Theorem (Monotone convergence)

Suppose f, $(f_n)_{n\geq 1}$ are all positive, measurable real valued functions with $f_n \uparrow f$

$f_1 \leq f_2 \leq \dots \quad f_n \to f$ pointwise

then $\mu(f_n) \uparrow \mu(f) \qquad \mu(f) = \int f(x)\,\mu(dx)$

Proved in video 1

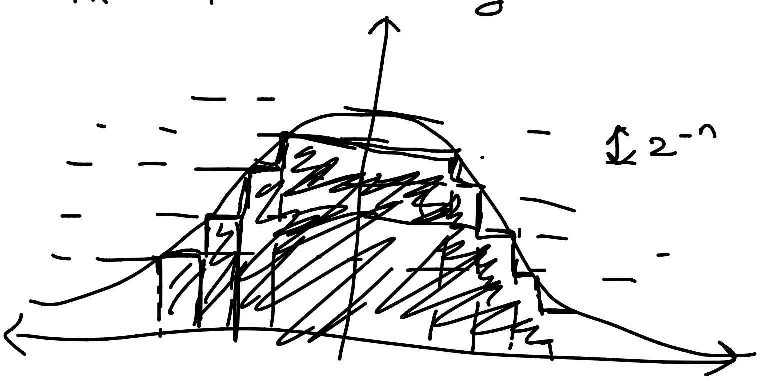
Corollary (Which can also be thought of as a def^n)

If f is a non-negative real valued meas function and $f_n = (\underbrace{2^{-n}\lfloor 2^n f \rfloor}) \wedge n$ then f_n is a

biggest number of the form $k2^{-n}$ that is smaller than $f(x)$.

simple function and $\mu(f) = \lim_{n\to\infty} \mu(f_n)$

Pf $\quad f_n \uparrow f$ so by mon $\quad \mu(f) = \lim_{n\to\infty} \mu(f_n)$



Consequence 2

Prop Suppose f, g are non-negative, measurable real valued functions on $(E, \mathcal{E}, \mu)$ then

$$\begin{cases} \bullet \ \text{For every } \alpha > 0 \quad \mu(\alpha f) = \alpha \mu(f) \\ \bullet \ \mu(f+g) = \mu(f) + \mu(g) \end{cases}$$

- If $f \leq g$ then $\mu(f) \leq \mu(g)$
- $\mu(f) = 0$ iff $f = 0$ μ almost everywhere

~~Pf~~ Let f_n be a sequence of simple functions with $f_n \uparrow f$ then $\alpha f_n \uparrow \alpha f$

so $$\mu(\alpha f) \underset{\text{MON}}{=} \lim_n \mu(\alpha f_n) \underset{\substack{\text{linearity} \\ \text{for simple} \\ \text{func}^n}}{=} \lim_n \alpha \mu(f_n) \underset{\text{MON}}{=} \alpha \mu(f).$$

Let $g_n \uparrow g$ g_n a sequence of simple func^n

then $(f_n + g_n) \uparrow (f+g)$ and $f_n + g_n$ is simple

$$\mu(f+g) \underset{\text{MON}}{=} \lim_{n \to \infty} \mu(f_n + g_n) = \lim_{n \to \infty} (\mu(f_n) + \mu(g_n))$$

$$\underset{\text{MON}}{=} \mu(f) + \mu(g)$$

- If $f \leq g$ then recall the def^n

$h \leq f \}$

. If $f \leq g$ then recall the def:

$$\mu(f) = \sup\{\mu(h) : h \text{ simple } h \leq f\}$$

so if $h \leq f$ then $h \leq g$

$$\mu(g) = \sup\{\mu(h) : h \text{ simple } h \leq g\}$$

$$= \max\{\sup\{\mu(h) : h \text{ simple } h \leq f\}, \sup\{\mu(h) : h \text{ simple } h \nleq f, h \leq g\}\}$$

$$= \max\{\mu(f), \sup\{\mu(h) : h \text{ simple } h \nleq f, h \leq g\}\}$$

$$\geq \mu(f)$$

Now if $\mu(f) = 0$ iff $\sup\{\mu(h) : h \text{ simple } h \leq f\} = 0$

iff $\mu(h) = 0$ for every h with $h \leq f$, h simple

iff $h = 0$ almost everywhere for every h simple with $h \leq f$

and $f_n = (2^{-n}\lfloor 2^n f \rfloor) \wedge n$ is simple $\leq f$

and if a set of x, A, with $f(x) > 0$ with non-zero measure then $A = \bigcup_n \{x : f(x) > 2^{-n}\}$

so $\exists\, n$ s.t. $\{x : f(x) > 2^{-n}\}$ has non-zero measure so for this n $f_n(x) \neq 0$ almost everywhere so we've proved

$\mu(f) = 0$ iff $f = 0$ μ a.e.

Consequence 3 Beppo-Levi Lemma

Suppose f_n is a sequence of non-negative measurable, real valued functions then

$$\sum_n \mu(f_n) = \mu\left(\sum_n f_n\right)$$

Pf/ $g_n = \sum_{k=1}^{n} f_n$ then $g_n \uparrow \sum_{n=1}^{\infty} f_n$

$$\underline{\sum_n \mu(f_n)} = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(f_k)$$

using linearity $= \lim_{n\to\infty} \mu\left(\sum_{k=1}^{n} f_k\right) = \lim_{n\to\infty} \mu(g_n)$

using monotone convergence $= \underline{\mu\left(\sum_{n=1}^{\infty} f_n\right)}$ ✓