

# Relations and Functions

## Part 1

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Much of the mathematics is about finding a pattern, a recognizable link between quantities that change. In this chapter, we will see how to link two sets and thus introduce **Relations** between them. We also learn about special relations which will qualify to be **Functions**. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correlation between one quantity and another.

## 1 Cartesian Product of Sets

**Definition :** Given two non-empty sets  $A$  and  $B$ . The cartesian product  $A \times B$  is the set of all ordered pairs of elements from  $A$  and  $B$ . Denoted as  $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$ . That is, the set of all ordered pairs  $(a, b)$  and in that particular order, such that  $a$  belongs to set  $A$  and  $b$  belongs to set  $B$ .

For example,

Suppose  $A$  is a set of colors and  $B$  is a set of objects :

$$A = \{red, blue\} \text{ and } B = \{bag, coat\}$$

We see that four distinct ordered pairs can be obtained from the above sets :

$$A \times B = \{(red, bag), (red, coat), (blue, bag), (blue, coat)\}$$

### 1.1 Remarks

- The order of pairing of elements is crucial.  $(a, b)$  is not the same as  $(b, a)$ . That is why it is called an **ordered** pair.
- Two ordered pairs are equal only if their corresponding first and second elements are equal.
- If there are  $p$  elements in set  $A$  and  $q$  elements in set  $B$ , then there will be  $p \cdot q$  elements in the set  $A \times B$ . That is,  $n(A \times B) = p \cdot q$
- If either  $A$  or  $B$  is an empty set, then  $A \times B$  will also be empty set.
- If  $A$  and  $B$  are non-empty sets and either  $A$  or  $B$  is an infinite set, then so is  $A \times B$ .
- $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$ . Here  $(a, b, c)$  is an ordered triplet.

## 1.2 Example

If  $P = \{0, 1\}$ , form the set  $P \times P \times P$ .

$\Rightarrow$  Going step by first we can first calculate  $P \times P$

$\Rightarrow P \times P = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

$\Rightarrow$  And Therefore :

$$P \times P \times P = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$

## 1.3 Example

$\mathbf{R}$  is the set of real numbers, what do the cartesian products  $\mathbf{R} \times \mathbf{R}$  and  $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$  represent ?

$\Rightarrow$  The set  $\mathbf{R}$  represents the points on a line.

$\Rightarrow$  The set  $\mathbf{R} \times \mathbf{R} = \{(x, y) : x, y \in \mathbf{R}\}$  represents the coordinates of all the points in two-dimensional space.

$\Rightarrow$  The set  $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x, y, z) : x, y, z \in \mathbf{R}\}$  represents the coordinates of all the points in three-dimensional space.

## 1.4 Example

If  $A \times B = \{(p, q), (p, r), (m, q), (m, r)\}$ , find  $A$  and  $B$ .

$A =$  set of first elements  $= \{p, m\}$

$B =$  set of second elements  $= \{q, r\}$

## 2 Relations

**Definition :** A relation  $R$  from a non-empty set  $A$  to a non-empty set  $B$  is a subset of the cartesian product  $A \times B$ . This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in  $A \times B$ . The second element is called the image of of the first element.

The set of all first elements of the ordered pairs in relation  $R$  is called the *domain* of the relation and the set of all second elements is called the *range* of the relation  $R$ . The whole set  $B$  is called the codomain of the relation  $R$ .

**For example,**

Consider a set  $P = \{9, 4, 25\}$  and  $Q = \{5, 3, 2, 1, -2, -3, -5\}$

and the following relation  $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$

The above relation  $R$  which is a subset of  $P \times Q$  can also be written as :

$$R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$$

As per the definition,

The domain of the above relation is  $\{4, 9, 25\}$

Whereas, the range of this relation is  $\{-2, 2, -3, 3, -5, 5\}$

And of course, the set  $Q$  is the co-domain of this relation.

**Remark :** The above relation can also be visually represented using an arrow diagram representation. This can be seen in the reference text mentioned at the end of this chapter.

## 2.1 Note

**The total number of relations that can be defined from a set  $A$  to a set  $B$  is obviously the number of possible subsets of  $A \times B$ .**

**So, if  $n(A) = p$  and  $n(B) = q$ , then  $n(A \times B) = pq$ . Then the total number of relations is  $2^{pq}$ .**

**How come the number of relations or subsets is equal to  $2^{pq}$  ?**

Think of it in terms of binary digits. With one bit, we have two possible states - 0 & 1. With two bits, we have 4 possible states - (00, 01, 10, 11) and similarly 8 states with 3 bits, 16 states with 4 bits and so on.

The number of possible subsets for a set can be calculated in a similar fashion. For example, taking a set  $A = \{2, 3\}$ , the subsets can be computed as follows :

$\Rightarrow 00 =$  A subset with neither first nor second element  $= \{\emptyset\}$

$\Rightarrow 01 =$  A subset with only the second element  $= \{3\}$

$\Rightarrow 10 =$  A subset with only the first element  $= \{2\}$

$\Rightarrow 11 =$  A subset with both the elements  $= \{2, 3\}$

The number of subsets for a set or in this context, the number of relations can be calculated in the way shown above.

Thus, matching with the binary bits analogy above, when the set has only 1 element, we will have two possible subsets, for  $n(A) = 2$ , we will have 4 possible subsets, for  $n(A) = 3$  we have 8 possible subsets and thereby : In general for any set  $A$ , with  $n(A) = m$ , we will have in total  $2^m$  subsets or relations.

