

# Matrices

Aviral Janveja

2022

The concept of **Matrix** evolved through an attempt to obtain simpler and more compact methods of solving systems of linear equations.

Matrices are used as a representation for the coefficients in systems of linear equations. They simplify our work to a great extent and are therefore useful in various branches of science and mathematics from physics, cryptography, genetics, economics and so on.

## 1 What is a Matrix ?

**Definition :** A matrix is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

**For example,**

The situation that “Aviral has 1 pen and 2 note books, Siddharth has 2 pens and 1 notebook and Shreyas has 2 pens and 2 notebooks” can be represented in matrix form as follows :

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}$$

We denote matrices with capital letters. In the above matrix  $A$ , the horizontal lines of elements are called **rows** and the vertical lines of elements are called **columns**.

### 1.1 Order of a Matrix

**Definition :** A matrix having  $m$  rows and  $n$  columns is said to have order  $m \times n$ , read as “m by n” matrix.

In the general a  $m \times n$  matrix has the following representation :

$$X = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ a_{41} & a_{42} & a_{43} & \dots & a_{4n} \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

So, here we see that  $X$  is a matrix of order  $m \times n$ . The number of elements in an  $m \times n$  matrix will obviously be equal to  $m$  multiplied by  $n$ .

The individual elements of the matrix are represented by  $a$ , where  $a_{ij}$  refers to the element lying in the  $i^{th}$  row and  $j^{th}$  column.

**For example,**

$a_{43}$  is an element of  $X$  lying in the  $4^{th}$  row and  $3^{rd}$  column.

## 2 Types of Matrices

### 2.1 Column Matrix

**Definition :** A matrix is said to be a column matrix if it has only one column.

**For example,**

The following is a column matrix of order  $3 \times 1$  :

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

### 2.2 Row Matrix

**Definition :** A matrix is said to be a row matrix if it has only one row.

**For example,**

The following is said to be a row matrix of order  $1 \times 3$  :

$$A = [1 \quad 2 \quad 3]$$

### 2.3 Square Matrix

**Definition :** A matrix in which the number of rows and columns are equal is called a square matrix.

Thus, in case of a square matrix  $m = n$ . Hence, it will be simply called a square matrix of order  $n$ .

**For example,**

The following is a square matrix of order 3 :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

### 2.4 Diagonal Matrix

**Definition :** A square matrix is called a diagonal matrix if all its non diagonal elements are zero. meaning  $a_{ij} = 0$  when  $i \neq j$ .

**For example,**

The following is a diagonal matrix of order 3 :

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

## 2.5 Scalar Matrix

**Definition :** A diagonal matrix is called a scalar matrix if all its diagonal elements are equal.

For example,

The following is a scalar matrix of order 3 :

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

## 2.6 Identity Matrix

**Definition :** A scalar matrix is called an Identity matrix if all its diagonal elements are equal to 1. We denote the identity matrix by  $I$ , where the order of matrix is clear from the context :

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 2.7 Zero Matrix

**Definition :** A matrix is called a zero matrix or null matrix if all its elements are zero. We denote the zero matrix by  $O$ , where the order of matrix is clear from the context :

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 3 Equality of Matrices

**Definition :** Two matrices  $A$  and  $B$  are said to be equal if they are of the same order and each element of  $A$  is equal to the corresponding element of  $B$ . That is  $a_{ij} = b_{ij}$  for all  $i$  and  $j$ .

For example,

The following are an example of equal matrices :  $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ .

## 4 Operations on Matrices

### 4.1 Addition of Matrices

**Definition :** The sum of two matrices  $A$  and  $B$  is a matrix  $C$ , which is obtained by adding corresponding elements of  $A$  and  $B$ . That is,  $c_{ij} = a_{ij} + b_{ij}$  for all  $i$  and  $j$ . Furthermore, the two matrices have to be of the same order.

For example,  
Given,

$$A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & 1/2 \end{bmatrix}$$

Then,

$$C = A + B = \begin{bmatrix} \sqrt{3} + 2 & 1 + \sqrt{5} & 0 \\ 0 & 6 & 1/2 \end{bmatrix}$$

It is important to note that, if  $A$  and  $B$  are not of the same order, then  $A + B$  is not defined.

