Relations & Functions - 2

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Having defined the concepts of relations and functions in part 1, In this chapter we will study the different types of relations and functions, composition of functions, invertible functions and binary operations.

1 Types of Relations

We know that a relation on a set A is some subset of the cartesian product A $\times$ A. It could be either $\emptyset$ or A $\times$ A or some subset in between. This leads us to the following definitions :

1.1 Empty Relation

Relation R on a set A is called an empty relation, if no element of A is related to any element of A under R. That is R = $\emptyset$.

1.2 Universal Relation

Relation R on a set A is called universal relation, if each element of A is related to every element of A under R. That is R = A $\times$ A.
Both the empty relation and the universal relation are sometimes called **trivial relations**.

1.3 Reflexive Relation

Relation R on a set A is called reflexive if $(a, a) \in$ R for every $a \in$ A.

1.4 Symmetric Relation

Relation R on a set A is called symmetric if $(a, b) \in$ R implies that $(b, a) \in$ R for all $a, b \in$ A.

1.5 Transitive Relation

Relation R on a set A is called transitive if $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R for all $a, b, c \in$ A.

1.6 Equivalence Relation

Relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

1.7 Example

Let R be the relation defined on the set A = {1,2,3,4,5,6,7} by R = $\{(a, b)$: both a and b are either odd or even}. Show that R is an equivalence relation.
Solution :
⇒ Given any element a in A, it will either be odd or even. So $(a, a) \in$ R for all $a \in$ A. Thus R is reflexive.
⇒ Further, $(a, b) \in$ R implies both a and b must be either odd or even, thus $(b, a) \in$ R as well for any $a, b \in$ A. Thus R is symmetric.
⇒ Finally, $(a, b) \in$ R and $(b, c) \in$ R implies all elements a, b, c must be either even or odd simultaneously, hence implying that $(a, c) \in$ R for any $a, b, c \in$ A. Thus R is transitive.
Hence, we have shown that R is an equivalence relation.

2 Types of Functions

We have learned that a relation R : X → Y qualifies to be a function f : X → Y when every element of X has only one image in Y.

2.1 One-One (Injective)

Further, if every element in the range of f has only one pre-image in X, then the function qualifies to be a one-one function. Otherwise f is called **many-one**.
In other words, a function f : X → Y is said to be one-one if distinct elements of X have distinct images in Y. That is, for every x_1, $x_2 \in$ X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

2.2 Onto (Surjective)

A function f : X → Y is said to be onto, if every element in Y has a pre-image in X. That is, for every $y \in$ Y, there exists an element x in X such that $f(x) = y$.
This also implies that f : X → Y is onto if and only if Range of f = Y, which means that the range is equal to the codomain.

2.3 One-One & Onto (Bijective)

A function f : X → Y is bijective if every element in Y has one and only one pre-image in X. That is, the function is both one-one and onto.

2.4 Example

Show that the function f : N → N, given by $f(x) = 2x$ is one-one but not onto.
Solution : The function is one-one as all distinct natural numbers x under f will have a distinct image $y = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is not onto as $y = 1$ has no pre-image x in N under f. Naturally, as there is no natural number that can satisfy $y = 2x = 1$.

2.5 Example

Prove that the function f : R $\to$ R given by $f(x) = 2x$ is one-one and onto.
Solution : The function is one-one as distinct real numbers x under f will have distinct images $y = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
The function is onto as for any real number y, there exists x in R under f satisfying $y = 2x$.

2.6 Example

Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.
Solution : If f is not one-one, there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same.
Now, the image of 3 under f can be only one element. Therefore the range set can have at most two elements of the co-domain, showing that f is not onto, a contradiction. Hence f has to be one-one.

2.7 Example

Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.
Solution : Since f is one-one, the three elements of the domain must have three distinct images in the co-domain under f. And since the co-domain has only three elements which are all covered, hence f has to be onto.

> **Remark** : The results shown in the above two examples 2.6 and 2.7 are also true for an arbitrary finite set X. That is, a one-one function f : X $\to$ X is necessarily onto and an onto function f : X $\to$ X is necessarily one-one, for every finite set X.
> In contrast to this example 2.4 shows that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

3 Composition of Functions

Consider a school examination, where each student is assigned a unique roll number under the function f and further, each roll number is assigned a code number under the function g, in order to have confidentiality.
Thus, by a combination of these two functions, each student is eventually assigned a code number. This leads to the following definition :

> Let f : A $\to$ B and g : B $\to$ C be two functions. Then the composition of f and g is defined as gof : A $\to$ C where $gof(x) = g(f(x))$ for all $x \in$ A.

3.1 Example

$f : \mathbf{R} \to \mathbf{R}$ is a function defined as $f(x) = 2x$ and $g : \mathbf{R} \to \mathbf{R}$ is a function defined as $g(x) = x^2$. Find gof and fog.

We have $gof = g(f(x)) = g(2x) = (2x)^2 = 4x^2$. Similarly, $fog = f(g(x)) = f(x^2) = 2x^2$.

3.2 Remarks

- If f : A $\rightarrow$ B and g : B $\rightarrow$ C are one-one, then gof : A $\rightarrow$ C is also one-one.
- If f : A $\rightarrow$ B and g : B $\rightarrow$ C are onto, then gof : A $\rightarrow$ C is also onto.

