = 1.5156$$

Now, solving equation (3), (4), (5) and (6) using Guass-Siedel method.

h	$u_1=0.25(u_4+u_2)$	$u_2=0.25(u_1+u_3+2)$	$u_3=0.25(u_2+u_4+2)$	$u_4=0.25(u_1+u_3+2)$
0	2	2.25	2.0625	1.5156
1	0.9414	1.2509	1.1916	1.033
2	0.5710	0.9406	0.9934	0.8911
3	0.4579	0.8628	0.9384	0.8491
4	0.4280	0.8416	0.9227	0.8377
5	0.4198	0.8356	0.9183	0.8345
6	0.4175	0.8339	0.9171	0.8337
7	0.4169	0.8335	0.9168	0.8334
8	0.4167	0.8333	0.9167	0.8333

$\therefore$ $u_1 = 0.4267$

$u_2 = 0.8333$

$u_3 = 0.9167$

$u_4 = 0.833$

Calculator formula

$$A : B : C : D : A = 0.25(B + D) : B = 0.25\,(A + C + 2)$$
$$C = 0.25(B + D + 2) : D = 0.25\,(A + C + 2)$$

NOTE

You can also use symmetric property about the line in figure and get $u_2 = u_4$ and solve accordingly.

6. **Solve numerically the wave equation $f_{tt}(x, y) = 4f_{xx}(x, t)$ with the boundary conditions.**

$$f(0, t) \text{ and } f(5, t) = 0$$

and, initial values; $f(x, 0) = f(x) = x(5 - x)$

$$f_t(x, 0) = g(x) = 0 \qquad [2017]$$

Solution:

Here, for wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The differential equation is;

$$u_{i,j+1} = 2(1 - \alpha^2 c^2)\, u_{i,j} + \alpha^2 c^2 (u_{i-1,j} + u_{i+1,j}) - u_{i,j-1}$$

where, $\alpha = \dfrac{k}{h}$

Comparing $f_{tt}(x, y) = 2^2 f_{xx}(x, t)$ with standard equation, we get $c = 2$.

Hence the differential equation becomes;

$$u_{i,j+1} = 2(1 - 4\alpha^2)\, u_{i,j} + 4\alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad (1)$$

Taking $h = 1$, let us choose k so that coefficient of $u_{i,j}$ vanishes;

so, $1 - 4\left(\dfrac{k}{h}\right)^2 = 0$

or, $1 - 4\left(\dfrac{k}{1}\right)^2 = 0$

or, $4k^2 = 1$

or, $k = \dfrac{1}{\sqrt{4}} = 0.5$

$\alpha = \dfrac{k}{h} = \dfrac{0.5}{1} = \dfrac{1}{2}$ (<1) so solution converges and equation (1) becomes.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad (2)$$

Now,

$$u(0, t) = u(5, t) = 0$$

i.e., $u_{0,j} = 0$ and $u_{5,j} = 0$ for all values of j. (entries in first and last column is zero)

$$u(x, 0) = x(5 - x)$$
$$u(i, 0) = i(5 - i) = 4, 6, 6, 4, \text{ for } i = 1, 2, 3, 4 \text{ at } t = 0$$

(These are the entries for row)

$$[u_{0,0} = 0, u_{1,0} = 4, u_{2,0} = 6, u_{3,0} = 6, u_{4,0} = 4, u_{5,0} = 0]$$

Using standard formula

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0; \text{ since } u_t(x, 0) = 0. \quad i.e., j = 0$$

Above equation becomes;

$$u_{i,1} = u_{i,-1} \qquad (3)$$

Putting $j = 0$ in (2); we get

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

or, $u_{i,1} + u_{i,-1} = u_{i-1,0} + u_{i+1,0}$

$\therefore$ $u_{i,1} = \dfrac{1}{2}(u_{i-1,0} + u_{i+1,0})$ [by (3)]

Using $i = 1, 2, 3, 4$ successively; we get,

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0 + 6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0})$$

$$= \frac{1}{2}(4 + 6) = 5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 4) = 5$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(6 + 0) = 3$$

These are entries of second row.

Putting j = 1 in equation (2), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

Taking i = 1, 2, 3, 4 successively; we obtain,

$$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0}$$
$$= 0 + 5 - 4 = 1$$

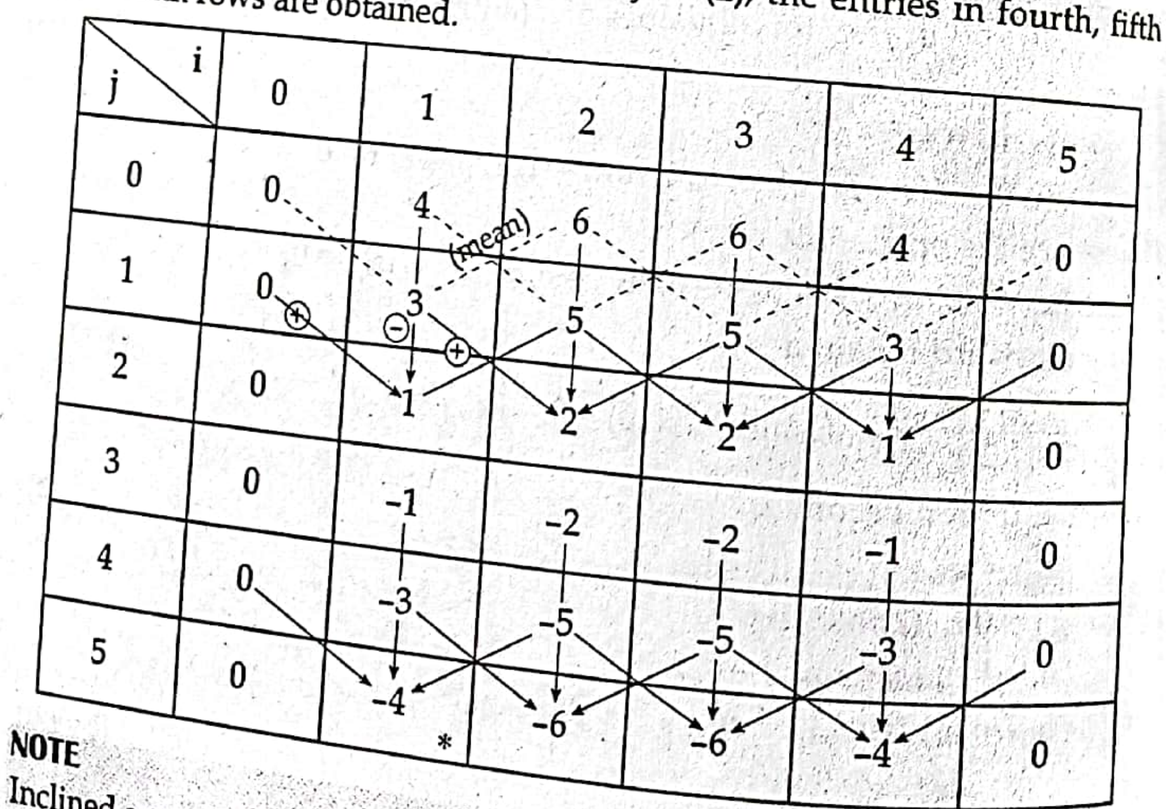
$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0}$$
$$= 3 + 5 - 6 = 2$$

$$u_{3,2} = u_{2,1} + u_{2,1} - u_{3,0}$$
$$= 5 + 3 - 6 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0}$$
$$= 5 + 0 - 4 = 1$$

These are entries of third row

Similarly, putting j = 2, 3, 4 successively in (2), the entries in fourth, fifth and sixth rows are obtained.

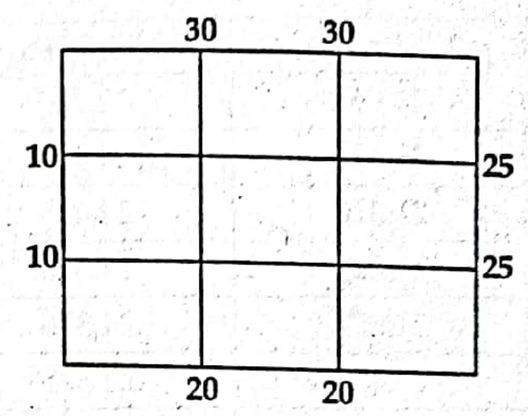


NOTE

Inclined arrows are added and is subtracted from straight arrow.

