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A branch-and-bound algorithm for a single machine sequencing to minimize the sum of maximum earliness and tardiness with idle insert

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Abstract

This paper presents the optimal sequence of a set of jobs for a single machine with idle insert, in which the objective function is to minimize the sum of maximum earliness and tardiness ($n/1/I/\mathrm{ET}_{\max}$). Since this problem tries to minimize and diminish the values of earliness and tardiness, the results can be useful for different production systems such as just in time (JIT). Special case of determining the optimal sequence, considering common due date, is investigated and the structure of optimal solution is introduced, using some simple orders. In the general case, the neighborhood conditions are developed and the dominant set for any optimal solution is determined. The branch-and-bound (B&B) method is used to solve the problem, and the proper upper and lower bounds are also

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introduced. To show the effectiveness of the proposed algorithm, 780 problems of different sizes, ranging from 5 to 20 jobs, are generated at random and then solved.

Keywords: Sequencing; Single machine; Branch-and-Bound method; Maximum earliness; Maximum tardiness; Idle insert

1. Introduction

Production planning is an important task in a manufacturing and/or service firm. The aim of production planning is to find optimal usage from resources of a firm depending on the suitability of schedule. In modern manufacturing systems, sequencing and scheduling play an important role requiring a special attention. Nowadays, only one objective or criterion is not good enough to make a decision. However, a multi-objective or criteria decision making (MODM or MCDM) is needed. Thus, several researchers proposed many algorithms to solve such difficult problems. A typical model for a single machine scheduling problem consists of the same ready times for all jobs. It is also assumed that all jobs have sequence-independent setup times. Thus, setup time is combined and designated as the processing time. Besides, idle insert is not allowable for each job and each machine. These assumptions exist in most papers related to this matter [1,2].

Various objective functions exist in the literature survey for a single machine scheduling problem with the above assumptions. Any of these functions presents a typical objective and makes an attempt for achieving the objectives. Most of these objective functions are introduced in terms of earliness and tardiness of jobs. Earliest due date (EDD) order is used for minimizing the maximum lateness ($L_{\max}$, i.e., difference between the due date and the completion time) as well as the maximum tardiness ($T_{\max}$) [1,2]. In most cases, the sum or mean of tardiness of all jobs is considered as a criterion to determine the job sequencing. The mean tardiness is represented by $\overline{T}$ and in overall, the traditional optimization methods like branch-and-bound (B&B) and dynamic programming (DP) are used for minimizing it [1,3].

These methods are usually very timely and only suitable for a small number of jobs. These methods are generally inefficient for solving large-scale problems. Emmons [4] has introduced essential conditions to find an optimal solution for $\overline{T}$ in a single machine problem, after proving some theorems. Sen and Borah [5] have reduced the set of feasible solutions in order to find the optimal solution using Emmons' theorems. They obtained the optimal solution for up to 30-jobs problems by the B&B method. Holsenback and Ressell [6] have introduced a heuristic method for minimizing $\overline{T}$ according to Emmons' theorems. This heuristic method has been considered as a base for further researches. Panwalker et al. [7] have proposed a heuristic algorithm, Ben-Daya and Al-Fawzan [3] have introduced an efficient method based on simulated annealing (SA) and Islam and Eksiglu [8] have proposed a method based on tabu search (TS) in order to minimize $\overline{T}$. These authors have compared their methods with Holsenback and Russell algorithm [6].

Most researchers have been interested in a multi-objective function for sequencing and scheduling problems to adapt and satisfy managers' requirements. One of the most important objectives is to minimize the sum (or weighted sum) of the earliness and tardiness of jobs. This matter is conformity to just-in-time (JIT) systems [9,10]. Tardiness and earliness cause penalties in losing customers and increasing inventory cost, respectively. Thus, none of these penalties is desirable. Most researchers are interested in various forms and with various assumptions for the due dates of jobs, allowing idle insert, and weighting of earliness and tardiness [11–13].

Ow and Morton [9] have first introduced this problem by assuming the difference between earliness and tardiness costs. They have also introduced an effective theorem for preceding and succeeding of jobs to each other. According to this theorem, some priority rules have been introduced in order to solve the problem, using a heuristic method. James and Buchanan [14] have proposed a tabu search (TS) method for this problem using the results of Ow and Morton. Zegordi et al. [10] used simulated annealing (SA) method for minimizing the weighted sum of earliness and tardiness of jobs.

There will be large values of earliness or tardiness for some jobs in results obtained from minimizing the sum (or weighted sum) of earliness and tardiness. Thus, this problem causes some difficulties in production systems. To identify this problem, consider a case that all jobs done on machines exit from a firm as the batches built-up many parts. If all jobs of a batch produce on time but a job has tardiness, then other jobs of the batch must wait. Thus, their on time production is not an advantage. In such situation, if the jobs are carried out earlier, they need some spaces and increase the inventory level. However, if there is earliness or tardiness, then their associated values should be almost the same for all jobs. In other word, if a job in a batch has earliness, then other jobs of the batch will have earliness. Likewise, if a job has tardiness, then other jobs will have tardiness. Thus, the interval time between earliness and tardiness must be approximately zero. This aim is fulfilled by minimizing the sum of the maximum earliness and tardiness. Another application of this objective function is the part feeding in an assembly line by a machine. In other word, the assembly line needs jobs in a known due date. If a job has much earliness or tardiness, then other jobs will not be used resulting imbalance in the assembly line. Thus, the objective is to minimize the sum of the maximum earliness and tardiness in such a way that the above-mentioned difficulty reduces. This objective function forces jobs not to be early and/or tardy.

