Relations and Functions
Part 1

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Much of the mathematics is about finding a pattern, a recognizable link between quantities that change. In this chapter, we will see how to link two sets and thus introduce **Relations** between them. We also learn about special relations which will qualify to be **Functions**. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correlation between one quantity and another.

1 Cartesian Product of Sets

> **Definition : Given two non-empty sets A and B. The cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B. Denoted as $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. That is, the set of all ordered pairs (a, b) and in that particular order, such that a belongs to set A and b belongs to set B.**

For example,
Suppose A is a set of colors and B is a set of objects :

$$A = \{red, blue\} \text{ and } B = \{bag, coat\}$$

We see that four distinct ordered pairs can be obtained from the above sets :

$$A \times B = \{(red, bag), (red, coat), (blue, bag), (blue, coat)\}$$

1.1 Remarks

- The order of pairing of elements is crucial. (a, b) is not the same as (b, a). That is why it is called an **ordered** pair.
- Two ordered pairs are equal only if their corresponding first and second elements are equal.
- If there are p elements in set A and q elements in set B, then there will be $p.q$ elements in the set $A \times B$. That is, $n(A \times B) = p.q$
- If either A or B is an empty set, then $A \times B$ will also be empty set.
- If A and B are non-empty sets and either A or B is an infinite set, then so is $A \times B$.
- $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here (a, b, c) is an ordered triplet.

1.2 Example

If $P = \{0, 1\}$, form the set $P \times P \times P$.
$\Rightarrow$ Going step by first we can first calculate $P \times P$
$\Rightarrow P \times P = \{(0,0), (0,1), (1,0), (1,1)\}$
$\Rightarrow$ And Therefore :

$$P \times P \times P = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

1.3 Example

$\mathbf{R}$ is the set of real numbers, what do the cartesian products $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ represent ?
$\Rightarrow$ The set $\mathbf{R}$ represents the points on a line.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$ represents the coordinates of all the points in two-dimensional space.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ represents the coordinates of all the points in three-dimensional space.

1.4 Example

If $A \times B = \{(p, q), (p, r), (m, q), (m, r)\}$, find A and B.
A = set of first elements = $\{p, m\}$
B = set of second elements = $\{q, r\}$

2 Relations

Definition : A relation R from a non-empty set A to a non-empty set B is a subset of the cartesian product $A \times B$. This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the image of of the first element.
The set of all first elements of the ordered pairs in relation R is called the $domain$ of the relation and the set of all second elements is called the $range$ of the relation R. The whole set B is called the codomain of the relation R.

For example,
Consider a set $P = \{9, 4, 25\}$ and $Q = \{5, 3, 2, 1, -2, -3, -5\}$
and the following relation $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$
The above relation R which is a subset of $P \times Q$ can also be written as :
$R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$
As per the definition,
The domain of the above relation is $\{4, 9, 25\}$
Whereas, the range of this relation is $\{-2, 2, -3, 3, -5, 5\}$
And of course, the set Q is the co-domain of this relation.
Remark : The above relation can also be visually represented using an arrow diagram representation. This can be seen in the reference text mentioned at the end of this chapter.

2.1 Note

> **The total number of relations that can be defined from a set A to a set B is obviously the number of possible subsets of $A \times B$.**
> **So, if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$. Then the total number of relations is 2^{pq}.**

How come the number of relations or subsets is equal to 2^{pq} ?
Think of it in terms of binary digits. With one bit, we have two possible states - 0 & 1. With two bits, we have 4 possible states - $(00, 01, 10, 11)$ and similarly 8 states with 3 bits, 16 states with 4 bits and so on.
The number of possible subsets for a set can be calculated in a similar fashion. For example, taking a set $A = \{2, 3\}$, the subsets can be computed as follows :
$\Rightarrow 00$ = A subset with neither first nor second element = $\{\emptyset\}$
$\Rightarrow 01$ = A subset with only the second element = $\{3\}$
$\Rightarrow 10$ = A subset with only the first element = $\{2\}$
$\Rightarrow 11$ = A subset with both the elements = $\{2, 3\}$
The number of subsets for a set or in this context, the number of relations can be calculated in the way shown above.
Thus, matching with the binary bits analogy above, when the set has only 1 element, we will have two possible subsets, for $n(A) = 2$, we will have 4 possible subsets, for $n(A) = 3$ we have 8 possible subsets and thereby : In general for any set A, with $n(A) = m$, we will have in total 2^m subsets or relations.

