Measure Theory: Exercises (not for credit)

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Question 1. Suppose that $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Show that the set $\mathcal{A} \subseteq \mathcal{E} \times \mathcal{F}$ with $\mathcal{A} = \{A \times B \,:\, A \in \mathcal{E}, B \in \mathcal{F}\}$ is a π-system.

Answer: We know that $\emptyset = \emptyset \times \emptyset$ so $\emptyset \in \mathcal{A}$. If we suppose that $C_1 = A_1 \times B_1$ and $C_2 = A_2 \times B_2$ then $C_1 \cap C_2 = (A_1 \cap A_2) \times (B_1 \cap B_2)$ (we can check this more precisely and you probably should in an exam). Therefore $\mathcal{A}$ is a π-system. $\square$

Question 2. Suppose that $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Let $\mathcal{A}_1 \subseteq \mathcal{E}$ and $\mathcal{A}_2 \subseteq \mathcal{F}$ be such that $\sigma(\mathcal{A}_1) = \mathcal{E}$ an $\sigma(\mathcal{A}_2) = \mathcal{F}$. Show that $\mathcal{E} \times \mathcal{F} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

Answer: Let us define $\mathcal{C} = \{A \times B \,:\, A \in \mathcal{A}_1, B \in \mathcal{F}\}$, then let us first show that $\mathcal{C} \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$. Let us fix $A \in \mathcal{A}_1$ and define $\tilde{C}_A = \{B \,:\, A \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ then we know that $\mathcal{A}_2 \in \tilde{C}_A$ and we can show that $\tilde{C}_A$ is a σ-algebra as it inherits the properties of $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$. Therefore $\tilde{C}_A \supseteq \sigma(\mathcal{A}_2)$ so $\tilde{C}_A = \mathcal{F}$. Therefore $\mathcal{C} \subseteq \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$. Now fix $B \in \mathcal{F}$ and define $\bar{C}_B = \{A \,:\, A \times B \in \sigma(\mathcal{A}_1 \times \mathcal{A}_2)\}$. Now since we know that $\mathcal{C} \subseteq \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ this means that $\mathcal{A}_1 \subseteq \bar{C}_B$ for each B. Then again $\bar{C}_B$ inherits the σ-algebra structure from $\sigma(\mathcal{A}_1 \times \mathcal{A}_2)$ and hence is a σ-algebra so $\bar{C}_B = \mathcal{E}$. Hence $\sigma(\mathcal{A}_1 \times \mathcal{A}_2) = \mathcal{E} \times \mathcal{F}$. $\square$

Question 3. Let $\mathcal{M}_1$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{M}_2$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^2$. Show that $\mathcal{M}_2 \neq \mathcal{M}_1 \times \mathcal{M}_1$.

Answer: Let A be a non-lebesgue measurable subset of $\mathbb{R}$. Then let $B = \{x\} \times A$, this is contained in a line in $\mathbb{R}^2$ so is a null set therefore $B \in \mathcal{M}_2$ however $B_x = A \notin \mathcal{M}_1$. If $\mathcal{M}_2$ was equal to $\mathcal{M}_1 \times \mathcal{M}_1$ then every set C in $\mathcal{M}_2$ would have $C_x \in \mathcal{M}_1$ for every x. Therefore $\mathcal{M}_2 \neq \mathcal{M}_1 \times \mathcal{M}_1$. $\square$

Question 4. Let μ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (the measure that counts how many elements there are in a set) and let λ be Lebesgue measure on $\mathbb{R}$. Let f be the indicator function of the set $\{(x, x) \,:\, x \in \mathbb{R}\}$. Show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \mu(\mathrm{d}x) \lambda(\mathrm{d}y) \neq \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \lambda(\mathrm{d}x) \mu(\mathrm{d}x).$$

What part of the conditions of Fubini-Tonelli theorem doesn't hold to mean this can happen?

Answer: In the displayed equation $LHS = \infty$ and $RHS = 0$. This is possible because $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is not σ-finite. $\square$

Question 5. Let $f(x,y) = 1_{x\geq 0}(1_{y\in[x,x+1)} - 1_{y\in[x+1,x+2)})$ Show that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\mathrm{d}x\mathrm{d}y \neq \int_{\mathbb{R}} f(x,y)\mathrm{d}y\mathrm{d}x.$$

What part of the conditions of Fubini-Tonelli theorem doesn't hold to allow this to happen?

Answer: We can compute with a bit of work and get $LHS = 1$ and $RHS = 0$. This is possible because $|f(x,y)|$ is not integrable. $\square$

Question 6. Let A be a bounded Borel subset of $\mathbb{R}$ with $\lambda(A) > 0$ show that the function $x \mapsto \lambda(A\cap(x+A))$ is continuous and is non-zero on some open interval containing 0. Define $diff(A) = \{z : z = x - y, x \in A, y \in A\}$ show that if A is a Borel subset of $\mathbb{R}$ with non-zero measure then $diff(A)$ contains some open interval around 0.

Answer: Let $f(x) = 1_A(x), g(x) = 1_A(-x)$ then $f*g = \int_{\mathbb{R}} 1_A(y)1_A(-x+y)\mathrm{d}y = \int_{\mathbb{R}} 1_A(y)1_{x+A}(y)\mathrm{d}y = \int_{\mathbb{R}} 1_{A\cap(x+A)}(y)\mathrm{d}y = \lambda(A\cap(x+A))$. Now for any p, q we have that $f \in L^p, g \in L^q$ so by a result in lectures $f * g$ is continous. We also have $f * g(0) = \lambda(A) > 0$. So there must be some open interval U containing 0 such that if $x \in U$ then $\lambda(A\cap(x+A)) > 0$. Then for each $x \in U$ there exists $y, z \in A$ such that $z = y + x$. Therefore $x \in diff(A)$ for every $x \in U$. $\square$