

Theorem 2.4.1 (Monotone Convergence). Let f and f_1, f_2, \dots be $[0, +\infty]$ valued measurable functions on X . Suppose

$$f_1(x) \leq f_2(x) \leq \dots$$

and

$$f(x) = \lim_{n \rightarrow \infty} f_n(x).$$

Then
$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Proof. As in our previous versions of this proposition one inequality is easy.

$$f_1 \leq f_2 \leq f_3 \leq f_4 \dots \leq f$$

implies that

$$\int f_1 d\mu \leq \int f_2 d\mu \leq \int f_3 d\mu \leq \dots \leq \int f d\mu$$

so the left hand side has a limit (perhaps $+\infty$)
and

$$\lim_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu.$$

Now we look at the reverse inequality.

We have proved a version of this theorem

before (Prop. 2.3.3) when the f_n are simple functions.

We would like to "replace" the f_n by simple functions. We know that each f_n is a monotone limit of simple functions so introduce simple functions $g_{n,k}$ with $\dots \leq g_{n,k} \leq g_{n,k+1} \leq \dots$

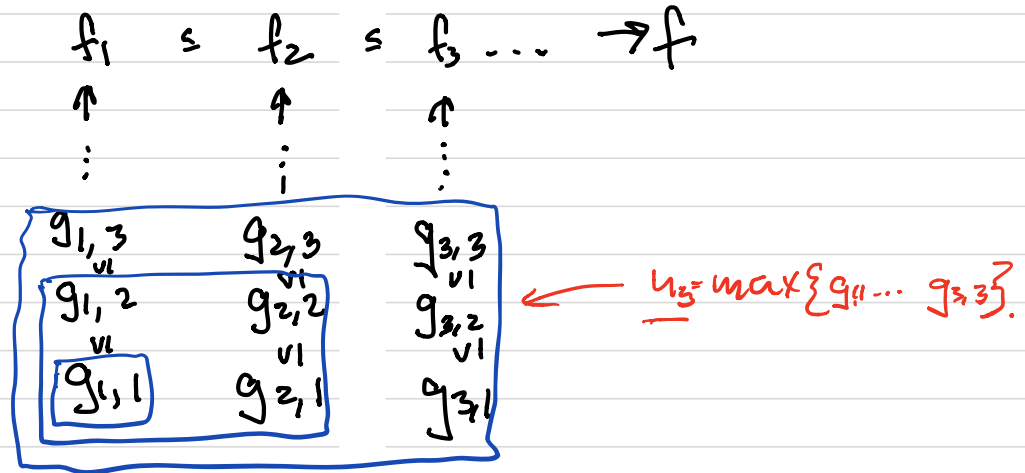
and
$$f_n = \lim_{k \rightarrow \infty} g_{n,k}.$$

$$\begin{array}{ccc}
 f_1 & f_2 & \cdots \rightarrow f \\
 \uparrow & \uparrow & \\
 \vdots & \vdots & \\
 g_{1,3} & g_{2,3} & \\
 g_{1,2} & g_{2,2} & \\
 g_{1,1} & g_{2,1} & \cdots
 \end{array}$$

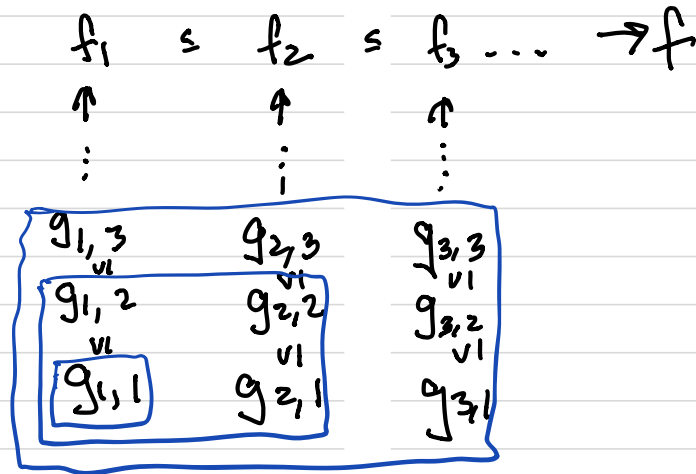
Order
properties:

$$\begin{array}{ccccc}
 f_1 & \leq & f_2 & \leq & f_3 \dots \rightarrow f \\
 \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots \\
 g_{1,3} & & g_{2,3} & & g_{3,3} \\
 \quad \downarrow & & \quad \downarrow & & \quad \downarrow \\
 g_{1,2} & & g_{2,2} & & g_{3,2} \\
 \quad \downarrow & & \quad \downarrow & & \quad \downarrow \\
 g_{1,1} & & g_{2,1} & & g_{3,1}
 \end{array}$$

