BAYESIAN ESTIMATION OF RISK-NEUTRAL PROBABILITY

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IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

$$S = \sum_{s=1}^{K} \phi(s)q(s)$$

$$\phi^* \stackrel{d}{=} \frac{\phi(s)}{\sum_{s=1}^{K} \phi(s)} = \frac{\phi(s)}{1/(1+r)} = (1+r)\phi(s)$$

$$S = \frac{1}{1+r} \sum_{s=1}^{K} \phi^*(s)q(s) = \frac{1}{1+r}\mathbb{E}^*[q(s)].$$

IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

For TAS utility, the state-prices are

$$\phi(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]},$$

$\rightsquigarrow$ implied risk-neutral probability is

$$\phi^*(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]}(1+r) = \pi \frac{u'[c(s)]}{u'[c(0)]} \approx \pi,$$

for u' is constant (risk-neutrality) or $c(s) \approx c(0)$.

BREEDEN-LITZENBERGER (1978) FORMULA

For short-term, $r \approx 0$.

$$\begin{aligned}
C(S,t) &= \mathbb{E}^* \left[[S(T) - \mathcal{K}]^+ \mid S(t) = S \right] \\
&= \int_0^{+\infty} (x - \mathcal{K})^+ \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K} \int_0^{+\infty} \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K}(1 - P^*(\mathcal{K}))
\end{aligned}$$

$$\begin{aligned}
\frac{\partial C}{\partial \mathcal{K}} &= -\mathcal{K} p^*(\mathcal{K}) - 1 + P^*(\mathcal{K}) + \mathcal{K} p^*(\mathcal{K}) = P^*(\mathcal{K}) - 1 \\
\frac{\partial^2 C}{\partial \mathcal{K}^2} &= p^*(\mathcal{K}).
\end{aligned}$$

Ait-Sahalia and Duarte (2003)

BAYESIAN FORMULATION

Fisher (2016) proposes the following approach.

$$y_i = \lambda \sum_{j=1}^{K} X_{ij} \beta_j + \varepsilon_i,$$

where $\beta = (\beta_1, \ldots, \beta_K) \in \Delta^{K-1}$, Δ^{K-1} denotes the simplex of dimension $(K-1)$; X_{ij} is a payoff of derivative i in state j, and $\varepsilon_i \sim N(0, \sigma^2)$.

In vector form,

$$y = \lambda X \beta + \varepsilon.$$

In order to insure $\beta \in \Delta^{K-1}$, we select the (symmetric) Dirichlet prior:

$$p(\beta \mid \alpha) = \mathrm{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

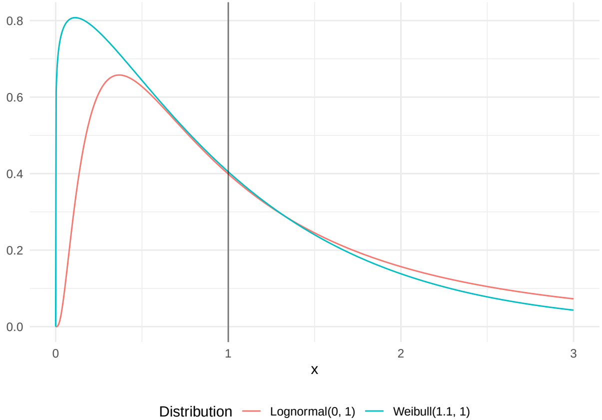
X consists of n payoffs of K derivatives, including the current price (you can think of it as call option with strike equal zero).

The state space is chosen to make

PRIOR FOR CONCENTRATION COEFFICIENT

α is a vector of concentration parameters for Dirichlet distribution.

It is reasonable to select a distribution that is relatively flat on $[0, 1]$ and assign a large portion of mass to $[0, 1]$. Fisher (2016) uses lognormal prior.



PRIORS AND POSTERIOR

$$p(\beta \mid \alpha) = \text{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

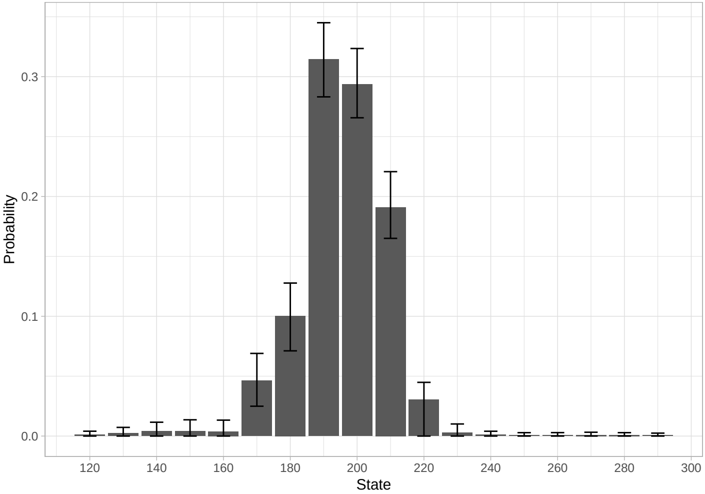
$$p(\alpha) = \text{Weibull}(1.1, 1)$$

$$p(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$p(\alpha, \lambda, \sigma, \beta \mid y) \propto N(y \mid \lambda X\beta, \sigma) \times \text{Dirichlet}(\beta \mid \alpha) \times \frac{1}{\sigma^2}$$

