Up to this point I have been asking you to believe in the value of limit theorems without too much evidence that they are important.

This week we used limit theorems to answer some natural questions.

Some of these questions were about compatibility between the Riemann integral and the Lebesgue integral.

Some of these questions were exclusively about the Riemann integral and how it works.

There are two ways to define the notion of Riemann integrability.

Given a bounded function f defined on $[a,b]$:

Def. 1. f is Riemann integrable if the supremum of the lower sums is equal to the infimum of the upper sums:

$$\sup_P \{ L(P,f) \} = \inf_P \{ U(P,f) \}.$$

Def 2.

There is a sequence of partitions P_n with $\mathrm{mesh}(P_n) \to 0$ for which

$$\lim_{n\to\infty} L(P_n,f) = \lim_{n\to\infty} U(P_n,f)$$

(where the mesh of a partition is the size of the largest interval in the partition.)

Cohn uses definition 1 and I used definition 2 in Wednesdays lecture.

It follows from the proof of our theorem that these definitions are equivalent.

After reviewing the proof I will explain the equivalence.

In my posted notes I use def. 1 (and follow Cohn's treatment).

Theorem. 2.5.4. Let $[a,b]$ be a closed bounded interval and let f be a bounded real valued function on $[a,b]$. Then

(a) if f is Riemann integrable then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.

(b) f is Riemann integrable if and only if it is continuous at almost every point of $[a,b]$.

Consider a function f that is Riemann integrable according to def 1.

According to def. 1 there are sequences P_n', P_n'' with

$$\lim_{n\to\infty} U(P_n', f) = \lim_{n\to\infty} L(P_n'', f).$$

By taking refinements we can find some single sequence of partitions

P_n so that

$$\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f).$$

(We really know nothing about this sequence.)

Given $f : [a,b] \to [-B,B] \subset \mathbb{R}$.

Def. 1 $\Leftrightarrow$ there is <u>some</u> sequence of partitions P_n with $\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f)$.

Def. 2 $\Leftrightarrow$ for <u>any</u> seq. of P_n with $\mathrm{mesh}(P_n) \to 0$
$$\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f).$$

Remark. f is Riem. int. according to Def 2 $\Rightarrow$ f is Riem. int. according to Def 1.

Assume f is Riem. int. according to def 1.

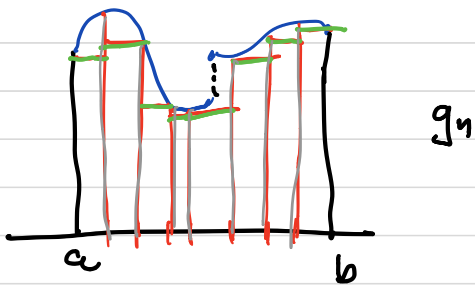
We can turn these partitions into functions.

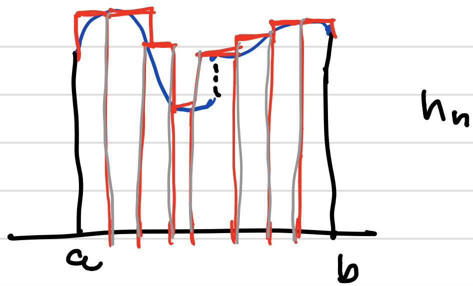
$$g_n \le f \le h_n$$

with

$$\int g_n \, d\mu = L(P_n, f)$$

$$\int h_n \, d\mu = U(P_n, f).$$





We turn the hypothesis about upper and lower sums into a hypothesis about approximating f from above and below by step functions.

$$g_n \to g \qquad h_n \to h$$

Pointwise convergence.

Using Dominated Convergence we get:

$$\int g \, d\mu = \lim \int g_n \, d\mu = \lim L(P_n, f) = \lim U(P_n, f) = \lim \int h_n \, d\mu = \int h \, d\mu$$

So $g \leq f \leq h$ and

$$\int g \, d\mu = \int h \, d\mu.$$

Since $h - g \geq 0$ and $\int h - g \, d\mu = 0$ this implies that $g = h$ a.e. so $g = f$ a.e.

Now g, h are limits of Borel functions so are Borel measurable (hence Lebesgue measurable).

It turns out that a function which is equal to a measurable function a.e. is also measurable but this is a little tricky and uses a special property of Lebesgue measure:

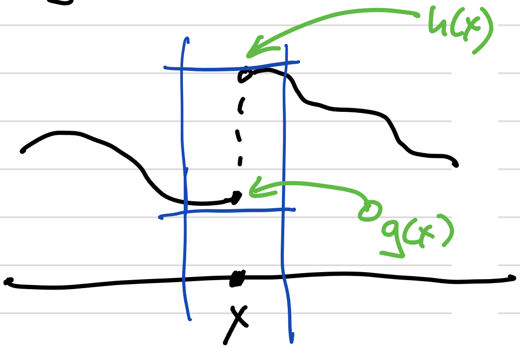
Lebesgue measure has the property that subsets of sets of measure zero are measurable.

