NEW LOWER BOUNDS ON THE MINIMUM SINGULAR VALUE OF A MATRIX*

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Abstract. The study of constraining the eigenvalues of the sum of two symmetric matrices, say $P + Q$, in terms of the eigenvalues of P and Q, has a long tradition. It is closely related to estimating a lower bound on the minimum singular value of a matrix, which has been discussed by a great number of authors. To our knowledge, no study has yielded a positive lower bound on the minimum eigenvalue, $\lambda_{\min}(P + Q)$, when $P + Q$ is symmetric positive definite with P and Q singular positive semi-definite. We derive two new lower bounds on $\lambda_{\min}(P + Q)$ in terms of the minimum positive eigenvalues of P and Q. The bounds take into account geometric information by utilizing the Friedrichs angles between certain subspaces. The basic result is when P and Q are two non-zero singular positive semi-definite matrices such that $P + Q$ is non-singular, then $\lambda_{\min}(P + Q) \geq (1 - \cos\theta_F) \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$, where $\lambda_{\min}$ represents the minimum positive eigenvalue of the matrix, and θ_F is the Friedrichs angle between the range spaces of P and Q. Such estimates lead to new lower bounds on the minimum singular value of full rank 1×2, 2×1, and 2×2 block matrices. Some examples provided in this paper further highlight the simplicity of applying the results in comparison to some existing lower bounds.

Key words. block matrix, eigenvalue, lower bound, Friedrichs angle, real symmetric matrices, singular value

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1. Introduction. The spectral problem of a symmetric matrix sum estimates the eigenvalues of a sum of two symmetric matrices $P + Q$, in terms of the eigenvalues of P and Q. Fundamental results, like Weyl's inequality in [25, p. 239], and several other works collected in [15], have addressed this problem. Another substantial contribution is Horn's conjecture proved in [31, 32]. The present work is focused on the case when P and Q are symmetric positive semi-definite (PSD) matrices, which impacts numerous areas-such as computational economics, graph theory, perturbation theory, semi-definite programming, spectrum of self-adjoint operators, among others. As variance-covariance matrices are PSD, this problem appears in statistics, and more recently in statistical machine learning and spectral methods for data science, discussed in [4] and [13], respectively.

Singular values have been investigated for more than a century. For a square real matrix, its minimum singular value is less than or equal to its absolute minimum eigenvalue. Thus, formulation of a lower bound for the minimum singular value is an influential problem appearing in several studies including the condition number estimates of a matrix, resonant frequencies, population dynamics, principal component analysis, etc. Since singular values of a matrix are square-root of eigenvalues of its corresponding gram matrix, the singular values of a general block matrix are associated to the spectral problem of a sum of symmetric matrices. Although a myriad of research has been done on these topics, however, when the symmetric matrices are both singular PSD, we could not find a result providing a positive lower bound even if their sum is non-singular.

In practice, we often come across symmetric positive definite (SPD) matrices represented as a sum of two singular PSD matrices. To illustrate, let us estimate the

minimum singular value of a full rank block column matrix, say $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ so that A_1 and A_2 are rank deficient. This problem is equivalent to finding the minimum eigenvalue of $A^T A = A_1^T A_1 + A_2^T A_2$, an SPD matrix which is a sum of two singular PSD matrices. We derive a positive lower bound on the minimum singular value of A in terms of the minimum positive singular values of A_1 and A_2 in Corollary 3.7.

In this work, we desire a positive lower bound on the minimum eigenvalue of an SPD matrix $P + Q$, where $P, Q \in \mathbb{R}^{n \times n}$ are PSD matrices. Two positive lower bounds on the smallest eigenvalue of $P + Q$, framed in terms of the smallest positive eigenvalues of P and Q, are presented in Theorems 3.1 and 3.5. These estimates of the minimum eigenvalue employ the Friedrichs angle between certain subspaces, i.e., some principal angle between them, as shown in Proposition 2.10. A notable application of principal angles is canonical correlations of matrix pairs given in [21], and in many other areas, namely eigenspaces, functional analysis, matrix perturbation theory, statistics, etc., are found in [33, 44, 16, 14], respectively. The spectral problem of a sum of two PSD matrices is closely related to the first aforementioned application of canonical correlations, and its dependence upon the Friedrichs angle elucidates geometric aspects of spectral theory. Moreover, the two new lower bounds lead to useful outcomes when applied to a 2×2 non-singular block matrix M as:

$$M^T M = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^T A & A^T B \\ B^T A & B^T B \end{bmatrix} + \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D \end{bmatrix}.$$

Here $M^T M$ is a full rank matrix expressed as a sum of two PSD matrices. Therefore, the above expression admits a lower bound on the minimum singular value of M, in terms of the minimum positive singular values of its blocks A, B, C, and D (Theorem 3.9). Finally, the above expression and MM^T are used again to get two lower bounds on other singular values of M in Theorem 3.13.

We now summarize the notation used in this paper. Let $A \in \mathbb{R}^{n \times n}$ be a PSD matrix and let $\Lambda(A)$ denote the spectrum of A, that is, the set of eigenvalue of A. If $r = \mathrm{rank}(A)$, then its eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A) \geq 0$, are such that $\lambda_r > 0$ and $\lambda_i = 0$ for all $r + 1 \leq i \leq n$. Moreover, we define the minimum positive eigenvalue of A as

$$\lambda_{\min}(A) := \begin{cases} \lambda_r(A), & \text{if } A \neq O, \\ \infty, & \text{if } A = O. \end{cases}$$

For a matrix $A \in \mathbb{R}^{m \times n}$, $\mathrm{rank}(A) = \mathrm{rank}(A^T A) = \mathrm{rank}(AA^T)$. The set of singular values of A is denoted by $\sigma(A)$. Let $r = \mathrm{rank}(A)$, then its singular values $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_{\min(m,n)}(A)$ are such that $\sigma_r(A) > 0$ and $\sigma_i = 0$ for all $r + 1 \leq i \leq \min(m, n)$. Again, we define the minimum positive singular value of A as

$$\sigma_{\min}(A) := \begin{cases} \sigma_r(A), & \text{if } A \neq O, \\ \infty, & \text{if } A = O. \end{cases}$$

The above expressions for $\lambda_{\min}$ and $\sigma_{\min}$ are defined for the convenience of notation for results derived in section 3 and their value is set as infinity for zero matrices to ignore the zeros while calculating the minimum as it was required by the formulations derived for zero matrices. The 2-norm of a vector is denoted by $|\cdot|$, whereas $\|\cdot\|$ represents 2-norm of a matrix. The range space and null space of A are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. For a scalar k, the constant vector of n components

equal to k is denoted by $\mathbf{k}_n$. Also, for $A_i \in \mathbb{R}^{n_i \times n_i}$ with $n_i \in \mathbb{N}$, $\mathrm{diag}(A_1, A_2, \ldots, A_k)$ denotes a square diagonal matrix of size $\sum_{i=1}^{k} n_i$ with diagonal blocks A_i, for $1 \leq i \leq k$. Let V be an inner-product space and $U \subseteq V$ be a subspace of V, then $U^{\perp} = \{v \in V \mid \langle v, u \rangle = 0, \ \forall\, u \in U\}$ is the orthogonal complement of U in V, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

There is an abundance of results related to these problems in the literature. We attempt to summarize some of the existing focal results related to the ones developed in section 3.

1.1. Minimum eigenvalue of sum of two PSD matrices. The problem of estimating a lower bound on the minimum eigenvalue of sum of two PSD matrices has been investigated for many years. One of the most fundamental results is Weyl's inequalities given by Theorem 2.1, which estimates the eigenvalues of a sum of symmetric matrices. R. Bhatia and F. Kittaneh established a lower bound on eigenvalues of sum of two PSD matrices (Theorem 2.3). They posed the question of a generalization of arithmetic-geometric mean inequality in [10], which stated that for two PSD matrices $P, Q \in \mathbb{R}^{n \times n}$, $\lambda_j(P + Q) \geq 2\sqrt{\sigma_j(PQ)}$, for all $j = 1, 2, \ldots, n$. It was proved by S. Drury in [17]. However, all these results give a trivial lower bound for the case in which $P + Q$ is non-singular, and both P and Q are rank deficient. Some additional properties for sum of two symmetric matrices are listed in [35].

1.2. Spectrum of saddle point matrices. One of the most commonly seen 2×2 block matrices of the form $M = \begin{bmatrix} A & B_1 \\ B_2^T & -C \end{bmatrix}$, where $A \in \mathbb{R}^{m \times m}$ and one or both of $B_1, B_2 \in \mathbb{R}^{m \times n}$ are non-zero, is called a saddle point matrix. See [5], for a good survey of results on saddle point matrices. In particular, see Theorem 3.5 in [5, p. 21], which estimates the spectrum for the case when A is SPD, $B_1 = B_2$ is full rank, and $C = O$. A noteworthy improvement was presented in the form of Theorem 1 in [2, p. 341] with a positive or negative semi-definite matrix C; however, it can be difficult to estimate the parameters defined in this theorem while being applied to some gram matrix of 2×2 block matrix. Another applicable result for the spectrum of a preconditioned saddle point matrix can be found in [42].

To formulate the spectrum of a more generalized saddle point matrix, several advancements have been considered, such as defining $B = B_1^T = -B_2$ in [6, 7, 41, 1]. Another step forward was to have a symmetric indefinite leading block A. The first of such case was proved in [22], by imposing the condition that A is SPD on $\mathcal{N}(B)$, which was eliminated in [3]. Recently, in [26], A has been considered to be a non-symmetric matrix with a positive definite symmetric part with $C = O$, which originates from discretized Navier-Stokes equations.

1.3. Lower bound on the minimum singular value. Several techniques are reported in the literature for formulating a lower bound on the minimum singular value of a particular type of matrices; however, we attempt to mention seminal contributions to this problem for a general non-singular matrix. An initial result for the special case of diagonally dominant matrices is derived in [45], and for a non-singular matrix a consequential approach is Gerschgorin-type lower bounds formulated in [40, 27]. The results evolved gradually into several stronger versions, as seen in [28, 36, 29, 50]. Also, [24] devised a lower bound in terms of the determinant, the 2-norms of the rows, and columns of the matrix. Some later advancements of this result include [47, 49, 37] and the references therein.

