Up to this point in this week's discussion we have assumed that our measures were finite.

We did this because finiteness played an important role in d-systems.

There is a weaker assumption which allows us to define product measures.

Given $(X, \mathcal{A}, \mu)$ we say μ is σ-finite if there is a sequence of sets $A_1, A_2 \dots$ with $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$. (Cohn p. 9)

Depending on the circumstance it may be convenient to choose our sequence of sets to be nested (take $B_n = \bigcup_{i=1}^{n} A_i$) or disjoint (take $C_1 = B_1$, $C_n = B_n - B_{n-1}$)

Lebesgue measure on $\mathbb{R}^n$ is an example of a σ-finite measure.

We have proved the following result in the finite measure case:

Cor. 1.6.3 (Agreement on generators). Let $(\mathbb{X}, \mathcal{A})$ be a measurable space and let $\mathcal{C}$ be a π-system s.t. $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν that satisfy $\mu(\mathbb{X}) = \nu(\mathbb{X}) < \infty$ and $\mu(C) = \nu(C)$ for $C \in \mathcal{C}$ then $\mu = \nu$.

Here is a version that works in a σ-finite setting:

Cor. 1.6.4. Let $(X, \mathcal{A})$ be a measurable space, let $\mathcal{C}$ be a π-system on X s.t. $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν are measures on $(X, \mathcal{A})$ that agree on $\mathcal{C}$ and if there is an increasing sequence of sets in $\mathcal{C}$ with $\mu(C_n) = \nu(C_n) < \infty$ so that $\bigcup_n C_n = X$ then $\mu = \nu$.

Proof. Let us choose an increasing sequence $C_n \in \mathcal{C}$ with $\bigcup_n C_n = X$ $\mu(C_n) = \nu(C_n) < \infty$.

Define measures μ_n and ν_n by

$\mu_n(A) = \mu(A \cap C_n)$ and $\nu_n(A) = \nu(A \cap C_n)$.

(Note that $\mu_n(X) = \mu_n(C_n) = \nu_n(C_n) = \nu_n(X) < \infty$.)

By Cor. 1.6.3 for each n we have $\mu_n = \nu_n$.

Since

$$\mu(A) = \mu\Big(\bigcup_n (A \cap C_n)\Big) = \lim_{n\to\infty} \mu(A \cap C_n) = \lim_{n\to\infty} \mu_n(A)$$

$$\nu(A) = \nu\Big(\bigcup_n (A \cap C_n)\Big) = \lim_{n\to\infty} \nu(A \cap C_n) = \lim_{n\to\infty} \nu_n(A)$$

we have $\mu_n = \nu_n \Rightarrow \mu = \nu$.

We proved the following formula in the finite measure case:

Prop. 5.1.3(a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function and $y \mapsto \mu(E_y)$ is $\mathcal{B}$-measurable.

Here is a σ-finite version:

Prop. 5.1.3(b) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function and $y \mapsto \mu(E^y)$ is $\mathcal{B}$-measurable.

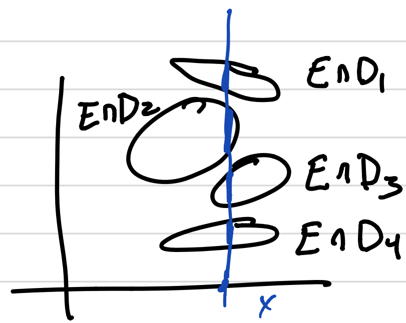
(Note that in this case the slice measure functions take values in $[0, +\infty]$.)

Proof. To show the measurability of $x \mapsto \nu(E_x)$ we use the σ-finiteness of ν. Let D_n be a sequence of disjoint subsets of Y with $\nu(D_n) < \infty$ and $\bigcup_n D_n = Y$.

Let $\nu_n(B) = \nu(B \cap D_n)$.

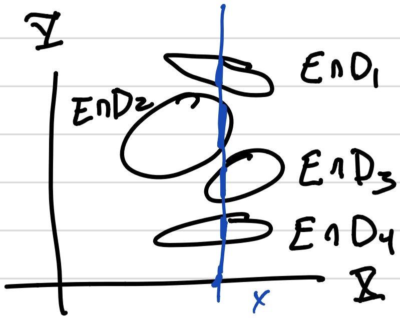
General remark:

$$\Big(E \cap \bigcup_n D_n\Big)_x = \bigcup_n (E \cap D_n)_x$$



$$\nu(E_x) = \nu\Big(\big(E \cap \bigcup_n D_n\big)_x\Big) = \nu\Big(\bigcup_n (E \cap D_n)_x\Big)$$

Disjoint subsets of Y.



$$= \sum_n \nu\big((E \cap D_n)_x\big)$$

$$= \sum_n \nu_n(E_x).$$

Since each function $x \mapsto \nu_n(E_x)$ is $\mathcal{A}$-measurable by 5.1.3(a), the function $x \mapsto \sum_n \nu_n(E_x)$ is $\mathcal{A}$-measurable.

