Measure Theory: Exercises (not for credit)

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Question 1. Find the σ-algebra on $\mathbb{R}$ which is generated by the collection of all one-point sets.

Question 2. Find an example to show that the union of a collection of σ-algebras is not necessarilly a σ-algebra.

Question 3. Prove that if $\mathcal{E}$ is both a d-system and π-system then it is a σ-algebra. Use this to prove Dynkin's π-system lemma that if $\mathcal{A}$ is a π-system then any d-system containing $\mathcal{A}$ also contains $\sigma(A)$. *Hint: Consider $\mathcal{D}$ the intersection of all d-systems containing $\mathcal{A}$, and $\mathcal{D}' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \forall A \in \mathcal{A}\}$ and $\mathcal{D}'' = \{B \in \mathcal{D} : B \cap A \in \mathcal{D} \forall A \in \mathcal{D}\}$ and show that they are all d-systems.*

Question 4. Suppose that $(E, \mathcal{E}, \mu)$ is a measure space. Prove the inclusion-exclusion formula

$$\mu(A_1 \cup A_2 \cup \dots A_n) = \sum_{k=1}^{n} \mu(A_k) - \sum_{k \neq j}^{n} \mu(A_k \cap A_j) + \dots + (-1)^n \mu(A_1 \cap A_2 \cap \dots A_n).$$

Question 5. Let μ be the measure on $\mathbb{R}$ defined by setting $\mu(A)$ to be the number of rationals in the set A (where $\mu(A) = \infty$ if there are infinitely many rationals). Show that μ is a σ-finite measure which gives every open interval infinite measure.

Question 6. Let μ be a finitely additive set function on a σ-algebra, $\mathcal{E}$ in a set E with $\mu(E) < \infty$. Show that μ is countably additive *if and only if* for any decreasing sequence of sets A_n with $\bigcap_n A_n = \emptyset$ and $\mu(A_1) < \infty$ then we have $\mu(A_n) \to 0$.

Question 7. Let $(E, \mathcal{E}, \mu)$ be a measure space. We call a set N a *null set* if there exists $B \in \mathcal{E}$ with $N \subseteq B$ and $\mu(B) = 0$. We write $\mathcal{N}$ for the collection of all null sets. Define the collection

$$\mathcal{E}^\mu = \{A \cup N : A \in \mathcal{E}, N \in \mathcal{N}\}.$$

Show that $\mathcal{E}^\mu$ is a σ-algebra and the extension of μ to $\mathcal{E}^\mu$ defined by $\mu(A \cup N) = \mu(A)$ is a measure. We call $\mathcal{E}^\mu$ the *completion of $\mathcal{E}$ with respect to μ.*