It is well-known that for a 2×2 block matrix, the maximum singular value is bounded above by 2-norm of the matrix consisting of 2-norms of its blocks, see

135 Theorem 1(f) in [43, p. 2630]. Thus, for $\mathcal{X} \in \mathbb{R}^{n \times n}$,

136
$$\mathcal{X} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad \|\mathcal{X}\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

137 Since $\sigma_{\min}(\mathcal{X}) = \sigma_{\max}(\mathcal{X}^{-1})^{-1}$, so on applying this result to $\mathcal{X}^{-1}$ calculated in
138 terms of its blocks, an estimate of $\sigma_{\min}(\mathcal{X})$ is obtained. One drawback of this method
139 is that the expression for $\mathcal{X}^{-1}$ can be quite problematic.

140 This paper is organized as follows. In section 2, some important results that
141 will be used to prove the main results are listed. In section 3, we prove some new
142 positive lower bounds. In addition, some examples and special cases for these results
143 are discussed. Finally, some ideas for the future work are mentioned in section 4.

144 **2. Existing Results.** This section reviews some essential facts about eigen-
145 values of PSD matrices, orthogonal projections, and angles between subspaces; the
146 Friedrichs angle will play a vital role in forming new estimates of the minimum eigen-
147 value of an SPD matrix.

148 THEOREM 2.1. *(Weyl's inequalities [25, p. 239]) Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric*
149 *matrices, then*

150
$$\lambda_j(A) + \lambda_n(B) \leq \lambda_j(A + B) \leq \lambda_j(A) + \lambda_1(B), \quad j = 1, 2 \ldots, n.$$

151 LEMMA 2.2. *([48, p. 57]) Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times m}$. Then AB and BA have*
152 *the same non-zero eigenvalues (multiplicities counted). If $m = n$, then eigenvalues of*
153 *AB and BA are the same.*

154 THEOREM 2.3. *([8, p. 904], [9]) Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{\ell \times n}$, then*

155
$$2\sigma_j(AB^T) \leq \lambda_j(A^T A + B^T B), \qquad j = 1, 2, \ldots, \min(\ell, m, n).$$

156 THEOREM 2.4. *([23]) Let $A_{ij} \in \mathbb{R}^{m \times n}$ for $i, j = 1, 2$, and*

157
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

158 *then*
159
$$\sigma_j(A) \geq \sqrt{2\sigma_j\left(A_{11}A_{12}^T + A_{21}A_{22}^T\right)}, \qquad j = 1, 2, \ldots, \min(m, n).$$

160 LEMMA 2.5. *([38, p. 409]) Let V be an inner-product space and U_1, U_2 be two*
161 *subspaces of V, then $U_1^{\perp} + U_2^{\perp} = (U_1 \cap U_2)^{\perp}$.*

162 DEFINITION 2.6. *(Principal angles [19]) Let $U, V \subseteq \mathbb{R}^n$ be subspaces with $p =$*
163 *$\dim(U) \geq \dim(V) = q \geq 1$. The principal angles $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_q \leq \frac{\pi}{2}$ between*
164 *U and V are recursively defined for $k = 1, 2, \ldots, q$ by*

165
$$\cos(\theta_k) = \max_{\substack{u \in U \\ |u|=1}} \max_{\substack{v \in V \\ |v|=1}} |u^T v| =: u_k^T v_k,$$

166 *subject to the constraints*

167
$$u_i^T u = 0, \quad v_i^T v = 0, \quad i = 1, 2, \ldots, k - 1.$$

168 *The vectors $\{u_1, \ldots, u_q\}$, $\{v_1, \ldots, v_q\}$ are called principal vectors of the pair of spaces.*
169 *The angle θ_1 is also called the minimal principal angle. Also, $\theta_1 = 0$ if and only if*
170 *$U \cap V \neq \{0\}$, and $\theta_1 = \frac{\pi}{2}$ if and only if $U \perp V$.*

DEFINITION 2.7. *(Orthogonal projection [20]) Let $U \subseteq \mathbb{R}^n$ be a subspace. A matrix $P \in \mathbb{R}^{n \times n}$ is the orthogonal projection onto U if $\mathcal{R}(P) = U$, $P^2 = P$, and $P^T = P$. Moreover, if $x \in \mathbb{R}^n$, then $Px \in U$ and $(I - P)x \in U^\perp$.*

THEOREM 2.8. *([46, 19]) Let $U, V \subseteq \mathbb{R}^n$ be subspaces with $p = \dim(U) \geq \dim(V) = q \geq 1$, θ_i be the principal angles between U and V, and r be the number of angles θ_i such that $0 < \theta_i < \frac{\pi}{2}$. Let $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{n \times n}$ be orthogonal projections onto $\mathcal{R}(U), \mathcal{R}(V)$, respectively, then the set of singular values of $\mathcal{P} \pm \mathcal{Q}$ are*

1. $\sigma(\mathcal{P} + \mathcal{Q}) = \{\mathbf{2}_k, 1 \pm \cos(\theta_{k+i})(i = 1, \ldots, r), \mathbf{1}_{n_1 + n_2}, \mathbf{0}_{n_3}\}$,
2. $\sigma(\mathcal{P} - \mathcal{Q}) = \{\mathbf{1}_{n_1 + n_2}, \mathbf{\sin(\theta_{k+i})}_2(i = 1, \ldots, r), \mathbf{0}_{k + n_3}\}$,

where $k = \dim(U \cap V)$, $n_1 = \dim(U \cap V^\perp) = p - k - r$, $n_2 = \dim(U^\perp \cap V) = q - k - r$, and $n_3 = \dim(U^\perp \cap V^\perp) = n - p - q + k$.

DEFINITION 2.9. *(Friedrichs angle [16]) The angle $\theta_F \in \left[0, \frac{\pi}{2}\right]$ between subspaces $U, V \subseteq \mathbb{R}^n$, whose cosine is defined by*

$$\cos \theta_F := \sup \left\{ |\langle x, y \rangle| \mid x \in U \cap (U \cap V)^\perp, |x| \leq 1, \ y \in V \cap (U \cap V)^\perp, |y| \leq 1 \right\},$$

is called the Friedrichs angle.

PROPOSITION 2.10. *Let $U, V \subseteq \mathbb{R}^n$ be subspaces, as defined in Theorem 2.8. Let $\mathcal{P}, \mathcal{Q}$ be orthogonal projections onto U and V, respectively, and let θ_F denote the Friedrichs angle between subspaces U and V, then the following results hold.*

1. $\theta_F = \theta_1 \left(U \cap (U \cap V)^\perp, V \cap (U \cap V)^\perp\right)$.
2. $\theta_F = \theta_1(U, V)$ *if and only if* $U \cap V = \{0\}$.
3. $\theta_F = \theta_1 \left(U, V \cap (U \cap V)^\perp\right) = \theta_1 \left(U \cap (U \cap V)^\perp, V\right)$.
4. $\cos \theta_F = \|\mathcal{P}\mathcal{Q} - \mathcal{P}_{U \cap V}\|$.
5. $\theta_F = \theta_{k+1}(U, V)$.

See [16, p. 110] for the first four properties of the above Proposition. A proof of Property 4 is also provided in [12, p. 1430], which along with $\|\mathcal{P}\mathcal{Q} - \mathcal{P}_{U \cap V}\| = \cos \theta_{k+1}$, proved in [19, p. 245], implies Property 5. Several more interesting results on the Friedrichs angle are stated in [18, p. 242].

3. Main results. As discussed in section 1, a positive lower bound on the minimum eigenvalue of a non-singular sum of two PSD matrices, say $P, Q \in \mathbb{R}^{n \times n}$, is the key tool for the development of a positive lower bound on the minimum singular value of some full rank block matrices. Note that $\mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$ when $P + Q$ is SPD, however, the range spaces of P and Q may intersect. Let $k = \dim(\mathcal{R}(P) \cap \mathcal{R}(Q))$, then the first k principal angles between $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ vanish: $\theta_i = 0$ for $i = 1, 2, \ldots, k$. Therefore, if θ_{k+1} exists then it could contribute in estimating the minimum eigenvalue of $P + Q$ in terms of the minimum positive eigenvalues of P and Q. Even when θ_{k+1} does not exist, this idea serves as a motivation for the following theorem for a pair of two PSD matrices with a non-singular sum.

THEOREM 3.1. *Let $P, Q \in \mathbb{R}^{n \times n}$ be PSD matrices of rank $p, q \leq n$, respectively, so that $P + Q$ is non-singular. Then*

$$\lambda_{\min}(P + Q) \geq c(P, Q) \min \{\lambda_{\min}(P), \lambda_{\min}(Q)\},$$

where $c(P, Q)$ is defined by

(3.1)
$$c(P, Q) = \begin{cases} 2, & r = 0, \ p + q = 2n, \\ 1, & r = 0, \ p + q < 2n, \\ 1 - \cos(\theta_{k+1}), & r > 0, \end{cases}$$

213   *where $k = p + q - n$, $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{\min(p,q)} \leq \frac{\pi}{2}$ represent the principal angles*
214   *between $\mathcal{R}(P), \mathcal{R}(Q) \subseteq \mathbb{R}^n$, and r is the number of angles θ_i so that $0 < \theta_i < \frac{\pi}{2}$, for*
215   *$1 \leq i \leq \min(p, q)$.*

216   *Proof.* Since P, Q are PSD matrices, there exist matrices $\mathcal{A}_1 \in \mathbb{R}^{p \times n}$ and $\mathcal{A}_2 \in$
217   $\mathbb{R}^{q \times n}$, so that $P = \mathcal{A}_1^T \mathcal{A}_1$, $Q = \mathcal{A}_2^T \mathcal{A}_2$. Moreover, $\mathcal{N}(P) = \mathcal{N}(\mathcal{A}_1)$ and $\mathcal{N}(Q) =$
218   $\mathcal{N}(\mathcal{A}_2)$. Define $M_1 := \mathcal{R}(P) = \mathcal{R}(P^T) = \mathcal{N}(P)^\perp = \mathcal{N}(\mathcal{A}_1)^\perp$ and similarly define
219   $M_2 := \mathcal{R}(Q) = \mathcal{N}(\mathcal{A}_2)^\perp$. Let $P_i \in \mathbb{R}^{n \times n}$ be the orthogonal projection on M_i, for
220   $i = 1, 2$. Therefore, $\mathcal{R}(P_i) = M_i$ and $\mathcal{R}(I - P_i) = M_i^\perp = \mathcal{N}(\mathcal{A}_i)$ for $i = 1, 2$. The
221   variational characterization of the smallest eigenvalue of a symmetric matrix implies

222
223
$$\lambda_{\min}(P + Q) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^T \mathcal{A}_1^T \mathcal{A}_1 x + x^T \mathcal{A}_2^T \mathcal{A}_2 x}{|x|^2}.$$

