Recording

Want to discuss Fubini's theorem and iterated integrals for Lebesgue measure and for general measures.

At some point in this discussion we want to use a new tool: Dynkin classes.

These have some interest in their own right so I will begin by discussing them.

Rather than thinking of these as only being a technique for the proof of the Fubini Thm. we will (like Cohn in Chapter 1) briefly give an independent discussion of Dynkin classes.

Say that you want to compare two measures μ and ν.

Naively we might want to say that if μ and ν agree on some "generating set $\mathcal{C}$ then they agree on the σ-algebra generated by $\mathcal{C}$.

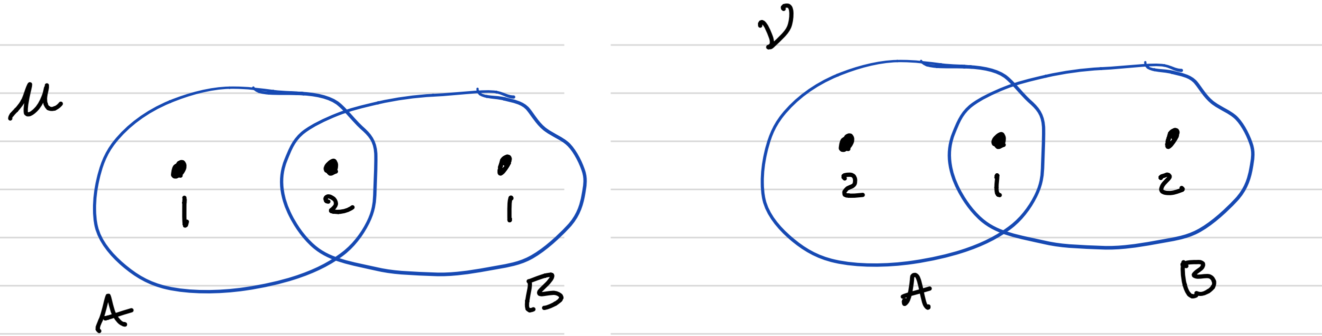
How would the argument go?

Recall our method of dealing with "generators of a σ-algebra":

$\mathcal{C}$ generates $\mathcal{A}$ if $\mathcal{A}$ is the smallest σ-algebra that contains $\mathcal{C}$.

Say μ and ν agree on $\mathcal{C}$ so they agree on some σ-algebra containing $\mathcal{C}$ so they agree on the smallest σ-algebra containing $\mathcal{C}$ so they are equal.

This argument fails since the collection of sets on which two measures agree need not be a σ-algebra.



$\mu(A)=3 \qquad \mu(B)=3 \qquad \nu(A)=3 \qquad \nu(B)=3$

$$2=\mu(A\cap B)\neq \nu(A\cap B)=3$$

The collection of sets where $\mu=\nu$ is not closed under intersection.

What structure does this collection of sets have?

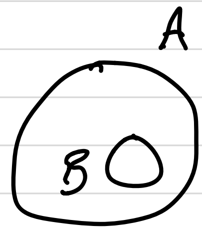
We make an assumption

$$\mu(X) = \nu(X) < \infty.$$

p.37

A collection $\mathcal{D}$ of subsets of X

is a d-system on X if



(a) $X \in \mathcal{D}$

(b) $A - B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$ and $A \supset B$

(c) $\bigcup_n A_n \in \mathcal{D}$ whenever A_n is an

increasing sequence of

sets in $\mathcal{D}$: $A_1 \subset A_2 \subset A_3 \cdots$

Lemma. Say that $(\mathfrak{X}, \mathcal{A})$ is a measurable space and μ and ν are measures defined on $\mathcal{A}$. Assume $\mu(\mathfrak{X}) = \nu(\mathfrak{X}) < \infty$.

Then the collection of sets A for which $\mu(A) = \nu(A)$ is a d-system.

Proof.

(a) $\mu(\mathfrak{X}) = \nu(\mathfrak{X})$ by assumption

(b) $\mu(A-B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A-B)$

(c) $\mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_n A_n\right)$.

Just as for σ-algebras the intersection of two d-systems is a d-system.