Recall that sets with Lebesgue outer measure 0 are measurable (this was our first example of a Lebesgue measurable set) and subsets of a set with outer measure 0 have outer measure 0.

At this point we have shown f Riem int. according to def. 1
$\Rightarrow$ f is Leb. int and

$\int f \, d\mu = \int_a^b f(x)\,dx.$

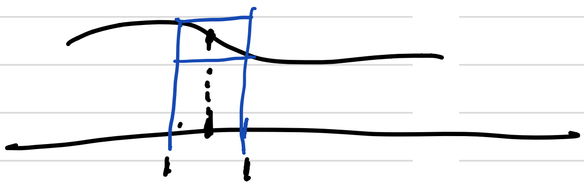
The next step in the analysis is to give a geometric interpretation of the condition that $g(x) = h(x)$ (at least when x is not in P_n for any n).



$g(x) < h(x)$

f is discontinuous at x

f is continuous at x.



$g(x) = h(x)$

$g(x) = h(x)$ at a point where f is continuous.

The last step in the proof of the theorem is to consider a sequence of partitions P_n (as in the 2nd definition) for which the mesh goes to 0.

(This is a different P_n than before. Before we knew about upper & lower sums $U(P_n, f)$, $L(P_n, f)$ now we only know about the mesh of P_n and we want to discover properties of $U(P_n, f)$, $L(P_n, f)$.)

We build g_n, h_n from P_n and construct g, h as limits.

In this case we get a converse: $g(x) = h(x)$ at x where f is continuous.

Assuming that f is Riem. integrable we get f is continuous a.e. so $g = h$ a.e.
This implies that

$$\lim L(P_n, f) \underset{\text{def}}{=} \lim \int g_n \, d\mu \underset{\text{DC}}{=} \int g \, d\mu = \int h \, d\mu \underset{\text{DC}}{=} \lim \int h_n \, d\mu \underset{\text{def}}{=} \lim U(P_n, f)$$

so $\lim L(P_n, f) = \lim U(P_n, f)$ and

f is Riem. int. according to def 2.

Scholium. For a bounded function $f:[a,b]\to\mathbb{R}$ Riem. int. according to def 1 is equivalent to Riem. integrability according to def. 2.

Integrability of f according to def 1

$\Rightarrow f$ is Leb. integrable and $\int f\,d\mu = \int_a^b f\,dx$

+ f is continuous a.e.

$\Rightarrow$ integrability according to def 2

$\Rightarrow$ Integrability according to def 1

① Almost everywhere convergence:

For a.e. $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$.

② Convergence in measure

f_n converges to f in measure

if for any $\varepsilon > 0$

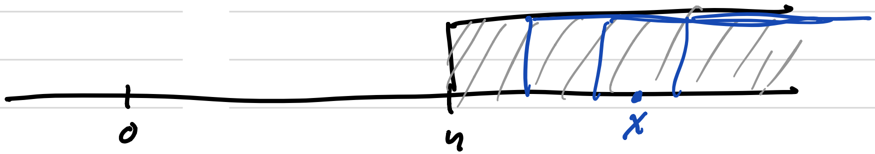
$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

③ Convergence in mean.

We say that f_n converges to f in mean if

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

Example 1.

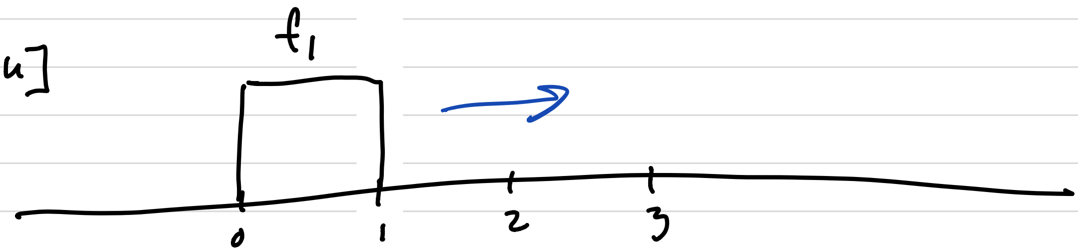


$$f_n = \chi_{[n,+\infty)}.$$

$f_n \to 0$ pointwise but not in measure.