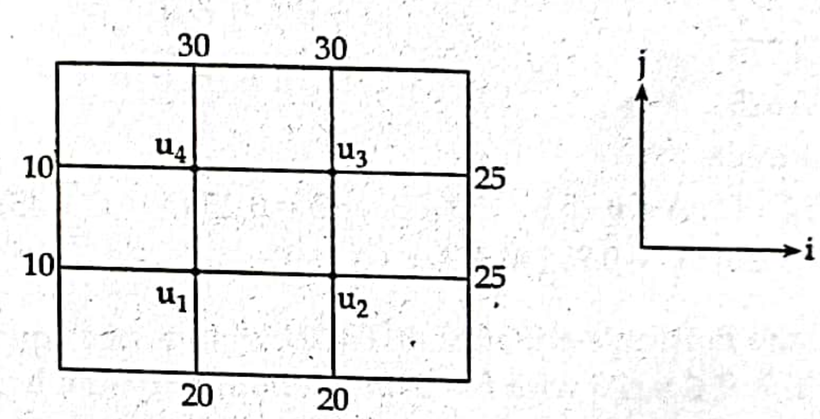
$u_{1,5} = * = 0 + (-5) - (-1) = -5 + 1 = -4$

7. **Solve the following Laplace equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < 1, 0 < y < 1$; where, $u(0, y) = 10$, $u(x, 0) = 20$, $u(1, y) = 25$, $u(x, 1) = 30$. Take $h = \frac{1}{3}$ as shown below in the figure.** **[2018]**



Solution:

Here;



Using standard five point formula for finite differential equation;

$$u_{xx} + u_{yy} = 0$$

$$u_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}]$$

so, $$u_1 = \frac{1}{4}(u_4 + 20 + 10 + u_2) = 0.25\,(30 + u_2 + u_4) \quad (1)$$

$$u_2 = \frac{1}{4}(u_1 + 25 + u_3 + 20) = 0.25\,(45 + u_1 + u_3) \quad (2)$$

$$u_3 = \frac{1}{4}(30 + u_2 + u_4 + 25) = 0.25\,(55 + u_2 + u_4) \quad (3)$$

$$u_4 = \frac{1}{4}(30 + u_1 + 10 + u_3) = 0.25\,(40 + u_1 + u_3) \quad (4)$$

Now, let use some guess values

$$u_1 = \frac{1}{4}(30 + 20 + 10 + 25) = 21.25 \text{ (using boundary value)}$$

$$u_2 = \frac{1}{4}(30 + 20 + 25 + 21.25) = 24.062$$

$$u_3 = \frac{1}{4}(10 + 30 + 25 + 24.062) = 22.765$$

$$u_4 = \frac{1}{4}(10 + 30 + 22.765 + 21.25) = 21$$

Now, let's use Gauss Siedel method to solve the equations (1), (2), (3) and (4) and find their values.

h	$u_1=0.25(u_2+u_4+30)$	$u_2=0.25(u_1+u_3+45)$	$u_3=0.25(55+u_2+u_4)$	$u_4=0.25(40+u_1+u_3)$
0	21.25	24.062	22.765	21
1	18.765	21.632	24.408	20.793
2	18.1065	21.878	24.418	20.631
3	18.127	21.886	24.379	20.626
4	18.128	21.877	24.376	20.626
5	18.125	21.875	24.375	20.625

$\therefore$ $u_1 = 18.125$

$u_2 = 21.875$

$u_3 = 24.375$

$u_4 = 20.625$

Calculator formula

A : B : C : D : A = 0.25 (B + D + 30) : B = 0.25 (A + C + 45) : C = 0.25 (B + D + 55) : D = 0.25 (40 + A + C)

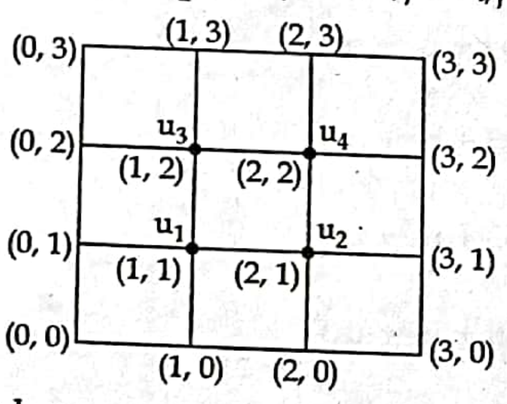
8. Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square domain $0 \le x \le 3, 0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$. [2019]

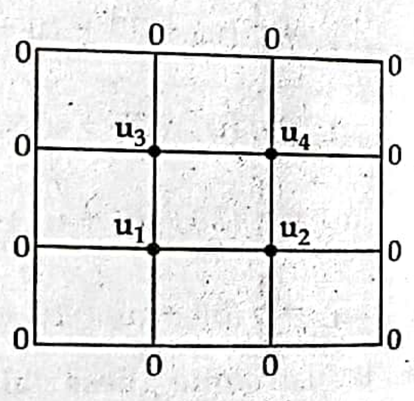
Solution:

Here,

Poisson's equation for finite difference is;

$$U_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f(ih, jh)]$$





We have,

$h = 1$

$f(x, y) = 2x^2y^2$

$\therefore$ $f(ih, jh) = 2(ih)^2(jh)^2 = 2i^2j^2h^2 \times h^2 = 2i^2j^2$

Hence, above equation becomes;

$$U_{i,j} = \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - (1)^2 \times 2i^2j^2]$$

$$= \frac{1}{4}[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 2i^2j^2] \quad (1)$$

i) For $i = j = 1$;

$$U_{i,1} = U_1 = \frac{1}{4}[u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} - 2 \times (1)^2 \times (1)^2]$$

$$= \frac{1}{4}[u_2 + 0 + u_3 + 0 - 2]$$

$$= \frac{u_2 + u_3 - 2}{4} \quad (2)$$

ii) For $i, j = 1, 2$;

$$U_{1,2} = U_3 = \frac{1}{4}[u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} - 2 \times (1)^2 \times (2)^2]$$

$$= \frac{1}{4}[u_4 + 0 + 0 + u_1 - 8]$$

$$= \frac{u_4 + u_1 - 8}{4} \quad (3)$$

iii) For $i, j = 2, 1$;

$$U_{2,1} = U_2$$

$$= \frac{1}{4}[u_1 + u_4 + 0 + 0 - 2 \times (2)^2 \times (1)^2]$$

$$= \frac{1}{4}[u_1 + u_4 - 8] \quad (4)$$

iv) For $i, j = 2, 2$;

$$U_{2,2} = U_4$$

$$= \frac{1}{4}[u_3 + u_2 + 0 + 0 - 2 \times (2)^2 \times (2)^2]$$

$$= \frac{1}{4}[u_3 + u_2 - 32] \quad (5)$$

From equation (3) and (4); we get,

$u_2 = u_3$

Hence, other equations are modified as;

$$U_1 = \frac{2u_2 - 2}{4} = \frac{u_2 - 1}{2} \quad (6)$$

$$U_2 = \frac{u_1 + u_4 - 8}{4} \quad (7)$$

$$U_4 = \frac{u_2 \times 2 - 32}{4} = \frac{u_2 - 16}{2} \quad (8)$$

Solving equation (6), (7) and (8) by Gauss Siedel method; we have,

Initial guess $U_1 = U_2 = U_4 = 0$

n	U_1	U_2	U_4
0	0	0	0
1	– 0.5	– 2.125	– 9.0625
2	– 1.5625	– 4.6562	– 10.3281
3	– 2.8281	– 5.2890	– 10.6445
4	– 3.14453	– 5.4472	– 10.7236
5	– 3.22363	– 5.4868	– 10.7434
6	– 3.2434	– 5.4967	– 10.7483
7	– 3.2483	– 5.4991	– 10.7495
8	– 3.2495	– 5.4997	– 10.7498
9	– 3.24989	– 5.4999	– 10.7499
10	– 3.25	– 5.5	– 10.75

$\therefore \quad U_1 = -3.25$

$U_2 = U_3 = -5.5$

$U_4 = -10.75$

NOTE:

Calculator code

$$A:B:C:A=\frac{B-1}{2}:B=\frac{A+C-8}{4}:C=\frac{B-16}{2}$$

Chapter 8

CODES

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8.1 BISECTION METHOD

Iterative Method

It is a method which generates a sequences of approximations which is supposed to converge to the real root of given equation. The process of iteration is stopped where we get desired rate of accuracy.

Bisection Method

Bisection method is the application of intermediate value theorem by bisecting an interval repeatedly which contains the root. "If a function $f(x) = 0$ is continuous in the interval [a, b] such that $f(a)$ and $f(b)$ are of opposite sign then there exists at least one or an odd number of roots between a and b" is statement of intermediate value theorem.

To solve the equation $f(x) = 0$ in the interval [a, b] by bisection method the intermediate values a and b must satisfy the equation.

