BAYESIAN ESTIMATION OF RISK-NEUTRAL PROBABILITY

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IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

$$S = \sum_{s=1}^{K} \phi(s) q(s)$$

$$\phi^* \stackrel{d}{=} \frac{\phi(s)}{\sum_{s=1}^{K} \phi(s)} = \frac{\phi(s)}{1/(1+r)} = (1+r)\phi(s)$$

$$S = \frac{1}{1+r} \sum_{s=1}^{K} \phi^*(s) q(s) = \frac{1}{1+r} \mathbb{E}^*[q(s)].$$

IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

For TAS utility, the state-prices are

$$\phi(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]},$$

$\rightsquigarrow$ implied risk-neutral probability is

$$\phi^*(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]}(1+r) = \pi \frac{u'[c(s)]}{u'[c(0)]} \approx \pi,$$

for u' is constant (risk-neutrality) or $c(s) \approx c(0)\ \forall s$.

BREEDEN-LITZENBERGER (1978) FORMULA

For short-term, $r \approx 0$.

$$
\begin{aligned}
C(S,t) &= \mathbb{E}^* \left[[S(T) - \mathcal{K}]^+ \mid S(t) = S \right] \\
&= \int_0^{+\infty} (x - \mathcal{K})^+ \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K} \int_0^{+\infty} \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K}(1 - P^*(\mathcal{K}))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C}{\partial \mathcal{K}} &= -\mathcal{K} p^*(\mathcal{K}) - 1 + P^*(\mathcal{K}) + \mathcal{K} p^*(\mathcal{K}) = P^*(\mathcal{K}) - 1 \\
\frac{\partial^2 C}{\partial \mathcal{K}^2} &= p^*(\mathcal{K}).
\end{aligned}
$$

Ait-Sahalia and Duarte (2003)

BAYESIAN FORMULATION

Fisher (2016) proposes the following approach.

$$y_i = \lambda \sum_{j=1}^{K} X_{ij} \beta_j + \varepsilon_i,$$

where $\beta = (\beta_1, \ldots, \beta_K) \in \Delta^{K-1}$, Δ^{K-1} denotes the simplex of dimension $(K-1)$; X_{ij} is a payoff of derivative i in state j, and $\varepsilon_i \sim N(0, \sigma^2)$.

In vector form,

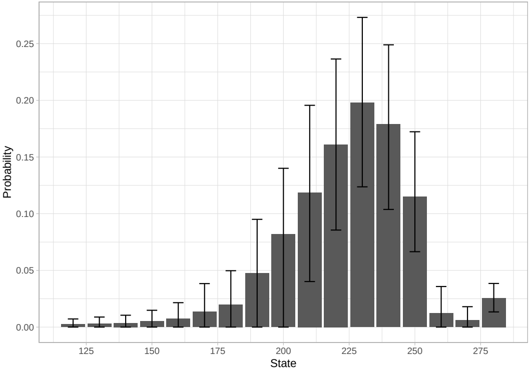
$$y = \lambda X \beta + \varepsilon.$$

In order to insure $\beta \in \Delta^{K-1}$, we select the (symmetric) Dirichlet prior:

$$p(\beta \mid \alpha) = \mathrm{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

X consists of n payoffs of K derivatives, including the payoffs price (you can think of it as call option with strike equal zero).

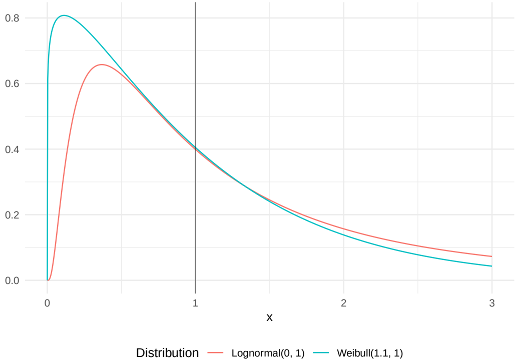
The state space is chosen to ensure that the first and last coefficients are near zero, so the list of possibilities is exhaustive.



PRIOR FOR CONCENTRATION COEFFICIENT

α is a vector of concentration parameters for Dirichlet distribution. The lower the value, the less 'flat' the distribution of risk-neutral density β.

It is reasonable to select a distribution that is relatively flat on $[0, 1]$ and assign a large portion of mass to $[0, 1]$. Fisher (2016) uses Lognormal prior, I propose Weibull. The result turns out to be almost identical.



