Obnoxious Facility Location Game On The Circle

Omer Kafri . Aviv Ghilai

August 26, 2017

Abstract

In this paper, we investigate the obnoxious facility location game on the circle. First, we prove an upper bound of $2(\sqrt{2}-1)$ on strategy-proof deterministic mechanisms. Then, we design a strategy-proof randomized mechanism, and discuss its properties.

1 Introduction

In a facility location game there are number of players located at different locations and a number of facilities needed to be located. Each player has a utility function based on his location and the facilities locations. A mechanism receives the players locations and decides the facilities locations while trying to maximize the social welfare: the sum of the players utilities. The players can lie about their location in order to increase their utility. A mechanism is called *Strategy-Proof* (SP) if each player can't increase his own utility by lying.

We consider the problem of obnoxious facility location games on the circle. Each player wants to be as far as possible from the facility, for example a garbage dump or antenna. We also can't take money from the players (in real life this can be because of ethical, legal or political reasons). Our goal is to characterize strategy-proof deterministic and randomized mechanisms and their approximation ratios.

1.1 Previous works

While maximizing the sum of players' utilities, if the players can be demanded payments, a truthful optimal mechanism is given by the Vickrey- Clarke-Groves (VCG) mechanism. But sometimes it isn't possible demanding money (e.g., in organs donation or in political decision making). researchers tried to come up with truthful but nontrivial (nondictorial) mechanisms without payments. Approximation mechanism design without money was first advocated by Procaccia and Tennenholts in facility location game on a real line. In facility location game, each agent wants to stay as close to the facility as possible, which means that the agents' preferences are single peaked. They characterized deterministic and randomized strategy-proof algorithms for the problem of one facility on the line, and showed an upper bound on deterministic algorithms of 2-appromixation. In obnoxious facility location game, players prefer to be away from the facility (such as a garbage dump). The obnoxious facility location game with maximizing social cost objective function was first investigated by Cheng et al. They designed a deterministic group strategy-proof 3-approximation mechanism and a randomized group strategy-proof $\frac{3}{2}$-approximation mechanism on a line. On a circle or a tree, they proposed a deterministic group strategy-proof 3-approximation mechanisms. Later, Ibara et al. proved that, in the line metric there does not exist strategy-proof mechanism if the range of mechanism contains more than two-facility location candidates and in general metric they completely characterized (group) strategy-proof mechanisms with exactly two candidates in the range.

1.2 Our results

We investigated the deterministic and randomized mechanisms in obnoxious facility game on the circle. We show that there is no deterministic $\alpha -$ appromixation strategy proof mechanism with $\alpha \gneqq 2(\sqrt{2}-1)$. We design a randomized strategy-proof mechanism and show that for 2 players it achieves a lower bound of $2(\sqrt{2}-1)$. We also show that as more players participate, the algorithm's approximation ratio becomes worse.

2 Settings & Preliminaries

The entire paper will focus on mechanisms that are choosing where to locate a single facility given the locations (may be misreported) of n players. All of the locations are on a circle with circumference 1.

Definition 1 (The circle) *A circle with circumference 1.*

Definition 2 (Distance between two points) *When we consider two points on the circle, they split the circle to two arcs. The distance between two points is defined to be the length of the shorter arc.*

The utility of each player is the distance between the player and the facility. The function any mechanism tries to maximize is minimum of all players' utilities, and we will call it the *social welfare*. We will use the term *opposite location* of a player to describe the location on the circle that is farthest from the player. A mechanism that guarantees a social welfare that is at least $\alpha \cdot \mathrm{OPT}$ for some constant $\alpha \in [0,1]$ will be called a $\alpha -$ appromixation.

3 Deterministic mechanisms

3.1 Trivial approach

The most trivial strategy proof mechanism will be one that chooses a fixed point on the circle without consideration of the players' locations. The mechanism is strategy proof, but might be infinitely far from the optimum.

3.2 Approximation upper bound

As we are working on a maximization problem, an upper bound m means that there is no $\alpha-$appromixation algorithm with $m \lneqq \alpha$.