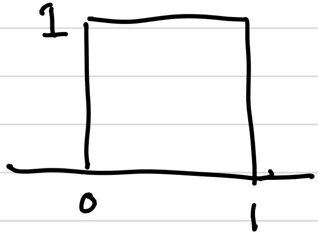
Sliding bump functions give the same conclusion

$$f_n = \chi_{[n-1,n]}$$

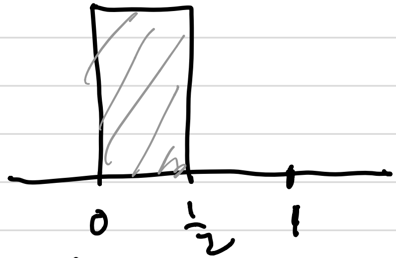


Example 2.

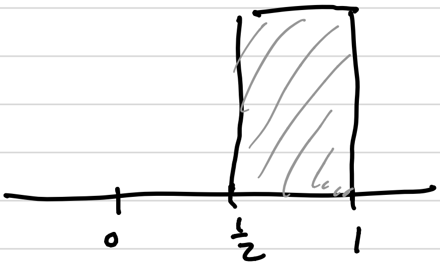
$f_1 =$



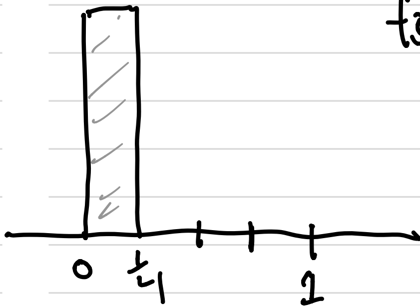
$f_2 =$



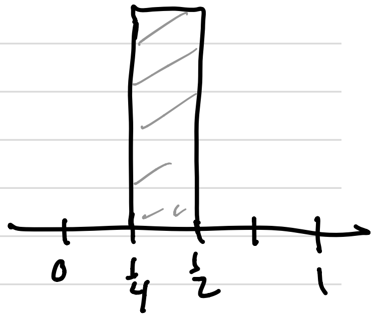
$f_3 =$



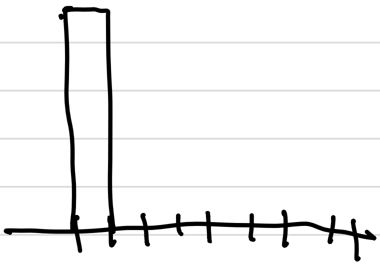
$f_4 =$



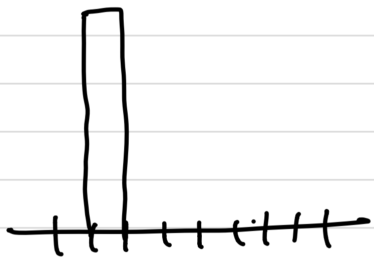
$f_5 =$



$f_8 =$

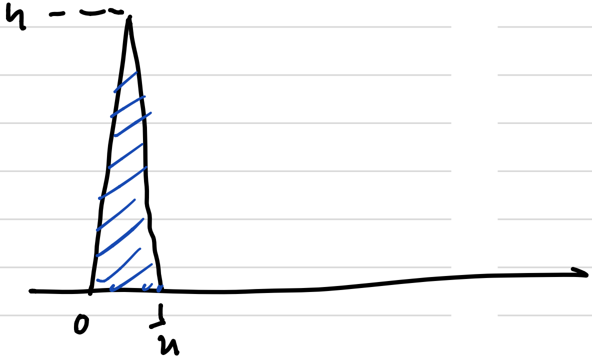


$f_9 =$



Example 3.

$f_n =$



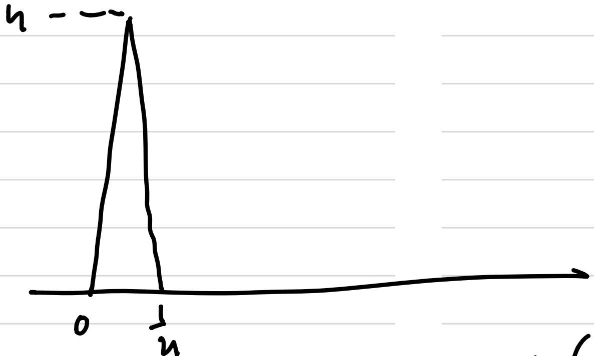
$$\int |f_n - f| \, dx = \frac{1}{2}$$

This does not go to 0

so f_n does not

converge in mean.

