The Vitali set.

I want to start by recalling that we can identify the quotient space $\mathbb{R}/\mathbb{Z}$ with the unit circle.

We can do this explicitly with the map $\phi : \mathbb{R} \longrightarrow \{z \in \mathbb{C} : |z| = 1\} = C$

$$\phi(r) = e^{2\pi i r}.$$

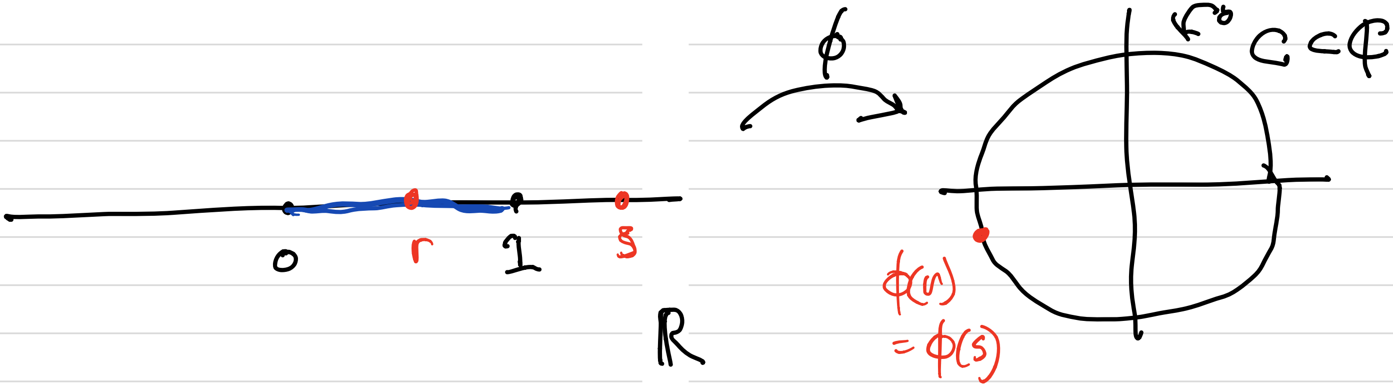
So if $\phi(r) = \phi(s)$ then

$$e^{2\pi i r} = e^{2\pi i s}$$

$$\left(e^{2\pi i r}\right) \cdot \left(e^{2\pi i s}\right)^{-1} = 1$$

$$e^{2\pi i (r-s)} = 1$$

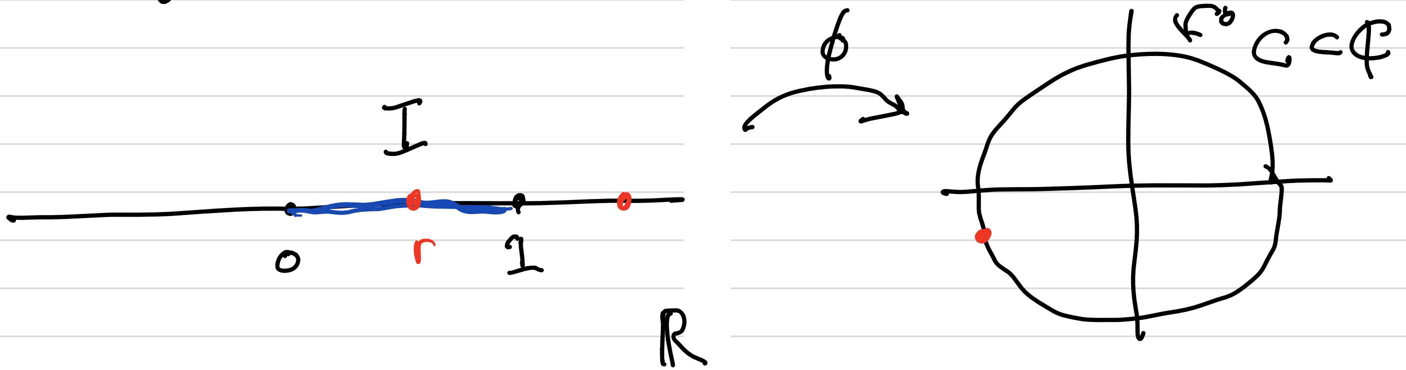
$$r - s \in \mathbb{Z}.$$



Let $I = [0,1) \subset \mathbb{R}$.

