Modelli Probabilistici per le Decisioni, Antonio Vivace

Kalman Filter

A background study and an
interactive, real time simulation

Overview

1. Introduction to the Kalman Filter
    a. Assumptions
    b. Model
2. Applications
    a. Historical Relevance
3. Demo
    a. Software stack, design choices
    b. Implementation and quirks
    c. Analysis of the results
4. Conclusions

Kalman Filter: Model and Assumptions

- Thiele-Swerling developed optimal estimation of orbits of satellites and trajectories of missiles, anticipating the development of the Kalman filter.
- Recursive algorithm (*estimator*) for dynamic systems (*Markov* and *Stationary* process) *minimum mean-square error* estimator

  Filtering Task: $P(X_t|e_{1-t})$
  Prediction Task: $P(X_{t+k}|e_{1:t})$ per un $k > 0$

- Model matching the real system?

- Gaussian Noises Q, R, independent and known

- *Markov Assumption*:
  Previous state knowledge (ONLY the previous state!)
  Transition model: $P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1})$
  Sensor model: $P(E_t \mid X_{0:t-1}, E_{0:t-1}) = P(E_t \mid X_t)$

- Practical implementation of the Kalman Filter is often difficult due to the difficulty of getting a good estimate of the noise covariance matrices $\mathbf{Q}_k$ and $\mathbf{R}_k$