The addition of matrices satisfies the following **properties** :

1. **Commutative** :  $A + B = B + A$
2. **Associative** :  $(A + B) + C = A + (B + C)$
3. **Additive Identity** :  $A + O = O + A = A$ . In other words, Null Matrix  $O$  is the additive identity of matrix addition.
4. **Additive Inverse** :  $A + (-A) = (-A) + A = O$ . So,  $-A$  is the additive inverse of  $A$ .

## 4.2 Scalar Multiplication of Matrices

**Definition** : Given a matrix  $A$  and a scalar  $k$  then  $kA$  is another matrix which is obtained by multiplying each element of  $A$  by the scalar  $k$ . That is  $a_{ij}$  becomes  $ka_{ij}$  for all  $i$  and  $j$ .

For example,  
If,

$$A = \begin{bmatrix} 3 & 1 & 1.5 \\ 5 & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$$

Then,

$$2A = \begin{bmatrix} 6 & 2 & 3 \\ 10 & 14 & -6 \\ 4 & 0 & 10 \end{bmatrix}$$

**Negative** of a matrix is denoted by  $-A$ . We define

$$-A = (-1).A$$

The scalar multiplication of matrix, satisfies the following **properties** :

1.  $k(A + B) = kA + kB$
2.  $(k + l)A = kA + lA$

Where,  $A$  and  $B$  are matrices of the same order whereas  $k$  and  $l$  are scalars.

### 4.3 Difference of Matrices

**Definition :** The difference of two matrices  $A$  and  $B$  is a matrix  $C$ , which is obtained by subtracting corresponding elements of  $A$  and  $B$ . That is,  $c_{ij} = a_{ij} - b_{ij}$  for all  $i$  and  $j$ . Furthermore, the two matrices have to be of the same order.

For example,

If,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

And

$$B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

Then,

$$2A - B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

### 4.4 Multiplication of Matrices

**Definition :** Firstly, The product of two matrices  $A$  and  $B$  is defined if the number of columns of  $A$  is equal to the number of rows of  $B$ . Then, the product of the matrices  $A$  and  $B$  is the matrix  $C$ , whose order is given by the number of rows of  $A$  and number of columns of  $B$ . Finally, to obtain the elements  $c_{ij}$  of  $C$ , we take the  $i^{th}$  row of  $A$  and  $j^{th}$  column of  $B$ , multiply them element-wise and take the sum these products.

For example,

If

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}_{2 \times 3}$$

And

$$B = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}_{3 \times 2}$$

$\Rightarrow$  Then, firstly product  $AB$  is defined as number of columns of  $A$  is same as the number of rows of  $B$ .

$\Rightarrow$  Secondly, the resultant matrix  $C$  has order  $2 \times 2$ , given by the number of rows of  $A$  and number of columns of  $B$ .

$\Rightarrow$  Finally, the resultant matrix can be written as :

$$AB = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}_{2 \times 2}$$

Where the elements  $c_{ij}$  of  $C$  can be computed as specified in the definition above.

$\Rightarrow$  For instance, to calculate  $c_{21}$ , we take the second row of  $A$  and the first column of  $B$ , multiply their corresponding elements and take the sum. The elements of  $C$  are computed accordingly and shown below for illustration :

$$c_{11} = (1)(2) + (-1)(-1) + (2)(5) = 13$$

$$c_{12} = (1)(7) + (-1)(1) + (2)(-4) = -2$$

$$c_{21} = (0)(2) + (3)(-1) + (4)(5) = 17$$

$$c_{22} = (0)(7) + (3)(1) + (4)(-4) = -13$$

Therefore,

$$AB = C = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}_{2 \times 2}$$

The multiplication of matrices satisfies the following **properties** :

1. **Non-commutative** : Even if products  $AB$  and  $BA$  are both defined, It is not necessary that  $AB$  equals  $BA$  that is  $AB \neq BA$ .  
**Remark** : Multiplication of diagonal matrices of the same order will be commutative.
2. **Associative** :  $(AB)C = A(BC)$ , whenever both sides of the equality are defined.
3. **Distributive** :  $A(B + C) = AB + AC$  and  $(A + B)C = AC + BC$ , whenever both sides of the equality are defined.
4. **Multiplicative Identity** : For every square matrix  $A$ , there exists an Identity matrix of the same order such that  $IA = AI = A$ .

