Measure Theory: Exercises (not for credit)

Josephine Evans

October 8, 2021

Question 1. Let C be a countable subset of $\mathbb{R}$. Show that $\lambda^*(C) = 0$.

Question 2. For each set $A \in \mathbb{R}$ show that there is a Borel subset, B, of $\mathbb{R}$ with $A \subseteq B$ such that $\lambda(B) = \lambda^*(A)$.

Question 3. Let B be a Borel subset of $[0,1]$ show that there exists a finite, disjoint union of half open intervals $A = I_1 \cup \cdots \cup I_n$ such that $\lambda(A \triangle B) \leq \epsilon$. Here $A \triangle B = (A^c \cap B) \cup (A \cap B^c)$.

Question 4. Let $(E, \mathcal{E}, \mu)$ be a finite measure space and let A_n be a sequence of measurable sets. Show that

$$\mu\left(\bigcup_n \bigcap_{m \geq n} A_m\right) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu\left(\bigcap_n \bigcup_{m \geq n} A_m\right).$$

Find an example to show that the last inequality is not necessarily true if μ is not finite.