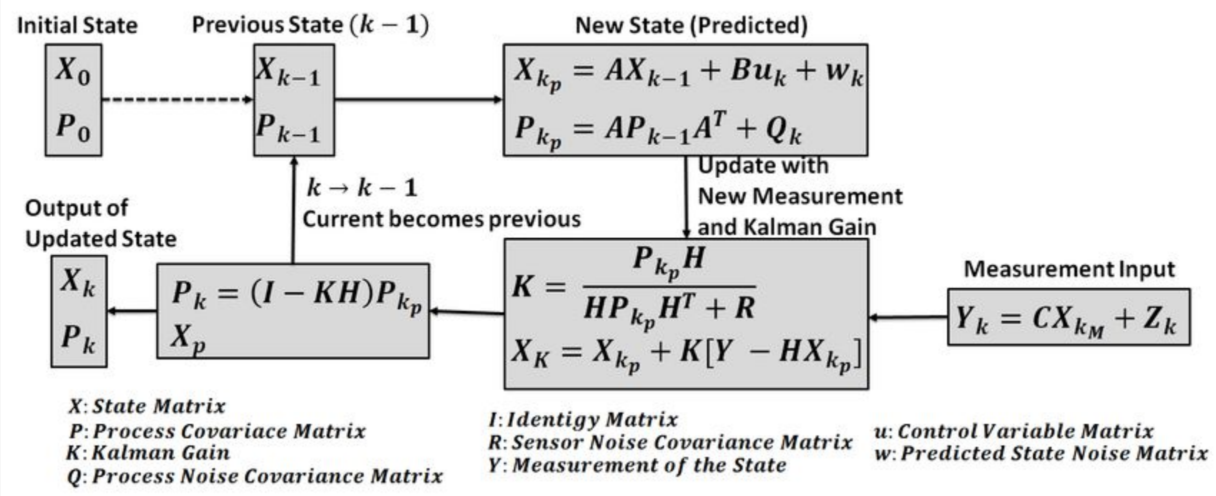
Introduction to the Kalman Filter

$$x_k = Ax_{k-1} + Bu_{k-1} + w_{k-1},$$

$$z_k = Hx_k + v_k.$$

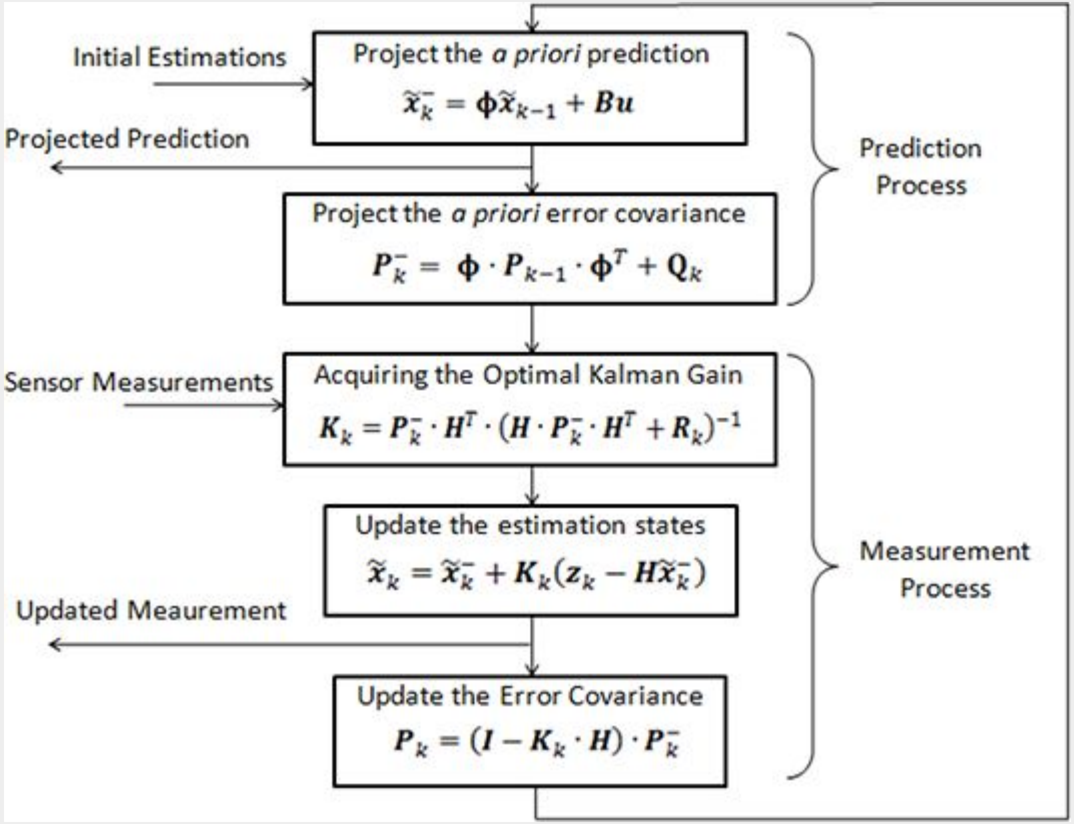
$$p(w) \sim N(0, Q),$$
$$p(v) \sim N(0, R).$$

Q: Process Noise, usually treated as a *tuning* parameter to adjust the gain of the Kalman filter to smooth either more or less the data
A (n x n): State Transition
B (n x l): Input Control
c: control vector, which would contain estimated changes from direct action commands
H (n x m): Measurement (stato/misurazione)
x in R^n, z in R^l, c in R^m.