Theorem 3 *There is no deterministic $\alpha-$ appromixation strategy proof mechanism with $\alpha \gneqq 2(\sqrt{2}-1)$.*

Proof: We will start with the proof for 2 players. Lets fix player-1 at some point on the circle¸ and let player-2 be on the circle such that the distance between them is $d = \frac{3-\sqrt{2}}{7}$. We will prove that player-2 can increase his utility by telling the mechanism that he is located at a distance of $2d$ to the same direction from player-1, therefore player-2 is motivated to lie. we will call the scenario in which both players tell the truth *truth scenario* and the scenario where player-1 tells the truth and player-2 lies that he is in distance of $2d$ *lie scenario*.

Lemma 4 *In this instance of the problem, whether player-2 lies or not¸ any deterministic $\alpha-$ appromixation mechanism must place the antenna on the long arc.*

Proof of Lemma 4:

Case1: In the *truth scenario*, optimal solution will be of course to put the antenna at the middle of the long arc¸ therefore it will have a social welfare of $\frac{1-d}{2} = \frac{4+\sqrt{2}}{14}$. Notice that the optimal location on the short arc is at the middle of it, with a social welfare of $\frac{d}{2}$. If$\alpha \gneqq 2(\sqrt{2}-1)$ it holds that $\alpha\frac{1-d}{2} \gneqq \frac{d}{2}$. It follows that any deterministic $\alpha-$appromixation algorithm must place the antenna on the long arc.

Case2: In the *lie scenario*, the optimal location for the antenna will again be on the middle of the long arc and have a social welfare of $\frac{1-2d}{2} = \frac{1+2\sqrt{2}}{14}$. The optimal social welfare on the short arc is d. If$\alpha \gneqq 2(\sqrt{2}-1)$ it holds that $\alpha\frac{1-2d}{2} \gneqq d$. Again, it follows that any deterministic $\alpha-$appromixation algorithm must place the antenna on the long arc.■

We will now make some calculations and see that the upper bound on the utility of player-2 in the *truth scenario* is lower than the lower bound on his utility in the *lie scenario*, therefore he is motivated to lie. In the *truth scenario* any deterministic $\alpha-$appromixation algorithm must place the antenna on the long arc, and not farther than $\frac{(1-\alpha)(1-d)}{2}$ from it's middle because the optimal location is the middle and $\frac{(1-\alpha)(1-d)}{2}$ is the difference between the optimal social welfare and the minimal social welfare. The best location player-2 can hope for is the one closest to it's opposite location, that is the point at distance $\frac{(1-\alpha)(1-d)}{2}$ from the middle of the long arc which is closest to the opposite location of player-2. This means that the utility of player-2 is at most $(1-\frac{\alpha}{2})(1-d)$. Now for the *lie scenario*, any deterministic $\alpha-$optimization algorithm must place the antenna on the long arc, and not farther than $\frac{(1-\alpha)(1-2d)}{2}$ from it's middle. Notice that the segment of possible locations for the antenna is between the opposite location of player-2 and it's false location. Therefore the utility of player-2 is at least $d+\frac{1-2d}{2}-\frac{(1-\alpha)(1-2d)}{2}=\frac{\alpha}{2}-\alpha d+d$ (the distance between player-2 and his false location + the distance between the false location and the optimal antenna location - the maximal distance of the antenna from the optimal location). If $\alpha \gneq 2(\sqrt{2}-1)$ then $\frac{\alpha}{2}-\alpha d+d>(1-\frac{\alpha}{2})(1-d)$ which means player-2 is motivated to lie and the algorithm is not strategy proof. If we have more than 2 players, the same proof holds when any other player is located at player-1's location. ■

Question Is there a deterministic algorithm that achieves an approximation of constant factor? or can we prove one does not exist?

