Quantum algorithm for linear systems of equations

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Discussion

I. Introduction

II - The HHL Algorithm

III - Concluding Remarks

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Intro - 1

Here, we look at the Harrow-Hassidim-Lloyd (HHL) algorithm for solving linear systems of equations.
A system of linear equations simply refers to two or more equations in one or more common variables.
For Example, This is a pair of linear equations in two variables :

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

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Intro - 2

The above pair of linear equations $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ can be represented in the matrix form like this :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The above matrix equation can be written as :

$$AX = B$$

Then "$X = A^{-1}B$" is the required solution. Given that matrix A is invertible(non-singular).

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Intro - 3

- In the above example, A is a matrix of order 2. However In many fields of science and engineering, it is required to solve large systems of linear equations, where the order of matrix A is going to be much larger.
- For a classical computer, even to approximate the solution of N linear equations in N unknowns, it required time that scales as least as N.
- Indeed, merely to write out the solution takes time of order N.

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Intro - 3

- However, there are times when one is not interested in the full solution vector to those equations, but rather in computing some function of that solution.
- For example, determining the total weight of some subset of the indices.
- Here we will see that under given conditions, the HHL algorithm can achieve up to an exponential speed up.
- That is, approximate the value of such a function(of the solution vector) in time, that scales logarithmically in N.

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HHL 1

- Presented here is a basic idea of the algorithm.
- What we start with is a hermitian matrix A (meaning, $A = A^{\dagger}$, conjugate transpose) of order N.
- unit vector $\vec{b}$ (meaning magnitude, norm $= 1$)
- $\vec{x}$ that satisfies : $A\vec{x} = \vec{b}$.

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HHL 2

- First we need to encode the above problem ($A\vec{x} = \vec{b}$) in its corresponding quantum mechanical form.
- This would look like $A\,|x\rangle = |b\rangle$
- $|x\rangle$ and $|b\rangle$ here are quantum states written in "ket" notation.
- This makes our final required state :

$$A^{-1}|b\rangle = |x\rangle$$

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HHL 3

- $\vec{b}$ to quantum state $|b\rangle$
- A quantum state is a super position of classical states,

$$|b\rangle = b_0|0\rangle + b_1|1\rangle + b_2|2\rangle .... + b_{N-1}|N-1\rangle$$

- Equivalently : $|b\rangle = \sum_{i=1}^{N} b_i|i\rangle$
- Here b_i is any complex number called the amplitude of $|i\rangle$ in $|b\rangle$.
- The quantum state can also be written as an N - dimensional vector of its amplitudes :

$$|b\rangle = \begin{bmatrix} b_0 \\ b_1 \\ . \\ . \\ b_{N-1} \end{bmatrix}$$

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HHL 4

We make the following restrictive assumptions that make the linear system suitable for the HHL algorithm :

- We have a Unitary that can prepare $\vec{b}$ to quantum state $|b\rangle$. (Unitary matrix meaning $U.U^{\dagger} = 1$, conjugate transpose is also its inverse. Important because they preserve the norm and thereby the probability amplitude of a quantum state.)
- The matrix A is s-sparse (at most "s" non zero elements per row) and well conditioned.
- well conditioned meaning, the condition number "k" is small.

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HHL 5

- k is the ratio between A's largest and smallest eigenvalues. ($A|a\rangle = \lambda|a\rangle$: where $|a\rangle$ is the eigenvector and λ is the corresponding eigenvalue)
- Meaning, all of A's eigenvalues lie in the interval [1/k,1].
- We assume these conditions so that A is well conditioned and therefore can be easily inverted.

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HHL 6

Now, Intuitively looking at the inversion process :

- If the hermitian matrix A has spectral decomposition $A = \sum_{j=0}^{N-1} \lambda_j a_j a_j^T$
- Then the function A^{-1} is the same as function $a_j \mapsto \frac{1}{\lambda_j} a_j$
- Meaning, we just multiply the eigenvector a_j with the scalar $1/\lambda_j$
- "b" can also be written as a linear combination of eigenvectors :

$$b = \sum_j \beta_j a_j$$

(We need not know the coefficients β_j for what follows.)