If we are given a collection $\mathcal{C}$ of subsets of X we can form the smallest d-system containing $\mathcal{C}$ and we call this the d-system generated by $\mathcal{C}$. (Cohn p. 37)

In sum σ-algebras are good for defining measures while d-systems are good for comparing measures.

We will return to d-systems in the course of analysing product measures.

Product measures and iterated integrals.

Given measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we want to define a product measure $\mu \times \nu$ on $X \times Y$.

A good example to keep in mind is

$X = \mathbb{R}$, $Y = \mathbb{R}$; $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathbb{R})$;

$\mu = \lambda$, $\nu = \lambda$ though as always we will work in greater generality.

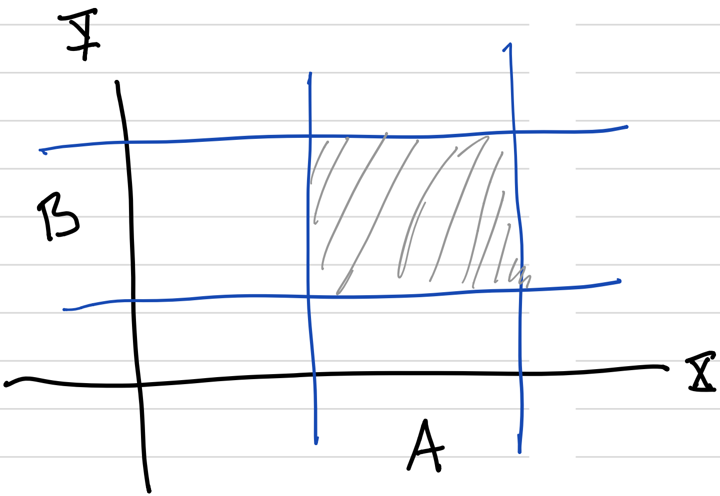
As always when we are defining a measure we need to pay attention to the question of the appropriate σ-algebra on which to define it.

So we start by defining a product σ-algebra.

Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{Y}, \mathcal{B})$ be measurable spaces.

A measurable rectangle is a set

$A \times B \subset \mathfrak{X} \times \mathfrak{Y}$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.



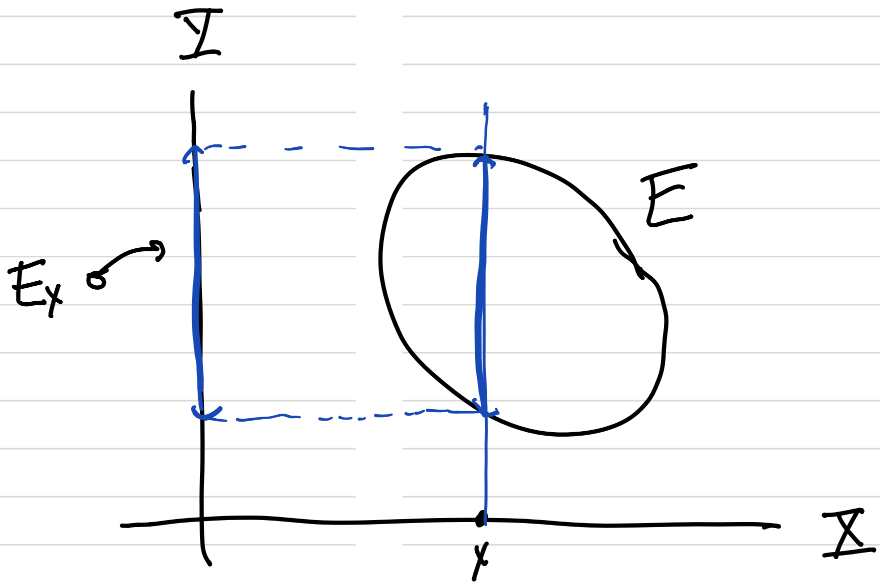
The product σ-algebra is generated by measurable rectangles is written: $\underline{\mathcal{A} \times \mathcal{B}}$
(Cohn p. 143)

Example 5.1.1. $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

Given a set $E \subset X \times Y$ we can define horizontal or vertical slices.

$$E_x = \{ y \in Y : (x,y) \in E \}$$

$$E^y = \{ x \in X : (x,y) \in E \}$$



Lemma 5.1.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces.

