Chaotic Scattering for a Sliding Mass in a Complex Topography

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CS 200

March 3, 2016

Abstract

Since the creation of the "Gapard-Rice" model, the study of the chaotic behavior in scattering problems has been shown to have significance for a multitude of physical situations. The aim of this paper is to provide a study of the field, by looking at a simulation of a particle, scattering due to the presence of four hills in its path. The simulation follows most classical scattering systems, and has one impact parameter that causes the particle to scatter. The trajectories of the impact parameters are generated by numerically integrating differential equations of motion using the four stage Runge-Kutta algorithm. While traversing across the hills, certain trajectories lead to the particle getting trapped for a certain period inside the system before exiting. The duration of this time, along with the length of the path of the particle and angle of exit from the system are compared to the impact parameter. It is found that the correlation between the parameters demonstrates fractal behavior.

Contents

1 Introduction

Nature is extraordinarily intricate, and we can only be certain of her unpredictability. Edward Lorenz, a meteorologist, created a computer simulation in 1961 to model weather [cite]**??**. He rounded an input parameter to three decimal places rather than six, and discovered that his tiny alteration utterly transformed his long term forecast [cite]**??**. This highlighted the fact that small uncertainties can yield diverging consequences. Hence predicting the patterns of complicated systems such as the migratory pattern of birds, spread of a virus across a continent, or even patterns in the heart are nearly impossible, as these require very high precision measurements. This gives rise to chaos theory, a field of study in mathematics that studies the behavior and condition of sensitive systems[cite]. This paper looks at a portion of chaos theory, called chaotic scattering, a topic with numerous applications in fields such as nuclear physics, celestial mechanics, and fluid mechanics[cite]. The paper discusses a simulation created to study scattering, which provides a new way of studying the topic. Before we begin discussing that, we will first review chaotic scattering.

1.1 What is Chaotic Scattering?

Chaotic scattering deals with the problem of finding a relationship between input variables and output variables, and then studying the chaotic nature of the relationship[cite].

1.2 Motivations

Following are some of the motivations to work on this project

1. Scattering has many direct applications, such as nuclear physics, hydrodynamical processes and even models of chemical reactions.
2. This project provides a better understanding of chaotic systems and hence might help us better predict weather, and other complicated systems.
3. Since a similar simulation has not been created, examining chaotic scattering in this way may give us new insights.
4. Theoretically, fractals may form in the graphs of the data for this study, we may improve our concepts behind the similar fractals that form on other scattering problems such as the Gaspard Rice model

1.3 Simple Example

A simple model to illustrate chaotic scattering could be that of a particle traveling through a region of potential $V(x)$. The potential is defined such that

$$V(x) = \begin{cases} 0, & \text{if } x \text{ is outside the region} \\ 0 < V < 1, & \text{otherwise} \end{cases} \tag{1}$$

where the size of the region in this example is a closed shape of area A. Figure 2 shows this shape, and the particle approaching the shape, in a trajectory parallel to the x axis. As the potential is zero outside the shape, the particle is unaffected and moves along its same initial path. However, inside the shape, as the potential varies from 0 and 1, the particle experiences different amounts of force during its transit. Hence, its path changes. The particle spends some time, T, in this region, before exiting at an angle θ_s, as seen in the figure. If these output parameters of T, and θ_s are compared with the initial impact parameter, b, the initial y position of the particle, we find graphs showing chaotic nature in some regions, and non chaotic in others. This is discussed in more detail in Section 5.2.

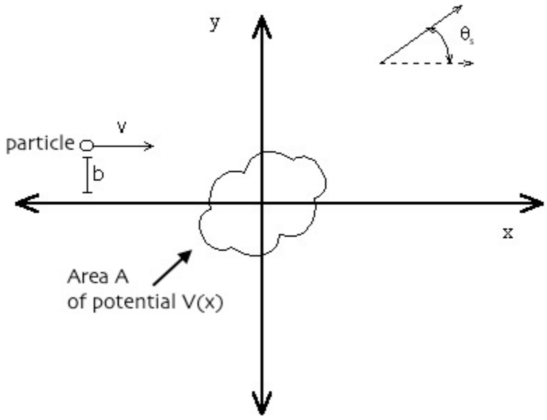

Figure 1: Schematic illustration of a scattering problem in two dimensions [cite the potentials source].

This is the main relationship that we study in this paper, but for a more complicated system. The system we study has four Gaussian potential hills, instead of the one arbitrary potential in the given example. Additionally, the Gaussian potentials are that of gravity, hence the model simulates a particle rolling through a plane with four identical hills placed side by side. The model is discussed in further detail in Section 2.

2 Physics Model

2.1 Description of the Model

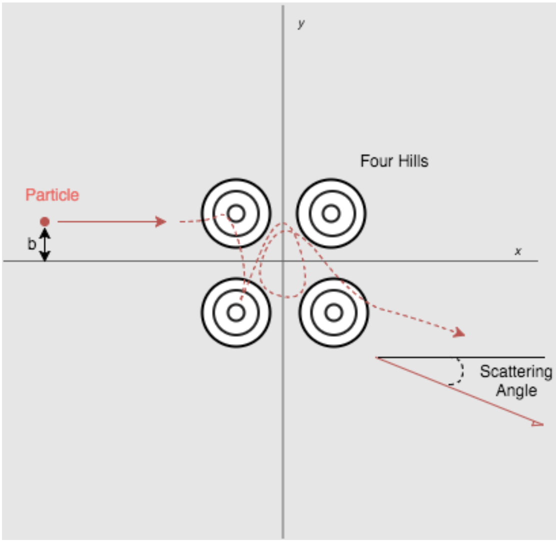

Figure 2: Schematic illustration of the scattering problem with four potential hills in the particle's path. b is the initial y position of the particle, and is known as the impact parameter. The scattering angle is made by the particle's trajectory with the horizontal axis.

