Prediction without Preclusion
Recourse Verification with Reachable Sets

Avni Kothari UC San Diego

Bogdan Kulynych EPFL

Tsui-Wei Weng UC San Diego

Berk Ustun UC San Diego

Abstract

Machine learning models can assign predictions without recourse – i.e., predictions that a person cannot change through actions on their features. We introduce a formal procedure called *recourse verification* to test if a given model assigns a prediction that can be changed through actions in feature space. We develop theory and methods to perform recourse verification with *reachable sets* – i.e., regions of feature space delineated by a user-defined collection of actionability constraints. We demonstrate how these tools can support model-agnostic recourse verification with respect to complex actionability constraints. We present a comprehensive empirical study on the infeasibility of recourse across actionability constraints and model classes in real-world lending datasets. Our results highlight how models can assign fixed predictions that permanently bar access, and motivate the need to properly account for actionability when developing models and providing recourse.

1 Introduction

Machine learning models routinely assign predictions to *people* – be it to approve an applicant for a loan [29], a job interview [5, 58], or a public benefit [15, 19, 75]. Models in such applications use features that capture individual characteristics without accounting for how individuals can change these characteristics. In turn, models may assign predictions that are not responsive to the *actions* of their decision subjects. In effect, even the most accurate model can assign a prediction that is *fixed* – i.e., a prediction that a person cannot change through actions on its features (see Fig. 1).

The responsiveness of machine learning models to actions is a vital aspect of their safety in modern applications. In fraud detection and content moderation [31, 39, 49], for example, models *should* assign fixed predictions to prevent malicious actors from circumventing detection. In lending and hiring, however, predictions should exhibit *some* sensitivity to actions. Otherwise, models that deny loans and interviews on can permanently preclude *access* to credit and employment – thus violating rights and regulations such as equal opportunity [4] and universal access [9].

There is a broad lack of awareness that models in lending and hiring may inadvertently assign fixed predictions. In this work, we propose to test for this effect by certifying the infeasibility of *recourse* [35, 65, 67]. In contrast to prior work on recourse [36, 68], our goal is *verification* – i.e., to check if a model assigns predictions that every decision subject can change through actions on its features. This procedure can return falsifiable information about a model by certifying that recourse is feasible or infeasible under a given set of actionability constraints. In practice, however, verification imposes a host of computational challenges as we need to avoid returning information that is "incorrect" or "inconclusive" – e.g., by encoding complex actionability constraints and ensuring that we can evaluate how the predictions of a model change over the space defined by these constraints.

We propose a model-agnostic approach for recourse verification with *reachable sets* – i.e., regions of feature space that are confined by actionability constraints. Reachable sets are a powerful tool for verification because they can be constructed directly from the actionability constraints and can be

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Features		Action Set	Reachable Set	Dataset		ERM	
has_phd	age$\geq$60	$A(x_1, x_2)$	$R_A(\boldsymbol{x})$	n^-	n^+	$\hat{f}$	$\hat{R}(\hat{f})$
0	0	$\{(1,1),(0,1),(1,0),(0,0)\}$	$\{(1,1),(0,1),(1,0),(0,0)\}$	10	25	+	10
0	1	$\{(1,0),(0,0)\}$	$\{(1,1),(0,0)\}$	11	25	+	11
1	0	$\{(0,1),(0,0)\}$	$\{(1,1),(0,0)\}$	12	25	+	12
1	1	$\{(0,0)\}$	$\{(0,0)\}$	27	15	-	15

Figure 1: Stylized classification task where the most accurate linear classifier assigns a prediction without recourse to a fixed point. We predict $y = \texttt{repay\_loan}$ using two features $(x_1, x_2) = (\texttt{has\_phd}, \texttt{age}{\geq}60)$ that can only change from 0 to 1. Here, $(\texttt{has\_phd}, \texttt{age}{\geq}60)$ $(1,1)$ is a *fixed point* that will be assigned a *prediction without recourse* by any classifier f such that $f(1,1) = -1$. We show that this point is assigned a prediction without recourse by $\hat{f}$, the empirical risk minimizer for a dataset with $n^- = 60$ negative examples and $n^+ = 90$ positive examples.

used to characterize the feasibility of recourse for *all possible models*. In particular, any model will assign a prediction without recourse if it fails to assign a target prediction over its reachable set. The main contributions of this work include:

1. We introduce a model-agnostic approach for recourse verification with reachable sets. We identify salient classes of reachable sets and develop theory to show how they can be used to design practical verification algorithms.

2. We develop fast algorithms to delineate reachable sets from complex actionability constraints. Our algorithms can be used to ensure that a model can provide recourse in model development or deployment, and are designed to abstain when they are unable to certify recourse in order to avoid incorrect outputs.

3. We present an empirical study of the infeasibility of recourse using several real-world datasets, realistic actionability constraints, and common model classes. Our results illustrate the prevalence of predictions without recourse in lending applications, and highlight the pitfalls in flagging these examples with recourse provision. Finally, we demonstrate how our methods can be used to ensure recourse in consumer-facing applications like lending and content moderation.

4. We provide a Python library that implements our tools, available in this anonymized repository.

Related Work Our work is related to research on *algorithmic recourse*, which studies how to change the prediction of a given model through *actions* in feature space [65, 67]. Much work on this topic develops methods for *provision* – i.e., to provide a person with an action to change the prediction of a given model [see e.g., 13, 30, 33, 36, 53, 59, 68, 69]. We focus instead on *verification* – i.e., to test if a model assigns predictions that each person can change using any action. Although actionability is a defining characteristic of recourse, the fact that it may induce infeasibility is not often discussed [see 11, 37, 65, for exceptions]. Our work motivates the need to study and test infeasibility by showing that recourse may not exist under realistic actionability constraints.

Our methods share the same low-level machinery as a number of methods to find recourse actions or counterfactual explanations for specific model classes [55, 63, 65]. These methods solve combinatorial optimization problems and thus are capable of providing proofs of infeasibility as well are capable of encoding the same actionability constraints as we do. In this paper, however, we show how this machinery can be used for model-agnostic verification. Our approach is valuable as it is enables to verify recourse feasibility for models that are hard to encode as part of combinatorial optimization.

We motivate the need for verification as a procedure to safeguard access in consumer-facing applications, and to operationalize recourse provision [61, 67]. Our motivation is shared by a stream of recent work on the robustness of recourse with respect to distributions shifts [21, 60], model updates [56, 64], group dynamics [3, 24, 54], and causal effects [35, 47, 70]. Testing infeasibility may be valuable for eliciting individual preferences over recourse actions [72, 76, 77], measuring the effort required to obtain a target prediction [22, 27], and building classifiers that incentivize improvement [2, 26, 40, 41, 62], or preventing strategic manipulation [7, 12, 17, 23, 25, 45, 50, 51]. Our tools can support other verification tasks that test the sensitivity of model predictions to perturbations in semantically meaningful feature spaces, such as ensuring adversarial robustness on tabular data [28, 34, 39, 46, 49, 71] or stress testing for counterfactual invariance [48, 57, 66].

2 Recourse Verification

We consider a standard classification task where we are given a model $f : \mathcal{X} \to \mathcal{Y}$ to predict a binary label $y \in \mathcal{Y} = \{-1, +1\}$ using a *feature vector* $\boldsymbol{x} = [x_1, \ldots, x_d]$ from a bounded feature space $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_d = \mathcal{X}$. We assume that the model assigns predictions to people with features $\boldsymbol{x}$, and refer to $y = +1$ as the *target (positive) outcome*. We study how each person can change the prediction of the model through actions in feature space. We represent each *action* as a vector $\boldsymbol{a} = [a_1, \ldots, a_d] \in \mathbb{R}^d$ that would shift features from $\boldsymbol{x}$ to $\boldsymbol{x} + \boldsymbol{a} = \boldsymbol{x}' \in \mathcal{X}$. We denote the set of all actions available from $\boldsymbol{x} \in \mathcal{X}$ as an *action set* $A(\boldsymbol{x})$ and assume that $\mathbf{0} \in A(\boldsymbol{x})$.

Semantically meaningful features are subject to hard actionability constraints. As shown in Table 1, we can specify assumptions about what actions are feasible in natural language and encode them as constraints. Once encoded, stakeholders (e.g., model designers) can scrutinize and contest these assumptions even without technical knowledge. Although in general there might be no consensus about which actionability assumptions are relevant, in any prediction setting there exists a set of *minimal* actionability constraints. Practitioners can obtain such minimal constraints from a data dictionary [47], which specifies how a feature is encoded or its physical limits (e.g., `work_hrs_per_week` ≤ 168).

Constraint Type	Sep.	Cvx.	Sample Constraint	Features	Encoding
Immutability	✓	✓	`n_dependents` should not change	$x_j =$ `n_dependents`	$a_j = 0$
Monotonicity	✓	✓	`prior_applicant` can only increase	$x_j =$ `prior_applicant`	$a_j \geq 0$
Integrality	✓	✗	`n_accounts` must be positive integer ≤ 10	$x_j =$ `n_accounts`	$a_j \in \mathbb{Z} \cap [0 - x_j, 10 - x_j]$
Encoding: Categorical Features	✗	✗	preserve one-hot encoding of `married`, `single`	$x_j =$ `married` $x_k =$ `single`	$a_j + a_k = 1,\ \{a_j, a_k\} \in \{0, 1\}$
Encoding: Ordinal Features	✗	✗	preserve one-hot encoding of `max_degree_BS`, `max_degree_MS`	$x_j =$ `max_degree_BS` $x_k =$ `max_degree_MS`	$a_j = x_j' - x_j \quad a_k = x_k' - x_k$ $x_j' + x_k' = 1 \quad x_k' \geq x_j'$, $\{x_j', x_k'\} \in \{0, 1\}$
Logical Implications	✗	✗	if `is_employed` = `TRUE` then `work_hrs_per_week` ≥ 0 else `work_hrs_per_week` $= 0$	$x_j =$ `is_employed`, $x_k =$ `work_hrs_per_week`	$a_j = x_j' - x_j \quad a_k = x_k' - x_k$ $x_j' \in \{0, 1\} \quad x_k' \in [0, 168]$, $x_k' \leq 168\, x_j'$
Deterministic Causal	✗	✗	if `years_at_residence` increases then `age` will increase commensurately	$x_j =$ `years_at_residence` $x_k =$ `age`	$a_j \leq a_k$

Table 1: Catalog of actionability constraints that we consider. We show whether a constraint is *separable* (can be specified for each feature independently) and *convex*. We show an example of each constraint type to show that they can be expressed in natural language and encoded as a constraint in a discrete optimization problem.

Recourse Verification Given a model $f : \mathcal{X} \to \mathcal{Y}$ and a point $\boldsymbol{x} \in \mathcal{X}$ with action set $\boldsymbol{a} \in A(\boldsymbol{x})$, the *recourse provision* problem aims to find an action that yields the target prediction by solving an optimization problem of the form:

$$\begin{aligned} \min \quad & \text{cost}(\boldsymbol{a} \mid \boldsymbol{x}) \\ \text{s.t.} \quad & f(\boldsymbol{x} + \boldsymbol{a}) = +1, \\ & \boldsymbol{a} \in A(\boldsymbol{x}), \end{aligned} \tag{1}$$

where $\text{cost}(\boldsymbol{a} \mid \boldsymbol{x}) \geq 0$ captures the cost of enacting $\boldsymbol{a}$ for a person with features $\boldsymbol{x}$. As action sets can be non-convex (see Table 1) and the predictive models can have highly nonlinear decisions boundaries (e.g., neural networks or random forests), this combinatorial optimization problem is hard in the general case. We study the problem of *recourse verification* in which we want to determine the feasibility of Eq. (2). Formally, we want to construct a function $\mathsf{Recourse}(\boldsymbol{x}, f, A)$ such that:

$$\mathsf{Recourse}(\boldsymbol{x}, f, A) = \begin{cases} \texttt{Yes}, & \text{only if there exists } \ \boldsymbol{a} \in A(\boldsymbol{x}) \text{ such that } f(\boldsymbol{x} + \boldsymbol{a}) = +1 \\ \texttt{No}, & \text{only if } f(\boldsymbol{x} + \boldsymbol{a}) = -1 \text{ for all } \boldsymbol{a} \in A(\boldsymbol{x}) \\ \perp, & \text{otherwise} \end{cases} \tag{2}$$

We say that the procedure *certifies feasibility* for $\boldsymbol{x}$ if it returns `Yes`, and *certifies infeasibility* if it returns `No`. The procedure also *abstains* by outputting the flag $\perp$ when it cannot evaluate the conditions to certify feasibility or infeasibility. This may occur, for example, when we call verification using an underspecified action set $\tilde{A}(\boldsymbol{x}) \subset A(\boldsymbol{x})$ and discover that $f(\boldsymbol{x} + \boldsymbol{a}) = -1$ for all $\boldsymbol{a} \in \tilde{A}(\boldsymbol{x})$.

