Matrices

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The concept of **matrix** evolved through an attempt to obtain simpler and more compact methods of solving systems of linear equations.
Matrices are used as a representation for the coefficients in systems of linear equations. They simplify our work to a great extent and are therefore useful in various branches of science and mathematics from physics, cryptography, genetics, economics and so on.

1 What is a Matrix ?

Definition : A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

For example,
The situation that "Aviral has 1 pen and 2 note books, Siddharth has 2 pens and 1 notebook and Shreyas has 2 pens and 2 notebooks" can be represented in matrix form as follows :

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$$

We denote matrices with capital letters. In the above matrix A, the horizontal lines of elements are called **rows** and the vertical lines of elements are called **columns**.

1.1 Order of a Matrix

Definition : A matrix having m rows and n columns is said to have order $m \times n$, read as "m by n" matrix.

In the general a $m \times n$ matrix has the following representation :

$$X = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & \dots & a_{4n} \\ . & & & & \\ . & & & & \\ . & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

So, here we see that X is a matrix of order $m \times n$. The number of elements in an $m \times n$ matrix will obviously be equal to m multiplied by n.
The individual elements of the matrix are represented by a, where a_{ij} refers to the element lying in the i^{th} row and j^{th} column.
For example,
a_{43} is an element of X lying in the 4^{th} row and 3^{rd} column.

2 Types of Matrices

2.1 Column Matrix

Definition : A matrix is said to be a column matrix if it has only one column.

For example,
The following is a column matrix of order 3×1 :

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2.2 Row Matrix

Definition : A matrix is said to be a row matrix if it has only one row.

For example,
The following is said to be a row matrix of order 1×3 :

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

2.3 Square Matrix

Definition : A matrix in which the number of rows and columns are equal is called a square matrix.

Thus, in case of a square matrix m = n. Hence, it will be simply called a square matrix of order n.
For example,
The following is a square matrix of order 3 :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

2.4 Diagonal Matrix

Definition : A square matrix is called a diagonal matrix if all its non diagonal elements are zero. meaning $a_{ij} = 0$ when $i \neq j$.

For example,
The following is a diagonal matrix of order 3 :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

2.5 Scalar Matrix

Definition : A diagonal matrix is called a scalar matrix if all its diagonal elements are equal.

For example,
The following is a scalar matrix of order 3 :

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

2.6 Identity Matrix

Definition : A scalar matrix is called an Identity matrix if all its diagonal elements are equal to 1. We denote the identity matrix by I, where the order of matrix is clear from the context :

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.7 Zero Matrix

Definition : A matrix is called a zero matrix or null matrix if all its elements are zero. We denote the zero matrix by O, where the order of matrix is clear from the context :

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3 Equality of Matrices

Definition : Two matrices A and B are said to be equal if they are of the same order and each element of A is equal to the corresponding element of B. That is $a_{ij} = b_{ij}$ for all i and j.

For example,
The following are an example of equal matrices : $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$.

4 Operations on Matrices

4.1 Addition of Matrices

Definition : The sum of two matrices A and B is a matrix C, which is obtained by adding corresponding elements of A and B. That is, $c_{ij} = a_{ij} + b_{ij}$ for all i and j. Furthermore, the two matrices have to be of the same order.

For example,
Given,

$$A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & 1/2 \end{bmatrix}$$

Then,

$$C = A + B = \begin{bmatrix} \sqrt{3}+2 & 1+\sqrt{5} & 0 \\ 0 & 6 & 1/2 \end{bmatrix}$$

It is important to note that, if A and B are not of the same order, then $A + B$ is not defined.
The addition of matrices satisfies the following **properties** :

1. **Commutative** : $A + B = B + A$
2. **Associative** : $(A + B) + C = A + (B + C)$
3. **Additive Identity** : $A + O = O + A = A$. In other words, Null Matrix O is the additive identity of matrix addition.
4. **Additive Inverse** : $A + (-A) = (-A) + A = O$. So, $-A$ is the additive inverse of A.

4.2 Scalar Multiplication of Matrices

Definition : Given a matrix A and a scalar k then kA is another matrix which is obtained by multiplying each element of A by the scalar k. That is a_{ij} becomes ka_{ij} for all i and j.

For example,
If,

$$A = \begin{bmatrix} 3 & 1 & 1.5 \\ 5 & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$$

Then,

$$2A = \begin{bmatrix} 6 & 2 & 3 \\ 10 & 14 & -6 \\ 4 & 0 & 10 \end{bmatrix}$$

Negative of a matrix is denoted by $-A$. We define

$$-A = (-1).A$$

The scalar multiplication of matrix, satisfies the following **properties** :

1. $k(A + B) = kA + kB$
2. $(k + l)A = kA + lA$

Where, A and B are matrices of the same order whereas k and l are scalars.

4.3 Difference of Matrices

Definition : The difference of two matrices A and B is a matrix C, which is obtained by subtracting corresponding elements of A and B. That is, $c_{ij} = a_{ij} - b_{ij}$ for all i and j. Furthermore, the two matrices have to be of the same order.

