# Relations & Functions - 1

Aviral Janveja

Much of the mathematics is about finding a pattern, a recognizable link between quantities that change. In this chapter, we will see how to link two sets and thus introduce **Relations** between them. We also learn about special relations which will qualify to be **Functions**. The concept of function is very important in mathematics since it captures the idea of a mathematically precise correlation between one quantity and another.

## 1 Cartesian Product of Sets

> **Definition : Given two non-empty sets A and B. The cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B. Denoted as $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. The $ordered\, pairs$ taken from any two sets $A$ and $B$ are pairs of elements written in brackets, grouped together in that particular order. That is, the set of all $(a, b)$ such that $a$ belongs to set $A$ and $b$ belongs to set $B$.**

**For example,**
Suppose $A$ is a set of colors and $B$ is a set of objects :

$$A = \{red, blue\} \text{ and } B = \{bag, coat\}$$

Then the cartesian product $A \times B$ is obtained as follows :

$$A \times B = \{(red, bag), (red, coat), (blue, bag), (blue, coat)\}$$

### 1.1 Remarks

- $(a, b)$ is not the same as $(b, a)$. Two ordered pairs are equal if and only if their corresponding first and second elements are equal. That is why it is called an **ordered** pair. Naturally, this implies that $A \times B \neq B \times A$.
- If there are $p$ elements in set $A$ and $q$ elements in set $B$, then there will be $p.q$ elements in the set $A \times B$. That is, $n(A \times B) = p.q$
- If either $A$ or $B$ is an empty set, then $A \times B$ will also be empty set.
- If $A$ and $B$ are non-empty sets and either $A$ or $B$ is an infinite set, then so is $A \times B$.
- $A \times A \times A = \{(a, b, c) : a, b, c \in A\}$. Here $(a, b, c)$ is an ordered triplet.

### 1.2 Example

If $P = \{0,1\}$, form the set $P \times P \times P$.
$\Rightarrow$ Going step by first we can first calculate $P \times P$
$\Rightarrow P \times P = \{(0,0),(0,1),(1,0),(1,1)\}$
$\Rightarrow$ And Therefore :

$$P \times P \times P = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

### 1.3 Example

$\mathbf{R}$ is the set of real numbers, what do the cartesian products $\mathbf{R} \times \mathbf{R}$ and $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ represent ?
$\Rightarrow$ The set $\mathbf{R}$ represents the points on a line.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} = \{(x,y) : x, y \in \mathbf{R}\}$ represents the coordinates of all the points in two-dimensional space.
$\Rightarrow$ The set $\mathbf{R} \times \mathbf{R} \times \mathbf{R} = \{(x,y,z) : x,y,z \in \mathbf{R}\}$ represents the coordinates of all the points in three-dimensional space.

### 1.4 Example

If $A \times B = \{(p,q),(p,r),(m,q),(m,r)\}$, find $A$ and $B$.
$A$ = set of first elements = $\{p, m\}$
$B$ = set of second elements = $\{q, r\}$

## 2 Relations

**Definition : A relation $R$ from a non-empty set $A$ to a non-empty set $B$ is a subset of the cartesian product $A \times B$. This subset is obtained by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the $image$ of of the first element.**
**The set of all first elements of the ordered pairs in relation $R$ is called the $domain$ of the relation and the set of all second elements is called the $range$ of the relation $R$. The whole set $B$ is called the $codomain$ of the relation $R$.**

**For example,**
Consider a set $P = \{9,4,25\}$ and $Q = \{5,3,2,1,-2,-3,-5\}$
and the following relation $R = \{(x,y) : x = y^2, x \in P, y \in Q\}$
The above relation $R$ which is a subset of $P \times Q$ can also be written as :
$R = \{(9,3),(9,-3),(4,2),(4,-2),(25,5),(25,-5)\}$
As per the definition,
The domain of the above relation is $\{4,9,25\}$
Whereas, the range of this relation is $\{-2,2,-3,3,-5,5\}$
And of course, the set $Q$ is the co-domain of this relation.
**Remark** :

1. A relation $R$ from set $A$ to $A$ is also stated in short as a “relation on set $A$” or a “relation in set $A$”.

2. A relation is essentially a certain set. Therefore just like any set, it may be represented by the roster or the set builder method.

3. Visually, an arrow diagram can be used to represent a relation. As shown in the reference text 1.

## 2.1 Note

**The total number of relations that can be defined from a set $A$ to a set $B$ is obviously the number of possible subsets of $A \times B$.**
**So, if $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$. Then the total number of relations is $2^{pq}$.**

**How come the number of relations or subsets is equal to $2^{pq}$ and how to write down these subsets ?**
Think of it in terms of binary digits. With one bit, we have two possible states - 0 & 1. With two bits, we have 4 possible states - $(00, 01, 10, 11)$ and similarly 8 states with 3 bits, 16 states with 4 bits and so on.
The number of possible subsets for a set can be calculated in a similar fashion. For example, taking a set $A = \{2, 3\}$ with two elements, the subsets can be computed as follows :
$\Rightarrow 00$ = A subset with neither first nor second element = $\{\emptyset\}$
$\Rightarrow 01$ = A subset with only the second element = $\{3\}$
$\Rightarrow 10$ = A subset with only the first element = $\{2\}$
$\Rightarrow 11$ = A subset with both the elements = $\{2, 3\}$
In general for any set $A$, with $n(A) = m$, we will have in total $2^m$ subsets or relations. The set containing all the $2^m$ subsets of $A$ is obviously called the **power set** of $A$ as seen in the chapter on sets. Finally, these subsets can be written down using bit analogy way shown above.

