Relations & Functions - 1

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1 Cartesian Products of Sets

Suppose A is a set of two colours and B is a set of two objects :

A = {red, black} and B = {bag, coat}

How many pairs of coloured objects can be made from these two sets ? Pairing the elements together in an orderly manner, we get the following four distinct pairs :

(red, bag), (red, coat), (black, bag), (black, coat)

The pairs formed above are called **ordered pairs**. These are pairs, of elements taken from two sets, paired together in brackets as shown above. Here the order of elements matters, hence (red, bag) is not the same as (bag, red).

Definition : Given two non-empty sets A and B. The cartesian product $A \times B$ is the set of all ordered pairs (x,y) such that x belongs to A and y belongs to B. That is,
$A \times B = \{ (x,y) : x \in A \text{ and } y \in B \}$

Naturally, In the example above, $A \times B$ = {(red, bag), (red, coat), (black, bag), (black, coat)} is the cartesian product of sets A and B.

1.1 Remarks

- The order of elements in an ordered pair matters. Meaning $(x,y) \neq (y,x)$, this implies that $A \times B \neq B \times A$.
- Two ordered pairs are equal if and only if their corresponding first and second elements are equal.
- If there are p elements in set A and q elements in set B, then there will be pq elements in $A \times B$.
- If either A or B is an empty set, then $A \times B$ will also be an empty set.
- If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.

1.2 Example

If $P = \{0,1\}$, form the set $P \times P \times P$.
$\Rightarrow$ Going step by step, first we calculate $P \times P$
$\Rightarrow P \times P = \{(0,0), (0,1), (1,0), (1,1)\}$
$\Rightarrow$ And therefore, we have $P \times P \times P = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

1.3 Example

$\mathbf{R}$ is the set of real numbers, what do the cartesian products $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ represent ?
$\Rightarrow$ The set $\mathbf{R}$ represents all the points on a line.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$ represents the coordinates of all the points in two-dimensional space.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ represents the coordinates of all the points in three-dimensional space.

1.4 Example

If A $\times$ B = {(p,q),(p,r),(m,q),(m,r)}, find A and B.
A = set of first elements = {p,m}
B = set of second elements = {q,r}

2 Relations

Consider two sets A = {a, b, c} and B = {Aviral, Bijoy, Chandra, Divya}
Their cartesian product can be written as A $\times$ B = {(a, Aviral),(a, Bijoy)...(c, Divya)}.
We can now obtain a subset of A $\times$ B by introducing a relation R between the first element x and the second element y of each ordered pair (x,y). For example, "x is the first letter of the name y".
What subset of A $\times$ B will be obtained when the above relation is applied ?
We obtain the following subset : {(a, Aviral), (b, Bijoy), (c, Chandra)}

> **Definition** : A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product A $\times$ B. This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in A $\times$ B.

In the above example, R = {(a, Aviral), (b, Bijoy), (c, Chandra)} was the relation obtained. It can also be written as R = {(x,y): x is the first letter of the name y, x $\in$ A, y $\in$ B}.

2.1 Remarks

- The second element of the ordered pairs in a relation R is called the **image** of the first element.
- The set of all first elements of the ordered pairs in a relation R is called the **domain** of the relation.
- The set of all second elements of the ordered pairs in a relation R is called the **range** of the relation.
- The whole set B is called the **codomain** of the relation.
- A relation R from set A to A is also stated in short as a "relation on A".

2.2 Example

Consider sets P = {9, 4, 25} and Q = {5, 3, 2, 1, -2, -3, -5}
and the following relation R = {(x,y) : $x = y^2$, x ∈ P, y ∈ Q}
The above relation can be written in roster form as : R = {(9,3), (9,-3), (4,2), (4,-2), (25,5), (25,-5)}
The domain of this relation is {4, 9, 25}
The range of this relation is {-2, 2, -3, 3, -5, 5}
Set Q is naturally the codomain of this relation.

2.3 Note

> The total number of relations that can be defined from set A to set B is equal to the number of possible subsets of A × B.
> We know that if n(A) = p and n(B) = q, then n(A × B) = pq. Therefore the number of possible relations, which is equal to the number of possible subsets will be 2^{pq}.