① ϕ restricted $[0,1)$ is a bijection.

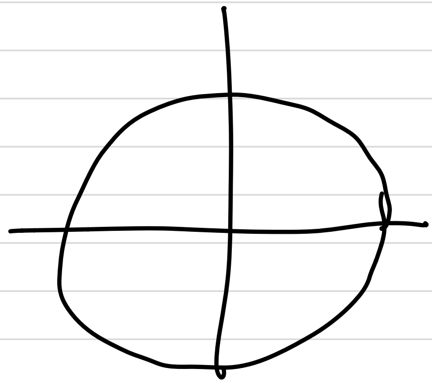
② Every $r \in \mathbb{R}$ is equivalent to a unique point in $[0,1)$ mod $\mathbb{Z}$.



Now let $\lambda = e^{2\pi i \theta}$ be a point in the unit circle. Here θ is real.

We define $R_\lambda : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$R_\lambda(z) = \lambda \cdot z,$$



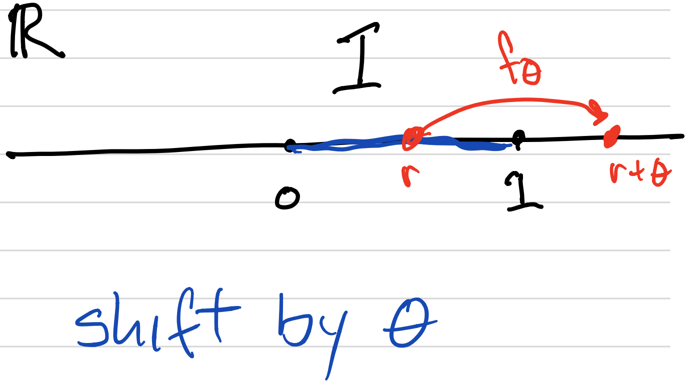
Rotation on $\mathbb{C}$ corresponds to addition on $\mathbb{R}$.

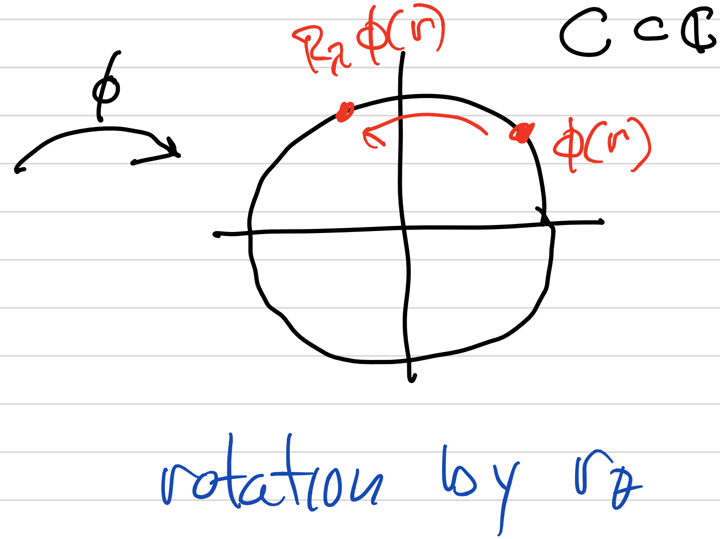
If we write $f_\theta(r) = r+\theta$ then we have

$$\phi \circ f_\theta = R_\lambda \circ \phi.$$

$$\begin{aligned}\phi \circ f_\theta(r) = \phi(r+\theta) &= e^{2\pi i(r+\theta)} \\ &= e^{2\pi i r} \cdot e^{2\pi i r\theta} \\ &= \phi(r) \cdot \lambda = R_\lambda(\phi(r))\end{aligned}$$