224   Since any $x \in \mathbb{R}^n \setminus \{0\}$ can be represented as $x = (I - P_i)x + P_i x$, for $i = 1, 2$, thus

225
$$x^T \mathcal{A}_i^T \mathcal{A}_i x = [\mathcal{A}_i((I - P_i)x + P_i x)]^T [\mathcal{A}_i((I - P_i)x + P_i x)]$$
226   (as $(I - P_i)x \in \mathcal{N}(\mathcal{A}_i)$)
$$= (\mathcal{A}_i P_i x)^T (\mathcal{A}_i P_i x)$$
227
228
$$= (P_i x)^T \mathcal{A}_i^T \mathcal{A}_i (P_i x),$$

229   therefore,

230   (3.2) $$\lambda_{\min}(P + Q) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{(P_1 x)^T \mathcal{A}_1^T \mathcal{A}_1 (P_1 x) + (P_2 x)^T \mathcal{A}_2^T \mathcal{A}_2 (P_2 x)}{|x|^2}.$$

231   Note that the minimum positive eigenvalue of $\mathcal{A}_i^T \mathcal{A}_i$ is identified by the variational
232   characterization as follows,

233
$$\lambda_{\min}(\mathcal{A}_i^T \mathcal{A}_i) = \inf_{x \in \mathcal{N}(\mathcal{A}_i)^\perp} \frac{x^T \mathcal{A}_i^T \mathcal{A}_i x}{|x|^2}.$$

234   Since for any $x \in \mathbb{R}^n \setminus \{0\}$, $P_i x \in M_i = \mathcal{N}(\mathcal{A}_i)^\perp$, therefore the above expression gives

235
$$(P_i x)^T \mathcal{A}_i^T \mathcal{A}_i (P_i x) \geq \lambda_{\min}(\mathcal{A}_i^T \mathcal{A}_i) |P_i x|^2,$$

236   hence (3.2) provides the following estimate

237
$$\lambda_{\min}(P + Q) \geq \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\lambda_{\min}(\mathcal{A}_1^T \mathcal{A}_1) |P_1 x|^2 + \lambda_{\min}(\mathcal{A}_2^T \mathcal{A}_2) |P_2 x|^2}{|x|^2}$$
238   (3.3)
$$= \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\lambda_{\min}(P) |P_1 x|^2 + \lambda_{\min}(Q) |P_2 x|^2}{|x|^2}$$
239
$$\geq \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\} \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_1 x|^2 + |P_2 x|^2}{|x|^2}$$
240   (3.4)
241
$$=: \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\} \inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x),$$

242   where $\Delta(x) := \dfrac{|P_1 x|^2 + |P_2 x|^2}{|x|^2}$, for $x \in \mathbb{R}^n \setminus \{0\}$. Since the parallelogram identity for
243   inner-product spaces states that

244   (3.5) $$|P_1 x|^2 + |P_2 x|^2 = \frac{1}{2} \left[|(P_1 + P_2)x|^2 + |(P_1 - P_2)x|^2 \right],$$

leading to a lower bound,

(3.6)
$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq \frac{1}{2} \left[\sigma_{\min}^2(P_1 + P_2) + \sigma_{\min}^2(P_1 - P_2) \right],$$

hence the set of singular values of $P_1 \pm P_2$ need to be analyzed. To this end, note that $M_1^\perp \cap M_2^\perp = \mathcal{N}(P) \cap \mathcal{N}(Q) = \{0\}$, as

$$x \in \mathcal{N}(P) \cap \mathcal{N}(Q) \quad \Leftrightarrow \quad (P + Q)x = 0 \quad \Leftrightarrow \quad x = 0,$$

since $P + Q$ is non-singular. Also, Lemma 2.5 gives

$$M_1 + M_2 = (M_1^\perp \cap M_2^\perp)^\perp = \{0\}^\perp = \mathbb{R}^n,$$

consequently, Theorem 2.8 gives the set of singular values of $P_1 \pm P_2$ as

(3.7)
$$\sigma(P_1 + P_2) = \{\mathbf{2}_k, 1 \pm \cos(\theta_{k+i})(i = 1, \ldots, r), \mathbf{1}_{n_1 + n_2}\},$$
$$\sigma(P_1 - P_2) = \{\mathbf{1}_{n_1 + n_2}, \mathbf{\sin(\theta_{k+i})}_2 (i = 1, \ldots, r), \mathbf{0}_k\},$$

where,

$$k = \dim(M_1 \cap M_2) = \dim(M_1) + \dim(M_2) - \dim(M_1 + M_2) = p + q - n,$$
$$n_1 = \dim(M_1 \cap M_2^\perp) = p - k - r,$$

(3.8)
$$n_2 = \dim(M_1^\perp \cap M_2) = q - k - r,$$
$$n_3 = \dim(M_1^\perp \cap M_2^\perp) = 0,$$
$$n = n_1 + n_2 + k + 2r = p + q - k.$$

Let us estimate (3.6), thus (3.4) in terms of the following cases.

Case 1: Suppose $r = 0$. Note that (3.8) implies $n_1 = p - k$, $n_2 = q - k$ for this case. Firstly, consider $k = 0$ and $n_1 = n_2 = 0$, then (3.8) implies $p = q = 0$, thus $P = Q = O$. This is rejected, since $P + Q$ is non-singular.

For $k = 0$ and $n_1 + n_2 > 0$, (3.7) and (3.8) yield $\sigma(P_1 \pm P_2) = \{\mathbf{1}_{n_1 + n_2}\} = \{\mathbf{1}_n\}$, hence (3.6) gives

(3.9)
$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq \frac{1}{2} \left[1^2 + 1^2 \right] = 1.$$

For $k > 0$ and $n_1 = n_2 = 0$, (3.8) implies that $k = p = q = n$ or both P and Q are non-singular. Therefore, $M_1 = M_2 = \mathbb{R}^n$, hence (3.4) gives

(3.10)
$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|x|^2 + |x|^2}{|x|^2} = 2.$$

For $k, n_1 > 0$, and $n_2 = 0$, (3.8) results in $p = n$ and $q = k$, that is, $M_2 \subseteq M_1 = \mathbb{R}^n$ or Q is non-singular. Hence, (3.4) becomes

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|x|^2 + |P_2 x|^2}{|x|^2} \geq 1.$$

Similarly, for $k, n_2 > 0$ and $n_1 = 0$, the same lower bound as above is derived which also coincides with (3.9). Finally, consider $k, n_1, n_2 > 0$, since $\dim(M_1 \cap M_2) = k > 0$,

therefore $M_1 \cap M_2$ is a non-trivial subspace. Define $M_3 := M_2 \cap (M_1 \cap M_2)^\perp$, then

$$\dim(M_3) = n - \dim(M_3^\perp)$$

(by Lemma 2.5)
$$= n - \dim(M_2^\perp + M_1 \cap M_2)$$

$$= n - (n - \dim(M_2)) - \dim(M_1 \cap M_2)$$

$$= n - (n - q) - k = q - k,$$

or $\dim(M_3) = q - k = n_2 > 0$, thus it is a non-trivial subspace. Let P_3 and P_U be the orthogonal projections onto the subspace M_3 and some subspace U of $\mathbb{R}^n$, respectively. The following was proved in [12, p. 1429],

$$P_3 = P_{M_2 \cap (M_1 \cap M_2)^\perp} = P_{M_2} P_{(M_1 \cap M_2)^\perp}$$

$$= P_{M_2} (I - P_{M_1 \cap M_2})$$

$$= P_{M_2} - P_{M_2} P_{M_1 \cap M_2}$$

$$= P_2 - P_{M_1 \cap M_2},$$

or $P_2 = P_{M_1 \cap M_2} + P_3$, which implies for any $x \in \mathbb{R}^n \setminus \{0\}$,

(3.11)
$$P_2 x = P_{M_1 \cap M_2} x + P_3 x.$$

By definition, $M_1 \cap M_2$ and M_3 are mutually orthogonal subspaces. Therefore, for any $x \in \mathbb{R}^n \setminus \{0\}$,

(3.12)
$$|P_2 x|^2 = |P_{M_1 \cap M_2} x|^2 + |P_3 x|^2,$$

and (3.4) becomes

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_1 x|^2 + |P_{M_1 \cap M_2} x|^2 + |P_3 x|^2}{|x|^2}$$

$$\geq \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_1 x|^2 + |P_3 x|^2}{|x|^2}$$

(by (3.5))
$$= \frac{1}{2} \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|(P_1 + P_3)x|^2 + |(P_1 - P_3)x|^2}{|x|^2}$$

(3.13)
$$\geq \frac{1}{2} \left[\sigma_{\min}^2(P_1 + P_3) + \sigma_{\min}^2(P_1 - P_3) \right].$$

By Theorem 2.8,

(3.14)
$$\sigma(P_1 + P_3) = \{ \mathbf{2}_{\tilde{k}}, 1 \pm \cos(\alpha_{\tilde{k}+i})(i = 1, \ldots, r), \mathbf{1}_{\tilde{n}_1 + \tilde{n}_2}, 0_{\tilde{n}_3} \},$$
$$\sigma(P_1 - P_3) = \{ \mathbf{1}_{\tilde{n}_1 + \tilde{n}_2}, \mathbf{sin}(\boldsymbol{\alpha}_{\tilde{\boldsymbol{k}}+\boldsymbol{i}})_2 (i = 1, \ldots, r), \mathbf{0}_{\tilde{k} + \tilde{n}_3} \},$$

where $0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\min(p, q-k)} \leq \frac{\pi}{2}$ represent the principal angles between the subspaces M_1 and M_3, and r is the number of principal angles that satisfy $0 < \alpha_i < \frac{\pi}{2}$. Observe that r is the same number for M_1 and M_2 by Definition 2.6, or see [19, p. 231] for more details. Thus, $r = 0$ and (3.8) gives the following parameters

(3.15)
$$\tilde{k} = \dim(M_1 \cap M_3) = \dim(M_1 \cap (M_2 \cap (M_1 \cap M_2)^\perp)) = 0,$$
$$\tilde{n}_1 = p - \tilde{k} - r = p,$$
$$\tilde{n}_2 = (q - k) - \tilde{k} - r = q - k,$$
$$\tilde{n}_3 = n - p - (q - k) + \tilde{k} = n - (p + q - k) = 0.$$

Hence, $\sigma_{\min}(P_1 \pm P_3) = 1$, therefore (3.13) implies that $\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq 1$, which coincides with (3.9).

In conclusion, for $r = 0$ and $p + q < 2n$, $\lambda_{\min}(P + Q) \geq \min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$ holds, or $c(P, Q) = 1$. Whereas, for $r = 0$ and $p + q = 2n$, $\lambda_{\min}(P + Q) \geq 2\min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$ holds, or $c(P, Q) = 2$.

