Relations and Functions
Part 1

Aviral Janveja

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Much of the mathematics is about finding a pattern, a recognizable link between quantities that change. In this chapter, we will see how to link two sets and thus introduce **Relations** between them. We also learn about special relations which will qualify to be **Functions**. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correlation between one quantity and another.

1 Cartesian Product of Sets

> **Definition : Given two non-empty sets A and B. The cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B. Denoted as $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. That is, the set of all ordered pairs (a, b) and in that particular order, such that a belongs to set A and b belongs to set B.**

For example,
Suppose A is a set of colors and B is a set of objects :

$$A = \{red, blue\} \text{ and } B = \{bag, coat\}$$

We see that four distinct ordered pairs can be obtained from the above sets :

$$A \times B = \{(red, bag), (red, coat), (blue, bag), (blue, coat)\}$$

1.1 Remarks

- The order of pairing of elements is crucial. (a, b) is not the same as (b, a). That is why it is called an **ordered** pair.
- Two ordered pairs are equal only if their corresponding first and second elements are equal.
- If there are p elements in set A and q elements in set B, then there will be $p.q$ elements in the set $A \times B$. That is, $n(A \times B) = p.q$
- If either A or B is an empty set, then $A \times B$ will also be empty set.
- If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.
- $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is an ordered triplet.

1.2 Example

If $P = \{0, 1\}$, form the set $P \times P \times P$.
$\Rightarrow$ Going step by first we can first calculate $P \times P$
$\Rightarrow P \times P = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$
$\Rightarrow$ And Therefore :

$$P \times P \times P = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

1.3 Example

$\mathbf{R}$ is the set of real numbers, what do the cartesian products $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ represent ?
$\Rightarrow$ The set $\mathbf{R}$ represents the points on a line.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$ represents the coordinates of all the points in two-dimensional space.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ represents the coordinates of all the points in three-dimensional space.

1.4 Example

If $A \times B = \{(p, q), (p, r), (m, q), (m, r)\}$, find A and B.
A = set of first elements = $\{p, m\}$
B = set of second elements = $\{q, r\}$

2 Relations

Definition : A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the *image* of of the first element.
The set of all first elements of the ordered pairs in relation R is called the *domain* of the relation and the set of all second elements is called the *range* of the relation R. The whole set B is called the *codomain* of the relation R.

For example,
Consider a set $P = \{9, 4, 25\}$ and $Q = \{5, 3, 2, 1, -2, -3, -5\}$
and the following relation $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$
The above relation R which is a subset of $P \times Q$ can also be written as :
$R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$
As per the definition,
The domain of the above relation is $\{4, 9, 25\}$
Whereas, the range of this relation is $\{-2, 2, -3, 3, -5, 5\}$
And of course, the set Q is the co-domain of this relation.
Remark : The above relation can also be visually represented using an arrow diagram representation. This can be seen in the reference text mentioned at the end of this chapter.

2.1 Note

The total number of relations that can be defined from a set A to a set B is obviously the number of possible subsets of $A \times B$.
So, if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$. Then the total number of relations is 2^{pq}.

How come the number of relations or subsets is equal to 2^{pq} ?
Think of it in terms of binary digits. With one bit, we have two possible states - 0 & 1. With two bits, we have 4 possible states - $(00, 01, 10, 11)$ and similarly 8 states with 3 bits, 16 states with 4 bits and so on.
The number of possible subsets for a set can be calculated in a similar fashion. For example, taking a set $A = \{2, 3\}$, the subsets can be computed as follows :
$\Rightarrow 00$ = A subset with neither first nor second element = $\{\emptyset\}$
$\Rightarrow 01$ = A subset with only the second element = $\{3\}$
$\Rightarrow 10$ = A subset with only the first element = $\{2\}$
$\Rightarrow 11$ = A subset with both the elements = $\{2, 3\}$
The number of subsets for a set or in this context, the number of relations can be calculated in the way shown above.
Thus, matching with the binary bits analogy above, when the set has only 1 element, we will have two possible subsets, for $n(A) = 2$, we will have 4 possible subsets, for $n(A) = 3$ we have 8 possible subsets and thereby : In general for any set A, with $n(A) = m$, we will have in total 2^m subsets or relations.