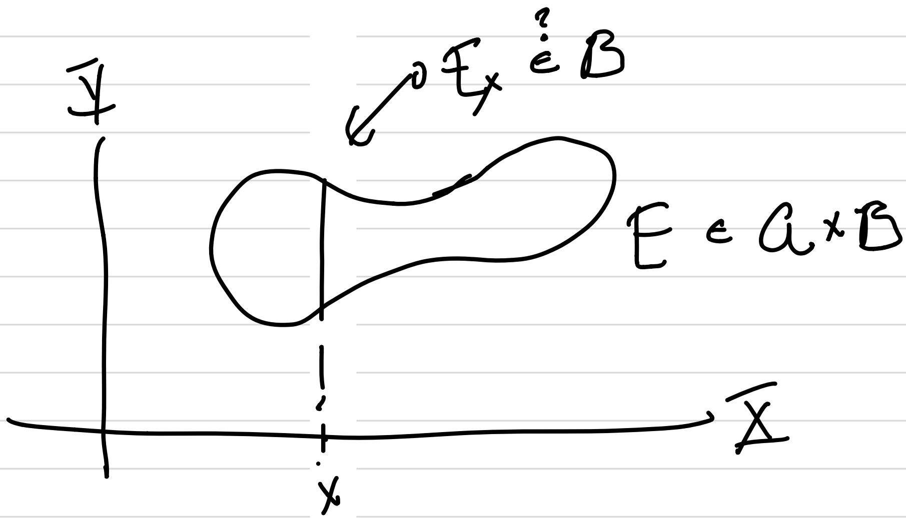
Say $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$

then $E_x \in \mathcal{B}$ for each $x \in X$.

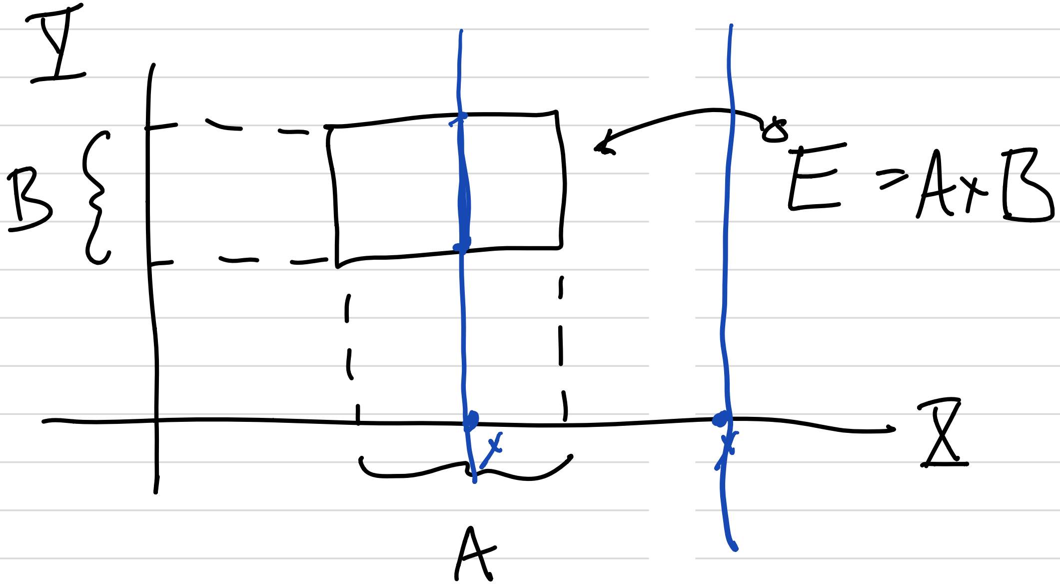
Proof discussion.

The strategy of the proof is somewhat indirect. This is forced on us because the definition of $\mathcal{A} \times \mathcal{B}$ is indirect.

Let $\mathcal{F}$ be the collection of all subsets E in $\mathcal{A} \times \mathcal{B}$ for which the conclusion of the lemma holds.



We check that measurable rectangles are in $\mathcal{A}$.