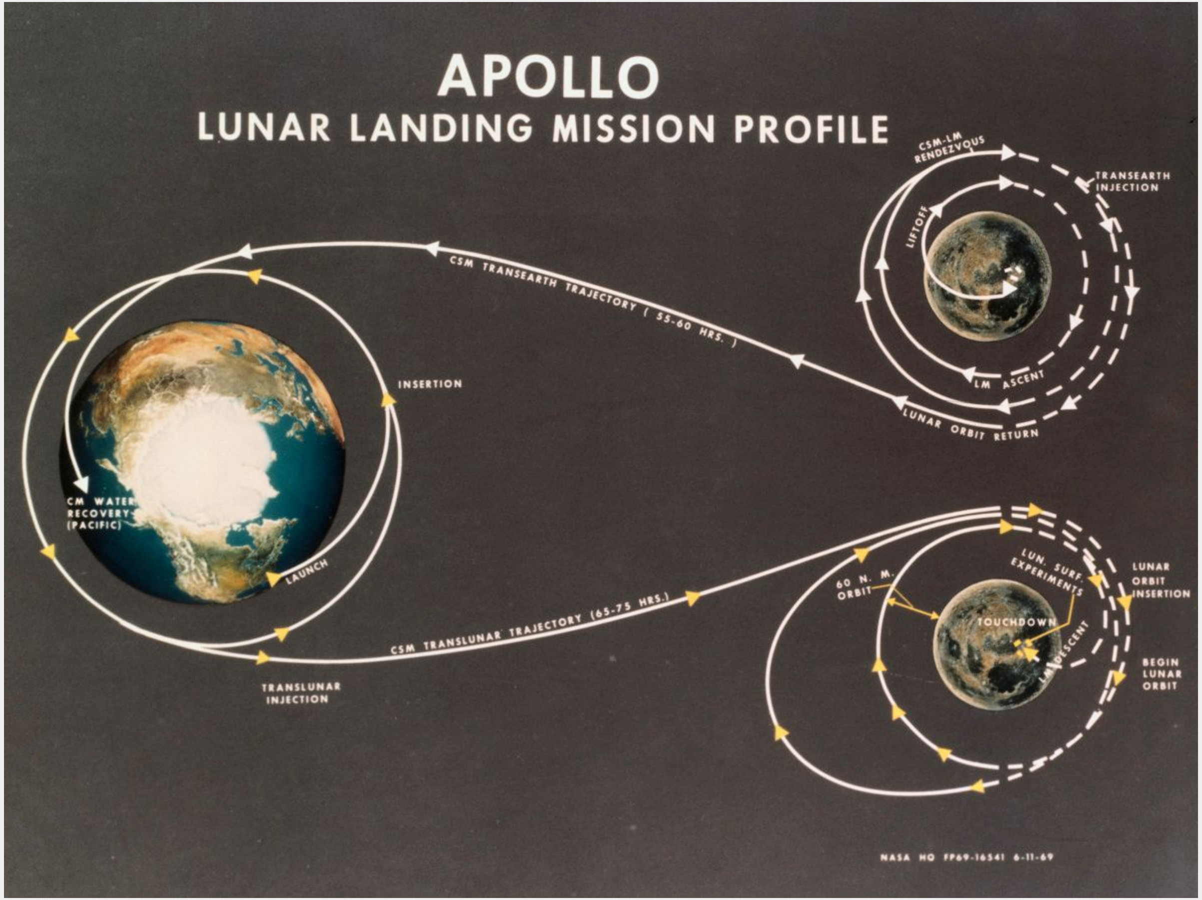


Introduction to the Kalman Filter

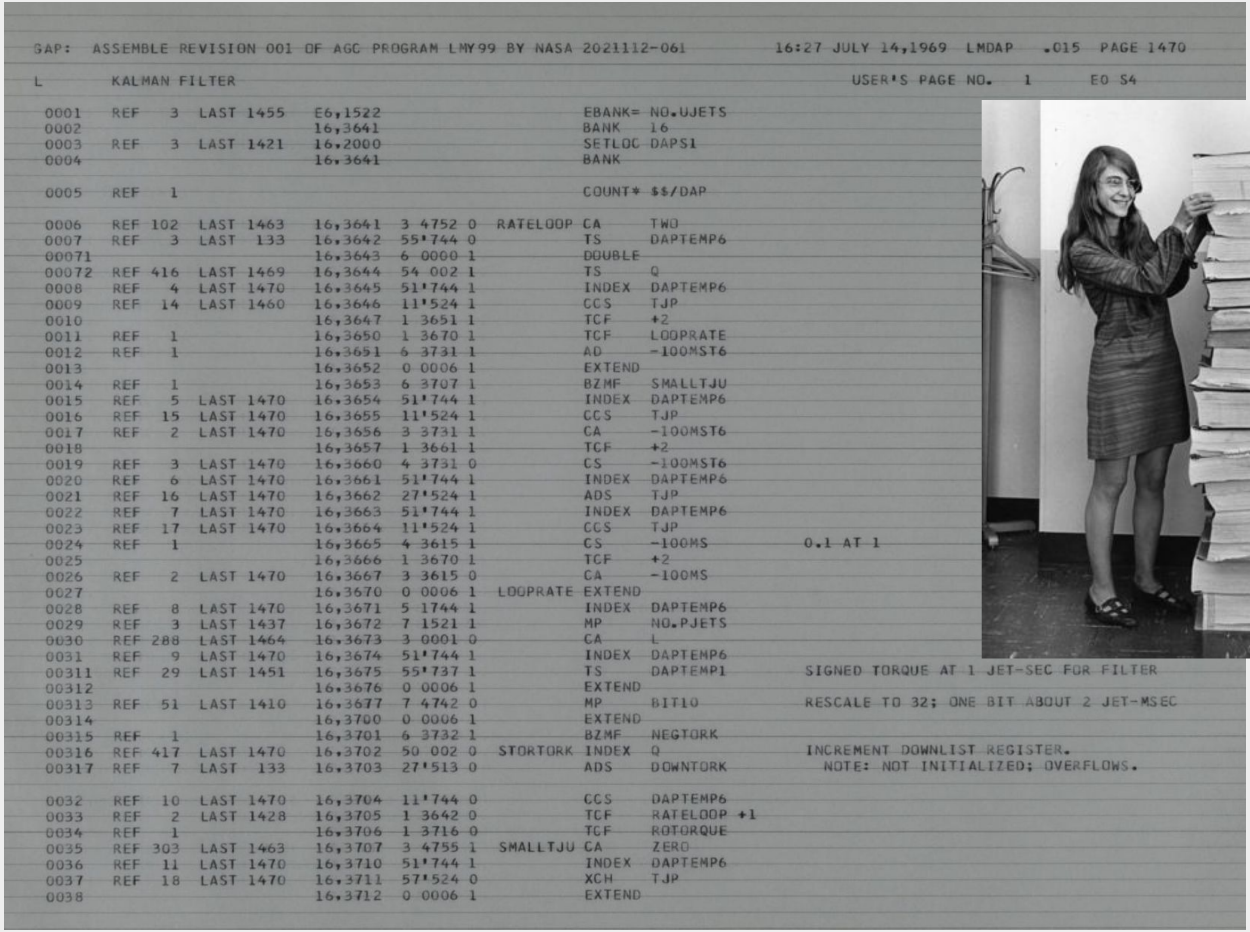
The Kalman gain is the **relative weight** given to the measurements and current state estimate, and can be "tuned" to achieve particular performance. With a high gain, the filter places more weight on the most recent measurements, and thus follows them more responsively. With a low gain, the filter follows the model predictions more closely. At the extremes, a high gain close to one will result in a more jumpy estimated trajectory, while low gain close to zero will smooth out noise but decrease the responsiveness.



