SUPPLEMENTARY MATERIALS: NEW LOWER BOUNDS ON THE MINIMUM SINGULAR VALUES OF A MATRIX

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This document elaborates on some minute yet significant details for the proof of Theorem 3.1 presented in [1], which will provide more clarity for the readers.

As discussed in Theorem 3.1, let $P, Q \in \mathbb{R}^{n\times n}$ be PSD matrices of ranks $p, q \le n$, respectively. Define $M_1 = \mathcal{R}(P)$, $M_2 = \mathcal{R}(Q)$, and $M_3 = M_2 \cap (M_1 \cap M_2)^\perp$ to be subspaces of $\mathbb{R}^n$. Additionally, define $M_4 = M_1 \cap M_2$ and $M_5 = M_1 \cap M_4^\perp$, also now $M_3 = M_2 \cap M_4^\perp$. Let P_i be the orthogonal projections onto M_i, for all $1 \le i \le 5$.

Recall that $\dim(M_1) = p$, $\dim(M_2) = q$, and as $k = \dim(M_1 \cap M_2) = \dim M_4$, thus $\dim(M_3) = q-k$, and $\dim(M_5) = p-k$. Also, $0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_{\min(p,q)} \le \frac{\pi}{2}$ represent the principal angles between M_1 and M_2, r is the number of angles θ_i so that $0 < \theta_i < \frac{\pi}{2}$, for $1 \le i \le \min(p,q)$, and the Friedrichs angle between M_1 and M_2 is $\theta_F = \theta_{k+1}$.

1. Special case. Here we discuss the need of distinguishing the case $r = 0$ from $r > 0$ in Theorems 3.1 and 3.5. Note that $r = 0$ means the principal angles between M_1 and M_2 are either 0 or $\frac{\pi}{2}$. Recall that $n_1 = \dim(M_1 \cap M_2^\perp)$, $n_2 = \dim(M_1^\perp \cap M_2)$, $n_3 = \dim(M_1^\perp \cap M_2^\perp) = 0$, and $M_1 + M_2 = \mathbb{R}^n$. On substituting $r = 0$ in (3.8), we get $n_1 = p-k$, $n_2 = q-k$, so we need to analyze eight cases as follows.

k	n_1	n_2	Interpretation
0	0	0	$M_1 = M_2 = O$ or $P = Q = 0$
0	0	+	$M_1 = O$ or $P = O$
0	+	0	$M_2 = O$ or $Q = O$
0	+	+	$M_1 = M_2^\perp$, e.g. $P = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$
+	0	0	$M_1 = M_2 = \mathbb{R}^n$, or both P and Q are non-singular
+	0	+	$M_1 \subseteq M_2 = \mathbb{R}^n$ or Q is non-singular
+	+	0	$M_2 \subseteq M_1 = \mathbb{R}^n$ or P is non-singular
+	+	+	$M_1 \cap M_2 \neq \{0\}$ and $M_3 = M_1^\perp$, e.g. $P = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

Table 1: Cases for $r = 0$

1. Suppose $k = n_1 = n_2 = 0$. Then $p = q = 0$, or M_1 and M_2 are zero subspaces. Thus, P and Q are zero matrices, and we do not consider this case.
2. Suppose $k = n_1 = 0$ and $n_2 > 0$. Then $p = 0$, thus M_1 is a zero subspace and P is a zero matrix. Therefore, all principal angles between M_1 and M_2 are $\frac{\pi}{2}$.
3. Suppose $k = n_2 = 0$ and $n_1 > 0$. Then $q = 0$, thus Q is a zero matrix. Therefore, all principal angles between M_1 and M_2 are $\frac{\pi}{2}$.
4. Suppose $k = 0$ and $n_1, n_2 > 0$. Then $n_1 = p$ and $n_2 = q$. Since $M_1 \cap M_2 = \{0\}$, $\dim(M_1 \cap M_2^\perp) = \dim(M_1)$, and $\dim(M_2 \cap M_1^\perp) = \dim(M_2)$, then $R^n = M_1 \oplus M_2$, thus $M_1 = M_2^\perp$. Therefore, all principal angles between them are $\frac{\pi}{2}$.
5. Suppose $k > 0$ and $n_1, n_2 = 0$. Then $p = q = k$, since $n = p + q - k$ or $n = k$, thus $p = q = n$. Therefore, $M_1 = M_2 = \mathbb{R}^n$ and P, Q are non-singular. Thus,

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all principal angles between them are zero.

6. Suppose $k, n_2 > 0$ and $n_1 = 0$. Then $p = k$, and $n = p + q - k$ implies $n = k + q - k$, then $n = q$. Thus, $M_2 = \mathbb{R}^n$ or Q is non-singular and clearly $M_1 \subseteq M_2$. Note that all principal angles are zero in this case as well.
7. Suppose $k, n_1 > 0$ and $n_2 = 0$. Similarly, to the above case $M_2 \subseteq M_1 = \mathbb{R}^n$, thus P is non-singular and all principal angles are zero in this case as well.
8. Suppose $k, n_1, n_2 > 0$. This is the most general case for the $r = 0$ setting. In this case, there is a non-trivial intersection between the subspaces, that is, $M_1 \cap M_2 \neq \{0\}$. Therefore, $r = 0$ means $M_3 = M_1^{\perp}$, so the first k principal angles are zero and rest of them are $\frac{\pi}{2}$.

