# Relations & Functions - 1

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## 1 Cartesian Products of Sets

Suppose A is a set of two colours and B is a set of two objects :

A = {red, black} and B = {bag, coat}

How many pairs of coloured objects can be made from these two sets ? Proceeding in an orderly manner, we can see that four distinct pairs will be possible :

(red, bag), (red, coat), (black, bag), (black, coat)

The pairs formed above are called **ordered pairs**, from two sets A and B. As the name suggests, they are pairs written together within brackets and their order matters.

**Definition** : Given two non-empty sets A and B. The cartesian product
A $\times$ B is the set of all ordered pairs (x,y) such that x belongs to A and y belongs to B. That is,
A $\times$ B = { (x,y) : x $\in$ A and y $\in$ B }

In the example above, A $\times$ B = {(red, bag), (red, coat), (black, bag), (black, coat)} is the cartesian product of sets A and B.

### 1.1 Remarks

- Two ordered pairs are equal if and only if their corresponding first and second elements are equal. Naturally, this implies that A $\times$ B $\neq$ B $\times$ A as (x,y) $\neq$ (y,x).
- If there are $p$ elements in set A and $q$ elements in set B, then there will be $pq$ elements in A $\times$ B.
- If either A or B is an empty set, then A $\times$ B will also be an empty set.
- If A and B are non-empty sets and either A or B is an infinite set, then so is A $\times$ B.

### 1.2 Example

If P = {0,1}, form the set P $\times$ P $\times$ P.
$\Rightarrow$ Going step by step, first we calculate P $\times$ P
$\Rightarrow$ P $\times$ P = {(0,0), (0,1), (1,0), (1,1)}
$\Rightarrow$ And therefore, we have P $\times$ P $\times$ P = {(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)}

### 1.3 Example

$\mathbf{R}$ is the set of real numbers, what do the cartesian products $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ represent ?
$\Rightarrow$ The set $\mathbf{R}$ represents all the points on a line.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$ represents the coordinates of all the points in two-dimensional space.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ represents the coordinates of all the points in three-dimensional space.

### 1.4 Example

If A × B = {(p,q),(p,r),(m,q),(m,r)}, find A and B.
A = set of first elements = {p,m}
B = set of second elements = {q,r}

## 2 Relations

Consider two sets A = {a, b, c} and B = {Aviral, Bijoy, Chandra, Divya}
Their cartesian product can be written as A × B = {(a, Aviral),(a, Bijoy).....(c, Divya)}.
We can now obtain a subset of A × B by introducing a relation R between the first element x and the second element y of each ordered pair (x,y). For example, "x is the first letter of the name y".
What subset of A × B will be obtained when the above relation is applied ?
We obtain the following subset : {(a, Aviral), (b, Bijoy), (c, Chandra)}

> **Definition** : A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A × B. This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in A × B.

In the above example, R = {(a, Aviral), (b, Bijoy), (c, Chandra)} was the relation obtained. It can also be written as R = {(x,y): x is the first letter of the name y, x ∈ A, y ∈ B}.

### 2.1 Remarks

- The second element of the ordered pairs in a relation R is called the **image** of the first element.
- The set of all first elements of the ordered pairs in a relation R is called the **domain** of the relation.
- The set of all second elements of the ordered pairs in a relation R is called the **range** of the relation.
- The whole set B is called the **codomain** of the relation.
- A relation R from set A to A is also stated in short as a "relation on A".

## 2.2 Example

Consider sets P = {9, 4, 25} and Q = {5, 3, 2, 1, -2, -3, -5}
and the following relation R = {(x,y) : $x = y^2$, x ∈ P, y ∈ Q}
The above relation can be written in roster form as : R = {(9,3), (9,-3), (4,2), (4,-2), (25,5), (25,-5)}
The domain of this relation is {4, 9, 25}
The range of this relation is {-2, 2, -3, 3, -5, 5}
Set Q is naturally the codomain of this relation.

## 2.3 Note

> The total number of relations that can be defined from set A to set B is equal to the number of possible subsets of A × B.
> We know that if n(A) = $p$ and n(B) = $q$, then n(A × B) = $pq$. Therefore the number of possible relations, which is equal to the number of possible subsets will be $2^{pq}$.

**How is the total number of relations or subsets equal to $2^{pq}$ and how do we write down all of these subsets ?**
Let us think of it, in terms of bits (0/1).
With one bit, we have two possible states 0 and 1.
With two bits, we have four possible states (00, 01, 10, 11)
8 possible states with three bits, 16 possible states with four bits and so on for $m$ bits, we have $2^m$ possible states.
Similarly, for any set A, with n(A) = $m$, we have $2^m$ possible subsets or relations.
And, these subsets can be computed in the same way as the bit states. For example, consider a set A = {2,3} with two elements, the four possible subsets can be computed as follows :
⇒ $00$ = subset with neither first nor second element = $\{\emptyset\}$
⇒ $01$ = subset with only the second element = $\{3\}$
⇒ $10$ = subset with only the first element = $\{2\}$
⇒ $11$ = subset with both the elements = $\{2, 3\}$
Further, the set containing all the $2^m$ subsets of A is naturally called the **power set** of A, as seen in the chapter on **Sets**.
Hence, all the possible subsets can be written down using the bit analogy as shown above.