Historical Relevance



Historical Relevance



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 0002                         16,3641                          BANK   16
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 0005    REF   1                                               COUNT* $$/DAP

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 00071                        16,3643  6 0000 1           DOUBLE
 00072   REF 416  LAST 1469   16,3644  54 002 1           TS     Q
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 0009    REF  14  LAST 1460   16,3646  11'524 1           CCS    TJP
 0010                         16,3647  1 3651 1           TCF    +2
 0011    REF   1              16,3650  1 3670 1           TCF    LOOPRATE
 0012    REF   1              16,3651  6 3731 1           AD     -100MST6
 0013                         16,3652  0 0006 1           EXTEND
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 0025                         16,3666  1 3670 1           TCF    +2
 0026    REF   2  LAST 1470   16,3667  3 3615 0           CA     -100MS
 0027                         16,3670  0 0006 1  LOOPRATE EXTEND
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 0029    REF   3  LAST 1437   16,3672  7 1521 1           MP     NO.PJETS
 0030    REF 288  LAST 1464   16,3673  3 0001 0           CA     L
 0031    REF   9  LAST 1470   16,3674  51'744 1           INDEX  DAPTEMP6
 00311   REF  29  LAST 1451   16,3675  55'737 1           TS     DAPTEMP1         SIGNED TORQUE AT 1 JET-SEC FOR FILTER
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 00313   REF  51  LAST 1410   16,3677  7 4742 0           MP     BIT10            RESCALE TO 32; ONE BIT ABOUT 2 JET-MSEC
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 00316   REF 417  LAST 1470   16,3702  50 002 0  STORTORK INDEX  Q                INCREMENT DOWNLIST REGISTER.
 00317   REF   7  LAST  133   16,3703  27'513 0           ADS    DOWNTORK           NOTE: NOT INITIALIZED; OVERFLOWS.

