Problem 1:

Let the two continuous random variables be X and Y

1) When X and y are independent

$$\text{Covariance}(X,Y) = E(X-\bar{X}).(Y-\bar{Y})$$

$$= E\{XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}\}$$

$$= E(XY) - \bar{Y}E(X) - \bar{X}E(Y) + \bar{X}\bar{Y}$$

$$= \bar{X}\bar{Y} - \bar{Y}\bar{X} - \bar{X}\bar{Y} - \bar{X}\bar{Y}$$

$$= 0$$

$$\text{Covariance}(X,Y) = 0$$

$$r_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_{xx}\,\sigma_y} = 0$$

Hence they are uncorrelated, since correlation coefficient is zero

11) When X and Y are uncorrelated

$X \sim N(0,1)$ $\quad Y = X^2$ $\quad$ Y is dependent on X

$E(X) = 0$ $\quad E(X^3) = 0$ $\quad \mu_3 = 0$ for Normal Distribution.

$$\text{Covariance}(X,Y) = E(X-\bar{X}) \cdot (Y-\bar{Y})$$
$$= E(XY) - E(X) \cdot E(Y)$$
$$= E(X^3) - E(X) \cdot E(X^2)$$
$$= 0 - 0 \cdot E(Y)$$
$$= 0$$

$$\text{Covariance}(X,Y) = 0$$

$$\text{correlation}_{xy} = \frac{\text{Cov}(x,y)}{\sigma x \cdot \sigma y} = 0$$

Even though they are uncorrelated they are not independent

Problem 3 :

$N(4,1) \Longrightarrow N($Mean, Standard deviation$)$

$P(x|w_1) = N(4,1)$ $\quad$ $P(x|w_2) = N(8,1)$

$$P(w_1) = 1 - P(w_2)$$
$$= 1 - 1/4 = 3/4$$

$$\lambda = \begin{bmatrix} 0_{11} & 1_{12} \\ 3_{21} & 0_{22} \end{bmatrix}$$

$$\frac{P(x|w_1)}{P(x|w_2)} > \frac{(\lambda_{12} - \lambda_{11})}{(\lambda_{21} - \lambda_{22})} \cdot \frac{P(w_2)}{P(w_1)}$$

$$\frac{P(x|w_1)}{P(x|w_2)} > \frac{1-0}{3-0} \times \frac{3}{\cancel{4}\,1} \times \frac{\cancel{4}}{1}$$