In the case of existing any earliness or tardiness, their values are approximately the same for jobs and large earliness and tardiness are not existed. This objective is associated with JIT production systems. The objective function value would be equal to zero in the optimal state. If whatever earliness and tardiness exist, then the value of the objective function will be greater than zero. In this case, the objective function tries to be reduced and this can be one of the main objectives in JIT production systems. For better understanding of the subject, consider the following example given in Table 1.

By solving the above example, the optimal sequence of 5-2-3-4-1 is obtained, in which the sum of earliness and tardiness is equal to 50. However, the sum of the maximum earliness and maximum tardiness of the above sequence is equal to 44. In other word, the interval between values of the maximum earliness and maximum tardiness is equal to 44 time units. Whereas, the minimum of the maximum earliness and maximum tardiness of the aforementioned problem with sequence 5-2-3-1-4 is equal to 33, in which about 25% of the interval is reduced. The sum of earliness and tardiness of this sequence is equal to 53.

Amin-Nayeri and Moslehi [15] have studied a single-machine sequencing problem to find an optimal sequence of jobs, in which the objective function is to minimize the maximum earliness and tardiness. In the above paper, some assumptions for the original and traditional model as well as the absence of idle insert for a job and a machine have been considered.

Since ET_{max} (i.e., the sum of the maximum earliness and tardiness) is an irregular criterion, then it is possible to eliminate the assumption of unallowable idle insert and to define a new problem. However, this paper introduces a new sequencing problem considering idle insert, namely $n/1/I/ET_{max}$, in which a search is carried out for finding the optimal sequence holding idle insert. If the objective function is to minimize the sum of the maximum earliness and maximum tardiness with allowing idle insert, then the objective is to find the best value of idle insert in a known sequence for improving the objective function. Tavakkoli-Moghaddam et al. [16] have proposed an optimal algorithm to obtain the best value of idle insert in the known sequence ($n/1/OI/ET_{max}$).

In the next section, we describe the symbols. In section 3, a special case of common due date is presented. Difference between $n/1/I/ET_{max}$ and $n/1//ET_{max}$

Table 1
Input data for a typical example

	Job				
	1	2	3	4	5
Processing time	18	16	14	11	12
Due date	40	30	40	55	25

considering idle insert is presented in Section 4. Neighboring conditions, upper and lower bounds for a B&B method are introduced in Sections 5 and 6, respectively. In Section 7, the proposed algorithm is presented. Computational results are reported in Section 8. Finally, Section 9 is a remarking conclusion.

2. Symbols

To explain the theorems and associated relationships, the general symbols are defined as follows and hereafter the terms of "job" and "part" have the same meaning. The number of jobs in a known sequence is n, in which the processing time and the due date of job i are represented by p_i and d_i, respectively. All jobs have two types of due date: equal and common due date. The former is known as a prior and the latter is resulting from the final problem solving. The completion time and the difference between the completion time and due date are represented by C_i and L_i, respectively. In a single machine sequencing, earliness (E_i) and tardiness (T_i) of job i, maximum earliness ($E_{\max}$), maximum tardiness ($T_{\max}$), and the sum of maximum earliness and tardiness ($\mathrm{ET}_{\max}$) in each sequence are obtained as follows:

$$E_i = \max(0, d_i - C_i), \tag{1}$$

$$T_i = \max(0, C_i - d_i), \tag{2}$$

$$E_{\max} = \max_{1 \leqslant i \leqslant n}\{E_i\}, \tag{3}$$

$$T_{\max} = \max_{1 \leqslant i \leqslant n}\{T_i\}, \tag{4}$$

$$\mathrm{ET}_{\max} = E_{\max} + T_{\max}. \tag{5}$$

Term id is the time of incremental idle insert in a sequence. The problem of an optimal sequence with the objective function $\mathrm{ET}_{\max}$ considering idle insert is represented by $n/1/I/\mathrm{ET}_{\max}$.

3. Special case of common due date

Following lemmas are corresponding with the same due dates of jobs versus sequence. These due dates may be assumed as input (same known due date) or output (same unknown due date) that are called the equal and the common due dates.

Lemma 1. *In the problem $n/1//\mathrm{ET}_{\max}$ with equal due date only one optimal sequence exists, in that a job with maximum processing time is positioned at the beginning of the sequence.*

Proof. If jobs 1 to n are arranged, in which they have the completion times $C_1, C_2, \ldots, C_n$, then the following cases are investigated:

(1) $d \leqslant C_1$: If the equal due date is smaller than or equal C_1, then all jobs have tardiness. The last job has always the maximum tardiness ($T_{\max}$). Thus, all sequences have the same value of $T_{\max}$.

(2) $C_1 < d < C_n$: If the equal due date is between C_1 ($C_{\min}$) and C_n ($C_{\max}$), then the first job and the last job have the maximum earliness and maximum tardiness, respectively. Thus, $\mathrm{ET}_{\max}$ is given by

$$\mathrm{ET}_{\max} = E_{\max} + T_{\max} = (d - C_1) + (C_n - d) = C_n - C_1. \tag{6}$$

Since the completion time of the last job (C_n) is always constant ($C_n = C_{\max}$), then for minimizing $\mathrm{ET}_{\max}$, only C_1 should be considered. Hence, $\mathrm{ET}_{\max}$ is minimized, when C_1 has the maximum possible value. This case is occurred, when a job with maximum processing time positions at the beginning of the sequence.