Suppose $f(a) < 0$ and $f(b) > 0$

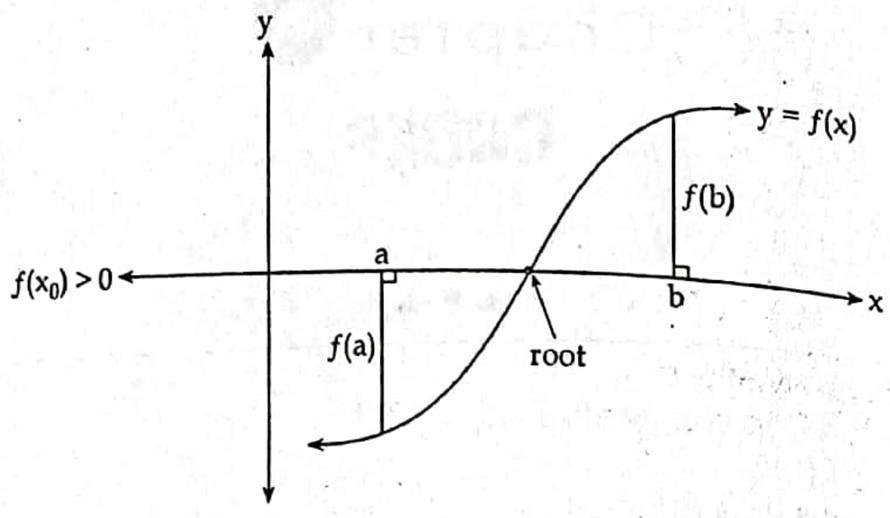
$$x_0 = \frac{a+b}{2}$$

If $f(x_0) > 0$ then replace b with x_0 and take mean

$$x_1 = \frac{a+x_0}{2}$$

Else, replace a and x_0 and take mean

$$x_1 = \frac{a + x_0}{2}$$



Pseudo Code

1. Define function F(x)
2. Input interval a and b
3. If (f(a) × f(b) > 0)
   exit
4. c = (a + b)/2
   if (f(a) × f(c) < 0)
   b = c
   else
   a = c
5. If | f(c) | < 0.0001
   print ("root = c")
   else
   repeat step (4)
6. End program

Code

```
# include <stdio.h>
# include <conio.h>
# include <math.h>
float func (float);
int main ( )
{
    float a, b, c;
    float func1, func2, func3;
    printf ("enter lower interval a = ");
    scanf("%f", & a);
    printf ("enter upper interval b = ");
```



```
scanf ("%f", & b);
    func1 = func (a);
    func2 = func (b);
    if(func1 * func2 > 0)
    {
        return;
    }
    do
    {
        c = (a + b)/2;
        func3 = func(c);
        if (func1 * func 3) < 0)
        {
            b = c;
        }
    else
    {
        a = c;
    }
} while (fabs (func3) > 0.0001)
    printf ("root = % f \ t error = % f", c, func3);
};
float func (float a)
{
    return (F(x));
};
```

Output

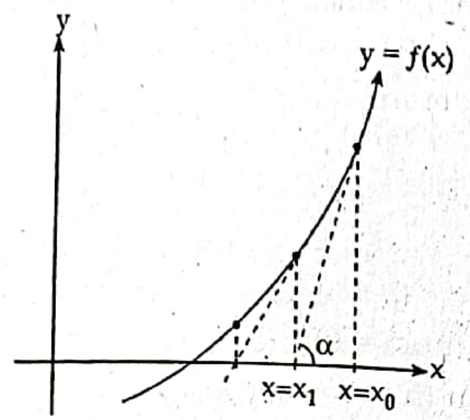
a) for f(x) = x * x – 4
enter lower interval a = 0
enter upper interval b = 5
root = 1.9999 error = 0.0001

b) for f(x) = x * x – 4 * x + 1
enter lower interval a = 0
enter upper interval b = 6
root = 0.2679 error = 0.0001

c) for f(x) = x * cos (x) + sin (x)
enter upper interval b = 5
enter lower interval a = 1
root = 2.0881 error = –0.000137

8.2 NEWTON RAPHSON METHOD

In Newton Raphson Method, the root is approximated by a tangent drawn at the point $f(x)$ cuts the curve.



Let us assume initial guess $x = x_0$ to find the root of equation $f(x) = 0$. Then

$$\tan\alpha = \frac{f(x)}{x_0 - x_1}$$

or, $$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

or, $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\because$ $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Pseudo Code

1. declare functions func and deri
2. declare variables x, a, b, y, h, n = 0
3. y = x
4. a = func (x)
   b = deri (x)
   h = a/b
   x = x – h
   n++
5. check if (x – y) > 0.0001 if no go to (3)
6. print x
7. stop

C Code

```
# include <stdio.h>
# include <math.h>
float func (float);
float deri (float);
```



```
float main ()
{
    float x, a, b, y, h, n = 0
    printf ("enter initial guess");
    scanf ("%f", & x);
    do
    {
        y = x;
        a = func (x);
        b = deri (x);
        h = a/b;
        x = x – h;
        n++;
    } while (fabs (x – y) > 0.0001 || n == 10);
    if (n == 10)
    {
        printf ("convergence error");
    }
    else
    {
        printf ("root = %f", x)
    }
    return 0;
}
float func (float x)
{
    return (x * x * x – 4 * x – 9);
}
float deri (float y)
{
    return (3 * y * y – 4);
}
```

Output

a) $f(x) = x * x * x - 4 * x - 9$
   root = 2.7065

b) $f(x) = x^2 - 4$
   root = 2.0000

c) $f(x) = x^3 - 27$
   root = 3.0000

8.3 CURVE FITTING

A. Curve Fitting of Equation y = a + bx

Theory

The given equation is y = a + bx

The normal equation (1) are

$$\Sigma y = na + b\Sigma x \quad (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad (2)$$

Writing in matrix from (2) and (3); we get, (3)

$$\begin{bmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma y \\ \Sigma xy \end{bmatrix}$$

Now,

$$\Delta = \begin{bmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix} = n\Sigma x^2 - (\Sigma x)^2$$

$$\Delta_1 = \begin{bmatrix} \Sigma y & \Sigma x \\ \Sigma xy & \Sigma x^2 \end{bmatrix} = \Sigma y \cdot \Sigma x^2 - \Sigma x \cdot \Sigma xy$$

$$\Delta_2 = \begin{bmatrix} n & \Sigma y \\ \Sigma x & \Sigma xy \end{bmatrix} = n\Sigma xy - \Sigma x \cdot \Sigma y$$

$$a = \frac{\Delta_1}{\Delta} = \frac{\Sigma y \cdot \Sigma x^2 - \Sigma x \cdot \Sigma xy}{n\Sigma x^2 - (\Sigma x)^2}$$

$$\therefore \quad b = \frac{\Delta_2}{\Delta} = \frac{n\Sigma xy - \Sigma x \,.\, \Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

Pseudo Code

1. declare n, x[10], y[10], sum x = 0, sum y = 0, sum x2 = 0, sum xy = 0, a, b, j = 0, i = 0
2. enter x [i]
3. i = i + 1
4. check if (i > 5) in no go to step (2).
5. enter y [j]
6. j = j + 1
   check i (j > 5) if no go to step (5)
7. i = 0 and j = 0
8. sum x = sum x + x[i]
   sum x2 = sum x2 + x[i] * x[i]
   sum y = sum y + y [i]
   sum xy = sum xy + x[i] * y[j]
9. check if (i > 5) if no then i = i + 1 and go to (8)
10. a = (sum y * sum x2 – sum x * sum xy)/(n * sum x2 – sum x * sum x)
    b = (n * sum xy – sum x * sum y)/(n * sum x2 – sum x * sum x)
11. Print a and b

C Code

```
# include <stdio.h>
# include <math.h>
float main ()
{
    float x[10], y[10], sum x = 0, sum y = 0, sum xy = 0, sum x2 = 0,
    a, b, n = 5;
    inti;
    for (i =0; i < 5; i ++)
    {
        printf ("x[%d] = ", i]
        scanf ("%f", & x [i];
    };
    printf ("enter the value of y");
    for (i = 0; i < 5; i++)
    {
        printf ("y[%d] = ", i];
        scanf ("%f", & y[i];
    };
    for (i = 0; i <-5; i++)
    {
        sum x = sum (x) + x[i];
        sum y = sum y + y[i];
        sum xy = sum xy + x[i] * y[i];
        sum x2 = sum x2 + x[i] * x[i];
    };
    a = (sum y * sum x2 – sum x * sum xy)/(n * sum x2 – sum x
                                                    * sum x);
    b = (n * sum xy – sum x * sum y)/(n * sum x2 – sum x * sum x);
    printf ("the reqd. result is .....");
    printf ("a = %f\n b = %f ", a, b);
    return 0;
}
```

