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Text books

→ E. Balgurusamy
→ B.S. Grewal

Curve Fitting

Curve Fitting - The process of determining the constants involved in a curve with the help of given data points by least square method is called curve fitting.

Least square Method - It is a mathematical technique that minimize the sum of square of the error.

Types of Curve

a) Linear Curve, $y = a+bx$
b) Quadratic curve, $y = a+bx+cx^2$
c) Power Curve, $y = ax^b$
d) Exponential curve, $y = ae^{bx}$, etc.

Fitting of Linear Curve or Straight line

Let $y = a+bx$ be a linear curve $\longrightarrow$ (i)
let S be the sum of square of errors.
Then,

$$S = \sum (y-a-bx)^2 \longrightarrow (ii)$$

For S to be minimum,

$$\frac{\partial s}{\partial a} = 0, \quad \frac{\partial s}{\partial b} = 0$$

when,

$$\frac{\partial s}{\partial a} = 0$$

or, $2\sum (y-a-bx) \times (-1) = 0$

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or, $-2\sum(y-a-bx)=0$

or, $\sum(y-a-bx)=0$

or, $\sum y - na - b\sum x = 0$ $\quad [\because \sum a = na]$

or, $\boxed{\sum y = na + b\sum x} \longrightarrow$ (iii)

When $\frac{\partial s}{\partial b} = 0$

$2\sum(y-a-bx)\times(-x)=0$

or, $-2\sum(xy-ax-bx^2)=0$

or, $\sum(xy-ax-bx^2)=0$

or, $\sum xy - a\sum x - b\sum x^2 = 0$

or, $\boxed{\sum xy = a\sum x + b\sum x^2} \longrightarrow$ (iv)

From eq^n (iii)

$$\frac{\sum y}{n} = \frac{na}{n} + \frac{b\sum x}{n}$$

or, $\bar{y} = a + b\bar{x}$

or, $\boxed{a = \bar{y} - b\bar{x}} \longrightarrow$ (v)

From eq^n (iv),

$\sum xy = a\sum x + b\sum x^2$

or, $\sum xy = (\bar{y} - b\bar{x})\sum x + b\sum x^2$

or, $\sum xy = \frac{\sum x \sum y}{n} - b\left(\frac{\sum x}{n}\right)(\sum x) + b\sum x^2$

or, $\sum xy - \frac{\sum x \sum y}{n} = b\left[\sum x^2 - \frac{(\sum x)^2}{n}\right]$

or, $\dfrac{n\sum xy - \sum x \sum y}{n} = b\left[\dfrac{n\sum x^2 - (\sum x)^2}{n}\right]$

or, $\boxed{b = \dfrac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}}$ $\longrightarrow$ (vi)

Program 1.

Write a program to fit the straight line for the given set of data points.

x : 2 5 7 10

y : 10 12 15 20

Also estimate the value of y when $x = 18$:

→

x	y	x^2	xy
2	10	4	20
5	12	25	60
7	15	49	105
10	20	100	200
$\sum x=$	$\sum y=$	$\sum x^2=$	$\sum xy=$

Variables used:

n	sum x, x
x, y	sum x, y
sum x	x mean
sum y	y mean

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Fitting of a quadratic curve or second degree parabola or parabolic curve:

Let $y = a + bx + cx^2$ be a quadratic curve. $\longrightarrow$ (i)

The normal eqns are:

$\sum y = na + b\sum x + c\sum x^2 \longrightarrow$ (ii)

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \longrightarrow$ (iii)

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \longrightarrow$ (iv)

Solving eqns (ii), (iii) and (iv) with the help of given data points

$a =$

$b =$

$c =$

Now putting the value of a, b and c in eqn (i) we get,

$y = (---) + (---)x + (---)x^2 \longrightarrow$ (v)

which is fitted quadratic curve.

Eg: Fit the quadratic curve to the following data points.

x:	0	1	2	3
y:	2	5	10	12

Let $y = a + bx + cx^2$ be a quadratic curve $\longrightarrow$ (i)

The normal eqns are: $\sum y = na + b\sum x + c\sum x^2 \longrightarrow$ (ii)

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \longrightarrow$ (iii)

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \longrightarrow$ (iv)

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x	y	x^2	x^3	x^4	xy	x^2y
0	2	0	0	0	0	0
1	5	1	1	1	5	5
2	10	4	8	16	20	40
3	12	9	27	81	36	108
$\Sigma x = 6$	$\Sigma y = 29$	$\Sigma x^2 = 14$	$\Sigma x^3 = 36$	$\Sigma x^4 = 98$	$\Sigma xy = 61$	$\Sigma x^2y = 153$

From eq^n (ii), (iii) and (iv),

$$4a + 6b + 14c = 29$$
$$6a + 14b + 36c = 61$$
$$14a + 36b + 98c = 153$$

Solving above three eq^{ns};

$$a = \frac{D_1}{D}, \quad b = \frac{D_2}{D}, \quad c = \frac{D_3}{D}$$

$$D = \begin{vmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{vmatrix} =$$

$$D_1 = \begin{vmatrix} 29 & 6 & 14 \\ 61 & 14 & 36 \\ 153 & 36 & 98 \end{vmatrix} =$$

$$D_2 = \begin{vmatrix} 4 & 29 & 14 \\ 6 & 61 & 36 \\ 14 & 153 & 98 \end{vmatrix} =$$

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$$D_3 = \begin{vmatrix} 4 & 6 & 29 \\ 6 & 14 & 61 \\ 14 & 36 & 153 \end{vmatrix} =$$

Putting the value of a, b, c in eqn (i), we get,

$y = (\text{- - - -}) + (\text{- - -})x + (\text{- - -})x^2$

which is fitted quadratic curve.

Fitting of an exponential curve, $y = ae^{bx}$

Given, $y = ae^{bx}$ ⟶ (i)

Taking log on both sides,

$\ln y = \ln a + bx \ln e$

i.e. $Y = A + BX$ which is linear. ⟶ (ii)

where,

$Y = \ln y$, $A = \ln a$, $B = b$, $X = x$, $\ln e = 1$

Then the normal eqns are:

$\Sigma Y = nA + B\Sigma X$ ⟶ (iii)

$\Sigma XY = A\Sigma x + B\Sigma X^2$ ⟶ (iv)

Solving eqn (iii), (iv) with the given data points,

a =

b =

Putting a, b in eqn (i),

$y = (\text{- - -})e^{(\text{- - -})x}$ which is fitted exponential curve

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Eg : Find the exponential curve, $y = ae^{bx}$ to the following data points.

x:	2	4	6	8	10
y:	1	3	5	9	12

x	y	$X = x$	$Y = \ln y$	X^2	XY
2	1	2	0		
4	3	4			
6	5	6			
8	9	8			
10	12	10			
		$\Sigma X =$	$\Sigma Y =$	$\Sigma X^2 =$	$\Sigma XY =$

Fitting of a power curve (geometric curve) $y = ax^b$

Given $y = ax^b \longrightarrow$ (i)
Taking log on both sides,

$$\ln y = \ln a + b \ln x$$

i.e. $Y = A + BX \longrightarrow$ (ii) which is linear
where, $Y = \ln y$, $A = \ln a$, $B = b$, $X = \ln x$

The normal equations are:

$$\Sigma Y = nA + B\Sigma X \longrightarrow \text{(iii)}$$
$$\Sigma XY = A\Sigma X + B\Sigma X^2 \longrightarrow \text{(iv)}$$

Solving equations (iii) & (iv) with the help of given data points,

$A = \text{-----}$ $\Rightarrow a = e^A = \text{-----}$
$B = \text{-----}$ $\Rightarrow b = B = \text{-----}$

Now,
Putting the value of a and b in eqn (i) we get

$$y = (---)x^{(---)}$$

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2012 Fit the least square geometric curve $y = ax^b$ to the following data:

x:	1	2	3	4	5
y:	0.5	2	4.5	8	12.5

Sol^n

Given $y = ax^b$ $\longrightarrow$ (i)

Taking log on both sides,

$$\ln y = \ln a + b \ln x$$

i.e. $Y = A + BX$ $\longrightarrow$ (ii) which is linear.

where,

$Y = \ln y$, $A = \ln a$, $B = b$, $X = \ln x$

The normal equations are:

$$\Sigma Y = nA + B\Sigma X \longrightarrow \text{(iii)}$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \longrightarrow \text{(iv)}$$

x	y	$X = \ln x$	$Y = \ln y$	X^2	XY
1	0.5				
2	2				
3	4.5				
4	8				
5	12.5	ΣX	ΣY	ΣX^2	ΣXY

From eq^n (iii) & (iv), $A =$ $B =$

$a =$ $b =$

$\therefore$ Eq^n is $y = (\text{---})x^{(\text{---})}$

Program WAP to fit an exponential curve $y = ae^{bx}$ to the given set of data points.

Soln

$A = \dots \Rightarrow a = e^A =$

$B = \dots \Rightarrow b = B =$

Interpolation :-

Let $y = f(x)$ be any function, where x is input and y is output. Then the process of estimating the output inside the given range of input is called interpolation.

When the output is estimated outside the given range of input then it is called extrapolation.

→ Types of interpolation:

1. Interpolation when inputs are at equal interval

→ Methods:

a) Newton's forward difference method.

b) Newton's backward " "

c) Gauss forward difference method

d) Gauss backward " "

2. Interpolation when inputs are not at equal intervals

Methods

a) Newton's divided difference method

b) Lagrange interpolation polynomial method

Newton's Forward Difference Method -

This method is appropriate when inputs are at equal interval and the interpolation is required in the beginning of the table.

This method uses forward difference method.

Forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		$y_1 - y_0 = \Delta y_0$			
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$		
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$		$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$
		$y_3 - y_2 = \Delta y_2$		$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	
x_3	y_3		$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
		$y_4 - y_3 = \Delta y_3$			
x_4	y_4				

where, Δ = del $\rightarrow$ forward difference operator

Formula:

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \dots\dots$$

where,

$u = \frac{x - x_0}{h}$, x = interpolating point

x_0 = inital point

h = equal interval (step size) of x

Eg: Estimate the production of electricity for the year 1997 from the given data below:

Year:	1995	2000	2005	2010	2015
production:	2	5	10	18	24

Forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1995	$2 = y_0$				
		$3 = \Delta y_0$			
2000	5		$2 = \Delta^2 y_0$		
		5		$1 = \Delta^3 y_0$	
2005	10		3		$-6 = \Delta^4 y_0$
		8		-5	
2010	18		-2		
		6			
2015	24				

$$\therefore y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

where,

$$u = \frac{x - x_0}{h}$$

$$= \frac{1997 - 1995}{5}$$

$$= 0.4$$

Now,

$y = 2 + (0.4)(3) + \frac{(0.4)(0.4-1)}{2}(2) + \frac{(0.4)(0.4-1)(0.4-}{6}$

$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)(-6)}{24}$

$=$

Newton's Backward Difference Method :-

This method is appropriate when inputs are at equal interval and interpolation is required in the end of the table. This method uses backward difference table.