Case 2: Suppose $r > 0$. For $k = 0$ and any $n_1, n_2 \geq 0$, (3.7) implies that $\sigma_{\min}(P_1 + P_2) = 1 - \cos\theta_1$ and $\sigma_{\min}(P_1 - P_2) = \sin\theta_1$. By (3.6),

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq \frac{1}{2}\left[(1 - \cos\theta_1)^2 + \sin^2\theta_1\right]$$

$$= 1 - \cos\theta_1.$$

Therefore, (3.4) gives $\lambda_{\min}(P + Q) \geq (1 - \cos\theta_1)\min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$.

For $k > 0$ and $n_1, n_2 \geq 0$, thus $M_1 \cap M_2$ is non-trivial. Consider the subspaces M_1 and M_3, as defined earlier for $k, n_1, n_2 > 0$ in Case 1. Note that M_3 is always non-trivial, as $n_1, n_2 \geq 0$ (3.8) implies $p \geq k + r$ and $q \geq k + r$, thus $\dim M_3 = q - k \geq r > 0$. Also, the set of parameters for M_1 and M_3 are given by (3.15) and $r > 0$ as follows,

(3.16)
$$\begin{aligned}
\tilde{k} &= \dim(M_1 \cap M_3) = 0, \\
\tilde{n}_1 &= p - \tilde{k} - r = p - r, \\
\tilde{n}_2 &= (q - k) - \tilde{k} - r = q - k - r, \\
\tilde{n}_3 &= n - p - (q - k) + \tilde{k} = 0.
\end{aligned}$$

Let θ_F be the Friedrichs angle between M_1 and M_2, then by Property 3 of Proposition 2.10, it is equal to the minimal angle between M_1 and M_3 in (3.14), that is,

(3.17)
$$\alpha_1 = \theta_F = \theta_{k+1},$$

where the last equality follows by Property 5 of Proposition 2.10. Therefore, (3.14) implies that $\sigma_{\min}(P_1 + P_3) = 1 - \cos\alpha_1 = 1 - \cos\theta_{k+1}$ and $\sigma_{\min}(P_1 - P_3) = \sin\alpha_1 = \sin\theta_{k+1}$, so (3.13) yields

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq \frac{1}{2}\left[(1 - \cos\theta_{k+1})^2 + \sin^2\theta_{k+1}\right]$$

(3.18)
$$= 1 - \cos\theta_{k+1}.$$

Thus, (3.4) implies $\lambda_{\min}(P + Q) \geq (1 - \cos\theta_{k+1})\min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$, or $c(P, Q) = 1 - \cos\theta_{k+1}$, which is consistent for $r > 0$ and $k = 0$. $\qquad\square$

Remark 3.2. Recall that θ_{k+1} is the Friedrichs angle between $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ as stated in Property 5 of Proposition 2.10. It can be easily calculated by using a result by A. Björck and G. Golub given in [11]. Let $Q_1 \in \mathbb{R}^{n \times p}$ and $Q_2 \in \mathbb{R}^{n \times q}$ represent a orthogonal bases for $\mathcal{R}(P)$ and $\mathcal{R}(Q)$, respectively. Define $M = Q_1^T Q_2$, then $\cos\theta_i = \sigma_i(M)$, where $i = 1, 2, \ldots, \min(p, q)$.

Remark 3.3. When $r = 0$ and $p + q = 2n$, which occurs when both P and Q are SPD, the term $c(P, Q) = 2$ is convenient for future use. It can be easily strengthened by using Theorem 2.1. The significance of Theorem 3.1 is that it gives a positive lower bound on $\lambda_{\min}(P + Q)$ for the case when both P and Q are rank deficient, for which the estimate of the theorem reads $\lambda_{\min}(P + Q) \geq (1 - \cos\theta_F)\min\{\lambda_{\min}(P), \lambda_{\min}(Q)\}$. Also, (3.18) is an optimal lower bound for $k = 0$, however, there is a scope of improvement when $k, r > 0$. This bound results from (3.12), which is carefully selected

through analysis described in section 2 of [30], which also summarizes the cases for $r = 0$ in section 1.

Definition (3.1) is also applicable to rectangular matrices as stated below.

PROPOSITION 3.4. *For distinct non-zero matrices $A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{n \times q}$*
  1. $c(A, B) = c(AA^T, BB^T)$.
  2. $c(A, B) = c(B, A)$.
  3. $c(A, O_{n \times k}) = 1$.
  4. $c(A, B) = 1 - \cos\theta_F$, *when both A, B are rank-deficient, where θ_F is the Friedrichs angle between $\mathcal{R}(A)$ and $\mathcal{R}(B)$.*

The following result is derived to complete the analysis of PSD matrices P, Q. It reduces to Theorem 2.1 when at least one of P and Q is SPD. However, the result may be weaker or stronger than the estimate given by Theorem 3.1.

THEOREM 3.5. *Let $P, Q \in \mathbb{R}^{n \times n}$ be PSD matrices of rank $p, q \leq n$, respectively, so that $P + Q$ is non-singular, then*

$$\lambda_{\min}(P + Q) \geq \psi(P, Q),$$

where, for $r = 0$

$$\psi(P, Q) := \begin{cases} a^2, & p = n, q < n, \\ b^2, & p < n, q = n, \\ a^2 + b^2, & p = q = n, \\ \min\{a^2, b^2\}, & otherwise, \end{cases}$$

and for $r > 0$,

$$\psi(P, Q) := \frac{1}{2}\left[a^2 + b^2 - \frac{1}{2}(a + b)\sqrt{(a + b)^2 - 4ab\sin^2\theta_{k+1}}\right.$$
$$\left. - \frac{1}{2}|a - b|\sqrt{(a - b)^2 + 4ab\sin^2\theta_{k+1}}\right],$$

where $a = \sqrt{\lambda_{\min}(P)}$, $b = \sqrt{\lambda_{\min}(Q)}$, $k = p+q-n$, $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{\min(p,q)} \leq \frac{\pi}{2}$ represent the principal angles between $\mathcal{R}(P), \mathcal{R}(Q) \subseteq \mathbb{R}^n$, and r is the number of principal angles θ_i so that $0 < \theta_i < \frac{\pi}{2}$, for $1 \leq i \leq \min(p, q)$.

Proof. Define $M_1 := \mathcal{R}(P)$, $M_2 := \mathcal{R}(Q)$ and $P_i \in \mathbb{R}^{n \times n}$ to be the orthogonal projection onto M_i, for $i = 1, 2$. On following the proof of Theorem 3.1, (3.3) gives

$$\lambda_{\min}(P + Q) \geq \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\lambda_{\min}(P)|P_1 x|^2 + \lambda_{\min}(Q)|P_2 x|^2}{|x|^2}$$

$$(3.19) \qquad = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{a^2|P_1 x|^2 + b^2|P_2 x|^2}{|x|^2}$$

$$\text{(by (3.5))} \qquad = \frac{1}{2}\inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|(aP_1 + bP_2)x|^2 + |(aP_1 - bP_2)x|^2}{|x|^2}$$

$$(3.20) \qquad \geq \frac{1}{2}\left[\sigma_{\min}^2(aP_1 + bP_2) + \sigma_{\min}^2(aP_1 - bP_2)\right].$$

The results given in [19, p. 247] and [19, p. 234-235] give the following expression,

$$aP_1 + bP_2 = Z\text{diag}\left((a + b)I_k, aS + bE(\theta_{k+i})(i = 1, \ldots, r), aI_{n_1}, bI_{n_2}, O_{n_3}\right)Z^T,$$

where Z is an orthogonal matrix, $S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$, and the expressions for k, n_1, n_2, n_3, n are given by (3.8). Thus, the set of singular values of $aP_1 \pm bP_2$ are

$$
\sigma(aP_1 + bP_2) = \left\{ \frac{1}{2} \left[(a+b) \pm \sqrt{(a+b)^2 - 4ab\sin^2\theta_{k+i}} \right] (i = 1, \dots, r), \right.
$$
$$
\left. (\boldsymbol{a}+\boldsymbol{b})_k, \boldsymbol{a}_{n_1}, \boldsymbol{b}_{n_2}, \boldsymbol{0}_{n_3} \right\},
$$

(3.21)

$$
\sigma(aP_1 - bP_2) = \left\{ \frac{1}{2} \left[\sqrt{(a-b)^2 + 4ab\sin^2\theta_{k+i}} \pm |a-b| \right] (i = 1, \dots, r), \right.
$$
$$
\left. |\boldsymbol{a}-\boldsymbol{b}|_k, \boldsymbol{a}_{n_1}, \boldsymbol{b}_{n_2}, \boldsymbol{0}_{n_3} \right\},
$$

resulting in the following cases.

Case 1: Suppose $r = 0$, then (3.8) yields $n_1 = p - k$ and $n_2 = q - k$ for this case. Firstly, consider $k = n_1 = 0$ and $n_2 > 0$, then (3.8) implies $p = 0$ and $q = n$. Thus, $P = O$ and $P + Q = Q$ is non-singular. Therefore, (3.19) leads to $\lambda_{\min}(P+Q) = \lambda_{\min}(Q) = b^2$. Similarly, $Q = O$ for $k = n_2 = 0$ and $n_1 > 0$, thus $\lambda_{\min}(P+Q) = \lambda_{\min}(P) = a^2$.

For $k = 0$ and $n_1, n_2 > 0$, (3.21) gives $\sigma_{\min}(aP_1 \pm bP_2) = \min\{a, b\}$. Thus, (3.20) gives $\lambda_{\min}(P+Q) \geq \min\{a^2, b^2\}$.