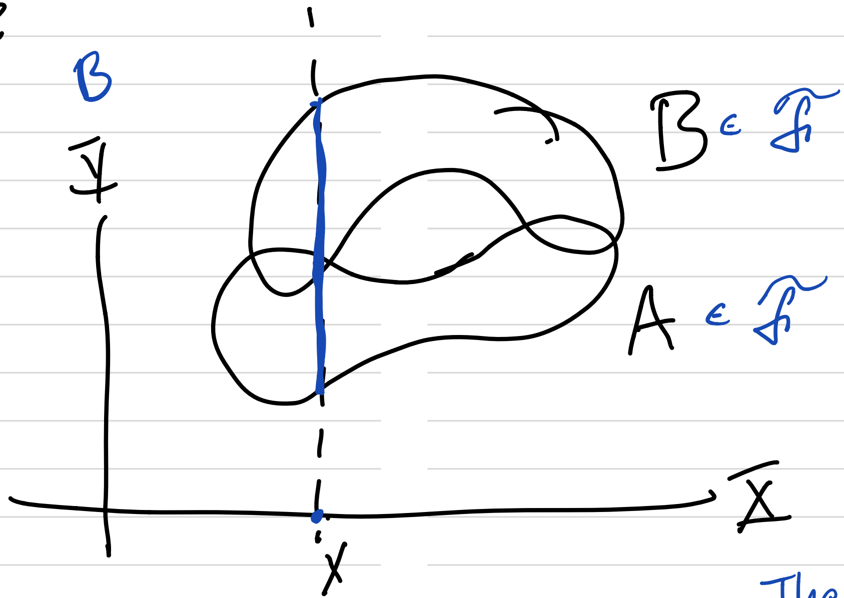
$A \in \mathcal{a}$

$B \in \underline{\mathcal{B}}$.

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A. \end{cases}$$

We check that $\tilde{\mathcal{F}}$ is a σ-algebra.

For example it is closed under unions.



$$(A \cup B)_x = A_x \cup B_x .$$

The slice of a union is the union of the slices.

We conclude that the result holds for the σ-algebra generated by measurable rectangles.

This is $\mathcal{A} \times \mathcal{B}$.

$\mathcal{F} \supset$ meas. rect.