Use Cases Suppose that the model f assigns a *prediction without recourse* to $\boldsymbol{x}$. Such predictions may be undesirable in tasks where we would like to ensure access (e.g., lending), and desirable in applications where we would like to safeguard against gaming (e.g., content moderation):

Recourse. We want to verify the model's violation of the right to access by checking whether the prediction is without recourse: $\mathsf{Recourse}(\boldsymbol{x}, f, A) = \texttt{No}$. We focus on a use case in which the action set $A(\boldsymbol{x})$ is the set of the minimal constraints that apply to every decision subject. In this case, we will flag predictions without recourse even under conservative assumptions as a basic test of infeasibility.

Robustness. We certify that the model f is not vulnerable to adversarial manipulation: $\mathsf{Recourse}(\boldsymbol{x}, f, A) = \texttt{No}$ where the true label of $\boldsymbol{x}$ is $y = -1$. We specify the action set $A(\boldsymbol{x})$ to capture illegitimate actions [e.g., 17] or to represent an adversary's threat model [e.g., 39].

We can minimize the chances that a model will assign predictions without recourse by calling verification routines at model development and deployment. In the former, we would test if a model assigns a prediction without recourse over training examples. In the latter, we would call the procedure for previously unseen points at prediction time to flag violations in deployment.

Pitfalls of Verification Methods for recourse provision may return results that are incorrect or inconclusive when used for verification unless they are designed to certify infeasibility. In practice, any procedure that can certify infeasibility should be able to: (1) account for constraints in an action set and (2) check every action in the action set. When methods fail to satisfy these requirements, they may exhibit two kinds of pitfalls:

Loopholes: A *loophole* is an instance where a method returns an action that violates actionability constraints. In practice, a method for recourse provision returns loopholes if it fails to account for a subset of actionability constraints. Formally, the procedure solves the recourse problem in (2) using a relaxed action set $A^{\text{ext}}(\boldsymbol{x}) \subseteq A(\boldsymbol{x})$. The result is an action that is not technically feasible.

Blindspots: A *blindspot* refers to an instance where a method fails to certify feasibility. Formally, a blindspot arises when a method for recourse provision overlooks certain actions in an action set. This may occur, for example, with methods that are designed to search for actions using a heuristic search.

3 Verification with Reachable Sets

We introduce a model-agnostic approach for recourse verification based on reachability. Our approach stems from the observation that the feasibility of recourse is determined by regions of feature space that are confined by actionability constraints. We delineate these regions through *reachable sets*:

Definition 1 (Reachable Set)**.** *Given a point $\boldsymbol{x}$ and action set $A(\boldsymbol{x})$, its* reachable set *contains all feature vectors that can be attained using an action $\boldsymbol{a} \in A(\boldsymbol{x})$: $R_A(\boldsymbol{x}) := \{\boldsymbol{x} + \boldsymbol{a} \mid \boldsymbol{a} \in A(\boldsymbol{x})\}$.*

Even though reachable sets theoretically map to action sets, action sets are easier to specify indirectly through constraints (see Section 2). Obtaining the reachable set from a specification of the action set is not trivial, and we provide methods for doing so in the next sections.

Given any model $f : \mathcal{X} \to \mathcal{Y}$, we can verify recourse for a point $\boldsymbol{x}$ by evaluating predictions over its a reachable set $R = R_A(\boldsymbol{x})$ or its interior approximation $R = R_A^{\text{int}} \subset R_A(\boldsymbol{x})$.

$$\mathsf{Recourse}(\boldsymbol{x}, f, R) = \begin{cases} \texttt{Yes}, & \text{if there exists } \ \boldsymbol{x}' \in R \text{ such that } f(\boldsymbol{x}') = +1 \\ \texttt{No}, & \text{if } f(\boldsymbol{x}') = -1 \text{ for all } \boldsymbol{x}' \in R = R_A(\boldsymbol{x}) \\ \perp, & \text{if } f(\boldsymbol{x}') = -1 \text{ for all } \boldsymbol{x}' \in R \subset R_A(\boldsymbol{x}) \end{cases} \tag{3}$$

The procedure in Eq. (3) has two key benefits:

1. **Model-Agnostic Certification**: It provides a way to *certify* infeasibility for any model. In this setting, a model-agnostic approach may simplify verification because it only considers actionability constraints. In contrast, a model-specific approach would have to solve an optimization problem with two kinds of challenging constraints as in Eq. (2): (i) prediction constraints $f(\boldsymbol{x} + \boldsymbol{a}) = +1$, and (ii) actionability constraints $\boldsymbol{a} \in A(\boldsymbol{x})$.

2. **Abstention**: It will abstain when it cannot certify recourse. For instance, suppose that we call the procedure using an interior approximation of the reachable set $R_A^{\text{int}}(\boldsymbol{x}) \subset R_A(\boldsymbol{x})$ and fail to find a point in $R_A^{\text{int}}(\boldsymbol{x})$ that achieves the target prediction. In this case, an abstention is valuable because it flags $\boldsymbol{x}$ as a *potential* prediction without recourse. In practice, this leads to practical benefits. For example, we can call $\mathsf{Recourse}(\boldsymbol{x}, f, R)$ with an approximate reachable set $R = R_A^{\text{int}}(\boldsymbol{x})$ to screen points that have recourse. We can then revisit those points on which the procedure abstained with either a better approximation or the full reachable set $R = R_A(\boldsymbol{x})$.

Certain classes of reachable sets have properties that facilitate verification:

Definition 2 (Fixed Point)**.** *A point $\boldsymbol{x}$ is* fixed *if its reachable set only contains itself: $R_A(\boldsymbol{x}) = \{\boldsymbol{x}\}$.*

Fixed points can occur in clusters. If the features of $A(\boldsymbol{x})$ are separable and some of them are immutable, then $\boldsymbol{x}$ has a set of *sibling points* $N_A(\boldsymbol{x})$: all points that contain modifications of the immutable features of $\boldsymbol{x}$. If $\boldsymbol{x}$ is a fixed point, then every sibling $\boldsymbol{x}' \in N_A(\boldsymbol{x}, A(\boldsymbol{x}))$ is also fixed.

Definition 3 (Fixed Region)**.** *A* fixed region *is a reachable set $R_A(\boldsymbol{x})$ such that for any $\boldsymbol{x}' \in R_A(\boldsymbol{x})$ we have $R_A(\boldsymbol{x}') \subseteq R_A(\boldsymbol{x})$.*

We refer to both fixed points and fixed regions as *confined regions*. Given a fixed point, we can determine if a model violates recourse by checking its prediction on a single point. Given a fixed region, we can verify recourse for all points within it without generating reachable sets.

Confined regions arise under common actionability constraints. For instance:

Proposition 4. *Any classification task with bounded features whose actions obey monotonicity constraints must contain at least one fixed point.*

Fixed points also exhibit compositional properties when new features are added to the dataset. Each fixed point leads to confined regions in a higher dimensional space:

Proposition 5. *Consider adding a new feature $\mathcal{X}_{d+1} \subseteq \mathbb{R}$ to a set of d features $\mathcal{X} \subseteq \mathbb{R}^d$. Let $A_{d+1}(x_{d+1})$ be the actions from a point with feature value $x_{d+1} \in \mathcal{X}_{d+1}$. Any fixed point $\boldsymbol{x} \in \mathcal{X}$ induces the following confined regions in the $(d+1)$-dimensional space:*

- *$|\mathcal{X}_{d+1}|$ fixed points feature space, if x_{d+1} is immutable.*
- *A fixed point $\boldsymbol{z}_0 := (\boldsymbol{x}, v_{d+1})$ where v_{d+1} is an extreme point of $\mathcal{X}_{d+1}$.*
- *A fixed region if $A_{d+1}(x_{d+1}) = A_{d+1}(x'_{d+1})$ for any $x_{d+1}, x'_{d+1} \in \mathcal{X}_{d+1}$.*

Proposition 5 provides a cautionary result. For instance, suppose that we have identified fixed points. To enable recourse for them, we could choose to source additional features. In this case, we must take care that these new features remove the fixed points and regions and not only propagate them.

Reachable sets let us verify recourse for an arbitrary classifier using a set of labeled examples.

Theorem 6 (Certification with Labeled Examples)**.** *Suppose we have a dataset of labeled examples $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$. Every model $f : \mathcal{X} \to \mathcal{Y}$ can provide recourse to $\boldsymbol{x}$ if:*

$$\mathsf{FNR}(f) < \frac{1}{n^+} \sum_{i=1}^{n} \mathbb{1}[\boldsymbol{x}_i \in R \ \wedge \ y_i = +1] \tag{4}$$

where $\mathsf{FNR}(f) := \frac{1}{n^+} \sum_{i=1}^{n} \mathbb{1}[f(\boldsymbol{x}_i) = -1 \ \wedge \ y_i = +1]$ is the false negative rate of f and where n^+ is number of positive examples, and $R \subseteq R_A(\boldsymbol{x})$ is any subset of the reachable set.

The proof of Theorem 6 relies on the pigeonhole principle, and is provided in Appendix B. The theorem states that given a reachable set $R_A(\boldsymbol{x})$, we immediately know that any model that is sufficiently accurate on positive examples in the dataset must provide recourse for $\boldsymbol{x}$. Conversely, having measured a model's false negative rate, we know that there exists recourse for reachable sets with a certain level of prevalence of positive examples.

4 Algorithms

In this section, we present algorithms for recourse verification with reachable sets. Our algorithms are designed to certify recourse in a way that minimizes abstentions – i.e., to cover the detection of as many predictions without recourse as possible. To this end, they can delineate fixed points in general feature spaces, construct reachable sets over discrete feature spaces, and identify predictions with recourse by testing reachability over samples.

Fixed Point Detection We start with a method to detect fixed points, which we also use as a building block in later methods. We verify if $\boldsymbol{x}$ is a fixed point by solving the optimization problem:

$$\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x})) \quad := \quad \text{argmin} \quad \|\boldsymbol{a}\| \ \text{ s.t. } \ \boldsymbol{a} \in A(\boldsymbol{x}) \setminus \{\boldsymbol{0}\}. \tag{5}$$

If $\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x}))$ is infeasible, we know that $\boldsymbol{x}$ is a fixed point. We formulate $\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x}))$ as a mixed integer program, and solve it with an off-the-shelf solver [see e.g., 20]. Once we know that $\boldsymbol{x}$ is fixed, we can certify $\mathsf{Recourse}(\boldsymbol{x}, f, A) = \texttt{Yes}$ if $f(\boldsymbol{x}) = +1$ and $\texttt{No}$ if $f(\boldsymbol{x}) = -1$. This approach avoids loopholes and blindspots by addressing the key requirements for verification. In particular, it supports a rich class of actionability constraints. We present a formulation that can encode all actionability constraints from Table 1 in Appendix C.

Identifying Reachable Samples The next method can certify recourse by testing if a point $\boldsymbol{x}$ can reach another point $\boldsymbol{x}'$ assigned a positive prediction:

$$\mathsf{IsReachable}(\boldsymbol{x}, \boldsymbol{x}', A(\boldsymbol{x})) \quad := \quad \min \; 1 \quad \text{s.t.} \quad \boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{a},\; \boldsymbol{a} \in A(\boldsymbol{x}) \tag{6}$$

As before, we formulate this problem as mixed integer program and solve it using an off-the-shelf solver. Given a set of positive samples S^+, we can apply this method for all $\boldsymbol{x}' \in S^+$ to maximize the chance of finding if the point $\boldsymbol{x}$ has recourse. If we have identified such reachable points using this method, we can certify $\mathsf{Recourse}(\boldsymbol{x}, f, A) = \texttt{Yes}$.