(3) $d \geqslant C_n$: If the equal due date is greater than or equal C_n, then all jobs have earliness, and $E_{\max}$ would be related with the first job. Thus, the best sequence is obtained by positioning the job with maximum processing time at the beginning of the sequence. □

Lemma 2. *In the problem* $n/1/I/ET_{\max}$ *with equal due date, considering idle insert, the objective function value is improved, if d is greater than* C_n. *In this case, an optimal solution is obtained by a sequence, in that a job with maximum processing time is positioned at the beginning of the sequence.*

Proof. Feasible cases of the equal due date are investigated as follows:

(1) $d \leqslant C_1$: In this case, all jobs with any sequence have tardiness and the maximum tardiness is obtained by the equation $T_{\max} = C_n - d$. Since completion time of each job is smaller than C_n ($C_i < C_n$), then the maximum tardiness is corresponding with the last job

$$C_i - d < C_n - d \quad \forall i \Rightarrow T_i < T_n, \;\; T_{\max} = T_n. \tag{7}$$

In this case, if the idle insert (*id*) is positioned before or after d, then the objective function $\mathrm{ET}_{\max}$ will be increased. Since the obtained function ($T'_{\max}$) is increased by *id* value, all sequences will have the same value of the maximum tardiness

$$T'_{\max} = T_{\max} + id > T_{\max}. \tag{8}$$

(2) $C_1 < d < C_n$: In this case, the following equations are given for the maximum earliness and tardiness

$$d - C_1 > d - C_i \quad \forall i \Rightarrow E_{\max} = E_1, \tag{9}$$

$$C_i - d < C_n - d \quad \forall i \Rightarrow T_{\max} = T_n. \tag{10}$$

If *id* is positioned before C_1, $T_{\max}$ is increased and $E_{\max}$ is decreased as shown by $T'_{\max} = T_{\max} + id$ and $E'_{\max} = E_{\max} - id$, respectively. The new objective function would be equal to the previous objective function

$$\mathrm{ET}'_{\max} = E'_{\max} + T'_{\max} = E_{\max} + T_{\max}. \tag{11}$$

Thus, the objective function remains constant. Now, if *id* is positioned after d and before C_n, then it causes increasing the maximum tardiness, but the maximum earliness remains constant, i.e. $T'_{\max} = T_{\max} + id, E'_{\max} = E_{\max}$. Finally, the new objective function is increased by $\mathrm{ET}'_{\max} = E_{\max} + T_{\max} + id$. In other word, this *id* adding changes the objective function. The above case is also occurred when *id* is positioned before d and after C_1. Thus in this case, the objective function cannot be improved by *id* adding. According to Lemma 1 in this case, if a job with maximum processing time positions at the beginning of the sequence, then the best value of the objective function is obtained.

(3) $d \geqslant C_n$: In this case, all jobs have earliness, in which their values are obtained by equation $E_{\max} = d - C_1$ and the maximum tardiness is zero. An increase in *id* before C_1 improves the objective function by $d - C_n$, i.e. $E'_{\max} = E_{\max} - id, T_{\max} = 0$. Since an increase in *id* is greater than $d - C_n$, then job n converts into a job with tardiness. If the term $id = d - C_n$ is true, then the maximum earliness is obtained as follows:

$$E_n = d - C_n. \tag{12}$$

If $id > d - C_n$, then $E_n = 0$, $T_n = 0$ and in this case, the problem is changed to 2.

If an optimal sequence is unknown, according to Lemma 1, the optimal sequence is obtained by positioning the job with maximum processing time at the beginning of sequence, and the best value *id* is $d - C_n$ that must be positioned before C_1. If *id* increases after C_1, then the job with maximum earliness does not change, i.e. $E'_{\max} = E_{\max}$ and the objective function will not be improved. □

Lemma 3. *In the problem $n/1/I/ET_{\max}$ with common due date, the best sequence is obtained when $d \geqslant C_n$ and the maximum processing time is positioned at the beginning of the sequence.*

Proof. Feasible cases of common due date are investigated as follows:

(1) $d \leqslant C_1$: In this case, all jobs have tardiness ($E_{\max} = 0$). Thus, *id* inserting does not improve the objective function and all sequences have the same value of $T_{\max}$.

(2) $C_1 < d < C_n$: In this case, the first and the last jobs have the maximum earliness and maximum tardiness, respectively. Thus, *id* increment does not improve the objective function. According to Lemma 1, the optimal sequence is obtained by positioning the job with maximum processing time at the beginning of the sequence.
(3) $d \geqslant C_n$: In this case, all jobs have earliness ($T_{\max} = 0$) and according to Lemma 1, an optimal sequence is obtained by positioning the job with the maximum processing time at the beginning of sequence. As proved in the third case of Lemma 2, the best *id* is found by $d - C_n$, in which this *id* inserts before the first job in a sequence. □

4. Difference between $n/1/I/\mathrm{ET}_{\max}$ and $n/1//\mathrm{ET}_{\max}$ with idle insert

In the problem $n/1//\mathrm{ET}_{\max}$, an optimal sequence of jobs is found by the proposed B&B algorithm [15], in which the objective function is to minimize $\mathrm{ET}_{\max}$. The concept "first, an optimal sequence of jobs with the absence of idle insert ($n/1//\mathrm{ET}_{\max}$) is found and then the best value of idle insert in the known optimal sequence for improving the objective function is obtained by the idle insert algorithm [16]" is different from $n/1/I/\mathrm{ET}_{\max}$. This subject is proved by a reversal example.

Jobs 1, 2, and 3 are considered as shown in Table 2.