Output

a)

x	5	6	7	8	9
y	10	12	13	14	19

a = –0.4 and b = 2

b)

x	1	2	3	4	5
y	1	2	3	4	5

a = 0 and b = 1

c)

x	50	55	60	65
y	100.1	115.5	116.3	118.2

a = 49.161 and b = 1.10198

B. Curve Fitting of Equation $y = ae^{bx}$

Theory

The given equation is $y = ae^{bx}$

Taking *ln* on both side

$$ly\ n = ln\ (a \cdot e^{bx}) = ln\ a + bx\ ln\ e$$

$$\therefore \quad ln\ y = ln\ a + bx$$

$$Y = A + bx$$

where,

$$Y = ln\ y \Rightarrow y = e^{Y} \qquad (1)$$

$$A = ln\ a \Rightarrow a = e^{A}$$

Writing normal equation for (1); we get,

$$\Sigma Y = nA + b\Sigma x \qquad (a)$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2 \qquad (b)$$

writing in matrix from

$$\begin{bmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} \Sigma Y \\ \Sigma xY \end{bmatrix}$$

$$\begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} n & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \Sigma Y \\ \Sigma xY \end{bmatrix}$$

Pseudo Code

1. declare n, i, x[10], y[10], A
2. initialize sum x = 0, sum y = 0, sum x2 = 0, sum xY = 0
3. i = 0
4. enter x[i], y[i]
5. check if (i > 5) if not (i = i + 1) and go to (4)
6. i = 0
7. sum x = x[i] + sum x
   sum Y = sum Y + log (y [i]):
   sum x2 = sum x2 + x[i] * x[i]
   sum xY = sum xY + x[i] * log (y[i]);
   if (i < 5) then i = i + 1 and go to (7)
8. A = (sum xY * sum x2 – sum x * sum xy)/(n*sum x2 – sum x*sum x)
   a = e^{A}
   b = (n * sum xy – sum x * sum y)/(n * sum x2 – sum x * sum x)
9. print a and b
10. stop

C Code

```
# include <stdio.h>
# include <math.h>
```



```
float main ()
{
    float x[10], y[10], sum x = 0, sum y = 0, sum x2 = 0, sum xy = 0,
    a, b, n = 5, A;
    int i;
    printf ("enter the values of x\n");
    for (i = 0; i < 5; i++)
    {
        printf ("x[%d]=", i);
        scanf ("%f", &x[i]);
    }
    printf ("enter the values of y/n");
    for (i = 0, i < 5; i++)
    {
        printf ("y[%d] = ", i);
        scanf ("%f", &y[i];
    }
    for (i = 0; i < 5; i++)
    {
        sum x = sum x + x[i];
        sum y = sum y + log(y[i]);
        sum xy = sum xy + x[i] * ln (y[i]);
        sum x2 = sum x2 + x[i] * x[i];
    }
    A = (sumy * sumx2 – sumx * sumxy)/(n * sumx2 – sumx * sum x);
    b = (n * sumxy – sumx * sumy)/(n * sumx2 – sumx * sumx);
    a = exp (A);
    printf ("a = %f and b = %f", a, b);
    return 0;
}
```

Output

a)

x	5	10	15	20	25
y	8	9	10	11	12

a = 7.30771 and b = 0.02023

b)

x	8	13	15	17
y	10	12	14	16

a = 6.46933 and b = 0.05152

c)

x	25	30	35	40	45
y	18	20	22	24	28

a = 10.4991 and b = 0.0213197

8.4 EULER AND RK-4 METHOD

A. Euler Method

Theory

Let us consider $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$

Slope of tangent at $(x_0, y_0) = f(x_0, y_0)$

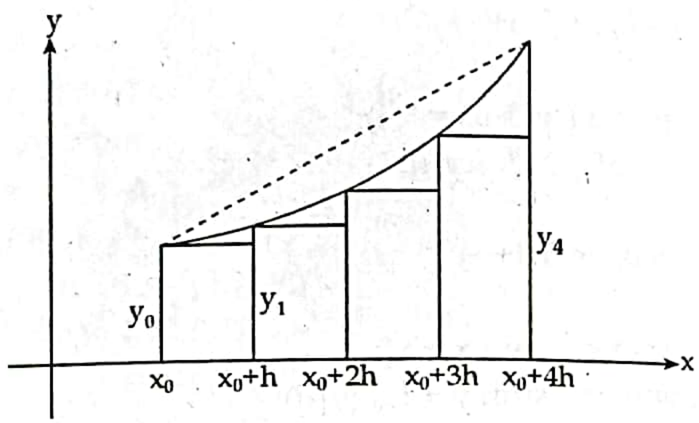
Equation of tangent at (x_0, y_0) is $(y - y_0) = m(x - x_0)$

$\therefore \quad y = y_0 + f(x_0, y_0) \cdot (x - x_0)$

Now, value of y at x_0 is given by

$$y_1 = y_0 + f(x_0, y_0) \times h$$

$$y_2 = y_1 + f(x_0 + h, y_1) \times h$$



Pseudo Code

1. Declare the variables a, b, m, inter
2. enter the values a, b, inter
3. loop:

    m = (b * b – a * a)/(b * b + a * a)

    print a, b, m

    b = b + m * inter

    a = a + inter

    if (a < 1) go to 3

Source Code

```
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <stdlib.h>
int main ()
{
        float a, b, inter, m;
        printf ("enter the value of x =");
        scanf ("%f", & a);
        printf ("enter the value of y = ");
```



```
        scanf ("%f", & b);
        printf ("enter interval")
        scanf ("%f", & inter);
        do
        {
                m = (b * b – a * a)/(b * b + a * a);
                printf ("x = %f and y = %f and m = %f\n", a, b, m);
                b = b + m * inter;
                a = a + inter;
        } while (a > 1);
}
```

Output

a)	a = 0	b = 1	m = 1	inter = 0.2
	a = 0.2	b = 1.2	m = 0.94595	
	a = 0.4	b = 1.38919	m = 0.84688	
	a = 0.6	b = 1.55857	m = 0.74186	
	a = 0.8	b = 1.70694	m = 0.63981	
	a = 1	b = 1.83490	m = 0.54200	

B. RK-4 Method (2nd order)

Theory

This method is most commonly used and is often referred to a Runge Kutta Method only.

$\frac{dy}{dx} = f(x, y); y(x_0) = y_0$ is as follows

$$m_1 = f(x_0, y_0)$$

$$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + m_1 \cdot \frac{h}{2}\right)$$

$$m_3 = f\left(x_0 + \frac{h}{2}, y_0 + m_2 \cdot \frac{h}{2}\right)$$

$$m_4 = f(x_0 + h, y_0 + m_3 \cdot h)$$

Now,

$$m = \frac{m_1 + 2m_2 + 2m_3 + m_4}{6}$$

then $\boxed{y = y_0 + mh}$

Pseudo Code

1. declare a function $f(x)$
2. declare variables a, b, m_1, m_2, m_3, m_4, m, h
3. $m_1 = f(a, b)$

    $$m_2 = f\left(a + \frac{h}{2}, b + m_1 \times \frac{h}{2}\right)$$

$m_3 = f\left(a + \frac{h}{2}, b + m_2 \times \frac{h}{2}\right)$

$m_4 = f(a + h, b + m_3 \times h)$

$m = (m_1 + 2 * m_2 + 2 * m_3 + m_4)/6$

$y = y_0 + m * h$

4. If a < 1 go to step 3.

Code

```
# include <stdio.h>
# include <conio.h>
# include <math.h>
# include <stdlib.h>
float slope (float, float);
int main ()
{
    float a, b, inter, m1, m2, m3, m4, m;
    printf ("enter value of x = ");
    scanf ("%f", & a);
    printf ("enter value of y = ");
    scanf ("%f", &b);
    printf ("enter interval = ");
    scanf("%f", & inter);
    do
    {
        m1 = (b * b – a * a)/(b * b + a * a);
        m2 = slope (a + inter/2, b + m1 * inter/2);
        m3 = slope (a + inter/2, b + m2 * inter/2);
        m4 = slope (a + inter, b + m3 * inter);
        m = (m1 + 2 * m2 + 2 * m3 + m4)/6;
        printf ("x = %f, y = %f and m = %f\n", a, b, m);
        b = b + m * inter;
        a = a + inter;
    } while (a < 1);
}
float slope (float x, float y)
{
    return (x * x – y * y)/(x * x + y * y)
}
```

Output

```
enter value of x = 0
enter value of y = 1
enter interval = 0.2
```


x = 0	y = 1	m = 0.979998
x = 0.2	y = 1.196	m = 0.896338
x = 0.4	y = 1.375267	m = 0.789236
x = 0.6	y = 1.533114	m = 0.680106
x = 0.8	y = 1.669135	m = 0.573811

8.5 INTEGRATION AND RK-4 METHOD (2nd Order O.D.E)

A. RK-4 method (2nd order O.D.E)

Theory

RK-4 method is used to solve 2nd order ordinary differential equation.