Backward Difference Table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		$y_1 - y_0 = \nabla y_1$			
x_1	y_1		$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
		$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
x_2	y_2		$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 =$
		$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
x_3	y_3		$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
		$y_4 - y_3 = \nabla y_4$			
x_4	y_4				

Here,

∇ = inverted del $\rightarrow$ backward difference operator

Formula:

$$y = y_4 + \frac{u}{1!}\nabla y_4 + \frac{u(u+1)}{2!}\nabla^2 y_4 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_4 + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4 + \cdots$$

where,

$u = \frac{x - x_4}{h}$, x = interpolating point

x_4 = end point of x

h = step size.

Eg: Estimate $y(9)$ from the table given below:

x:	2	4	6	8	10
y:	3	7	12	18	20

Sol^n Backward difference table:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
2	3				
		4			
4	7		1		
		5		0	
6	12		1		$-5 = \nabla^4 y_4$
		6		$-5 = \nabla^3 y_4$	
8	18		$-4 = \nabla^2 y_4$		
		$2 = \nabla y_4$			
10	$20 = y_4$				

we have,

$$y = y_4 + \frac{u}{1!}\nabla y_4 + \frac{u(u+1)}{2!}\nabla^2 y_4 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_4 + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4$$

where,

$$u = \frac{x - x_1}{h} = \frac{9 - 10}{2} = -0.5$$

$$\therefore y = 20 + (-0.5)(2) + \frac{(-0.5)(-0.5+1)}{2}(-4) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-5) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5}{24}(-5)$$

Lagrange Interpolation Polynomial Method-

This method is used even inputs are not at equal interval. Let $y = f(x)$ be any function, where x can take the values $x_0, x_1, x_2, \dots x_n$ and y can take the values $y_0, y_1, y_2, \dots y_n$. Then the lagrange interpolation polynomial is given by

x	y
x_0	y_0
x_1	y_1
x_2	y_2
$\vdots$	$\vdots$
x_n	y_n

$$y = P_n(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)\ldots(x_2-x_n)} y_2 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} y_n$$

Eg: Estimate $y(3)$ using Lagrange interpolation polynomial method for given table:

x:	2	5	10	12
y:	3	7	14	20

Sol:

x	$x_0 = 2$	$x_1 = 5$	$x_2 = 10$	$x_3 = 12$
y	$y_0 = 3$	$y_1 = 7$	$y_2 = 14$	$y_3 = 20$

We have,

$$y = P_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Here,

$x = 3$, $y = ?$

$$y = \frac{(3-5)(3-10)(3-12)}{(2-5)(2-10)(2-12)}(3) + \frac{(3-2)(3-10)(3-12)}{(5-2)(5-10)(5-12)}(7) +$$
$$\frac{(3-2)(3-5)(3-12)}{(10-2)(10-5)(10-12)}(14) + \frac{(3-2)(3-5)(3-10)}{(12-2)(12-5)(12-10)}(20)$$

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(2) Estimate the $e^{1.5}$ using interpolation polynomial b the following set of data:

$x: \; 0 \quad 1 \quad 2 \quad 3$

Sol^n $f(x) = e^{x} - 1$

$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$
$y_0 = 0$	$y_1 = 1.7183$	$y_2 = 6.3091$	$y_3 = 19.0855$

Sol

$e^{1.5} = ?$, $x = 1.5$

$$y = f(x) = e^{x} - 1$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(1.5-1)(1.5-2)(1.5-3)}{(0-1)(0-2)(0-3)} \cdot (0) + \frac{(1.5-0)(1.5-2)(1.5-3)}{(1-0)(1-2)(1-3)} (1.7183) + \frac{(1.5-0)(1.5-1)(1.5-3)}{(2-0)(2-1)(2-3)} (6.3091) + \frac{(1.5-0)(1.5-1)(1.5-2)}{(3-0)(3-1)(3-2)} (19.0855)$$

$=$

$e^{1.5} - 1 =$

$\therefore e^{1.5} =$

Try

Q. Find the cubic polynomial and hence estimate $f(7)$ from the data:

x:	5	10	12	15
y:	3	7	13	18

Numerical Differentiation:

Let us consider Newton's forward difference formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots \longrightarrow (i)$$

$y = f(u)$

where,

$$u = \frac{x - x_0}{h} \longrightarrow (ii) \qquad u = f(x)$$

Then,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0 + \frac{1}{6}(3u^2-6u+2)\Delta^3 y_0 + \frac{1}{24}(4u^3-18u^2+22u-6)\Delta^4 y_0 + \ldots\right]$$

$\longrightarrow (iii)$ $\frac{dy}{dx} = f(u)$

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right) \times \frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{1}{6}(6u-6)\Delta^3 y_0 + \frac{1}{24}(12u^2-36u+22)\Delta^4 y_0 + \ldots\right] \longrightarrow (iv)$$

when, $x = x_0$, $u = 0$, then,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \longrightarrow (v)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\Delta^2 y_0 + \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right] \longrightarrow (vi)$$

$$(u^2-u)(u-2)$$
$$= (u^3 - 3u^2 + 2u)(u-3)$$
$$= u^4 - 3u^3 + 2u^2 - 3u^3 + 9u^2 - 6u$$
$$= u^4 - 6u^3 + 11u - 6u$$

Eg: Find the first and second derivative at $x = 4$ from the table:

x:	2	4	6	8	10
y:	5	7	12	16	22

Soln

Forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	$5 = y_0$				
		$2 = \Delta y_0$			
4	7				
		5	$3 = \Delta^2 y_0$		
6	12			$-4 = \Delta^3 y_0$	
		4	-1		$7 = \Delta^4 y_0$
8	16			3	
		6	2		
10	22				

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From eq^n (iii)

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{1}{2}(2u-1)\Delta^2 y_0 + \frac{1}{6}(3u^2-6u+2)\Delta^3 y_0 + \frac{1}{24}(4u^3-18u^2+22u-6)\Delta^4 y_0\right.$$

$$= \frac{1}{2}\left[2 + \frac{1}{2}(2\times1-1)(3) + \frac{1}{6}(3\times1-6\times1+2)(-4) + \frac{1}{24}(4\times1-18\times1+22\times1-6)(7)\right]$$

$=$

From eq^n (iv)

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\Delta^2 y_0 + \frac{1}{6}(6u-6)\Delta^3 y_0 + \frac{1}{24}(12u^2-36u+22)\Delta^4 y_0\right]$$

$$= \frac{1}{4}\left[3 + \frac{1}{6}(6\times1-6)(-4) + \frac{1}{24}(12\times1-36\times1+22)(7)\right]$$

$=$

Try

Year	Electricity Consumption in Brt
1995	2
2000	5
2005	10
2010	18
2015	24

Find the first derivative for the year 2000 and 2010.

Numerical Integration

Let $y=f(x)$ be any function where x can take the values $x_0, x_1, x_2, \dots x_n$ and y can take the values $y_0, y_1, y_2, \dots y_n$. Then the integration of y from x_0 to x_n is given by

$$I = \int_{x_0}^{x_n} y\, dx$$

Let us consider Newton's forward difference formula

$$\int_{x_0}^{x_n} y\, dx = \int_{x_0}^{x_n} \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\right] dx$$

where, $u = \frac{x-x_0}{h}$

$\Rightarrow x - x_0 = hu$

$\Rightarrow dx = h\, du$

when $x = x_0$, $u=0$

$x = x_n$, $u = \frac{x_n - x_0}{h} = \frac{nh}{h} = n$

$$\therefore \int_{x_0}^{x} y\,dx$$

$$= \int_0^n \left[y_0 + u\Delta y_0 + \frac{1}{2}(u^2-u)\Delta^2 y + \frac{1}{6}(u^3 - 3u^2 + 2u)\Delta^3 y_0 + \dots\right] h\,du$$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{u^4}{4} - \frac{3u^3}{3} + \frac{2u^2}{2}\right)\Delta^3 y_0 + \dots\right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \dots\right] \to (1)$$

which is known as Newton-Cote's quadrature formulae for integration

Trapezoidal Rule:

putting $n=1$ in eq^n (1)

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\right)\cancel{\Delta^2 y_0}^{\,0} + \dots\right]$$

$$= h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= \frac{h}{2}(y_0 + y_1)$$

x	y	Δy
x_0	y_0	$y_1 - y_0 = \Delta y_0$
x_1	y_1	
x_2	y_2	
x_3	y_3	
|	|	
x_{n-1}	y_{n-1}	
x_n	y_n	

piecewise integration

Similarly

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1 + y_2)$$

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$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

We know,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \cdots \int_{x_{n-1}}^{x_n} y\,dx$$

$$= \frac{h}{2}\left[y_0 + y_1 + y_1 + y_2 + \cdots + y_{n-1} + y_n\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

which is known as Trapezoidal rule for integration.

Eg: Evaluate $\int_0^6 x^2\,dx$ using Trapezoidal rule.

Soln Given, $y = x^2$, $x_0 = 0$, $x_n = 6$

$$x_n - x_0 = nh$$

$$\therefore h = \frac{x_n - x_0}{n} = \frac{6-0}{6\ (\text{let})} = 1$$

Here, n = no. of strips (5 to 10)

x: x_0 1 x_1 2 x_2 3 x_3 4 x_4

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	$x_4 = 4$	$x_5 = 5$	$x_6 = 6$
$y = x^2$	$y_0 = 0$	$y_1 = 1$	$y_2 = 4$	$y_3 = 9$	$y_4 = 16$	$y_5 = 25$	$y_6 = 36$

$$\therefore \int_0^6 x^2\,dx = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(0+36) + 2(1+4+9+16+25)\right]$$

$$=$$

Simpson's $\frac{1}{3}$ Rule :-

Putting $n=2$ in eqn (i)

$$\int_{x_0}^{x_2} y\,dx = h\left[2y_0 + 2\Delta y_0 + \frac{1}{2}\left(\frac{8}{3}-2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(\Delta y_1 - \Delta y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{1}{3}(y_2 - y_1 - y_1 + y_0)\right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\right]$$

x	y	Δy	$\Delta^2 y$
x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$
x_1	y_1	$y_2 - y_1 = \Delta y_1$	
x_2	y_2		
x_3	y_3		
x_4	y_4		
x_5	y_5		
x_6	$\vdots$		
$\vdots$	$\vdots$		

Similarly $\int_{x_2}^{x_4} y\,dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$

. . . .

