BAYESIAN ESTIMATION OF RISK-NEUTRAL PROBABILITY

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IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

$$S = \sum_{s=1}^{K} \phi(s)q(s)$$

$$\phi^* \stackrel{d}{=} \frac{\phi(s)}{\sum_{s=1}^{K} \phi(s)} = \frac{\phi(s)}{1/(1+r)} = (1+r)\phi(s)$$

$$S = \frac{1}{1+r} \sum_{s=1}^{K} \phi^*(s)q(s) = \frac{1}{1+r}\mathbb{E}^*[q(s)].$$

IMPLIED RISK-NEUTRAL PROBABILITY DENSITY

For TAS utility, the state-prices are

$$\phi(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]},$$

$\rightsquigarrow$ implied risk-neutral probability is

$$\phi^*(s) = \pi \frac{\beta u'[c(s)]}{u'[c(0)]}(1+r) = \pi \frac{u'[c(s)]}{u'[c(0)]} \approx \pi,$$

for u' is constant (risk-neutrality) or $c(s) \approx c(0)\ \forall s$.

BREEDEN-LITZENBERGER (1978) FORMULA

For short-term, $r \approx 0$.

$$
\begin{aligned}
C(S,t) &= \mathbb{E}^* \left[[S(T) - \mathcal{K}]^+ \mid S(t) = S \right] \\
&= \int_0^{+\infty} (x - \mathcal{K})^+ \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K} \int_0^{+\infty} \, dP^*(x) \\
&= \int_{\mathcal{K}}^{+\infty} x \, dP^*(x) - \mathcal{K}(1 - P^*(\mathcal{K}))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C}{\partial \mathcal{K}} &= -\mathcal{K}p^*(\mathcal{K}) - 1 + P^*(\mathcal{K}) + \mathcal{K}p^*(\mathcal{K}) = P^*(\mathcal{K}) - 1 \\
\frac{\partial^2 C}{\partial \mathcal{K}^2} &= p^*(\mathcal{K}).
\end{aligned}
$$

Ait-Sahalia and Duarte (2003)

BAYESIAN FORMULATION

Fisher (2016) proposes the following approach.

$$y_i = \lambda \sum_{j=1}^{K} X_{ij}\beta_j + \varepsilon_i,$$

where $\beta = (\beta_1, \ldots, \beta_K) \in \Delta^{K-1}$, Δ^{K-1} denotes the simplex of dimension $(K-1)$; X_{ij} is a payoff of derivative i in state j, and $\varepsilon_i \sim N(0, \sigma^2)$.

In vector form,

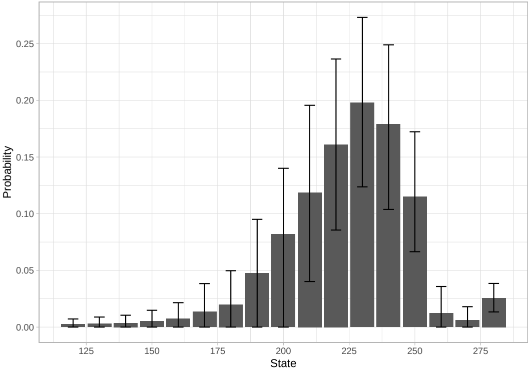
$$y = \lambda X\beta + \varepsilon.$$

In order to insure $\beta \in \Delta^{K-1}$, we select the (symmetric) Dirichlet prior:

$$p(\beta \mid \alpha) = \mathrm{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

X consists of n payoffs of K derivatives, including the payoffs price (you can think of it as call option with strike equal zero).

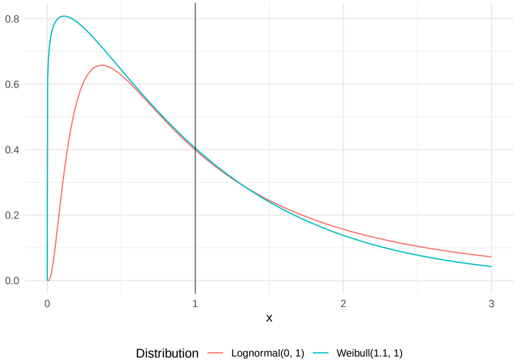
The state space is chosen to ensure that the first and last coefficients are near zero, so the list of possibilities is exhaustive.



PRIOR FOR CONCENTRATION COEFFICIENT

α is a vector of concentration parameters for Dirichlet distribution.

It is reasonable to select a distribution that is relatively flat on $[0, 1]$ and assign a large portion of mass to $[0, 1]$. Fisher (2016) uses Lognormal prior, I propose Weibull. The result turns out to be almost identical.



