Mikowski's Inequality - Triangle inequality for the L^p norm

Theorem

Let $(E, \mathcal{E}, \mu)$ be a measure space and

$f, g \in L^p(E, \mathcal{E}, \mu)$ $\quad \int |f|^p d\mu < \infty$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Pf/ First when $p = \infty$

$$|f(x)+g(x)| \leq |f(x)| + |g(x)|$$

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$ as supremum of a sum $\leq$ sum of sups

$p \in [1, \infty)$ choose q $\quad \frac{1}{p} + \frac{1}{q} = 1$ $\quad \frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$ so $q = \frac{p}{p-1}$

Then $|f+g|^{p-1} \in L^q$

we know $|f+g| \in L^p$ $\quad \mu(|f+g|^{q(p-1)}) = \mu(|f+g|^{\frac{p}{p-1}(p-1)})$

$$= \mu(|f+g|^p) < \infty$$

$$\| |f+g|^{p-1} \|_q = \mu(|f+g|^p)^{1/q}$$

$$= \|f+g\|_p^{p/q} = \|f+g\|_p^{p-1}$$

$$\|f+g\|_p^p = \mu(|f+g|^p) = \mu(|f+g| |f+g|^{p-1})$$

$$\le \mu\big(\underbrace{|f|}_{\in L^p}\,\underbrace{|f+g|^{p-1}}_{L^q}\big) + \mu\big(\underbrace{|g|}_{L^p}\,\underbrace{|f+g|^{p-1}}_{L^q}\big)$$

$$\overset{\text{Hölder}}{\le} \|f\|_p \, \| |f+g|^{p-1} \|_q + \|g\|_p \, \| |f+g|^{p-1} \|_q$$

$$= (\|f\|_p + \|g\|_p) \, \|f+g\|_p^{p-1}$$

So we've proved

$$\|f+g\|_p^p \le (\|f\|_p + \|g\|_p) \underbrace{\|f+g\|_p^{p-1}}_{\text{divide by this}}$$

$$\|f+g\|_p \le \|f\|_p + \|g\|_p \quad \checkmark$$