How is the total number of relations or subsets equal to 2^{pq} and how do we write down all of these subsets ?
Let us think of it, in terms of bits (0/1).
With one bit, we have two possible states 0 and 1.
With two bits, we have four possible states (00, 01, 10, 11)
8 possible states with three bits, 16 possible states with four bits and so on for m bits, we have 2^m possible states.
Similarly, for any set A, consisting of m elements, that is n(A) = m, we have 2^m possible subsets or relations.
And, these subsets can be computed in the same way as the bit states. For example, consider a set with two elements, A = {2,3}. The four possible subsets can be computed as follows :
$\Rightarrow 00$ = subset with neither first nor second element = $\{\emptyset\}$
$\Rightarrow 01$ = subset with only the second element = $\{3\}$
$\Rightarrow 10$ = subset with only the first element = $\{2\}$
$\Rightarrow 11$ = subset with both the elements = $\{2, 3\}$
Hence, all the possible subsets can be written down using the bit analogy as shown above. Further, the set containing all the 2^m subsets of A is naturally called the **power set** of A, as seen in the chapter on **Sets**.

3 Functions

A function is a special type of relation. Terms such as **map** or **mapping** are also used to denote a function.

> A relation from a set A to a set B is said to be a function if **every** element of set A has **one** and only one image in set B.

If f is a function from set A to set B, denoted by $f : A \to B$ and $(x, y) \in f$ then $f(x) = y$ such that $x \in A$ and $y \in B$.

3.1 Example

N is the set of natural numbers. A relation R is defined on **N** such that R = {(x,y) : y = 2x where x, y ∈ **N**}. What is the domain, codomain and range of R ? Is this relation a function ?
The domain of R is **N**.
The codomain is also **N**.
The range is the set of even natural numbers.
And since every natural number has one and only one image as per the defined relation, therefore this relation is a function.

> **Note** : A function whose range is real valued is called a **real valued function**. Further, if its domain is also real valued, it is called a **real function**.

4 Some Common Functions

4.1 Identity Function

Let $\mathbf{R}$ be the set of real numbers. The identity function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = x$ for each $x \in \mathbf{R}$. The $domain$ and $range$ of f are $\mathbf{R}$.

4.2 Constant Function

The constant function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = c$ and c is a constant. Here the $domain$ of f is $\mathbf{R}$ and its $range$ is $\{c\}$.

4.3 Polynomial Function

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be a polynomial function defined as $y = f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where n is a non-negative integer and $a_0, a_1, a_2...a_n \in \mathbf{R}$.

4.4 Rational Function

These are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.
For example, consider the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by :

$$f(x) = \frac{1}{x}$$

4.5 Modulus Function

The function $f : \mathbf{R} \to \mathbf{R}$ denoted by $f(x) = |x|$ is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers $\mathbf{R}$, whereas the range is the set of all non-negative real numbers.

5 Algebra of Real Functions

1. **Addition** : Let f and g be two functions, then we define $(f+g)(x) = f(x) + g(x)$
2. **Subtraction** : Let f and g be two functions, then we define $(f-g)(x) = f(x) - g(x)$
3. **Multiplication by Scalar** : Let f be a function and α be a real number. Then we define $(\alpha f)(x) = \alpha f(x)$
4. **Multiplication** : Let f and g be two functions, then we define $(f.g)(x) = f(x).g(x)$
5. **Quotient** : Let f and g be two functions, then quotient f by g is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

6 Solved Exercises

6.1

Let $f(x) = x^2$ and $g(x) = 2x + 1$
Find : $(f+g)(x)$, $(f-g)(x)$, $(f.g)(x)$ and $\frac{f}{g}(x)$
Solution :
$(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$
$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq -1/2$

6.2

Let R be a relation on the set of rational numbers $\mathbf{Q}$ defined by :

$$R = \{(a,b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1. $(a,a) \in R$ for all $a \in \mathbf{Q}$
2. $(a,b) \in R$ implies that $(b,a) \in R$
3. $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$

Solution :

1. For any rational number a, $a - a = 0$, which is obviously an integer. Therefore, it follows that $(a,a) \in R$ for all $a \in \mathbf{Q}$
2. $(a,b) \in R$ implies that $a - b \in \mathbf{Z}$. So, If $a - b$ is an integer then it follows naturally that $b - a$ which is simply $-(a-b)$ is an integer as well. Therefore $(b,a) \in R$
3. $(a,b) \in R$ and $(b,c) \in R$ implies that $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. The sum of two integers will be an integer as well. Adding the two $(a-b)+(b-c) = a-c \in \mathbf{Z}$. Therefore, $(a,c) \in R$

6.3

Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a linear function on the set of integers **Z**, then find $f(x)$.
Solution : Since f is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where m and c are constants. Also since $(1,1) \in f$ and $(0,-1) \in f$, substituting these values (x,y) in $y = f(x)$ we have

$$1 = f(1) = m(1) + c$$

$$-1 = f(0) = m(0) + c$$

Solving the two equations $(1 = m + c)$ and $(-1 = c)$ from above, we get $m = 2$, $c = -1$ and thus $f(x)$ is found :

$$f(x) = 2x - 1$$

7 References

1. Class 11 - Chapter 2 : Relations and Functions.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.