4 Invertible Function

Continuing with the school examination analogy above. After the answer sheets are examined, the examiner assigns marks to all the code numbers. However, we need to assign the marks back to each student.
Therefore, in a process reverse to g, roll numbers are assigned back to the code numbers. Further, in a process reverse to f, student names are assigned to the roll numbers. Thus, all students are assigned the marks that they scored. This leads us to the following definition :

> A function f : X $\rightarrow$ Y is defined to be invertible, if there exists a function g : Y $\rightarrow$ X such that $g(f(x)) = x$ for all x in X. And $f(g(y)) = y$ for all y in Y. The function g is called the inverse of f and is denoted as f^{-1}.

4.1 Remarks

- f is invertible if and only if it is bijective (one-one and onto).
- The inverse of f if it exists, is unique. Such that $f^{-1} = g$ and $g^{-1} = f$.
- This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, especially when the actual inverse of f need not be determined.

4.2 Example

Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function defined as $f(x) = 2x$. Show that f is invertible and find the inverse.
$\Rightarrow$ As per the given function, $f(x) = 2x$.
$\Rightarrow y = 2x$
$\Rightarrow x = y/2$
Now, let us define $g : \mathbf{R} \rightarrow \mathbf{R}$ such that $g(y) = y/2$. It can be easily verified that $gof(x) = x$ and $fog(y) = y$ for all $x, y \in \mathbf{R}$. Hence, f is invertible and g is the inverse of f.

4.3 Theorem 1

If f : A $\rightarrow$ B, g : B $\rightarrow$ C and h : C $\rightarrow$ D are functions then $ho(gof) = (hog)of = h(g(f(x)))$ for all $x \in$ A.

4.4 Theorem 2

Let f : A $\rightarrow$ B and g : B $\rightarrow$ C be two invertible functions, then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$.

> **Note** : Observe that while composing f and g to get gof, first f and then g was applied. However, when inverting the composite gof, first g^{-1} and then f^{-1} is applied.

5 Binary Operations

"Binary" means two. A binary operation is an operation which can be applied to only two elements at a time to produce another element. We have looked at four fundamental operations since childhood, namely addition, subtraction, multiplication and division.
It is to be noted that only two numbers can be added or multiplied at a time. When we need to add three numbers, we add the first two and the result is then added to the third number. Thus, addition, multiplication, subtraction and division are examples of binary operations. This gives rise to a general definition as follows :

> A binary operation on a set X is a function f : X $\times$ X $\rightarrow$ X. We denote $f(a, b)$ by afb.

5.1 Example

Show that addition, subtraction and multiplication are binary operations on $\mathbf{R}$, but division is not a binary operation on $\mathbf{R}$. Further, show that division is a binary operation on the set $\mathbf{R} - \{0\}$.
Solution :
Addition, $+ : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ given by $+(a, b) \rightarrow a + b$
Subtraction, $- : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ given by $-(a, b) \rightarrow a - b$
Multiplication, $\times : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ given by $\times(a, b) \rightarrow a \times b$
are valid functions and hence by definition binary operations on $\mathbf{R}$, since they carry each pair (a, b) to a unique element in $\mathbf{R}$.
However, Division, $\div : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ given by $\div(a, b) \rightarrow a \div b$
is not a valid function on $\mathbf{R}$, as for $b = 0$, $\dfrac{a}{b}$ is not defined.
However, for $\mathbf{R} - \{0\}$, division is a valid function and hence a binary operation.

5.2 Commutative

A binary operation $*$ on a set X is called commutative, if $a * b = b * a$, for every $a, b \in$ X. Addition and multiplication are commutative binary operations, whereas subtraction and division are not commutative.

5.3 Associative

A binary operation $* :$ X $\times$ X $\rightarrow$ X is said to be associative if $(a * b) * c = a * (b * c)$ for all $a, b, c \in$ X. Addition and multiplication are associative binary operations, however subtraction and division are not associative.

5.4 Identity

Given a binary operation $* : \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{X}$, an element $e \in \mathrm{X}$, if it exists, is called identity for the operation if $a * e = a = e * a$ for all $a \in \mathrm{X}$.
0 is the identity for addition on $\mathbf{R}$ and 1 is the identity for multiplication on $\mathbf{R}$. However, subtraction and division do not have an identity element in $\mathbf{R}$.

5.5 Inverse

Given a binary operation $* : \mathrm{X} \times \mathrm{X} \rightarrow \mathrm{X}$ with the identity element e in X, an element $a \in \mathrm{X}$ is said to be invertible with respect to the operation $*$, if there exists an element b in X such that $a * b = e = b * a$. Here b is called the inverse of a and is denoted by a^{-1}.
$-a$ is the inverse of a for the addition operation on $\mathbf{R}$ and $\dfrac{1}{a}$ is the inverse of a for the multiplication operation on $\mathbf{R} - \{0\}$.

6 Solved Exercises

6.1 Exercise

If R1 and R2 are equivalence relations on set A, show that R1 $\cap$ R2 is also an equivalence relation.
Solution :
Since R1 and R2 are both equivalence relations on set A, it means R1 and R2 are reflexive, symmetric and transitive on set A.
Therefore, for all $a, b, c \in A$,
$(a, a) \in$ R1 and $(a, a) \in$ R2
$(a, b), (b, a) \in$ R1 and R2.
$(a, b), (b, c), (a, c) \in$ R1 and R2.
Naturally, it follows that :
$(a, a) \in$ R1 $\cap$ R2
$(a, b), (b, a) \in$ R1 $\cap$ R2.
$(a, b), (b, c), (a, c) \in$ R1 $\cap$ R2 for all $a, b, c \in$ A.
Hence proved that R1 $\cap$ R2 is an equivalence relation on A.

7 References

1. Class 12 - Chapter 1 : Relations and Functions.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.