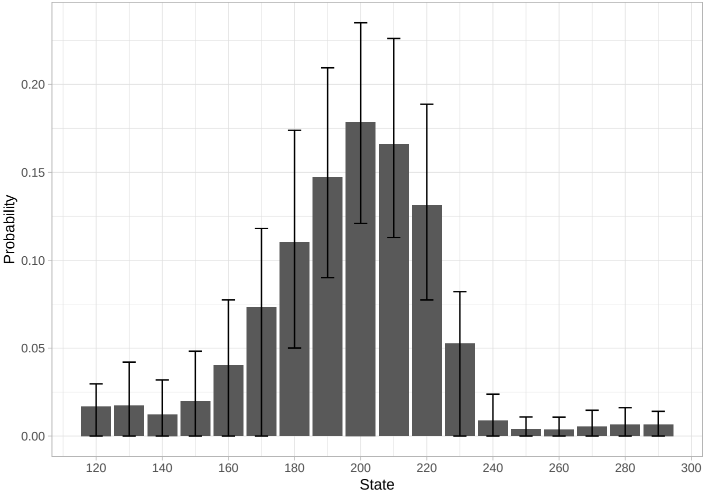
RISK NEUTRAL DENSITY FOR AAPL I

Calculated in 2025-05-23 with expiration date 2025-06-13

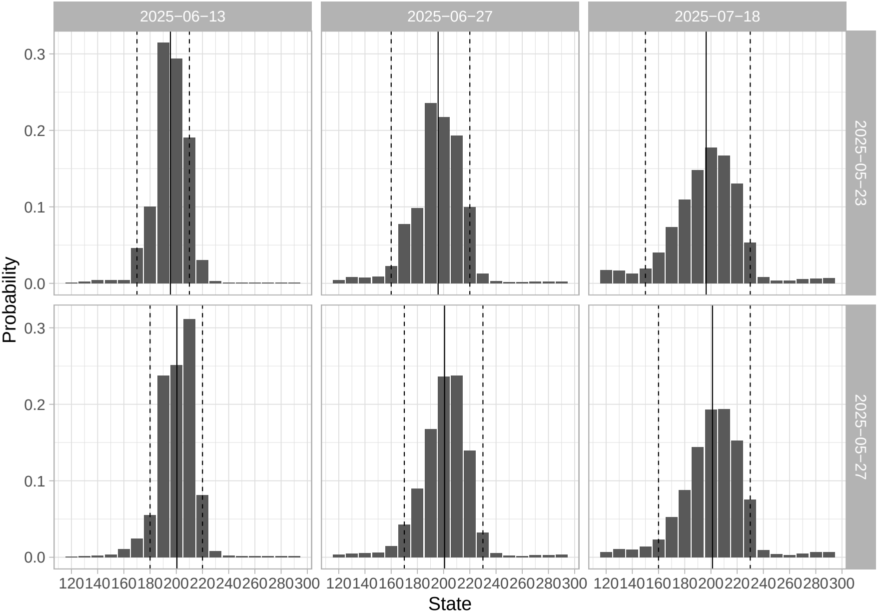


RISK NEUTRAL DENSITY FOR AAPL II

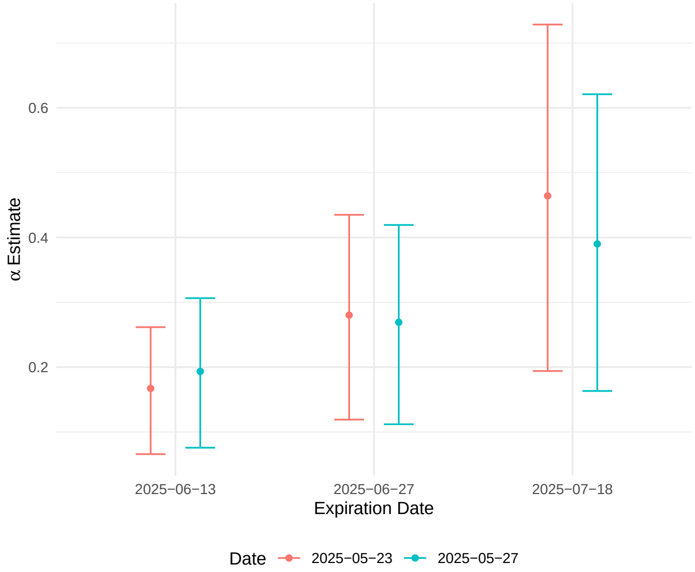
Calculated in 2025-05-23 (same) with expiration date 2025-07-18 (+35 days).



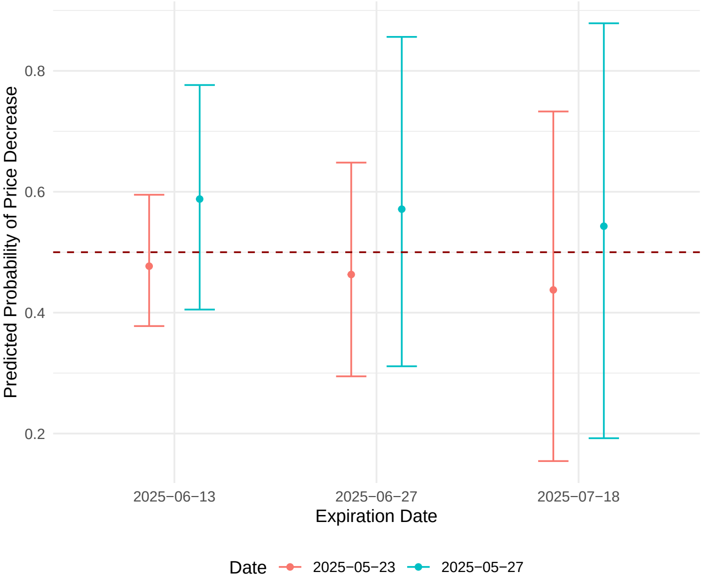
ESTIMATES OF IMPLIED RISK-NEUTRAL DENSITY



CONCENTRATION PARAMETER ESTIMATES



MODEL SANITY CHECK



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