Reachable Set Generation Our next method can certify feasibility and infeasibility in a discrete feature space by constructing a reachable set. We present the procedure for generating a reachable set of a given point $\boldsymbol{x}$ in Algorithm 1. For this, we repeatedly solve $\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x}))$ (line 3), while removing the previous solution from the considered action set at every next step (line 5). The procedure continues until the problem becomes infeasible or another stopping condition is met. For example, as described in Section 3, we might be interested in generating only a subset of the reachable set $R_A^{\text{int}}(\boldsymbol{x}) \subset R_A(\boldsymbol{x})$. In this case, the stopping condition could be that the algorithm has identified a certain minimum number of points in the reachable set.

Algorithm 1 GetReachableSet

Require: $\boldsymbol{x} \in \mathcal{X}$, where $\mathcal{X}$ is discrete; $A(\boldsymbol{x})$
Require: Action Sets $A(\boldsymbol{x})$

```
R ← {x}, F ← A(x)
repeat
    if FindAction(x, F) is feasible then
        a* ← FindAction(x, F)
        R ← R ∪ {x + a*}
        F ← F \ {a*}
until stopping condition
```

Output $R \subseteq R_A(\boldsymbol{x})$

Full Sample Audit Generating reachable sets for multiple points in a dataset could pose computational challenges. However, the properties of fixed points and fixed regions outlined in Section 3 enable us to optimize this process by reducing the number of points in a dataset for which we need to generate the reachable set. For instance, if a reachable set is a fixed region, then by definition we do not need to generate reachable sets for any other point in the fixed region. We detail the full auditing procedure with optimizations in Algorithm 2. Finally, in order to verify recourse we call the verification procedure in Eq. (3) for each point and its reachable set. In certain use cases, we can do this without the need to get the reachable sets of all points in a dataset. For example, if we run the audit during model development in order to ensure recourse feasibility, then we can stop once we find any prediction without recourse.

Algorithm 2 SampleAudit

Require: Sample $S = \{\boldsymbol{x}_i\}_{i=1}^n$; $A(\cdot)$

```
ℂ ← { }
repeat
    x_i ← Pop(S)
    R_i ← GenReachableSet(x_i, A(x_i))
    if R_i = {x_i} then    (for separable action sets)
        S ← S \ N_A(x_i, A(x_i))
    else if R_i is a fixed region then
        S ← S \ R_i
    ℂ ← ℂ ∪ {R_i}
until no points remain in S
```

Output $\mathbb{C}$, collection of reachable sets for $\boldsymbol{x}_i$

5 Experiments

We present an empirical study of infeasibility in recourse. Our goal is to study the prevalence of predictions without recourse under actionability constraints, and to characterize the reliability of verification using existing methods. We include code to reproduce our results in our anonymized repository and additional details in Appendix D.

			Simple			Separable			Non-Separable		
Dataset	Model Type	Metrics	Reach	AR	DiCE	Reach	AR	DiCE	Reach	AR	DiCE
german	LR	Recourse	90.8%	99.1%	94.7%	91.9%	95.5%	82.7%	94.9%	22.4%	11.3%
Dua and Graff [14]		No Recourse	0.0%	0.9%	—	0.4%	4.5%	—	2.1%	4.5%	—
$n = 1,000$		Abstain	9.2%	—	—	7.7%	—	—	3.0%	—	—
$d = 24$		Loopholes	—	0.0%	0.0%	—	0.0%	0.0%	—	73.1%	71.4%
		Blindspots	—	0.0%	5.3%	—	0.0%	17.3%	—	0.0%	17.3%
	XGB	Recourse	90.4%		94.7%	91.9%		84.5%	94.9%		21.3%
		No Recourse	0.0%		—	0.4%		—	2.1%		—
		Abstain	9.6%	—	—	7.7%	—	—	3.0%	—	—
		Loopholes	—		0.0%	—		0.0%	—		62.8%
		Blindspots	—		5.3%	—		15.5%	—		16.0%
givemecredit	LR	Recourse	100.0%	100.0%	100.0%	99.9%	99.9%	99.9%	99.9%	64.9%	53.6%
Kaggle [32]		No Recourse	0.0%	0.0%	—	0.0%	0.1%	—	0.1%	0.0%	—
$n = 8,000$		Abstain	0.0%	—	—	0.1%	—	—	0.1%	—	—
$d = 13$		Loopholes	—	0.0%	0.0%	—	0.0%	0.0%	—	35.1%	46.4%
		Blindspots	—	0.0%	0.0%	—	0.0%	0.1%	—	0.0%	0.0%
	XGB	Recourse	100.0%		100.0%	99.9%		99.9%	99.9%		28.6%
		No Recourse	0.0%		—	0.0%		—	0.1%		—
		Abstain	0.0%	—	—	0.1%	—	—	0.1%	—	—
		Loopholes	—		0.0%	—		0.0%	—		71.4%
		Blindspots	—		0.0%	—		0.1%	—		0.0%
heloc	LR	Recourse	24.7%	85.2%	51.7%	29.0%	62.2%	48.2%	57.3%	23.5%	20.4%
FICO [16]		No Recourse	0.0%	14.8%	—	0.0%	37.8%	—	41.9%	37.8%	—
$n = 3,184$		Abstain	75.3%	—	—	71.0%	—	—	0.8%	—	—
$d = 29$		Loopholes	—	0.0%	0.0%	—	0.0%	0.0%	—	38.8%	27.8%
		Blindspots	—	0.0%	48.3%	—	0.0%	51.8%	—	0.0%	51.8%
	XGB	Recourse	84.4%		99.8%	35.1%		57.5%	73.2%		23.4%
		No Recourse	0.0%		—	0.0%		—	26.4%		—
		Abstain	15.6%	—	—	64.9%	—	—	0.5%	—	—
		Loopholes	—		0.0%	—		0.0%	—		34.1%
		Blindspots	—		0.2%	—		42.5%	—		42.5%

Table 2: Reliability of recourse of verification over datasets, model classes, and actionability constraints. We determine the feasibility of recourse for training examples using our method (Reach), and use them to evaluate the reliability of baseline methods for verification. We report the following metrics for each method and model type: *Recourse* – % of points where a method certifies that recourse exists, *No Recourse* – % of points where the method certifies no recourse exists, *Abstain* – % of points in which Reach cannot determine if recourse exists, *Loopholes* – % of points whose actions are inactionable, *Blindspots* – % of points where a method fails to return an action.

Setup We work with three publicly available lending datasets in Table 2. Each dataset pertains to a task where predictions without recourse preclude access to credit and recourse provision. We use each dataset to fit a classifier using *logistic regression* (LR) and *XGBoost* (XGB). We construct reachable sets for each training example using Algorithm 1 in CPLEX v22.1 on a 3.2GHz CPU with 8GB RAM. We benchmark our method (Reach) against baseline methods for recourse provision: AR [65], a *model-specific* method that can certify infeasibility for linear models with separable actionability constraints; DiCE [52], a *model-agnostic* method that supports separable actionability constraints. We evaluate the feasibility of recourse under nested action sets: Non-Separable, which includes constraints on immutability, monotonicity, and non-separable constraints such as causal relations; Separable, which includes constraints on immutability and monotonicity; Simple, which includes constraints on immutability. To certify feasibility, we use standard discretization into bins of continuous features. We summarize the reliability of recourse verification for each dataset, method, and model class in Table 2. In what follows, we discuss these results in detail.

On Predictions without Recourse Our results show how recourse may not exist under actionability constraints. Recourse is generally feasible under simple constraints such as immutability and integrality, with infeasibility mainly arising as we consider more complex constraints. On german, for example, LR only assigns predictions without recourse under Separable and Non-Separable constraints to 0.4% and 2.1% of the data, respectively.

We find minor variations in the prevalence of predictions without recourse across model types. Given that the reachable sets do not change under a given dataset and actionability constraints, these differences stem from differences at the prediction level. We observe that predictions without recourse can drastically change across model types that are equally accurate at a population level. In Non-Separable heloc, for example, we observe a 15.5% difference in predictions without recourse between LR and XGB even though both classifiers have similar performance in terms of the area under the ROC curve (AUC) on the test dataset (0.729 vs 0.737). This highlights the potential to choose between models to ensure recourse.

On Pitfalls of Verification Our results highlight common failure modes in using methods for recourse provision for verification as described in Section 2. In heloc, for example, we observe 26.4% of points without recourse with XGB. In cases where recourse is feasible, methods may fail to return any actions. However, we may be unable to tell if a point has recourse or if a method was unable to generate it in the first place. For example, DiCE fails to find actions for 42.5% of points. However, there may exist feasible actions. This effect highlights the potential failure to account for actionability constraints by, e.g., post-hoc filtering [44]. In heloc, DiCE is unable handle the deterministic causality constraints between features and increasing monotonic thermometer encoding. Although the violations of these constraints are sometimes possible to detect by inspection of the actions, they will be more difficult to detect in cases with multiple constraints.

On the Value of Model-Agnostic Verification Although methods such as AR are capable of providing a certificate of infeasibility for linear models under separable constraints, they start yielding loophole actions as soon as the constraints become more complex. For instance, in heloc, AR flags 37.8% of predictions as without recourse with Separable, but fails to certify additional 4.1% of predictions without recourse under Non-Separable constraints. Reach does have limitations in verifying recourse when it hits a predefined hard limit for building out the full reachable that we set to 25 points for these experiments. Although the number of abstentions can be reduced by increasing the hard limit, we can also certify more points by using the methods in conjunction with one another. For instance, for points where Reach abstains, AR can be used in non-separable cases to determine the feasibility of a point, and if DiCE can find actions for an individual, Reach can determine if the action is feasible. In cases where DiCE or AR cannot find actions for constraints that are non-separable, Reach can be used to determine if the point is fixed or if we indeed observe recourse infeasibility.

6 Demonstration

We now demonstrate how our methods can ensure recourse in lending. We include an additional study to show how we can evaluate adversarial robustness in content moderation in Appendix E, and code to reproduce these results in our anonymized repository.

Setup We perform recourse verification on the FICO heloc dataset [16], which covers $n = 3\,184$ consumers and $d = 29$ features about their credit history. Here, $y_i = +1$ if a consumer i has duly repaid a home equity loan. Our goal is to ensure that each training example has recourse – so that we can flag models that permanently deny access to credit [8, 38], and use recourse provision methods to produce adverse action notices [1]. We work with a domain expert in the U.S. credit industry to identify common constraints on features. Our final action set includes 24 constraints, both separable (e.g., RevolvingTradesWBalance is a positive integer, MostRecentTradeLastYear can only increase), and non-separable (e.g., RevolvingDebtBurdenLeq30, RevolvingDebtBurdenGeq60).

Results We generate reachable sets for all points in the training data using Algorithm 1 and use them to perform recourse verification for LR and XGB classifiers. We summarize the feasibility of recourse across the 3179 reachable sets in Fig. 2. Our results reveal 733 predictions without recourse for LR, and 453 for XGB. In this case, we find 5 fixed points that are assigned positive predictions. Thus, all predictions without recourse stem from a generalized reachable set. The mix of individual feature constraints and constraints on the interactions between features causes fixed points and reachable sets with no recourse.