We discussed a scattering model with one potential in Section 1.3. Now, let us consider a particle moving towards a potential $V(x, y)$ without friction. This potential is different from the previous, as it is comprised of four hills that have peaks at $(x, y) = (\pm 1, \pm 1)$. We can examine the system in figure 2. As seen from the figure, the particle starts at a fixed horizontal distance from the potential hills. Its vertical distance from the x axis is denoted as b, and called the 'impact parameter'.

It is expected that if the energy of the particle E, is greater then the maximum potential energy at the hill peaks E_m, then the particle should be able to pass through the hilly region without much alteration in its trajectory, and the scattering is non-chaotic. On decreasing the E below the threshold energy E_m and altering b in small increments, we expect the scattering angle to vary rapidly, demonstrating the chaotic nature of the pile.

2.2 Physical Theory

Reference for the potential - Dr.Lindner, and Chaos: An Inter- disciplinary Journal of Nonlinear Science

The Potential mapped by a four hill system:

$$\vec{V}(x,y) = x^2y^2\ e^{-x^2-y^2} \tag{2}$$

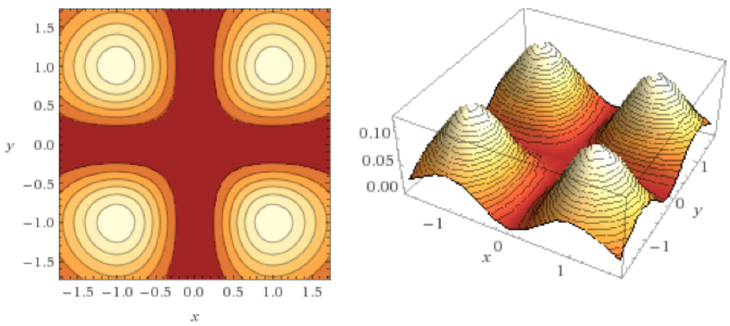

Figure 3: A 3D and a contour plot of the potential hills created by Wolfram Alpha

We clearly see, that the potential has four maxima, in the coordinates (1,1) , (-1,-1) , (-1,1) , (1,-1). At this maxima the potential energy has a value of $V(x,y) = 1/e^2 = E_{max}$. Next, we can find that the scattering is related to the ratio E/E_{max}, E being the particle's energy

The force experienced in this four hill system:

$$\begin{aligned}\vec{F}/m &= -\bigtriangledown\vec{V} \\ &= -\{2e^{(-x^2-y^2)}x(-1+x^2)y^2, 2e^{(-x^2-y^2)}x^2y(-1+y^2)\}\end{aligned} \tag{3}$$

Hence, with this, we are able to use the Runge-Kutta formulae to plot the trajectory of a particle in a four hill region.

Explain how and why we are taking F/m instead of F, and what equations we need to get the acceleration.

2.3 Integration Accuracy Checks

Talk about how integration accuracy can be checked in the system by studying the energy of the particle in different regions of the system. Energy should be constant in all positions of the system.

3 Numerical Integration

Here the various integration techniques is introduced and discussed.

3.1 Euler-Cromer

Discussion about what the Euler-Cromer method is, the algorithm, and how it is useful. Talk about the need to get new integration method.

3.2 Four Stage Runge-Kutta

Talk about what this is, and its algorithm. Talk about why it is better than Euler-Cromer algorithm.

3.3 Adaptive Runge-Kutta

Talk about how we need to alter the time step, and make it smaller in certain regions where the particle may be traveling fast. Discuss what happens if we don't do this, and how this is helpful compared to the normal method, in terms of energy comparisons.

4 Software

Here, we discuss the software used to implement the model of the simulation.

4.1 Programming Language and APIs

The Mac OS X application was made using Objective-C, and the Cocoa API. Discuss why this was used.

4.2 Nifty Code Segments

Put implementation of the Runge-Kutta Algorithm here, and discuss.

4.3 Interface Design and Interacting with the Program

Talk about the features of the application and how to use it. Include screen shots.

5 Results and Analysis

5.1 Chaotic and Non Chaotic Regions

1. Display graphs of the impact parameter against the scattering angle, and the time duration of the path.
2. Discuss how even when the $E > E_m$, there are some regions on the graphs where the scattering angle does not vary much, until changing b slightly causing the scattering angle to become drastically different. This is how the system is chaotic.
3. Discuss how the same patterns can be seen on the plots of the time against the scattering angle.
4. Talk about how these patterns are the same as seen by the Gaspard Rice Model.
5. Discuss how we see that on the plots of the scattering angle against the impact parameter, there are certain speckled regions. Explain this with a graph, by enlarging the plot. When the horizontal scale for the speckled regions are enlarged, the speckled regions still remain. Even further enlarging of those speckled regions does not resolve the function, suggesting the fractal nature of these graphs. This last paragraph will act as a good transition to the next analysis section.

5.2 Fractal Nature of Data

1. Provide plots showing the fractal nature, and discuss when they form.
2. Include a paragraph on the fractal nature of the time versus b as well.

6 Conclusion

Recognizing the chaotic, fractal nature of our world can give us new insight, power, and wisdom.

References

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