# 3 Functions

**A relation from a set $A$ to a set $B$ is said to be a function if every element of set $A$ has one and only one image in set $B$. The function $f$ from set $A$ to set $B$ is denoted by $f : A \to B$ where $f(a) = b$ such that $a \in A$ and $b \in B$. Here, $b$ is called the image of $a$ under $f$.**

In other words, a function $f$ is a special type of relation for which, the domain is set $A$ (every element in set $A$) and no two distinct ordered pairs $(a, b) \in f$ have the same first element (one and only one image in set $B$). The term **map** or **mapping** is sometimes used to denote a function.

## 3.1 Example

**N** is the set of natural numbers. A relation $R$ is defined on **N** such that $R = \{(x, y) : y = 2x \text{ where } x, y \in \mathbf{N}\}$
**Question** : What is the domain, codomain and range of $R$ ? Is this relation a function ?
**Solution** : The Domain of $R$ is the set of natural numbers. The codomain is also **N**.

The range is the set of even natural numbers. Since every natural number has one and only one image as per the defined relation, therefore this relation is a function.
**Note** : A $function$ whose $range$ is real valued is called a $real\ valued\ function$. Further, if its $domain$ is also real valued, it is called a $real\ function$.

# 4 Some Common Functions

## 4.1 Identity Function

Let $\mathbf{R}$ be the set of real numbers. The identity function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = x$ for each $x \in \mathbf{R}$. The $domain$ and $range$ of $f$ are $\mathbf{R}$.

## 4.2 Constant Function

The constant function is defined as $f : \mathbf{R} \to \mathbf{R}$, where $y = f(x) = c$ and $c$ is a constant. Here the $domain$ of $f$ is $\mathbf{R}$ and its $range$ is $\{c\}$.

## 4.3 Polynomial Function

A function $f : \mathbf{R} \to \mathbf{R}$ is said to be a polynomial function defined as $y = f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $n$ is a non-negative integer and $a_0, a_1, a_2...a_n \in \mathbf{R}$.

## 4.4 Rational Function

These are functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of $x$ defined in a domain, where $g(x) \neq 0$.
**For example**, consider the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by :

$$f(x) = \frac{1}{x}$$

## 4.5 Modulus Function

The function $f : \mathbf{R} \to \mathbf{R}$ denoted by $f(x) = |x|$ is called the modulus function. It is defined as :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain of the modulus function is the set of real numbers $\mathbf{R}$, whereas the range is the set of all non-negative real numbers.

# 5 Algebra of Real Functions

1. **Addition** : Let $f$ and $g$ be two functions, then we define $(f + g)(x) = f(x) + g(x)$

2. **Subtraction** : Let $f$ and $g$ be two functions, then we define $(f - g)(x) = f(x) - g(x)$

3. **Multiplication by Scalar** : Let $f$ be a function and $\alpha$ be a real number. Then we define $(\alpha f)(x) = \alpha f(x)$

4. **Multiplication** : Let $f$ and $g$ be two functions, then we define $(f.g)(x) = f(x).g(x)$

5. **Quotient** : Let $f$ and $g$ be two functions, then quotient $f$ by $g$ is defined by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$.

# 6 Solved Exercises

## 6.1

Let $f(x) = x^2$ and $g(x) = 2x + 1$
Find : $(f + g)(x)$, $(f - g)(x)$, $(f.g)(x)$ and $\frac{f}{g}(x)$
**Solution** :
$(f + g)(x) = f(x) + g(x) = x^2 + 2x + 1$
$(f - g)(x) = f(x) - g(x) = x^2 - 2x - 1$
$(f.g)(x) = f(x).g(x) = 2x^3 + x^2$
$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2}{2x+1}, x \neq -1/2$

## 6.2

Let $R$ be a relation on the set of rational numbers $\mathbf{Q}$ defined by :

$$R = \{(a, b) : a - b \in \mathbf{Z} \text{ where } a, b \in \mathbf{Q}\}$$

Show that :

1. $(a, a) \in R$ for all $a \in \mathbf{Q}$

2. $(a, b) \in R$ implies that $(b, a) \in R$

3. $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

**Solution** :

1. For any rational number $a$, $a - a = 0$, which is obviously an integer. Therefore, it follows that $(a, a) \in R$ for all $a \in \mathbf{Q}$

2. $(a, b) \in R$ implies that $a - b \in \mathbf{Z}$. So, If $a - b$ is an integer then it follows naturally that $b - a$ which is simply $-(a - b)$ is an integer as well. Therefore $(b, a) \in R$

3. $(a, b) \in R$ and $(b, c) \in R$ implies that $a - b \in \mathbf{Z}$ and $b - c \in \mathbf{Z}$. The sum of two integers will be an integer as well. Adding the two $(a - b) + (b - c) = a - c \in \mathbf{Z}$. Therefore, $(a, c) \in R$

### 6.3

Let $f = \{(1,1),(2,3),(0,-1),(-1,-3)\}$ be a linear function on the set of integers **Z**, then find $f(x)$.
**Solution** : Since $f$ is a **linear function** which by definition are of the form :

$$f(x) = mx + c$$

Where $m$ and $c$ are constants. Also since $(1,1) \in f$ and $(0,-1) \in f$ , substituting these values $(x,y)$ in $y = f(x)$ we have

$$1 = f(1) = m(1) + c$$

$$-1 = f(0) = m(0) + c$$

Solving the two equations $(1 = m + c)$ and $(-1 = c)$ from above, we get $m = 2$, $c = -1$ and thus $f(x)$ is found :

$$f(x) = 2x - 1$$

# 7 References


1. Class 11 - Chapter 2 : Relations and Functions.
   NCERT Mathematics Textbook, Version 2020-21.
   As per Indian National Curriculum Framework 2005.