1 Measure Theory: Assignment - Constructing new measurable functions from old measurable functions

First let us have a definition

Definition We call ϕ a step function if we can write ϕ in the form $\phi(x) = \sum_{k=1}^{n} a_k 1_{(c_k,d_k]}$ where a_k, c_k and d_k are real numbers.

The goal of this sheet is to show we can approximate functions in $L^p(\mathbb{R})$ by step functions. We do this in three steps, one for each question.

Question 1.1. Let f be a non-negative measurable function in $L^p(\mathbb{R})$ for $p \in [1, \infty)$ by considering the functions $f_{n,m}(x) = f(x)1_{|x|\leq n}1_{f(x)\leq m}$, or otherwise, show that for every $\epsilon > 0$ there is a function g where g is a bounded, non-negative, measurable function that is 0 outside some closed bounded interval and $\|g - f\|_p \leq \epsilon$. *7 marks*

Question 1.2. Let g be a non-negative, bounded, measurable function whose support is contained inside $[-M, M]$ for some $M < \infty$. Show that for any fixed $\epsilon > 0$ there is a *simple function*, h whose support is inside a closed bounded interval, such that $\|g - h\|_p < \epsilon$. *7 marks*

Question 1.3. Suppose that A is a bounded Lebesgue measurable set, show that, given $\epsilon > 0$ there is a finite collection of disjoint, half open intervals I_k such that $\|1_A - \sum_{k=1}^{n} 1_{I_k}\|_p < \epsilon$ *Hint: look at the non credit exercise sheet from week 2.* Use this to show that if h is a simple function whose support is contained in $[-M, M]$ then there is a *step function*, ϕ, such that $\|h - \phi\|_p < \epsilon$. You may use Minkowski's inequality which says that $\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p$. *7 marks*

Question 1.4. Now show that for any (not necessarily non-negative) function $f \in L^p(\mathbb{R})$ with $p \in [1, \infty)$, and for any ϵ there exists a step function ϕ with $\|\phi - f\|_p \leq \epsilon$. (Note that step functions don't have to be positive). *4 marks*