A correct proof of Lemma 5.1 from InfoGAN*

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The InfoGAN paper[1] has the following lemma:

Lemma 5.1. *For random variables X, Y and function $f(x, y)$ under suitable regularity conditions:* $\mathbb{E}_{x\sim X, y\sim Y|x}[f(x, y)] = \mathbb{E}_{x\sim X, y\sim Y|x, x'\sim X|y}[f(x, y)]$.

The proof in the paper seems wrong – here's a step where x mysteriously becomes x':

$$\begin{aligned} &\int_x \int_y P(x, y) \color{red}{f(x, y)} \int_{x'} P(x'|y) dx' dy dx \\ = &\int_x P(x) \int_y P(y|x) \int_{x'} P(x'|y) \color{red}{f(x', y)} dx' dy dx \end{aligned}$$

After consulting with others, we couldn't fix this proof. Instead, Nic Ford found the following proof:

Proof.

$$\begin{aligned}
&\mathbb{E}_{x\sim X, y\sim Y|x}[f(x, y)] = && \text{make expectations explicit...} \\
&\mathbb{E}_{x\sim P(X)}\left[\mathbb{E}_{y\sim P(Y|X=x)}[f(x, y)]\right] = && \text{by definition of } P(Y|X = x)\text{...} \\
&\mathbb{E}_{x,y\sim P(X,Y)}[f(x, y)] = && \text{by definition of } P(X|Y = y)\text{... ...} \\
&\mathbb{E}_{y\sim P(Y)}\left[\mathbb{E}_{x\sim P(X|Y=y)}[f(x, y)]\right] = && \text{rename } x \text{ to } x'\text{...} \\
&\mathbb{E}_{y\sim P(Y)}\left[\mathbb{E}_{x'\sim P(X|Y=y)}[f(x', y)]\right] = && \text{by the law of total expectation...} \\
&\mathbb{E}_{x\sim P(X)}\left[\mathbb{E}_{y\sim P(Y|X=x)}\left[\mathbb{E}_{x'\sim P(X|Y=y)}[f(x', y)]\right]\right] = && \text{make expectations implicit...} \\
&\mathbb{E}_{x\sim X, y\sim Y|x, x'\sim X|y}[f(x', y)]
\end{aligned}$$

□

*Read and comment on the latest version of this note at http://aoliver.org/correct-proof-of-infogan-lemma

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[1] https://arxiv.org/pdf/1606.03657.pdf