. . . .

$$\int_{x_{n-2}}^{x_n} dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

we know that,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \dots \int_{x_{n-2}}^{x_n} y\,dx$$

$$= \frac{h}{3}[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + \dots + y_{n-2} + 4y_{n-1} + y_n]$$

$$= \frac{h}{3}[(y_0 + y_n) + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1})] \rightarrow \text{(iii)}$$

which is known as Simpson's $\frac{1}{3}$ Rule.

Eg: Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ Rule. Also obtain the value of π and error.

sol^n

Given $y = \frac{1}{1+x^2}$, $x_0 = 0$, $x_n = 1$

$\therefore h = \frac{x_n - x_0}{n} = \frac{1-0}{n}$

let $n = 8$, Then $h = 1/8 = 0.125$

x	$x_0=0$	$x_1=0.125$	$x_2=0.25$	$x_3=0.375$	$x_4=0.5$	$x_5=0.625$	$x_6=0.75$	$x_7=0.875$	$x_8=1$
y	$y_0=1$	$y_1=0.98$	$y_2=0.94$	$y_3=0.87$	$y_4=0.8$	$y_5=0.72$	$y_6=0.64$	$y_7=0.56$	$y_8=0.5$

$$\therefore \int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_8)+2(y_2+y_4+y_6)+4(y_1+y_3+y_5+y_7)\right]$$

$$= \frac{0.125}{3}\left[(1+0.5)+2(0.94+0.8+0.64)+4(0.98+0.87+0.72+0.56)\right]$$

$$= 0.7825$$

Actual Integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$
$$= \tan^{-1}(\tan \pi/4) = \pi/4$$

$\therefore \pi/4 = 0.7825$

$\pi = 4 \times 0.7825 = 3.13$

Error = |Actual Integration − estimated integration|

$= \left|\frac{\pi}{4} - 0.7825\right|$

$= \left|\frac{3.14}{4} - 0.7825\right|$

$= 0.0025$

Try ① Evaluate $\int_0^1 \frac{dx}{1+x}$ using

a. Trapezoidal Rule

b. Simpson's $\frac{1}{3}$ Rule.

c. Estimate the value of $\ln(2)$ and error in both a. and b

② Evaluate $\int_0^{\pi/2} \sin x \, dx$ using Simpson's $\frac{1}{3}$ Rule taking 5 strips.

③ Evaluate $\int_0^1 (e^x + \cos x)dx$ using Trapezoidal rule.

Simpson's 3/8 Rule :-

$$\int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots\ldots\right] \longrightarrow (i)$$

putting $n=3$ in eq^n (i)

$$\int_{x_0}^{x_n} y\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{1}{2}\left(9 - \frac{9}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{81}{4} - 27 + 9\right)\Delta^3 y_0\right]$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

$$\int_{x_0}^{x_3} y\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(\Delta y_1 - \Delta y_0) + \frac{3}{8}(\Delta^2 y_1 + \Delta^2 y_0)\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_2) + \frac{3}{4}(y_2 - y_1 - y_1 + y_0) + \frac{1}{8}(\Delta y_2 - \Delta y_1 - \Delta y_1 + \Delta y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - y_2 - 2y_2 + 2y_1 + y_1 - y_0\right]$$

$$= \frac{3h}{8}\left[8y_1 + 12y_1 - 12y_0 - 6y_2 - 12y_1 + 6y_0 + y_3 - y_2 - 3y_2 + 3y_1 - y_0\right]$$

$$\int_{x_0}^{x_3} y\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly,

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 - 3y_5 + y_6\right]$$

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$$\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_n + y_n\right]$$

We know that,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_3} y\,dx + \int_{x_3}^{x_6} y\,dx + \cdots\cdots + \int_{x_{n-3}}^{x_n} y\,dx$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3 + 3y_4 + 3y_5 + y_6 + \cdots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1})\right] \quad \text{(iv)}$$

This is known as Simpson's 3/8 Rule of Integration

2018/5) Given, $y = 1/1+x^2$, $x_0 = 0$, $x_n = 1$

$h = \frac{x_n - x_0}{n} = \frac{1-0}{n} = \frac{1}{n}$

let $n=8$ then $h = 0.125$

x	$x_0=0$	$x_1=0.125$	$x_2=0.25$	$x_3=0.375$	$x_4=0.5$	$x_5=0.625$	$x_6=0.75$
y	$y_0=1$	$y_1=0.985$	$y_2=0.94$	$y_3=0.876$	$y_4=0.8$	$y_5=0.715$	$y_6=0.645$

$x_7=0.875$	$x_8=1$
$y_7=0.566$	$y_8=0.5$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}\left[(y_0+y_8)+2(y_3+y_6)+3(y_1+y_2+y_4+y_5+y_7)\right]$$

$$= \frac{3\times0.125}{8}\left[(1+0.5)+2(0.876+0.645)+3(0.985+0.941+0.8+0.719+0.566)\right]$$

$$= 0.77695 \longrightarrow (i)$$

Actual Integration $\pi = ?$

$$\int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 = \tan^{-1}\pi/4 = \pi/4 \longrightarrow (ii)$$

From (i) & (ii)

$\frac{\pi}{4} = 0.77695$

$\pi = 4\times0.77695$

$\therefore \pi = 3.1078$

Error $= |\text{actual} - \text{estimated}|$

$= \left|\frac{\pi}{4} - 0.77695\right|$

$= \left|\frac{3.14}{4} - 0.77695\right|$

$= 0.008$ //

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Try 20 5(b)
2015 (4)

Romberg's Integration

Romberg purified the integration obtained by Trapezoidal rule.

Computation of Romberg integration.

Interval	Trapezoidal integration	Romberg Integration	Final Romberg integration
h	I_1		
		$I_1 = I_2 + \frac{1}{3}(I_2 - I_1)$	
$\frac{1}{2}h$	I_2		$I = I_2 + \frac{1}{3}(I_2 - I_1)$
		$I_2 = I_3 + \frac{1}{3}(I_3 - I_2)$	
$\frac{1}{4}h$	I_3		

Eg: Evaluate: $\int_0^1 \frac{1}{1+x^2}\,dx$ using Romberg method.

soln Given,

$$y = \int_0^1 \frac{1}{1+x^2}\,dx \quad , x_0 = 0, \quad x_n = 1$$

(a) At $h = \frac{x_n - x_0}{n} = \frac{1-0}{2} = 0.5$ [Put $n=2$ for Romberg

x	$x_0=0$	$x_1=0.5$	$x_2=1$
y	$y_0=1$	$y_1=0.8$	$y_2=0.5$

$$I_1=\frac{h}{2}\left[(y_0+y_2)+2y_1\right]=\frac{0.5}{2}\left[(1+0.5)+2\times0.8\right]$$

$$=$$

(b) $h=\frac{1}{2}h=\frac{0.5}{2}=0.25$

x	$x_0=0$	$x_1=0.25$	$x_2=0.5$	$x_3=0.75$	$x_4=1$
y	$y_0=1$	$y_1=0.94$	$y_2=0.84$	$y_3=0.64$	$y_4=0.5$

$$\therefore I_2=\frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$=\frac{0.25}{2}\left[(1+0.5)+2(0.94+0.8+0.64)\right]$$

$$=0.7$$

(c) $h=\frac{h}{4}=\frac{0.5}{4}=$

x	$x_0=0$	$x_1=0.125$	$x_2=0.25$	$x_3=0.375$	$x_4=0.5$	$x_5=0.625$	$x_6=0.75$
y	$y_0=1$	$y_1=0.98$	$y_2=0.94$	$y_3=0.9$	$y_4=0.8$	$y_5=0.71$	$y_6=0.64$

$x_7=0.875$	$x_8=1$
$y_7=0.59$	$y_8=0.5$

$$I_3 = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{0.125}{2}\left[(1+0.5)+2(0.98+0.94+0.9+0.8+0.71+0.64+0.5)\right]$$

$$= 0.78875$$

Computation of Romberg Integration :-

Interval	Trapezoidal Integration	Romberg Integration	Final Romberg integration
h	$I_1 = 0.775$		
		$I_1 = 0.7825 + \frac{1}{3}(0.7825 - 0.775)$ =	
$\frac{1}{2}h$	$I_2 = 0.7825$		$I = 0.7908 + \frac{1}{3}(0.7908 - 0.785)$ =
		$I_2 = 0.78875 + \frac{1}{3}(0.78875 - 0.7825)$ =	
$\frac{1}{4}h$	$I_3 = 0.78875$		

Solution of System of Linear Equations :

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{system of linear eqns.}$$

Methods:

1. Direct Method
   (a) Gauss Elimination method
   (b) Gauss Jordan method
   (c) Factorization method
   (d) Matrix Inversion method
   (e) Cramer's method / Rule
   etc.