By solving the above example with the absence of idle insert, the optimal sequence 1-2-3 is obtained, in which the sum of the maximum earliness and maximum tardiness is equal to 6 ($n/1//\mathrm{ET}_{\max}$). By using the idle insert algorithm [16], the objective function reduces one unit and improves to 5, whereas sequence 2-1-3 with the objective function 4 is obtained by a complete enumeration of $n/1/I/\mathrm{ET}_{\max}$.

In Table 3, all feasible sequences are given. As seen, the first sequence with the objective function value 6 has the best value of $\mathrm{ET}_{\max}$ in a problem $n/1//\mathrm{ET}_{\max}$. In Table 4, the improvement values of the objective function for all feasible sequences are given. As seen, only in cases 1 and 4, the objective function can be improved by using the idle insert. In cases 1 and 4, the objective function can be improved 1 and 3 units, respectively.

Table 2
Input data for the reversal example

	Job		
	1	2	3
Processing time	1	6	2
Due date	12	5	4

Table 3
Calculation of maximum earliness and maximum tardiness for all feasible sequences

Sequence	Earliness of job 1	Tardiness of job 1	Earliness of job 2	Tardiness of job 2	Earliness of job 3	Tardiness of job 3	Maximum earliness	Maximum tardiness	Objective function value
1-2-3	2	0	0	3	3	0	3	3	6
1-3-2	2	0	9	0	0	4	9	4	13
2-3-1	0	1	5	0	0	5	5	5	10
2-1-3	0	1	0	4	3	0	3	4	7
3-1-2	11	0	1	0	0	4	11	4	15
3-2-1	11	0	0	2	0	5	11	5	16

Table 4
Comparison between "considering idle insert" and "without idle insert"

Sequence	Objective function value considering idle insert	Objective function value without idle insert	Best objective function value
1-2-3	5	6	5
1-3-2	13	13	13
2-3-1	10	10	10
2-1-3	4	7	4
3-1-2	15	15	15
3-2-1	16	16	16
Best value	4	6	4

As mentioned above, the best sequence for $n/1/I/\mathrm{ET}_{\max}$ is 2-1-3, in which the objective function value is equal to 4. Whereas, the best sequence for $n/1//\mathrm{ET}_{\max}$ is 1-2-3, in which the idle insert improves 1 unit and the associated objective function value is transformed into 4.

5. Neighborhood conditions

In this section, some lemmas, which are the basis of the proposed B&B algorithm, are presented. Lemmas 4,5 are used for determining the dominant set in the B&B method. Before determining the dominant set by using the following notes, it provides an answer to this question; "when does the idle insert improve the objective function of the sequence?". These four notes have been taken from Tavakkoli-Moghaddam et al. [16].

Note 1. In a known sequence, if a job with $E_{\max}$ is positioned before a job with $T_{\max}$, then the idle insert does not improve the objective function.

Note 2. In a known sequence, if a job with $T_{\max}$ is positioned before a job with $E_{\max}$ and idle insert is considered in the set B (set of jobs, which are positioned between the job with $T_{\max}$ and the job with $E_{\max}$), then the idle insert may improve the objective function.

Note 3. If all jobs in a known sequence do not have earliness (they have tardiness greater than or equal zero), then the objective function may not be improved by considering the idle insert.

Note 4. If the whole jobs in the known sequence have earliness, then the objective function is improved by considering the idle insert.

Lemma 4. *In the problem $n/1/I/ET_{max}$, if the sequence is arranged in order of longest slack time (LST) and the last job in the sequence has tardiness, then positioning the idle insert in this sequence does not improve the objective function.*

Proof. It is supposed that T_n is the tardiness of job n or tardiness of the last job in the sequence with n jobs

$$T_n = C_n - d_n = (C_{n-1} + p_n) - d_n. \tag{13}$$

According to the LST order, the last job has a minimum $d - p$ and according to the assumption, the last job has tardiness, i.e. $C_n > d_n$. Thus, Eq. (14) is obtained

$$d_n - p_n < d_{n-1} - p_{n-1} < \cdots < d_1 - p_1. \tag{14}$$

If the elements of Eq. (14) are multiplied by (-1), then Eq. (15) is given

$$p_n - d_n > p_{n-1} - d_{n-1} > \cdots > p_1 - d_1. \tag{15}$$

By adding the elements of the following relation $C_n > C_{n-1} > \cdots > C_1$ to Eq. (15), Eq. (16) is resulted

$$C_{n-1} + p_n - d_n > C_{n-2} + p_{n-1} - d_{n-1} > \cdots > p_1 - d_1 \Rightarrow T_n > T_{n-1} > \cdots > T_1. \tag{16}$$

Eq. (16) results that the maximum tardiness is equal to tardiness of job n ($T_{\max} = T_n$). Since $T_{\max} = T_n$, then all jobs have tardiness or a job with $E_{\max}$ is positioned before a job with $T_{\max}$. According to Notes 3 and 1, if all jobs have tardiness or a job with $E_{\max}$ is positioned before a job with $T_{\max}$, then the idle insert does not improve the objective function $ET_{\max}$. □

Lemma 5. *In the problem $n/1/I/ET_{\max}$, if the sequence is arranged in order of LST and the last job in sequence has earliness, then all jobs in every sequence will* *have earliness and the minimum earliness is obtained for the last job in the sequence with LST order.*

Proof. It is supposed that E_n is the earliness of job n or earliness of the last job in a sequence with n jobs

$$E_n = d_n - C_n = d_n - (C_{n-1} + p_n). \tag{17}$$

According to LST order, the last job has minimum $d - p$ and according to the assumption, the last job has earliness, i.e. $C_n < d_n$. Thus, Eq. (14) is resulted.