For $k > 0$ and $n_1 = n_2 = 0$, (3.8) implies $k = p = q = n$, that is, both P and Q are non-singular. By (3.21), $\sigma_{\min}(aP_1 + bP_2) = a + b$, $\sigma_{\min}(aP_1 - bP_2) = |a - b|$, therefore (3.20) gives

$$
\lambda_{\min}(P+Q) \geq \frac{1}{2} \left[(a+b)^2 + (a-b)^2 \right] = a^2 + b^2.
$$

For $k, n_2 > 0$ and $n_1 = 0$, (3.8) gives $p = k$ and $q = n$, which imply $M_1 \subseteq M_2 = \mathbb{R}^n$, or Q is non-singular. Thus, (3.19) becomes

$$
\lambda_{\min}(P+Q) = \inf_{x \in \mathbb{R}^n \backslash \{0\}} \frac{a^2|P_1 x|^2 + b^2|x|^2}{|x|^2} \geq b^2 = \lambda_{\min}(Q),
$$

similarly $k, n_1 > 0$ and $n_2 = 0$ gives $\lambda_{\min}(P+Q) \geq a^2 = \lambda_{\min}(P)$. Finally, consider $k, n_1, n_2 > 0$, since $\dim(M_1 \cap M_2) = k > 0$, then $M_1 \cap M_2$ is a non-trivial subspace. Consider $M_3 = M_2 \cap (M_1 \cap M_2)^\perp \neq \{0\}$ as defined in the proof of Theorem 3.1, then (3.12) in (3.19) gives

$$
\lambda_{\min}(P+Q) = \inf_{x \in \mathbb{R}^n \backslash \{0\}} \frac{a^2|P_1 x|^2 + b^2|P_{M_1 \cap M_2} x|^2 + b^2|P_3 x|^2}{|x|^2}
$$

$$
\geq \inf_{x \in \mathbb{R}^n \backslash \{0\}} \frac{a^2|P_1 x|^2 + b^2|P_3 x|^2}{|x|^2}
$$

(by (3.5))
$$
= \frac{1}{2} \inf_{x \in \mathbb{R}^n \backslash \{0\}} \frac{|(aP_1 + bP_3)x|^2 + |(aP_1 - bP_3)x|^2}{|x|^2}
$$

(3.22)
$$
\geq \frac{1}{2} \left[\sigma_{\min}^2(aP_1 + bP_3) + \sigma_{\min}^2(aP_1 - bP_3) \right].
$$

The set of singular values of $aP_1 \pm bP_3$ are given by (3.21) as follows

$$\sigma(aP_1 + bP_3) = \left\{ \frac{1}{2} \left[(a+b) \pm \sqrt{(a+b)^2 - 4ab\sin^2\alpha_{\tilde{k}+i}} \right] (i = 1, \ldots, r),$$

$$(\boldsymbol{a+b})_{\tilde{k}}, \boldsymbol{a}_{\tilde{n}_1}, \boldsymbol{b}_{\tilde{n}_2}, \boldsymbol{0}_{\tilde{n}_3} \right\},$$

(3.23)

$$\sigma(aP_1 - bP_3) = \left\{ \frac{1}{2} \left[\sqrt{(a-b)^2 + 4ab\sin^2\alpha_{\tilde{k}+i}} \pm |a-b| \right] (i = 1, \ldots, r),$$

$$|\boldsymbol{a-b}|_{\tilde{k}}, \boldsymbol{a}_{\tilde{n}_1}, \boldsymbol{b}_{\tilde{n}_2}, \boldsymbol{0}_{\tilde{n}_3} \right\},$$

where the parameters are the same as (3.15) with $r = 0$. Thus, $\sigma_{\min}(aP_1 \pm bP_3) = \min\{a, b\}$. By (3.22), $\lambda_{\min}(P + Q) \geq \min\{a^2, b^2\}$.

Case 2: Suppose $r > 0$. Then for $k = 0$ and $n_1, n_2 \geq 0$, by (3.21)

$$\sigma_{\min}(aP_1 + bP_2) = \frac{1}{2} \left[(a+b) - \sqrt{(a+b)^2 - 4ab\sin^2\theta_1} \right],$$

$$\sigma_{\min}(aP_1 - bP_2) = \frac{1}{2} \left[\sqrt{(a-b)^2 + 4ab\sin^2\theta_1} - |a-b| \right].$$

Thus, (3.20) gives

$$\lambda_{\min}(P + Q) \geq \frac{1}{2} \left[a^2 + b^2 - \frac{1}{2}(a+b)\sqrt{(a+b)^2 - 4ab\sin^2\theta_1} \right.$$

$$\left. - \frac{1}{2}|a-b|\sqrt{(a-b)^2 + 4ab\sin^2\theta_1} \right].$$

For $k > 0$ and $n_1, n_2 \geq 0$, (3.16), (3.17), and (3.23) yield

$$\sigma_{\min}(aP_1 + bP_3) = \frac{1}{2} \left[(a+b) + \sqrt{(a+b)^2 - 4ab\sin^2\theta_{k+1}} \right],$$

$$\sigma_{\min}(aP_1 - bP_3) = \frac{1}{2} \left[\sqrt{(a-b)^2 + 4ab\sin^2\theta_{k+1}} - |a-b| \right],$$

hence (3.22) implies

$$\lambda_{\min}(P + Q) \geq \frac{1}{2} \left[a^2 + b^2 - \frac{1}{2}(a+b)\sqrt{(a+b)^2 - 4ab\sin^2\theta_{k+1}} \right.$$

$$\left. - \frac{1}{2}|a-b|\sqrt{(a-b)^2 + 4ab\sin^2\theta_{k+1}} \right]. \qquad \square$$

In the proofs of Theorems 3.1 and 3.5, a technique similar to the case of $k = 0$ and $n_1, n_2 > 0$ can be applied to the cases $k = n_1 = 0$ and $n_2 > 0$, and $k = n_2 = 0$ and $n_1 > 0$, to get another positive lower bound; however, they turn out to be weaker than the stated results. On combining Theorems 3.1 and 3.5, another positive lower bound on $\lambda_{\min}(P + Q)$ is given as follows.

COROLLARY 3.6. *Let $P, Q \in \mathbb{R}^{n \times n}$ be PSD matrices of rank $p, q \leq n$, respectively, so that $P + Q$ is non-singular. Then*

$$\lambda_{\min}(P + Q) \geq \max \left[c(P, Q) \min \{\lambda_{\min}(P), \lambda_{\min}(Q)\}, \psi(P, Q) \right].$$

Example 1.   Here, we consider four pairs of PSD matrices P, Q, so that $P + Q$ is SPD, to present simple illustrations of the above results. The exact value of $\lambda_{\min}(P + Q)$ is compared with the lower bounds given by Theorems 3.1 and 3.5. The existing results in the literature give a trivial lower bound. See Remark 3.2 for the definition of matrix M used in these examples.

1. Let $P = \mathrm{diag}\,(5,0,0)$, $Q = \mathrm{diag}\,(0,4,9)$, so that $\mathrm{rank}\,P = 1$ and $\mathrm{rank}\,Q = 2$, and $P + Q = \mathrm{diag}\,(5,4,9)$ is SPD. Note that $\mathcal{R}(P) = \mathcal{R}(Q)^{\perp}$, thus $k = r = 0$ and $p + q = 3 < 6$, so that $c(P,Q) = 1$ and thus, Theorems 3.1 and 3.5 give the same lower bound $4 = \lambda_{\min}(P + Q) \geq 4$.

2. Let $P = \mathrm{diag}\,(1,1,0)$, $Q = \mathrm{diag}\,(0,1,3)$, so that $\mathrm{rank}\,P = \mathrm{rank}\,Q = 2$, and $P + Q = \mathrm{diag}\,(1,2,3)$ is SPD. Note that principal angles between $\mathcal{R}(P)$ and $\mathcal{R}(Q)$ are $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, so that $k = 1$, $r = 0$ and $p + q = 4 < 6$, thus $c(P,Q) = 1$. Therefore, Theorems 3.1 and 3.5 give the same lower bound, that is, $1 = \lambda_{\min}(P + Q) \geq 1 \cdot 1 = 1$.

3. Let $P = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $Q = \mathrm{diag}\,(6,0)$, so that they are PSD with $\mathrm{rank}\,P = \mathrm{rank}\,Q = 1$ and $P + Q = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}$ is SPD. Since the eigen-decomposition of $P = \mathcal{E}\Lambda\mathcal{E}^T$ and $Q = IQI^T$, where $\mathcal{E} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \mathrm{diag}\,(4,0)$, we get $M = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{2}$. Note that $k = 0$ and $r > 0$, thus $\cos\theta_1 = \sigma_{\max}(M) = \frac{\sqrt{2}}{2}$ and Theorem 3.1 implies that $1.3944 \approx \lambda_{\min}(P + Q) \geq (1 - \frac{\sqrt{2}}{2}) \cdot 4 \approx 1.1716$, which is stronger than the lower bound obtained by applying Theorem 3.5, $\lambda_{\min}(P + Q) \geq 1.1270$. Thus, Corollary 3.6 gives the former result, that is, $\lambda_{\min}(P + Q) \geq 1.1716$.

4. Let $P = \mathrm{diag}\,(10,5,0)$, $Q = \begin{bmatrix} 12 & & \\ & 3 & 9 \\ & 9 & 27 \end{bmatrix}$, so that $\mathrm{rank}\,P = \mathrm{rank}\,Q = 2$, and $P + Q = \begin{bmatrix} 22 & & \\ & 8 & 9 \\ & 9 & 27 \end{bmatrix}$. Note that $k = r = 1 > 0$, so we use orthonormal bases for P and Q to find $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{10}} & 0 \end{bmatrix}$. Thus, $c(P,Q) = 1 - \cos\theta_2 = 1 - \frac{1}{\sqrt{10}}$. Therefore, Theorem 3.1 implies that $4.4137 \approx \lambda_{\min}(P + Q) \geq (1 - \frac{1}{\sqrt{10}}) \cdot 5 \approx 3.4189$, which is weaker than the lower bound obtained by applying Theorem 3.5, $\lambda_{\min}(P+Q) \geq 3.7770$. Thus, Corollary 3.6 gives $\lambda_{\min}(P + Q) \geq 3.7770$.

For convenience of notation, define the function Ψ for matrices $A \in \mathbb{R}^{n \times p}$, $B \in \mathbb{R}^{n \times q}$:

$$(3.24) \qquad \Psi(A, B) = \sqrt{\psi(AA^T, BB^T)},$$

where ψ is defined by Theorem 3.5. Note that $\psi(AA^T, BB^T)$ is a function defined in terms of $a = \sqrt{\lambda_{\min}(AA^T)} = \sigma_{\min}(A)$, $b = \sqrt{\lambda_{\min}(BB^T)} = \sigma_{\min}(B)$, and principal angles between $\mathcal{R}(AA^T) = \mathcal{R}(A)$ and $\mathcal{R}(BB^T) = \mathcal{R}(B)$. Thus, $\Psi(A, B)$ is a function defined in terms of positive singular values of A and B, and principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(B)$.

472   A positive lower bound, defined by Corollary 3.6, could be useful in several cir-
473 cumstances, such as for a full rank block 2×1 matrix, with rank deficient sub-blocks.
474 Hence, the following applications are presented.

