Theorem 2.4.1 (Monotone Convergence). Let f and $f_1, f_2 \ldots$ be $[0, +\infty]$ valued measurable functions on X. Suppose

$$f_1(x) \leq f_2(x) \leq \ldots$$

and

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Proof. As in our previous versions of this proposition one inequality is easy.

$$f_1 \le f_2 \le f_3 \le f_4 \dots \qquad \le f$$

implies that

$$\int f_1\,d\mu \le \int f_2\,d\mu \le \int f_3\,d\mu \le \dots \qquad \le \int f\,d\mu$$

so the left hand side has a limit (perhaps $+\infty$)

and

$$\lim_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu.$$

Now we look at the reverse inequality.

We have proved a version of this theorem before (Prop. 2.3.3) when the f_n are simple functions.

We would like to "replace" the f_n by simple functions. We know that each f_n is a monotone limit of simple functions so introduce simple functions $g_{n,k}$ with $\dots \le g_{n,k} \le g_{n,k+1} \le \dots$

and
$$f_n = \lim_{k \to \infty} g_{n,k}.$$

$$\begin{array}{ccccc}
f_1 & f_2 & \dashrightarrow & \to & f \\
\uparrow & \uparrow & & & \\
\vdots & \vdots & & & \\
g_{1,3} & g_{2,3} & & & \\
g_{1,2} & g_{2,2} & & & \\
g_{1,1} & g_{2,1} & \dots & &
\end{array}$$

Order properties:

$$
\begin{array}{ccccccc}
f_1 & \leq & f_2 & \leq & f_3 & \dots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \\
\rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & &
\end{array}
$$

We would like to do some sort of diagonalisation procedure that gives us a monotone sequence converging to f. We do not have any relations between say $g_{1,1}$ and $g_{2,1}$.

$$
\begin{array}{ccccccc}
f_1 & \le & f_2 & \le & f_3 & \cdots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \leftarrow \ h_3 = \max\{g_{1,1} \ldots g_{3,3}\}. \\
\rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & &
\end{array}
$$

Let h_n be the max of all the functions in the $n \times n$ box. If $n < m$ then h_m is the max of a larger collection of functions than h_n so $h_n \le h_m$.

$$\begin{array}{ccccccc}
f_1 & \le & f_2 & \le & f_3 & \cdots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \\
\rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \rotatebox[origin=c]{90}{$\leq$} & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & &
\end{array}$$

Note also that f_n dominates all the functions in the $n \times n$ box so it dominates their maximum and $h_n \le f_n$.

$$
\begin{array}{ccccccccc}
f_1 & \leq & f_2 & \leq & f_3 & \cdots & \leq f_n & \cdots & f \\
\uparrow & & \uparrow & & \uparrow & & \vdots & & \\
\vdots & & \vdots & & \vdots & & g_{n,k} & & \\
 & & & & & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & g_{n,3} & & \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & g_{n,2} & & m \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & & g_{n,1} & &
\end{array}
$$

m

Claim that $h_n \to f$ pointwise. If we fix an $x \in X$ and an $\varepsilon > 0$ we can find an n so that $f(x) - f_n(x) < \frac{\varepsilon}{2}$ and a k so that $f_n(x) - g_{n,k}(x) < \frac{\varepsilon}{2}$. If $m = \max(n, k)$ then $f(x) - h_m(x) < \varepsilon$.

Note that h_m is also the max of the top row of the $m \times m$ box.

$$\begin{array}{ccccc}
f_1 & \leq & f_2 & \leq & f_3 \cdots \to f \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
g_{1,3} & & g_{2,3} & & g_{3,3} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
g_{1,2} & & g_{2,2} & & g_{3,2} \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\boxed{g_{1,1}} & & g_{2,1} & & g_{3,1}
\end{array}$$

Now since h_n is a non-decreasing sequence of simple $[0, +\infty)$ valued measurable functions it follows from Prop 2.3.3 that

$$\int f \, d\mu = \lim_{n \to \infty} \int h_n \, d\mu$$

and since $f_n \geq h_n$ we have $\int f_n \, d\mu \geq \int h_n \, d\mu$ and

$$\int f \, d\mu = \lim_{n \to \infty} \int h_n \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu .$$

Combined with the previous inequality

$$\left(\int f \, d\mu \geq \lim_{n \to \infty} \int f_n \, d\mu \right)$$

we have
$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu .$$