If the elements of Eq. (14) are multiplied by (-1), then by adding the elements of relation $C_n > C_{n-1} > \cdots > C_1$ to the transformed equation, the following equation is resulted

$$d_n - p_n - C_{n-1} < \cdots < d_1 - p_1. \tag{18}$$

Since the last job in the sequence has earliness, i.e. $d_n - p_n - C_{n-1} < 0$, then other jobs in sequence have earliness too

$$E_n < E_{n-1} < \cdots < E_2 < E_1. \tag{19}$$

Thus, the last job in the mentioned sequence with LST order will have the minimum earliness. □

By using Lemmas 4 and 5, the following five principles can be used as a dominant set in the B&B method. Before introducing these five principles, a definition of the partial sequence σ and set σ' are presented. The associated positions of these two sets are shown in Fig. 1. Elements of σ' are positioned before the elements of σ. The number of elements of each of these sets is equal or smaller than the total number of jobs, that is $\sigma + \sigma'$ is equal to n.

Principle 1. According to Lemma 4, if the last job in a sequence arranged by LST rule has tardiness, then the last job has maximum tardiness. Hence, if in the first sequence arrangement with LST rule, the last job has tardiness, then this job will have the maximum tardiness. Thus, the investigation of some sequences, in that the last job is the last job of LST rule and has tardiness is avoided (see Notes 1 and 3).

$\sigma = \{i \mid \text{order of job } i \text{ is specified}\}$
$\sigma' = \{i \mid \text{order of job } i \text{ is not specified}\}$

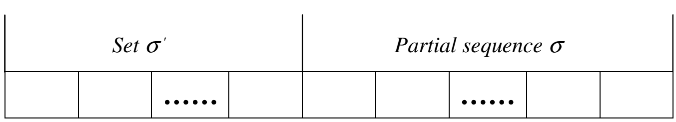

Fig. 1. Situations of the partial sequence σ and set σ'.

Principle 2. If all elements of partial sequence σ have tardiness and the last job of set σ' arranged with LST rule has tardiness, then the investigation of some sequences that all elements of partial sequence σ have tardiness and the last job of set σ' is the last job of LST arrangement and it also has tardiness are avoided (see Notes 1 and 3). The raeson is that all jobs until the end of the sequence have tardiness and according to Lemma 4, the remainder of jobs until the beginning of the sequence has a tardiness smaller than tardiness of the last job of LST rule for set σ'.

Principle 3. If all elements of partial sequence σ have tardiness and the last job of set σ' arranged with LST rule has earliness, then maximum tardiness is positioned after maximum earliness and according to Note 1, investigation of these sequences are avoided. Based on Lemma 5, the remainder of elements of set σ' with any arrangement will not have an earliness greater than the earliness of the last job of LST rule of set σ' until the beginning of the sequence.

Principle 4. If all elements of partial sequence σ have earliness and tardiness and the last job of set σ' arranged with LST rule has earliness and maximum earliness of partial sequence σ is smaller than earliness of the last job of LST rule of set σ', then according to Lemma 5, any earliness smaller than earliness of the last job of LST rule of set σ' with any arrangement is not existed in set σ'. When maximum earliness of partial sequence σ is smaller than earliness of the last job of LST rule of set σ', this means that the job with maximum earliness is positioned before a job with maximum tardiness, and according to Note 1, the investigation of these sequences is avoided.

Principle 5. If all elements of partial sequence σ have earliness and tardiness and the last job of set σ' arranged with LST rule has earliness and the maximum earliness of minimum slack time (MST) rule of set σ' is greater than the maximum earliness of partial sequence σ, then according to Note 1, the minimum earliness of set σ', which is obtained by MST rule, is greater than the maximum earliness of partial sequence σ and this means that a job with maximum earliness is positioned before a job with maximum tardiness. Therefore, the investigation of these sequences is avoided.

6. Upper and lower bounds for the objective function value

In this section, Lemmas 6 and 7 are presented to determine the proper upper and lower bounds, respectively.

Lemma 6. *In the problem $n/1/I/ET_{\max}$, the improved objective function for a solution obtained from the problem $n/1//ET_{\max}$ by idle insert is an upper bound for $n/1/I/ET_{\max}$.*

Proof. By using the B&B algorithm for determining the optimal sequence of $n/1//\mathrm{ET}_{\max}$, the problem is solved, first. Then, by using the idle insert algorithm, the objective function of the mentioned optimal sequence is improved. The B&B algorithm and the idle insert algorithm are taken from Amin-Nayeri and Moslehi [15] and Tavakkoli-Moghaddam et al. [16] respectively. This sequence is a feasible solution and consequently it can be used as an upper bound. □

Lemma 7. *In the problem $n/1/I/\mathrm{ET}_{\max}$, the lower bound includes maximum earliness and maximum tardiness. Maximum earliness of a lower bound is $E_{\max\sigma id}$ (i.e., maximum earliness of partial sequence σ after considering idle insert) and maximum tardiness of a lower bound is the maximum of $T_{max\sigma id}$ (i.e., maximum tardiness of partial sequence σ) as well as $T_{\max\sigma pEDD}$ (i.e., maximum tardiness of the order EDD of σ').*