4 Randomized mechanisms

4.1 Coin flipping mechanism

As mentioned before, the optimal location for the facility is at the middle of the longest arc. To make this mechanism strategy proof we add the following adjustment: flip a fair coin and choose a random arc (either the long arc or the short one), then locate the facility at the middle of this arc. The mechanism chooses the optimal solution in probability $\frac{1}{2}$, and therefore is at least $\frac{1}{2}$ approximation. In the case where both of the people are located at the same spot, at probability $\frac{1}{2}$ the facility would be located on their exact location and therefore we get a social welfare of 0. That's why the approximation ratio exactly$\frac{1}{2}$. Notice that this mechanism also strategy proof for n players when we uniformly choose a middle of a random arc, and achieves an approximation factor of $\frac{1}{n}$. We won't prove that the mechanism is strategy proof in this paper as we provide a better approximation algorithm and prove it to be strategy proof.

4.2 Weights mechanism

This mechanism is similar to the coin flipping mechanism mentioned before, but the arcs aren't chosen uniformly. Instead, the probability to choose an arc depends on it's length. An arc of length d would be chosen at probability d, and then the facility would be located at it's middle point. We need to prove it's strategy proof and analyze the approximation factor. We will show that it is strategy proof for n player, and achieves an approximation of $2(\sqrt{2}-1)$ for two players.

4.3 Weights mechanism approximation factor

We'll focus on a scenario where the algorithm is far from optimal. It would give an upper bound on the algorithm performance for n players, and it would be tight for 2 players. Consider the following case:

Let d be the length of the longest arc on the circle, and p_0, p_1 be the players creating the arc. We spread the rest of the $n-2$ players uniformly on the complementary arc (which is $1-d$ long). Lets analyze the social welfare of the algorithm's output:

$$SW = d \cdot \frac{d}{2} + (1-d) \cdot \frac{1-d}{2(n-1)} = \frac{nd^2 - 2d + 1}{2(n-1)}$$

The optimal solution would be locating the facility on the middle of (p_0, p_1), and then the SW is $\frac{d}{2}$. Lets calculate the ratio:

$$f_{ratio}(d) = \frac{\frac{nd^2-2d+1}{2(n-1)}}{\frac{d}{2}} = \frac{nd^2 - 2d + 1}{d(n-1)}$$

We want to find a minimum to this function (a small ratio means the algorithm is bad). Lets find the derivative:

$$f'_{ratio}(d) = \frac{nd^2 - 1}{(n-1)d^2} = 0$$

Notice the denominator is always bigger than 0. therefore we got $d = \frac{1}{\sqrt{n}}$. Finding the ratio in this case:

$$f_{ratio}(\frac{1}{\sqrt{n}}) = 2(\sqrt{2}-1)$$

This is an upper bound on the algorithm. Notice for example that when $n = 9$, we got an upper bound of $\frac{1}{2} - \textit{appromixation}$. For $n = 2$, we found the worst possible scenario for the algorithm (because there are no more players to locate) and therefore got a tight approximation bound of $2(\sqrt{2}-1) \sim 0.828$. A question rises - whether this bound is tight for n players also or can you locate the players differently to make the algorithm behave even worse. We won't discuss this question any further in this article.

4.4 Weights mechanism is strategy proof

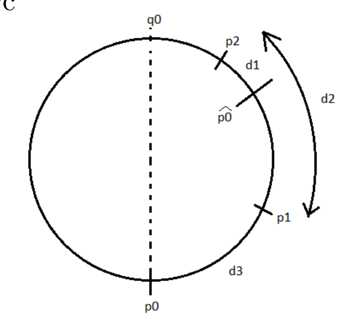
We now provide a proof that the algorithm is SP. This means the players can't benefit from misreporting their location. Let p_0 be the player location (located at the bottom of the circle for convenience), q_0 the opposite location of p_0, $\hat{p_0}$ is the player's misreported location. Let p_1, p_2 be the location of the players closest to $\hat{q_0}$ clockwise and counter-clockwise respectively (not including p_0 itself).

Lemma 5 *For each location on the circle $\hat{p_0}$, there exists a location of a player p_i, such that if player p_0 would say his location is p_i, his utility would be bigger than saying his location is $\hat{p_0}$. This means the optional places for a lie are only the places of the players.*

Proof: We split the proof into cases. in each case we look only on the arc that contains the lie $\hat{p_0}$, and analyze the player's utility in the case that the arc is chosen by the algorithm. Then we show that one of the arc's end points are at least as good places to lie as any point on the arc itself.

case 1: $\hat{p_0}$ lies on a different arc than p_0 and q_0.

