Relations & Functions - 2

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Having defined the concepts of relations and functions in part 1, In this chapter we will study the different types of relations and functions, composition of functions, invertible functions and binary operations.

1 Types of Relations

We know that a relation on a set A is some subset of the cartesian product A $\times$ A. It could be either $\emptyset$ or A $\times$ A or some subset in between. This leads us to the following definitions :

1.1 Empty Relation

Relation R on a set A is called an empty relation, if no element of A is related to any element of A under R. That is R = $\emptyset$.

1.2 Universal Relation

Relation R on a set A is called universal relation, if each element of A is related to every element of A under R. That is R = A $\times$ A.
Both the empty relation and the universal relation are sometimes called **trivial relations**.

1.3 Reflexive Relation

Relation R on a set A is called reflexive if $(a, a) \in$ R for every $a \in$ A.

1.4 Symmetric Relation

Relation R on a set A is called symmetric if $(a, b) \in$ R implies that $(b, a) \in$ R for all $a, b \in$ A.

1.5 Transitive Relation

Relation R on a set A is called transitive if $(a, b) \in$ R and $(b, c) \in$ R implies that $(a, c) \in$ R for all $a, b, c \in$ A.

1.6 Equivalence Relation

Relation R on a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

1.7 Example

Let R be the relation defined on the set A = {1,2,3,4,5,6,7} by R = $\{(a, b)$: both a and b are either odd or even$\}$. Show that R is an equivalence relation.
Solution :
$\Rightarrow$ Given any element a in A, it will either be odd or even. So $(a, a) \in$ R for all $a \in$ A. Thus R is reflexive.
$\Rightarrow$ Further, $(a, b) \in$ R implies both a and b must be either odd or even, thus $(b, a) \in$ R as well for any $a, b \in$ A. Thus R is symmetric.
$\Rightarrow$ Finally, $(a, b) \in$ R and $(b, c) \in$ R implies all elements a, b, c must be either even or odd simultaneously, hence implying that $(a, c) \in$ R for any $a, b, c \in$ A. Thus R is transitive.
Hence, we have shown that R is an equivalence relation.

2 Types of Functions

2.1 One-One

We have learned that a relation R : X $\rightarrow$ Y qualifies to be a function f : X $\rightarrow$ Y when every element of X has only one image in Y.
Further, if every element in the range of f has only one pre-image in X, then the function qualifies to be a **one-one** function. Otherwise f is called **many-one**.
In other words, a function f : X $\rightarrow$ Y is said to be one-one (also injective) if distinct elements of X have distinct images in Y. That is, for every $x_1, x_2 \in$ X, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

2.2 Onto

A function f : X $\rightarrow$ Y is said to be onto (also surjective) if every element in Y has a pre-image in X. That is, for every $y \in$ Y, there exists an element x in X such that $f(x) = y$.
This also implies that f : X $\rightarrow$ Y is onto if and only if Range of f = Y, which means that the range is equal to the codomain.

2.3 One-One & Onto

For a function f : X $\rightarrow$ Y, If the range of f is equal to its codomain and further every element in Y has only one pre-image in X, then that function is both one-one and onto.
A function that is both one-one and onto is also called bijective.

2.4 Example

Show that the function f : N $\rightarrow$ N, given by $f(x) = 2x$ is one-one but not onto.
Solution : The function is one-one as all distinct natural numbers x under f will have a distinct image $y = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2$, implying $x_1 = x_2$.
The function is not onto as $y = 1$ has no pre-image x in N under f. Naturally, as there is no natural number that can satisfy $y = 2x = 1$.

2.5 Example

Prove that the function f : R $\rightarrow$ R given by $f(x) = 2x$ is one-one and onto.
Solution : The function is one-one as distinct real numbers x under f will have distinct images $y = 2x$. Further, $f(x_1) = f(x_2)$ implies $2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
The function is onto as for any real number y, there exists x in R under f satisfying $y = 2x$.

2.6 Example

Show that an onto function $f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ is always one-one.
Solution : If f is not one-one, there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same.
Now, the image of 3 under f can be only one element. Therefore the range set can have at most two elements of the co-domain, showing that f is not onto, a contradiction. Hence f has to be one-one.

2.7 Example

Show that a one-one function $f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ must be onto.
Solution : Since f is one-one, the three elements of the domain must have three distinct images in the co-domain under f. And since the co-domain has only three elements which are all covered, hence f has to be onto.

> **Remark** : The results shown in the above two examples 2.6 and 2.7 are also true for an arbitrary finite set X. That is, a one-one function f : X $\rightarrow$ X is necessarily onto and an onto function f : X $\rightarrow$ X is necessarily one-one, for every finite set X.
> In contrast to this example 2.4 shows that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

3 Composition of Functions

4 Invertible Function