3 Functions

> **A relation from a set A to a set B is said to be a function if every element of set A has one and only one image in set B. The function f from set A to set B is defined as by $f : A \to B$ where $f(a) = b$ such that $a \in A$ and $b \in B$. Here, b is called the image of a under f.**

In other words, a function f is a special type of relation for which, the domain is set A (every element in set A) and no two distinct ordered pairs $(a, b) \in f$ have the same first element (one and only one image in set B). The term **map** or **mapping** is sometimes used to denote a function.

3.1 Example

N is the set of natural numbers. A relation R is defined on **N** such that $R = \{(x, y) : y = 2x \text{ where } x, y \in \mathbf{N}\}$
Question : What is the domain, codomain and range of R ? Is this relation a function ?
Solution : The Domain of R is the set of natural numbers. The codomain is naturally also **N**. The range is the set of even natural numbers.
Since every natural number has one and only one image as per the defined relation, therefore this relation is a function.
Note : A $function$ whose $range$ is real valued is called a $real\ valued\ function$. Further, if its $domain$ is also real valued, it is called a $real\ function$.

4 Some Common Functions

4.1 Identity Function

Let $\mathbf{R}$ be the set of real numbers. The identity function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = x$ for each $x \in \mathbf{R}$. The $domain$ and $range$ of f are $\mathbf{R}$.

4.2 Constant Function

The constant function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = c$ and c is a constant. Here the $domain$ of f is $\mathbf{R}$ and its $range$ is $\{c\}$.

4.3 Polynomial Function

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be a polynomial function defined as $y = f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2 ... a_n \in \mathbf{R}$.

4.4 Rational Function

These are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.
For example, consider the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by :

$$f(x) = \frac{1}{x}$$

4.5 Modulus Function

The function $f : \mathbf{R} \to \mathbf{R}$ denoted by $f(x) = |x|$ is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers $\mathbf{R}$, whereas the range is the set of all non-negative real numbers.

5 Algebra of Real Functions

1. **Addition** : Let f and g be two functions, then we define $(f + g)(x) = f(x) + g(x)$
2. **Subtraction** : Let f and g be two functions, then we define $(f - g)(x) = f(x) - g(x)$
3. **Multiplication by Scalar** : Let f be a function and α be a real number. Then we define $(\alpha f)(x) = \alpha f(x)$
4. **Multiplication** : Let f and g be two functions, then we define $(f.g)(x) = f(x).g(x)$
5. **Quotient** : Let f and g be two functions, then quotient f by g is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

6 Solved Exercises

6.1

Let $f(x) = x^2$ and $g(x) = 2x + 1$
Find : $(f + g)(x)$, $(f - g)(x)$, $(f.g)(x)$ and $\frac{f}{g}(x)$
Solution :
$(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f - g)(x) = f(x) - g(x) = x^2 - 2x - 1$
$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}$, $x \neq -1/2$

6.2

Let R be a relation on the set of rational numbers $\mathbf{Q}$ defined by :

$$R = \{(a, b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1. $(a, a) \in R$ for all $a \in \mathbf{Q}$
2. $(a, b) \in R$ implies that $(b, a) \in R$
3. $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

Solution :

1. For any rational number a, $a - a = 0$, which is obviously an integer. Therefore, it follows that $(a, a) \in R$ for all $a \in \mathbf{Q}$
2. $(a, b) \in R$ implies that $a - b \in \mathbf{Z}$. So, If $a - b$ is an integer then it follows naturally that $b - a$ which is simply $-(a - b)$ is an integer as well. Therefore $(b, a) \in R$
3. $(a, b) \in R$ and $(b, c) \in R$ implies that $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. The sum of two integers will be an integer as well. Adding the two $(a - b) + (b - c) = a - c \in \mathbf{Z}$. Therefore, $(a, c) \in R$

6.3

Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a linear function on the set of integers $\mathbf{Z}$, then find $f(x)$.
Solution : Since f is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where m and c are constants. Also since $(1, 1) \in f$ and $(0, -1) \in f$, we have

$$f(1) = 1 = m + c$$

$$f(0) = -1 = c$$

Solving the two equations, we get $m = 2$, $c = -1$ and thus :

$$f(x) = 2x - 1$$

7 References

1. Class 11 - Chapter 2 : Relations and Functions.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.