 0032    REF  10  LAST 1470   16,3704  11'744 0           CCS    DAPTEMP6
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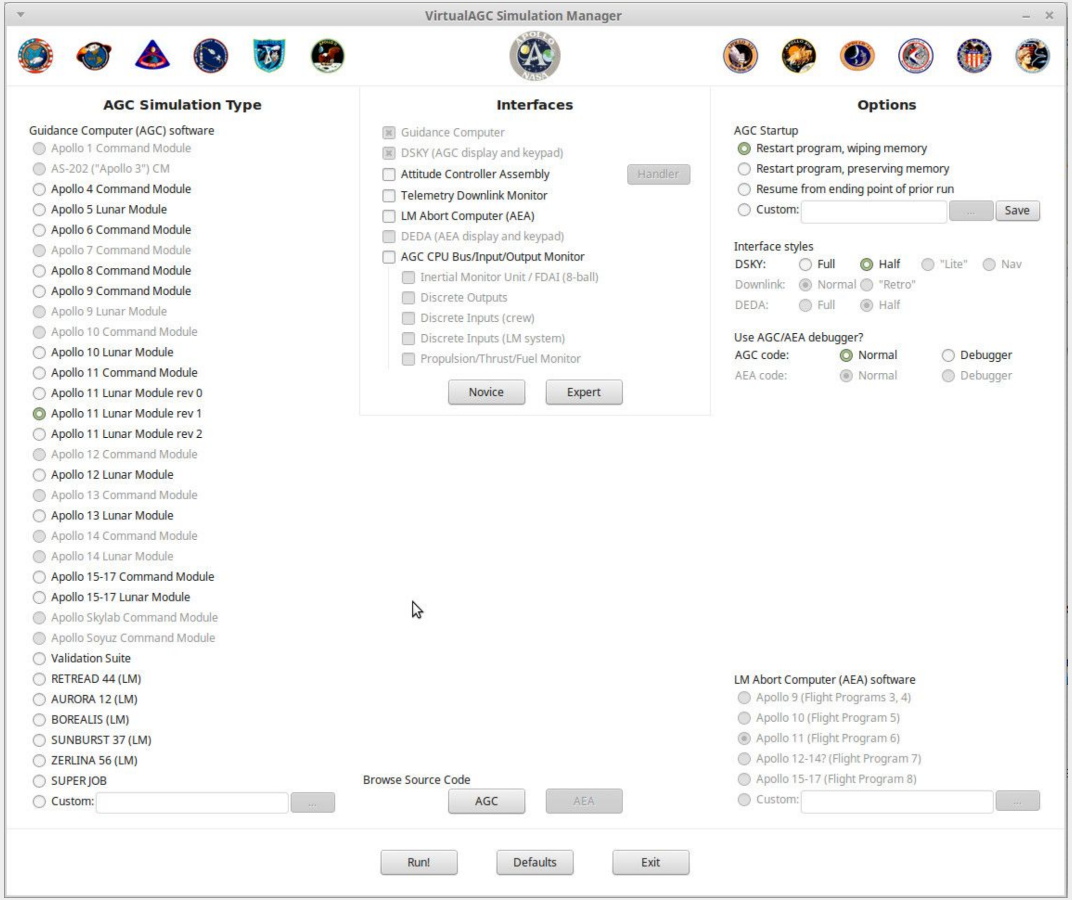
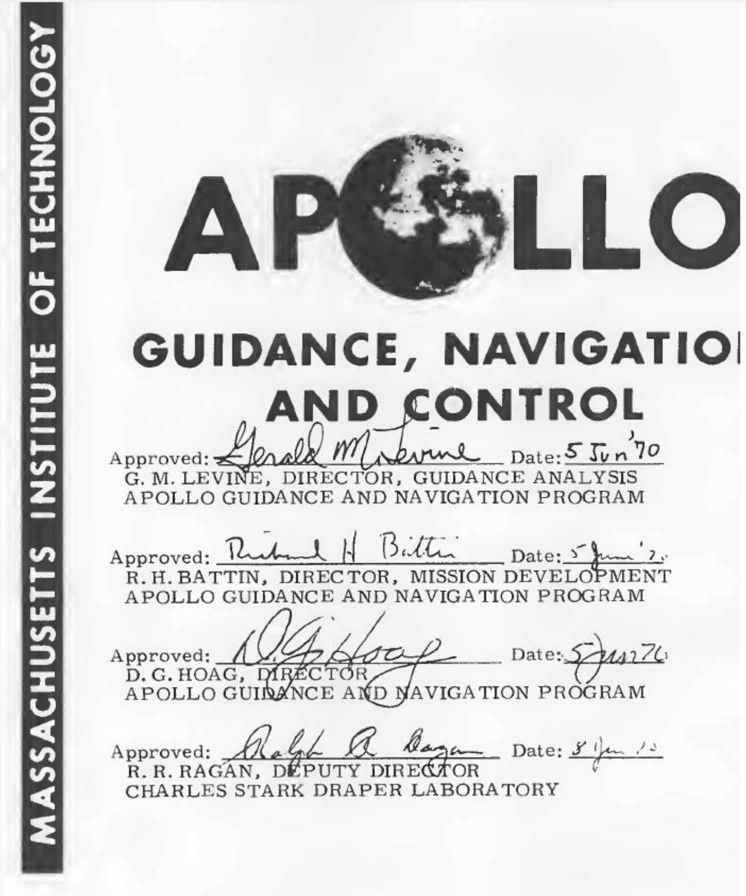
```

Historical Relevance




MASSACHUSETTS INSTITUTE OF TECHNOLOGY

APOLLO

GUIDANCE, NAVIGATIO[N]

AND CONTROL

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THE APOLLO RENDEZVOUS NAVIGATION FILTER THEORY, DESCRIPTION AND PERFORMANCE

Historical Relevance

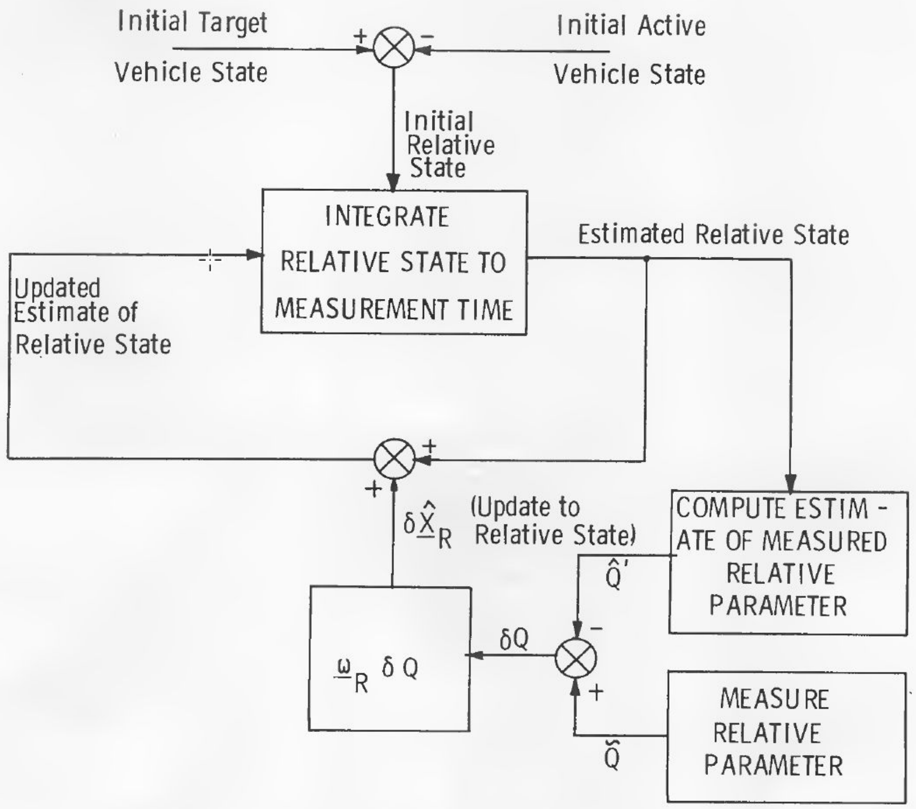


Fig. II-1 Rendezvous Navigation System Using Recursive Kalman Filter

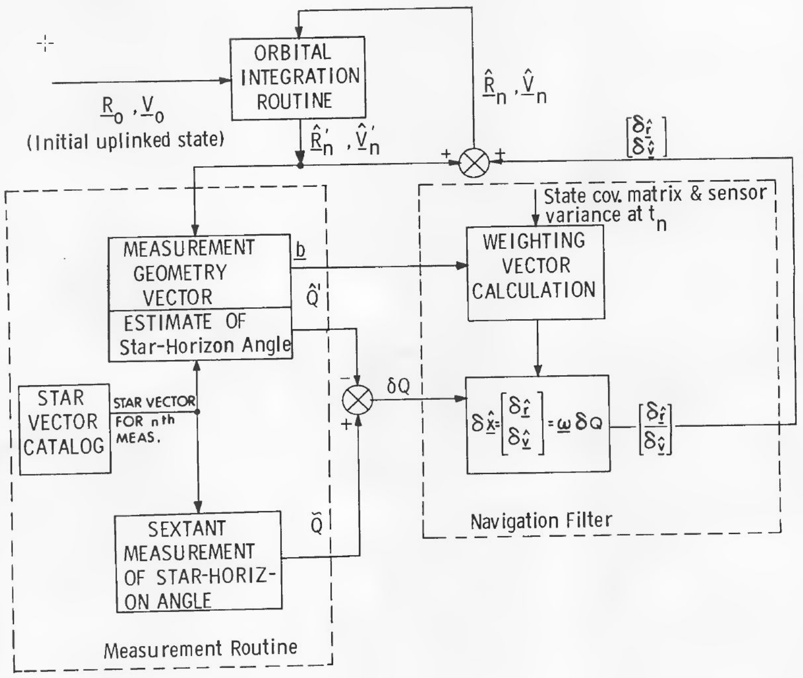


Fig. I-1 Cislunar Recursive Navigation System

Applications

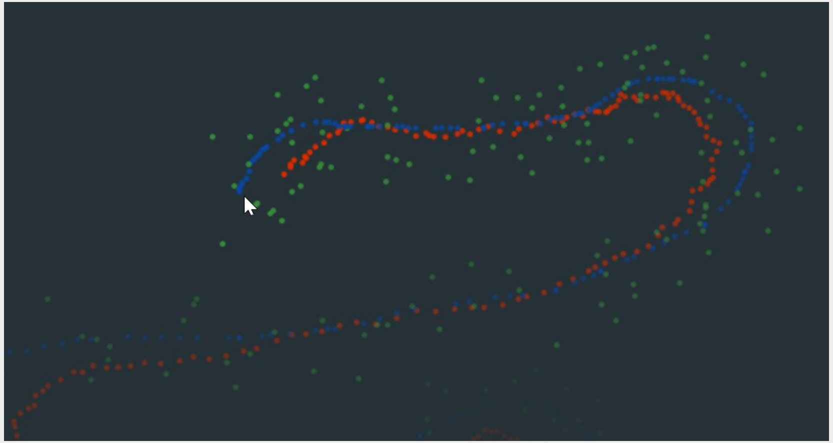
1. Navigation (Apollo 11!)
2. Seismology
3. Autopilot
4. Vehicle control (Sensorless control of AC motor variable-frequency drives)
5. many more!

