Probabilistic Ramsey Numbers

Avi Semler

July 2022

1 Definitions

Let Ω_n be the set of all graphs on n vertices.

Let $G_n \in \Omega_n$ be a graph randomly selected by an Erdős–Rényi model $G(n, 0.5)$.

Let $Y_k : \Omega_n \to \mathbb{N}$ be a random variable counting the number of k-cliques in the selected graph.

Let $Z_l : \Omega_n \to \mathbb{N}$ be a random variable counting the number of l-independent sets in the selected graph.

Define the probabilistic Ramsey number $R_p(k, l)$ as the least value for n such that when selecting with $G(n, 0.5)$ we have

$$\mathbb{P}(\{Y_k > 0\} \cup \{Z_l > 0\}) \geq p \tag{1}$$

i.e. the necessary size of a graph in order that the probability of there being either a k-clique or an l-independent set is at least p.

Observe that by setting $p = 1$ we obtain the Ramsey number $R(k, l)$ In this sense, probabilistic Ramsey numbers are a generalisation of the standard variety.

2 Bounds

There is a trivial upper bound:

$$R_p(k, l) \leq R(k, l)$$

and generally, for $p < p'$:

$$R_p(k, l) \leq R_{p'}(k, l)$$

Theorem 2.1 (Lower bound). *If $p \in (0, 1)$ and $k \geq 1$,*

$$R_p(k, k) \geq \left(k! \cdot p \cdot 2^{\binom{k}{2} - 1}\right)^{\frac{1}{k}}$$

Proof. The probability that a fixed subset of k vertices is either a k-clique or a k-independent set is

$$\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \tag{2}$$

Using this and the union bound, we can put a bound on the probability that the entire graph contains a k-clique or k-independent set (where n is the number of vertices):

$$\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \geq p \tag{3}$$

Then the result follows by rearranging and using $\frac{n^k}{k!} \geq \binom{n}{k}$:

□

3 Estimation of some low numbers

These numbers were generated by randomly sampling graphs to estimate the probability that they statisfy the constraints. Only the upper portion of the tables are shown because they are symmetrical.

$p = 0.5$:

$k \backslash l$	1	2	3	4	5
1	1	1	1	1	1
2		2	2	2	2
3			4	5	5
4				7	9
5					12