Relations & Functions - 2

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1 Types of Relations

We know that a relation in a set A is a subset of A $\times$ A. It could be $\emptyset$ or A $\times$ A or some subset in between. This leads us to the following definitions :

1.1 Empty Relation

A relation R in a set A is called an empty relation, if no element of A is related to any element of A. That is R = $\emptyset$.

1.2 Universal Relation

A relation R in a set A is called universal relation, if each element of A is related to every element of A. That is R = A $\times$ A.

> Both the empty relation and the universal relation are sometimes called **trivial relations**.

1.3 Reflexive Relation

A relation R in a set A is called reflexive if $(a, a) \in$ R for every $a \in$ A.

1.4 Symmetric Relation

A relation R in a set A is called symmetric if $(a, b) \in$ R implies that $(b, a) \in$ R for all $a, b \in$ A.

1.5 Transitive Relation

A relation R in a set A is called transitive if $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R for all $a, b, c \in$ A.

1.6 Equivalence Relation

A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

1.7 Example

Let R be the relation defined in the set A = {1,2,3,4,5,6,7} by R = $\{(a, b)$: both a and b are either odd or even$\}$. Show that R is an equivalence relation.
Solution :
$\Rightarrow$ Given any element a in A, it will either be odd or even. So $(a, a) \in$ R for all $a \in$ A. Thus R is reflexive.
$\Rightarrow$ Further, $(a, b) \in$ R implies both a and b must be either odd or even, thus $(b, a) \in$ R as well for any $a, b \in$ A. Thus R is symmetric.
$\Rightarrow$ Finally, $(a, b) \in$ R and $(b, c) \in$ R implies all elements a, b, c must be either even or odd simultaneously, hence implying that $(a, c) \in$ R for any $a, b, c \in$ A. Thus R is transitive.
Hence, we have shown that R is an equivalence relation.

2 Types of Functions

2.1 One-One

A function f : X $\rightarrow$ Y is said to be one-one (or **injective**) if distinct elements of X under f have distinct images in Y. That is, for every x_1, $x_2 \in$ X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
Otherwise, if multiple elements in X have the same image in Y, then f is called **many-one**.

2.2 Onto

A function f : X $\rightarrow$ Y is said to be onto (or **surjective**) if every element of Y is the image of some element of X under f. That is, for every $y \in$ Y, there exists an element x in X such that $f(x) = y$.
This implies that f : X $\rightarrow$ Y is onto if and only if range of f = Y = codomain.

2.3 One-One & Onto

A function f : X $\rightarrow$ Y is said to be **bijective** if it is both one-one and onto.

2.4 Example

Show that the function f : N $\rightarrow$ N, given by $f(x) = 2x$ is one-one but not onto.
Solution : The function is one-one as all distinct natural numbers x under f will have a distinct image $y = f(x) = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is not onto as $y = 1$ is not an image of any element x in N under f. Meaning, for $y = 1$, there does not exist any natural number x, satisfying $y = f(x) = 2x = 1$.

2.5 Example

Prove that the function f : R $\rightarrow$ R given by $f(x) = 2x$ is one-one and onto.
Solution : The function is one-one as all distinct real numbers x under f will have

a distinct image $y = f(x) = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is onto as for any real number y, there exists x in R under f satisfying $y = f(x) = 2x$.

2.6 Example

Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.
Solution : If f is not one-one, there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same.
Now, the image of 3 under f can be only one element. Therefore the range set can have at most two elements of the co-domain, showing that f is not onto, a contradiction. Hence f has to be one-one.

2.7 Example

Show that a one-one function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.
Solution : Since f is one-one, the three elements of the domain must have three distinct images in the co-domain under f. And since the co-domain has only three elements which are all covered, hence f has to be onto.

> **Remark** : The results shown in the above two examples 2.6 and 2.7 are also true for an arbitrary finite set X. That is, a one-one function f : X $\to$ X is necessarily onto and an onto function f : X $\to$ X is necessarily one-one, for every finite set X.
> In contrast to this example 2.4 shows that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

3 Composition of Functions

4 Invertible Function