Some Lemmas Used in the Proofs for Graphs

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Let $G = (V, E)$ be a simple graph, and $B = (L \cup R, E)$ be a bipartite graph.

A walk is any sequence of vertices connected by edges.

A trail is a walk with no repeating edges - when closed it's a circuit. A path is a trail with no repeating vertices - when closed its a cycle.

1 Misc

1.1 Number of incident edges must be less than or equal to degree sum

If $W \subset V$ and F is the set of edges incident on any $w \in W$, then $|F| \leq \Sigma_{w \in W} degree(w)$.

1.2 Either G or its complement is connected

If G is not connected, then $\overline{G}$ is connected.

Proof: Let $u, v \in V$. If they are in different connected components of G, then no edge connects them in G. Hence they are connected by a an edge (and therefore also a path) in $\overline{G}$.

If they are in the same connected component of G, then pick a vertex w from another connected component. There is then a path from u to w to v in $\overline{G}$.

2 Bipartite Graphs

2.1 Bipartite if no odd cycles

Assume G is connected (if not apply this argument to each connected component). Let $u \in V$ Define $A = \{u\} \cup \{v : d(u, v) \text{ is even}\}$, and $B = V \setminus A$ (i.e. vertices with odd distance from u). Now, all edges are between A and B, because if there was an edge with both endpoints in A or B, there would be an odd cycle.

2.2 Sum of degrees of partitions are equal

$$\Sigma_{l \in L} degree(l) = \Sigma_{r \in R} degree(r) \tag{1}$$

Proof: By induction on the number of edges. Each edge added adds one to the degree sum of both L and R.

2.3 k-Regular implies that the partitions are equal in size

By 2.2:

$$|L| = \frac{1}{k}\Sigma_{l \in L} degree(l) = \frac{1}{k}\Sigma_{r \in R} degree(r) = |R|$$

2.4 k-Regular implies existence of matching of size $|L|$

3 Matchings

3.1 Degree 1 implies matching

If for all $v \in V$, $degree(v) \leq 1$, then E is a matching.

Proof: no two edges have the same vertex as an endpoint, or that vertex would have degree at least 2.

3.2 Unmatched vertices in a maximal matching

If $v \in V$ is does not have an edge in maximal (including maximum) matching M incident on it (i.e. unmatched), then v can only be adjacent to matched vertices.

Proof: if v is adjacent to an unmatched vertex w, the matching can be extented by including the edge that connects v and w.

3.3 Maximum matching when a vertex is removed

Seems like there should be a result analogous to 3.1 but I can't find it.

3.4 Symmetric difference of matchings

Let M and N be matchings. Then, the graph $(V, (M \cup N) \setminus (M \cap N))$ is composed of disjoint paths and cycles. Moreover, every cycle is even and there is at least one path that starts and ends with edges from the larger matching if there is one.

Proof: if any vertex has degree greater than 2, there must be at least 2 incident edges from the same matching - contradiction.

4 Vertex Covers

4.1 Minimum vertex cover when a vertex is removed

Let $v \in V$. Then, $\tau(G - \{v\}) \geq \tau(G) - 1$.

Proof: if instead $\tau(G - \{v\}) < \tau(G) - 1$, we could find a smaller vertex cover for G by adding v to the vertex cover for $G - \{v\}$.

4.2 Disjoint neighbourhood

Let S be a vertex cover, and $X \subseteq V$ with $X \cap S = \emptyset$. Then, if $S \subseteq N(X)$, $S = N(X)$.

Proof: Let $v \in N(X)$. Assume $v \notin S$. Consider the edge $\{x, v\}$ that connects some $x \in X$ to v (this exists because $v \in N(X)$). Because X and S are disjoint, x is not in S. By assumption v is not in S either. So $\{x, v\}$ has neither endpoint in S - contradiction, so instead $v \in S$.

Therefore, we also have that $N(V) \subseteq S$, so the sets are equal.