Predictions without recourse may have serious implications for consumers attempting to acquire a loan. A specific example of this is a consumer with 10 years account history and 15 open credit and fixed loans with a majority of them paid off. Among their open fixed loans, there is a substantial remaining balance that needs to be paid, and they experienced a delinquent trade within the past year. They are denied by both classifiers. This consumer has the ability to reach 7 other points by reducing

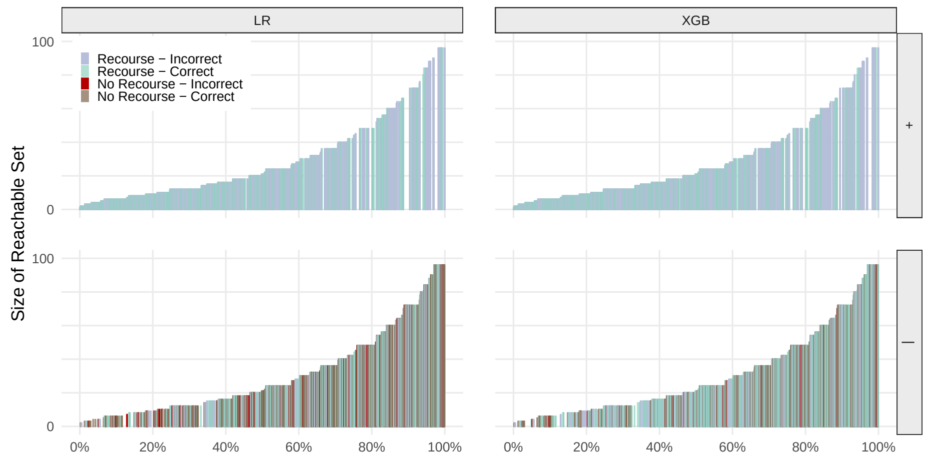

Figure 2: Composition of reachable sets for the `heloc` for LR and XGB. Each plot shows the size of reachable sets for each training example delineated by Algorithm 1. The top row displays sizes of reachable sets for samples with negative predictions and the bottom row for samples with positive predictions. Correct/incorrect denotes where the true label does/does not equal the predicted label and Recourse/No Recourse denotes if recourse is feasible/infeasible. We highlight predictions without recourse for both correctly classified and incorrectly classified negative points. We can see predictions without recourse are prevalent with all reachable set sizes and can be drastically different between classifiers.

their one credit card loan with balance and increasing the number of years they have open and active loans; however, even if with all these changes, they will still be denied approval.

These results can guide interventions in model development to ensure recourse. A simple solution is to use the information for model selection, e.g., choosing models that limit the number of predictions without recourse. Here, for all instances where XGB has no recourse, LR has no recourse as well. Surprisingly, both classifiers have similar performance in terms of AUC (see Section 5), but XGB has 280 fewer predictions without recourse. Additionally, immutable features can have a significant impact on recourse feasibility. These features cannot be omitted as they can carry important information such as `MaxDelqEver` (maximum duration of a delinquency). However, if the immutable features are assigned high importance by the classifiers whereas the mutable features are assigned low importance, the changes in the mutable features may not have enough weight to overturn the model's prediction. This can result in large regions without recourse.

7 Concluding Remarks

Our work highlights the inherent value of infeasibility in recourse – i.e., as a tool to ensure access in consumer-facing applications. The normative basis for access stems from principles of equality of opportunity (e.g., in hiring) and universal access (e.g., in affordable healthcare). In such settings, we would a model to ensure access – even if this comes at a cost – because it reflects the kind of society we want to build. Ensuring access may benefit all stakeholders in consumer-facing applications. In lending, for example, we only observe labels for consumers who are assigned a specific prediction [due to selective labeling 10, 43, 73]. Thus, consumers who are assigned predictions without recourse cannot generate labeled examples to signal their creditworthiness [8]. In the United States, such effects have led to a loss in credit access for large consumer segments whose creditworthiness is not known [see, e.g., 26M "credit invisibles" 74].

Limitations Our treatment of infeasibility does not consider probabilistic causal effects – e.g., when altering a feature may induce changes in other features as per a probabilistic causal model [see e.g., 11, 37, 42, 70]. Our machinery may be useful for modeling complex interactions in such settings [11]. However, infeasibility requires a probabilistic definition that can characterize the "reducible" uncertainty due to model development from the uncertainty in the underlying causal process. Finally, given that actionability varies across individuals as well as subpopulations, recourse verification should be used as a basic test to flag models that will preclude access, rather than as a tool to "rubber-stamp" for safety.

References

[1] 2013. URL https://www.consumercomplianceoutlook.org/2013/second-quarter/adverse-action-notice-requirements-under-ecoa-fcra/.

[2] Alon, Tal, Magdalen Dobson, Ariel Procaccia, Inbal Talgam-Cohen, and Jamie Tucker-Foltz. Multiagent evaluation mechanisms. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 1774–1781, 2020.

[3] Altmeyer, Patrick, Giovan Angela, Aleksander Buszydlik, Karol Dobiczek, Arie Deursenvan , and Cynthia Liem. Endogenous macrodynamics in algorithmic recourse. In *First IEEE Conference on Secure and Trustworthy Machine Learning*.

[4] Arneson, Richard. Equality of Opportunity. In Zalta, Edward N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Summer 2015 edition, 2015.

[5] Bogen, Miranda and Aaron Rieke. Help wanted: An examination of hiring algorithms, equity, and bias. *Upturn, December*, 7, 2018.

[6] Calzavara, Stefano, Claudio Lucchese, Gabriele Tolomei, Seyum Assefa Abebe, and Salvatore Orlando. Treant: training evasion-aware decision trees. *Data Min. Knowl. Discov.*, 2020.

[7] Chen, Yatong, Jialu Wang, and Yang Liu. Strategic recourse in linear classification. *arXiv preprint arXiv:2011.00355*, 2020.

[8] Chien, Jennifer, Margaret Ew Roberts, and Berk Ustun. Learning through Recourse under Censoring. *NeurIPS Workshop on Learning and Decision-Making with Strategic Feedback*, 2021.

[9] Daniels, Norman. Equity of access to health care: some conceptual and ethical issues. *The Milbank Memorial Fund Quarterly. Health and Society*, pages 51–81, 1982.

[10] De-Arteaga, Maria, Artur Dubrawski, and Alexandra Chouldechova. Learning under selective labels in the presence of expert consistency. *arXiv preprint arXiv:1807.00905*, 2018.

[11] Dominguez-Olmedo, Ricardo, Amir H Karimi, and Bernhard Schölkopf. On the adversarial robustness of causal algorithmic recourse. In *International Conference on Machine Learning*, pages 5324–5342. PMLR, 2022.

[12] Dong, Jinshuo, Aaron Roth, Zachary Schutzman, Bo Waggoner, and Zhiwei Steven Wu. Strategic classification from revealed preferences. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 55–70. ACM, 2018.

[13] Downs, Michael, Jonathan L Chu, Yaniv Yacoby, Finale Doshi-Velez, and Weiwei Pan. Cruds: Counterfactual recourse using disentangled subspaces. *ICML WHI*, 2020:1–23, 2020.

[14] Dua, Dheeru and Casey Graff. UCI machine learning repository, 2017. URL http://archive.ics.uci.edu/ml.

[15] Eubanks, Virginia. *Automating inequality: How high-tech tools profile, police, and punish the poor*. St. Martin's Press, 2018.

[16] FICO. Fico heloc, 2018. URL https://community.fico.com/s/explainable-machine-learning-challenge.

[17] Ghalme, Ganesh, Vineet Nair, Itay Eilat, Inbal Talgam-Cohen, and Nir Rosenfeld. Strategic classification in the dark. In *International Conference on Machine Learning*, pages 3672–3681. PMLR, 2021.

[18] Gilani, Zafar, Ekaterina Kochmar, and Jon Crowcroft. Classification of twitter accounts into automated agents and human users. In *Proceedings of the 2017 IEEE/ACM international conference on advances in social networks analysis and mining 2017*, pages 489–496, 2017.

[19] Gilman, Michele E. Poverty lawgorithms: A poverty lawyer's guide to fighting automated decision-making harms on low-income communities. *Data & Society*, 2020.

[20] Gleixner, Ambros, Gregor Hendel, Gerald Gamrath, Tobias Achterberg, Michael Bastubbe, Timo Berthold, Philipp Christophel, Kati Jarck, Thorsten Koch, Jeff Linderoth, et al. Miplib 2017: data-driven compilation of the 6th mixed-integer programming library. *Mathematical Programming Computation*, 13(3):443–490, 2021.

[21] Guo, Hangzhi, Feiran Jia, Jinghui Chen, Anna Squicciarini, and Amulya Yadav. Rocoursenet: Distributionally robust training of a prediction aware recourse model. *arXiv preprint arXiv:2206.00700*, 2022.

[22] Gupta, Vivek, Pegah Nokhiz, Chitradeep Dutta Roy, and Suresh Venkatasubramanian. Equalizing recourse across groups. *arXiv preprint arXiv:1909.03166*, 2019.

[23] Hardt, Moritz, Nimrod Megiddo, Christos Papadimitriou, and Mary Wootters. Strategic classification. In *Proceedings of the 2016 ACM Conference on Innovations in Theoretical Computer Science*, pages 111–122. ACM, 2016.

[24] Hardt, Moritz, Eric Mazumdar, Celestine Mendler-Dünner, and Tijana Zrnic. Algorithmic collective action in machine learning. *arXiv preprint arXiv:2302.04262*, 2023.

[25] Harris, Keegan, Hoda Heidari, and Steven Z Wu. Stateful strategic regression. *Advances in Neural Information Processing Systems*, 34:28728–28741, 2021.

[26] Harris, Keegan, Valerie Chen, Joon Kim, Ameet Talwalkar, Hoda Heidari, and Steven Z Wu. Bayesian persuasion for algorithmic recourse. *Advances in Neural Information Processing Systems*, 35:11131–11144, 2022.

[27] Heidari, Hoda, Vedant Nanda, and Krishna Gummadi. On the long-term impact of algorithmic decision policies: Effort unfairness and feature segregation through social learning. In *International Conference on Machine Learning*, pages 2692–2701. PMLR, 2019.

[28] Hein, Matthias and Maksym Andriushchenko. Formal guarantees on the robustness of a classifier against adversarial manipulation. In *NIPS*, 2017.

[29] Hurley, Mikella and Julius Adebayo. Credit scoring in the era of big data. *Yale JL & Tech.*, 18:148, 2016.

[30] Joshi, Shalmali, Oluwasanmi Koyejo, Warut Vijitbenjaronk, Been Kim, and Joydeep Ghosh. Towards realistic individual recourse and actionable explanations in black-box decision making systems. *arXiv preprint arXiv:1907.09615*, 2019.

[31] Jurgovsky, Johannes, Michael Granitzer, Konstantin Ziegler, Sylvie Calabretto, Pierre-Edouard Portier, Liyun He-Guelton, and Olivier Caelen. Sequence classification for credit-card fraud detection. *Expert Systems with Applications*, 100:234–245, 2018.

[32] Kaggle. Give Me Some Credit. http://www.kaggle.com/c/GiveMeSomeCredit/, 2011.

[33] Kanamori, Kentaro, Takuya Takagi, Ken Kobayashi, and Yuichi Ike. Counterfactual explanation trees: Transparent and consistent actionable recourse with decision trees. In *International Conference on Artificial Intelligence and Statistics*, pages 1846–1870. PMLR, 2022.

[34] Kantchelian, Alex, J. D. Tygar, and Anthony D. Joseph. Evasion and hardening of tree ensemble classifiers. In *ICML*, 2016.

[35] Karimi, Amir-Hossein, Julius Von Kügelgen, Bernhard Schölkopf, and Isabel Valera. Algorithmic recourse under imperfect causal knowledge: a probabilistic approach. *Advances in neural information processing systems*, 33:265–277, 2020.

[36] Karimi, Amir-Hossein, Gilles Barthe, Bernhard Schölkopf, and Isabel Valera. A survey of algorithmic recourse: definitions, formulations, solutions, and prospects. 2021.

[37] Karimi, Amir-Hossein, Bernhard Schölkopf, and Isabel Valera. Algorithmic recourse: from counterfactual explanations to interventions. In *Proceedings of the 2021 ACM Conference on Fairness, Accountability, and Transparency*, pages 353–362, 2021.

[38] Kilbertus, Niki, Manuel Gomez-Rodriguez, Bernhard Schölkopf, Krikamol Muandet, and Isabel Valera. Improving consequential decision making under imperfect predictions. *arXiv preprint arXiv:1902.02979*, 2019.

