Borel sets are Lebesgue measurable

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$\mathcal{M}$ - the set of Lebesgue measurable subsets of $\mathbb{R}$

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$

All Borel sets are Lebesgue measurable.

<u>Lemma</u>

$(-\infty, b]$ is Lebesgue measurable for any b.

For any set $B \subseteq \mathbb{R}$

$$\lambda^*(B \cap (-\infty, b]) + \lambda^*(B \cap (b, \infty)) = \lambda^*(B)$$

Pf/ Take any sequence $I_1, I_2, I_3 \ldots$ of half open intervals $(c, d]$

s.t. $B \subseteq \bigcup_{n=1}^{\infty} I_n$

lets define some other half open intervals

$$I_k^l = (-\infty, b] \cap I_k$$

$$I_k^r = (b, \infty) \cap I_k$$

I_k^l and I_k^r are both (possibly empty) half open intervals

$$(-\infty, b] \cap (c, d] = (c, b] \text{ if } c < b \leq d \text{ or } \emptyset \text{ o/w}$$

$$(b, \infty) \cap (c, d] = (b, d] \text{ if } c \leq b < d$$

or ϕ o/w

We also have $B \cap (-\infty, b] \subseteq \bigcup_n I_n^l$

$$B \cap (b, \infty) \subseteq \bigcup_n I_n^r$$

$$\sum_n \lambda(I_n^l) + \sum_n \lambda(I_n^r) = \sum_n \lambda(I_n^*)$$

$$\lambda^*(B \cap (-\infty, b]) \leq \sum_n \lambda(I_n^l) \quad \text{by def}^n \ \lambda^*$$

$$\lambda^*(B \cap (b, \infty)) \leq \sum_n \lambda(I_n^r)$$

$$\lambda^*(B \cap (-\infty, b]) + \lambda^*(B \cap (b, \infty)) \leq \sum_n \lambda(I_n^l) + \sum_n \lambda(I_l^r)$$

$$= \sum_n \lambda(I_n)$$

take the infimum over all possible sequences I_n

to get

$$\lambda^*(B \cap (-\infty, b]) + \lambda^*(B \cap (b, \infty)) \leq \lambda^*(B)$$

By countable subadditivity

$$\lambda^*(B) \leq \lambda^*(B \cap (-\infty, b]) + \lambda^*(B \cap (b, \infty))$$

so $\lambda^*(B) = \lambda^*(B \cap (-\infty, b]) + \lambda^*(B \cap (b, \infty))$

$$(-\infty, b] \in \mathcal{M}$$

<u>Cor</u> $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$

Pf/ $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that contains

Pf/ $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra that contains every set of the form $(-\infty, b]$

(proved in video 1 week 1.)

$\mathcal{M}$ is a σ-algebra containing all set $(-\infty, b]$

so $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$