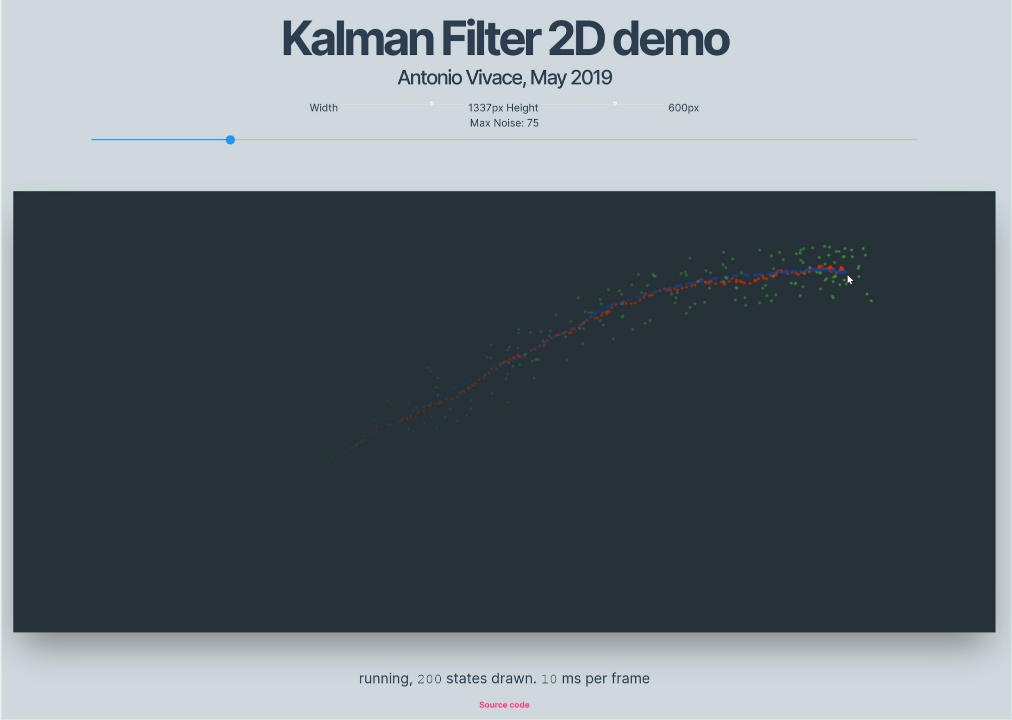
- Attitude and heading reference systems
- Autopilot
- Battery state of charge (SoC) estimation[57][58]
- Brain-computer interface
- Chaotic signals
- Tracking and vertex fitting of charged particles in particle detectors[59]
- Tracking of objects in computer vision
- Dynamic positioning
- Economics, in particular macroeconomics, time series analysis, and econometrics[60]
- Inertial guidance system
- Nuclear medicine – single photon emission computed tomography image restoration[61]
- Orbit Determination
- Power system state estimation
- Radar tracker
- Satellite navigation systems
- Seismology[62]
- Sensorless control of AC motor variable-frequency drives
- Simultaneous localization and mapping
- Speech enhancement
- Visual odometry
- Weather forecasting
- Navigation system
- 3D modeling
- Structural health monitoring
- Human sensorimotor processing[63

Demo (1)

Software stack:

- Javascript
- VueJS
- **SylvesterJS**
- MuseUI
- **CanvasRenderingContext2D web API**





Why:

- Cross platform
- Reactive: realtime editable parameters
- Presets
- Paths
- Customisable canvas and stable performance
- Working on touchscreen devices
- Fully client side

Demo (2)

m = [measuredX, measuredY, xVelocity, yVelocity] ← *measurement vector*

c = [0, 0, 0, 0] ← *control vector*

Prediction Step:

x = (A * x) + (B * c)

P = (A * P * A^T) + Q ← *A^T is the matrix transpose of A*

Correction Step:

S = (H * P * H^T) + R ← *H^T is the matrix transpose of H*

K = P * H^T * S^{-1} ← *S^{-1} is the matrix inverse of S*

y = m - (H * x)

x = x + (K * y)

P = (**I** - (K * H)) * P ← ***I** is the Identity matrix*

predX = x

*predX = (A * predX) + (B * c)*

x ← [xPos, yPos, xVel, yVel]

Demo (3)

Drawing

