Relations & Functions - 2

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Having defined the concepts of relation and function in part 1, In this chapter we will study different types of relations and functions, composition of functions, invertible functions and binary operations.

1 Types of Relations

We know that a relation in a set A is a subset of A $\times$ A. It could be $\emptyset$ or A $\times$ A or some subset in between. This leads us to the following definitions :

1.1 Empty Relation

A relation R in a set A is called an empty relation, if no element of A is related to any element of A. That is R = $\emptyset$.

1.2 Universal Relation

A relation R in a set A is called universal relation, if each element of A is related to every element of A. That is R = A $\times$ A.

> Both the empty relation and the universal relation are sometimes called **trivial relations**.

1.3 Reflexive Relation

A relation R in a set A is called reflexive if $(a, a) \in$ R for every $a \in$ A.

1.4 Symmetric Relation

A relation R in a set A is called symmetric if $(a, b) \in$ R implies that $(b, a) \in$ R for all $a, b \in$ A.

1.5 Transitive Relation

A relation R in a set A is called transitive if $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R for all $a, b, c \in$ A.

1.6 Equivalence Relation

A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

1.7 Example

Let R be the relation defined in the set A = {1,2,3,4,5,6,7} by R = $\{(a, b)$: both a and b are either odd or even$\}$. Show that R is an equivalence relation.
Solution :
$\Rightarrow$ Given any element a in A, it will either be odd or even. So $(a, a) \in$ R for all $a \in$ A. Thus R is reflexive.
$\Rightarrow$ Further, $(a, b) \in$ R implies both a and b must be either odd or even, thus $(b, a) \in$ R as well for any $a, b \in$ A. Thus R is symmetric.
$\Rightarrow$ Finally, $(a, b) \in$ R and $(b, c) \in$ R implies all elements a, b, c must be either even or odd simultaneously, hence implying that $(a, c) \in$ R for any $a, b, c \in$ A. Thus R is transitive.
Hence, we have shown that R is an equivalence relation.

2 Types of Functions

2.1 One-One

A function f : X $\rightarrow$ Y is said to be one-one (or **injective**) if distinct elements of X under f have distinct images in Y. That is, for every x_1, $x_2 \in$ X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
Otherwise, if multiple elements in X have the same image in Y, then f is called **many-one**.

2.2 Onto

A function f : X $\rightarrow$ Y is said to be onto (or **surjective**) if every element of Y is the image of some element of X under f. That is, for every $y \in$ Y, there exists an element x in X such that $f(x) = y$.
This implies that f : X $\rightarrow$ Y is onto if and only if range of f = Y, which is the codomain.

2.3 One-One & Onto

A function f : X $\rightarrow$ Y is said to be **bijective** if it is both one-one and onto.

2.4 Example

Show that the function f : N $\rightarrow$ N, given by $f(x) = 2x$ is one-one but not onto.
Solution : The function is one-one as all distinct natural numbers x under f will have a distinct image $y = f(x) = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is not onto as $y = 1$ is not an image of any element x in N under f. Meaning, for $y = 1$, there does not exist any natural number x, satisfying $y = f(x) = 2x = 1$.

2.5 Example

Prove that the function f : R $\to$ R given by $f(x) = 2x$ is one-one and onto.
Solution : The function is one-one as all distinct real numbers x under f will have a distinct image $y = f(x) = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is onto as for any real number y, there exists x in R under f satisfying $y = f(x) = 2x$.

2.6 Example

Show that an onto function f : $\{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.
Solution : If f is not one-one, there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same.
Now, the image of 3 under f can be only one element. Therefore the range set can have at most two elements of the co-domain, showing that f is not onto, a contradiction. Hence f has to be one-one.

2.7 Example

Show that a one-one function f : $\{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.
Solution : Since f is one-one, the three elements of the domain must have three distinct images in the co-domain under f. And since the co-domain has only three elements which are all covered, hence f has to be onto.

> **Remark** : The results shown in the above two examples 2.6 and 2.7 are also true for an arbitrary finite set X. That is, a one-one function f : X $\to$ X is necessarily onto and an onto function f : X $\to$ X is necessarily one-one, for every finite set X.
> In contrast to this example 2.4 shows that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

3 Composition of Functions

4 Invertible Function