Proof. Maximum tardiness of an optimal sequence for $n/1/I/\mathrm{ET}_{\max}$ is always equal or greater than maximum tardiness of partial sequence σ after *id* inserting

$$T_{\max\sigma id} \leqslant T^*_{\max}. \tag{20}$$

Maximum earliness of an optimal sequence for $n/1/I/\mathrm{ET}_{\max}$ is always equal or greater than maximum earliness of partial sequence σ after *id* inserting

$$E_{\max\sigma id} \leqslant E^*_{\max}. \tag{21}$$

Maximum tardiness of an optimal sequence for $n/1/I/\mathrm{ET}_{\max}$ is always equal or greater than maximum tardiness of EDD order for set σ'

$$T_{\max\sigma pEDD} \leqslant T^*_{\max}. \tag{22}$$

Considering Eqs. (20) and (22), the following equation is obtained:

$$\max\{T_{\max\sigma id}, T_{\max\sigma pEDD}\} \leqslant T^*_{\max}. \tag{23}$$

Considering Eqs. (21) and (23), the following equation is obtained:

$$\mathrm{ET}^*_{\max} \geqslant \max\{T_{\max\sigma id}, T_{\max\sigma pEDD}\} + E_{\max\sigma id}. \tag{24}$$

Finally, the lower bound is given by

$$\mathrm{LB} = \max\{T_{\max\sigma id}, T_{\max\sigma pEDD}\} + E_{\max\sigma id}, \tag{25}$$

$T_{\max\,\sigma id}$: maximum tardiness of partial sequence σ after considering *id*, $E_{\max\,\sigma id}$: maximum earliness of partial sequence σ after considering *id*, $T_{\max\sigma pEDD}$: maximum tardiness of order EDD for set σ'. □

7. Optimal algorithm for minimizing $\mathbf{ET_{max}}$ considering idle insert

In this section, by combining the presented lemmas and a B&B method, an algorithm is proposed for minimizing $\mathrm{ET}_{\max}$ considering idle insert ($n/1/I/\mathrm{ET}_{\max}$). Useful items from the previous lemmas are divided into four stages as follows:

Stage 1. *Computing the upper bound*: In this stage, a feasible solution is presented as the upper bound. As shown in Notes 1 and 3, if all jobs do not have earliness (they have tardiness greater than or equal zero), or a job with $E_{\max}$ is positioned before a job with $T_{\max}$, then the idle insert does not improve the objective function. According to the mentioned notes, explained dominant principles try to eliminate sequences that cause to create two mentioned cases. Considering the property of dominant principles, it is possible that optimal solution is not searched. This subject is occurred, when two conditions are done simultaneously. First, the solution of $n/1/I/\mathrm{ET}_{\max}$ is the same solution of $n/1//\mathrm{ET}_{\max}$ and second, in the obtained optimal sequence of $n/1/I/\mathrm{ET}_{\max}$ and $n/1//\mathrm{ET}_{\max}$, which are the same, a job with $E_{\max}$ is positioned before a job with $T_{\max}$. For avoiding the elimination of the optimal solution with the B&B method in this case, the solution of $n/1//\mathrm{ET}_{\max}$ is considered as an upper bound for the problem $n/1/I/\mathrm{ET}_{\max}$. Thus, two properties of feasibility and correspondence with dominant principle are satisfied. After obtaining the optimal sequence of problem $n/1//\mathrm{ET}_{\max}$ by using the algorithm [15], the improvement value is obtained by using the idle insert algorithm [16]. Finally, the improved value of the objective function is considered as an upper bound.

Stage 2. *Using the dominant principle*: In this stage, the sequences satisfying the dominant principle are not searched. As proved in Lemma 4, if set σ' is arranged with LST rule, in that the last job has tardiness, then by increasing the idle insert in the sequence, the objective function value is not improved. In other word, if the last job in LST rule of set σ' has tardiness, then all jobs will not have earliness or a job with $E_{\max}$ is positioned before a job with $T_{\max}$. This lemma is used as a dominant lemma. If a partial sequence σ is empty, then Principle 1 is used. If a partial sequence σ is not empty, then Principles 2–5 are utilized. Five principles resulted from Lemmas 4 and 5 try to eliminate some sequences from the complete enumeration, which satisfy the dominant principle. Thus, the speed of the B&B method in achieving an optimal solution increases.

Stage 3. *Computing the lower bound*: Considering the dominate principle for each sequence, the maximum improvement is created in the objective function of partial sequence σ by using the idle inset algorithm [16]. Then, the lower bound is computed from Eq. (25).

Stage 4. If the lower bound for each node is smaller than the upper bound, then the upper bound is converted into the lower bound for this node. Otherwise, if the lower bound for each node is equal or greater than the upper bound, then the algorithm desists from continuing the node and positioning the arranged last job in the partial sequence σ.

According to the above four stages, the steps of the proposed algorithm are listed as below:

Step 1. Obtain the optimal sequence of the problem $n/1//\mathrm{ET}_{\max}$ without considering the idle insert, using the optimal algorithm [15], and compute the objective function value.

Step 2. Create the maximum improvement in the obtained objective function value, using the optimal algorithm [16], and assign the objective function value to the upper bound.

Step 3. Assign the obtained optimal sequence in step 1 to set U.

Step 4. Divide the jobs of set U into two sections; set σ' and the partial sequence σ. i is the first job of partial sequence σ, and j is an element of set σ', which is created in the new branch. Ji is a partial sequence and j is positioned before i.

Step 5. If all branches are investigated, then the upper bound would be the final solution, and the problem solving is terminated. Thus, go to step 14. Otherwise, create a separation in branch j, i.e. job j is entered from non-arranged set σ' to the arranged jobs set namely partial sequence σ.

