Measure Theory: Exercises (not for credit)

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Question 1. Use the inequality $(x-y)^2 \geq 0$ to show that $xy \leq (x^2+y^2)/2$.

Question 2. Draw a graph of the function $t = s^{p-1}$. Let L be the are underneath the graph and above the s-axis and U be the area above the graph and to the right of the t-axis. Compute the areas of $L \cap \{s : 0 \leq s \leq x\}$ and $U \cap \{t : 0 \leq t \leq x\}$ and draw these two sets on your picture. Use this to show that $xy \leq x^p/p + y^q/q$.

Question 3. Let $(E, \mathcal{E}, \mu)$ be a finite measure space and let $p_1 \leq p_2$ show that if $f \in L^{p_2}$ then $f \in L^{p_1}$.

Question 4. Let X be a random variable. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p x^{p-1} \mathbb{P}(|X| > x) \mathrm{d}x.$$

You may switch the order of integration without justification (we'll do it next week). Hence show that if for all $q > p$ we have $\mathbb{P}(|X| > x) = O(x^{-q})$ as $x \to \infty$ then $X \in L^p$. (Recall that X is a measurable function from a probability space to $\mathbb{R}$, $\mathbb{P}$ is the measure on this space and $\mathbb{E}$ is the notation for integrating with respect to $\mathbb{P}$.