Consider,

$$\frac{dy}{dx} = y' = z = f(x, y, z)$$

$$\frac{dz}{dx} = z' = y'' = \phi(x, y, z)$$

Starting at (x_0, y_0, z_0) and taking the step sizes for x, y, z to be h, k, l respectively, the Runge Kutta method gives:

$k_1 = h \cdot f(x_0, y_0, z_0)$

$l_1 = h \cdot \phi(x_0, y_0, z_0)$

$k_2 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$l_2 = h \cdot \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = h \cdot f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$l_3 = h \cdot \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3, z_0 + l_3)$

$l_4 = h \cdot \phi(x_0 + h, y_0 + k_3, z_0 + l_3)$

$$k = \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$$

$$l = \frac{l_1 + 2l_2 + 2l_3 + l_4}{6}$$

$x_1 = x_0 + h;$

$y_1 = y_0 + k;$

$z_1 = z_0 + l$

and to compute x_2, y_2, z_2 simply x_0, y_0, z_0 are replaced by x_1, y_1, z_1 in above formula.

Pseudo Code

1. Define function $f(x)$ and $g(x)$ such that
   $f(x) = y' = z$
   $g(x) = z' = y''$
2. Read value x_0, y_0, z_0, x_n, h
3. $k_1 = h*f(x_0, y_0, z_0)$
   $l_1 = h*g(x_0, y_0, z_0)$
   $k_2 = h*f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
   $l_2 = h*f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
   $k_3 = h*f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
   $l_3 = h*\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
   $k_4 = h.f\,(x_0 + h, y_0 + k_3, z_0 + l_3)$
   $l_4 = h.g\,(x + h, y + k_3, z_0 + l_3)$
   $k = \frac{1}{6} \cdot (k_1 + 2k_2 + 2k_3 + k_4)$
   $l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$
4. $y = y_0 + k$
   $y' = z = z_0 + l$
   $x_0' = x_0 + h$
5. if $x_0 \neq x_n$ go to step 4
6. Display x_0, y, z
7. stop

Source Code

```
# include <stdio.h>
float f(float x, float y, float z)
{
    return z;
    {
        float g (float x, float y, float z)
        {
            return (x * z * z - y * y);
        }
    }
    int main ()
    {
    float x0, y0, h, xn, z0, x y, z, k1, k2, k3, k4, k, l1, l2, l3, l4, l;
```



```
printf ("enter values of x0, y0, z0");
scanf ("%f%f%f", & x0, & y0, & z0);
printf ("enter the values of xn and h: ");
scanf ("%f %f," & xn, h);
    x = x0;
    y = y0;
    z = z0;
while (x! = xn)
{
k1 = h*f(x0, y0, z0); l1 = h*g(x0, y0, z0);
k2 = h*f(x0 + h/2, y0 + k1/2, z0 + l1/2);
l2 = h*g(x0 + h/2, y0 + k1/2, z0 + l1/2);
k3 = h*f(x0 + h/2, y0 + k2/2, z0 + l2/2);
l3 = h*g(x0 + h/2, y0 + k2/2, z0 + l2/2);
k4 = h*f(x0 + h, y0 + k3, z0 + l3);
l4 = h*g(x0 + h, y0 + k3, z0 + l3);
k = (k1 + 2k2 + 2k3 + k4)/6;
l = (l1 + 2l2 + 2l3 + l4)/6;
x = x + h;
y = y + k;
z = z + l;
printf ("when x = % 0.4f y = %0.4f z =  %0.4f", x, y, z);
}
return 0;
}
```

Output

a) for $y'' = xy^2 - y^2$
enter values of $x_0, y_0, z_0, x_n, h = 0\ \ 1\ \ 0\ \ 0.2\ \ 0.1$
when x = 0.100 y = 0.9950 z = -0.0996
when x = 0.200 y = 0.9900 z = -0.1993

b) for $y'' + y^2y' = x^3$
enter values of $x_0, y_0, z_0, x_n, h = 1, 1, 1, 1.2, 0.1$
when x = 1.1000 y = 1.1002 z = 1.0055
when x = 1.2000 y = 1.2004 z = 1.0110

c) for $y'' = ex + x + 2$

enter values of $x_0, y_0, z_0, x_n, h = 0, 0, 1, 2, 1$

when $x = 1.0000$ $y = 2.8829$ $z = 5.2189$

when $x = 2.0000$ $y = 11.7154$ $z = 13.3912$

Integration

Evaluate definite integral by

i) Trapezoidal rule

ii) Simpson's (1/3) rule

iii) Simpson's (3/8) rule

Theory

Let $I = \int_a^b f(x)\,dx$

where, $f(x)$ takes the value $y_0, y_1, y_2, \ldots\ldots\ldots$ for $x = x_0, x_1, x_2, \ldots\ldots\ldots$, etc.

Let us divide the interval (a, b) into n sub-intervals with 'h' so that

$$I = \int_{x_0}^{x_0+n\cdot h} f(x)\,dx = h\cdot\int_0^n f(x_0 + n\cdot h)\,dx$$

Putting $x = x_0 + n\cdot h \Rightarrow dx = h\cdot dn$

$$\therefore \quad I = h\cdot\int_0^n \left[y_0 + n\cdot\Delta y_0 + \frac{n\cdot(n-1)}{2!}\Delta^2 y_0 + \frac{n\cdot(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots\ldots\right] dn$$

$$= h\cdot\left[n\cdot y_0 + \frac{n^2}{2}\cdot\Delta y_0 + \frac{n^2\cdot(2n-3)}{12}\Delta^2 y_0 + \frac{n^2\cdot(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\right]$$

$$= h\cdot n\left[y_0 + \frac{n}{2}\cdot\Delta y_0 + \frac{n\cdot(2n-3)}{12}\Delta^2 y_0 + \frac{n\cdot(n-2)^2}{24}\Delta^3 y_0 + \ldots\ldots\right]$$

This is Newton cotes quadratic formula

Trapezoidal rule

Putting n = 1

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Also

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_0+(n-1)h}^{x_0+n\cdot h} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding them we get;

$$\int_{x_0}^{x_0+n\cdot h} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2.(y_1 + y_2 + \ldots\ldots\ldots y_{n-1})]$$

This is trapezoidal rule.

a) For Trapezoidal rule:

$$I = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots\ldots y_{n-1})]$$

b) For Simpson's $\frac{1}{3}$ rule:

$$I = \frac{h}{3}[y_0 + y_n + 2(y_2 + y_4 + \ldots\ldots\ldots) + 4(y_3 + y_5 + \ldots\ldots\ldots)]$$

c) For Simpson's $\frac{3}{8}$ rule:

$$I = \frac{3h}{8}[y_0 + y_n + 2(y_3 + y_6 + \ldots\ldots\ldots + y_{n-3}) + 3(y_1 + y_2 + \ldots\ldots\ldots y_{n-1})]$$

B. Trapezoidal Rule

Pseudo Code

1. declare a function $f(x)$
2. enter the values x_0, x_n, h
3. Initialize sum = 0
4. $n = \frac{(x_n - x_0)}{h}$
5. for (i = 1; i <= n; i++)
   $x_i = x_0 + hx_i$
   $y_i = f(x_i)$
   end i
6. for (i = 1; i < n; i++)
   sum = sum + y_i
   end i
   area = $\frac{h}{2}\cdot[y_0 + y_n + 2\,.\,\text{sum}]$
7. print area

Source Code

```
# include <stdio.h>
# include <math.h>
float func. (float a)
{
        return (1/(1 + a * a));
}
int main ()
{
        float x0, y0, xn, h, n, x[20], y[20], sum = 0 area;
        int i, j, n;
        printf ("enter the values of x0, xn, h");
```

```
        scanf ("%f%f%f", &x0, &xn, &h);
        n = (xn - x0)/h;
        for (i = 1; i < = n; i++)
        {
            x[i] = x0 + i * h;
            y[i] = func . (x[i]);
        }
        for (j = 1; j < n; j++)
        {
            sum = sum + y[j];
        }
        area = (2*sum + y[0] + y[n])*(h/2);
        printf ("the area = %f", area);
    }
```

Output

i) enter the values of x_0, x_n, h : 0, 6, 1.
area = 1.41079

ii) for function = $1 + x^2$
enter values of x_0, x_n, h: 0, 1, 0.5
area = 1.3334

iii) for function = x sin x
enter values of x_0, x_n, h: 0, 5, 2.5
area = –2.37724

