In this short note I want to expand on comments I made in the first lecture relating to pointwise limits and Riemann integrability.

Uniform limits of Riemann integrable functions are Riemann integrable.

A uniform limit is a pointwise limit but a pointwise limit need not be a uniform limit.

I will give an example of a sequence of Riemann integrable functions f_n which converge pointwise to a function f which is not Riemann integrable.

To make this more concrete let me give a sequence of functions that converges pointwise but not uniformly.

This example is also discussed in Cohn's book.

The collection of rational numbers in the unit interval is countable.

We can list them:

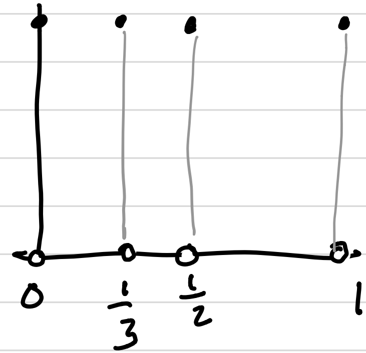
$$r_1 = 0, \quad r_2 = 1, \quad r_3 = \frac{1}{2},$$

$$r_4 = \frac{1}{3}, \quad r_5 = \frac{2}{3}, \quad r_6 = \frac{1}{4} \dots$$

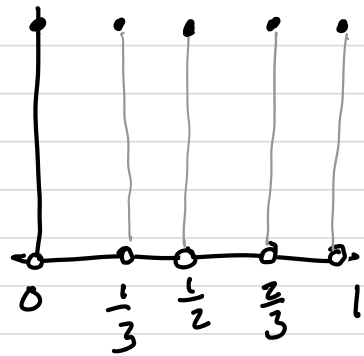
Define

$$f_N(x) = \begin{cases} 1 & \text{if } x \in \{r_1 \dots r_N\} \\ 0 & \text{otherwise} \end{cases}$$

f_4



f_5



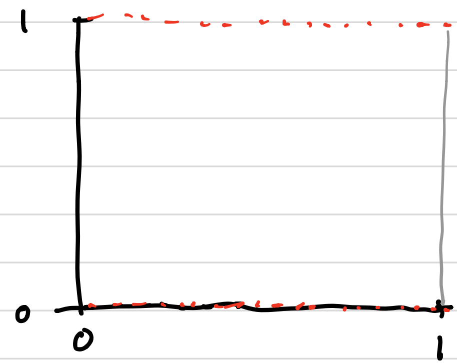
f_N is Riemann integrable since it has a finite set of discontinuities and

$\int f_N \, dx = 0$.

Define

$$f_\infty(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Graph of f_∞:



(the Dirichlet function)

f_N converges pointwise to f_∞:

If x is irrational then $f_N(x) = 0 = f_\infty(x)$.

If x is rational then $x = r_m$ for some m so $f_N(x) = 0$ for $N < m$ and $f_N(x) = 1$ for $N \geq m$. In particular $\lim_{N\to\infty} f_N(x) = 1 = f_\infty(x)$.

Convergence is not uniform.

f_∞ is a pointwise limit of Riemann integrable functions it is however not Riemann integrable.

Recall that Riemann integrability of a function f on $[a,b]$ is defined in terms of

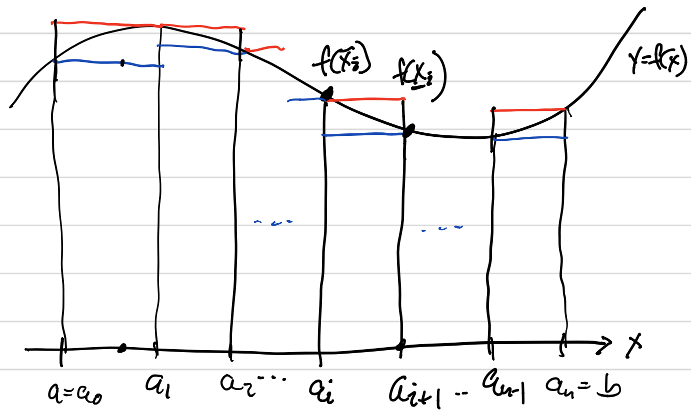
(1) a partition of the domain

$$P_j = (a = a_0 < a_1 < a_2 \dots < a_n = b)$$

(2) upper sums:

