Bonus Question

Problem 1:

Let x^t be a 2x1 vector containing x_1 and x_2 as the binary input at time t.
Let h^t be a 3x1 vector containing h_1, h_2 and h_3 at time t
Let $y^{(t)}$ be a scalar output binary digit.

$$U = \begin{Bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{Bmatrix} \quad W = \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} \quad b_h = \begin{Bmatrix} 0.5 \\ -1.5 \\ -2.5 \end{Bmatrix} \quad V = \begin{Bmatrix} 1 & -1 & 1 \end{Bmatrix}$$

$$b_y = -0.5$$

Then for all $t \geq 1$

$$h^{(t)} = Ux^t + Wh^{t-1} + b_h \quad \ldots\ldots \quad (1)$$
$$y^{(t)} = V^{h(t)} + b_y$$

Expanding equation (1) we get,

$$h_1^{(t)} = x_1^{(t)} + x_2^{(t)} + h_1^{(t-1)} - 0.5$$
$$h_2^{(t)} = x_1^{(t)} + x_2^{(t)} + h_2^{(t-1)} - 1.5$$
$$h_3^{(t)} = x_1^{(t)} + x_2^{(t)} + h_3^{(t-1)} - 0.5$$
$$y^{(t)} = h_1^{(t)} - h_2^{(t)} + h_3^{(t)} - 0.5.$$

This satisfies the truth table.

x_1	x_2	$h_1^{(t-1)}$	$h_1^{(t)}$	$h_2^{(t-1)}$	$h^{(t)}$	$h_3^{(t-1)}$	$h_3^{(t)}$	y
0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	1	0	1
0	1	0	1	0	0	0	0	1
0	1	1	1	1	1	1	0	0
1	0	0	1	0	0	0	0	1
1	0	1	1	1	1	1	0	0
1	1	0	1	0	1	0	0	0
1	1	1	1	1	1	1	1	1

Problem 2

1. $$\overline{h^{(t)}} = 1 + \overline{i^{(t+1)}} \frac{\partial i^{(t+1)}}{\partial h^{(t+1)}} + \overline{f^{(t+1)}} \frac{\partial f^{(t+1)}}{\partial h^{(t+1)}} + \overline{o^{t+1}} \frac{\partial o^{(t+1)}}{\partial h^{(t+1)}}$$
$$+ \overline{g^{(t+1)}} \frac{\partial g^{(t+1)}}{\partial h^{(t+1)}}$$

$$\overline{c^{(t)}} = \overline{h^{(t)}} \frac{\partial h^{(t)}}{\partial c^{(t)}} + \overline{c^{t+1}} \frac{\partial c^{(t+1)}}{\partial c^{(t)}}$$

$$= \overline{h^{(t)}}\, o^{(t)} \tanh^{-1}(c^{(t)}) + \overline{c^{t+1}} f^{(t)}$$

$$\overline{g^{(t)}} = \overline{c^{(t)}} \frac{\partial c^{(t)}}{\partial g^{(t)}}$$

$$= \overline{c^{(t)}}\, i^{(t)}$$

$$\overline{o^{(t)}} = \overline{h^{(t)}} \frac{\partial h^{(t)}}{\partial o^{(t)}}$$

$$= \overline{h^{(t)}} \tanh(c^{(t)})$$

$$\overline{f^{(t)}} = \overline{c^{(t)}} \frac{\partial c^{(t)}}{\partial f^{(t)}}$$

$$= \overline{c^{(t)}}\, c^{(t-1)}$$

$$= \overline{i^{(t)}} = \overline{c^{(t)}}\, g^{(t)}$$

2. $\overline{W_{ix}} = \sum_t \overline{i^{(t)}}\, \sigma^{-1}\left(W_{ix} x^{(t)} + W_{ih} h^{(t)}\right) x^{t}$

3. This is because when $f^{(t)} = 1$

$i^{(t)} = 0 \qquad o^{(t)} = 0$

$$\overline{c^{(t)}} = \overline{h^{(t)}}\, o^{(t)} \tanh^{-1}(c^{(t)}) + \overline{c^{(t+1)}} f^{(t)}$$
$$= \overline{c^{(t+1)}}$$

$$\overline{g^{(t)}} = \overline{c^{(t)}}\, i^{(t)}$$
$$= 0$$

$$\overline{o^{(t)}} = \overline{h^{(t)}} \tanh(c^{(t)})$$

$$\overline{f^{(t)}} = \overline{c^{(t)}}\, c^{(t-1)}$$

$$\overline{i^{(t)}} = \overline{c^{(t)}} \frac{\partial c^{(t)}}{\partial i^{(t)}}$$
$$= \overline{c^{(t)}}\, g^{(t)}$$

then $\overline{c^{(t)}} = \overline{c^{(t+1)}}$ stay the same.