2. Indirect Method or Iterative Method
   (a) Gauss Jacobi iterative method
   (b) Gauss Sidel iterative method

Gauss Elimination Method :-

Let us consider system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The given system of equations can be written as

$$AX = B$$

where,

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

The augmented matrix is

$$(A,B) \sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

when, $(A, B) \sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3' & d_3' \end{pmatrix}$ by row operation

Multiplying the upper triangular matrix with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then,

$a_1x + b_1y + c_1z = d_1 \Rightarrow x =$

$b_2'y + c_2'z + d_2' \Rightarrow y =$

$c_3'z = d_3' \Rightarrow z =$

Eg: Solve the system of linear eqns by Gauss Elimination method:

$$x + 3y + 5z = 7$$
$$2x + 4y + 10z = 12$$
$$3x + 5y + 7z = 14$$

Soln

The given system of eqns can be written as $AX = B$ where,

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 10 \\ 3 & 5 & 7 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 12 \\ 14 \end{pmatrix}$$

The augmented matrix is

$(A, B) \sim \begin{pmatrix} 1 & \cancel{4}3 & 5 & 7 \\ 2 & \cancel{3}4 & 10 & 12 \\ 3 & 5 & 7 & 14 \end{pmatrix}$

By $R_2 \to R_2 - 2R_1$, and $R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & 0 & -2 \\ 0 & -4 & -8 & -7 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -8 & -3 \end{pmatrix}$$

Multiplying above with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

$\therefore \quad x + 3y + 5z = 7$

$-2y = -2 \quad \Rightarrow y = 1$

$-8z = -3 \quad \Rightarrow z = 3/8$

solving

$x + 3 \cdot 1 + 5 \cdot \frac{3}{8} = 7$

$\therefore x = 47/8$

Q. Solve :

$x + 2y + 3z + 5w = 10$

$2x + 5y + 7z + 12w = 15$

$3x + 7y - 9z + 10w = 18$

$4x - 8y + 10x - 12w = 20$

Gauss Jordan Method-

The augmented matrix

$$(A,B) \sim \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$

$$(A,B) \sim \begin{pmatrix} 1 & 0 & 0 & d_1' \\ 0 & 1 & 0 & d_2' \\ 0 & 0 & 1 & d_3' \end{pmatrix}$$ by row operation.

Then $x = d_1'$, $y = d_2'$, $z = d_3'$

Eg: Solve the system by Gauss Jordan Method-

$$2x + 3y + 5z = 6$$
$$3x + 5y + 7z = 10$$
$$5x - 7y + 10z = 12$$

Factorization or Decomposition or Triangularization Method-

In this method the co-efficient matrix A is factorized into unit lower triangular and upper triangular matrix i.e.

$$A = LU$$

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}_{3\times3} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}_{3\times3} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}_{3\times3}$$

$$= \begin{pmatrix} \\ \\ \end{pmatrix}_{3\times3}$$

Now,

L and U are known

We know that,

$AX = B$

$LUX = B$

Let $UX = Y \longrightarrow$ (i)

$LY = B \longrightarrow$ (ii)

Solving eqn (ii) for Y

$$Y = \begin{pmatrix} \\ \\ \end{pmatrix}$$

Solving eqn (i) for X

$$X = \begin{pmatrix} \\ \\ \end{pmatrix}$$

Note -

1. The co-efficient matrix A is factorized into unit lower triangular matrix and upper triangular matrix, then it is called Dolittle's factorization

2. If co-efficient matrix A is factorized into lower triangular matrix and unit upper triangular matrix, then it is called Crout's factorization.

Solve the system by factorization method-

$$2x+y+z=5$$
$$3x+5y+2z=15$$
$$2x+y+4z=8$$

The given system of equation can be written as $AX=B$ where,

$$A=\begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}, \quad X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B=\begin{pmatrix} 5 \\ 15 \\ 8 \end{pmatrix}$$

In this method, the co-efficient matrix A is factorized in Unit lower triangular and upper triangular matrix

i.e.

$$A=LU$$

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$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}_{3\times3} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}_{3\times3}$$

$$= \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}+u_{21}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{pmatrix}_{3\times3}$$

Now,

$\therefore u_{11}=2$, $u_{12}=1$, $u_{13}=1$

$l_{21}u_{11}=3$

$\therefore l_{21}=\frac{3}{2}$

$l_{21}u_{12}+u_{22}=5$

$u_{22}=5-l_{21}u_{12}$

$=5-\frac{3}{2}\times1$

$=7/2$

$l_{21}u_{13}+u_{23}=2$

$u_{23}=2-l_{21}u_{13}$

$=2-\frac{3}{2}\times1$

$=1/2$

$l_{31}u_{11}=2$

$\therefore l_{31}=2/2=1$

$l_{31}u_{12}+l_{32}u_{22}=1$

$l_{32}=\frac{1-l_{32}u_{12}}{u_{22}}$

$=\frac{1-(1\times1)}{7/2}$

$=0$

$l_{31}l_{13}+l_{32}u_{23}+u_{33}=4$

$u_{33}=4-l_{31}l_{13}-l_{32}u_{23}$

$=4-1\cdot1-0\cdot\frac{1}{2}$

$=3$

So,

$$L=\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad U=\begin{pmatrix} 2 & 1 & 1 \\ 0 & 7/2 & 1/2 \\ 0 & 0 & 3 \end{pmatrix}$$

We know that,

$$AX = B$$

ie. $LUX = B$

let, $UX = Y \longrightarrow (i)$

Then,

$$LY = B \longrightarrow (ii)$$

Solving eq^n (ii)

$$\begin{pmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ 3/2\, y_1 + y_2 \\ y_1 + y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 8 \end{pmatrix}$$

Comparing, we get,

$\therefore y_1 = 5$

$\frac{3}{2} y_1 + y_2 = 15$

$y_2 = 15 - \frac{3}{2} \cdot 5$

$\therefore y_2 = 15/2$

$y_1 + y_3 = 8$

$\therefore y_3 = 8 - 5 = 3$

$$\therefore Y = \begin{pmatrix} 5 \\ 15/2 \\ 3 \end{pmatrix}$$

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Solving eq^n (i)

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 7/2 & 1/2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 15/2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2x+y+5 \\ 7/_2\, y + 1/_2 z \\ 3z \end{pmatrix} = \begin{pmatrix} 5 \\ 15/2 \\ 3 \end{pmatrix}$$

Comparing we get,

$3z = 3$

$\therefore z = 1$

$7/_2\, y + 1/_2\, z = 15/_2$

$\therefore y = \left(\frac{15}{2} - \frac{1}{2}\right) / \frac{7}{2} = 2$

$2x + y + z = 5$

or, $2x = 5 - 2 - 1$

$\therefore x = 1$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Iterative Method-

Let us consider the system

$$a_1 x + b_1 y + c_1 z = d_1 \longrightarrow (i)$$
$$a_2 x + b_2 y + c_2 z = d_2 \longrightarrow (ii)$$
$$a_3 x + b_3 y + c_3 z = d_3 \longrightarrow (iii)$$

From eq^n (i)

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \longrightarrow (iv)$$

From eq^n (ii)

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \longrightarrow (v)$$

From eq^n (iii)

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \longrightarrow (vi)$$

(a) Gauss Jacobi Iterative Method-

First Iteration [$x = y = z = 0$]

$$\left.\begin{aligned} x &= \frac{d_1}{a_1} = a \\ y &= \frac{d_2}{b_2} = b \\ z &= \frac{d_3}{c_3} = c \end{aligned}\right\} \text{Rough values}$$

Second iteration,

$x = \frac{1}{a_1}(d_1 - b_1 b - c_1 c) = d \neq a$

$y = \frac{1}{b_2}(d_2 - a_2 a - c_2 c) = e \neq b$

$z = \frac{1}{c_3}(d_3 - a_3 a - b_3 b) = f \neq c$

Third iteration,

$x = \frac{1}{a_1}(d_1 - b_1 e - c_1 f) = g \neq d$

$y = \frac{1}{b_2}(d_2 - a_2 d - c_2 f) = h \neq e$

$z = \frac{1}{c_3}(d_3 - a_3 d - b_3 e) = i \neq f$

The process will be continued until it get ~~to~~ the same values in last two iteration.

(b) Gauss Sidel Iterative Method

First iteration $[y = z = 0]$

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1) = a \\ y &= \frac{1}{b_2}(d_2 - a_2 a) = b \\ z &= \frac{1}{c_3}(d_3 - a_3 a - b_3 b) = c \end{aligned}\right\} \text{Rough values}$$

Second iteration.

$x = \frac{1}{a_1}(d_1 - b_1 b - c_1 c) = d \neq a$

$y = \frac{1}{b_2}(d_2 - a_2 d - c_2 c) = e \neq b$

$z = \frac{1}{c_3}(d_3 - a_3 d - b_3 e) = f \neq c$

Third iteration,

$x = \frac{1}{a_1}(d_1 - b_1 e - c_1 f) = g \neq d$

$y = \frac{1}{b_2}(d_2 - a_2 g - c_2 f) = h \neq e$

$z = \frac{1}{c_3}(d_3 - a_3 g - b_3 h) = i \neq f$

2022 4) Given,

$x + y + z = 9 \longrightarrow (i)$

$2x + 3y + 4z = 13 \longrightarrow (ii)$

$3x + 4y + 5z = 40 \longrightarrow (iii)$

From eq^n (i)

$x = 9 - y - z \longrightarrow (iv)$

From eq^n (ii)

$y = \frac{1}{3}(13 - 2x - 4z) \longrightarrow (v)$

From eq^n (iii)

$z = \frac{1}{5}(40 - 3x - 4y) \longrightarrow (vi)$

First iteration $[y = z = 0]$

$x = 9$

$y = \frac{1}{3}(13 - 2\times9) = 1.667$

$z = \frac{1}{5}(40 - 3\times9 - 4\times1.667) =$

Second iteration

$x = 9 - 1.667 - 1.266 = 6.067$

$y = \frac{1}{-3}(13 - 2\times6.067 - 4\times1.266) = 1.399$

$z = \frac{1}{5}(40 - 3\times6.067 - 4\times1.399) = 3.241$

Third iteration,

$x = 9 - 1.399 - 3.241 = 4.361$

$y = \frac{1}{-3}(13 - 2\times4.361 - 4\times3.241) = 2.895$

$z = \frac{1}{5}(40 - 3\times4.361 - 4\times2.895)$

$= 3.067$

2016

7) Given,
$$12x - 3y + z = 10 \quad \text{(i)}$$
$$2x + 11y - 4z = 9 \quad \text{(ii)}$$
$$4x - 3y + 13z = 14 \quad \text{(iii)}$$

From eq^n (i)
$$x = \frac{1}{12}(10 + 3y - z) \quad \text{(iv)}$$

From eq.^{n} (ii)
$$y = \frac{1}{11}(9 - 2x + 4z) \quad \text{(v)}$$

From eq^n (iii)
$$z = \frac{1}{13}(14 - 4x + 3y) \quad \text{(vi)}$$

First Iteration $[x = y = z = 0]$
$$x = \frac{10}{12} = 0.833$$
$$y = 9/11 = 0.818$$
$$z = 14/13 = 1.077$$

Second iteration
$$x = \frac{1}{12}(10 - 3 \times 0.818 - 1.077)$$
$$y = \frac{1}{11}(9 - 2 \times 0.833 + 4 \times 1.077)$$
$$z = \frac{1}{13}(14 - 4 \times 0.833 + 3 \times 0.818)$$

Continue yourself.....