C. Simpson's $\frac{1}{3}$ Rule

Pseudo Code

1. declare function $f(x)$
2. enter the values of x_0, x_n, h
3. $n = (x_n - x_0)/h$
4. if (n/2) rem = 1 odd; then $n \rightarrow n-1$, $h = (x_n - x_0)/n$
5. display ("modified n and h are: ", n and h)
   i = 0 to n step 1
   $x_i = x_0 + i \cdot h$
   $y_i = f(x_i)$
6. sumeven = 0, sumodd = 0
7. for i = 1 to n step 1
   if rem (i/2) = 1
   so, sumodd = sumodd + y_i
   else
   sumeven = sumeven + y_i
8. sum = $y_0 + y_n$ + 4 sumodd + 2 sumeven
9. print sum

Source Code

```
# include <stdio.h>
# include <math.h>
float func (float a)
{
    return (1/(1 + a*a));
}
int main ()
{
    float sumeven , sumodd, x0, y0, xn h, x[20], y[20], sum = 0;
    int i, j, n;
    printf ("enter the values of x0, xn, h");
    scanf ("%f%f%f", &x0, &xn, &h);
    n = (xn - x0)/h;
    if (n%2 ==1)
    {
        n = n - 1;
        h = (xn - x0)/n;
    }
    printf ("modified values of n and h are % d and %f", n , h);
    for (i = 0; i < = n; i++)
    {
        x[i] = x0 + i * h;
        y[i] = func (x[i]);
    }
    sumeven = 0;
    sumodd = 0;
    for (i = 1; i < n; i++)
    {
        if (i%2 ==1)
        {
            sumodd = sumodd + y[i];
        }
        else
    {
        sumeven = sumeven + y[i];
    }
    sum = sum + y[n] + y[0] + 4 * sumodd + 2 * sumeven;
    area = sum * (h/3);
    printf ("the area = %f", area);
}
```

Output

a) enter the values of x_0, x_n, h: 0, 6, 1
area = 1.366173

b) for function = $1 + x^2$
enter values of x_0, x_n, h = 0, 1, 0.125
area = 1.333334

c) for function = x sin x
enter values of x_0, x_n, h: 0, 5, 0.5
area = –2.382506

D. Simpson's $\frac{3}{8}$ Rule

Pseudo Code

1. declare function $f(x)$
2. read x_0, x_n
3. read number of strips (n)
4. $h = (x_n - x_0)/n$
5. if (n%3) = 1, n = n + 2 → $h = (x_n - x_0)/n$
   if (n%3) = 2, n = n + 1 → $h = (x_n - x_0)/n$
6. print n and b
7. for (i = 0, i < = n; i++)
   read y[i]
8. sumeven = 0, sumodd = 0
9. i = 1 to n – 1, step 1
   if (i%3) = 0 sumodd = sumodd + 2(y[i]);
   else
   sumeven = sumeven + 3(y[i]);
   sum = $(y_0 + y_n$ + sumodd + sumeven) * (h/8) * 3
   print sum

Source Code

```
# include <stdio.h>
# include <conio.h>
float func (float a)
{
    return (1/(1 + a*a));
}
int main ()
{
    float sumodd, sumeven, x0, xn, h, x[20], y[20], sum;
    int i, j, n;
    print (enter x0, xn, n);
    scanf ("%f%f%d", &x0, &xn, n);
```



```
    h = (xn – x0)/n;
    if (n%3 == 1)
    {
        n = n + 1;
        h = (xn – x0)/n;
    }
    if (n%3 == 2)
    {
        n = n + 2
        h = (xn – x0)/n;
        printf ("modified values of n and h are %d and %f", n, h);
        for (i = 0; i < = n; i++)
        {
            printf (" enter y");
            scanf ("%f", &y[i]);
        }
        sumodd = 0, sumeven = 0;
        for (i = 1; i <= n – 1; i++)
        {
            if ((i%3) ==0)
            {
                sumodd = sumodd + 2 * y [i];
            }
            else
        {
            sumeven = sumeven + 3*y[i];
        }
        sum = (y[0] + y[n] + sumodd + sumeven) * 3 * (h/8);
        printf ("Area = %f", sum);
    }
```

Output

```
enter x0, x, n; 0   6   6
modified values of n and h are 6 and 1
enter y[0] = 4
enter y[1] = 2
enter y[2] = 6
enter y[3] = 3
enter y[4] = 6
enter y[5] = 3
enter y[6] = 6
sum = 25.13
```

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8.6 LAGRANGE'S INTERPOLATION AND GUASS ELIMINATION METHOD

A. Guass Elimination

Theory

Gaussian elimination is also known as row reduction is used for solving system of linear equations. In guassian elimination method; a sequence of operation is performed on corresponding coefficient matrix to convert it into upper triangular matrix.

Pseudo Code

```
1.    declare variables a[10][10], x[10], temp, temp1, = 1 ratio, j, i, k, n
2.    enter order of equation 'n'
      for (i = 1; i < = n; i++)
      for (j = 1; j < = n + 1; j++)
      enter a[i][j]
            end j
      end i
4.    for (j = 1; j < = n - 1; j++)
      for ( i = j + 1; i < =n; i++)
      ratio = a[i][j]/a[j][j];
      for (k = 1; k < = n + 1; k++)
      a[i][k] = a [i][k] - ratio * a[j][k];
            end k
            end j
      end i
            x[n] = a[n] [n + 1]/a[n][n];
      for (i = n - 1; i > = 1; i--)
      {
            temp = 0;
            for (j = i + 1; j < = n; j++)
                  temp + = a[i] [j] * x[j]
            end j
                  x[i] = (a[i] [n + 1] - temp)/a[i][i];
            end i
                  for (i = 1; i < = n; i++)
                  print x[i]
            end i
      end
```

Source Code

```
# include <stdio.h>
# include <math.h>
```



```
int main ()
{
      float a[10][10], x[10], temp, temp1 = 1, ratio
      int i, j, k, n;
      printf ("enter the order of equation");
      scanf ("%d", &n);
      for (i = 1; i < = n; i++)
      {
            for (j = 1; j < = n + 1; j++)
            {
            scanf ("%f", &a[i][j]);
            }
      }
      for (j = 1, j < = n - 1; j++)
      {
            for (i = j + 1; i <= n; i++)
            {
                  ratio = a[i][j]/a[j][j];
                  for (k = 1; k <= n + 1; k ++)
                  {
                        a[i][k] = a[i][k] - ratio * a[j] [k];
                  }
            }
      }
      printf ("/n");
      x[n] = a[n] [n + 1]/a[n] [n];
      for (i = n - 1; i > = 1; i--)
      {
            temp = 0;
            for (j = i + 1; j < = n; j++)
            {
                  temp + = a[i][j] * x[j];
            }
            x[i] = (a[i] [n + 1] - temp)/a[i][i];
      }
      for (i = 1; i < = n; i++)
      {
            printf ("x[i] =%f", x[i]);
      }
      return 0 ;
}
```

Result

1. enter the order of equation = 2

1	1	4
1	–1	2

x[1] = 3
x[2] = 1

2. enter the order of equation = 3

20	1	–2	19
3	20	–1	–18
2	–3	20	25

x[1] = 1.09955
x[2] = –1.01554
x[3] = 0.98770

3. enter the order of equation = 3

1	2	3	5
6	9	5	3
1	5	6	9

x[1] = 0.03334
x[2] = –0.966667
x[3] = 2.3

B. Lagrange's Interpolation

Theory

Interpolation means finding the unknown value of y for given 'x' using the data given Lagrange interpolation is used when the intervals are unequally spaced.