PRIORS AND POSTERIOR

$$p(\beta \mid \alpha) = \mathrm{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

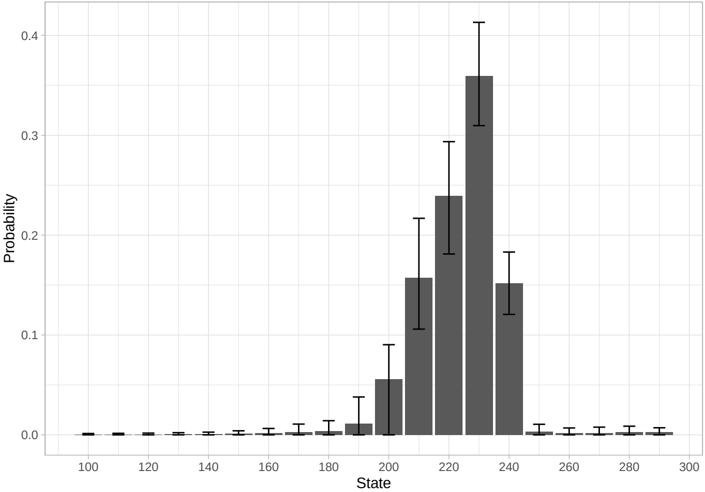
$$p(\alpha) = \mathrm{Weibull}(1.1, 1)$$

$$p(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$p(\alpha, \lambda, \sigma, \beta \mid y) \propto N(y \mid \lambda X\beta, \sigma) \times \mathrm{Dirichlet}(\beta \mid \alpha) \times \mathrm{Weibull}(1.1, 1) \times \frac{1}{\sigma^2}$$

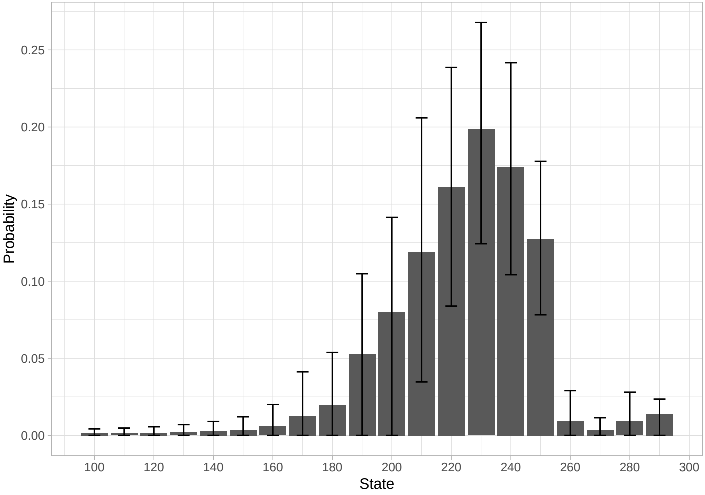
RISK NEUTRAL DENSITY FOR AAPL I

Calculated in 2025-04-01 with expiration date 2025-04-17 (+16 days). Wiskers are 90% HPD interval.