$$\frac{P(x|w_1)}{P(x|w_2)} \Rightarrow 1$$

$$P(X|W_1) = \frac{1}{2\pi} e^{-\frac{(x-4)^2}{2}}$$

$$P(X|W_2) = \frac{1}{2\pi} e^{-\frac{(x-8)^2}{2}}$$

$$\therefore \frac{e^{-\frac{(x-4)^2}{2}}}{e^{-\frac{(x-8)^2}{2}}} > 1$$

$$e^{-\frac{(x-4)^2}{2}} > e^{-\frac{(x-8)^2}{2}}$$

taking log on both sides

$$-(x-4)^2 > -(x-8)^2$$

$$-(x^2 - 8x + 16) > -(x^2 - 16x + 64)$$

$$\cancel{-x^2} + 8x - 16 > \cancel{-x^2} + 16x - 64$$

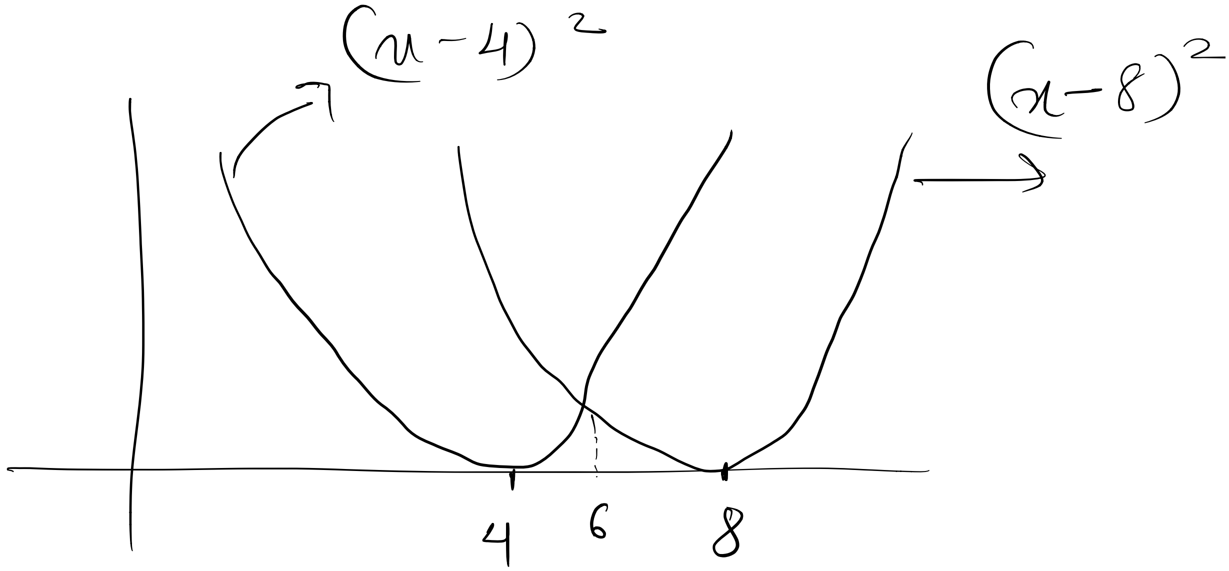
$64 - 16 > 8x$

$x < 6$

Therefore the decision Rule goes by :

Choose ω_1 if $x < 6$

else choose ω_2



Problem 5:

Given

$$p(x|\eta) = h(x) \exp\{n^T T(x) - A(\eta)\}$$

1) Expression of $A(n)$ in terms of $T(x)$ and $h(x)$

Taking Integral

$$\int p(x|\eta) = \int h(x)\, e^{n^T T(x) - A(n))} dx$$

$$1 = \int h(x) e^{n^T T(x)} - e^{-A(\eta)} dx$$

$$e^{-A(n)} = \int h(x)\, e^{n^T T(x)} dx$$

$$A(\eta) = \log\left(\int h(x) e^{n^T T(x)} dx\right)$$

2. To show that $\frac{\partial}{\partial \eta} A(\eta) = E_{\eta} T(x)$

$$\frac{\partial}{\partial \eta} A(\eta) = \frac{\partial}{\partial \eta}\left(\log \int h(x) e^{\eta^T T(x)} dx\right)$$

$$= \frac{1}{\int h(x) e^{\eta^T T(x)} dx} \times \int h(x) e^{\eta^T T(x)} T(x) dx$$

$$= \int \frac{h(x) e^{\eta^T T(x)} dx}{e^{A(\eta)}}$$

$$RHS = \int \underbrace{h(x) e^{\eta^T T(x) - A(\eta)}}_{P(x|\eta)} T(x) dx$$

$$\Rightarrow \int P(x|\eta) \, T(x) \, dx \quad (1)$$

$$E\left[f(x)\right] = \int p(x|\omega) f(x)\, dx$$

$$\frac{\partial}{\partial \eta} A(\eta) = E_\eta\left[T(x)\right]$$

3. $P(x_1, x_2 \ldots x_n | \eta) =$

$\prod P(x_i|\eta)$ $\quad X = x_1, x_2 \ldots x_n$

$$\prod_{i=1}^{n} P(x_i|\eta) = \prod_{i=1}^{n} u(x_i)\, e^{\eta^T \sum_{i=1}^{n} T(x_i) - nA(\eta)}$$

$\therefore$ log likelihood

$$l(\eta|X) = \log P(X|\eta)$$

$$= \log\left(\prod_{i=1}^{n} h(x_i)\right) + \eta^T \sum_{i=1}^{n} T(x_i) -$$

$$n\, A(\eta)$$

Global Maxima $\Longrightarrow \frac{\partial}{\partial \eta} l(\eta/X) = 0$

$$\frac{\partial}{\partial \eta} \log\left(\prod_{i=1}^{n} h(x_i)\right) + \frac{d\eta^T}{\partial \eta} \sum_{i=1}^{n} T(x_i)$$

$$- n\, A(\eta)$$

$$\sum_{i=1}^{n} T(x_i) - n \frac{\partial A(\eta)}{\partial \eta} = 0$$

$$n \frac{\partial A(\eta)}{\partial \eta} = \sum_{i=1}^{n} T(x_i)$$

$$E_\eta [T(x)] = \frac{1}{n} \sum_{i=1}^{n} T(x_i)$$

Problem 4:

