Good Morning!

Will start at 9.05 Remind me to record!

Reminder: Assignment 4 is out and is due in 2^{nd} December 12pm. Thursday of week 9.

We introduced L^p spaces so this is the vector space of measurable functions with in $(E, \mathcal{E}, \mu)$

$$\mu(|f|^p) < \infty$$

and equipped with the norm $\|f\|_p = \mu(|f|^p)^{1/p}$

* We showed $L^p(E)$ is a vector space
* In order to show its a normed space we need to show $\underbrace{\|f+g\|_p \leq \|f\|_p + \|g\|_p}$

This is called Minkowski's inequality

The first inequality is just about real numbers

Young's inequality lots of things ~~are~~ are called Young's inequality

If $x, y \geq 0$ and $p \in (1, \infty)$ then choose q so that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

$\sigma \qquad \rho \qquad v$

Pf/ the case where $x=0$ or $y=0$ is immediate.

If $x,y>0$ then rename our variables

write $u = x^p \quad v = y^q$

our inequality we want to prove becomes

$$u^{1/p} v^{1/q} \leq \frac{u}{p} + \frac{v}{q}$$

This is true iff

$$(u/v)^{1/p} \leq \frac{(u/v)}{p} + \frac{1}{q}$$

(by dividing both sides by v)

This is true if for all $t>0$

$$t^{1/p} \leq \frac{t}{p} + \frac{1}{q}$$

if $\frac{t}{p} + \frac{1}{q} - t^{1/p} \geq 0$

We are going to prove this by differentiation

$$\frac{d}{dt}\left(\frac{t}{p} + \frac{1}{q} - t^{1/p} \right) = \frac{1}{p}\left(1 - t^{-1/q}\right)$$

$$\frac{d^2}{dt^2}\left(\frac{t}{p} + \frac{1}{q} - t^{1/p} \right) = \frac{1}{pq} t^{-1/q - 1} > 0$$

so $\frac{t}{p} + \frac{1}{q} - t^{1/p}$ is convex and achieves a unique minimum when $\frac{1}{p}\left(1 - t^{-1/q}\right) = 0$

so when $t = 1$.

$$\left(\frac{t}{p} + \frac{1}{q} - t^{1/p}\right)\Bigg|_{t=1} = \frac{1}{p} + \frac{1}{q} - 1 = 0$$

so $\frac{t}{p} + \frac{1}{q} - t^{1/p} \geq 0$

Corollary for any $\eta > 0$

$$xy \leq \frac{x^p \eta^p}{p} + \frac{y^q}{\eta^q q}$$

~~Proof~~ write $xy = (x\eta)(y/\eta)$ then apply Young's inequality

$$\boxed{t\log(t) - t + 1 \geq 0}$$

Hölder's inequality $(E, \mathcal{E}, \mu)$ is a measure space and $f \in L^p(E)$ $g \in L^q(E)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $fg \in L^1$ and we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Remark It's an example of an interpolation inequality something that comes up in PDEs a lot it tells us that if $f \in L^p(E)$ and $f \in L^q(E)$

it tells you that if $f \in L^p(E)$ and $f \in L^q(E)$

then $f \in L^2$

$$\|f\|_2^2 \le \|f\|_p \|f\|_q$$

so if $p > 2$ then $q < 2$.

<u>Proof of Hölder</u>:

First let $p = 1$ $q = \infty$

then wlog g is bounded by $\|g\|_\infty$

$$|f(x) g(x)| \le |f(x)| \, \|g\|_\infty$$

integrating this gives $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$

When $p \in (1, \infty)$ then I'm going to give two proofs

First proof (ugly one):

$$|f(x) g(x)| \le \eta^p \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{\eta^q q}$$

integrating gives $\|fg\|_1 \le \dfrac{\eta^p \|f\|_p^p}{p} + \dfrac{\|g\|_q^q}{\eta^q q}$

Optimise over η to make RHS as small as possible

$$\frac{d}{d\eta} \left(\frac{\eta^p}{p} \|f\|_p^p + \frac{1}{\eta^q q} \|g\|_q^q \right) = \eta^{p-1} \|f\|_p^p - \eta^{-q-1} \|g\|_q^q$$

$$\frac{d}{d\eta}\left(\frac{\eta^p}{p}\|f\|_p^p + \frac{1}{\eta^q q}\|g\|_q^q\right) = \eta \ldots$$

$$\frac{d^2}{d\eta^2}\left(\text{———}\right) > 0$$

so we can find a minimum when

$$\eta^p \|f\|_p^p = \eta^{-q} \|g\|_q^q$$

$$\eta = \frac{\|g\|^{q/p+q}}{\|f\|^{p/p+q}}$$ sub this in

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Second proof of Hölder

First suppose $\|f\|_p = 1$ and $\|g\|_q = 1$

then $|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$

integrating $\|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1$

Now notice that $\left\|\frac{f}{\|f\|_p}\right\|_p = 1$ and $\left\|\frac{g}{\|g\|_q}\right\|_q = 1$

so $\left\|\frac{f}{\|f\|_p}\frac{g}{\|g\|_q}\right\|_1 \leq 1$

$$\frac{1}{\|f\|_p\|g\|_q}\|fg\|_1 \leq 1 \qquad \|fg\|_1 \leq \|f\|_p\|g\|_q$$

$$\ldots (1-t)y^q) \geq t\log(x^p) + (1-t)\log(y^q)$$

$$\log(tx^p + (1-t)y^q) \geq t\log(x^p) + (1-t)\log(y^q)$$

$$= tp\log(x) + (1-t)q\log(y)$$

$t = \frac{1}{p}$ $(1-t) = \frac{1}{q}$ $= \log(x) + \log(y) = \log(xy)$

$$\log\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \geq \log(xy)$$ exponentiate.

Cauchy-Schwartz normally about an inner product space

$$\langle x, y\rangle \leq \|x\|\|y\|$$

$L^2(E)$ is an inner product space

Hölder's inequality when $p=q=2$ is often called Cauchy-Schwartz.

$$\|fg\|_1 \leq \|f\|_2\|g\|_2$$

<u>Minkowski's inequality</u> for $p \in [1,\infty)$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Pf For $p=1$

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

integrating this gives $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

$p \in (1,\infty)$ let q be s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$p \in (1,\infty)$ let q ...

then $|f+g|^{p-1} \in L^q(E)$ as

$$|f+g|^{(p-1)q} = |f+g|^p$$ and we showed on Friday that $f+g \in L^p$

$$\| |f+g|^{p-1} \|_q = \|f+g\|_p^{p/q}$$

$$\|f+g\|_p^p = \mu(|f+g|^p) = \mu(|f+g|\,|f+g|^{p-1})$$

$$\leq \mu(|f|\,|f+g|^{p-1}) + \mu(|g|\,|f+g|^{p-1})$$

$$\overset{\text{Hölder}}{\leq} \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \, \| |f+g|^{p-1} \|_q$$

$$= (\|f\|_p + \|g\|_p) \, \|f+g\|_p^{p/q}$$

Divide by $\|f+g\|_p^{p/q}$

$$\|f+g\|_p^{p(1-\frac{1}{q})} \leq \|f\|_p + \|g\|_p \qquad 1-\frac{1}{q} = \frac{1}{p}$$

so we have $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.