2080-8-10 After Chhath

Solution of Non-Linear Equation

Algebraic non-linear eqⁿ (quadratic, cubic, etc.)

Transcendental non-linear eqⁿ (logarithmic, trigonometric exponential)

Methods
1. Bisection Method
2. Newton-Raphson Method
3. Secant Method
4. Horner's method (only algebraic)
5. Fixed Point Iteration Method

① Bisection Method-

Let $f(x)=0$ be a non-linear equation

let $f(a) = -ve$

let $f(b) = +ve$

$$f(a).f(b) < 0$$

Then, there exist a root between a and b. Let that root be x_0. Then $x_0 = \frac{a+b}{2}$

Now,

i) If $f(x_0) = 0$, then x_0 be the root of $f(x)$.

ii) If $f(a) f(x_0) < 0$, then a root lies betⁿ a and x_0.

iii) If $f(b) f(x_0) < 0$, then a root lies betⁿ b and x_0.

Eg: Find the positive root of $x^3 - x - 1 = 0$, correct to two decimal places by bisection method

Sol^n;

Let $f(x) = x^3 - x - 1$

Here, $f(0) = -ve$

$f(1) = -ve$

$f(2) = +ve$

Since, $f(1)f(2) < 0$, hence there exist a root between 1 and 2.

i.e. $x_0 = \frac{1+2}{2} = 1.5$

$f(x_0) = f(1.5) = (1.5)^3 - 1.5 - 1$

$= 0.875 = +ve$

~~f(0.875)±~~

$\therefore$ a root lies bet^n 1 and 1.5.

i.e. $x_1 = \frac{1+1.5}{2} = 1.25$

$f(x_1) = f(1.25) = -ve$

$\therefore$ a root lies between 1.25 and 1.5,

i.e. $x_2 = \frac{1.25 + 1.5}{2} = 1.375$

$f(1.375) = +ve$

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$f(a) = -ve$ $\qquad a = 1 = -ve$

$f(b) = +ve$ $\qquad b = 2 = +ve$

$x_0 = \frac{a+b}{2}$ $\qquad x_0 = 1.5 = +ve$

$f(x_0) = -ve$

$f(x_0) = +ve$

root lies betn 1.25 and 1.375.

i.e $x_3 = \frac{1.25 + 1.375}{2} = 1.3125$

$f(x_3) = f(1.3125) = -ve$

$\therefore$ a root lies between 1.3125 and 1.375

i.e. $x_4 = \frac{1.3125 + 1.375}{2} = 1.34375$

$f(1.34375) = +ve$

$\therefore$ a root lies between 1.3125 and 1.34375

i.e. $x_5 = \frac{1.3125 + 1.34375}{2} = 1.328125$

$f(1.32815) = +ve$

$\therefore$ a root lies between 1.3125 and 1.328125

i.e. $x_6 = \frac{1.3125 + 1.328125}{2} = 1.3203125$

Hence, the positive root of $x^3 - x - 1 = 0$, correct to two decimal places is 1.32.

② Newton Raphson Method-

$$x_{n+1} = x_n - \frac{F(x_n)}{F^{-1}(x_n)} \quad , \quad n = 0, 1, 2, \ldots$$

Eg. Find the positive root of $x^3 - x - 1 = 0$, correct to three decimal places by Newton Raphson Method.

Solⁿ

let $f(x) = x^3 - x - 1$

then, $f'(x) = 3x^2 - 1$

we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad , \quad n = 0, 1, 2, \ldots,$$

$$= x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

$$= \frac{3x_n^3 - x_n - x_n^3 + x_n + 1}{3x_n^2 - 1}$$

$$= \frac{2x_n^3 + 1}{3x_n^2 - 1} \longrightarrow ①$$

Putting $n = 0$, $x_1 = \dfrac{2x_0^3 + 1}{3x_0^2 - 1}$

$f(x) = x^3 - x - 1$
$f(0) = -1 = -ve$
$f(1) = -1 = -ve$
$f(2) = 5 = +ve$
$x_0 = \frac{1+2}{2}$.

Let, $x_0 = \frac{1+2}{2} = 1.5$

$$x_1 = \frac{2\times 1.5^3 + 1}{3\times 1.5^2 - 1} = 1.3478$$

$$x_2 = \frac{2x_1^3+1}{3x_1^2-1} = \frac{2\times 1.3478^3+1}{3\times 1.3478^2-1} = 1.30252$$

$$x_3 = \frac{2x_2^3+1}{3x_2^2-1} = \frac{2\times 1.3252^3+1}{3\times 1.3252^2-1} = 1.3247$$

$$x_4 = \frac{2x_3^3+1}{3x_2^2-1} = \frac{2\times 1.3247^3+1}{3\times 1.3247^2-1} = 1.3247$$

$\therefore$ Positive root of $x^3-x-1=0$, correct to three decimal places is 1.325 //

Try Q. Find Cube root of 5 by Newton Raphson Method correct to 3 decimal places.

2019 1.b. Solve eqn $e^x - \cos x - 1.2 = 0$ using Newton Raphson.

2022 1.b. For the non-linear eqn $f(x) = 3x + \sin x - e^{-x} = 0$, Find soln by bisection method. Take $e = 0.0001$.

2021 1.b. Solve $x^2 + \tan x + e^x = 0$ using bisection method correct to 4 decimal places.

2018 1b. Find the soln of $x^2 = 4x - 10$ using Newton Raphson Method.

(3) Secant Method-

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 0, 1, 2, \ldots$$

Q. Find the root of $x^3 - x - 1 = 0$ using secant Method.

soln

let $f(x) = x^3 - x - 1$

we have,

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 0, 1, 2, \ldots \rightarrow (1)$$

Putting $n = 0$, in above eqn

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$f(x) = x^3 - x - 1$
$f(0) = -ve$
$f(1) = -ve$
$f(2) = +ve$
$\therefore x_{-1} = 1$
$\therefore x_0 = 2$

Let, $x_0 = 2$

$x_{-1} = 1$

$f(x_{-1}) = f(1) = \cancel{1} - \cancel{1} - 1 = -1$

$f(x_0) = f(2) = 8 - 2 - 1 = 5$

Now,

$$x_1 = 2 - \frac{5(2-1)}{5-(-1)} = 1.1667$$

$f(x_1) = 1.1667^3 - 1.1667 - 1 = -0.5787$

Put $n = 1$, in eqn (1).

$$x_2 = x_1 - \frac{f(x_1) - (x_1 - x_0)}{f(x_1) - f(x_0)}$$

~~x1 = 1.1667~~

$\cancel{f(x_1) = f(-0.5787) = (-0.5787)^3 + 0.5787 - 1}$

$$= 1.1667 - \frac{(-0.5787) - (1.1667 - 2)}{(-0.5787 - 5)}$$

$= 1.2531$

$f(x_2) = f(1.2531) = 1.2531^3 - 1.2531 - 1$

$= -0.2854$

Put $n = 2$ in eqn (2),

$$x_3 = x_2 - \frac{f(x_2) - (x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$= 1.2531 - \frac{(-0.2854) - (1.2531 - 1.1667)}{-0.2854 + 0.5787}$$

$= 1.3372$

$f(x_3) = f(1.3372) = 1.3372^3 - 1.3372 - 1 = 0.05385$

Putting $n = 3$ in eqn (2)

$$x_4 = x_3 - \frac{f(x_3) - (x_3 - x_2)}{f(x_3) - f(x_2)}$$

$$= 1.3372 - \frac{0.05385 - (1.3372 - 1.2531)}{0.05385 + 0.2854}$$

$= 1.3288$

$f(x_4) = f(1.3238) = 1.3238^3 - 1.3238 - 1$
$= -0.0037$

$$x_5 = x_4 - \frac{f(x_4)(x_4 - x_3)}{f(x_4) - f(x_3)}$$

$$= 1.3238 - \frac{(-0.0037)(1.3238 - 1.3372)}{-0.0037 - 0.05385}$$

$= 1.3247$

Hence, the root of $x^3 - x - 1 = 0$ correct to 2 decimal places is 1.32

self study

P.134 #	Convergence of Bisection Method		Question in Youtube
P.147 #	" " Newton Raphson Method		
P.155 #	" " Secant "		

E. Balagurusamy

Try

Q. Find cube root of 12 by Bisection Method.

Q. Use secant method to estimate the root of $x^2 - 4x - 10 = 0$ with initial estimate of $x_1 = 4$ & $x_2 = 2$.

4. Horner's Method:

Algorithm-

$P_n = a_n$

$P_{n-1} = P_n x + a_{n-1}$

- - - - - - -

$P_j = P_{j+1} x + a_j$

- - - - - - -

$P_1 = P_2 x + a_1$

$f(x) = P_0 = P_1 x + a_0$

Q. Evaluate the polynomial $f(x) = x^3 - 4x^2 + x + 6$ using Horner's rule at $x = 2$.

Soln

$f(x) = P_3(x) = x^3 - 4x^2 + x + 6 \longrightarrow$ (i)

Comparing eqn (i) with cubic polynomial.

$P_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

we get,

$n = 3,\ a_0 = 6,\ a_1 = 1,\ a_2 = -4,\ a_3 = 1$

$P_3 = a_3 = 1$

$P_2 = P_3 x + a_2 = 1 \times 2 + (-4) = -2$

$P_1 = P_2 x + a_1 = (-2)\cdot(2) + 1 = -3$

$P_0 = P_1 x + a_0 = (-3)(2) + 6 = 0$

$\therefore f(x) = f(2) = P_0 = 0$

Solution of Ordinary Differential Equation

1. Initial value Problem
2. Boundary value problem

1) Initial Value Problem –
A differential eqⁿ with initial condition is called initial value problem.

Eg. $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

2) Boundary Value Problem –
A differential eqⁿ with boundary conditions boundary value problem.

Eg, $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$, $y(x_n) = y_n$

Solving by Initial Value Problem –
Methods are:–

i) Euler's Method
ii) Four order Runge-Kutta (RK) method
iii) Heun's Method
iv) Taylor Series Method.