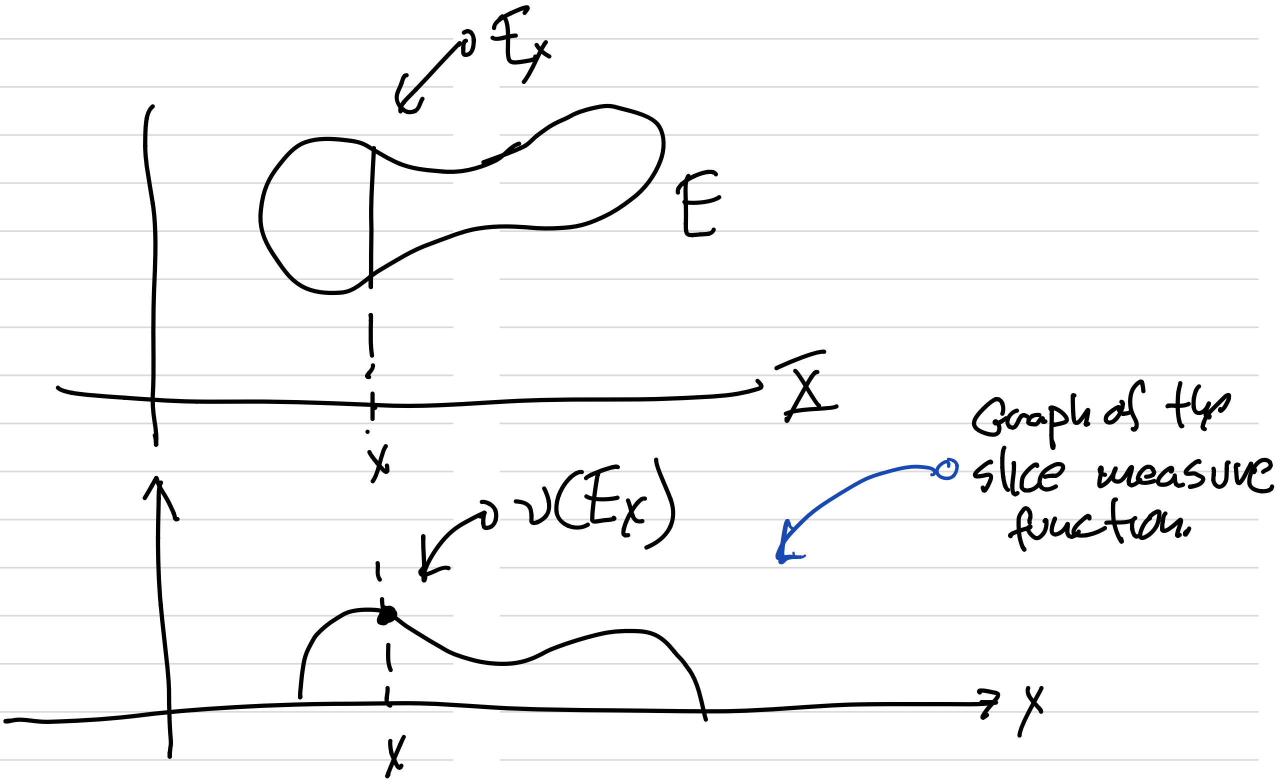
$\mathcal{F}$ is a σ-algebra

$\mathcal{F}$ contains the smallest σ-algebra containing meas. rects.

$\mathcal{A} \times \mathcal{B}$.

Since E_x is $\mathcal{B}$ measurable for each x the function $x \mapsto \nu(E_x)$ is well defined. What more can we say?

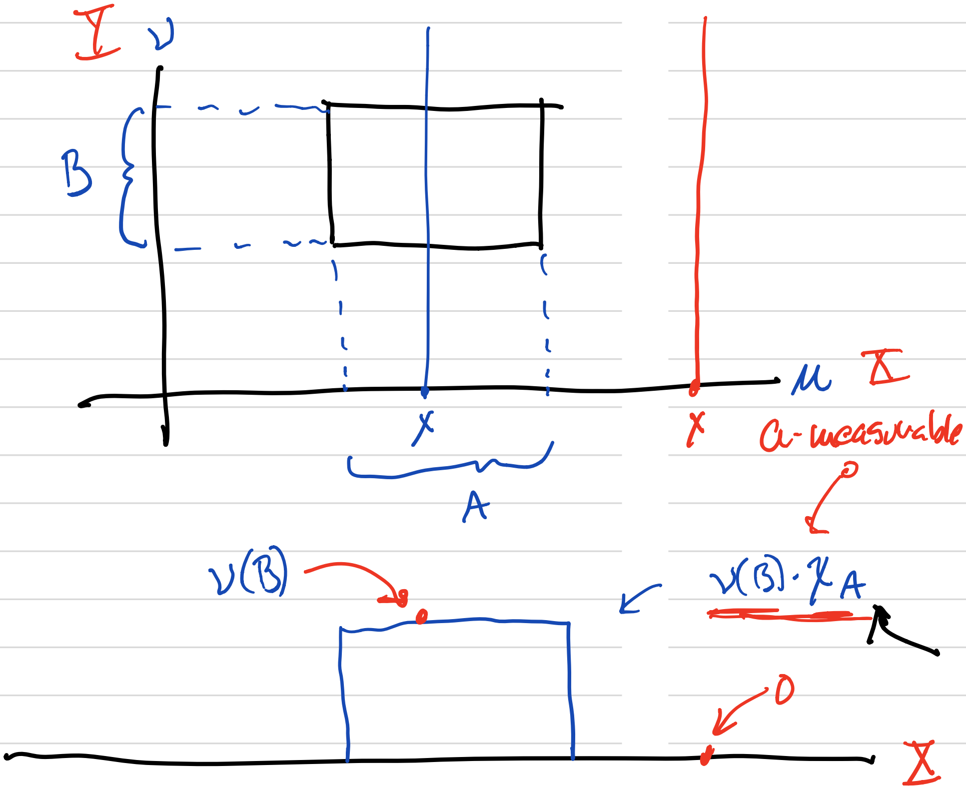
Call $x \mapsto \nu(E_x)$ a "slice measure function".



Prop. 5.1.3. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function.

Proof discussion. Let $\mathcal{F}$ be the collection of sets E for which the slice measure function is $\mathcal{A}$-measurable.

