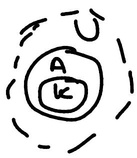
Regularity of Lebesgue measure

Prop^n λ is regular if $A \in \mathcal{M}$ then

a) $\lambda(A) = \inf\{\lambda(U) \mid U \text{ open } A \subseteq U\}$

b) $\lambda(A) = \sup\{\lambda(K) \mid K \text{ compact } K \subseteq A\}$



~~Pf~~ Start with (a)

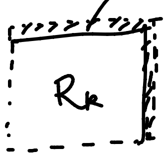
By monotonicity $\lambda(A) \leq \inf\{\lambda(U) : U \text{ open } A \subseteq U\}$

If $A \subseteq U$ then $\lambda(A) \leq \lambda(U)$

Given $\varepsilon > 0$ $\exists$ a sequence of half open rectangles R_n

s.t. $A \subseteq \bigcup_n R_n$ and $\sum_n \lambda(R_n) \leq \lambda(A) + \varepsilon$

Lets define a new sequence of rectangles $\tilde{R}_n$ making R_n slightly larger and $\tilde{R}_n$ open rectangles



measure of this strip $\leq \varepsilon 2^{-k}$

$$A \subseteq \bigcup_n \tilde{R}_n$$

$$\sum_n \lambda(\tilde{R}_n) \leq \lambda(A) + 2\varepsilon$$

The $\tilde{R}_n$ are open so $\bigcup_n \tilde{R}_n$ is open

$$\lambda\left(\bigcup_n \tilde{R}_n\right) \leq \sum_n \lambda(\tilde{R}_n) \leq \lambda(A) + 2\varepsilon$$

so $\inf\{\lambda(U) \mid U \text{ open } A \subseteq U\} \leq \lambda(A) + 2\varepsilon$

since ε is arbitrary $\inf\{\lambda(U) \mid U \text{ open } A \subseteq U\} \leq \lambda(A)$

Proof of (b) First lets assume $\exists$ some B compact with $A \subseteq B$ and $\lambda(B) < \infty$

Then by the first part $\exists$ U s.t. $B \setminus A \subseteq U$

(applied part (a))

Then by the first part $\exists\ U$ s.t. B ...

and $\lambda(U) \leq \lambda(B \setminus A) + \varepsilon$ (applied part (a) to $B \setminus A$)

$K = B \setminus U$ by countable additivity

$\lambda(K) = \lambda(B) - \lambda(U) \geq \lambda(B) - \lambda(B \setminus A) - \varepsilon$

$= \lambda(B) - \lambda(B) + \lambda(A) - \varepsilon$

so

we can only do this $= \lambda(A) - \varepsilon$

because $\lambda(B) < \infty$ and K, A, $B \setminus A$ are all contained in B so have finite measure

$\lambda(K) \geq \lambda(A) - \varepsilon$

so $\lambda(A) \leq \sup\{\lambda(K) \mid K \text{ compact } K \subseteq A\} + \varepsilon$

and ε is arbitrary so

$\lambda(A) \leq \sup\{\lambda(K) \mid K \text{ compact}, K \subseteq A\}$

By monotonicity, $\lambda(A) \geq \sup\{\lambda(K) \mid K \text{ compact}, K \subseteq A\}$

What if A isn't contained in some compact set

$A_n = A \cap B_n$ where B is the closed ball of radius n

given $\varepsilon > 0$

Then $\forall n$ $\exists$ $K_n \subseteq A_n$ s.t. $\lambda(K_n) \geq \lambda(A_n) - \varepsilon$

By continuity of measure $\lambda(A_n) \to \lambda(A)$

If $\lambda(A) = \infty$ then $\lambda(A_n) \to \infty$ so $\lambda(K_n) \to \infty$

If $\lambda(A) < \infty$ then since $\lambda(A_n) \to \lambda(A)$

$\exists$ N s.t. $\forall$ $n \geq N$ $\lambda(A_n) \geq \lambda(A) - \varepsilon$

so $\lambda(K_N) \geq \lambda(A_n) - \varepsilon \geq \lambda(A) - 2\varepsilon$

so $\sup_K \{\lambda(K) \mid K \text{ compact}, K \subseteq A\} \geq \lambda(A) - \cancel{2\varepsilon}$

ε is arbitrary

so $\sup_{K} \{ \lambda(K) \mid K$ [illegible]

and since ε is arbitrary

the other ineq follows by monotonicity

so

$$\sup \{ \lambda(K) \mid K \text{ compact}, K \subseteq A \} = \lambda(A).$$