$d_1 = dist(p_2, \hat{p_0})$, $d_2 = dist(p_1, p_2)$, and $d3 = dist(p_0, p_1)$. Given the arc (p_1, p_2) is chosen, The utility of the player is:



$$u_0 = \frac{d1}{d2} \cdot (d_3 + d_2 - \frac{d1}{2}) + (1 - \frac{d1}{d2}) \cdot (d3 + \frac{d2 - d1}{2}) =$$

$$= \frac{d1}{d2} \cdot \frac{d2}{2} + d_3 + \frac{d2 - d1}{2} =$$

$$= d_3 + \frac{d2}{2}$$

Therefore d_1 has no effect on the player's utility, which means the utility would be the same if the player said his location is p_1 or p_2.
The rest of the cases would be shown at the appendix to keep the flow of the article.

■

Theorem 6 *p_0 is the best location for the player to report, meaning the algorithm is* SP.

Proof:

We showed that the only possible places for a lie are the players' locations. Now we need to show that the best location of all is p_0. Notice that if the player lies and says his location is p_i, $i \neq 0$, then it is the same as if he didn't participate at all. That's because the algorithm doesn't care if more than one person is in the same spot, because the *Social Welfare* function is based only on the minimum distance, and not the sum of distances, for example. Therefore, there exists some constant $C = u_{0,\hat{p_0}=p_i}$ for every $i \neq 0$. We just need to show that $u_{0,\hat{p_0}=p_0} > C$.

Notice case 3 and 4 are the only two cases where the player could say his location is his real location. In those cases, if the player said his location is p_1 or p_2, his utility would be C. using the notations of those cases and assuming the arc (p_1, p_2) was chosen, his utility would be $d_3 - \frac{d_2}{2}$. In addition, if the player hadn't lie his utility would be $\frac{d_3^2}{d_2} - d_3 + \frac{d_2}{2}$. Let's see the difference between these values:

$$u_{0,\hat{p_0}=p_0} - C = \frac{d_3^2}{d_2} - d_3 + \frac{d_2}{2} - (d_3 - \frac{d_2}{2}) =$$

$$= \frac{d_3^2 - 2d_3 + d_2^2}{d_2} = \frac{(d_3 - d_2)^2}{d_2} \geq 0$$

We showed the only possible places for a misreported location are the players' locations, and in each of them the value is a constant C. We also showed that the utility if the player doesn't lie is larger than C. Therefore, the algorithm is *SP*. ■

5 Summary

We considered the mechanisms that solve the problem of obnoxious facility location games on the circle without using money. We have shown an upper bound of $2(\sqrt{2} - 1)$ on the approximation factor of deterministic mechanisms. The question whether there is a deterministic mechanisms that achieves a constant approximation factor remains unanswered, we believe that there is no such mechanism. Later, we presented a strategy-proof random mechanism and proved that it is a $2(\sqrt{2} - 1) - $ *approximation* when there are only 2 players.We think that the proof of the upper bound for deterministic mechanisms can be adjusted to random mechanisms as well. If so, it will mean that we have found the best possible random SP mechanism for 2 players. Another question is what upper bound applies to random mechanisms with more players? We have showen that our mechanism become worse as the amount of players grows .

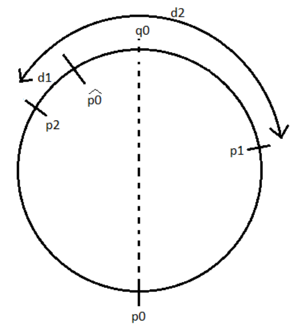
6 Appendix

We now show the rest of the proof of Lemma 1. **Proof:** We continue with the cases:

case 2: $\hat{p_0}$ lies on a different arc than p_0 but on the same arc as q_0.