Assume $0<\theta<1$



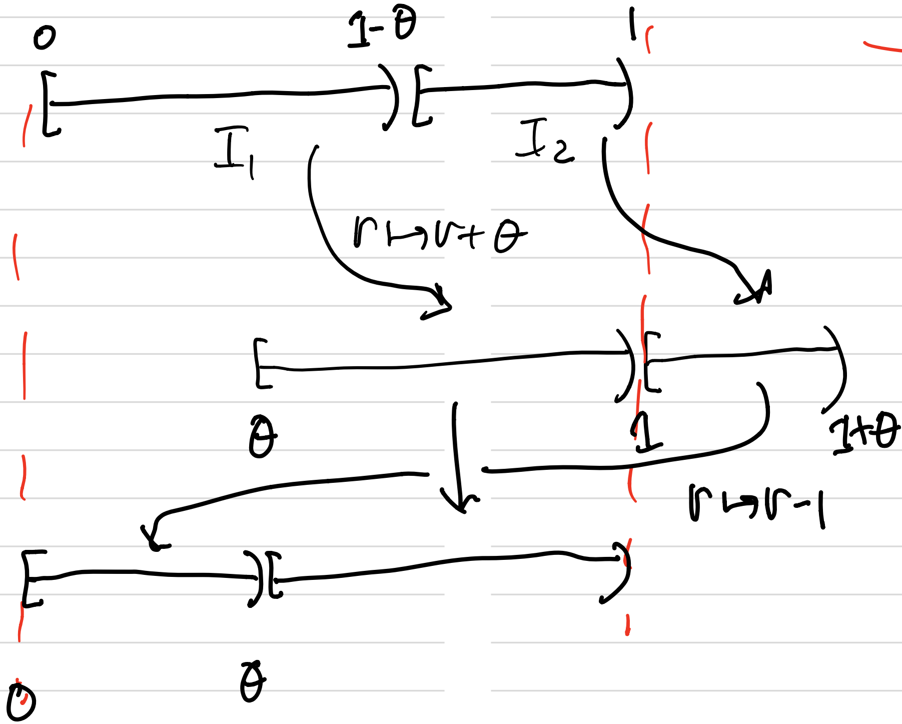


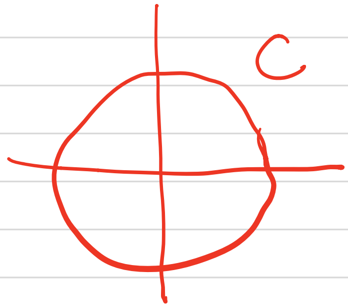
R_λ induces a map on I.

Let us figure out what this looks like.

Let $I = [0, 1)$.

Say $0 < \theta < 1$.

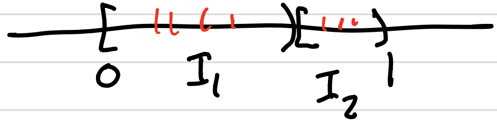




$\phi : I \longrightarrow S_1$ is a bijection. The rotation R_θ induces a map on I.

$$f(r) := \begin{cases} r+\theta & \text{if } 0 \le r < 1-\theta \\ r+\theta-1 & \text{if } 1-\theta \le r < 1 \end{cases}$$

The map $f : I \longrightarrow I$ preserves Lebesgue measure (and measurability).



If $B \subset [0,1)$ then

$$\lambda^*(f(B)) = \lambda^*\big(f((B \cap I_1) \cup (B \cap I_2))\big)$$

$$= \lambda^*(f(B \cap I_1)) + \lambda^*(f(B \cap I_2))$$

$$= \lambda^*(B \cap I_1) + \lambda^*(B \cap I_2)$$

$$= \lambda^*(B).$$

As in "Translation invariance" if we show that outer measure is preserved

It follows that measurability is preserved and measure is preserved.

$$f^m = \overbrace{f \circ f \circ f \circ \dots \circ f}^{m}$$

We define an equivalence relation on I. We say $r \sim s$ if $f^n(r) = f^m(s)$ for some m and n in $\mathbb{Z}$. This is equivalent to saying $f^{n-m}(r) = s$.

We call the equivalence classes for this equivalence relation "orbits".

Lemma. Assume that θ is irrational. If $f^n(r) = r$ then $n = 0$.

Proof. $f^n(r) = r + n\theta \mod \mathbb{Z}$.