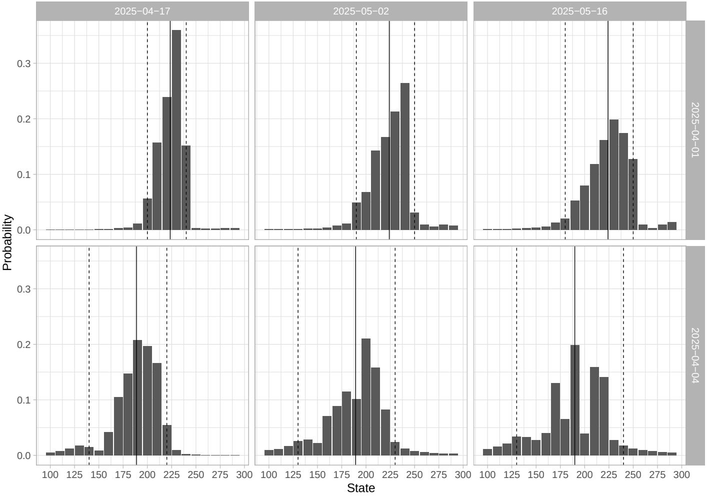


RISK NEUTRAL DENSITY FOR AAPL II

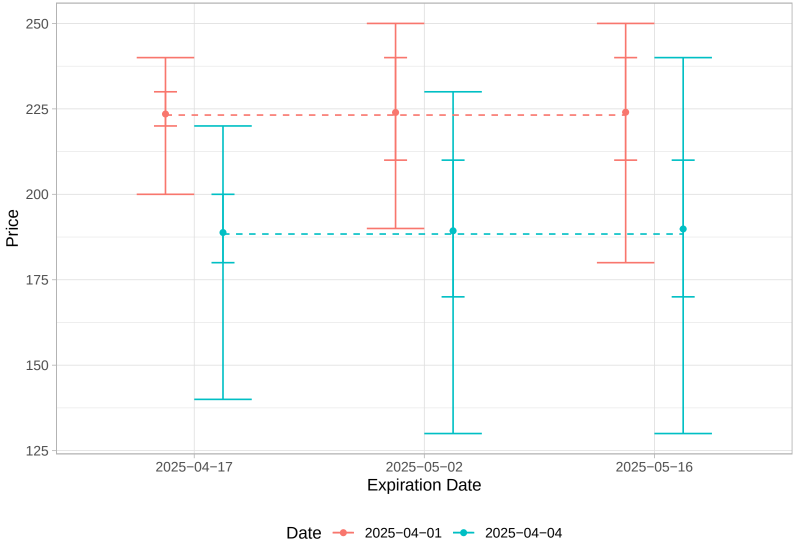
Calculated in 2025-04-01 (same) with expiration date 2025-07-18 (+45 days).



DENSITY BEFORE AND AFTER THE “LIBERATION DAY” 2025-04-02

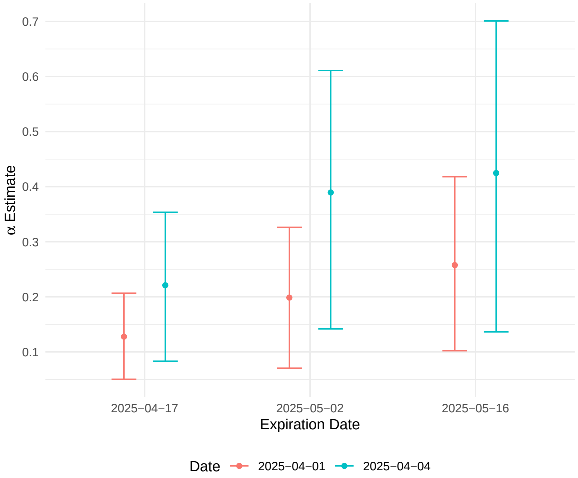


SUMMARY STATISTICS & SANITY CHECK



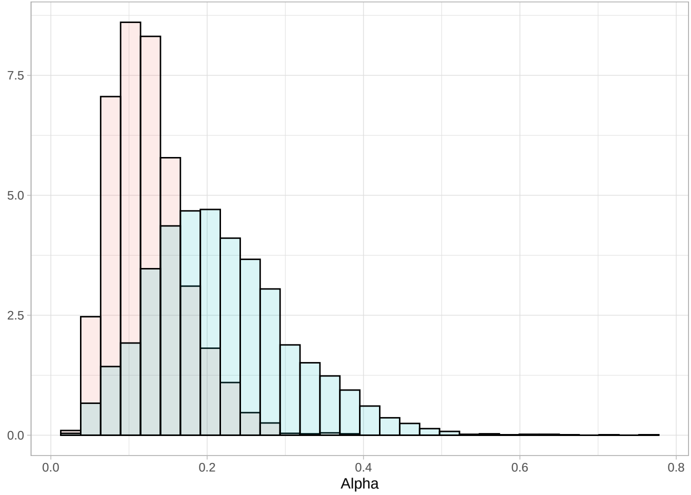
CONCENTRATION PARAMETER POSTERIOR DISTRIBUTION

Concentration parameter, α, is a measure of uncertainty in the risk-neutral probability density. The lower the value, the less 'flat' the distribution.



CONCENTRATION PARAMETER POSTERIOR DISTRIBUTION

The posterior distribution of the concentration parameter, α, for 2025-05-16.



CONCENTRATION PARAMETER AND STATE SPACE

The concentration parameter is dependent on the state space.

CONCLUSIONS AND FURTHER WORK

- We have constructed a way to measure
- The liberation day (April 2) increased the uncertainty of expected AAPL price.

- This method can be applied to derivatives of rate of interest (e.g SOFR options by CME).
- Concentration parameter then measures whether the forward guidance policy communication is perceived by the market.

REFERENCES [1]

Ait-Sahalia, Yacine and Jefferson Duarte (Sept. 2003). “Nonparametric Option Pricing under Shape Restrictions”. In: *Journal of Econometrics* 116(1), pp. 9–47. ISSN: 0304-4076. DOI: 10.1016/S0304-4076(03)00102-7. (Visited on 05/31/2025).

Breeden, Douglas T. and Robert H. Litzenberger (1978). “Prices of State-Contingent Claims Implicit in Option Prices”. In: *The Journal of Business* 51(4), pp. 621–651. ISSN: 0021-9398. JSTOR: 2352653. (Visited on 05/31/2025).

Fisher, Mark (May 2016). *Simplex Regression*.