475   COROLLARY 3.7. *For $m \geq n$, let $A = \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix} \right] \in \mathbb{R}^{m \times n}$ be full rank, then*

476
$$\sigma_{\min}(A) \geq \max \left[\sqrt{c(A_1^T, A_2^T)} \min \left\{ \sigma_{\min}(A_1), \sigma_{\min}(A_2) \right\}, \Psi(A_1^T, A_2^T) \right].$$
477

478   *Proof.* Since A is a full rank matrix, $\sigma_{\min}^2(A) = \lambda_n(A^T A) = \lambda_{\min}(A_1^T A_1 + A_2^T A_2)$,
479 thus Corollary 3.6 implies

480    $\sigma_{\min}^2(A) \geq \max \left[c(A_1^T A_1, A_2^T A_2) \min \left\{ \sigma_{\min}^2(A_1), \sigma_{\min}^2(A_2) \right\}, \psi \left(A_1^T A_1, A_2^T A_2 \right) \right],$

481 which gives the desired result after applying Property 1 of Proposition 3.4 and (3.24). □

482   COROLLARY 3.8. *For $m \leq n$, let $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \in \mathbb{R}^{m \times n}$ be full rank, then*

483
$$\sigma_{\min}(A) \geq \max \left[\sqrt{c(A_1, A_2)} \min \left\{ \sigma_{\min}(A_1), \sigma_{\min}(A_2) \right\}, \Psi(A_1, A_2) \right].$$

484   After securing the above lower bounds, our subsequent aim is to extend them to
485 a non-singular 2×2 block matrix. While it could be tedious to estimate the singular
486 values of 2×2 block matrices, it is easier to find the singular values of its blocks which
487 are of smaller size. Thus, another significant application of Theorems 3.1 and 3.5 is
488 the following result, which give four estimates on the minimum singular value of a
489 non-singular matrix.

490   THEOREM 3.9. *For a non-singular matrix*

491
$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right] \in \mathbb{R}^{n \times n},$$

492 *where $A_{11} \in \mathbb{R}^{p \times k}$, $A_{22} \in \mathbb{R}^{q \times \ell}$, for $1 \leq p, q, k, \ell \leq n$, the following hold*

493 (3.25a) $\sigma_{\min}(A) \geq \sqrt{1 - \cos\theta} \cdot \min \left\{ \sigma_{\min}\left([A_{11}, A_{12}]\right), \sigma_{\min}\left([A_{21}, A_{22}]\right) \right\},$

494 (3.25b) $\sigma_{\min}(A) \geq \Psi \left([A_{11}, A_{12}]^T, [A_{21}, A_{22}]^T \right),$

495
496 (3.25c) $\sigma_{\min}(A) \geq \sqrt{1 - \cos\theta} \cdot \min \left\{ r_1, r_2 \right\},$

497 *where*

498
$$r_1 := \max \left[c_1 \min \left\{ \sigma_{\min}(A_{11}), \sigma_{\min}(A_{12}) \right\}, \Psi \left(A_{11}, A_{12} \right) \right],$$
499
500
$$r_2 := \max \left[c_2 \min \left\{ \sigma_{\min}(A_{21}), \sigma_{\min}(A_{22}) \right\}, \Psi \left(A_{21}, A_{22} \right) \right],$$

501 *where $c_1 = \sqrt{c(A_{11}, A_{12})}$, $c_2 = \sqrt{c(A_{21}, A_{22})}$, and $\theta \in \left(0, \frac{\pi}{2}\right]$ is the minimum princi-*
502 *pal angle between $\mathcal{R}\left([A_{11}, A_{12}]^T\right), \mathcal{R}\left([A_{21}, A_{22}]^T\right) \subseteq \mathbb{R}^n$. Moreover,*

503 (3.26) $\sigma_{\min}(A) \geq \sqrt{1 - \cos\theta} \cdot \min \left\{ c_1, c_2 \right\} \cdot \min_{1 \leq i,j \leq 2} \left\{ \sigma_{\min}(A_{ij}) \right\}.$

504   *Proof.* Since $A_{11} \in \mathbb{R}^{p \times k}$, $A_{22} \in \mathbb{R}^{q \times \ell}$, then $p + q = n = k + \ell$. Let $R_1 =$
505 $\begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \in \mathbb{R}^{p \times n}$, and $R_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{q \times n}$, then by a direct calculation

506 (3.27) $A^T A = \begin{bmatrix} R_1^T & R_2^T \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = R_1^T R_1 + R_2^T R_2.$
507

508 Note that $\operatorname{rank}(R_1) \leq p$ and $\operatorname{rank}(R_2) \leq q$, thus $\operatorname{rank}(R_1) + \operatorname{rank}(R_2) \leq p + q = n$,
509 also

$$\operatorname{rank}(R_1) + \operatorname{rank}(R_2) = \operatorname{rank}(R_1^T R_1) + \operatorname{rank}(R_2^T R_2)$$

$$\geq \operatorname{rank}(R_1^T R_1 + R_2^T R_2)$$

$$= \operatorname{rank}(A^T A) = \operatorname{rank}(A) = n.$$

514 Therefore, $\operatorname{rank}(R_1) + \operatorname{rank}(R_2) = n$, which implies that $\operatorname{rank}(R_1) = p$ and
515 $\operatorname{rank}(R_2) = q$, that is, R_1 and R_2 are full rank matrices. And, $A^T A$ is an SPD
516 matrix expressed a sum of two singular PSD matrices, thus by Theorem 3.1

$$\sigma_{\min}^2(A) = \lambda_{\min}(A^T A)$$

$$\geq c(R_1^T R_1, R_2^T R_2) \min \left\{ \lambda_{\min}(R_1^T R_1), \lambda_{\min}(R_2^T R_2) \right\}$$

(3.28) $$= c(R_1^T, R_2^T) \min \left\{ \sigma_{\min}^2(R_1), \sigma_{\min}^2(R_2) \right\},$$

521 where the last equality results from Property 1 of Proposition 3.4. Let θ be the
522 minimum principal angle between $\mathcal{R}(R_1^T)$ and $\mathcal{R}(R_2^T)$. Since A is non-singular, (3.27)
523 gives $\mathcal{N}(R_1) \cap \mathcal{N}(R_2) = \{0\}$, thus Lemma 2.5 implies $\mathcal{R}(R_1^T) + \mathcal{R}(R_2^T) = \mathcal{N}(R_1)^\perp +$
524 $\mathcal{N}(R_2)^\perp = (\mathcal{N}(R_1) \cap \mathcal{N}(R_2))^\perp = \{0\}^\perp = \mathbb{R}^n$. Therefore,

$$\dim(\mathcal{R}(R_1^T) \cap \mathcal{R}(R_2^T)) = \dim(\mathcal{R}(R_1^T)) + \dim(\mathcal{R}(R_2^T)) - \dim(\mathcal{R}(R_1^T) + \mathcal{R}(R_2^T))$$

$$= \operatorname{rank}(R_1) + \operatorname{rank}(R_2) - \dim(\mathbb{R}^n)$$

$$= p + q - n = 0,$$

529 or, $\mathcal{R}(R_1^T) \cap \mathcal{R}(R_2^T) = \{0\}$. Hence, $\mathbb{R}^n = \mathcal{R}(R_1^T) \oplus \mathcal{R}(R_2^T)$, that is, $\mathcal{R}(R_1^T)$ and $\mathcal{R}(R_2^T)$
530 are complementary subspaces so that $0 < \theta \leq \frac{\pi}{2}$, which implies if $r = 0$ then $\theta = \frac{\pi}{2}$.
531 Therefore, (3.1) can simply be expressed as $c(R_1^T, R_2^T) = 1 - \cos\theta$, thus (3.28) yields

$$\sigma_{\min}^2(A) \geq (1 - \cos\theta) \min \left\{ \sigma_{\min}^2(R_1), \sigma_{\min}^2(R_2) \right\},$$

533 which leads to (3.25a). Also, applying Theorem 3.5 to (3.27) implies

$$\sigma_{\min}^2(A) = \lambda_{\min}\left(R_1^T R_1 + R_2^T R_2\right) \geq \psi\left(R_1^T R_1, R_2^T R_2\right),$$

535 which by (3.24) gives (3.25b). Since R_1 has full rank, Corollary 3.8 for estimating
536 $\sigma_{\min}(R_1)$ leads to

$$\sigma_{\min}(R_1) \geq \max\left[c_1 \min \left\{\sigma_{\min}(A_{11}), \sigma_{\min}(A_{12})\right\}, \Psi\left(A_{11}, A_{12}\right)\right] =: r_1,$$

538 where $c_1 = \sqrt{c(A_{11}, A_{12})}$. Similarly, a lower bound on $\sigma_{\min}(R_2)$ is

$$\sigma_{\min}(R_2) \geq \max\left[c_2 \min \left\{\sigma_{\min}(A_{21}), \sigma_{\min}(A_{22})\right\}, \Psi\left(A_{21}, A_{22}\right)\right] =: r_2,$$

540 where $c_2 = \sqrt{c(A_{21}, A_{22})}$, and hence (3.25a) is expressed as

$$\sigma_{\min}(A) \geq \sqrt{1 - \cos\theta} \cdot \min\left\{r_1, r_2\right\},$$

543 which on further simplification gives,

$$\sigma_{\min}(A) \geq \sqrt{1 - \cos\theta} \cdot \min\left[c_1 \min \left\{\sigma_{\min}(A_{11}), \sigma_{\min}(A_{12})\right\},\right.$$

$$\left. c_2 \min \left\{\sigma_{\min}(A_{21}), \sigma_{\min}(A_{22})\right\}\right]$$

$$\geq \sqrt{1 - \cos\theta} \cdot \min\left\{c_1, c_2\right\} \cdot \min_{1 \leq i,j \leq 2} \left\{\sigma_{\min}(A_{ij})\right\}. \qquad \square$$

As $\sigma_{\min}(A) = \sigma_{\min}(A^T)$, the above estimates yield the following result framed in terms of block columns of A.