$f_n =$



$$\mu(\{x \in X : |f_n(x) - f(x)|\}) \le \frac{1}{n}$$

so goes to 0.

Graph of non-implications:

$$
\begin{tikzcd}
\text{Conv. a.e.} \arrow[rr, bend left, "\text{Ex1}"] \arrow[dr, "\text{Ex3}"] & & \text{Conv. in measure} \arrow[ll, bend left, "\text{Ex2}"] \arrow[dl, bend left, "\text{Ex3}"] \\
& \text{Conv. in mean} \arrow[ul, bend left, "\text{Ex 2}"] \arrow[ur, "3.1.5"] &
\end{tikzcd}
$$

On the other hand with additional hypotheses we do have some implications.

General tricks:

Choosing subsequences carefully (so that you control the behaviour at most points).

Controlling the size of sets where the behaviour of the sequence is <u>not</u> controlled.

Prop. 3.1.2 If $(X, \mathcal{A}, \mu)$ is a finite measure space (if $\mu(X) < \infty$) then convergence a.e. implies convergence in measure.

Proof. Let $\varepsilon > 0$. We need to show that

$$\lim_{n \to \infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

Let $A_k = \{x \in X : |f_k(x) - f(x)| > \varepsilon\}$

Let

$$B_n = \bigcup_{k=n}^{\infty} A_k.$$

This is the set of x which are in some A_k for $k \geq n$.

$$B_\infty = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

B_∞ is the set of x which lie in ∞ many A_k's.

On the one hand convergence a.e. implies $\mu(B_\infty) = 0$. On the other hand

$$\mu(B_\infty) = \lim_{n\to\infty} \mu(B_n)$$

since B_n is a decreasing sequence of sets <u>and</u> $\mu(B_1)$ is finite.

Follows that $\mu(B_n) \to 0$. Since

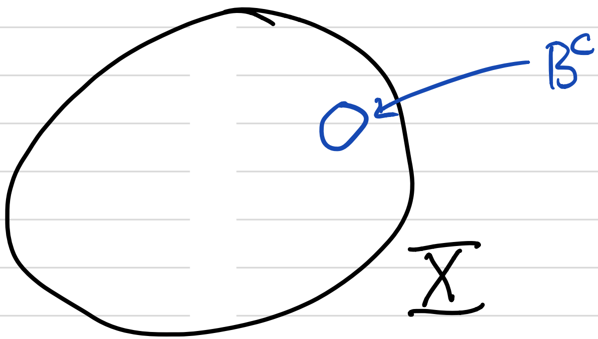
$A_n \subset B_n$ $\mu(A_n) \to 0$.

Uniform convergence is a very strong form of convergence.

In the context that we are considering it is very rare.

Never the less it is present in some sense even when we are considering a pointwise convergent sequence.

Prop. 3.1.4. (Egoroff's Thm). If f_n converges pointwise to f and μ is finite then for any $\varepsilon > 0$ there is a subset $B \subset X$ with $\mu(B^c) < \varepsilon$ so that f_n restricted to B converges uniformly to f restricted to B.



Note that in this theorem we are not throwing away a set of measure 0.
We are throwing away a set of small but positive measure.

Recall the definition of uniform convergence

$f_n \to f$ uniformly if for each $\varepsilon > 0$ there is an n with $|f_m(x) - f_{m'}(x)| \leq \varepsilon$ for $m, m' \geq n$.

Proof of Egoroff's Theorem.

Let $g_n(x) = \sup_{j \geq n} |f_j(x) - f(x)|$.

We have $g_n(x) \longrightarrow 0$ for a.e. x.

Uniform convergence means that for any $\varepsilon > 0$ there is an n with $g_n(x) \leq \varepsilon$.

We want this to hold for a set x of large measure.

In the finite measure case pointwise convergence of g to 0 implies convergence in measure.

This means given $\varepsilon, \delta > 0$ we can choose n so that:

$$\mu\Big(\underbrace{\{x \in X : g_n(x) > \varepsilon\}}_{\text{set of } x \text{ where we do \underline{not} have control}}\Big) < \delta$$

The essence of the proof is choosing the S's carefully so that after we throw out all sets where we don't have control the remaining set is still large.

$$\boxed{\mu\left(\left\{x\in X : g_n(x) > \varepsilon\right\}\right) < \delta}$$

We choose a sequence of n_j's for which $\delta_j = \frac{\varepsilon}{2^j}$ for some fixed ε. This means that the total measure of the sets we throw out is bounded by $\sum \frac{\varepsilon}{2^j} = \varepsilon$.

(Choosing δ_j very small means n_j could be very large which means that the uniform convergence could be very slow.)