Step 6. If j is not the last job of order LST of set σ', then go to step 12. Otherwise, go to step 7.

Step 7. If the partial sequence σ is empty or all jobs have tardiness, then eliminate the branch ij and go to step 8. Otherwise, go to step 9.

Step 8. If set σ' is empty, then assign the lower bound to the upper bound and go to step 5. Otherwise, go to step 5, directly.

Step 9. If j does not have earliness, then go to step 12. Otherwise, go to step 10.

Step 10. If the maximum earliness of partial sequence σ is smaller than the earliness of j, then eliminate branch ij and go to step 8. Otherwise, go to step 11.

Step 11. If the maximum earliness of set σ' with MST rule is greater than the maximum earliness of partial sequence σ, then go to step 12. Otherwise, eliminate branch ij and go to step 8.

Step 12. Create the maximum improvement in the objective function of partial sequence σ by using the idle insert. Consider the maximum of $T_{\max}$ of partial sequence σ and $T_{\max}$ of EDD rule of set σ' as the maximum tardiness of the lower bound. Moreover, consider the maximum earliness of partial sequence σ, after inserting idle insert, as the maximum earliness of the lower bound. The lower bound would be equal to the summation of the maximum earliness and the maximum tardiness of the lower bound.

Step 13. If the lower bound is smaller than the upper bound, then go to step 8. Otherwise, eliminate branch *ij* and go to step 8.

Step 14. Stop.

8. Efficiency of the proposed method

To show the efficiency of the B&B method in a single machine sequencing with idle insert ($n/1/I/\mathrm{ET}_{\max}$), it is necessary to design problems showing the strength or weakness of the proposed algorithm. In this section, a set of problems [15] is solved and the computational results are presented.

8.1. Design of the problem

Many researchers have used random samples for test problems in the field of job earliness and tardiness. These researchers have considered two significant factors in these problems. The first factor is the tardiness represented by τ. This factor specifies the proportion of the average due dates of jobs to the sum of processing times in single machine problem. Ow and Morton [9], Kim and Yano [17], Yano and Kim [11] and James and Buchanan [14] have considered the above two factors and presented the following equation for τ:

$$\bar{d} = (1-\tau)\sum_{j=1}^{n} p_j, \tag{26}$$

where $\bar{d}$ is the average due dates of jobs and p_j represents the processing time of job j. Processing times and τ are known as a priori and then $\bar{d}$ is obtained, accordingly. The second factor is the range of due date. According to Zegordi et al. [10], processing times are generated by an uniform distribution in the range [5,25]. $\bar{d}$ is obtained from Eq. (26), first. Then, the due dates of jobs are defined by an uniform distribution presented as follows:

$$[\bar{d}(1-R/2), \bar{d}(1+R/2)]. \tag{27}$$

In the above equation, R is the range of due date and its value is known. Ow and Morton [9] and Zegordi et al. [10] have considered τ = 0.2, 0.6 and R = 0.6, 1.6. These standard values are used by most researches. Researchers use these values for generating test problems at random.

8.2. Design of experiments

To show the efficiency of the B&B method, four different types of problems are generated by combining two factors of tardiness and the range of due date.

These four types are first with $\tau = 0.2$, $R = 1.6$, second with $\tau = 0.6$, $R = 1.6$, third with $\tau = 0.2$, $R = 0.6$ and fourth with $\tau = 0.6$, $R = 0.6$. In each type, problems in sizes 5, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20 are considered. 15 iterations of each size in every type are solved. Thus, 195 problems ($15 \times 13 = 195$) for each type and 780 problems for four types ($4 \times 195 = 780$) are generated. In each size, if more than 80% of problems (iterations) cannot be solved in a reasonable time, then the bigger sizes in that type are not generated. These problems are solved on a personal computer Pentium IV 1.2 GHz Processor.

8.3. Computational results

Tables 5–8 show the computational results for type one, two, three and four, respectively. If the proposed B&B method succeeds to achieve the optimal sequence in time equal or smaller than 180 s, then the state of the solution is represented by "PBB". If the computational time exceeds the given time (180 s), then the algorithm is interrupted and the best solution up to this time is introduced. This state of the solution is represented by "BF". The content of "time average of algorithm running" is the arithmetic mean of 15 iterations.

In Table 5, 32 and 163 problems have the states BF and PBB, respectively. As shown in Table 5, in problems of type one, all iterations until 14 jobs have PBB state. BF state is existed from 15 jobs upward, as for the problems with 15, 16, 17, 18, 19, and 20 jobs, the number of BF states is 3, 4, 6, 7, 7, and 10 items, respectively. These numbers show that with increasing the number of jobs, the efficiency of the proposed method reduces, as in problems with

Table 5
Computational results of a B&B method in type 1 ($\tau = 0.2$, $R = 1.6$)

Number of jobs	Number of problems	Time average of algorithm running (s)	Branch and bound with BF state	Branch and bound with PBB state
5	15	0.00	0	15
7	15	0.00	0	15
10	15	0.11	0	15
11	15	0.59	0	15
12	15	1.48	0	15
13	15	6.02	0	15
14	15	13.74	0	15
15	15	44.53	3	12
16	15	90.00	4	11
17	15	98.70	6	9
18	15	116.80	7	8
19	15	128.71	7	8
20	15	152.00	10	5
Total	195		37	168

Table 6
Computational results of a B&B method in type 2 ($\tau = 0.6$, $R = 1.6$)