COROLLARY 3.10. *For a non-singular matrix*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \in \mathbb{R}^{n\times n},$$

where $A_{11} \in \mathbb{R}^{p\times k}$, $A_{22} \in \mathbb{R}^{q\times \ell}$, for $1 \le p,q,k,\ell \le n$,

(3.29a) $\qquad \sigma_{\min}(A) \ge \sqrt{1-\cos\theta} \cdot \min\left\{\sigma_{\min}\left(\left[\begin{smallmatrix} A_{11} \\ A_{21}\end{smallmatrix}\right]\right), \sigma_{\min}\left(\left[\begin{smallmatrix} A_{12} \\ A_{22}\end{smallmatrix}\right]\right)\right\},$

(3.29b) $\qquad \sigma_{\min}(A) \ge \Psi\left(\left[\begin{smallmatrix} A_{11} \\ A_{21}\end{smallmatrix}\right], \left[\begin{smallmatrix} A_{12} \\ A_{22}\end{smallmatrix}\right]\right),$

$\qquad \sigma_{\min}(A) \ge \sqrt{1-\cos\theta} \cdot \min\left\{s_1, s_2\right\},$

where

$$s_1 := \max\left[c_1 \min\left\{\sigma_{\min}(A_{11}), \sigma_{\min}(A_{21})\right\}, \Psi\left(A_{11}^T, A_{21}^T\right)\right],$$

$$s_2 := \max\left[c_2 \min\left\{\sigma_{\min}(A_{12}), \sigma_{\min}(A_{22})\right\}, \Psi\left(A_{12}^T, A_{22}^T\right)\right],$$

where $c_1 = \sqrt{c(A_{11}^T, A_{21}^T)}$, $c_2 = \sqrt{c(A_{12}^T, A_{22}^T)}$, and $\theta \in \left(0, \frac{\pi}{2}\right]$ is the minimum principal angle between $\mathcal{R}\left(\left[\begin{smallmatrix} A_{11} \\ A_{21}\end{smallmatrix}\right]\right), \mathcal{R}\left(\left[\begin{smallmatrix} A_{12} \\ A_{22}\end{smallmatrix}\right]\right) \subseteq \mathbb{R}^n$. Moreover,

$$\sigma_{\min}(A) \ge \sqrt{1-\cos\theta} \cdot \min\left\{c_1, c_2\right\} \cdot \min_{1\le i,j\le 2}\left\{\sigma_{\min}(A_{ij})\right\}.$$

Remark 3.11. The estimate given by (3.25a) is stronger than (3.25c) and (3.26), however, the sharpness of (3.25b) varies for different matrices (see Example 5). The inequality (3.26) gives a lower bound on the minimum singular value of a non-singular 2×2 block matrix in terms of the minimum positive singular value of its blocks. The estimates from Theorem 3.9 and Corollary 3.10 may differ, so in practice, one may use the maximum of all of the bounds obtained from both of them. A MATLAB® implementation for any 2×2 block matrix is given in [30].

Now, we simplify Theorem 3.9 for the special case of a saddle point matrix as follows.

COROLLARY 3.12. *For a non-singular saddle point matrix*

$$M = \begin{bmatrix} A & B \\ B^T & O \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)},$$

where $A \in \mathbb{R}^{m\times m}$ is non-singular and $B \in \mathbb{R}^{m\times n}$ is full rank,

$$\sigma_{\min}(M) \ge \sqrt{1-\cos\theta} \cdot \min\left\{\sigma_{\min}(A), \sigma_{\min}(B)\right\},$$

where θ is the minimum principal angle between $\mathcal{R}\left([A,B]^T\right)$ and $\mathcal{R}\left([B^T,O]^T\right)$.

Proof. Since M is non-singular, according to (3.25a),

$$\sigma_{\min}(M) \ge \sqrt{1-\cos\theta} \cdot \min\left\{\sigma_{\min}([A,B]), \sigma_{\min}([B^T,O])\right\}.$$

Since $\sigma_{\min}^2([A,B]) = \lambda_{\min}([A,B][A,B]^T) = \lambda_{\min}(AA^T + BB^T) \ge \sigma_{\min}^2(A)$, and similarly $\sigma_{\min}^2([B^T,O]) = \sigma_{\min}^2(B)$, hence the desired result. ☐

Example 2.   (Block diagonal matrix) For a non-singular block diagonal matrix $D = \left[\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right]$, use (3.25a) to get

$$\sigma_{\min}(D) \geq \sqrt{1 - \cos\theta} \cdot \min\{\sigma_{\min}([A, O]), \sigma_{\min}([O, B])\},$$

where θ is the minimum angle between $\mathcal{R}\left([A, O]^T\right)$ and $\mathcal{R}\left([O, B]^T\right)$. Note that $\sigma_{\min}([A, O]) = \sigma_{\min}(A)$ and $\sigma_{\min}([O, B]) = \sigma_{\min}(B)$, and it is straightforward to see that $\theta = \frac{\pi}{2}$. Therefore, the result becomes $\sigma_{\min}(D) \geq \min\{\sigma_{\min}(A), \sigma_{\min}(B)\}$. In fact, this inequality is an equality, thus the lower bound is sharp. Similar results can be obtained for a non-singular block anti-diagonal square matrix.

Example 3.   (Block triangular matrix) For a non-singular block upper triangular matrix $U = \left[\begin{smallmatrix} A_{11} & A_{12} \\ O & A_{22} \end{smallmatrix}\right]$, the most simplified result of Theorem 3.9 is given by (3.26):

$$\sigma_{\min}(U) \geq \sqrt{1 - \cos\theta} \cdot \min_{i=1,2}\{c_i\} \cdot \min\{\sigma_{\min}(A_{11}), \sigma_{\min}(A_{12}), \sigma_{\min}(O), \sigma_{\min}(A_{22})\},$$

where θ is the minimum angle between $\mathcal{R}\left([A_{11}, A_{12}]^T\right)$ and $\mathcal{R}\left([O, A_{22}]^T\right)$, $c_1 = \sqrt{c(A_{11}, A_{12})}$, and $c_2 = \sqrt{c(O, A_{22})} = 1 \geq c_1$ by Property 3 of Proposition 3.4. Also, $\sigma_{\min}(O) = \infty$ by section 1. Hence,

(3.30) $$\sigma_{\min}(U) \geq \sqrt{1 - \cos\theta} \cdot c_1 \cdot \min\{\sigma_{\min}(A_{11}), \sigma_{\min}(A_{12}), \sigma_{\min}(A_{22})\}.$$

Similarly, an estimate for a non-singular block lower triangular matrix can be derived.

When every block is a square matrix, then [8, p. 352] gives the following expression

$$U^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1} A_{12} A_{22}^{-1} \\ O & A_{22}^{-1} \end{bmatrix},$$

whose maximum singular value is $(\sigma_{\min}(U))^{-1}$. This task could be challenging to perform due to the presence of the term $A_{11}^{-1} A_{12} A_{22}^{-1}$. Note that the blocks need not to be square for (3.30). Let us consider $U = \begin{bmatrix} 10 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 1 & 6 \end{bmatrix}$, then [50] gives $\sigma_{\min}(U) \geq$ 1.7087. Whereas, on placing the partitions on U to make it a 2×2 block matrix so that its $(1, 1)$ block is either of size 2×1 or 2×2, (3.25b) gives a stronger result $1.7473 \leq 1.8285 \approx \sigma_{\min}(U)$. A MATLAB® implementation is given in [30].

Example 4.   Here, we explain the use of the new lower bounds with the help of two 2×2 block matrices denoted by A, where its (i, j)-th block is denoted by A_{ij} and its i-th block row is denoted by R_i, where $1 \leq i, j \leq 2$. Also, M represents the matrix defined in Remark 3.2. Most of the existing results do not provide a bound in terms of the blocks of the matrices considered in this example.

1. Consider $A = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$ which is non-singular such that its every

   block is singular. Note that $\sigma_{\min}(A) = 1 = \sigma_{\min}(A_{ij}) = \sigma_{\min}(R_i)$. It is straight forward to see that a basis for $\mathcal{R}(R_1^T)$ is $\{[0\ 1\ 0\ 0]^T, [0\ 0\ 0\ 1]^T\}$ and for $\mathcal{R}(R_2^T)$ is $\{[1\ 0\ 0\ 0]^T, [0\ 0\ 1\ 0]^T\}$. Since $\theta = \frac{\pi}{2}$ or $r = 0$, (3.25a) implies $1 = \sigma_{\min}(A) \geq (1 - 0) \cdot \min\{1, 1\} = 1$, and (3.25b) yields $1 = \sigma_{\min}(A) \geq$

$\sqrt{\min\{1,1\}} = 1$. Moreover, the same result is obtained on applying (3.25c), as $r = 0$ for both the pairs A_{1i} and A_{2i} for $i = 1, 2$, thus $1 = \sigma_{\min}(A) \geq (1 - 0)\min\{1,1\} = 1$. In order to use (3.26), note that a basis for $\mathcal{R}(A_{11}), \mathcal{R}(A_{21})$ is $\{[0\ 1]^T\}$, and for $\mathcal{R}(A_{12}), \mathcal{R}(A_{22})$ is $\{[1\ 0]^T\}$, therefore $c_1 = c_2 = 1$. Thus, this inequality yields $1 = \sigma_{\min}(A) \geq (1 - 0)\min\{1,1\}\min\{1,1,1,1\} = 1$.

2. Consider $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, which is a non-singular non-symmetric sad-

dle point matrix with an indefinite matrix as its leading block. For us-

ing (3.25a), $M = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix}$. Thus, $\cos\theta =$

$\sigma_{\max}(M) = \sqrt{\frac{2}{3}} \approx 0.8165$. Therefore, this inequality implies $0.4450 \approx \sigma_{\min}(A) \geq \sqrt{1 - 0.8165}\cdot\min\{1,1\} \approx 0.4284$. Moreover, the same lower bound is obtained on applying other results. Since $k = \mathrm{rank}(R_1^T) + \mathrm{rank}(R_2^T) - 3 = 0$, and $r = 1$ with $\cos\theta$ derived as above, (3.25b) implies $\sigma_{\min}(A) \geq$

$\sqrt{\frac{1}{2}\left[2 - \sqrt{4(1 - \sin^2\theta)}\right]} = \sqrt{1 - 0.8165} \approx 0.4284$. Also, (3.25c) results

in $\sigma_{\min}(A) \geq \sqrt{1 - 0.8165} \cdot \min\{1,1\} \approx 0.4284$. For using (3.26), observe that $c_1 = c_2 = 1$ and $\sigma_{\min}(A_{11}) = 1$, $\sigma_{\min}(A_{12}) = \sqrt{2}$, $\sigma_{\min}(A_{21}) = 1$ and $\sigma_{\min}(A_{22}) = \infty$. Therefore, the inequality gives $0.4450 \approx \sigma_{\min}(A) \geq \sqrt{1 - 0.8165} \cdot \min\{1,1\} \cdot \min\{\min\{1, \sqrt{2}\}, \min\{1, \infty\}\} \approx 0.4284$.

