Measure Theory: Exercises (not for credit)

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This is a shorter sheet because last week was longer and you have the assignment as well. I think the most useful question is 3.

Question 1. Let A_n be a sequence of measurable sets. Show that 1_{A_n} converges to 0 in measure if and only if $\mu(A_n) \to 0$. Furthermore show that 1_{A_n} converges almost everywhere if and only if $\mu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) = 0$.

Question 2. Let $(f_n)_{n\geq 1}, f, g$ be Borel measurable functions from $\mathbb{R} \to \mathbb{R}$. Suppose further that g is continuous. Show that if $f_n \to f$ almost everywhere then $g \circ f_n \to g \circ f$ almost everywhere. Can the conclusion fail if g is only continuous almost everywhere.

Question 3. Suppose that f is a measurable function from $(E, \mathcal{E}, \mu)$ to $(F, \mathcal{F})$ show that the image measure defined by $\nu(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{F}$ is indeed a measure.

Question 4. Suppose that $(E, \mathcal{E}, \mu)$ is a σ-finite measure space and f_n is a sequence of real valued measurable functions on E. Suppose that $f_n \to f$ almost everywhere. Show that there exists a sequence of sets A_n and a set B such that $E = \bigcup_n A_n \cup B$ and $\mu(B) = 0$ and f_n converges uniformly on each A_n.