3 Functions

A relation from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. The function f from set A to set B is defined as by $f : A \rightarrow B$ where $f(a) = b$ such that $a \in A$ and $b \in B$. Here, b is called the image of a under f.

In other words, a function f is a special type of relation for which, the domain is set A (every element in set A) and no two distinct ordered pairs $(a, b) \in f$ have the same first element (one and only one image in set B). The term **map** or **mapping** is sometimes used to denote a function.

3.1 Example

N is the set of natural numbers. A relation R is defined on **N** such that $R = \{(x, y) : y = 2x \text{ where } x, y \in \mathbf{N}\}$
Question : What is the domain, codomain and range of R ? Is this relation a function ?
Solution : The Domain of R is the set of natural numbers. The codomain is naturally also **N**. The range is the set of even natural numbers.
Since every natural number has one and only one image as per the defined relation, therefore this relation is a function.
Note : A $function$ whose $range$ is real valued is called a $real\ valued\ function$. Further, if its $domain$ is also real valued, it is called a $real\ function$.

4 Some Common Functions

4.1 Identity Function

Let $\mathbf{R}$ be the set of real numbers. The identity function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = x$ for each $x \in \mathbf{R}$. The *domain* and *range* of f are $\mathbf{R}$.

4.2 Constant Function

The constant function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = c$ and c is a constant. Here the *domain* of f is $\mathbf{R}$ and its *range* is $\{c\}$.

4.3 Polynomial Function

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be a polynomial function defined as $y = f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where n is a non-negative integer and $a_0, a_1, a_2...a_n \in \mathbf{R}$.

4.4 Rational Function

These are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.
For example, consider the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by :

$$f(x) = \frac{1}{x}$$

4.5 Modulus Function

The function $f : \mathbf{R} \to \mathbf{R}$ denoted by $f(x) = |x|$ is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers $\mathbf{R}$, whereas the range is the set of all non-negative real numbers.

5 Algebra of Real Functions

1. **Addition** : Let f and g be two functions, then we define $(f + g)(x) = f(x) + g(x)$
2. **Subtraction** : Let f and g be two functions, then we define $(f - g)(x) = f(x) - g(x)$
3. **Multiplication by Scalar** : Let f be a function and α be a real number. Then we define $(\alpha f)(x) = \alpha f(x)$
4. **Multiplication** : Let f and g be two functions, then we define $(f.g)(x) = f(x).g(x)$
5. **Quotient** : Let f and g be two functions, then quotient f by g is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

6 Solved Exercises

6.1

Let $f(x) = x^2$ and $g(x) = 2x + 1$
Find : $(f + g)(x)$, $(f - g)(x)$, $(f.g)(x)$ and $\frac{f}{g}(x)$
Solution :
$(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f - g)(x) = f(x) - g(x) = x^2 - 2x - 1$
$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq -1/2$

6.2

Let R be a relation on the set of rational numbers $\mathbf{Q}$ defined by :

$$R = \{(a, b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1. $(a, a) \in R$ for all $a \in \mathbf{Q}$
2. $(a, b) \in R$ implies that $(b, a) \in R$
3. $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

Solution :

1. For any rational number a, $a - a = 0$, which is obviously an integer. Therefore, it follows that $(a, a) \in R$ for all $a \in \mathbf{Q}$
2. $(a, b) \in R$ implies that $a - b \in \mathbf{Z}$. So, If $a - b$ is an integer then it follows naturally that $b - a$ which is simply $-(a - b)$ is an integer as well. Therefore $(b, a) \in R$
3. $(a, b) \in R$ and $(b, c) \in R$ implies that $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. The sum of two integers will be an integer as well. Adding the two $(a-b)+(b-c) = a-c \in \mathbf{Z}$. Therefore, $(a, c) \in R$

6.3

Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a linear function on the set of integers $\mathbf{Z}$, then find $f(x)$.
Solution : Since f is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where m and c are constants. Also since $(1, 1) \in f$ and $(0, -1) \in f$, we have

$$f(1) = 1 = m + c$$

$$f(0) = -1 = c$$

Solving the two equations, we get $m = 2$, $c = -1$ and thus :

$$f(x) = 2x - 1$$

7 References

1. Class 11 - Chapter 2 : Relations and Functions.
NCERT Mathematics Textbook, Version 2020-21.
As per Indian National Curriculum Framework 2005.