## 3 Functions

**A relation from a set  $A$  to a set  $B$  is said to be a function if every element of set  $A$  has one and only one image in set  $B$ . The function  $f$  from set  $A$  to set  $B$  is defined as by  $f : A \rightarrow B$  where  $f(a) = b$  such that  $a \in A$  and  $b \in B$ . Here,  $b$  is called the image of  $a$  under  $f$ .**

In other words, a function  $f$  is a special type of relation for which, the domain is set  $A$  (every element in set  $A$ ) and no two distinct ordered pairs  $(a, b) \in f$  have the same first element (one and only one image in set  $B$ ). The term **map** or **mapping** is sometimes used to denote a function.

### 3.1 Example

$\mathbb{N}$  is the set of natural numbers. A relation  $R$  is defined on  $\mathbb{N}$  such that  $R = \{(x, y) : y = 2x \text{ where } x, y \in \mathbb{N}\}$

**Question :** What is the domain, codomain and range of  $R$  ? Is this relation a function ?

**Solution :** The Domain of  $R$  is the set of natural numbers. The codomain is naturally also  $\mathbb{N}$ . The range is the set of even natural numbers.

Since every natural number has one and only one image as per the defined relation, therefore this relation is a function.

**Note :** A *function* whose *range* is real valued is called a *real valued function*. Further, if its *domain* is also real valued, it is called a *real function*.

## 4 Some Common Functions

### 4.1 Identity Function

Let  $\mathbf{R}$  be the set of real numbers. The identity function is defined as  $f : \mathbf{R} \rightarrow \mathbf{R}$ , where  $y = f(x) = x$  for each  $x \in \mathbf{R}$ . The *domain* and *range* of  $f$  are  $\mathbf{R}$ .

### 4.2 Constant Function

The constant function is defined as  $f : \mathbf{R} \rightarrow \mathbf{R}$ , where  $y = f(x) = c$  and  $c$  is a constant. Here the *domain* of  $f$  is  $\mathbf{R}$  and its *range* is  $\{c\}$ .

### 4.3 Polynomial Function

A function  $f : \mathbf{R} \rightarrow \mathbf{R}$  is said to be a polynomial function defined as  $y = f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ , where  $n$  is a non-negative integer and  $a_0, a_1, a_2, \dots, a_n \in \mathbf{R}$ .

### 4.4 Rational Function

These are functions of the type  $\frac{f(x)}{g(x)}$ , where  $f(x)$  and  $g(x)$  are polynomial functions of  $x$  defined in a domain, where  $g(x) \neq 0$ .

**For example**, consider the real valued function  $f : \mathbf{R} - \{0\} \rightarrow \mathbf{R}$  defined by :

$$f(x) = \frac{1}{x}$$

### 4.5 Modulus Function

The function  $f : \mathbf{R} \rightarrow \mathbf{R}$  denoted by  $f(x) = |x|$  is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers  $\mathbf{R}$ , whereas the range is the set of all non-negative real numbers.

## 5 Algebra of Real Functions

1. **Addition** : Let  $f$  and  $g$  be two functions, then we define  $(f + g)(x) = f(x) + g(x)$
2. **Subtraction** : Let  $f$  and  $g$  be two functions, then we define  $(f - g)(x) = f(x) - g(x)$
3. **Multiplication by Scalar** : Let  $f$  be a function and  $\alpha$  be a real number. Then we define  $(\alpha f)(x) = \alpha f(x)$
4. **Multiplication** : Let  $f$  and  $g$  be two functions, then we define  $(f.g)(x) = f(x).g(x)$
5. **Quotient** : Let  $f$  and  $g$  be two functions, then quotient  $f$  by  $g$  is defined by  $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ , provided  $g(x) \neq 0$ .

## 6 Examples

### 6.1 Example 1

Let  $f(x) = x^2$  and  $g(x) = 2x + 1$

Find :  $(f + g)(x)$ ,  $(f - g)(x)$ ,  $(f.g)(x)$  and  $\frac{f}{g}(x)$

**Solution :**

$$(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$$

$$(f - g)(x) = f(x) - g(x) = x^2 - 2x - 1$$

$$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$$

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq -1/2$$

### 6.2 Example 2

Let  $R$  be a relation on the set of rational numbers  $\mathbf{Q}$  defined by :

$$R = \{(a, b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1.  $(a, a) \in R$  for all  $a \in \mathbf{Q}$
2.  $(a, b) \in R$  implies that  $(b, a) \in R$
3.  $(a, b) \in R$  and  $(b, c) \in R$  implies that  $(a, c) \in R$

**Solution :**

1. For any rational number  $a$ ,  $a - a = 0$ , which is obviously an integer. Therefore, it follows that  $(a, a) \in R$  for all  $a \in \mathbf{Q}$
2.  $(a, b) \in R$  implies that  $a - b \in \mathbf{Z}$ . So, If  $a - b$  is an integer then it follows naturally that  $b - a$  which is simply  $-(a - b)$  is an integer as well. Therefore  $(b, a) \in R$
3.  $(a, b) \in R$  and  $(b, c) \in R$  implies that  $a - b \in \mathbf{Z}$  and  $b - c \in \mathbf{Z}$ . The sum of two integers will be an integer as well. Adding the two  $(a - b) + (b - c) = a - c \in \mathbf{Z}$ . Therefore,  $(a, c) \in R$

### 6.3 Example 3

Let  $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$  be a linear function on the set of integers  $\mathbf{Z}$ , then find  $f(x)$ .

**Solution :** Since  $f$  is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where  $m$  and  $c$  are constants. Also since  $(1, 1) \in f$  and  $(0, -1) \in f$ , we have

$$f(1) = 1 = m + c$$

$$f(0) = -1 = c$$

Solving the two equations, we get  $m = 2$ ,  $c = -1$  and thus :

$$f(x) = 2x - 1$$

## 7 References

- Mathematics - Class 11, Chapter 2 - Relations and Functions. NCERT Text-book India, version 2020-21. As per National Curriculum Framework 2005.