[39] Kireev, Klim, Bogdan Kulynych, and Carmela Troncoso. Adversarial robustness for tabular data through cost and utility awareness. In *Network and Distributed System Security (NDSS) Symposium*, 2023.

[40] Kleinberg, Jon and Manish Raghavan. How Do Classifiers Induce Agents To Invest Effort Strategically? *ArXiv e-prints*, art. arXiv:1807.05307, July 2018.

[41] Kleinberg, Jon and Manish Raghavan. How do classifiers induce agents to invest effort strategically? *ACM Transactions on Economics and Computation (TEAC)*, 8(4):1–23, 2020.

[42] König, Gunnar, Timo Freiesleben, and Moritz Grosse-Wentrup. Improvement-focused causal recourse (icr). *arXiv preprint arXiv:2210.15709*, 2022.

[43] Lakkaraju, Himabindu, Jon Kleinberg, Jure Leskovec, Jens Ludwig, and Sendhil Mullainathan. The selective labels problem: Evaluating algorithmic predictions in the presence of unobservables. In *Proceedings of the 23rd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 275–284, 2017.

[44] Laugel, Thibault, Adulam Jeyasothy, Marie-Jeanne Lesot, Christophe Marsala, and Marcin Detyniecki. Achieving diversity in counterfactual explanations: a review and discussion. *arXiv preprint arXiv:2305.05840*, 2023.

[45] Levanon, Sagi and Nir Rosenfeld. Strategic classification made practical. In *International Conference on Machine Learning*, pages 6243–6253. PMLR, 2021.

[46] Lowd, Daniel and Christopher Meek. Adversarial learning. In *ACM SIGKDD international conference on Knowledge discovery in data mining*, pages 641–647, 2005.

[47] Mahajan, Divyat, Chenhao Tan, and Amit Sharma. Preserving causal constraints in counterfactual explanations for machine learning classifiers. *arXiv preprint arXiv:1912.03277*, 2019.

[48] Makar, Maggie, Ben Packer, Dan Moldovan, Davis Blalock, Yoni Halpern, and Alexander D'Amour. Causally motivated shortcut removal using auxiliary labels. In *International Conference on Artificial Intelligence and Statistics*, pages 739–766. PMLR, 2022.

[49] Mathov, Yael, Eden Levy, Ziv Katzir, Asaf Shabtai, and Yuval Elovici. Not all datasets are born equal: On heterogeneous tabular data and adversarial examples. *Knowledge-Based Systems*, 242:108377, 2022.

[50] Miller, John, Smitha Milli, and Moritz Hardt. Strategic classification is causal modeling in disguise. In *International Conference on Machine Learning*, pages 6917–6926. PMLR, 2020.

[51] Milli, Smitha, John Miller, Anca D Dragan, and Moritz Hardt. The social cost of strategic classification. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pages 230–239, 2019.

[52] Mothilal, Ramaravind K, Amit Sharma, and Chenhao Tan. Explaining machine learning classifiers through diverse counterfactual explanations. In *Proceedings of the 2020 conference on fairness, accountability, and transparency*, pages 607–617, 2020.

[53] Nguyen, Duy, Ngoc Bui, and Viet Anh Nguyen. Feasible recourse plan via diverse interpolation. In *International Conference on Artificial Intelligence and Statistics*, pages 4679–4698. PMLR, 2023.

[54] O'Brien, Andrew and Edward Kim. Toward multi-agent algorithmic recourse: Challenges from a game-theoretic perspective. In *The International FLAIRS Conference Proceedings*, volume 35, 2022.

[55] Parmentier, Axel and Thibaut Vidal. Optimal counterfactual explanations in tree ensembles. In *International Conference on Machine Learning*, pages 8422–8431. PMLR, 2021.

[56] Pawelczyk, Martin, Teresa Datta, Johannes Heuvelvan-den , Gjergji Kasneci, and Himabindu Lakkaraju. Algorithmic recourse in the face of noisy human responses. *arXiv preprint arXiv:2203.06768*, 2022.

[57] Quinzan, Francesco, Cecilia Casolo, Krikamol Muandet, Niki Kilbertus, and Yucen Luo. Learning counterfactually invariant predictors. *arXiv preprint arXiv:2207.09768*, 2022.

[58] Raghavan, Manish, Solon Barocas, Jon Kleinberg, and Karen Levy. Mitigating bias in algorithmic hiring: Evaluating claims and practices. In *Proceedings of the 2020 conference on fairness, accountability, and transparency*, pages 469–481, 2020.

[59] Ramakrishnan, Goutham, Yun Chan Lee, and Aws Albarghouthi. Synthesizing action sequences for modifying model decisions. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 34, pages 5462–5469, 2020.

[60] Rawal, Kaivalya, Ece Kamar, and Himabindu Lakkaraju. Algorithmic recourse in the wild: Understanding the impact of data and model shifts. *arXiv preprint arXiv:2012.11788*, 2020.

[61] Ross, Alexis, Himabindu Lakkaraju, and Osbert Bastani. Learning models for actionable recourse. *Advances in Neural Information Processing Systems*, 34:18734–18746, 2021.

[62] Shavit, Yonadav, Benjamin Edelman, and Brian Axelrod. Causal strategic linear regression. In *International Conference on Machine Learning*, pages 8676–8686. PMLR, 2020.

[63] Tolomei, Gabriele, Fabrizio Silvestri, Andrew Haines, and Mounia Lalmas. Interpretable predictions of tree-based ensembles via actionable feature tweaking. In *Proceedings of the 23rd ACM SIGKDD international conference on knowledge discovery and data mining*, pages 465–474, 2017.

[64] Upadhyay, Sohini, Shalmali Joshi, and Himabindu Lakkaraju. Towards robust and reliable algorithmic recourse. *arXiv preprint arXiv:2102.13620*, 2021.

[65] Ustun, Berk, Alexander Spangher, and Yang Liu. Actionable recourse in linear classification. pages 10–19. doi: 10.1145/3287560.3287566.

[66] Veitch, Victor, Alexander D'Amour, Steve Yadlowsky, and Jacob Eisenstein. Counterfactual invariance to spurious correlations: Why and how to pass stress tests. *arXiv preprint arXiv:2106.00545*, 2021.

[67] Venkatasubramanian, Suresh and Mark Alfano. The philosophical basis of algorithmic recourse. In *Proceedings of the 2020 Conference on Fairness, Accountability, and Transparency*, pages 284–293, 2020.

[68] Verma, Sahil, John Dickerson, and Keegan Hines. Counterfactual explanations for machine learning: A review, 2020.

[69] Verma, Sahil, Keegan Hines, and John P Dickerson. Amortized generation of sequential algorithmic recourses for black-box models. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 8512–8519, 2022.

[70] Kügelgen, Juliusvon , Amir-Hossein Karimi, Umang Bhatt, Isabel Valera, Adrian Weller, and Bernhard Schölkopf. On the fairness of causal algorithmic recourse. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 9584–9594, 2022.

[71] Vos, Daniël and Sicco Verwer. Efficient training of robust decision trees against adversarial examples. In Meila, Marina and Tong Zhang, editors, *ICML*, 2021.

[72] Wang, Zijie J, Jennifer Wortman Vaughan, Rich Caruana, and Duen Horng Chau. Gam coach: Towards interactive and user-centered algorithmic recourse. In *Proceedings of the 2023 CHI Conference on Human Factors in Computing Systems*, pages 1–20, 2023.

[73] Wei, Dennis. Decision-making under selective labels: Optimal finite-domain policies and beyond. In *International Conference on Machine Learning*, pages 11035–11046. PMLR, 2021.

[74] Wilhelm, Colin. Big data and the credit gap. https://www.politico.com/agenda/story/2018/02/07/big-data-credit-gap-000630, February 2018.

[75] Wykstra, S. Government's use of algorithm serves up false fraud charges. undark, 6 january, 2020.

[76] Yadav, Prateek, Peter Hase, and Mohit Bansal. Inspire: A framework for integrating individual user preferences in recourse.

[77] Yadav, Prateek, Peter Hase, and Mohit Bansal. Low-cost algorithmic recourse for users with uncertain cost functions. *arXiv preprint arXiv:2111.01235*, 2021.

Prediction without Preclusion
Recourse Verification with Reachable Sets

Supplementary Material

A Notation

Symbol	**Meaning**
$\mathcal{X} \subseteq \mathbb{R}^d$	feature space
$\mathcal{Y} = \{-1, +1\}$	label space
$\mathcal{X}_j \subseteq \mathbb{R}$	feature space for feature j
$\boldsymbol{a} \in A(\boldsymbol{x})$	action vector from $\boldsymbol{x} \in \mathcal{X}$
$A(\boldsymbol{x})$	action set for $\boldsymbol{x} \in \mathcal{X}$. $\mathbf{0} \in A(\boldsymbol{x})$
$A_j(\boldsymbol{x})$	set of feasible actions for feature j from $\boldsymbol{x}$
$R_A(\boldsymbol{x}) := \{\boldsymbol{x} + \boldsymbol{a} \mid \boldsymbol{a} \in A(\boldsymbol{x})\}$	reachable set from $\boldsymbol{x}$
$R_A^{\text{int}}(\boldsymbol{x}) \subset R_A(\boldsymbol{x})$	inner approximation of a reachable set from $\boldsymbol{x}$
$N_A(\boldsymbol{x})$	set of sibling points from $\boldsymbol{x}$
$f : \mathcal{X} \to \mathcal{Y}$	classification model
$S^+ := \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n^+}$ s.t. $y_i = +1$	a set of positive examples
$S^- := \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n^+}$ s.t. $y_i = -1$	a set of negative examples
$n^+ := \lvert S^+ \rvert$	number of positive examples in a sample
$n^- := \lvert S^- \rvert$	number of negative examples in a sample
$[k] := \{1, \ldots, k\}$	set of positive integers from 1 to k

Table 3: Table of Notation

B Proofs

In this Appendix, we present proofs of our claims in Section 3.

B.1 Proof of Theorem 6

Theorem 6. *Suppose we have a dataset of labeled examples $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$. Every model $f : \mathcal{X} \to \mathcal{Y}$ can provide recourse to $\boldsymbol{x}$ if:*

$$\mathsf{FNR}(f) < \frac{1}{n^+} \sum_{i=1}^{n} \mathbb{1}[\boldsymbol{x}_i \in R \ \wedge \ y_i = +1] \tag{7}$$

where $\mathsf{FNR}(f) := \frac{1}{n^+} \sum_{i=1}^{n} \mathbb{1}[f(\boldsymbol{x}_i) = -1 \ \wedge \ y_i = +1]$ is the false negative rate of f and where n^+ is number of positive examples, and $R \subseteq R_A(\boldsymbol{x})$ is any subset of the reachable set.

Proof. The proof is based on an application of the pigeonhole principle over the positive examples $S^+ := \{\boldsymbol{x}_i \mid y_i = +1, i \in [n]\}$. Given a classifier f, denote the total number of true positive and false negative predictions over S^+ as:

$$\mathsf{TP}(f) := \sum_{i=1}^{n} \mathbb{1}[f(\boldsymbol{x}_i) = +1 \wedge y_i = +1] \qquad \mathsf{FN}(f) := \sum_{i=1}^{n} \mathbb{1}[f(\boldsymbol{x}_i) = -1 \wedge y_i = +1].$$

Say that for a given point $\boldsymbol{x}$ with a reachable set $R = R_A(\boldsymbol{x})$, the classifier obeys:

$$\mathsf{TP}(f) > n^+ - |S^+ \cap R|.$$

In other words, the number of correct positive predictions exceeds the number of positive examples outside R. In this case, by the pigeonhole principle, the classifier f must assign a correct prediction to at least one of the positive examples in R – i.e., there exists a point $\boldsymbol{x}' \in S^+ \cap R$ such that $f(\boldsymbol{x}') = y_i = +1$. Given $R \subseteq R_A(\boldsymbol{x})$, we have that $\boldsymbol{x} \in R_A(\boldsymbol{x})$. Thus, we can reach $\boldsymbol{x}'$ from $\boldsymbol{x}$ by performing the action $\boldsymbol{a} = \boldsymbol{x}' - \boldsymbol{x}$ – i.e., we can change the prediction from $f(\boldsymbol{x}) = -1$ to $f(\boldsymbol{x} + \boldsymbol{a}) = +1$.

