A Bound on Probabilistic Ramsey Numbers

Avi Semler

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Abstract

This essay considers an generalisation of Ramsey numbers to consider not only the size of a graph that ensures with certainty the presence of a particular subgraph, but also the size of a graph that ensures the presence of a subgraph merely with probability p. We first define probabilistic Ramsey (p-Ramsey) numbers, and prove a lower bound. Then, then the values of some probabilistic Ramsey numbers are approximated using a Monte Carlo method.

1 Definitions

Let Ω_n be the set of all simple undirected graphs on n vertices, and consider the probability space $(\Omega_n, \mathcal{P}(\Omega_n), \mathbb{P})$ where $\mathbb{P}$ is the probability measure corresponding to the uniform distribution over Ω_n.

Let $Y_k : \Omega_n \to \mathbb{N}$ be a random variable counting the number of k-cliques in the graph, and $Z_l : \Omega_n \to \mathbb{N}$ be a random variable counting the number of l-independent sets.

Definition 1.1. *For $p \in (0, 1)$ and $k, l \in \mathbb{N}$, define the probabilistic Ramsey number $R_p(k, l)$ as the least value for n such that*

$$\mathbb{P}(\{Y_k > 0\} \cup \{Z_l > 0\}) \geq p \tag{1}$$

That is, the least number of vertices in a graph such that the probability of there being at least one k-clique or at least one l-independent set is at least p. (This definition could be generalised further to graph colourings in the usual way - here we are considering the case of 2-colouring with edges being either present or absent.)

Observe that by setting $p = 1$ we obtain the Ramsey number $R(k, l)$ In this sense, probabilistic Ramsey numbers are a generalisation of the standard variety.

2 Bounds

There is a trivial bound:

$$1 \leq R_p(k, l) \leq R(k, l) \quad \forall p \in (0, 1)$$

so by the well-ordering principle, $R_p(k, l)$ is well-defined.

Theorem 2.1 (Lower bound for diagonal numbers)**.** *If $p \in (0,1)$ and $k \geq 1$,*

$$R_p(k,k) \geq \left(k! \cdot p \cdot 2^{\binom{k}{2}-1}\right)^{\frac{1}{k}}$$

Proof. Consider a graph with n vertices, $G = (V, E)$ with $|V| = n$. The probability that a fixed subset of k vertices is either a k-clique or a k-independent set is

$$\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \tag{2}$$

as there are $\binom{K}{2}$ edges between the k vertices and they are either present or absent with (independent) probabilities of 0.5.

Let $S_1, \ldots, S_{\binom{n}{k}} \subset V$ be the sets of vertices of size k. Now using the union bound, we can put a bound on the probability that the entire graph contains a k-clique or k-independent set:

$$\begin{aligned} &\mathbb{P}(\text{There is a k-clique or a k-independent set}) && (3)\\ =&\mathbb{P}\Big(\bigcup_{S \in \{S_1, \dots\}} \{S \text{ is a k-clique or k-independent set}\}\Big) && (4)\\ \geq&\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} && (5) \end{aligned}$$

Since when $n = R_p(k,k)$ this probability is at least p, we have

$$\binom{R_p(k,k)}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \geq p \tag{6}$$

Then the result follows by rearranging and using $\frac{n^k}{k!} \geq \binom{n}{k}$.

□

3 Constant parameters

By holding some parameters (p, and one of k or l) constant, we obtain a sequence.

Theorem 3.1 (Convergence of $a_j := R_p(k,j)$)**.** *For $p \neq 1$ and fixed k, there exists $M \in \mathbb{N}$ such that*

$$\lim_{j \to \infty} R_p(k, j) = M$$

for all $k \in N$.

Proof. By the Erdős-Frankl-Rödl bound, the probability of G_n having a k-clique is lower-bounded by

$$1 - \frac{2^{\left(1 - \frac{1}{k-1} + o(1)\right)\frac{n^2}{2}}}{2^{\binom{n}{2}}} \tag{7}$$

Since this quantity tends to 1 as n tends to infinity, in particular there exists a lowest n such that it exceeds p. Then, set $M = \lceil n \rceil$ for that n. □

4 Estimation of some low numbers

These numbers were generated by randomly sampling graphs to estimate the probability that they statisfy the constraints. Only the upper portion of the tables are shown because they are symmetrical.

$p = 0.5$:

$k \backslash l$	1	2	3	4	5
1	1	1	1	1	1
2		2	2	2	2
3			4	5	5
4				7	9
5					12