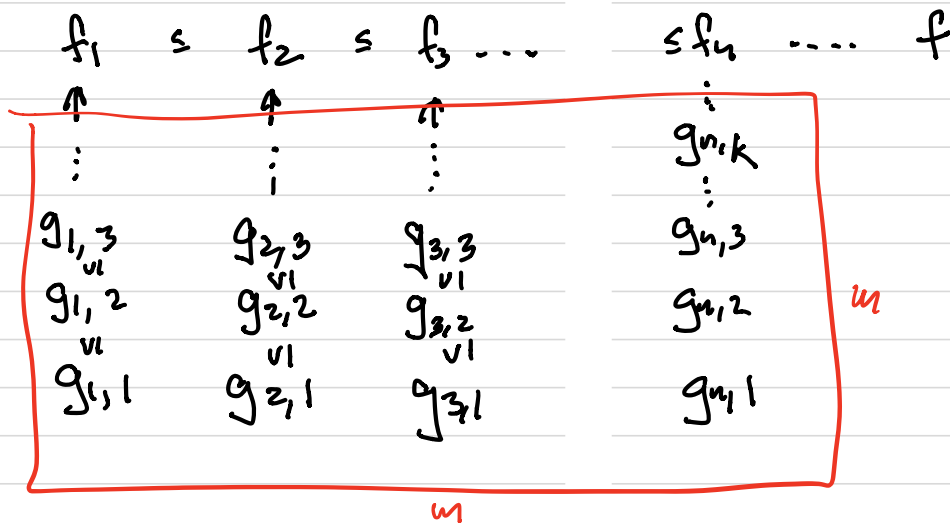
We would like to do some sort of diagonalisation procedure that gives us a monotone sequence converging to f . We do not have any relations between say $g_{1,1}$ and $g_{2,1}$.



Let h_n be the max of all the functions in the $n \times n$ box. If $\underline{u} < \underline{u}$ then h_m is the max of a larger collection of functions than h_n so $\underline{h_n} \leq \underline{h_m}$.

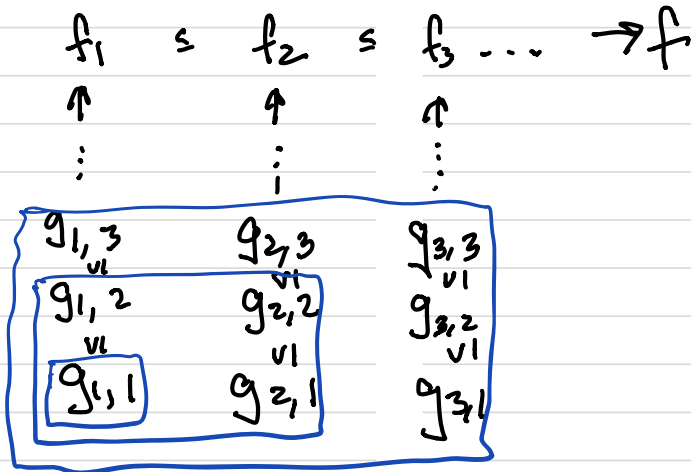


Note also that f_n dominates all the functions in the $n \times n$ box so it dominates their maximum and $h_n \leq f_n$.



Claim that $h_n \rightarrow f$ pointwise. If we fix
 an $x \in \mathbb{X}$ and an $\varepsilon > 0$ we can find an
 n so that $f(x) - f_n(x) < \frac{\varepsilon}{2}$ and a k so that
 $f_n(x) - g_{n,k}(x) < \frac{\varepsilon}{2}$. If $m = \max(n, k)$ then
 $f(x) - h_m(x) < \varepsilon$.

Note that h_m is also the max of the top row of the $m \times m$ box.



Now since h_n is a non-decreasing sequence of simple $[0, +\infty)$ valued measurable functions it follows from Prop 2.3.3 that

$$\int f d\mu = \lim_{n \rightarrow \infty} \int h_n d\mu$$

and since $f_n \geq h_n$ we have $\int f_n d\mu \geq \int h_n d\mu$
and

$$\int f d\mu = \lim_{n \rightarrow \infty} \int h_n d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Combined with the previous inequality

$$\left(\int f d\mu \geq \lim_{n \rightarrow \infty} \int f_n d\mu \right)$$

we have $\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$