KALMAN FILTER NOTES

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1. Kalman filtering of a simple linear system

Before going into much detail about the filter itself let's see what it does. Let's start with a simple dynamical system which, for instance, may be taken as representing the position and velocity of a mass on a spring:

$$\begin{aligned} \dot{y}_1 &= -2\,y_1 + 2\,y_2 + 1 \\ \dot{y}_2 &= -4\,y_1 + 2\,y_2 \end{aligned} \tag{1}$$

The analytical solution for the system of equations is:

$$\begin{aligned} y_1 &= 1.5\,\sin 2\,t + 1.5\,\cos 2\,t - 0.5 \\ y_2 &= 3\,\cos 2\,t - 1 \end{aligned} \tag{2}$$

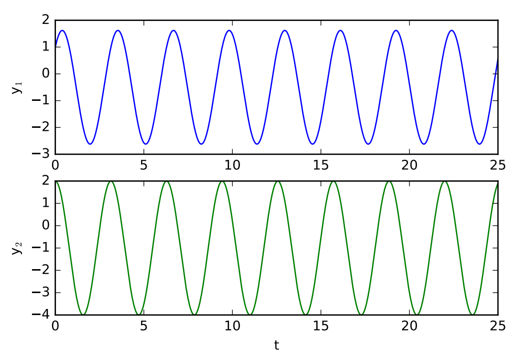

Figure 1. Plot of analytical solution (Equation 2) for simple dynamical system (Equation 1).

Let us further assume that our model of the system is imperfect. There are nonlinearities in the spring, there is air resistance, wind, and a myriad of other unaccounted for effects. If these effects are uncorrelated then their sum total will have a Gaussian distribution. We can represent this uncertainty by adding noise to our initial value and then at every simulation step to the dynamical system (Figure 2).

Note that now the model does not belong anymore to the familiar deterministic models we are used to working with, but rather is now stochastic. Each time the model is simulated (each realization) the result will be different, and this randomness is an inherent characteristic of the *model*, so that no amount of numerical improvement (reduction in step size, etc.) can result in a more accurate output. For non-linear systems the realizations may be wildly different from each other. Nonetheless, for linear systems, the statistical properties of the stochastic system are guaranteed to converge to those of the analytical solution, e.g. the mean of the ensemble at any time converges to the true mean.

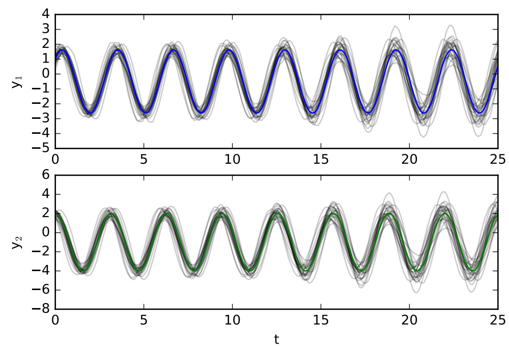

FIGURE 2. Plot of dynamical system with normally distributed noise added to it (gray lines). Blue and green lines are analytical solution.

A second source of uncertainty may come from our observations which may themselves be imprecise. For instance, as geochemists are well aware, repeated weighing of a small amount of material will inevitably result in slightly different values, though the amount of material being weighed remains constant. If the sources of error are uncorrelated then this uncertainty too can be represented by a noise term with a Gaussian distribution, which may be different from the uncertainty related to the model (Figure 3).

The question is then, given these uncertainties can one recover a precise estimate of the states of the system? The Kalman filter does precisely that, and, for linear systems and Gaussian noise, is guaranteed to provide an estimate that is as close as possible to the true (analytical) solution, and is *better than either model or observations alone*. In Figure 4 the output of the Kalman Filter is plotted alongside the noisy model and noisy data.

The particular variant of filter that I use (RTS smoother) takes takes advantage of the fact that we have the entire time series in our possession and so "know the

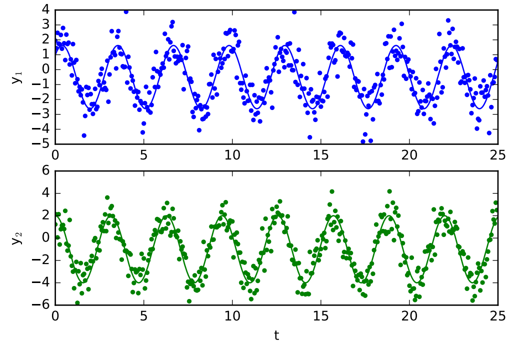

FIGURE 3. Analytical solution plus sensor noise.