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Q. Solve the following ordinary differential equation for $y(2)$.

$\frac{dy}{dx} = x+y$, $y(0) = 2$

Solution

From Euler's Method –
Formula:–

$$y_{n+1} = y_n + hf(x_n, y_n) \quad , n = 0, 1, 2, \ldots$$

Given differential eqⁿ

$$\frac{dy}{dx} = f(x,y) = x+y$$

$h = \frac{x_n - x_0}{n}$

$y(0) = 2$, $0 \to x_0$, $2 \to y_0$

$h = \frac{2-0}{4} = 0.5$

let, $h = \Delta x = 0.5$

x	y
$x_0 = 0$	$y_0 = 2$
$x_1 = 0.5$	$y_1 = 3$
$x_2 = 1$	$y_2 =$
$x_3 = 1.5$	$y_3 =$
$x_4 = 2$	$y_4 =$

We have,

$y_{n+1} = y_n + hf(x_n, y_n)$

$\therefore y_1 = y_0 + hf(x_0, y_0)$
$= y_0 + hf(0, 2)$
$= 2 + 0.5(0+2)$
$= 3$

$y_2 = y_1 + hf(x_1, y_1)$
$= 3 + 0.5 f(0.5, 3)$
$= 3 + 0.5(0.5+3)$
$=$

$y_3 = y_2 + hf(x_2, y_2)$
$=$

$y_4 = y_3 + hf(x_3, y_3)$
$= \quad + 0.5 f(1.5,$
$=$
$=$

2020
(6) $y = ?$ when $x = 2$
Given, $\frac{dy}{dx} = f(x, y) = x^2 + y^2$, $h = 0.5$

Sol^n

For y_1

x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.5$	$y_1 =$
$x_2 = 1$	$y_2 =$
$x_3 = 1.5$	$y_3 =$
$x_4 = 2$	$y_4 =$

$K_1 = f(x_0, y_0) = f(0,0) = 0+0 = 0$
$K_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1)$
$= f(0.25, 0) = 0.25^2 + 0^2 = 0.0625$
$K_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2)$
$= f(0.25, 0.015625)$
$= 0.25^2 + 0.015625^2 = 0.062744$

$K_4 = f(x_0 + h, y_0 + hK_3) = f(0.5, 0.03137)$
$= 0.5^2 + 0.03137^2 = 0.25098$

$\therefore y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$
$= 0 + \frac{0.5}{6}(0 + 2\times 0.0625 + 2\times 0.62744 + 0.25098)$
$= 0.048 = 0.042$

For y_2
$K_1 = f(x_1, y_1) = f(0.5, 0.042) = 0.5^2 + 0.042^2 = 0.252$
$K_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_1) = f(0.75, 0.0735) = 0.568$
$K_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_2) = f(0.75, 0.184) = 0.596$
$K_4 = f(x_1 + h, y_1 + hK_3) = f(1, 0.34) = 1.1156$

$\therefore y_2 = y_1 + \frac{h}{6}[K_1 + 2K_2 + 2K_3 + K_4]$
$= 0.042 + \frac{0.5}{6}[0.252 + 2\times 0.568 + 2\times 0.596 + 1.1156]$
$= 0.35$

For y_3
$K_1 = f(x_2, y_2) = f(1, 0.35) = 1^2 + 0.35^2 =$

CamScanner

For y_4

do urself

Q. 2021 (6) Given $\frac{dy}{dx} = f(x,y) = x^2+y^2$, $y(0)=0.5$, $h=0.2$

$y=?$ when $x=0.4$

Sol^n

x	y
$x_0=0$	$y_0=0.5$
$x_1=0.2$	$y_1=?$
$x_2=0.4$	$y_2=?$

Fourth order Runge-Kutta (R.K.) Method.

We have,

$$y_1 - y_0 = \Delta y$$
$$\Rightarrow y_1 = y_0 + \Delta y$$
$$\Rightarrow y_1 = y_0 + \frac{h}{6}(K_1+2K_2+2K_3+K_4) \longrightarrow (1)$$

where,

$$K_1 = f(x_0, y_0)$$
$$K_2 = f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}K_1\right)$$
$$K_3 = f\left(x_0+\frac{h}{2}, y_0+\frac{h}{2}K_2\right)$$
$$K_4 = f(x_0+h, y_0+hK_3)$$

Also, we have,

$$y_2 - y_1 = \Delta y$$
$$\Rightarrow y_2 = y_1 + \Delta y$$
$$\Rightarrow y_2 = y_1 + \frac{h}{6}(K_1+2K_2+2K_3+K_4)$$

where,

$$K_1 = f(x_1, y_1)$$
$$K_2 = f\left(x_1+\frac{h}{2}, y_1+\frac{h}{2}K_1\right)$$
$$K_3 = f\left(x_1+h/2, y_1+\frac{h}{2}K_2\right)$$
$$K_4 = f(x_1+h, y_1+hK_3)$$

and so on.

2020 (6) $y=?$ when $x=2$

Given, $\frac{dy}{dx} = f(x,y) = x^2+y^2$, $h=0.5$

x	y
$x_0 = 0$	$y_0 = 0$
$x_1 = 0.5$	$y_1 = ?$
$x_2 = 1$	$y_2 = ?$
$x_3 = 1.5$	$y_3 = ?$
$x_4 = 2$	$y_4 = ?$

For y_1

$K_1 = f(x_0, y_0) = f(0,0) = 0+0 = 0$

$K_2 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1) = f(0.25, 0) = 0.0625$

$K_3 = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2) = f(0.25, 0.015625)$

$= 0.062744$

$K_4 = f(x_0 + h, y_0 + hK_3) = f(0.5, 0.03137)$

$= 0.25098$

$\therefore y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 3K_3 + K_4)$

$= 0 + \frac{0.5}{6}[0 + 2\times 0.0625 + 2\times 0.062744 + 0.25098]$

For y_2

$K_1 = f(x_1, y_1) = f(0.5, 0.042) = 0.252$

$K_2 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_1) = f(0.75, 0.0735) = 0.568$

$K_3 = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_2) = f(0.75, 0.184) = 0.596$

$K_4 = f(x_1 + h, y_1 + hK_3) = f(1, 0.34) = 1.1156$

$y_2 = y_1 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 0.042 + \frac{0.5}{6}(0.252 + 2\times 0.568 + 2\times 0.596 + 1.1156)$

$= 0.35$

2021 (6) Given, $\frac{dy}{dx} = f(x,y) = x^2 + y^2$, $y(0) = 0.5$, $h = 0.2$

$y = ?$ when $x = 0.4$

x	y
$x_0 = 0$	$y_0 = 0.5$
$x_1 = 0.2$	$y_1 = ?$
$x_2 = 0.4$	$y_2 = ?$

$h = \frac{x_n - x_0}{n} = \frac{0.4 - 0}{2} = 0.2$

2020 6) Given $\frac{dy}{dx} = f(x,y) = x^2 + y^2$, $y(0) = 0.5$, $h = 0.25$

x_0	y
$x_0 = 0$	y_1
$x_1 = 0.25$	y_2
$x_2 = 0.5$	y_3
$x_3 = 0.75$	y_4
$x_4 = 1$	y_5

2018 6) Given, $\frac{dy}{dx} = x + y$, $h = 0.5$, $y = ?$ when $x = 1$

x	y
$x_0=0$	$y_0=1$
$x_1=0.5$	$y_1=?$
$x_2=1$	$y_2=?$

2016 (i) $\frac{dy}{dx}=f(x,y)=\frac{4y}{x}$

$y(2)=?$, $y(0)=1$, $h=1$

x	y
$x_0=0$	$y_0=1$
$x_1=1$	$y_1=?$
$x_2=2$	$y_2=?$

$y(2)=?$, $y(1)=2$, $h=1$

x	y
$x_0=1$	$y_0=2$
$x_1=2$	$y_1=?$

Heun's Method–

Formula–

$$y_{n+1}=y_n+\frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y^e_{n+1})\right]$$

where, $n=0,1,2,\ldots$

$$y^e_{n+1}=y_n+hf(x_n,y_n) \longrightarrow \text{Euler's Formula}$$

Q. Given Eq^n

$$y'(x)=\frac{2y}{x} \text{ with } y(1)=2$$

estimate $y(2)$ using Heun's Method.

Sol^n Given, $y'(x)=f(x,y)=\frac{2y}{x}$

let $h=0.25$

$$\boxed{h=\frac{x_n-x_0}{n}=\frac{2-1}{4}=0.25}$$

x	y
$x_0=1$	$y_0=2$
$x_1=1.25$	$y_1=?$
$x_2=1.5$	$y_2=?$
$x_3=1.75$	$y_3=?$
$x_4=2$	$y_4=?$

We have,

$$y_{n+1}=y_n+\frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y_{n+1})\right] \longrightarrow (i)$$

$$y^e_{n+1}=y_n+hf(x_n,y_n) \longrightarrow (ii)$$

Put $n=0$ in (1)

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^e)\right]$$

where,

$$y_1^e = y_0 + h f(x_0, y_0) = 2 + 0.25\, f(1, 2)$$
$$= 2 + 0.25 \times \frac{2\times 2}{1} = 3$$

~~From (1), put $n=3$~~
~~$y_4^e = y_3 + hf(x_3, y_3)$~~

$$y_1 = 2 + \frac{0.25}{2}\left[f(1,2) + f(1.25, 3)\right]$$
$$= 2 + \frac{0.25}{2}\left[\frac{2\times 2}{1} + \frac{2\times 3}{1.25}\right] = 3.1$$

putting $n=1$ in eq^n (11)

$$y_2^e = y_1 + hf(x_1, y_1) = 3.1 + 0.25\, f(1.25, 3.1)$$
$$= 3.1 + 0.25 \times \frac{2\times 3.1}{1.25} = 4.34$$

$$\therefore y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^e)\right]$$
$$= 3.1 + \frac{0.25}{2}\left[f(1.25, 3.1) + f(1.5, 4.34)\right]$$
$$= 3.1 + \frac{0.25}{2}\left[\frac{2\times 3.1}{1.25} + \frac{2\times 4.34}{1.5}\right] = 4.44$$