# 3 Functions

> **A relation from a set $A$ to a set $B$ is said to be a function if every element of set $A$ has one and only one image in set $B$. The function $f$ from set $A$ to set $B$ is denoted by $f : A \rightarrow B$ where $f(a) = b$ such that $a \in A$ and $b \in B$. Here, $b$ is called the image of $a$ under $f$.**

In other words, a function $f$ is a special type of relation for which, the domain is set $A$ (every element in set $A$) and no two distinct ordered pairs $(a, b) \in f$ have the same first element (one and only one image in set $B$). The term **map** or **mapping** is sometimes used to denote a function.

### 3.1 Example

$\mathbf{N}$ is the set of natural numbers. A relation $R$ is defined on $\mathbf{N}$ such that $R = \{(x, y) : y = 2x \text{ where } x, y \in \mathbf{N}\}$
**Question** : What is the domain, codomain and range of $R$ ? Is this relation a function ?
**Solution** : The Domain of $R$ is the set of natural numbers. The codomain is also $\mathbf{N}$. The range is the set of even natural numbers. Since every natural number has one and only one image as per the defined relation, therefore this relation is a function.
**Note** : A $function$ whose $range$ is real valued is called a $real\ valued\ function$. Further, if its $domain$ is also real valued, it is called a $real\ function$.

## 4 Some Common Functions

### 4.1 Identity Function

Let $\mathbf{R}$ be the set of real numbers. The identity function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = x$ for each $x \in \mathbf{R}$. The $domain$ and $range$ of $f$ are $\mathbf{R}$.

### 4.2 Constant Function

The constant function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = c$ and $c$ is a constant. Here the $domain$ of $f$ is $\mathbf{R}$ and its $range$ is $\{c\}$.

### 4.3 Polynomial Function

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be a polynomial function defined as $y = f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $n$ is a non-negative integer and $a_0, a_1, a_2...a_n \in \mathbf{R}$.

### 4.4 Rational Function

These are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of $x$ defined in a domain, where $g(x) \neq 0$.
**For example**, consider the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by :

$$f(x) = \frac{1}{x}$$

### 4.5 Modulus Function

The function $f : \mathbf{R} \to \mathbf{R}$ denoted by $f(x) = |x|$ is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers $\mathbf{R}$, whereas the range is the set of all non-negative real numbers.

# 5 Algebra of Real Functions

1. **Addition** : Let $f$ and $g$ be two functions, then we define $(f+g)(x) = f(x) + g(x)$
2. **Subtraction** : Let $f$ and $g$ be two functions, then we define $(f-g)(x) = f(x) - g(x)$
3. **Multiplication by Scalar** : Let $f$ be a function and $\alpha$ be a real number. Then we define $(\alpha f)(x) = \alpha f(x)$
4. **Multiplication** : Let $f$ and $g$ be two functions, then we define $(f.g)(x) = f(x).g(x)$
5. **Quotient** : Let $f$ and $g$ be two functions, then quotient $f$ by $g$ is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

# 6 Solved Exercises

## 6.1

Let $f(x) = x^2$ and $g(x) = 2x + 1$
Find : $(f+g)(x)$, $(f-g)(x)$, $(f.g)(x)$ and $\frac{f}{g}(x)$
**Solution** :
$(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$
$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq -1/2$

## 6.2

Let $R$ be a relation on the set of rational numbers $\mathbf{Q}$ defined by :

$$R = \{(a,b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1. $(a,a) \in R$ for all $a \in \mathbf{Q}$
2. $(a,b) \in R$ implies that $(b,a) \in R$
3. $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$

**Solution** :

1. For any rational number $a$, $a - a = 0$, which is obviously an integer. Therefore, it follows that $(a,a) \in R$ for all $a \in \mathbf{Q}$
2. $(a,b) \in R$ implies that $a - b \in \mathbf{Z}$. So, If $a - b$ is an integer then it follows naturally that $b - a$ which is simply $-(a-b)$ is an integer as well. Therefore $(b,a) \in R$
3. $(a,b) \in R$ and $(b,c) \in R$ implies that $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. The sum of two integers will be an integer as well. Adding the two $(a-b)+(b-c) = a-c \in \mathbf{Z}$. Therefore, $(a,c) \in R$

### 6.3

Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a linear function on the set of integers **Z**, then find $f(x)$.
**Solution** : Since $f$ is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where $m$ and $c$ are constants. Also since $(1,1) \in f$ and $(0,-1) \in f$ , substituting these values $(x,y)$ in $y = f(x)$ we have

$$1 = f(1) = m(1) + c$$

$$-1 = f(0) = m(0) + c$$

Solving the two equations $(1 = m + c)$ and $(-1 = c)$ from above, we get $m = 2$, $c = -1$ and thus $f(x)$ is found :

$$f(x) = 2x - 1$$

# 7 References


1. Class 11 - Chapter 2 : Relations and Functions.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.