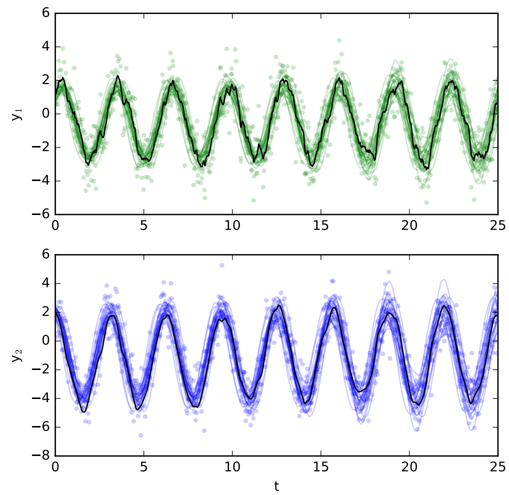

FIGURE 4. The filtered data (black lines) compared with the noisy model (blue and green lines) and noisy data (blue and green dots).

future". By integrating backwards the filter can better distinguish between signal and noise. We see that the filter identifies the noise remarkably well resulting in an output that is in fact extremely close to the analytical solution (Figure 5).

There are two more features of the Kalman filter that make it very handy. First, the states are not required to be directly observable. For instance, we may be able to see only a subset of the states, or some function of them (e.g. only position, rather than position and velocity). So for our model, here written in matrix form, we might

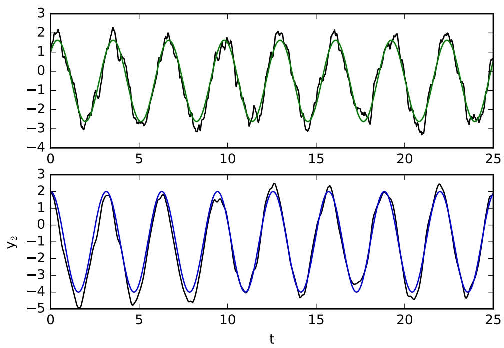

FIGURE 5. The filtered data (black lines) plotted alongside the analytical solution.

have a measurement function $h = \begin{bmatrix} 1 & 0 \end{bmatrix}$ that maps the states y to the observations z.

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3}$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{4}$$

The second useful feature of the filter is that it allows for straightforward parameter estimation. Any parameters that are part of the system and whose values are imperfectly known can be designated as state variables, and so have the filter estimate their value alongside those of the actual states. For instance, in the above example, suppose the value of the forcing function was some unknown value a rather than 1. Figure 6 shows how the filter converges onto the right value for the parameter, despite a far off initial guess. It also shows that we are provided an estimate for y_2 despite it being a latent state.

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{a} \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ a \end{bmatrix} \tag{5}$$

$$z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ a \end{bmatrix} \tag{6}$$

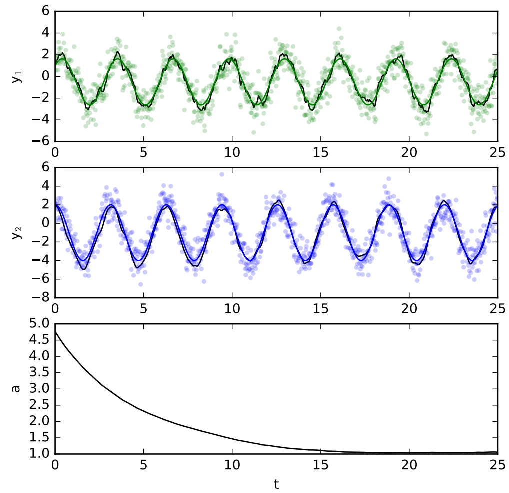

FIGURE 6. The filtered data and estimated parameter (black lines) plotted alongside the noisy observations and analytical solution.

2. Filtering of carbon cycle data