Assume W.L.G $dist(p_2, q_0) < dist(p_1, q_0)$, and denote $d_3 = dist(p_2, q_0)$. We'll split the proof for this case into two sub cases: $d_1 \leq 2 \cdot d_3$ or $d_1 > 2 \cdot d_3$. In the first case, the utility of the player is:

$$u_0 = \frac{d_1}{d_2} \cdot (\frac{1}{2} - (d_3 - \frac{d_1}{2})) + (1 - \frac{d_1}{d_2}) \cdot (\frac{1}{2} - (\frac{d_2 - d_1}{2} - (d_3 - d_1))) =$$



$$= \frac{1}{d_2} d_1^2 - 2\frac{d_3}{d_2} d_1 + \frac{1 - d_2}{2} + d_3$$

We look for a maximum for the equation. The equation is a U-shape parabola, which means the maximum values are at the edges. Let's check the utility:

$u_{0,d_1=0} = \frac{1-d_2}{2} + d_3$

$u_{0,d_1=2d_3} = \frac{(2d_3)^2}{d_2} - \frac{(2d_3)^2}{d_2} + \frac{1-d_2}{2} + d_3 = \frac{1-d_2}{2} + d_3$

Therefore if the player says his location is the same as p_2, his utility would be biggest.

Now we'll solve the second case: $d_1 > 2 \cdot d_3$. The player's utility is:

$$u_0 = \frac{d_1}{d_2} \cdot (\frac{1}{2} - (\frac{d_1}{2} - d_3)) + (1 - \frac{d_1}{d_2}) \cdot (\frac{1}{2} - (\frac{d_2 - d_1}{2} - (d_3 - d_1))) =$$

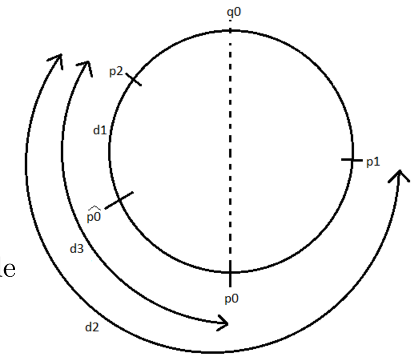
$$2d_1 + \frac{1}{2} - \frac{d_2}{2} - d_3 =$$

This is a linear equation in d_1, which means it's largest value is when $d_1 = d_2$. The player's best lie is saying his location is the same as p_1.

case 3: $\hat{p_0}$ lies on a the same arc as p_0 but on a different arc than q_0.

We change notations and now p_1 is the closest player to $\hat{p_0}$ counter-clockwise, and p_2 clockwise. Assume W.L.G that $d_3 > \frac{d_2}{2}$ which means p_2 is farther from p_0 than p_1. We need to separate the calculation into two cases: $d_1 \leq 2d_3 - d_2$ and $d_1 > 2d_3 - d_2$.

$d_1 \leq 2d_3 - d_2$:



$$u_0 = \frac{d_1}{d_2} \cdot (d_3 - \frac{d_1}{2}) + (1 - \frac{d_1}{d_2}) \cdot (d_3 - d_1 - (\frac{d_2 - d_1}{2})) =$$

$$= d_3 - \frac{d_2}{2}$$

The utility doesn't depend on d_1, and therefore p_2 is one of the favorable places.

$d_1 > 2d_3 - d_2$:

$$u_0 = \frac{d_1}{d_2} \cdot (d_3 - \frac{d_1}{2}) + (1 - \frac{d_1}{d_2}) \cdot (\frac{d_1}{2} + \frac{d_2}{2} - d_3) =$$

$$= -\frac{1}{d_2} d_1^2 + \frac{2d_3}{d_2} d_1 + \frac{d_2}{2} - d_3$$

The equation is $\cap$-shaped parabola, we look for the maximum:

$$u_0^{'} = -\frac{2}{d_2} d_1 + \frac{2d_3}{d_2} = 0$$

$$d_1 = d_3$$

When $d_1 = d_3$, the player's location is the true location: p_0. Let's see what is the utility in that place (It would help us at the second part of the proof).

$$u_{0,d_1=d_3} = -\frac{1}{d_2} d_3^2 + \frac{2d_3^2}{d_2} + \frac{d_2}{2} - d_3$$

