Will start at 9.05

Remind me to record!

Assignment 3 is up! Due in 18^{th} Thursday w7.
at 12 noon

About the exam: Every past paper except last year has bookwork questions

* Look at them when revising
* Questions on non-examinable ex sheets are also a good revision resource

Three elements 1. Bookwork 2. Stuff you've seen before
3. Unseen problems

Does bookwork include really long proofs
They are included in point 2 but not point 1.

Assignment 1: Q2 Because I confused this in lectures!

Define Lebesgue outer measure on $\mathbb{R}^d$ by

$$\lambda^*(A) = \inf\left\{\sum_n \lambda(R_n) : R_n \text{ are half open rectangles and } A \subseteq \bigcup_n R_n\right\}$$

$\lambda(R_n) = vol(R_n)$

Question is if R is a half open rectangle show that $\lambda^*(R) = \lambda(R)$.

easier

$\lambda^*(R) \leq \lambda(R)$

Classic trick in measure theory

show that $\lambda^*(R) = \lambda(R)$.

Classic trick in measure theory
$\lambda^*(R) \leq \lambda(R)$
$\underline{\underline{\lambda(R) \leq \lambda^*(R)}}$

<u>Broad strategy</u>

1. Take $R \subseteq \bigcup_n R_n$ $\quad R, R_n$ all half open rectangles

2. Produce more rectangles R' and $(\tilde{R}_n)_{n \geq 1}$ with R' closed and $\tilde{R}_n$ open for each n and $R' \subseteq \bigcup_n \tilde{R}_n$ and

   R' almost the same as R
   $\tilde{R}_n$ almost the same as R_n

3. Apply the fact that any open cover of a compact set has a finite subcover and R' is closed and bounded so compact to find N s.t.

$$R' \subseteq \bigcup_{n=1}^{N} \tilde{R}_n$$

4. Now you've got a <u>finite</u> number of rectangles we can safely compare volumes to get $\text{vol}(R') \leq \sum_{n=1}^{N} \text{vol}(\tilde{R}_n)$

5. Relax our approximations to get

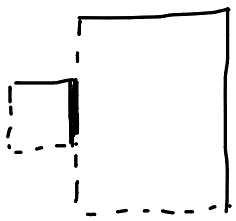
$$\lambda(R) \leq \sum_{n=1}^{\infty} \lambda(R_n)$$

then take the infimum over all possible to get $\lambda(R) \leq \lambda^*(R)$

then take the infimum over all ... covers to get $\lambda(R) \leq \lambda^*(R)$

Problems come from exactly how to form R' and $\tilde{R}_n$.

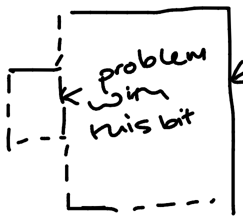
In the lectures I said you could use $\tilde{R}_n = \text{interior}(R_n)$



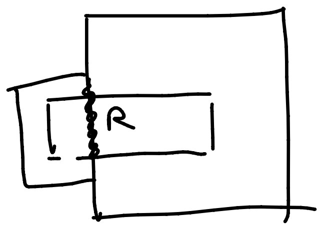
([|

don't want to miss this point in 1d.

If you try and stick them together



not a rectangle



Because I did this proof in lectures I only took 1/2 a mark if you did it this way

---

Another way people made a mistake is to enlarge each rectangle in the cover by the same amount.

$$[a,b] \subseteq (c_1,d_1] \cup (c_2,d_2] \cup \cdots$$

$$[a+\varepsilon, b-\varepsilon] \subseteq (c_1-\delta, d_1+\delta) \cup (c_2-\delta, d_2+\delta) \cup \cdots$$

Get $\exists N$ s.t.

$$[a+\varepsilon, b-\varepsilon] \subseteq (c_1-\delta, d_1+\delta) \cup \dots \cup (c_N-\delta, d_N+\delta)$$

comparing volumes

$$b-a-2\varepsilon \leq \sum_{k=1}^{N} (d_k - c_k) + 2N\delta$$

so we want to send ε and δ to zero

but N the size of the finite cover depends on δ so we can't be sure $N\delta \to 0$ as $\delta \to 0$

so instead enlarge by $(c_k - \delta 2^{-k}, d_k + \delta 2^{-k})$

Q3 Some people made this answer much longer than it neede to be.

Q4 Some people are a bit confused about how big the Borel σ-algebra is.

$$\underbrace{\bigcup_n \bigcap_m \bigcup_k \cdots \bigcup_r}_{\text{very long}} A_{n,m,k,r}$$

Almost every set you can think of will be a Borel set.

Almost every function thats ever going to come up will be Borel measurable.

Construction of Lebesgue measure

Big steps in the construction

Big steps in the construction

1. First we define what we want the Lebesgue measure of an interval to be:

We look at the set $\mathcal{I}$ of half open intervals $(a,b]$ we say $\lambda((a,b]) = b-a$.

=

In 1D you can use open intervals instead (thats what they do in Cohn)

or closed intervals, or all intervals

In higher dimension you can use half-open rectangles or open rectangles or open balls.

=

Most people use open balls / rectangles or half-open rectangles.

Intersection of two half open intervals is another half open interval so they form a π-system.

2. Construct an outer measure by

$$\lambda^*(A) = \inf\left\{\sum_n \lambda(I_n) : I_n \in \mathcal{I} \quad A \subseteq \bigcup_n I_n\right\}$$

check its an outer measure and check

$$\lambda^*(I) = \lambda(I) \quad \text{for} \quad I \in \mathcal{I}$$

3. Our outer measure is defined on $\mathcal{P}(\mathbb{R})$ but it isn't a true measure in particular

but it isn't a true measure in particular because it is not countably additive.

Define $\mathcal{M}$, the Lebesgue measurable sets by saying $B \in \mathcal{M}$ iff $\forall A$

$$\lambda^*(A) = \lambda^*(B \cap A) + \lambda^*(B^c \cap A)$$

$\hookrightarrow$ This would be true if λ^* was <u>finitely additive</u> on A, $B \cap A$ and $B^c \cap A$.

4. Show that $\mathcal{M}$ is a σ-algebra and that $\lambda^*|_{\mathcal{M}}$ is a true measure

Note: You can do steps 3 & 4 with any outer measure

5. Show that $\mathcal{I} \subseteq \mathcal{M}$ and therefore that $\sigma(\mathcal{I}) \subseteq \mathcal{M}$.