Number of jobs	Number of problems	Time average of algorithm running (s)	Branch and bound with BF state	Branch and bound with PBB state
5	15	0.00	0	15
7	15	0.00	0	15
10	15	0.43	0	15
11	15	0.63	0	15
12	15	13.79	0	15
13	15	41.89	3	12
14	15	83.00	5	10
15	15	103.60	7	8
16	15	155.70	12	3
Total	195		27	108

Table 7
Computational results of a B&B method in type 3 ($\tau = 0.2$, $R = 0.6$)

Number of jobs	Number of problems	Time average of algorithm running (s)	Branch and bound with BF state	Branch and bound with PBB state
5	15	0.00	0	15
7	15	0.03	0	15
10	15	13.66	0	15
11	15	86.96	1	14
12	15	180.00	15	0
Total	195		16	59

Table 8
Computational results of a B&B method in type 4 ($\tau = 0.6$, $R = 0.6$)

Number of jobs	Number of problems	Time average of algorithm running (s)	Branch and bound with BF state	Branch and bound with PBB state
5	15	0.00	0	15
7	15	0.02	0	15
10	15	14.03	0	15
11	15	152.94	3	12
12	15	180.00	15	0
Total	195		18	57

20 jobs only 1/3 of problems are achieved to a solution in a computational time less than 180 s. The computational time increases quickly, when the number of jobs increases, as shown in the mean computational time of Fig. 2.

In the problems of type two, all iterations until 14 jobs have PBB state, but for the problems with 13, 14, 15, and 16 jobs, the number of BF states is 3, 5, 7,

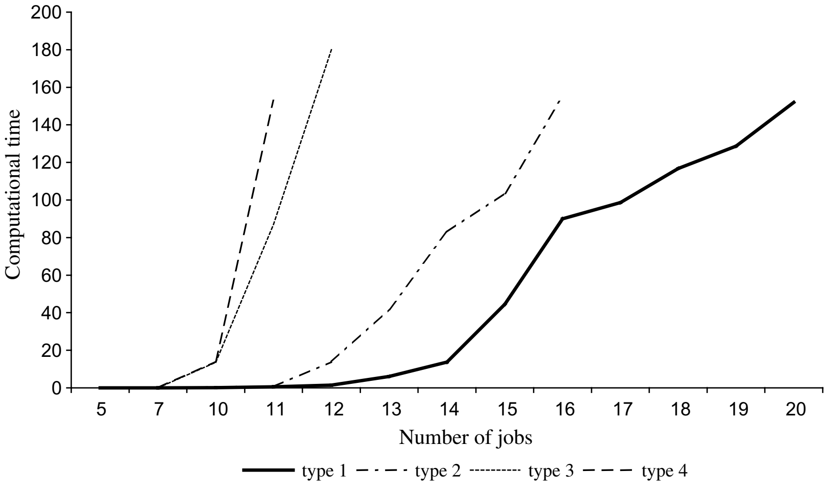

Fig. 2. Diagram of computational time respect to the problem size using B&B method.

and 12 items, respectively. Thus, from 16 jobs upward, the efficiency of the proposed method reduces. In this type, the average of computational times for each size is greater than type one.

In the problems of type three, all iterations until 10 jobs have PBB state, but for the problems with 11 and 12 jobs, the number of BF states is 1 and 15 items, respectively. Thus, from 12 jobs upward, the efficiency of the proposed method reduces. In this type, the average of computational times for each size is greater than types one and two.

In the problems of type four, all iterations until 10 jobs have PBB state, but for the problems with 11 and 12 jobs, the number of BF states is 3 and 15 items, respectively. Thus, from 12 jobs upward, the efficiency of the proposed method reduces. In this type, the average of computational times for each size is greater than all previous types.

The sensitivity of the computational time respect to the problem size, in these four types, is shown in Fig. 2.

According to Fig. 2, it is concluded that the slope of chart is increased from type one to type four, and the speed of increasing in computational time is increased too, i.e. problem solving from type one to type four will be further difficult by using the proposed B&B method. In every type (with constant τ and R), the computational time increases, when the number of jobs increases. Among different types, the problem solving would be more difficult, when R reduces. Furthermore, the difficulty of the proposed B&B method has a reverse relation with R and a direct relation with τ.

9. Conclusion

In this paper, the objective function is to minimize the sum of maximum earliness and tardiness (ET_{max}). This objective can be adapted by any production system, in which the optimal sequencing of a set of jobs is presented for a single machine with maximum earliness and tardiness considering the idle insert. Special case of determining a sequence with common due date were studied and the structure of optimal solution was introduced, using some simple orders. In the general case, for one machine and n jobs considering idle insert ($n/1/I/ET_{max}$), the neighborhood conditions were developed and the dominant set for the optimal solution was determined. The B&B method was also used to solve the problem. For evaluating the efficiency of the proposed algorithm, 780 problems in four types were designated and then solved.

Future researches can include other applications of this objective function (ET_{max}) in other types of sequencing problems such as job shop, flow shop, and so forth, faster and more effective solving methods, as well as whatever changing in assumptions. Furthermore, the other subjects are listed as follows:

- Considering the utilization of the first depth in the B&B method. The utilization of the jump method can also be considered.
- Utilizing the other optimization methods such as dynamic programming.
- Utilizing a heuristic method such as SA, TS, and GA.
- Considering the utilization of the idle insert algorithm, using B&B method and also generating the optimal sequence with B&B method without using the idle insert, and finally comparing the results by obtaining the value of improvement in the objective function, using the idle insert algorithm.

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