Risk $\rightarrow$ $R(\alpha_i | x) = \sum_{j=1}^{c} \lambda(\alpha_i | w_j) P(w_j | x)$

for $i = 1 \ldots\ldots c$

$$R(\alpha_i | x) = \lambda_c \sum_{j=1}^{c} P(w_j | x)$$

$$= \lambda_c [1 - P(w_i | x)] \text{ when } j \neq i$$

for $i = c+1$

$$R(\alpha_{c+1} | x) = \lambda_r$$

1) Minimal Risk if $R(\alpha_i|x) \leq R(\alpha_{c+1}|x)$

$$\lambda_s [1 - P(\omega_i|x)] \leq \lambda_r$$

Choosing ω_i so that, $P(\omega_i|x) \geq 1 - \frac{\lambda_r}{\lambda_s}$

2) If $\lambda_r = 0$, we will have to reject the $(c+1)$ action because the loss incurred is too high

3) If $\lambda_r > \lambda_s$ there will be no rejection in terms of choosing the action.

Problem 2:

Given:

$$\sum_{i=1}^{c} P(w_i|x) = 1 \quad \text{—— (1)}$$

Given $P(w_{max}|x) \geq P(w_i|x)$

Using equation (1)

$$\sum_{i=1}^{c} P(w_{max}|x) \geq \sum_{i=1}^{c} P(w_i|x) = 1$$

$$\Rightarrow \; c\, P(w_{max}|x) \geq 1$$

$$P(w_{max}|x) \geq \frac{1}{c}$$

2) Probability of error is given by P(error)

$$P(error) = \int P(error, x)\, dx$$

a)

$$P(error) = \int P(error/x)\, P(x)\, dx$$

b)

$$P(error/x) = 1 - P(\omega_{max}/x)$$

from a and b

$$P(error) = \int [1 - P(\omega_{max}/x)]\, P(x)\, dx$$

$$= 1 - \int P(\omega_{max} | x) P(x) \, dx$$

3) $$P(\text{error}) = 1 - \int P(\omega_{max} | x) P(x) \, dx$$

$$P(\omega_{max} | x) \geq \frac{1}{C} \quad (1)$$

$$P(\text{error}) \leq 1 - \int \frac{1}{C} P(x) \, dx$$

$$P(\text{error}) \leq 1 - \frac{1}{C}$$

$$P(\text{error}) \leq \frac{C-1}{C}$$

Hence proved.

Problem 6:

$$L(w_i x_i, t_i) =$$

$$- \left[t_i \log \sigma(w^T x) + (1-t_i) \log (1 - \sigma(w^T x)) \right]$$

$$\frac{\partial L}{\partial w} = - \frac{\partial}{\partial w} \left[t_i \log \sigma(w^T x) + (1-t_i) \log (1 - \sigma(w^T x)) \right]$$

$$\frac{\partial \sigma(w^T x)}{\partial X} = \sigma(w^T x) \cdot [1 - \sigma(w^T x)] \cdot X$$

$$= \frac{\partial}{\partial w} \left[t \cdot \frac{1}{\cancel{\sigma(w^T x)}} \cdot \cancel{\sigma(w^T x)} \right.$$

$$1 - \sigma(w^T x) X$$

$$+ (1-t_i) \cdot \frac{1}{1 - \cancel{\sigma(w^T x)}} (-1) \neq$$

$$\neq \sigma(w^T x) \; (\cancel{1 - \sigma(w^T x)} \cdot$$

$$X \,]$$

$$= t \cdot (1 - \sigma(w^T x) + (1-t) - \sigma(w^T x) \cdot X$$

$$= w^T - \eta_T \left(\sigma w^{(T)T} x_j - t_i \right) X_j$$