We recover the condition in the statement of the Theorem as follows:

$$\mathsf{TP}(f) > n^+ - |S^+ \cap R| \tag{8}$$

$$\mathsf{FN}(f) < |S^+ \cap R|, \tag{9}$$

$$\mathsf{FNR}(f) < \frac{1}{n^+} \sum_{i=1}^{n} \mathbb{1}[\boldsymbol{x}_i \in R \ \wedge \ y_i = +1] \tag{10}$$

Here, we proceed from Eqn. (8) to Eqn. (9) by using the fact that $\mathsf{TP}(f) = n^+ - \mathsf{FN}(f)$, and from Eqn. (9) to (10) by dividing both sides by $\frac{1}{n^+}$ and applying the definition of the false negative rate. □

B.2 Proof of Proposition 4

Proposition 4. *Any classification task with bounded features whose actions obey monotonicity constraints must contain at least one fixed point.*

Proof. Consider a set of d features $(x_1, \ldots, x_d) = \boldsymbol{x} \in \mathcal{X}$ over a bounded feature space. Let l_j and u_j denote the lower and upper bounds on feature j, so that $x_j \in [l_j, u_j]$ for all $j \in [d]$

We will proceed to construct a fixed point over $\mathcal{X}$ under the following conditions: (i) each feature is monotonically increasing, so that $a_j \geq 0$ for all $j \in [d]$; (ii) each feature is monotonically decreasing, so that $a_j \leq 0$ for all $j \in [d]$; (iii) each feature is either monotonically increasing or monotonically decreasing so that $a_j \geq 0$ or $a_j \leq 0$ for all $j \in [d]$.

In the case of (i), the fixed point corresponds to a feature vector $\boldsymbol{x} \in \mathcal{X}$ such that $x_j = u_j$ for all $j \in [d]$. We proceed by contradiction. Suppose $\boldsymbol{x}$ is not a fixed point, then there exists an action

$\boldsymbol{a}' \in A(\boldsymbol{x})$ such that $\boldsymbol{a}' \neq \{\boldsymbol{0}\}$. In turn, there exists $a'_i > 0$. Let $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a}'$ then $x'_i = x_i + a'_i > u_i$ which violates our initial assumption that $x'_i \in [l_i, u_i]$. Thus, $\boldsymbol{x}$ must be a fixed point.

In the case of (ii), the fixed point corresponds to a feature vector $\boldsymbol{x} \in \mathcal{X}$ such that $x_j = l_j$ for all $j \in [d]$. We proceed by contradiction. Suppose $\boldsymbol{x}$ is not a fixed point, then there exists an action $\boldsymbol{a}' \in A(\boldsymbol{x})$ such that $\boldsymbol{a}' \neq \{\boldsymbol{0}\}$. In turn, there exists $a'_j < 0$. Let $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{a}'$ then $x'_j = x_j + a'_j < l_j$ which violates our initial assumption that $x'_j \in [l_j, u_j]$. Thus, $\boldsymbol{x}$ must be a fixed point.

In the case of (iii), by combining the above, it can be seen that as long as x_j satisfies monotonicity constraints which can either be increasing or decreasing there must contain at least one fixed point

$$x_i = \begin{cases} u_j, \text{if } j \in J_+ \\ l_j, \text{if } j \in J_- \end{cases}$$

where J_+ is the set of indices with monotonically increasing constraints and J_- is the set of indices with monotonically decreasing constraints.

□

B.3 Proof of Proposition 5

Proposition 5. *Consider adding a new feature $\mathcal{X}_{d+1} \subseteq \mathbb{R}$ to a set of d features $\mathcal{X} \subseteq \mathbb{R}^d$. Any fixed point $\boldsymbol{x} \in \mathcal{X}$ induces the following confined regions in the $(d+1)$-dimensional space:*

- *$|\mathcal{X}_{d+1}|$ fixed points in the $(d+1)$-dimensional feature space, if x_{d+1} is immutable.*
- *A fixed point $\boldsymbol{z}_0 := (\boldsymbol{x}, x_{d+1})$ where x_{d+1} is an* extreme point *of $\mathcal{X}_{d+1}$, that is, $x_{d+1} := \max \mathcal{X}_{d+1}$ or $x_{d+1} := \min \mathcal{X}_{d+1}$ if $(d+1)$-th feature is monotonically increasing (respectively, decreasing) in the action set $A(\boldsymbol{z}_0)$, and the constraints in $A(\boldsymbol{z}_0)$ are separable.*
- *A fixed region if $R_A(x_1, x_2, \ldots, x_d, x_{d+1}) = R_A(x_1, x_2, \ldots, x_d, x'_{d+1})$ for any two $x_{d+1}, x'_{d+1} \in \mathcal{X}_{d+1}$.*

Proof. Let us denote the $(d+1)$-dimensional feature space as $\bar{\mathcal{X}} := \mathcal{X}_1 \times \ldots \times \mathcal{X}_d \times \mathcal{X}_{d+1}$.

- Suppose a point $\boldsymbol{x}' \in \bar{\mathcal{X}}$ has the same feature values as $\boldsymbol{x}$ in its first d dimensions. As x_{d+1} is immutable, the only feasible action for x'_{d+1} is $a_{d+1} = 0$. This holds for any possible value of x'_{d+1}. This implies that for all feature values of the $(d+1)$-th feature, $\boldsymbol{x}'$ remains a fixed point. Therefore, there must exist $|\mathcal{X}_{d+1}|$ fixed points.
- Observe that if v_{d+1} is an extreme point, then the only possible action is $a_{d+1} = 0$ because the $d+1$-th feature must satisfy a monotonicity constraint. As the constraints in $A(\boldsymbol{z}_0)$ are separable by assumption, and $A(\boldsymbol{x}) = \{\boldsymbol{0}\}$, $\boldsymbol{z}_0$ must also have only one possible action $A(\boldsymbol{z}_0) = \{\boldsymbol{0}\}$.
- Given any $\boldsymbol{x}' \in \bar{\mathcal{X}}$ where the first d dimensions are the same as in $\boldsymbol{x}$, we have $R_A(\boldsymbol{x}') = R_A(\boldsymbol{x})$. As any other $\boldsymbol{x}'' \in R_A(\boldsymbol{x}')$ also shares the first d dimensions and is also $\boldsymbol{x}'' \in R_A(\boldsymbol{x})$, we have that $R_A(\boldsymbol{x}') \subseteq R_A(\boldsymbol{x})$.

□

C Supplement to Section 4 – Algorithms

In this Appendix, we describe how to formulate and solve the optimization problems in Section 4 as mixed-integer programs. We start by presenting a MIP formulation for the optimization problem solved in the $\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x}))$ routine. We then describe how this formulation can be extended to an optimization problem in the $\mathsf{IsReachable}(\boldsymbol{x}, \boldsymbol{x}', A(\boldsymbol{x}))$ routine. Finally, we describe how this formulation can be extended to the complex actionability constraints in Table 1.

C.1 MIP Formulation for FindAction

Given a point $\boldsymbol{x} \in \mathcal{X}$, an action set $A(\boldsymbol{x})$, and a set of previous optima $\mathcal{A}^{\text{opt}}$, we can formulate $\mathsf{FindAction}(\boldsymbol{x}, A(\boldsymbol{x}))$ as the following mixed-integer program:

$$\begin{aligned}
\min_{\boldsymbol{a}} \quad & \sum_{j \in [d]} a_j^+ + a_j^- & & \\
\text{s.t.} \quad & a_j^+ \geq a_j & j \in [d] \quad & \textit{positive component of } a_j && \text{(11a)}\\
& a_j^- \geq -a_j & j \in [d] \quad & \textit{negative component of } a_j && \text{(11b)}\\
& a_j = a_{j,k} + \delta_{j,k}^+ - \delta_{j,k}^- & j \in [d], \boldsymbol{a}_k \in \mathcal{A}^{\text{opt}} \quad & \textit{distance from prior actions} && \text{(11c)}\\
& \varepsilon_{\text{min}} \leq \sum_{j \in [d]} (\delta_{j,k}^+ - \delta_{j,k}^-) & \boldsymbol{a}_k \in \mathcal{A}^{\text{opt}} \quad & \textit{any solution is } \varepsilon_{min} \textit{ away from } \boldsymbol{a}_k && \text{(11d)}\\
& \delta_{j,k}^+ \leq M_{j,k}^+ u_{j,k} & j \in [d], \boldsymbol{a}_k \in \mathcal{A}^{\text{opt}} \quad & \delta_{j,k}^+ > 0 \implies u_{j,k} = 1 && \text{(11e)}\\
& \delta_{j,k}^- \leq M_{j,k}^- (1 - u_{j,k}) & j \in [d], \boldsymbol{a}_k \in \mathcal{A}^{\text{opt}} \quad & \delta_{j,k}^- > 0 \implies u_{j,k} = 0 && \text{(11f)}\\
& a_j \in A_j(\boldsymbol{x}) & j \in [d] \quad & \textit{separable actionability constraints on } j && \text{(11g)}\\
& \delta_{j,k}^+, \delta_{j,k}^- \in \mathbb{R}_+ & j \in [d] \quad & \textit{signed distances from } a_{j,k} && \text{(11h)}\\
& u_{j,k} \in \{0, 1\} & j \in [d] \quad & u_{j,k} := 1[\delta_{j,k}^+ > 0] && \text{(11i)}
\end{aligned}$$

The formulation finds action in the set $\boldsymbol{a} \in A(\boldsymbol{x})/\mathcal{A}^{\text{opt}}$ by combining two classes of constraints: (i) constraints to restrict actions $\boldsymbol{a} \in A(\boldsymbol{x})$ and (ii) constraints to rule out actions in $\boldsymbol{a} \in \mathcal{A}^{\text{opt}}$.

The formulation encodes the separable constraints in $A(\boldsymbol{x})$ – i.e., a constraint that can be enforced for each feature. The formulation must be extended with additional variables and constraints to handle constraints as discussed in Appendix C.3. These constraints are handled through the $a_j \in A_j(\boldsymbol{x})$ conditions in Constraint 11g. This constraint can handle a number of actionability constraints that can be passed solver when defining the variables a_j, including *bounds* (e.g., $a_j \in [-x_j, 10 - x_j]$), *integrality* (e.g., $a_j \in \{0, 1\}$ or $a_j \in \{L - x_j, L - x_j + 1, \ldots, U - x_j\}$), and *monotonicity* (e.g., $a_j \geq 0$ or $a_j \leq 0$).

The formulation rules out actions in $\boldsymbol{a} \in \mathcal{A}^{\text{opt}}$ through the "no good" constraints in Constraints (11c) to (11f). Here, Constraint (11d) ensures feasible actions from previous solutions by at least ε_{min}. We set to a sufficiently small number $\varepsilon_{\text{min}} := 10^-6$ by default, but use larger values when working with discrete feature sets (e.g., $\varepsilon_{\text{min}} = 1$ for cases where every actionable feature is binary or integer-valued). Constraints (11e) and (11f) ensure that either $\delta_{j,k}^+ > 0$ or $\delta_{j,k}^- > 0$. These are "Big-M constraints" where the Big-M parameters can be set to represent the largest value of signed distances. Given an action $a_j \in [a_j^{\text{LB}}, a_j^{\text{UB}}]$, we can set $M_{j,k}^+ := |a_j^{\text{UB}} - a_{j,k})$ and $M_{j,k}^- := |a_{j,k} - a_j^{\text{LB}}|$.

The formulation chooses each action in $\boldsymbol{a} \in A(\boldsymbol{x})/\mathcal{A}^{\text{opt}}$ to minimize the L_1 norm. We compute the L_1-norm component-wise as $|a_j| := a_j^+ + a_j^-$ where the variables a_j^+ and a_j^- are set to the positive and negative components of $|a_j|$ in Constraints (11a) and (11b). This choice of objective is meant to induce sparsity among the actions we recover by repeatedly solving Algorithm 1. Given that the objective function does not affect the feasibility of the optimization problem, one could set the objective to 1 when solving the problem for fixed-point detection.