Putting, $n=2$, in eq^n (11)

$$y_3^e = y_2 + hf(x_2, y_2) = 4.44 + 0.25\, f(1.5, 4.44)$$
$$= 4.44 + 0.25 \times \frac{2\times 4.44}{1.5} = 5.92$$

$$\therefore y_3 = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^e)\right]$$
$$= 4.44 + \frac{0.25}{2}\left[f(1.5, 4.44) + f(1.75, 5.92)\right]$$
$$= 4.44 + \frac{0.25}{2}\left[\frac{2\times 4.44}{1.5} + \frac{2\times 5.92}{1.75}\right]$$
$$= 6.025$$

Putting $n=3$ in eq^n (ii)

$$y_4^e = y_3 + hf(x_3, y_3)$$
$$= 6.025 + 0.25\, f(1.75, 6.025)$$
$$= 6.025 + \frac{0.25 \times 2 \times 6.025}{1.75}$$
$$= 7.75$$

From (i)

$$y_4 = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^e)\right]$$

$$= 7.75 + \frac{0.25}{2}\left[f(1.75, 6.025) + f(2, 7.75)\right]$$

$$= 7.75 + 0.125\left[\frac{2 \times 6.025}{1.75} + \frac{2 \times 7.75}{2}\right]$$

$$= 7.855$$

$$\therefore \quad y(2) = 7.855$$

Solution of Partial differential Equation–

Laplace Eqⁿ

An equation of the form

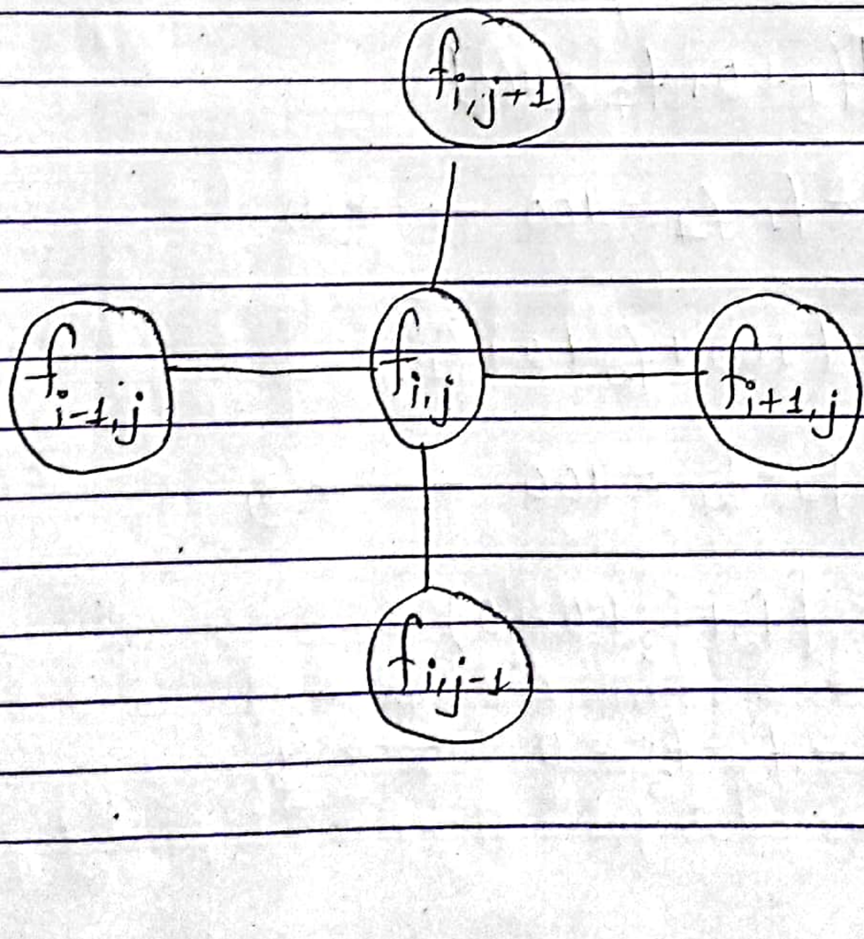
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \underline{OR}$$

$$f_{xy} + f_{yy} = 0, \quad \underline{OR}$$

$$\nabla^2 f = 0 \quad \text{where, } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{is}$$

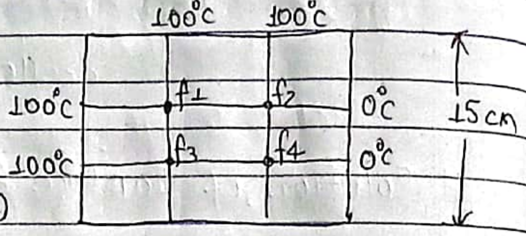
called Laplace equation.

Solution of Laplace Equation, Five-Point Formula



$$f_{i,j} = f(i,j) = \frac{1}{4}\left[f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1}\right]$$

Q. Consider a steel plate of size 15x15 cm. If two sides are held at 100°C and the other two sides are held at 0°C. What are the steady state temperature at interior point assuming a grid size of 5x5 cm.



Sol^n

$$f_1 = \frac{1}{4}(100 + f_2 + f_3 + 100)$$

or, $4f_1 - f_2 - f_3 = 200 \rightarrow (1)$

$$f_2 = \frac{1}{4}[f_1 + 0 + f_4 + 100]$$

or, $4f_2 - f_1 - f_4 = 100 \longrightarrow (2)$

$$f_3 = \frac{1}{4}(100 + f_2 + 0 + f_1)$$

or, $4f_3 - f_1 - f_2 = 100 \longrightarrow (3)$

$$f_4 = \frac{1}{4}[f_2 + f_3 + 0 + 0)$$

or, $4f_4 - f_2 - f_3 = 0 \longrightarrow (4)$

do used

Solving Eq^n (1), (2), (3), (4)

Poisson Equation.

An Equation of the form

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = g(x,y)$$

or,

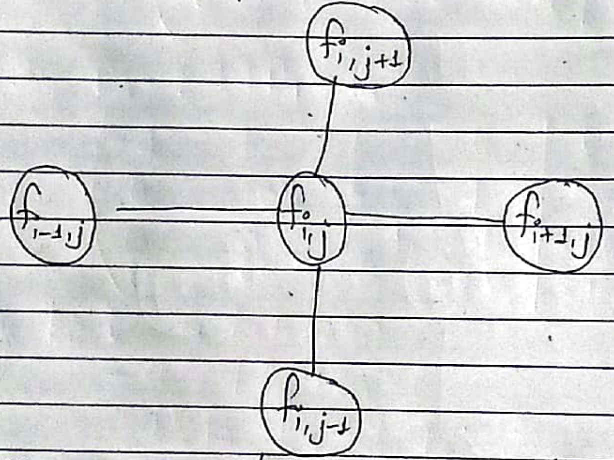
$$f_{xx} + f_{yy} = g(x,y) \quad \text{or}$$

$\nabla^2 f = g(x,y)$ where,

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called poisson Equation.

Solution of Poisson Equation.

Five Point Formula



At point 1; $(h = \Delta x, h = \Delta y)$

$$f_{i-1,j} + f_{i+1,j} + f_{i,j-1} + f_{i,j+1} - 4f_{i,j} = h^2 g_{i,j}$$

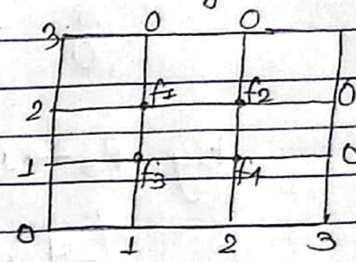
Eg. Solve the Poisson Equation

$$\nabla^2 f = 2x^2y^2 \; (g_{ij}) - g(x,y)$$

Over the square domain $0 \le x \le 3$ and $0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

Sol^n

$$h = \frac{x_n - x_0}{n} = \frac{3-0}{3} = 1$$



At point 1,

$0 + f_2 + f_3 - 0 - 4f_1 = 1^2 \times 2 \times 1^2 \times 2^2$

or, $f_2 + f_3 - 4f_1 = 8 \longrightarrow (1)$

At point 2,

$f_1 + 0 + f_4 + 0 - 4f_2 = 1^2 \times 2 \times 2^2 \times 2^2$

or, $f_1 + f_4 - 4f_2 = 32 \longrightarrow (2)$

At point 3,

$0 + f_4 + 0 + f_1 - 4f_3 = 1^2 \times 2 \times 1^2 \times 1^2$

or, $f_1 - 4f_3 + f_4 = 2 \longrightarrow (3)$

At point 4,

$f_3 + 0 + 0 + f_2 - 4f_4 = 1^2 \times 2 \times 2^2 \times 1^2$

or, $f_2 - f_3 - 4f_4 = 8 \longrightarrow (4)$

Solving Eq^n 1, 2, 3, 4 by Gauss Elimination

$$(A, B) \sim \left(\begin{array}{cccc|c} 4 & 1 & 1 & 0 & 8 \\ 1 & -4 & 0 & 1 & 32 \\ 1 & 0 & -4 & 1 & 2 \\ 0 & 1 & 1 & -4 & 8 \end{array}\right)$$

By $R_2 \longrightarrow 4R_2 + R_1$
$R_3 \longrightarrow 4R_3 + R_1$

$$\sim \left(\begin{array}{cccc|c} -4 & 1 & 1 & 0 & 8 \\ 0 & -15 & 1 & 4 & 136 \\ 0 & 1 & -15 & 4 & 16 \\ 0 & 1 & 1 & -4 & 8 \end{array}\right)$$

By $R_3 \to 15R_3 + R_2$ and $R_2 \to 15R_4 + R_2$

$$\sim \left(\begin{array}{cccc|c} -4 & 1 & 1 & 0 & 8 \\ 0 & -15 & 1 & 4 & 136 \\ 0 & 0 & -224 & 64 & 376 \\ 0 & 0 & 16 & -56 & 256 \end{array}\right)$$

By $R_4 \to 14R_1 + R_3$

$$\left(\begin{array}{cccc|c} -4 & 1 & 1 & 0 & 8 \\ 0 & -15 & 1 & 4 & 136 \\ 0 & 0 & -224 & 64 & 376 \\ 0 & 0 & 0 & -720 & 3960 \end{array}\right)$$