PRIORS AND POSTERIOR

$$p(\beta \mid \alpha) = \text{Dirichlet}(\beta \mid \alpha) \propto \prod_{i=1}^{K} x_i^{\alpha-1}.$$

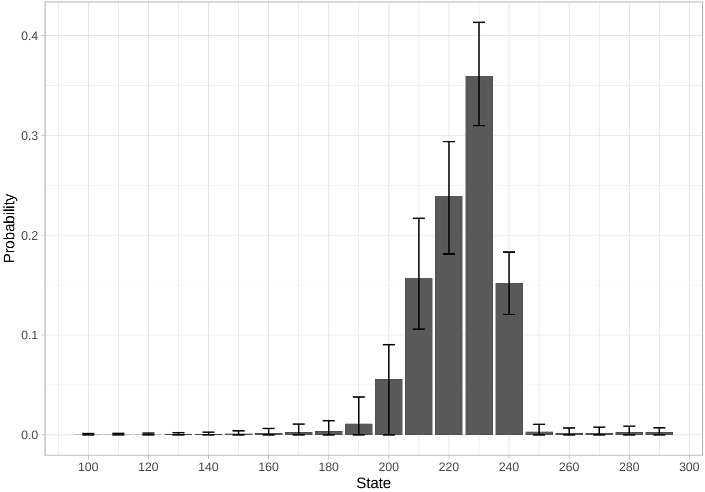
$$p(\alpha) = \text{Weibull}(1.1, 1)$$

$$p(\sigma^2) \propto \frac{1}{\sigma^2}$$

$$p(\alpha, \lambda, \sigma, \beta \mid y) \propto N(y \mid \lambda X \beta, \sigma) \times \text{Dirichlet}(\beta \mid \alpha) \times \text{Weibull}(1.1, 1) \times \frac{1}{\sigma^2}$$

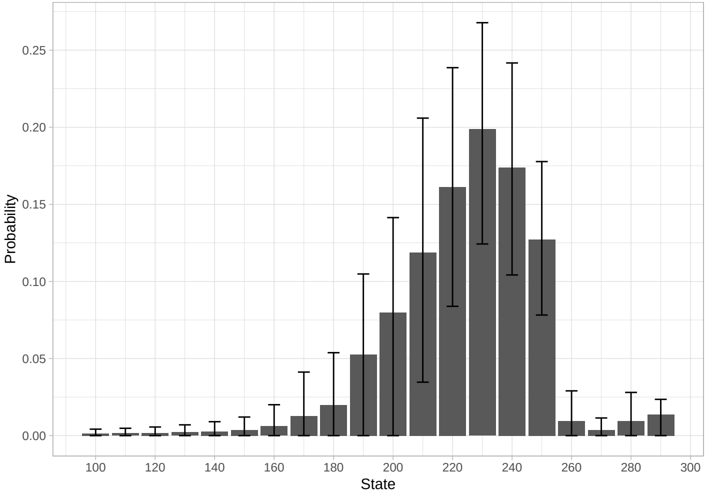
RISK NEUTRAL DENSITY FOR AAPL I

Calculated in 2025-04-01 with expiration date 2025-04-17 (+16 days). Wiskers are 90% HPD interval.