Then;

$$y = \frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots\cdots(x_1-x_n)}y_1 \vdots \vdots \frac{(x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots\cdots(x_n-x_{n-1})}y_n$$

In general

$$y(x) = \sum_{i=0}^{n}\left[y_i \times \prod_{j=0}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)\right] \text{ (if } i \neq j)$$

Pseudo Code

1. declare variables x[10], y[10], a, temp, temp1 = 1, b = 0, temp2, i, j, n
2. enter the number of terms 'n'


```
3.  for (i = 0; i < n; i++)
    {
        enter x[i]
        enter y [i]
    }
    end i
4.  enter value of a
5.  for (i = 0; i < n; i++)
    {
        temp1 = 1;
        for (j = 0; j < n; j++)
        {
            if (j! = i)
            {
                temp = [(a – x[j])/(x[i] – x[j]);
                temp 1 = temp 1 * temp
            }
        }
        end j
        temp2 = temp1 * y[i];
        b = b + temp2
    }
    end i
    print b
```

Source Code

```
# include <stdio.h>
# include <math.h>
int main ()
{
    float x[10]; y[10], a, temp, temp1 = 1; b = 0, temp 2;
    int i, j, n;
    printf ("enter the number of terms");
    scanf ("%d", &n);
    printf("enter the data");
    for (i = 0, i < n; i++)
    {
        scanf ("%f", &x[i]);
        scanf ("%f", &y[i]);
    }
    printf ("enter the value of a = ");
    scanf ("%f" &a);
```

```
        for (i = 0; i < n; i++)
        {
            temp1 = 1;
            for (j = 0; j < n; j++)
            {
                if (j! = i)
                {
                    temp = (a - x[j])/(x[i] - x[j]);
                    temp1 = temp1 * temp;
                }
            }
            temp2 = temp1 * y[i];
            b = b + temp2;
        }
        printf ("the value of b = %f", b);
        return 0;
    }
```

Output

a) n = 4

enter the values:

1	1
2	4
4	16
6	36

a = 3

b = 9

b) n = 4

5	35
7	49
9	63
11	77

a = 8

b = 56

c) n = 5

5	150
7	392
11	1452
13	2366
17	5202

a = 9

b = 810

8.7 GAUSS JORDAN METHOD AND MATRIX INVERSION METHOD

A. Gauss Jordan Method

Theory

The Guass-Jordan method is very useful in solving a linear system of equations. It is a technique in which a system of linear equations is resolved by the means of matrices. In Guass-Jordan method; given matrix can be fetched to row echelon form and made simpler. The transformation is done by row operation. *i.e.*, in general following elimination is followed:

For a_{ij}

where, i = 1 to n

j = 1 to n

$i \neq j$

$$R_i \leftarrow R_i - \frac{a_{ij}}{a_{jj}} \cdot R_j$$

Pseudo Code

```
1.  enter the number of equations 'n'
2.  create array a[10][10], x[10]
3.  define i, j, k, r
4.  for i = 1 to n
        for j = 1 to (n + 1)
            input a_ij
        end j
    end i
    for j = 1 to n
        for i = 1 to n
            if (i! = j)
            r = a_ij/a_jj
            for k = 1 to n + 1
                a_ik = a_ik - c * a_jk
            end k
        end i
    end j
        for i = 1 to n
            x_i = a_i(n+1)/a_ii
            print x_i
        end i
    end
```

Source Code

```
# include <stdio.h>
# include <math.h>
int main ()
{
    int i, j, k, n;
    float x[10], a[10][10], r;
    printf ("enter the order of equation");
    scanf ("%d", &n);
    for (i = 1; i < = n; i++)
    {
        for (j = 1; j <= (n + 1); j++)
        {
            scanf ("%f", &a[i][j]);
        }
    }
    for (j = 1; j <= n; j++)
    {
        for (i = 1; i <= n; i++)
        {
            if (i! = j)
            {
                r = (a [i][j]/a[j][j]);
            }
            for (k = 1; k <=(n + 1); k++)
            {
                a[i][k] = a[i][k] – r * a[j][k];
            }
        }
    }
    printf ("the results are");
    for (i = 1; i <= n; i++)
    {
        x[i] = a[i] [n + 1]/a[i] [i];
        printf ("%f", x[i]);
    }
    return 0;
}
```


Output

a) enter the order of equation = 2

```
10  5  11
6   9  13
```

the results are

```
0.56667
1.066667
```

b) enter the order of equation = 3

```
3   2   6   4
-5  -7  2   -8
-6  7   1   -9
```

the results are

```
1.52
0.03
0.10
```

c) enter the order of equation = 4

```
-5   4   2   -6   -9
4    -5  6   20   -50
-11  -13 14  25   30
21   42  30  49   -111
```

the results are

```
43.20
-85.48
35.80
-55.61
```

B. Matrix Inversion Method

Theory

First of all the equations are written augmented with the equivalent order of unit matrix

$$\begin{bmatrix} a_{11} & a_{12} & : & 1 & 0 \\ a_{21} & a_{22} & : & 0 & 1 \end{bmatrix}$$

Then, it is solved to make the matrix on left to unit matrix; then augmented matrix at right is the unit matrix of the given matrix.

$$\begin{bmatrix} 1 & 0 & : & a_{13} & a_{14} \\ 0 & 1 & : & a_{23} & a_{24} \end{bmatrix}$$

Pseudo Code

1. declare variable i, j, k, r, d and arrays
   a[10][10], x[10]
2. for (i = 1 to n)
   for (j = 1 to n)

```
            enter a_ij
        end j
    end i
3.  for (i = 1 to n)
        for (j = n + 1 to 2n)
            if (j - i) == n
            then
                a_ij = 1
            else
                a_ij = 0
        end j
    end i
4.  for (i = 1 to n)
        for (j = 1 to n)
            if (i! = j) then
                r = a_ij/a_jj
            for (k = 1 to 2n)
                a_ik = a_ik - r * a_jk
            end k
        end j
    end i
5.  for (i = 1 to n)
        d = a_ii
        for (j = 1 to 2n)
            a_ij = a_ij/d
        end j
    end i
6.  for (i = 1 to n)
        for (j = n + 1  to 2n)
            print a_ij
        end j
    end i
7.  end
```

Source Code

```
# include <stdio.h>
# include <math.h>
int main ()
{
    int i, j, k, n;
    float a[10][10], r, x[10], d;
```



```
    printf ("enter the order of equation:");
    scanf ("%d", &n);
    for (i = 1, i <= n; i++)
    {
        for (j = i, j <= n+1; j++)
        {
            scanf ("%f", &a[i][j]);
        }
    }
    for (i = 1; i <= n; i++)
    {
        for (j = n + 1; j <= 2*n; j++)
        {
            if (j - i == n)
            {
                a[i] [j] = 1;
            }
            else
            {
                a[i][j] = 0;
            }
        }
    }
    for (j = 1; j <= n; j ++)
    {
        for (i = 1; i <= n; i++)
        {
            if (i! = j)
            {
                r = (a [i][j]/a[j][j]);
            }
            for (k = 1; k <= 2 * n; k++)
            {
                a[i][k] = a[i][k] - r * a[j][k];
            }
        }
    }
    for (i = 1; i <= n; i++)
    {
        d = a[i][i];
```

```
        for (j = 1; j <= 2 * n; j++)
        {
            a[i][j] = a[i][j]/d;
        }
    }
    printf ("results:");
    for (i = 1; i <=n; i++)
    {
        for (j = n + 1; j <= 2 * n; j++)
        {
            printf ("%0.3f\t", a[i][j]);
        }
        printf ("\n");
    }
    return 0;
}
```

Output

a) enter the order of equation = 2

```
3   1
5   2
results
3   -1
-5   2
```

b) enter the order of equation = 3

```
1   1   2
5   3   4
3   2   -6
results
-1.440   2.330   8.880
0.360   -8.670   8.860
-0.110   -8.330   -8.110
```

c) enter the order of equation = 3

```
1   4   4
2   5   6
2   8   9
results
-3   0   1
-4   -3   4
4   2   -3
```

8.8 EIGEN VALUES AND QUADRATIC CURVE FITTING

A. Power Method

Theory

Power method is the iterative process to find the dominant eigen values and its corresponding eigen vector. It starts from an initial guess of the column vector 'X'.

Let 'A' be the square matrix

Find $AX = \lambda X$

$\lambda^{(1)} X^{(1)}$ is the first approximation of the eigen value $\lambda^{(1)}$ and eigen vector $X^{(1)}$. Again, find $AX^{(1)}$ and represent it by $X^{(2)}$ $\lambda^{(2)}$ as the second approximation after normalization and so on. The process is stopped if $X^r - X^{r-1}$ is negligible then; X^r is largest eigen vector and λ is largest eigen value.