```
for (let i = this.states.length - 1; i > 0; --i) {
      let state = this.states[i];
      state.display();
      state.update();
      if (state.dead) {
            this.states.splice(i, 1);
      }
}
```

Keeping the simulation framerate costant:

```
// See ya in 1000/desiredFramerate milliseconds
this.lastCalledTime = performance.now();
let ms = performance.now() - start;
this.ms = ms
setTimeout(this.frame, 1000 / this.framerate - ms);
```

Parameter Setting

```
// State Transition
this.A = m([
        [1, 0, 0.2, 0],
        [0, 1, 0, 0.2],
        [0, 0, 1, 0],
        [0, 0, 0, 1]
]);
```

```
this.H = m([
        [1, 0, 0, 0],
        [0, 1, 0, 0],
        [0, 0, 1, 0],
        [0, 0, 0, 1]
]);
```

```
this.Q = m([
        [0.001, 0, 0, 0],
        [0, 0.001, 0, 0],
        [0, 0, 0, 0],
        [0, 0, 0, 0]
]);
```

```
this.R = m([
        [0.1, 0, 0, 0],
        [0, 0.1, 0, 0],
        [0, 0, 0.1, 0],
        [0, 0, 0, 0.1]
]);
```

Demo (4)

Kalman Filter implementation (simple, ignoring c and B)

```
// Adding artificial and gaussian noise
// m = [noisyX, noisyY, deltaX, deltaY]
let deltaX = noisyX - this.lastPoint.elements[0];
let deltaY = noisyY - this.lastPoint.elements[1];
let measurement = v([noisyX, noisyY, deltaX, deltaY]);

// PREDICTION step
// x = (A * x) + (B * c)
var x = this.A.multiply(this.lastPoint)
// P = (A * P * AT) + Q
var P = ((this.A.multiply(this.last)).multiply(this.A.transpose())).add(this.Q);

// CORRECTION step
// S = (H * P * HT) + R
var S = ((this.H.multiply(P)).multiply(this.H.transpose())).add(this.R);
// K = P * HT * S-1
var K = (P.multiply(this.H.transpose())).multiply(S.inverse());
// y = m - (H * x)
var y = measurement.subtract(this.H.multiply(x));
// x = x + (K * y)

//  this is the final filtered point for this iteration
this.lastPoint = x.add(K.multiply(y));
// P = (I - (K * H)) * P
this.last = ((window.Matrix.I(4)).subtract(K.multiply(this.H))).multiply(P);
```

Demo: experimentation and conclusions

1. Presets
2. Paths
3. Mouse input
4. Random Path
5. Changing R during the simulation
6. H and c
7. Q and R estimation
   Autocovariance Least-Squares (ALS) Package