If $f^n(r) = r$ then $r + n\theta = r \mod \mathbb{Z}$

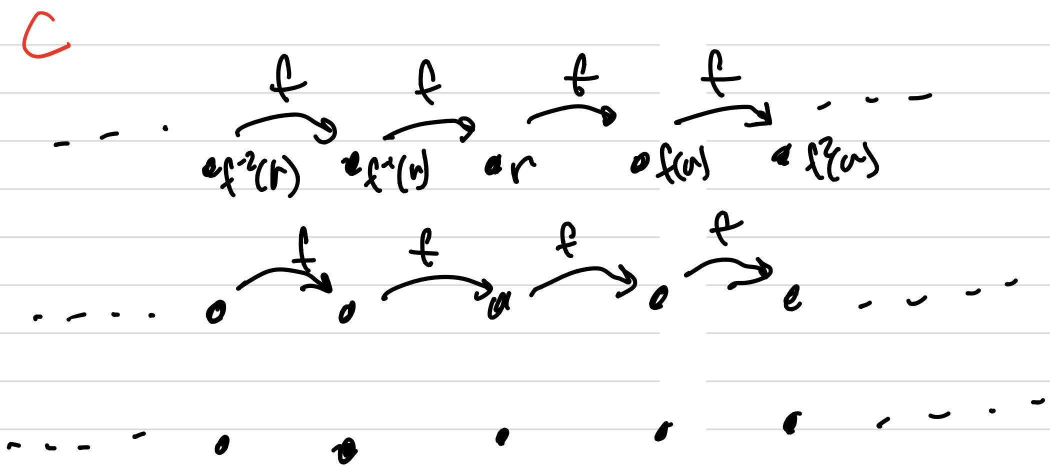
or $n\theta = 0 \mod \mathbb{Z}$

or $n\theta = m \quad m \in \mathbb{Z}$;

If $n \neq 0$ then $\theta = \frac{m}{n}$.

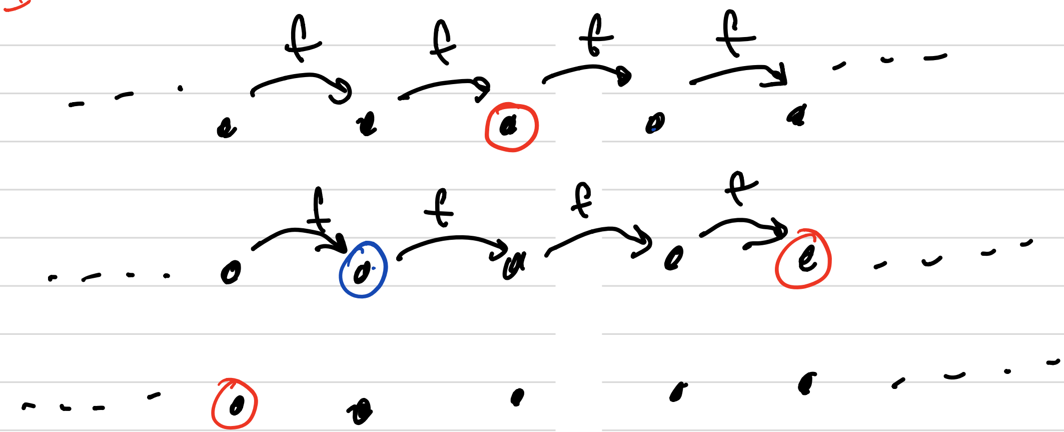
If $n \neq 0$ then $\theta = m/n$ and we derive a contradiction. QED

Now think about the circle as a union of orbits



Let $J \subset I$ be the set constructed by picking 1 point from each orbit.

$C = I$



By construction D is disjoint from $f^n(D)$ if $n \neq 0$. (1)

Also by construction the sets

$$\bigcup_{n \in \mathbb{Z}} f^n(D) = I. \qquad (2)$$

We claim that D is <u>not</u> measurable.

Assume that D is measurable.

In particular $\lambda(D)$ exists.

$$0 \leq \lambda(D) \leq 1.$$

Now:

$$1 = \lambda(I) = \lambda\left(\bigcup_{n\in\mathbb{Z}} f^n(D)\right) = \sum_{n\in\mathbb{Z}} \lambda(f^n(D))$$

$$= \sum_{n\in\mathbb{Z}} \lambda(D)$$

$$1 = \sum_{n \in \mathbb{Z}} \lambda(D)$$

If $\lambda(D) > 0$ we get $1 = \infty$. ✗

If $\lambda(D) = 0$ we get $1 = 0$. ✗

In either case we have a contradiction.