C.2 MIP Formulation for IsReachable

Given a point $\boldsymbol{x} \in \mathcal{X}$, an action set $A(\boldsymbol{x})$, we can formulate the optimization problem for $\mathsf{IsReachable}(\boldsymbol{x}, \boldsymbol{x}', A(\boldsymbol{x}))$ as a special case of the MIP in (11) in which we set $\mathcal{A}^{\text{opt}} = \emptyset$ and include the constraint $\boldsymbol{a} = \boldsymbol{x} - \boldsymbol{x}'$. In this case, any feasible solution would certify that $\boldsymbol{x}'$ can be attained from $\boldsymbol{x}$ using the actions in $A(\boldsymbol{x})$. Thus, we can return $\mathsf{IsReachable}(\boldsymbol{x}, \boldsymbol{x}', A(\boldsymbol{x})) = 1$ if the MIP is feasible and $\mathsf{IsReachable}(\boldsymbol{x}, \boldsymbol{x}', A(\boldsymbol{x})) = 0$ if it is infeasible.

C.3 Encoding Actionability Constraints

We describe how to extend the MIP formulation in (11) to encode salient classes of actionability constraints. Our software includes an ActionSet API that allows practitioners to specify these constraints across each MIP formulation.

C.3.1 Encoding Preservation for Categorical Features

Many datasets contain subsets of features that reflect the underlying value of a categorical attribute. For example, a dataset may encode a categorical attribute with $K = 3$ categories such `marital_status` $\in \{single, married, other\}$ using a subset of $K - 1 = 2$ features such as `married` and `single`. In such cases, actions on these features must obey non-separable actionability constraints to preserve the encoding – i.e., to ensure that a person cannot be `married` and `single` at the same time.

We can enforce these conditions by adding the following constraints to the MIP Formulation in (11):

$$L \leq \sum_{j \in \mathcal{J}} x_j + a_j \leq U \tag{12}$$

Here, $\mathcal{J} \subseteq [d]$ is the index set of features with encoding constraints, and L and U are lower and upper limits on the number of features in $\mathcal{J}$ that must hold to preserve an encoding. Given a standard one-hot encoding of a categorical variable with K categories, $\mathcal{J}$ would contain the indices of $K-1$ features (i.e., dummy variables for the $K-1$ categories other than the reference category). We would ensure that all actions preserve this encoding by setting $L = 0$ and $U = 1$.

C.3.2 Logical Implications & Deterministic Causal Relationships

Datasets often include features where actions on one feature will induce changes in the values and actions for other features. For example, in Table 1, changing `is_employed` from `FALSE` to `TRUE` would change the value of `work_hrs_per_week` from 0 to a value ≥ 0.

We capture these conditions by adding variables and constraints that capture logical implications in action space. In the simplest case, these constraints would relate the values for a pair of features $j, j' \in [d]$ through an if-then condition such as: "if $a_j \geq v_j$ then $a'_j = v_{j'}$". In such cases, we could capture this relationship by adding the following constraints to the MIP Formulation in (11):

$$\begin{aligned} Mu &\geq a_j - v_j && (13)\\ M(1-u) &\geq v_j - a_j && (14)\\ uv_{j'} &= a_{j'} && (15)\\ u &\in \{0,1\} \end{aligned}$$

The constraints shown above capture the "if-then" condition by introducing a binary variable $u := \mathbb{1}[a_j \geq v_j]$. The indicator is set through the Constraints (13) and (14) where $M := a_j^{\text{UB}} - v_j$. If the implication is met, then $a_{j'}$ is set to $v_{j'}$ through Constraint (15). We apply this approach to encode a number of salient actionability constraints shown in Table 1 by generalizing the constraint shown above to a setting where: (i) the "if" and "then" conditions to handle subsets of features, and (ii) the implications link actions on mutable features to actions on an immutable feature (i.e. so that actions on a mutable feature `years_since_last_application` will induce changes in an immutable feature `age`).

C.3.3 Custom Reachability Conditions

We now describe a general-purpose solution to specify "reachable" values for a subset of discrete features. These constraints can be used when we need to encode constraints that require fine-grained control over the actionability of different features. For example, when specifying actions over one-hot encoding of ordinal features (e.g., `max_degree_BS` and `max_degree_MS` as in Table 1) or as 'thermometer encoding" (e.g.,`monthly_income_geq_2k`, `monthly_income_geq_5k`, monthly_income_geq_10k). In such cases, we can formulate a set of custom reachability constraints over these features given the following inputs:

- index set of features $\mathcal{J} \subset [d]$,
- V, a set of all valid values that can be realized by the features in $\mathcal{J}$.
- $E \in \{0,1\}^{k \times k}$, a matrix whose entries encode the reachability of points in V: $e_{i,j} = 1$ if and only if point v_i can reach point v_j for $v_i, v_j \in V$.

Given these inputs, we add the following constraints for each $j \in \mathcal{J}$ to the MIP Formulation in (11):

$$a_j = \sum_{k \in E[i]} e_{i,k} a_{j,k} u_{j,k} \tag{16}$$

$$1 = \sum_{k \in E[i]} u_{j,k} \tag{17}$$

$$u_{j,k} \leq e_{i,k} \tag{18}$$
$$u_{j,k} \in \{0,1\}$$

Here, $u_{j,k} := \mathbb{1}[\boldsymbol{x}' \in V]$ indicates if we choose an action to attain point $\boldsymbol{x}' \in$. Constraint (16) defines the set of reachable points from i, while Constraint (16) ensures that only one such point can be selected. Here, $e_{i,k}$ is a parameter obtained from the entries of E for point i, and the values of $a_{j,k}$ are set as the differences from x_j to x'_j where $\boldsymbol{x}, \boldsymbol{x}' \in V$. We present examples of how to use these constraints to preserve a one-hot encoding over ordinal features in Fig. 3, and to preserve a thermometer encoding in Fig. 4.

monthly_income_geq_10k)

V			
IsEmployedLeq1Yr	IsEmployedBt1to4Yrs	IsEmployedGeq4Yrs	E
0	0	0	$[1,1,0,0]$
1	0	0	$[0,1,1,0]$
0	1	0	$[0,0,1,1]$
0	0	1	$[0,0,0,1]$

Figure 3: Here V denotes valid combinations of features in columns 1 - 3. E in column 4 and shows which points can be reached. For example, $[1,1,0,0]$ represents point $[0,0,0]$ can be reached and point $[1,0,0]$ can be reached, but no other points can be reached.

V			
NetFractionRevolvingBurdenGeq90	NetFractionRevolvingBurdenGeq60	NetFractionRevolvingBurdenLeq30	E
0	0	0	$[1,1,0,0]$
1	0	0	$[0,1,0,0]$
0	1	0	$[1,1,1,0]$
0	1	1	$[1,1,1,1]$

Figure 4: Here V denotes valid combinations of features in columns 1 - 3. For these features, we wanted to produce actions that would reduce NetFractionRevolvingBurden for consumers. E in column 4 and shows which points can be reached. For example, $[1,1,0,0]$ represents point $[0,0,0]$ can be reached, and point $[1,0,0]$ can be reached, but no other points can be reached.

D Supplement to Section 5 – Experiments

For each dataset, the Simple action set contains only immutability features and integrality constraints. The Separable action set contains the same actionability constraints as simple and adds monotonicity constraints. The Non-Separable action set contains all the actionability constraints as separable and adds non-separable constraints.

D.1 Actionability Constraints for the `heloc` Dataset

We use the action set shown in Table 7. The non-separable constraints are described in Appendix E.

D.2 Actionability Constraints for the `givemecredit` Dataset

We show a list of all features and their separable actionability constraints in Table 4. The non-separable actionability constraints for this dataset include:

1. Logical Implications on `AnyRealEstateLoans` and `MultipleRealEstateLoans`. Here, if `AnyRealEstateLoans` changes from 1 to 0, then `MultipleRealEstateLoans` must also change from 1 to 0.
2. Logical Implications on `AnyOpenCreditLinesAndLoans` and `MultipleOpenCreditLinesAndLoans`. Here, if `AnyOpenCreditLinesAndLoans` changes from 1 to 0, then `MultipleOpenCreditLinesAndLoans` must also change from 1 to 0.
3. Custom Constraints to Preserve Thresholds for features `MonthlyIncomeIn1000sGeq2`, `MonthlyIncomeIn1000sGeq5`, `MonthlyIncomeGeq7K`. An example can be found in Fig. 4. Here the feasible actions increase the consumer's `MonthlyIncome` and the maximum value a user can have is where `MonthlyIncomeGeq2K` = 1, `MonthlyIncomeGeq5K` = 1, and `MonthlyIncomeIn1000sGeq7` = 1
4. Custom Constraints to Preserve Thresholds for features `TotalCreditBalanceGeq1K`, `TotalCreditBalanceGeq2K`, `TotalCreditBalanceGeq5K`. An example can be found in figure 4. Here the feasible actions decrease the consumer's `TotalCreditBalance` and the minimum value a consumer can have is where `TotalCreditBalanceGeq1K` = 0, `TotalCreditBalanceGeq2K` = 0, and `TotalCreditBalanceGeq5K` = 0

Feature Name	LB	UB	Actionable	Monotonicity
`Age`	21	90	F	
`NumberOfDependents`	0	10	F	
`DebtRatioGeq1`	0	1	F	
`MonthlyIncomeGeq2K`	0	1	T	0
`MonthlyIncomeGeq5K`	0	1	T	0
`MonthlyIncomeGeq7K`	0	1	T	0
`TotalCreditBalanceGeq1K`	0	1	T	0
`TotalCreditBalanceGeq2K`	0	1	T	0
`TotalCreditBalanceGeq5K`	0	1	T	0
`AnyRealEstateLoans`	0	1	T	0
`MultipleRealEstateLoans`	0	1	T	0
`AnyOpenCreditLinesAndLoans`	0	1	T	0
`MultipleOpenCreditLinesAndLoans`	0	1	T	0

Table 4: Overview of Separable Actionability Constraints for the `givemecredit` dataset.

D.3 Actionability Constraints for the german Dataset

We show a list of all features and their separable actionability constraints in Table 5. The non-separable actionability constraints for this dataset include:

1. One Hot Encoding for features savings_acct_le_100, savings_acct_bt_100_499, savings_acct_bt_500_999, savings_acct_ge_1000 An example of this can be found in Appendix C.3.1. Here, actions must restrict only one category to be selected.

Feature	LB	UB	Actionable	Monotonicity
age	19	75	F	
is_male	0	1	F	
is_foreign_worker	0	1	F	
has_liable_persons	1	1	F	
max_approved_loan_duration_geq_10_m	0	1	F	
max_approved_loan_amt_geq_10k	0	1	F	
max_approved_loan_rate_geq_2	0	1	F	
credit_history_no_credits_taken	0	1	F	
credit_history_all_credits_paid_till_now	0	1	F	
credit_history_delay_or_critical_in_payment	0	1	F	
loan_required_for_car	0	1	F	
loan_required_for_home	0	1	F	
loan_required_for_education	0	1	F	
loan_required_for_business	0	1	F	
loan_required_for_other	0	1	F	
max_val_checking_acct_ge_0	0	1	T	+
max_val_savings_acct_ge_0	0	1	T	+
years_at_current_home_ge_2	0	1	T	+
employed_ge_4_yr	0	1	T	+
savings_acct_le_100	0	1	T	0
savings_acct_bt_100_499	0	1	T	0
savings_acct_bt_500_999	0	1	T	0
savings_acct_ge_1000	0	1	T	0
has_history_of_installments	0	1	T	+

Table 5: Overview of Separable Actionability Constraints for the german dataset.

D.4 Overview of Model Performance

Dataset	Model Type	Sample	AUC
heloc	LR	Train	0.738
heloc	LR	Test	0.730
heloc	XGB	Train	0.733
heloc	XGB	Test	0.737
givemecredit	LR	Training	0.653
givemecredit	LR	Test	0.644
givemecredit	XGB	Training	0.651
givemecredit	XGB	Test	0.640
german	LR	Training	0.752
german	LR	Test	0.690
german	XGB	Training	0.753
german	XGB	Test	0.690

Table 6: Performance of LR and XGB models for all 3 datasets. We show the performance of each model on the training dataset and a held-out dataset. We perform a random grid search to tune the hyperparameters for each model and split the train and test by 80%/ 20%. We use the entire dataset to calculate the number of predictions without recourse.