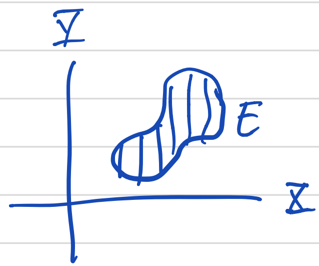
Thm. 5.1.4. Let $(\mathfrak{X}, \mathcal{A}, \mu)$ and $(\mathfrak{Y}, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then there is a unique measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$ such that

$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

for each measurable rectangle $A \times B$.

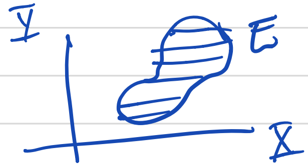
Furthermore for $E \in \mathcal{A} \times \mathcal{B}$

$$\mu \times \nu(E) = \int_{\mathfrak{X}} \nu(E_x)\, d\mu(x)$$



"vertical slice formula"

and $\mu \times \nu(E) = \int_Y \mu(E^y)\, d\nu(y).$



"horizontal slice formula"

(Note that I insert an x in $d\mu(x)$ to indicate the variable of integration. Cohn writes $\mu(dx)$.)

Def. $\mu \times \nu$ is the product of μ and ν.

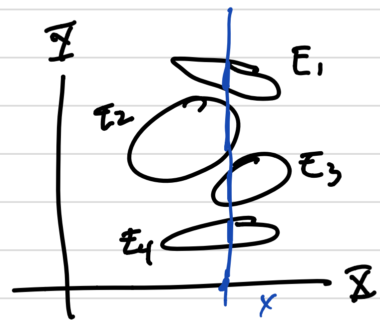
Proof. We use the vertical slice formula to define a function on sets E in $\mathcal{A} \times \mathcal{B}$.

$$\text{Define } (\mu \times \nu)_1 (E) = \int_X \nu(E_x)\, d\mu(x).$$

We claim that $(\mu \times \nu)_1$ is a measure. We have to check that it (1) takes the value 0 on the empty set and (2) is countably additive.

① $(\mu\times\nu)_1(\emptyset) = \int_X \nu(\emptyset)\, d\mu(x) = \int_X 0\, d\mu = 0.$

② Now let E_n be a sequence of disjoint sets in $\mathcal{A}\times\mathcal{B}$. Write $E = \bigcup_n E_n$.



Our earlier General Remark on slices gives: $\left(\bigcup_n E_n\right)_x = \bigcup_n (E_n)_x$.

For each $x \in X$ ν is a measure on Y and $(E_n)_x$ are disjoint subsets of Y so

$$\nu\left(\bigcup_n (E_n)_x\right) = \sum_n \nu((E_n)_x). \qquad (*)$$

Putting these observations together with the Beppo-Levy limit theorem we have:

$$(\mu\times\nu)_1(E) = \int \nu\Big(\big(\bigcup_n E_n\big)_x\Big)\, d\mu \qquad \text{Def. of } (\mu\times\nu)_1$$

$$= \int \nu\Big(\bigcup_n (E_n)_x\Big)\, d\mu \qquad \text{General remark}$$

$$= \int \sum_n \nu\big((E_n)_x\big)\, d\mu \qquad \begin{array}{l}\nu \text{ is a measure}\\ (E_n)_x \text{ disjoint, meas.}\end{array}$$

$$= \sum_n \int \nu\big((E_n)_x\big)\, d\mu \qquad \text{Beppo-Levy 2.4.2}$$

$$= \sum_n (\mu\times\nu)_1(E_n). \qquad \text{Def. of } (\mu\times\nu)_1$$

This shows countable additivity.

Thus $(\mu\times\nu)_1$ is a measure on $\mathcal{A}\times\mathcal{B}$.
A similar argument shows that $(\mu\times\nu)_2$ is a measure on $\mathcal{A}\times\mathcal{B}$.

Let us check the values of these measures on measurable rectangles:

$$(\mu\times\nu)_1(A\times B) = \int_X \nu(B)\cdot \chi_A \, d\mu(x)$$

$$= \nu(B)\cdot\mu(A).$$

$$(\mu \times \nu)_2 (A \times B) = \int_Y \mu(A) \cdot \chi_B \, d\nu(y)$$

$$= \mu(A) \cdot \nu(B).$$

We want to apply Cor. 1.6.4 to conclude that the two measures agree.

We construct sets C_n as in the Corollary.

Let A_n be a nested sequence with $\bigcup A_n = X$. $\mu(A_n) < \infty$

Let B_n be a nested sequence with $\bigcup B_n = Y$. $\nu(B_n) < \infty$

Let $C_n = A_n \times B_n$. $\bigcup_n C_n = X \times Y$.

$$(\mu \times \nu)_1 (C_n) = (\mu \times \nu)_2 (C_n) = \mu(A_n) \cdot \nu(B_n) < \infty.$$

Thus $(\mu \times \nu)_1 = (\mu \times \nu)_2$.

Let's drop the subscript and write $\mu \times \nu$.

Furthermore we can apply Cor. 1.6.4 to any measure agreeing with $\mu \times \nu$ on meas. rectangles to see that it is in fact $\mu \times \nu$. This establishes uniqueness.