Example 5.   Let us consider two different partitions on the same matrix as follows,

$$A = \begin{bmatrix} t & 10 & 0 \\ 3 & 2 & -2 \\ 2 & 0 & 6 \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} t & 10 & 0 \\ 3 & 2 & -2 \\ 2 & 0 & 6 \end{bmatrix}, \qquad \text{where } t = 1, 2, \ldots, 30.$$

Figure 1 displays the result of best lower bounds from Theorem 3.9 and Corollary 3.10 for both partitions, along with the exact value of $\sigma_{\min}(A)$.

Figure 1a shows that Theorem 3.9 provides a decent estimate of $\sigma_{\min}(A)$. The best lower bound on $\sigma_{\min}(A)$ is given by (3.25b) for $t = 1$, and by (3.25a) for $t = 2, 3, \ldots 30$. For $\tilde{A}$, the largest lower bound is given by (3.25b) for $t = 1, 2, \ldots 5$, and by (3.25a) for $t = 6, 7, \ldots, 30$. On increasing the value of t, the lower bound obtained from (3.25a) improves up to $t = 18$, for which the absolute error in approximation for A is 0.008452 and for $\tilde{A}$ is 0.008389. The results from $\tilde{A}$ appear to be overall sharper than A. Thus, the sharpness of results may vary for distinct partitions of the same matrix.

Similarly, the trends for Corollary 3.10 are depicted by Figure 1b. It is observed that Corollary 3.10 gives identical results for both matrix A and $\tilde{A}$. The minimum absolute error in approximation is 0.01469, which occurs when $t = 18$. The best lower bound is given by the first inequality of Corollary 3.10 for all t except for $t = 1$, for which it was obtained from the second inequality of Corollary 3.10. A MATLAB® implementation is given in [30].

Example 6.   (M- and H-matrices) The following matrices are considered in [39],

$$A = \begin{bmatrix} 8 & -2 & -1 \\ -5 & 7 & -3 \\ -3 & -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -3 & -2 \\ -2 & 5 & -1 \\ -3 & -4 & 9 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 2 & -4 \\ 3 & -6 & -2 \\ -1 & -4 & -8 \end{bmatrix},$$
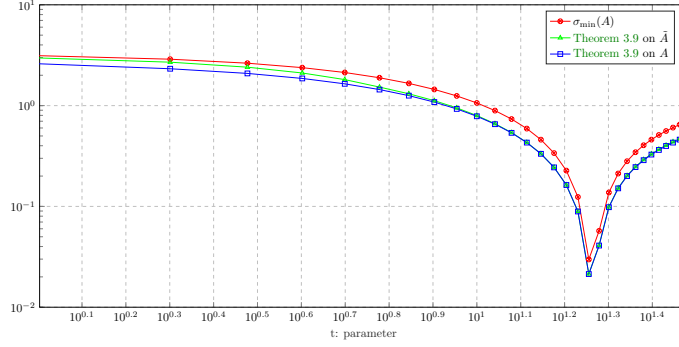
(a) Graph for Theorem 3.9
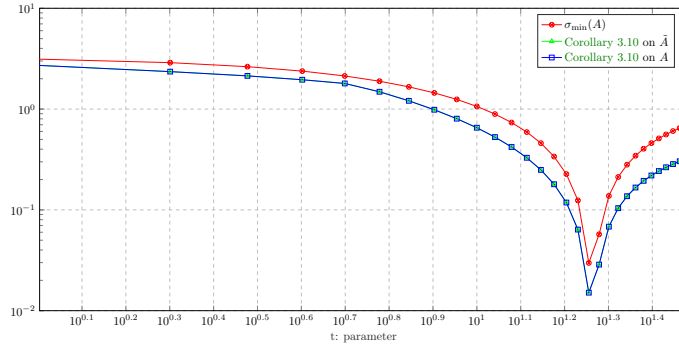
(b) Graph for Corollary 3.10

Fig. 1: Estimates of $\sigma_{\min}(A)$ for Example 5

where A and B are M-matrices and C is an H-matrix, for which upper bounds on the minimum singular values were devised. We calculate the best lower bounds secured from Theorem 3.9 and Corollary 3.10 in Table 1 through a MATLAB® implementation given in [30]. In Table 1, the size leading block refers to the size of $(1,1)$ block of the matrix specifying the partition being placed, and more than one partition means that the same lower bound is obtained in all cases. It is evident that our results provide a good estimate for M- and H-matrices.

Matrix	$\sigma_{\min}$	Best lower bound	Size of leading block
A	0.7744	(3.29a):0.7354	1×2 or 2×2
B	1.8830	(3.29a):1.5855	1×2 or 2×2
C	0.9015	(3.29a):0.8770	1×1 or 2×1

Table 1: Lower bounds for M- and H- matrices

Example 7.   In this example, we compare our results to some well-known existing results that give a lower bound on the minimum singular value of a matrix. The

Matrix	$\sigma_{\min}$	Existing	New Result	Size of leading block
D	9.8608	9.6389	(3.25b):9.6932	1×2 or 2×1 or 2×2
E	9.0409	8.0731	(3.25b):8.1814	2×1 or 2×2
F	7.6233	5.6070	(3.25a):6.0553	1×1 or 1×2
G	6.7547	5.2107	(3.25a):5.2728	1×1 or 1×2
H	1.9619	1.4142	(3.29b):1.8651	1×1 or 2×1
I	1.0677	0.7898	(3.25a):0.8996	1×1 or 1×2
J	3.0786	2.2303	(3.25b):2.8220	1×1 or 1×2
K	2.5146	2.2170	(3.29b):2.3847	1×1 or 2×1

Table 2: Comparison of new lower bounds with the existing results

following matrices are strictly diagonally dominant (SDD) matrices, for which several lower bounds were analyzed in [39],

$$D = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 20 & 1 \\ 1 & 1 & 30 \end{bmatrix}, \ E = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 20 & 1 \\ 10 & 1 & 30 \end{bmatrix}, \ F = \begin{bmatrix} 10 & 1 & 1 \\ 1 & 20 & 1 \\ 20 & 1 & 30 \end{bmatrix}, \ G = \begin{bmatrix} 10 & 1 & 1 \\ 10 & 20 & 1 \\ 20 & 1 & 30 \end{bmatrix}.$$

Also, some lower bounds for following matrices were compared in [37],

$$H = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 9 & 5 \\ 0 & 5 & 7 \end{bmatrix}, \quad I = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ -4 & -4 & 5 \end{bmatrix}, \quad J = \begin{bmatrix} 5 & 0 & 0 \\ -4 & 9 & 4 \\ -1 & 7 & 9 \end{bmatrix}, \quad K = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 5 & 0 \\ 0 & 5 & 4 \end{bmatrix}.$$

The third column of Table 2 states the best among all lower bounds evaluated for the above matrices in [39, 37]. The findings mentioned in Table 2 indicate that the results obtained from Theorem 3.9 and Corollary 3.10 provide a sharper lower bound on the minimum singular value of all SDD matrices considered in [39], albeit they may not be optimal for all SDD matrices. A MATLAB® implementation is given in [30]. In the above examples, we have listed the partitions that lead to best estimates, which may not be feasible if the matrix is large. Based on our numerical experiments, choosing the leading block of the matrix to be a square matrix of a suitable size often results in a partition that gives fine results.

After discussing lower bounds on the minimum singular value of a non-singular 2×2 block matrix, we divert our attention to constructing a lower bound on some other singular values. One such bounds is Theorem 2.4, which was given in [23]. In the following theorem, we provide a simpler proof for two similar lower bounds with more general sizes for its sub-blocks.

THEOREM 3.13. *For a block matrix*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right],$$

where $A_{11} \in \mathbb{R}^{m \times \ell}$ and $A_{22} \in \mathbb{R}^{n \times p}$, then

$$\sigma_j(A) \geq \sqrt{2\sigma_j \left(A_{11} A_{21}^T + A_{12} A_{22}^T \right)}, \qquad j = 1, 2, \ldots, \min(m, n, \ell + p),$$

also,

$$\sigma_j(A) \geq \sqrt{2\sigma_j\left(A_{11}^T A_{12} + A_{21}^T A_{22}\right)}, \qquad j = 1, 2, \ldots, \min(m+n, \ell, p).$$

Proof. Let $R_1 = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}$ and $R_2 = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix}$, so that $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. For $j = 1, 2, \ldots, \min(m+n, \ell+p)$, $\sigma_j^2(A) = \lambda_j(A^T A) = \lambda_j(R_1^T R_1 + R_2^T R_2)$. Therefore, by Theorem 2.3, for $j = 1, 2, \ldots, \min(m, n, \ell+p)$

$$\sigma_j^2(A) \geq 2\sigma_j(R_1 R_2^T) = 2\sigma_j\left(A_{11} A_{21}^T + A_{12} A_{22}^T\right).$$

Also, let $C_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$ and $C_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$, so that $A = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, then $\sigma_j^2(A) = \lambda_j(AA^T) = \lambda_j(C_1 C_1^T + C_2 C_2^T)$, for $j = 1, 2, \ldots, \min(m+n, \ell+p)$. Therefore, by Theorem 2.3, for $j = 1, 2, \ldots, \min(m+n, \ell, p)$

$$\sigma_j^2(A) \geq 2\sigma_j(C_1^T C_2) = 2\sigma_j\left(A_{11}^T A_{12} + A_{21}^T A_{22}\right). \qquad \Box$$

COROLLARY 3.14. *For a saddle point matrix*

$$M = \begin{bmatrix} A & B \\ B^T & O \end{bmatrix},$$

where $A \in \mathbb{R}^{m \times m}$ is non-singular and $B \in \mathbb{R}^{m \times n}$ is full rank,

$$\sigma_j(M) \geq \sqrt{2\sigma_j\left(AB\right)}, \qquad j = 1, 2, \ldots, \min(m, n).$$

4. Future Work. The lower bounds described by Theorems 3.1 and 3.5 are sharp for the case of $\mathcal{R}(P) \cap \mathcal{R}(Q) = \{0\}$, so that the parameter $k = 0$. However, there is a scope for improvement in these results for the case when $r, k > 0$. It may be possible to incorporate $P_{M_1 \cap M_2}$ in (3.22) to improve the lower bound.

In Example 5, it was observed that changing the partition of a matrix changes the lower bounds given by Theorem 3.9 and Corollary 3.10. Thus, one can try to determine the best partition for a certain class of matrices yielding optimal estimates on the minimum singular value.

Some techniques exist for calculating the principal angles between two subspaces, see [34]. A more efficient algorithm can be designed for calculating the Friedrichs angle between two subspaces. Since it is defined for subspaces of a Hilbert space in Definition 2.9, it may allow us to extend the main results to a more general setting.

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