E Supplement to Section 6 – Demonstrations

E.1 Actionability Constraints for the `heloc` Dataset

We show a list of all features and their separable actionability constraints in Table 7. The non-separable actionability constraints for this dataset include:

1. Logical Implications on `MostRecentTradeInLastYear` and `MostRecentTradeInLast2Years` is explained in section Appendix C.3.2. Here, if `MostRecentTradeInLastYear` changes from 0 to 1 then `MostRecentTradeInLast2Years` must also change from 0 to 1.
2. Custom Constraints to Preserve Thresholds for features `NetFractionRevolvingBurdenGeq90`, `NetFractionRevolvingBurdenGeq60`, `NetFractionRevolvingBurdenLeq30`. An example can be found in figure 4. Here, feasible actions must decrease the consumer's `NetFractionRevolvingBurden`. Therefore, the lowest category a consumer can reach is `NetFractionRevolvingBurdenLeq30` = 1.

Feature	LB	UB	Actionable	Monotonicity
`AvgYearsInFileGeq3`	0	1	T	0
`AvgYearsInFileGeq5`	0	1	T	0
`AvgYearsInFileGeq7`	0	1	T	0
`AvgYearsInFileGeq9`	0	1	T	0
`ExternalRiskEstimate`	36	89	F	
`InitialYearsOfAcctHistory`	0	2	T	+
`ExtraYearsOfAcctHistory`	0	48	F	
`MostRecentTradeWithinLastYear`	0	1	T	+
`MostRecentTradeWithinLast2Years`	0	1	T	+
`AnyDerogatoryComment`	0	1	F	
`AnyDelTradeInLastYear`	0	1	F	
`AnyTrade120DaysDelq`	0	1	F	
`AnyTrade90DaysDelq`	0	1	F	
`AnyTrade60DaysDelq`	0	1	F	
`AnyTrade30DaysDelq`	0	1	F	
`NumInstallTrades`	0	55	F	
`NumInstallTradesWBalance`	1	23	F	
`NumRevolvingTrades`	1	85	F	
`NumRevolvingTradesWBalance`	0	32	T	-
`NetFractionInstallBurdenGeq90`	0	1	F	
`NetFractionInstallBurdenGeq70`	0	1	F	
`NetFractionInstallBurdenGeq50`	0	1	F	
`NetFractionInstallBurdenGeq30`	0	1	F	
`NetFractionInstallBurdenGeq10`	0	1	F	
`NetFractionInstallBurdenEq0`	0	1	F	
`NetFractionRevolvingBurdenGeq90`	0	1	T	0
`NetFractionRevolvingBurdenGeq60`	0	1	T	0
`NetFractionRevolvingBurdenLeq30`	0	1	T	0
`NumBank2NatlTradesWHighUtilizationGeq2`	0	1	T	-

Table 7: Overview of Separable Actionability Constraints for the `heloc` dataset.

E.2 Prototypes of Predictions without Recourse

Feature	$\boldsymbol{x}$	Actions $\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{a}_7$
AvgYearsInFileGeq3	1	-	-	-	-	-	-	-
AvgYearsInFileGeq5	0	1	-	1	1	1	1	1
AvgYearsInFileGeq7	0	-	-	1	-	1	1	1
AvgYearsInFileGeq9	0	-	-	-	-	-	1	1
ExternalRiskEstimate	59	-	-	-	-	-	-	-
InitialYearsOfAcctHistory	2	-	-	-	-	-	-	-
ExtraYearsOfAcctHistory	8	-	-	-	-	-	-	-
MostRecentTradeWithinLastYear	1	-	-	-	-	-	-	-
MostRecentTradeWithinLast2Years	1	-	-	-	-	-	-	-
AnyDerogatoryComment	0	-	-	-	-	-	-	-
AnyDelTradeInLastYear	1	-	-	-	-	-	-	-
AnyTrade120DaysDelq	0	-	-	-	-	-	-	-
AnyTrade90DaysDelq	0	-	-	-	-	-	-	-
AnyTrade60DaysDelq	1	-	-	-	-	-	-	-
AnyTrade30DaysDelq	0	-	-	-	-	-	-	-
NumInstallTrades	8	-	-	-	-	-	-	-
NumInstallTradesWBalance	2	-	-	-	-	-	-	-
NumRevolvingTrades	7	-	-	-	-	-	-	-
NumRevolvingTradesWBalance	1	-	-1	-	-1	-1	-	-1
NetFractionInstallBurdenGeq90	0	-	-	-	-	-	-	-
NetFractionInstallBurdenGeq70	1	-	-	-	-	-	-	-
NetFractionInstallBurdenGeq50	1	-	-	-	-	-	-	-
NetFractionInstallBurdenGeq30	1	-	-	-	-	-	-	-
NetFractionInstallBurdenGeq10	1	-	-	-	-	-	-	-
NetFractionInstallBurdenEq0	0	-	-	-	-	-	-	-
NetFractionRevolvingBurdenGeq90	0	-	-	-	-	-	-	-
NetFractionRevolvingBurdenGeq60	0	-	-	-	-	-	-	-
NetFractionRevolvingBurdenLeq30	1	-	-	-	-	-	-	-
NumBank2NatlTradesWHighUtilizationGeq2	0	-	-	-	-	-	-	-

Table 8: This prototype is presented in Section 6. This specific example was chosen because the consumer has negative classifications for both LR and XGB for all possible actions. Although this consumer has feasible actions they are still unable to obtain recourse since every reachable point is negatively classified. In this demo, there are 453 examples of consumers that may have feasible actions, but they are still predictions without recourse by both LR and XGB. In this table, $\boldsymbol{x}$ represents all the feature values for this consumer. $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_7$ represent all the feasible actions in $A(\boldsymbol{x})$ for this consumer. Features that do not have actions are either immutable or the consumer is at the bounds for the features that have increasing or decreasing monotonicity constraints. Features that are able to change are `AvgYearsInFileGeq3`, `AvgYearsInFileGeq5`, `AvgYearsInFileGeq7`, `AvgYearsInFileGeq9` and `NumRevolvingTradesWBalance`. `AvgYearsInFile` represents the average length of loans that are open and active. The feasible actions for these must only increase the consumer's `AvgYearsInFile`. `NumRevolvingTradesWBalance` has a decreasing montonicity constraint, so $\{0, -1\}$ are the only feasible actions for this feature. These actions obey the actionability constraints displayed in table 7.

E.3 Certifying Adversarial Robustness in Bot Detection on Social Media Platforms

Our machinery can also certify adversarial robustness to manipulations that normally cannot be captured by traditional threat models such as perturbations within an L_p ball [see, e.g., 6, 28, 34, 49, 71]. In this demonstration, we show that our methods enable to reason about the behavior of arbitrary models under semantically meaningful adversarial manipulations of the feature vectors. Specifically, we do so by building action sets that encode constraints from Table 1. In what follows, we showcase this by evaluating the adversarial robustness of a bot detector on a social media platform.

Feature	LB	UB	Actionable
source_automation	0	1	F
source_other	0	1	F
source_branding	0	1	F
source_mobile	0	1	F
source_web	0	1	F
source_app	0	1	F
follower_friend_ratio	0.83	1.16×10^5	F
age_of_account_in_days_geq_365	0	1	T
age_of_account_in_days_geq_730	0	1	T
age_of_account_in_days_le_365	0	1	T
user_replied_geq_10	0	1	T
user_replied_geq_100	0	1	T
user_replied_le_10	0	1	T
user_favourited_geq_1000	0	1	T
user_favourited_geq_10000	0	1	T
user_favourited_le_1000	0	1	T
user_retweeted_geq_1	0	1	T
user_retweeted_geq_10	0	1	T
user_retweeted_geq_100	0	1	T
user_retweeted_le_1	0	1	T

Table 9: Features used for the Twitter bot detector. The groups of features age_of_account_⋆, user_replied_⋆, user_favourited_⋆, and user_retweeted_⋆ are non-separable thermometer-encoded.

Setup We use the dataset of Twitter accounts from April 2016 annotated by experts [18] as genuine ("human") labeled as $y = +1$ or those representing inauthentic behavior ("bot") labeled as $y = -1$. As before, we consider a processed version with $n = 1\,438$ accounts and $d = 20$ features on their account history and activity (e.g., age of account, number of tweets, re-tweets, replies, use of apps), listed in Table 9. As in Section 6, we train a logistic regression and an XGBoost model. We set aside 287 accounts (20%) as a held-out test dataset.

Our goal is to demonstrate the use of Algorithm 2 for evaluating the robustness of a detector to adversarial manipulations. We assume that the adversary starts with a bot account that is correctly detected as bot, and aims to modify the features of the account until it is classified as human. The capabilities of the adversary include procuring additional tweets, retweets, and replies; waiting to increase the account age, and adding tweets from previously unused categories of apps. As this is a complex model of adversarial capabilities which includes non-separable constraints, it cannot be captured by the commonly considered box constraints or L_p distances.

To evaluate adversarial robustness, we perform the following procedure. We run Algorithm 2 to generate reachable sets for all correctly classified bot accounts. We then evaluate the prediction of the detector on each of the points in the corresponding reachable set. Second, we measure adversarial robustness through a version of the *robust error* metric [as per 39]: the proportion of the bot accounts from the test set that are correctly classified as bots yet can have their predictions altered through adversarial actions. Formally, for a set of correctly predicted bot examples $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^m$ from the test data, i.e., such that every $y_i = -1$ ("bot") and $f(\boldsymbol{x}_i) = -1$, we define the robust error as:

$$\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}[\exists \boldsymbol{x}' \in R_A(\boldsymbol{x}_i) \;\text{ s.t. }\; f(\boldsymbol{x}') = +1]. \tag{19}$$

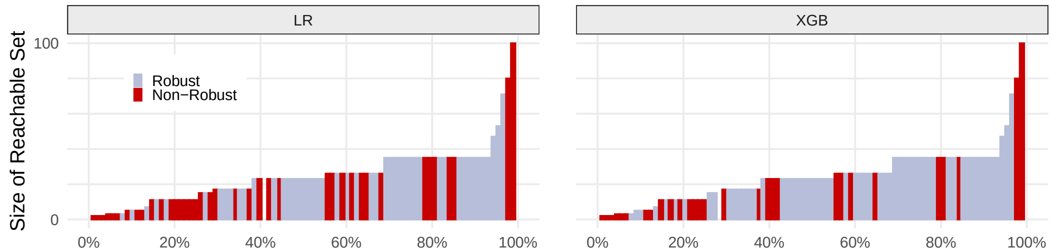

Figure 5: Composition of reachable sets for Twitter bot detection with LR and XGB models. Each plot shows the size of reachable sets generated by Algorithm 1 for every correctly classified bot account in the test set. Robust/Non-Robust denotes if the bot example can flip the prediction via manipulations within action set or not.

Model Type	AUC	Error	Robust Error
LR	0.697	34.1%	44.8%
XGB	0.698	34.5%	**33.3%**

Table 10: Robust error and performance of LR and XGB models trained for Twitter inauthentic behavior detection task. All metrics are computed on the test data.

Results In our test data, we have 88 (out of 287 total accounts) bot accounts that are correctly classified as bots. We generate the 88 corresponding reachable sets for each account, and evaluate the predictions in each. Fig. 5 shows the distribution of reachable set sizes.

To evaluate the robustness of classifiers, in Table 10, we show the performance metrics of the classifiers along with the computed robust error. We find that for the majority of bots it is not possible to flip their prediction with any possible action within the adversarial model, with the robust error being approximately 33.3% for XGB and 44.82% for LR. Despite both classifier attaining similar error and AUC, XGB is more robust to adversarial manipulations.

In summary, our method enables to find adversarial examples, thus evaluate adversarial robustness, in tabular domains under a complex model of adversarial capabilities.