$$U(P_j) = \sum_i f(\bar{x}_i)(a_{i+1} - a_i)$$

and



(3) lower sums:

$$L(P_j) = \sum_i f(\underline{x}_i)(a_{i+1} - a_i).$$

The Riemann integral $\int_a^b f\,dx$ exists if

$$\lim_j L(P_j) = \lim_j U(P_j).$$

Let's apply this criterion to f_{∞}. Fix a partition P.

To evaluate the lower sums

$$L(P_j) = \sum_i f_{\infty}(\underline{x}_i)(a_{i+1} - a_i)$$

we choose $\underline{x}_i$ where f achieves its minimum.

Take $\underline{x}_i$ irrational in $[a_{i+1} - a_i]$.

$f_{\infty}(\underline{x}_i) = 0$. $L(P_j) = 0$.

(Every interval contains an irrational point.)

To evaluate the upper sums

$$U(P_j) = \sum_i f_{\infty}(\overline{x}_i)(a_{i+1} - a_i)$$

we choose $\underline{x}_i$ where f achieves its maximum.

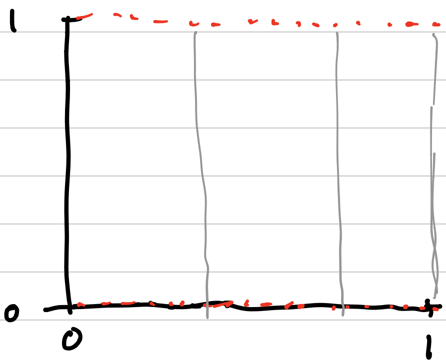
Take $\underline{x}_i$ rational in $[a_{i+1} - a_i]$.

$f_{\infty}(\underline{x}_i) = 1$. $U(P_j) = 1$.

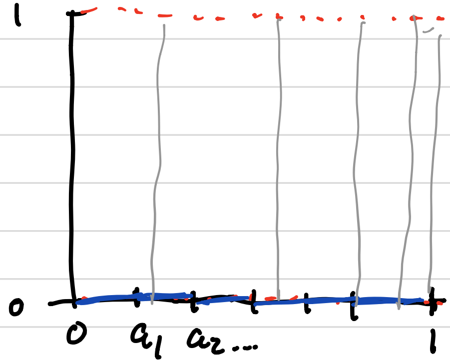
(Every interval contains a rational point.)

$$0 = L(P_j) \neq U(P_j) = 1.$$

$f(x)$

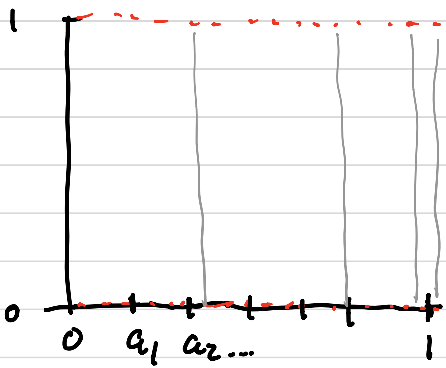


Lower sum = 0

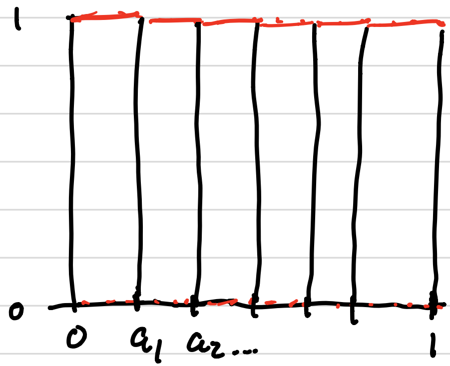


"Rectangles" of height 0

Choose a partition:



Upper sum = 1



Rectangles of height 1

Since $0 \neq 1$ $f(x)$ is <u>not</u> Riemann integrable.