Pseudo Code

```
1.  input order of matrix as 'n'; tolerance as 't'.
2.  for i = 1 to n
        for j = 1 to n
            input a_ij
        end j
    end i
3.  for i = 1 to n
        input guess vector as x_i
4.  do
        for i = 1 to n
            z_i = 0
        for j = 1 to n
            z_i = z_i + a_ij * x_j
        end j
    end i
5.  z_max = |z_1|
        for i = 2 to n
            if (z_max < |z_i|)
                z_max = |z_i|
6.  for i = 1 to n
        y_i = z_i/z_max
        E_i = |y_i - x_i|
        x_i = y_i
7.  e_max = |E_1|
        for i = 2 to n
            if (e_max < |E_i|)
                e_max = |E_i|
```

8. while (e_{max} > t)
9. print z_{max} and y_i

Code

```
# include <stdio.h>
# include <math.h>
main ()
{
    float a[10][10], x[10], z[10], y[10], zmax, d[10], dmax;
    int i, j, k, n;
    printf ("enter the order of matrix");
    scanf ("%d", &n);
    printf ("enter a square matrix");
    for (i = 1; i < n; = i++)
    {
        for (j = 1; j <= n; j++)
        {
            scanf ("%f", &a[i][j]);
        }
    }
    printf ("enter a column matrix");
    for (i = 1, i <= n; i++)
    {
        scanf ("%f", &x[i]);
    }
    do
    {
        for (i = 1; i <= n; i++)
        {
            z[i] = 0;
            for (j = 1; j <= n; j++)
            {
                z[i] = z[i] + a[i][j] * x[j];
            }
        }
        zmax = 0;
        for (i = 1; i <= n; i++)
        {
            if (fabs (z[i]) > zmax)
            {
```



```
                zmax = fabs (z[i]);
            }
        }
        for (i = 1; i <= n; i++)
        {
            y[i] = z[i]/zmax;
        }
        for (i = 1; i <= n; i++)
        {
            d[i] = fabs (y[i] - x[i]);
        }
        dmax = 0;
        for (i = 1; i <= n; i++)
        {
            if (fabs (d[i]) dmax)
            {
                dmax = d[i];
            }
        }
        for (i = 1; i <= n; i++)
        {
            x[i] = y[i];
        }
    }
    while (dmax > 0.0005);
    printf ("zmax = %f\n", zmax);
    for (i = 1; i <= n; i++)
    {
        printf ("z[i] = %f\n", x[i]);
    }
    return 0;
}
```

Output

1. enter the size of matrix 3
   1
   -2
   3
   4
   -6

7
8
9
enter column vector
x[1] = 1
x[2] = 0
x[3] = 0
z_{max} = 12.868185
x[1] = 6.200933
x[2] = 0.307574
x[3] = 1.000000

2. Enter the size of matrix 2
6
9
1
2
enter col. matrix;
x[1] = 1
x[2] = 0
z_{max} = 7.605263
x[1] = 1.000000
x[2] = 0.178393

B. Second Degree Curve Fitting

Theory

The process of finding an equation of best fit is known as curve fitting.

Consider a 2nd degree equation:

$$y = ax^2 + bx + c$$

Normal equations are:

$$\Sigma y = a\Sigma x^2 + b\Sigma x + c \cdot n$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2$$

These constants are solved using Gauss-Jordan method.

Pseudo Code

1. input 'n' number of data
2. for i = 1 to n
   input x_i, y_i
3. initialize all sum to zero (0)

4. for i = 1 to n
   sumx + = x_i
   sumx2 + = $x_i * x_i$
   sumx3 + = $x_i * x_i * x_i$
   sumx4 + = $x_i * x_i * x_i * x_i$
   sumxy + = $x_i * y_i$
   sumx2y + = $x_i * x_i * y_i$
   sumy + = y_i
5. Assign values as;
   $a_{11} = a_{22} = a_{33}$ = sumx2
   $a_{12} = a_{23}$ = sumx
   a_{24} = sumxy
   a_{34} = sumx2y
   a_{31} = sumx4
   $a_{21} = a_{32}$ = sumx3
   a_{14} = sumy
   a_{13} = n
6. for j = 1 to 3
   for i = 1 to 3
   if ($i \neq j$)
   $r = a_{ij}/a_{jj}$;
   for k = 1 to 4
   $a_{ik} = a_{ik} - r * a_{jk}$
7. for i = 1 to 3
   $x_i = a_{i4}/a_{ii}$
8. print values of a, b, c as x_1, x_2, x_3.

Source Code

```
# include <stdio.h>
# include <math.h>
int main ()
{
    int i, j, k, n, d;
    float r, sum1, sum2, x[10], y[10], A[10][10];
    printf ("enter the degree of equation");
    scanf ("%d", &d);
    printf ("enter the number of data entries");
    scanf ("%d", &n);
    for (i = 0; i < n; i++)
    {
        printf ("x[%d] = ", i);
```

```
        scanf ("%f", &[i]);
    }
    for (i = 0; i < n; i++)

    {
        printf ("y[%d]", i);
        scanf ("%f", &y[i]);
    }
    for (i = 0; i <= d; i++)
    {
        for (j = 0; j <= d; j++)
        {
            sum1 = 0;
            for (k = 0; k <= m - 1; k++)
            {
                sum1 + = pow (x[k], i + j);
            }
            A[i][j] = sum 1;
        }
    }
    for (i = 0; i <= d; i++)
    {
        for (j = 0; j <= d + 1; j++)
        {
            sum2 = 0;
            for (k = 0; k <= n - 1; k++)
            {
                sum2 + = pow (x[k], i) * y[k]);
            }
            A[i] [d + 1] = sum2;
        }
    }
    for (j = 0; j <= d; j++)
    {
        for (i = 0; i <= d; i++)
        {
            if (i! = j)
            {
                r = A[i][j]/A[j][j];
                for (k = 0; k <= d + 1; k++)
                {
                    A[i][k] = r * A[j][k];
                }
            }
        }
    }
    for (i = 0; i <= d;    )
    {
    printf ("\n% · 2f", A[i] [d + 1]/A[i][i]);
    }
}
```

Output

a) enter degree of equation 2
enter number of data entries 3
x[0] = 1
x[1] = 12
x[2] = 19
y[0] = 4
y[1] = 13
y[2] = 17
3.08
0.99
-0.01

b) enter the degree of equation 1
enter the number of entries 2
x[0] = 1
x[1] = 2
y[0] = 1
y[1] = 4
-2
3

SOLVED NUMERICAL (PURBANCHAL UNIVERSITY)

1. **Write a program to find the root of non-linear equation using bisection method.** [2015]

Solution: See code section 8.1.

2. **Write a program to fit the straight-line for the given set of data points.** [2015]

Solution: See code section 8.3-(A).

3. **Write a program to evaluate the value of a function at specified point, using Lagrange interpolation polynomial method.** [2015]

Solution: See code section 8.6-(B).

4. **Write an algorithm and program to find the root of non-linear equation using regular falsi method.** [2016]

Solution:

```
# include <stdio.h>
# include <conio.h>
# include <math.h>
float f(float x)
{
    return(2 * x - log10 (x) - 6);
}
void main()
{
    float x, x1, a, b, err = 0.00005;
    int itr = 1, n;
    clrscr ();
    printf ("enter the values of a,b and maximum iterations\n");
    scanf ("%f %f %d", & a, & b, & n);
    x = a - (((b - a)/(f(b) - f(a))) * f(a));
    printf ("iteration number %d\tx = %f \ n", itr, x);
    itr ++;
    while (itr < n)
    {
        if (f(a) * f(x) < 0)
        b = x;
        else
        a = x;
        x = a - (((b - a)/(f(b) - f(a))) * f(a));
        printf ("iteration number %d \ tx = %f \ n", itr, x);
        if (fabs (x1 - x) < err)
        {
        printf ("\nafter %d iterations, the value of root is
        %f \ n", itr, x);
        break;
        }
        else
        itr ++;
        x1 = x;
    }
    if (itr == n)
    printf ("\nsolution does not exist as iterations are not
    sufficient");
    getch ();
    return 0;
}
```

5. **Write a program in any high level language to find $\int_a^b (1 + x^3)\, dx$ using Simpson's $\frac{3}{8}$ rule.** [2016]

Solution: See code section 8.5-(D)

NOTE: Replace (1/(1 + a*)) by (1/(1 + a * a *)).

6. **Write a program in any high level Language to solve the differential equation using Euler's method.** [2016]

Solution: See code section 8.4-(A)

7. **Write a program to solve a non-linear equation using bisection method.** [2017]

Solution: See code section 8.1

8. **Write an algorithm and program for Lagrange interpolation.** [2017]

Solution: See code section 8.6-(B)

9. **Write a program to solve differential equations using Euler's method.** [2017]

Solution: See code section 8.4-(A)

10. **Write a c-program of linear curve for determining the constants of linear equation.** [2018]

Solution: See code section 8.3-(B)

11. Write a c-program of Lagrange interpolation for fitting a polynomial in a given set of data points. [2018]

Solution: See code section 8.6-(B)

12. Write an algorithm of Gauss elimination method for solving linear system. [2018]

Solution: See code section 8.6-(A)

13. Write a program to compute the interpolation value at a specified point for a given set of data points using Lagrange method. [2019]

Solution: See code section 8.6-(B)

14. Write down the algorithm and program to find a root of a non-linear equation using bisection method. [2019]

Solution: See code section 8.1

15. Write a program to solve gauss elimination method. [2019]

Solution: See code section 8.6-(A)

CamScanner