By Back Substitution

$$-4f_1 + f_2 + f_3 = 8$$
$$-15f_2 + f_3 + 4f_4 = 136$$
$$-224f_3 + 64f_4 = 376$$
$$-720f_4 = 3496$$

do urself

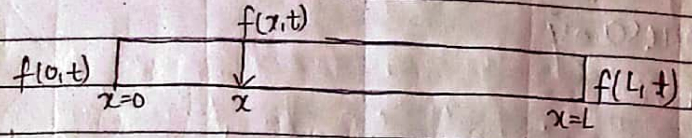
Pr 2019 (7), 2021 (7)

E-Balgurusamy
Program

Title	Program	Page. No.
Bisection	2	537
Newton-Raphson	4	541
Gauss-Elimination	10	557
Factorization	12	562
Lagrange	15	571
Newton Forward	16	572
Straight line	18	577
Trapezoidal	21	583
Simpson's 1/3	22	585
Euler's	26	551
R.K.	29	595

Parabolic Equations:

Heat Equation–



The temperature $f(x,t)$ in the rod at position x and time t is governed by the heat equation

$$K_1 \frac{\partial^2 f}{\partial x^2} = K_2 K_3 \frac{\partial f}{\partial t} \longrightarrow (i)$$

where,

K_1 = co-efficient of thermal conductivity
K_2 = specific heat
K_3 = density of material

Equation (i) may be written as

$k f_{xx}(x,t) = f_t(x,t)$, where $k = \frac{K_1}{K_2 K_3}$

Solution of Heat Equation–

(i) $\tau \le \frac{h^2}{2k}$ where, $h = \Delta x$
$\tau = \Delta t$

(ii) $f_{i,j+1} = \frac{1}{2}\left[f_{i-1,j} + f_{i+1,j}\right]$ $\quad$ i = position, j = time

This is Bender Schmidt recurrance equation

2015 (7) Given,

$2 f_{xx}(x,t) = f_t(x,t)$, $\quad 0 \le t \le 1.5$
and $\quad 0 < x < 4$

Initial condition

$f(x,0) = 50(4-x)$, $\quad 0 \le x \le 4$

Boundary condition

$f(0,t) = 0$, $\quad 0 \le t \le 1.5$
$f(4,t) = 0$, $\quad 0 \le t \le 1.5$

Here,

$K = 2$. Let $h = \Delta x = 1$

$\tau \le \frac{h^2}{2k} = \frac{1^2}{2 \times 2} = 0.25$

i.e. $\tau = \Delta t = 0.25$

$h = \frac{x_n - x_0}{n}$ $\quad$ n → choose yourself

$= \frac{4-0}{4} = 1$

$\left(\frac{\text{left}+\text{right}}{2}\right)$

$x \rightarrow$ / $t \downarrow$	0	1	2	3	4
	$f(0,t)=0 \downarrow$	$50(4-1)$	$50(4-2)$	$50(4-3)$	$f(4,t)=0 \downarrow$
0	0	150	100	50	0
0.25	0	$\frac{0+100}{2}$ 50	$\frac{150+50}{2}$ 100	$\frac{100+0}{2}$ 50	0
0.5	0	$\frac{0+100}{2}$ 50	$\frac{50+50}{2}$ 50	$\frac{100+0}{2}$ 50	0
0.75	0	$\frac{0+50}{2}$ 25	$\frac{50+50}{2}$ 50	$\frac{50+0}{2}$ 25	0
1	0	$\frac{0+50}{2}$ 25	$\frac{25+25}{2}$ 25	$\frac{50+0}{2}$ 25	0
1.25	0	$\frac{0+25}{2}$ 12.5	$\frac{25+25}{2}$ 25	$\frac{25+0}{2}$ 12.5	0
1.5	0	$\frac{0+25}{2}$ 12.5	$\frac{12.5+12.5}{2}$ 12.5	$\frac{25+0}{2}$ 12.5	0

Hyperbolic Equations

Wave Equation

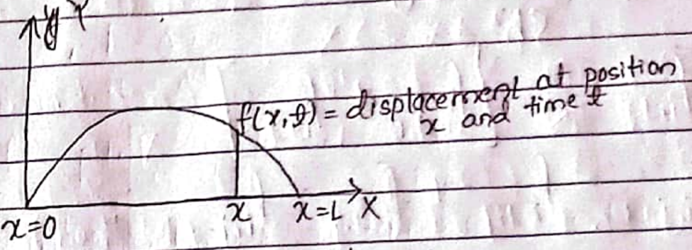


Fig: Displacement of vibrating string

The displacement $f(x,t)$ of a vibrating string at position x and time t is governed by the wave equation

$$T\frac{\partial^2 f}{\partial x^2} = \rho\frac{\partial^2 f}{\partial t^2} \longrightarrow (i)$$

where, T be the tension in the string and ρ be the mass per unit length.

Eq^n (i) may be written

$$T f_{xx}(x,t) = \rho f_{tt}(x,t) \longrightarrow (ii)$$

$$h = \frac{x_n - x_0}{n} = \frac{5-0}{5} = 1$$

Solution of wave Equations -

1. $1 - \frac{T\tau^2}{\rho h^2} = 0$

2. $f_{i,1} = \frac{1}{2}\left[f_{i-1,0} + f_{i+1,0}\right]$

3. $f_{i,j+1} = f_{i-1,j} + f_{i+1,j} - f_{i,j-1}$

Q. Solve numerically the wave equation
$f_{tt}(x,t) = 4 f_{xx}(x,t)$, $0 \le x \le 5$
$0 \le t \le 2.5$

with boundary conditions.
$f(0,t) = 0$, $0 \le t \le 2.5$
$f(5,t) = 0$, $0 \le t \le 2.5$

and,
initial values
$f(x,0) = f(x) = x(5-x)$

solution;

Let $h = \Delta x = 1$, Here, $T = 4$, $\rho = 1$

Now,
$$1 - \frac{T\tau^2}{\rho h^2} = 0$$
$$\Rightarrow 1 - \frac{4\tau^2}{1 \times 1^2} = 0$$
$$\Rightarrow 1 - 4\tau^2 = 0$$
$$\Rightarrow 4\tau^2 = 1$$
$$\therefore \tau = \tfrac{1}{2} = 0.5$$

$\therefore \tau = \Delta t = 0.5$

	$t \downarrow$ \ $x \rightarrow$	0 $f(0,t)=0$	1 $x(5-x)$	2	3	4	5 $f(5,t)=0$	
$j-1$ t_0	0	0	4	6	6	4	0	← Initial condition
j t_1	0.5	0	3	5	5	3	0	$f_{i,1}$
$j+1$	1	0	1	2	2	1	0	$f_{i,j+1}$
	1.5	0	−1	−2	−2	−1	0	
	2	0	−3	−5	−5	−3	0	
	2.5	0	−4	−6	−6	−4	0	

Eigen Value and Eigen Vector

Let A be the square matrix and X be the column vector such that

$$AX = \lambda X$$

where λ be the scalar known as eigen value and the column vector X is called eigen vector.

Power Method (Iterative Method)-

Eg: Find the largest eigen value (λ_1) and corresponding eigen vector V_1 of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ using the power method.

solution;

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and

$X^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Note: We can assume any 3x1 matrix as $X^{(0)}$

$$AX^{(0)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = 2X^{(1)}$$

Note: ठूलो No. लाई बाहिर निकालने so that we can find largest eigen value.

$$AX^{(1)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2.5 \\ 0 \end{pmatrix} = 2.5 \begin{pmatrix} 0.8 \\ 1 \\ 0 \end{pmatrix}$$

$$= 2.5 X^{(2)}$$

$$AX^{(2)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0.8 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 2.6 \\ 0 \end{pmatrix} = 2.8 \begin{pmatrix} 1 \\ 0.93 \\ 0 \end{pmatrix}$$

$$= 2.8 X^{(3)}$$

$$AX^{(3)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0.93 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.86 \\ 2.93 \\ 0 \end{pmatrix} = 2.93 \begin{pmatrix} 0.98 \\ 1 \\ 0 \end{pmatrix}$$

$$= 2.93 X^{(4)}$$

$$AX^{(4)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0.98 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.98 \\ 2.96 \\ 0 \end{pmatrix} = 2.98 \begin{pmatrix} 1 \\ 0.99 \\ 0 \end{pmatrix}$$

$$= 2.98 X^{(5)}$$

$$AX^{(5)} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0.99 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.98 \\ 2.99 \\ 0 \end{pmatrix} = 2.99 \begin{pmatrix} 0.99 \\ 1 \\ 0 \end{pmatrix}$$

$\approx 3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

= 2.99 is largest eigen value ≈ 3

$\therefore$ Largest Eigen value = 3

Corresponding Eigen Vector = $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Convergence of Newton Raphson Method

Let x_n be an estimate of a root ~~of~~ at the function $f(x)$. Let x_n and x_{n+1} are close to each other. Then by using Taylor series

$$f(x_{n+1}) = f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{f''(R)}{2}(x_{n+1} - x_n)^2 \longrightarrow (i)$$

Let x_r be the exact root of $f(x)$. Then $x_{n+1} = x_r$ and hence $f(x_{n+1}) = 0$

From eq^n (i),

$$0 = f(x_n) + f'(x_n)(x_r - R_n) \frac{f''(R)(x_r - x_n)^2}{2} \longrightarrow (ii)$$

We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow \frac{f(x_n)}{f'(x_n)} = (x_n - x_{n+1})$$

$$\Rightarrow f(x_n) = f'(x_n)(x_n - x_{n+1})$$

From eq^n (ii)

$$0 = f'(x)(x_n - x_{n+1}) + f'(x_n)(x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2$$

$x-3=0$ ~ linear

$x^2 - 4x = 0$ ~ quadratic

If power is 1.618 ~ super linear

or, $0 = f'(x_n)(x_n - x_{n+1} + x_r - x_n) + \frac{f''(R)}{2}(x_r - x_n)^2$

or, $0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(R)}{2}(x_r - x_n)^2 \longrightarrow (iii)$

We know

$$x_r - x_{n+1} = e_{n+1}$$

$$x_r - x_n = e_n$$

From eq^n (iii)

$$0 = f'(x_n)e_{n+1} + \frac{f''(R)}{2}e_n^2$$

$$\Rightarrow e_{n+1} = \frac{-f''(R)}{2f'(x_n)} e_n^2 \longrightarrow (iv)$$

which shows that Newton Raphson method is said to have quadratic convergence.

$$f'(R_n) = \frac{f(x_n) - f(x_r)}{x_n - x_r}$$

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(n) - f(x_{n-1})}$$