$\mathcal{F}$ contains the collection of measurable rectangles.



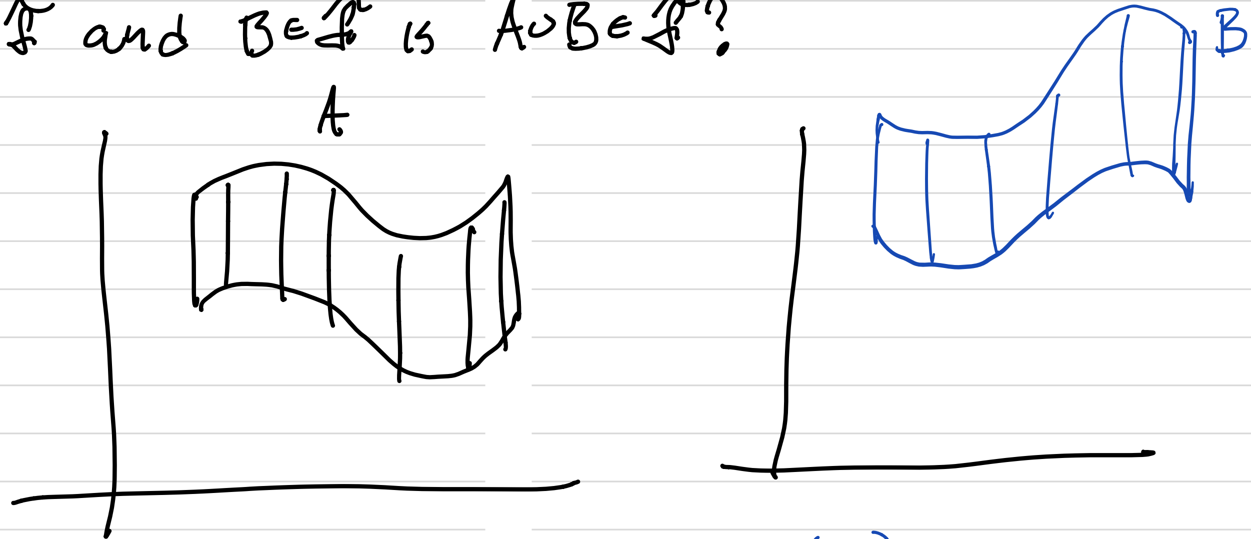
slice measure function.

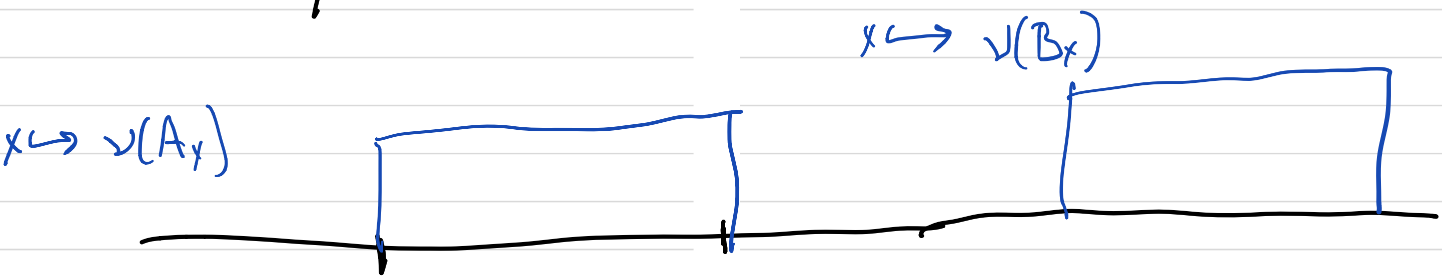
The slice measure function is $\mathcal{A}$ measurable: $A \in \mathcal{A}$ so χ_A is $\mathcal{A}$-measurable