For cases 5-7, the Friedrichs angle between M_1 and M_2 is 0, so that $\theta_F = 0$ which gives $1 - \cos\theta_F = 1 - \cos 0 = 0$. Therefore, we define $c(P, Q)$ separately for the cases $r = 0$ and $r > 0$ in the proof of the theorem. All the above cases are mentioned in Table 1, where '+' represents that the corresponding parameter is considered to be positive.

2. The choice of subspaces. In this section, we discuss the choice of decomposition (3.12), particularly for the case $r, k > 0$, which states $|P_2x|^2 = |P_{M_1 \cap M_2}x|^2 + |P_3x|^2 = |P_4x|^2 + |P_3x|^2$.

Note that, this decomposition of M_2, in terms of M_3 and M_4, is considered for the case when $M_4 = M_1 \cap M_2 \neq \{0\}$. It is because when we try to estimate (3.6), which is

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \geq \frac{1}{2}\left[\sigma^2_{\min}(P_1 + P_2) + \sigma^2_{\min}(P_1 - P_2)\right],$$

then since $k, r > 0$, Property 2 of Theorem 2.8 implies that $P_1 - P_2$ is singular, thus

$$\begin{aligned}\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) &\geq \frac{1}{2}\left[\sigma^2_{\min}(P_1 + P_2) + 0\right], \\ &= \frac{1}{2}\left(1 - \cos\theta_{k+1}\right)^2,\end{aligned}$$

where the last equality holds by (3.7). In order to improve this lower bound, we consider various combinations of two subspaces to estimate a lower bound on the infimum of $\Delta(x)$, as defined by (3.4). We restrict to two subspaces to facilitate analysis. The investigation of the case with three subspaces is a topic for future research. Note that $n_1, n_2 \geq 0$ and (3.8) imply $p, q \geq k + r$, thus $\dim(M_3) = q - k \geq r > 0$ and $\dim(M_5) = p - k \geq r > 0$, in other words, M_3 and M_5 are non-trivial whenever $r, k > 0$. Since (3.12) gives $|P_2x|^2 = |P_4x|^2 + |P_3x|^2$, and similarly, $|P_1x|^2 = |P_4x|^2 + |P_5x|^2$, thus the following approaches are established.

$$\begin{aligned}\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) &= \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_1x|^2 + |P_2x|^2}{|x|^2} \\ &= \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_1x|^2 + |P_4x|^2 + |P_3x|^2}{|x|^2} \\ &= \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_5x|^2 + 2|P_4x|^2 + |P_3x|^2}{|x|^2} \\ &= \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_5x|^2 + |P_4x|^2 + |P_2x|^2}{|x|^2}.\end{aligned}$$

By considering two terms at once in the above set of equations and applying (3.5), we can get a lower bound in terms of an expression of the form

$$\frac{1}{2}\left[\sigma^2_{\min}(P_i+P_j)+\sigma^2_{\min}(P_i-P_j)\right],$$

where $i \neq j$.

For subspaces $U, V \subseteq \mathbb{R}^n$, let P_U and P_V be orthogonal projections onto U and V, respectively. Properties 1 and 2 of Theorem 2.8 imply that $P_U + P_V$ is non-singular if and only if $\dim(U^\perp \cap V^\perp) = 0$, and $P_U - P_V$ is non-singular if and only if $\dim(U \cap V) = \dim(U^\perp \cap V^\perp) = 0$. Therefore, we construct Table 2, which describes the sum and difference of two orthogonal projections P_i, for $1 \le i \le 5$. Since $n = p$

U	V	$\dim(U \cap V)$	$\dim(U^\perp \cap V^\perp)$	Conclusion
M_1	M_3	0	0	both $P_1 \pm P_3$ are non-singular
M_1	M_4	k	$n-p$	both $P_1 \pm P_4$ are singular
M_2	M_4	k	$n-q$	both $P_2 \pm P_4$ are singular
M_2	M_5	0	0	both $P_2 \pm P_5$ are non-singular
M_3	M_4	0	$n-q$	both $P_3 \pm P_4$ are singular
M_3	M_5	0	k	both $P_3 \pm P_5$ are singular
M_4	M_5	0	$n-p$	both $P_4 \pm P_5$ are singular

Table 2: Combinations of two subspaces

or $n = q$ occur when $r = 0$ as discussed in the cases 5-7 of the last section, therefore $n-p, n-q > 0$ whenever $r, k > 0$. Thus, only two of the combinations, M_1 and M_3, and M_2 and M_5, give non-singular sum and difference of their orthogonal projections, that is, providing a non-trivial lower bound for both sum and difference of their orthogonal projections. The proof of Theorem 3.1 is completed by considering M_1 and M_3. It is easy to see that if we consider M_2 and M_5, then on following the proof to analyze

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \ge \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|P_5 x|^2 + |P_2 x|^2}{|x|^2},$$

we get the exact same lower bound as M_1 and M_4, that is,

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \Delta(x) \ge 1 - \cos\theta_F = 1 - \cos\theta_{k+1},$$

which is optimal for our analysis.

REFERENCES

[1] A. Kaur and S. H. Lui, *New lower bounds for the minimum singular value of a matrix*, preprint, (2022).