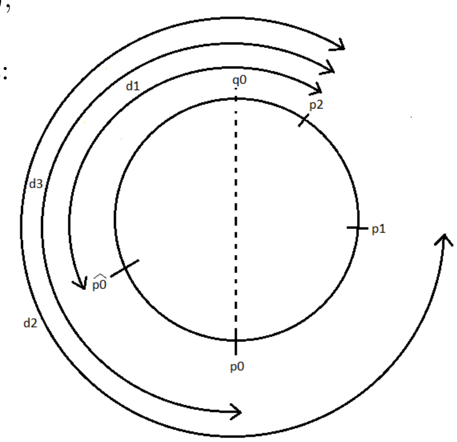
$$= \frac{d_3^2}{d_2} - d_3 + \frac{d_2}{2}$$

case 4: $\hat{p_0}$ lies on a the same arc as p_0 and q_0.

One can see that either $p_1 \leq \frac{1}{2}$ or $p_2 \leq \frac{1}{2}$, because the whole circle perimeter is 1. W.L.G assume $p_1 \leq \frac{1}{2}$. because $\hat{q_0}$is on the same arc as q_0, $dist(p_0, p_2) = d_3 \geq \frac{1}{2}$.

We split the proof of this case into three sub cases: $d_1 \leq 2(d_3 - \frac{1}{2})$, $2(d_3 - \frac{1}{2}) < d_1 \leq 2d_3 - d_2$ and $d_1 < 2d_3 - d_2$.

1. $d_1 \leq 2(d_3 - \frac{1}{2})$: The player's utility when the chosen arc is (p_1, p_2) is:



$$u_0 = \frac{d_1}{d_2} \cdot (1 - d_3 + \frac{d_1}{2}) + (1 - \frac{d_1}{d_2}) \cdot (d_3 - \frac{d_1}{2} - \frac{d_2}{2}) =$$

$$= \frac{1}{d_2} d_1^2 + \frac{1 - 2d_3}{d_2} d_1 + d_3 - \frac{d_2}{2}$$

The maximum is on the edges:

$u_{0,d_1=0} = d_3 - \frac{d_2}{2}$

$u_{0,d_1=2(d_3-\frac{1}{2})} = \frac{(2d_3-1)^2}{d_2} - \frac{(2d_3-1)^2}{d_2} + d_3 - \frac{d_2}{2} = d_3 - \frac{d_2}{2}$

And therefore p_2 is one of the favorable locations.

2. $2(d_3 - \frac{1}{2}) < d_1 \leq 2d_3 - d_2$: The player's utility when the chosen arc is (p_1, p_2) is:

$$u_0 = \frac{d_1}{d_2} \cdot (d_3 - \frac{d_1}{2}) + (1 - \frac{d_1}{d_2}) \cdot (d_3 - \frac{d_1}{2} - \frac{d_2}{2}) = d_3 - \frac{d_2}{2}$$

The utility doesn't depend on d_1. We've seen that $u_{0,d_1=2(d_3-\frac{1}{2})} = u_{0,d_1=0}$, and therefore p_2 is still one of the favorable locations.

3. $d_1 > 2d_3 - d_2$: The player's utility when the chosen arc is (p_1, p_2) is:

$$u_0 = \frac{d_1}{d_2} \cdot (d_3 - \frac{d_1}{2}) + (1 - \frac{d_1}{d_2}) \cdot (\frac{d_1 + d_2}{2} - d_3) = -\frac{1}{d_2} d_1^2 + \frac{2d_3}{d_2} d_1 - d_3 + \frac{d_2}{2}$$

The equation is $\cap$-shaped parabola, we look for the maximum:

$$u_0^{'} = -2\frac{1}{d_2} d_1 + \frac{2d_3}{d_2} = 0 \rightarrow d_1 = d_3$$

Again, the best location in this case is p_0. The value of the function at this location is:

$$u_{0,d_1=d_3} = \frac{d_3^2}{d_2} - d_3 + \frac{d_2}{2}$$

■