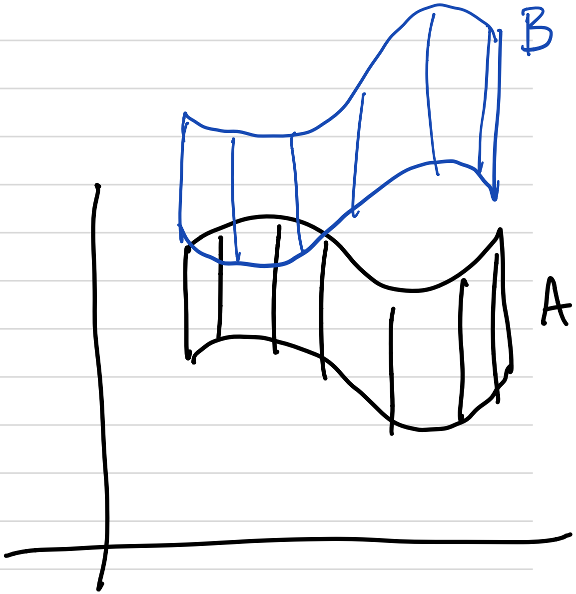
so $\nu(B) \cdot \chi_A$ is $\mathcal{A}$-measurable.

If the slice measure functions for A and B are $\mathcal{A}$-measurable then what can you say about the slice measure function for $A \cup B$?

If $A \in \mathcal{F}$ and $B \in \mathcal{F}$ is $A \cup B \in \mathcal{F}$?

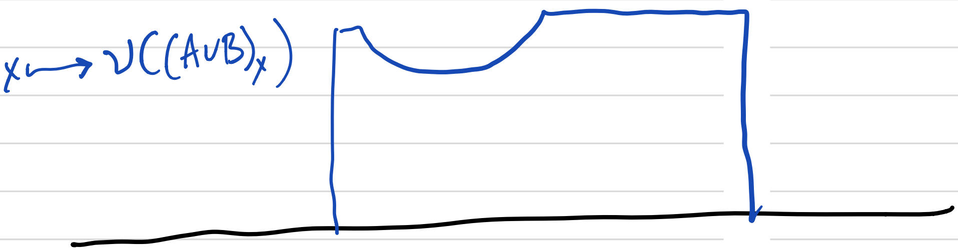




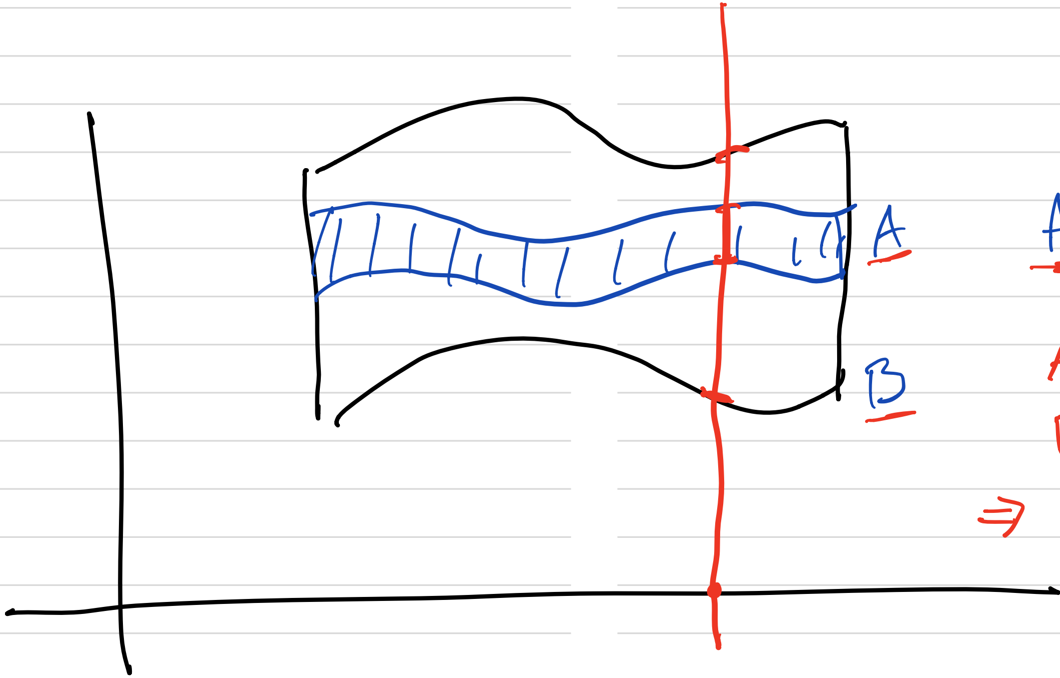


There is no particular relation between the slice measure functions for A, B and $A \cup B$.