## 5 Transpose of a Matrix

**Definition** : Let  $A$  be a  $m \times n$  matrix, then the matrix obtained by interchanging the rows and columns of  $A$  is called the transpose of  $A$ . That is  $a_{ij} \rightarrow a_{ji}$  for all  $i$  and  $j$ . This new  $n \times m$  matrix is denoted by  $A'$  or  $A^T$ .

For example,

If

$$A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & -1 \end{bmatrix}_{3 \times 2}$$

Then

$$A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & -1 \end{bmatrix}_{2 \times 3}$$

The transpose of matrices, satisfies the following **properties** :

1.  $(A')' = A$
2.  $(kA)' = kA'$  (where  $k$  is any constant)
3.  $(A + B)' = A' + B'$
4.  $(AB)' = B'A'$

## 6 Symmetric and Skew Symmetric Matrices

**Definition :** A square matrix  $A$  is said to be a symmetric if  $A' = A$ , that is  $a_{ij} = a_{ji}$  for all  $i, j$ . Further, a square matrix  $A$  is said to be skew-symmetric if  $A' = -A$ , that is  $a_{ij} = -a_{ji}$  for all  $i, j$ .

For example, the following is a symmetric matrix :

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & -3 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

Whereas, the following is a skew-symmetric matrix :

$$B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$$

Notice, as per the definition of a skew-symmetric matrix,  $a_{ij} = -a_{ji}$  for all  $i, j$ .

Therefore, when  $i = j$  :

$\Rightarrow$  We have  $a_{ii} = -a_{ii}$

$\Rightarrow 2a_{ii} = 0$

$\Rightarrow a_{ii} = 0$  for all  $i$

This means that the diagonal elements of a skew-symmetric matrix are always zero.

### 6.1 Theorem 1

**For any square matrix  $A$  with real number entries,  $A + A'$  is a symmetric matrix and  $A - A'$  is a skew symmetric matrix.**

**Proof :**

The first part of the above theorem is proven if we show  $(A + A')' = A + A'$

Taking  $(A + A')'$

$= A' + (A')'$

$= A' + A$

$= A + A'$

For the second part, we need to show that  $(A - A')' = -(A - A')$

Let us take  $(A - A')'$

$= A' - (A')'$

$= A' - A$

$= -(A - A')$

Hence Proved.

### 6.2 Theorem 2

**Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.**

**Proof :**

Let  $A$  be a square matrix, we can thus write -

$$2A = A + A$$

$$2A = A + A + O$$

$$2A = A + A + A' - A'$$

$$2A = (A + A') + (A - A')$$

$2A$  has been thus expressed as the sum of a symmetric and a skew-symmetric matrix and thus  $A$  as well, can be simply written as -

$$A = 1/2(A + A') + 1/2(A - A')$$

Hence Proved.

## 7 Elementary Transformations of a Matrix

There are three main elementary transformations of a matrix :

1. The interchange of any two rows or two columns. Denoted as  $R_i \leftrightarrow R_j$  or  $C_i \leftrightarrow C_j$
2. The multiplication of elements of any row or column by a non-zero number. Denoted by  $R_i \rightarrow kR_i$  or  $C_i \rightarrow kC_i$  where  $k \neq 0$ .
3. The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non-zero number. Denoted by  $R_i \rightarrow R_i + kR_j$  or  $C_i \rightarrow C_i + kC_j$  where  $k \neq 0$ .

## 8 Invertible Matrices

**Definition :** If  $A$  is a square matrix and if there exists another square matrix  $B$  of the same order, such that  $AB = BA = I$ , then  $B$  is called the inverse of  $A$  and is denoted by  $A^{-1}$ . In that case  $A$  is said to be invertible.

For example,

Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

We see that  $AB = BA = I$ . Thus  $B$  is the inverse of  $A$  and  $A$  is the inverse of  $B$ , that is  $A = B^{-1}$  and  $B = A^{-1}$ .

The following points are to be noted regarding inverse of a matrix :

- A rectangular matrix does not possess an inverse.
- Inverse of a square matrix, if it exists, is unique.
- If  $A$  and  $B$  are invertible matrices of the same order, then  $(AB)^{-1} = B^{-1}A^{-1}$ .

## 9 References

- Mathematics - Class 12, Chapter 3 - Matrices. NCERT Textbook India, version 2020-21. As per National Curriculum Framework 2005.