A useful example of an application of dominated convergence.

Thm Let $(E, \mathcal{E}, \mu)$ be a measure space and $U \subseteq \mathbb{R}$ open and let $f: U \times E \to \mathbb{R}$ be a function with

$x \mapsto f(t,x)$ is measurable, integrable for every t

$t \mapsto f(t,x)$ is differentiable for every x.

Suppose $\exists$ an integrable function $g(x)$ s.t.

$$\left| \frac{\partial f(t,x)}{\partial t} \right| \leq g(x) \quad \forall\ t \in U$$

Then $x \mapsto \frac{\partial f}{\partial t}$ is integrable and measurable

and $F(t) = \int_E f(t,x)\,\mu(dx)$ is differentiable with

$$\frac{dF}{dt}(t) = \int_E \frac{\partial f}{\partial t}(t,x)\,\mu(dx).$$

Pf/ Let ε_n be an arbitrary null sequence

Let $$g_n(t,x) = \frac{f(t+\varepsilon_n, x) - f(t,x)}{\varepsilon_n} - \frac{\partial f}{\partial t}(t,x)$$

Then $g_n \to 0$ everywhere as f is differentiable

and $\frac{f(t+\varepsilon_n, x) - f(t,x)}{\varepsilon_n}$ is measurable

so $\frac{\partial f}{\partial t}$ is the limit of measurable functions so measurable

$$\frac{f(t+\varepsilon_n, x) - f(t,x)}{\varepsilon_n} = \frac{\partial f}{\partial t}(t+\theta_n, x) \quad \text{for some } \theta_n \in (t, t+\varepsilon_n)$$

$$|g_n(t,x)| = \left| \frac{f(t+\varepsilon_n, x) - f(t,x)}{\varepsilon_n} - \frac{\partial f}{\partial t}(t,x) \right| \leq 2g$$

so By dominated convergence $\int g_n(t,x)\mu(dx) \to 0$

so $$\frac{\int f(t+\varepsilon_n, x)\mu(dx) - \int f(t,x)\mu(dx)}{\varepsilon_n} \longrightarrow \int \frac{\partial f}{\partial t}(t,x)\mu(dx)$$

this is true for any null sequence ε_n so

if $F(t) = \int f(t,x)\mu(dx)$ then

$$\lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = \int \frac{\partial f}{\partial t}(t,x)\mu(dx).$$

<u>Example</u> Laplace transforms Let f be integrable $f: \mathbb{R} \to \mathbb{R}$ and non-negative then define

$$\ell(\lambda) = \int_{\mathbb{R}} e^{\lambda x} f(x)\,dx$$

And suppose that there exists $a > 0$ s.t. $\ell(\lambda) < \infty$ for $\lambda \in (-a, a)$

then $\int x f(x)\,dx < \infty$.

Pf/ Look at $h(\lambda, x) = e^{\lambda x} f(x)$

fix ε smaller than a then $\frac{\partial h}{\partial \lambda} = x e^{\lambda x} f(x)$

and there exists a C s.t. $x \leq C e^{\varepsilon/2 x}$

prove this by differentiating $x e^{-\varepsilon/2}$ and finding its maximum.

so $\frac{\partial h}{\partial \lambda} \leq C e^{\varepsilon/2 x} e^{(a-\varepsilon)x} f(x)$ for $\lambda \in (-a, a-\varepsilon)$

$\frac{\partial h}{\partial \lambda} \leq C e^{(a-\varepsilon/2)x} f(x)$

and $a - \varepsilon/2 < \lambda$ so $\int e^{(a-\varepsilon/2)x} f(x)\,dx < \infty$

We can apply our differentiation through the integral theorem with $U = (-a, a-\varepsilon)$ and

$g(x) = C e^{(a-\varepsilon/2)x} f(x)$

If we go back to the proof of the theorem we see that

$$\frac{h(\lambda+\varepsilon_n, x) - h(\lambda, x)}{\varepsilon_n} \leq g(x)$$ provided $\lambda, \lambda+\varepsilon_n \in (-a, a-\varepsilon)$

so $$\frac{\ell(\lambda+\varepsilon_n) - \ell(\lambda)}{\varepsilon_n} \leq \int g(x)\,\mu(dx) = C\,\ell(a-\tfrac{\varepsilon}{2})$$

$$\frac{d\ell}{d\lambda} \leq C\,\ell(a-\tfrac{\varepsilon}{2}) < \infty$$ for all $\lambda \in (-a, a-\varepsilon)$

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so in particular $\left.\frac{d\ell}{d\lambda}\right|_{\lambda=0} \leq C\ell(a-\frac{\varepsilon}{2}) < \infty$

$$\left.\frac{d\ell}{d\lambda}\right|_{\lambda=0} = \left.\int x e^{\lambda x} f(x)\,dx\right|_{\lambda=0} = \int x\, f(x)\,dx$$

therefore $\int x f(x)\,dx < \infty$ You can similarly prove that $\int x^n f(x)\,dx < \infty$.