There is no reason for $A \cup B$ to be in $\mathcal{A}$.



If $A \subset B$ however then the situation is different.



$A \subset B$

$A \in \mathcal{F}$

$B \in \mathcal{F}$

$\Rightarrow B - A \in \mathcal{F}$

$$\nu((B-A)_x) = \nu(B_x) - \nu(A_x).$$

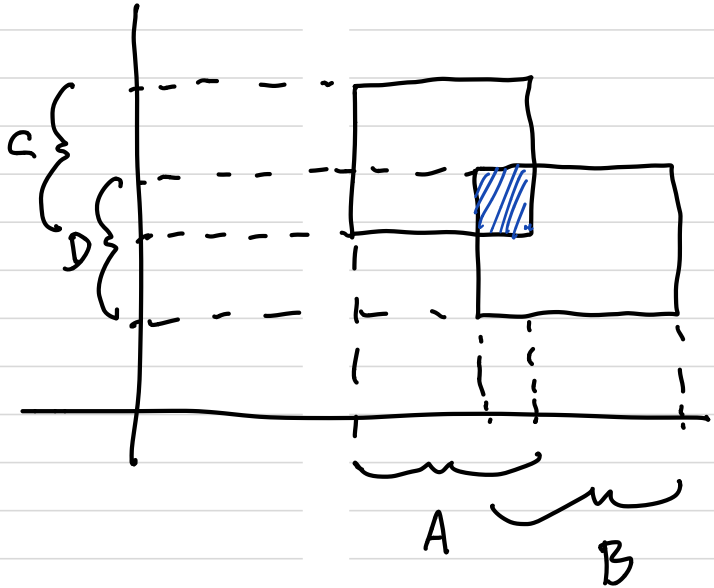
The collection of sets for which the conclusion holds is a d-system.

This set contains measurable rectangles and hence the d-system generated by measurable rectangles.

On the other hand we want to know the conclusion for the σ-algebra generated by measurable rectangles.

We say that a collection of sets $\mathcal{C}$ is a <u>π-system</u> if it is closed under taking finite unions. (p. 37)

The collection of measurable rectangles is a π-system.



The collect. of meas. rects. is a π system

$$(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D).$$

(Dynkin)

Thm. 1.6.2. Let X be a set and let $\mathcal{C}$ be a π-system on X. Then the $\underline{\sigma\text{-algebra}}$ generated by $\mathcal{C}$ coincides with the $\underline{d\text{-system}}$ generated by $\mathcal{C}$.

$$\mathcal{F} \supset \mathcal{A} \times \mathcal{B}$$

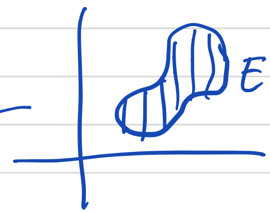
Using this theorem we complete the proof of Prop. 5.1.3.

Thm. 5.1.4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.
Then there is a unique measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$ such that

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

for each measurable rectangle

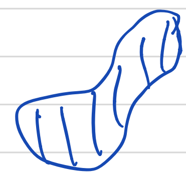
Furthermore for $E \in \mathcal{A} \times \mathcal{B}$



$$\underline{\mu \times \nu (E)} = \int_X \nu(E_x)\, d\mu(x) = \int_Y \mu(E^y)\, d\nu(y)$$

Def. $\mu\times\nu$ is the product of μ and ν.

"Proof" We use one of these expressions



$$\mu\times\nu_1(E) = \int_X \nu(E_x)\, d\mu(x)$$

to define a potential measure.

We use the Beppo-Levi Theorem to show that this function is countably additive hence a measure.

If we had chosen the other expression

$$\mu \times \nu_2 (E) = \int_Y \mu(E^y)\, d\nu(y) \text{ we would}$$

also get a measure.

It is easy to see that these two measures agree on measurable rectangles so they agree on the d-system generated by meas. rects. so according to Dynkin's Theorem they agree on the σ-alg. generated by meas. rects. which is $\mathcal{A} \times \mathcal{B}$.

This is what we wanted to show.