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HHL 7

- We would like to apply A^{-1} to b to obtain the quantum state :

$$A^{-1}|b\rangle = \sum_j \beta_j \frac{1}{\lambda_j} a_j$$

- Unfortunately, the functions A and A^{-1} are not unitary(unless all $\lambda_j = 1$). So we cannot directly apply A^{-1} as a quantum operation to state $|b\rangle$ to get the state $|x\rangle$.

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HHL 8

- However, under the taken assumptions, we can convert A to a Unitary U_A of the form e^{iAt} which we can apply at will.
- This is possible due to hamiltonian simulation.
- The only way in which a quantum system can change is by multiplication with a unitary matrix.
- The hamiltonian of the system decided which unitary will actually occur in the given physical environment, which is the observable H corresponding to the total energy in the system.

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HHL 9

- One can think of the Hamiltonian H as describing the physical characteristics of the system. And it comes via the Schroedinger Equation, which is a linear differential equation that relates the time derivative of the current state to that state itself and to the Hamiltonian :

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

- Then, if we start in some state $|\psi(0)\rangle$, the solution to this differential equation is the following unitary evolution of the state :

$$|\psi(t)\rangle = U|\psi(0)\rangle, \ \textit{where } U = e^{-iHt}$$

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HHL 10

- So, t time-steps of evolution induced by Hamiltonian H, corresponds to applying the unitary matrix e^{-iHt} t times.
- For simplicity, we will ignore the minus sign in hamiltonian simulation, implementing e^{iHt} instead.

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HHL 11

- Given, an N $\times$ N unitary matrix, we can interpret it as a quantum operation, mapping an N- dimensional vector of amplitudes to another N-dimensional vector of amplitudes. This is called the quantum Fourier transform (QFT).
- An application of QFT is phase estimation : Suppose we can apply a unitary U and we are given an eigenvector $|\psi\rangle$ of U with corresponding unknown eigenvalue $\lambda(U|\psi\rangle = \lambda|\psi\rangle)$, and we would like to compute or at least approximate the λ.
- Here, without going into the phase estimation algorithm in detail, we utilize it as a black box that furnishes the estimate λ_j associated with eigenvectors $|a_j\rangle$.(with some approximation error, that we will ignore here.)

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HHL 12

- Conditioned on our estimate of λ_j, we can then rotate an auxiliary $|0\rangle$ qubit to $\frac{1}{k|\lambda_j\rangle}|0\rangle + \sqrt{1 - \frac{1}{(k|\lambda_j)^2}}|1\rangle$.
- Next we undo the phase estimation to set the register that contained the estimate back to $|0\rangle$.
- Suppressing the auxiliary qubits containing temporary results of the phase estimation, we have now unitarily mapped :

$$|a_j\rangle|0\rangle \mapsto |a_j\rangle\left(\frac{1}{k|\lambda_j\rangle}|0\rangle + \sqrt{1 - \frac{1}{(k|\lambda_j)^2}}|1\rangle\right)$$

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HHL 13

- When we prepare a copy of $|b\rangle$ and apply the above unitary function to it, we obtain the required :

$$\frac{1}{k}\sum_j \beta_j \frac{1}{\lambda_j}|a_j\rangle + |\psi\rangle|1\rangle$$

- Where we don't care about the state $|\psi\rangle$ and thus the state $|x\rangle$ is prepared as intended.

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slide 1

- Even though the algorithm involves a lot caveats with respect to assumptions with regards to matrix A and state $\vec{b}$. Also, the resultant state is not the vector $\vec{x}$ exactly but an estimation of the state $|x\rangle$ so on.
- HHL and the later algorithms represent real advances in the theory of quantum algorithms.
- The End.

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