For example,
If,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

Then,

$$2A - B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4.4 Multiplication of Matrices

**Definition : Firstly, The product of two matrices A and B is defined if the number of columns of A is equal to the number of rows of B.
Then, the product of the matrices A and B is the matrix C, whose order is given by the number of rows of A and number of columns of B.
Finally, to obtain the elements c_{ij} of C, we take the i^{th} row of A and j^{th} column of B, multiply them element-wise and take the sum these products.**

For example,
If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}_{2\times 3}$$

And

$$B = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}_{3\times 2}$$

⇒ Then, firstly product AB is defined as number of columns of A is same as the number of rows of B.
⇒ Secondly, the resultant matrix C has order 2×2, given by the number of rows of A and number of columns of B.
⇒ Finally, the resultant matrix can be written as :

$$AB = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2\times 2}$$

Where the elements c_{ij} of C can be computed as specified in the definition above.
⇒ For instance, to calculate c_{21}, we take the second row of A and the first column

of B, multiply their corresponding elements and take the sum. The elements of C are computed accordingly and shown below :

$$c_{11} = (1)(2) + (-1)(-1) + (2)(5) = 13$$

$$c_{12} = (1)(7) + (-1)(1) + (2)(-4) = -2$$

$$c_{21} = (0)(2) + (3)(-1) + (4)(5) = 17$$

$$c_{22} = (0)(7) + (3)(1) + (4)(-4) = -13$$

Therefore,

$$AB = C = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}_{2\times 2}$$

The multiplication of matrices satisfies the following **properties** :

1. **Non-commutative** : Even if products AB and BA are both defined, It is not necessary that AB equals BA that is $AB \neq BA$.
   Remark : Multiplication of diagonal matrices of the same order will be commutative.

2. **Associative** : $(AB)C = A(BC)$, whenever both sides of the equality are defined.

3. **Distributive** : $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$, whenever both sides of the equality are defined.

4. **Multiplicative Identity** : For every square matrix A, there exists an Identity matrix of the same order such that $IA = AI = A$.

5 Transpose of a Matrix

Definition : Let A be a $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. That is $a_{ij} \to a_{ji}$ for all i and j. This new $n \times m$ matrix is denoted by A' or A^T.

For example,
If

$$A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & -1 \end{bmatrix}_{3\times 2}$$

Then

$$A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & -1 \end{bmatrix}_{2\times 3}$$

The transpose of matrices, satisfies the following **properties** :

1. $(A')' = A$

2. $(kA)' = kA'$ (where k is any constant)

3. $(A + B)' = A' + B'$

4. $(AB)' = B'A'$

6 Symmetric and Skew Symmetric Matrices

Definition : A square matrix A is said to be a symmetric if $A' = A$, that is $a_{ij} = a_{ji}$ for all i, j. Further, a square matrix A is said to be skew-symmetric if $A' = -A$, that is $a_{ij} = -a_{ji}$ for all i, j.

For example, the following is a symmetric matrix :

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -3 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

Whereas, the following is a skew-symmetric matrix :

$$B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$$

Notice, as per the definition of a skew-symmetric matrix, $a_{ij} = -a_{ji}$ for all i, j. Therefore, when $i = j$:
$\Rightarrow$ We have $a_{ii} = -a_{ii}$
$\Rightarrow 2a_{ii} = 0$
$\Rightarrow a_{ii} = 0$ for all i
This means that the diagonal elements of a skew-symmetric matrix are always zero.

6.1 Theorem 1

For any square matrix A with real number entries, $A + A'$ is a symmetric matrix and $A - A'$ is a skew symmetric matrix.

Proof :
The first part of the above theorem is proven if we show $(A + A')' = A + A'$
Taking $(A + A')'$
$= A' + (A')'$
$= A' + A$
$= A + A'$

For the second part, we need to show that $(A - A')' = -(A - A')$
Let us take $(A - A')'$
$= A' - (A')'$
$= A' - A$
$= -(A - A')$
Hence Proved.

6.2 Theorem 2

Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Proof :
Let A be a square matrix, we can thus write -

$2A = A + A$
$2A = A + A + O$
$2A = A + A + A' - A'$
$2A = (A + A') + (A - A')$
$2A$ has been thus expressed as the sum of a symmetric and a skew-symmetric matrix and thus A as well, can be simply written as -
$A = 1/2(A + A') + 1/2(A - A')$
Hence Proved.

7 Elementary Transformations of a Matrix

There are three main elementary transformations of a matrix :

1. The interchange of any two rows or two columns. Denoted as $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
2. The multiplication of elements of any row or column by a non-zero number. Denoted by $R_i \to kR_i$ or $C_i \to kC_i$ where $k \neq 0$.
3. The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number. Denoted by $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$ where $k \neq 0$.

8 Invertible Matrices

Definition : If A is a square matrix and if there exists another square matrix B of the same order, such that $AB = BA = I$, then B is called the inverse of A and is denoted by A^{-1}. In that case A is said to be invertible.

For example,
Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

We see that $AB = BA = I$. Thus B is the inverse of A and A is the inverse of B, that is $A = B^{-1}$ and $B = A^{-1}$.
The following points are to be noted regarding inverse of a matrix :

- A rectangular matrix does not possess an inverse.
- Inverse of a square matrix, if it exists, is unique.
- If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

9 References

1. Class 12 - Chapter 3 : Matrices.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.