Define the set $\mathcal{M} \to$ the Lebesgue measurable sets to be all subsets, A, of $\mathbb{R}$ satisfying

$\forall\, B \subseteq \mathbb{R}$

$$\lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(A^c \cap B)$$

where λ^* is Leb outer measure defined in Lecture 3

Prop (1.6) $\mathcal{M}$ is a σ-algebra

Pf/ First show $\mathcal{M}$ is an algebra

The defn of Leb meas is symmetric in A and A^c

$A \in \mathcal{M}$ iff $A^c \in \mathcal{M}$

$\lambda^*(\emptyset) = 0$ and $B \cap \emptyset = \emptyset$, $B \cap \mathbb{R} = B$ for every $B \subseteq \mathbb{R}$

so $$\lambda^*(B) = \lambda^*(\emptyset) + \lambda^*(B) = \lambda^*(\emptyset \cap B) + \lambda^*(B \cap \mathbb{R})$$

so $\emptyset, \mathbb{R} \in \mathcal{M}$

Now we show $A_1, A_2 \in \mathcal{M} \Rightarrow A_1 \cup A_2 \in \mathcal{M}$

$$\lambda^*(B \cap (A_1 \cup A_2)) \overset{\text{use } A_1 \in \mathcal{M}}{=} \lambda^*(B \cap (A_1 \cup A_2) \cap A_1) + \lambda^*(B \cap (A_1 \cup A_2) \cap A_1^c)$$

$(A_1 \cup A_2) \cap A_1 = A_1$ $\qquad$ $(A_1 \cup A_2) \cap A_1^c = A_2 \cap A_1^c$

$$\lambda^*(B \cap (A_1 \cup A_2)) = \lambda^*(B \cap A_1) + \lambda^*(B \cap A_2 \cap A_1^c)$$

Now we want to look at $\lambda^*(B \cap (A_1 \cup A_2)^c) = \lambda^*(B \cap A_1^c \cap A_2^c)$

$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$$\lambda^*(B \cap (A_1 \cup A_2)) + \lambda^*(B \cap (A_1 \cup A_2)^c) = \lambda^*(B \cap A_1) + \lambda^*(B \cap A_2 \cap A_1^c) + \lambda^*(B \cap A_1^c \cap A_2^c)$$

$$= \lambda^*(B \cap A_1^c)$$

$= \lambda^*(B\cap A_i^c)$

use the fact that $A_2 \in \mathcal{M}$

use identit w/ $B = B\cap A_i^c$

$$\lambda^*(B\cap(A_1\cup A_2)) + \lambda^*(B\cap(A_1\cup A_2)^c) = \lambda^*(B\cap A_1) + \lambda^*(B\cap A_1^c)$$

using $A_1 \in \mathcal{M}$ $\quad = \lambda^*(B)$

this shows that $A_1\cup A_2 \in \mathcal{M}$.

Now we want to show this for countable unions

take $C_1, C_2, C_3 \ldots \in \mathcal{M}$ we can turn it into a sequence of disjoint sets in $\mathcal{M}$ with finite set operations

$A_1 = C_1 \quad A_2 = C_2 \setminus (C_1\cap C_2), \quad A_3 = C_3 \setminus ((C_1\cup C_2)\cap C_3) \ldots$

so the $A_i \in \mathcal{M}$ for every i and are pairwise disjoint and $\bigcup_n A_n = \bigcup_n C_n$

want to show $\bigcup_n A_n \in \mathcal{M}$

1. $\lambda^*(B) \leq \lambda^*(B\cap \bigcup_n A_n) + \lambda^*(B\cap(\bigcup_n A_n)^c)$

and

2. $\lambda^*(B) \geq \lambda^*(B\cap \bigcup_n A_n) + \lambda^*(B\cap(\bigcup_n A_n)^c)$

1. follows from countable subadditivity of λ^*

2. Is the hard one

We can show by induction that

$$\lambda^*(B) = \sum_{i=1}^{n} \lambda^*(B\cap A_i) + \lambda^*(B\cap(\bigcap_{i=1}^{n} A_i^c))$$

basically this follows from the fact that

$\bigcup_{i=1}^{n} A_i$ is in $\mathcal{M}$

clear in the case $n=1$ (just the fact $A_1 \in \mathcal{M}$)

Induction step:

$$\lambda^*(B) = \sum_{i=1}^{n-1} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right)$$

$$\lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right) = \lambda^*\left(B \cap \underbrace{A_n \cap \bigcap_{i=1}^{n-1} A_i^c}\right) + \lambda^*\left(B \cap A_n^c \cap \bigcap_{i=1}^{n-1} A_i^c\right)$$

$A_n \in \mathcal{M}$

as A_i are disjoint

$A_n \subseteq \bigcap_{i=1}^{n-1} A_i^c$ so

$$\lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right) = \lambda^*(B \cap A_n) + \lambda^*\left(B \cap \bigcap_{i=1}^{n} A_n^c\right)$$

---

since $\bigcap_{i=1}^{n} A_i^c \supseteq \bigcap_{i=1}^{\infty} A_i^c$ and λ^* is monotone

$$\lambda^*(B) \geq \sum_{i=1}^{n} \lambda^*(A_i \cap B) + \lambda^*\left(B \cap \bigcap_{i=1}^{\infty} A_i^c\right)$$

let $n \to \infty$ to get

$$\lambda^*(B) \geq \sum_{i=1}^{\infty} \lambda^*(A_i \cap B) + \lambda^*\left(B \cap \bigcap_{i=1}^{\infty} A_i^c\right)$$

$$\geq \lambda^*\left(\bigcup_n (A_n \cap B)\right) + \lambda^*\left(B \cap \bigcap_{i=1}^{\infty} A_i^c\right)$$

using countable subadditivity

$$= \lambda^*\left(B \cap \bigcup_n A_n\right) + \lambda^*\left(B \cap \left(\bigcup_n A_n\right)^c\right)$$

$\Rightarrow \bigcup_n A_n \in \mathcal{M}$

Prop 1.7 The restriction of λ^* to $\mathcal{M}$ is a measure

~~Pf~~ We just need to show countable additivity.

If $A_1, A_2, \ldots$ is a sequence of disjoint set in $\mathcal{M}$

then $\lambda^*\left(\bigcup_n A_n\right) = \sum_n \lambda^*(A_n)$

By countable subadditivity $\lambda^*\left(\bigcup_n A_n\right) \leq \sum_n \lambda^*(A_n)$

In our last proof we showed

$$\lambda^*(B) \geq \sum_{i=1}^{\infty} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right)$$

for any $B \subseteq \mathbb{R}$

use this with $B = \bigcup_{i=1}^{\infty} A_i$ to get

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \lambda^*\left(\bigcup_{i=1}^{\infty} A_i \cap A_i\right) + \lambda^*\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right)$$

$$= \sum_{i=1}^{\infty} \lambda^*(A_i) + \lambda^*(\emptyset)$$

$$= \sum_{i=1}^{\infty} \lambda^*(A_i)$$

so we know $\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i)$