

# A correct proof of Lemma 5.1 from InfoGAN \*

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The InfoGAN paper<sup>1</sup> has the following lemma:

**Lemma 5.1.** *For random variables  $X, Y$  and function  $f(x, y)$  under suitable regularity conditions:  $\mathbb{E}_{x \sim X, y \sim Y|x}[f(x, y)] = \mathbb{E}_{x \sim X, y \sim Y|x, x' \sim X|y}[f(x, y)]$ .*

The proof in the paper seems wrong – here’s a step where  $x$  mysteriously becomes  $x'$ :

$$\begin{aligned} & \int_x \int_y P(x, y) f(x, y) \int_{x'} P(x'|y) dx' dy dx \\ &= \int_x P(x) \int_y P(y|x) \int_{x'} P(x'|y) f(x', y) dx' dy dx \end{aligned}$$

After consulting with others, we couldn’t fix this proof. Instead, Nic Ford found the following proof:

*Proof.*

$$\begin{aligned} \mathbb{E}_{x \sim X, y \sim Y|x}[f(x, y)] &= && \text{make expectations explicit...} \\ \mathbb{E}_{x \sim P(X)}[\mathbb{E}_{y \sim P(Y|X=x)}[f(x, y)]] &= && \text{by definition of } P(Y|X=x) \dots \\ \mathbb{E}_{x, y \sim P(X, Y)}[f(x, y)] &= && \text{by definition of } P(X|Y=y) \dots \dots \\ \mathbb{E}_{y \sim P(Y)}[\mathbb{E}_{x \sim P(X|Y=y)}[f(x, y)]] &= && \text{rename } x \text{ to } x' \dots \\ \mathbb{E}_{y \sim P(Y)}[\mathbb{E}_{x' \sim P(X|Y=y)}[f(x', y)]] &= && \text{by the law of total expectation...} \\ \mathbb{E}_{x \sim P(X)}[\mathbb{E}_{y \sim P(Y|X=x)}[\mathbb{E}_{x' \sim P(X|Y=y)}[f(x', y)]]] &= && \text{make expectations implicit...} \\ \mathbb{E}_{x \sim X, y \sim Y|x, x' \sim X|y}[f(x', y)] \end{aligned}$$

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\*Read and comment on the latest version of this note at <http://aoliver.org/correct-proof-of-infogan-lemma>

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<sup>1</sup><https://arxiv.org/pdf/1606.03657.pdf>