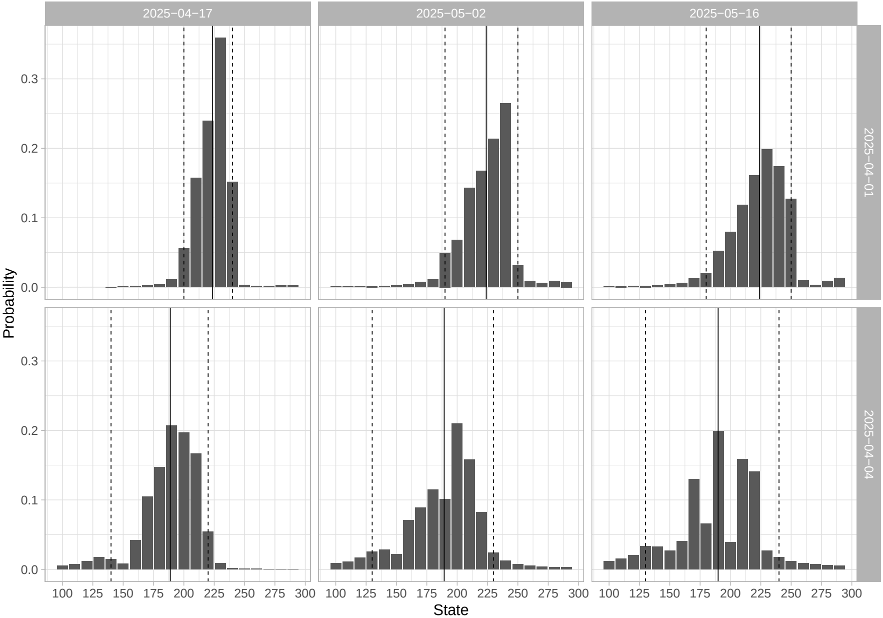


RISK NEUTRAL DENSITY FOR AAPL II

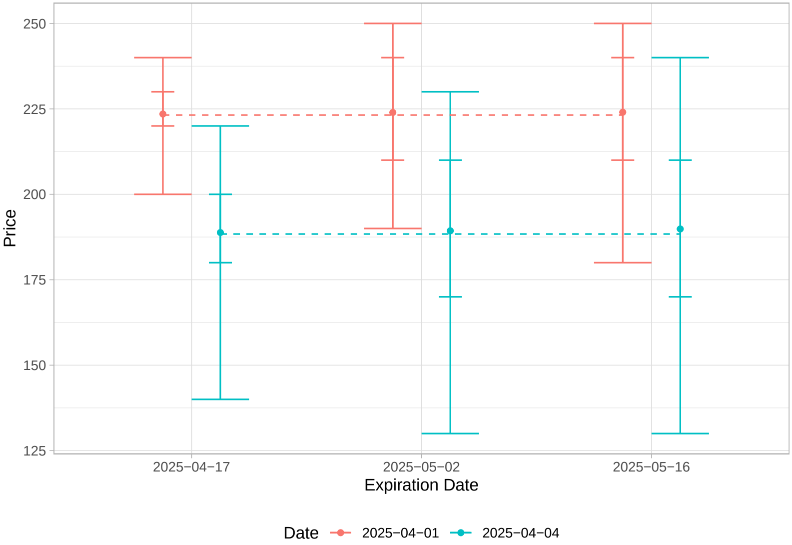
Calculated in 2025-04-01 (same) with expiration date 2025-07-18 (+45 days).



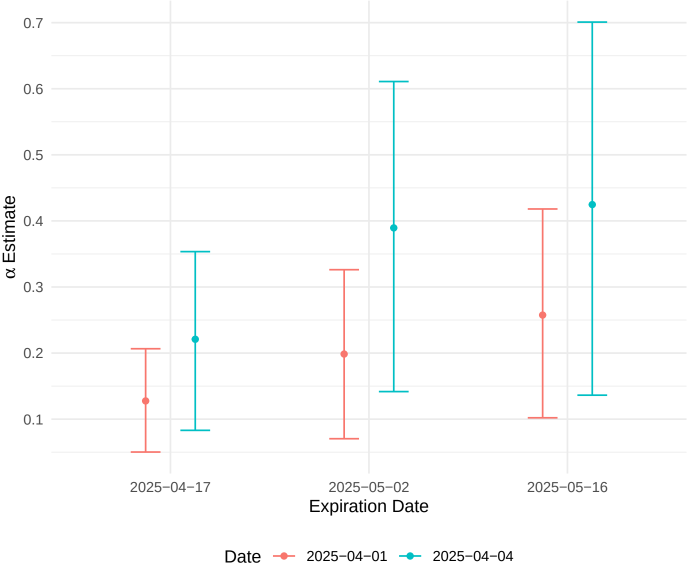
UNCERTAINTY BEFORE AND AFTER "LIBERATION DAY" (APRIL 2)



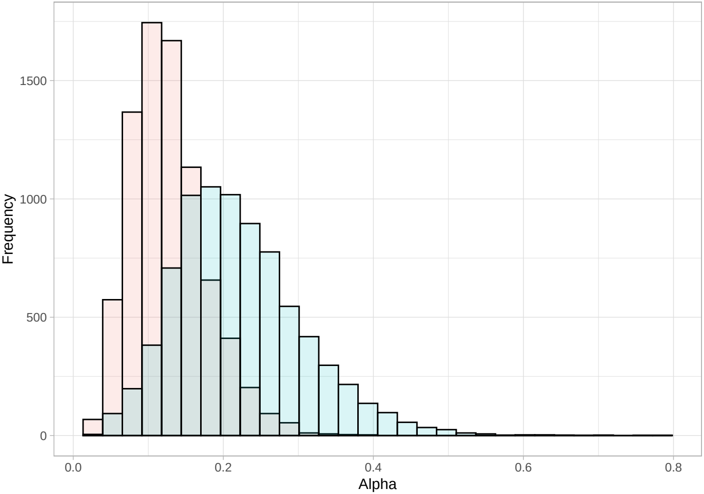
SUMMARY STATISTICS & SANITY CHECK



CONCENTRATION PARAMETER ESTIMATES



ALPHA DISTRIBUTION



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