07 December 2021 09:02

Good morning! Will start at 9.05

Remind me to record!

Final feedback survey is open! on moodle

Signed measures

We will show that we can write as the difference of two normal measures.

Defn Finite signed measure

$(E, \mathcal{E})$ a measurable space $\nu : \mathcal{E} \to \mathbb{R}$ is a finite signed measure if

- $\nu(\emptyset) = 0$
- If $(A_n)_{n \geq 1}$ is a collection of disjoint set $\nu(\bigcup_n A_n) = \sum_n \nu(A_n)$

It satisfies the axioms of a measure except positivity

Note signed measures are not necessarily monotone and countably subadditive.

Example If $(E, \mathcal{E}, \mu)$ is a measure space and $f \in L^1(E)$ not necessarily non-negative then $\nu(A) = \mu(f \mathbb{1}_A)$ defines a signed measure.

We wrote f a function as $f = f_+ - f_-$

supported where f is +ve

supported where f is -ve.

we want to do something similar with a signed measure $\nu = \nu_+ - \nu_-$

Defn If $(E, \mathcal{E})$ is a measurable space, ν a finite signed measure. Then we call A a negative set for ν if for every $B \subseteq A$ $\nu(B) \leq 0$.

We define a positive set analagously.

Lemma Suppose $(E, \mathcal{E})$ measure space, ν finite signed measure and $A \in \mathcal{E}$ with $\nu(A) < 0$ then there exists a negative set B with $B \in A$ and $\nu(B) \leq \nu(A)$.

Pf/ We construct this set itteratively by removing sets of positive measure (under ν) from A until we can't any more.

Let $\delta_1 = \sup\{\nu(C) : C \subseteq A\}$

then as $\emptyset \subseteq A$ $\delta_1 \geq \nu(\emptyset) = 0$.

If $\delta_1 \neq 0$ then we can define C_1 to be a set with $\nu(C_1) \geq \min\{\delta_1/2, 1\}$

we just take min incase $\delta_1 = \infty$.

Then we define δ_n, C_n sequences itteratively by

$$\delta_n = \sup\{\nu(C) : C \subseteq A \setminus \bigcup_{k=1}^{n-1} C_k\}$$

C_n a set s.t. $\nu(C_n) \geq \min\{\delta_n/2, 1\}$

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Now, $C_\infty = \bigcup_n C_n$ and $B = A \setminus C_\infty$

then we want to show B has the required properties

$$\nu(A) = \nu(B) + \nu(C_\infty) \geq \nu(B)$$

as C_∞ is a disjoint union of sets of positive measure

$\nu(C_\infty) = \sum_n \nu(C_n) \geq 0.$

Then as ν is finite we must have $\nu(C_\infty) < \infty$

so $\sum_n \nu(C_n) < \infty$ and $\nu(C_n) > 0$ for each n

so $\nu(C_n) \to 0$ as $n \to \infty$

as $\nu(C_n) \geq \min\{\frac{\delta_n}{2}, 1\}$ we must have

$\delta_n \to 0$ as $n \to \infty$.

So if $D \subseteq B$ then we must have $\nu(D) \leq \delta_n$ for each n (as $D \subseteq A \setminus \bigcup_{k=1}^{n-1} C_k$ and δ_n is defined by $\sup\{\nu(C) : C \subseteq A \setminus \bigcup_{k=1}^{n-1} C_k\}$)

so as $\delta_n \to 0$ we have $\nu(D) \leq 0$.

Therefore B is a negative set.

<u>Theorem</u> Hahn Decomposition

Let $(E, \mathcal{E})$ be a measurable space, ν a finite signed measure. Then there exists a decomposition of E, (P, N) where $E = P \cup N$ $P \cap N = \emptyset$ and

ν

of E, (P,N) where $E = P \cup N$ $P \cap N = \phi$ and P is a positive set for ν, N is a negative set for ν.

Remark: This stands in for our "places where ν is positive" and "places where ν is negative".

Proof Let $L = \inf \{ \nu(A) : A \text{ is a negative set for } \nu \}$

claim.
$L > -\infty$

If not $\exists A_n$ with $\nu(A_n) \leq -n$ A_n negative then let $A_\infty = \bigcup_n A_n$ this will still be a negative set (prove in a minute) and so $\nu(\bigcup_n A_n) = -\infty$ which is a contradiction to ν being finite.

Claim: if A is negative and $B \subseteq A$ then B is negative (if $C \subseteq B$ then $C \subseteq A$ so $\nu(C) \leq 0$)

If $(A_n)_{n \geq 1}$ is a sequence of negative sets then let $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ then $B_n \subseteq A_n$ so $(B_n)_{n \geq 1}$ is a disjoint sequence of negative sets and $\bigcup_n A_n = \bigcup_n B_n$

We also have $\bigcup_n A_n$ is a negative set as

If $C \subseteq \bigcup_n A_n$ then $C = \bigcup_n (C \cap B_n)$ $(C \cap B_n)_{n \geq 1}$ is a disjoint sequence of sets $C \cap B_n \subseteq A_n$ so $\nu(C \cap B_n) \leq 0$

and $\nu(C) = \sum_n \nu(C \cap B_n) \leq 0$.

and $\quad \nu(\bigcup_n A_n) \leq \nu(A_n)$ for any n

as
$$\begin{aligned}\nu(\bigcup_n A_n) &= \nu\left(A_n \cup \left(\bigcup_n A_n \setminus A_n\right)\right) \\ &= \nu(A_n) + \nu\left(\bigcup_n A_n \setminus A_n\right) \\ &\leq \nu(A_n).\end{aligned}$$

Now we introduce a sequence A_n

$\nu(A_n) \leqslant L + \frac{1}{n}$, A_n is negative

then let $N = \bigcup_n A_n$. Then N is the union of negative sets, so negative and $\nu(N) \leqslant \nu(A_n) \leqslant L + \frac{1}{n}$ $\forall n$

so $\nu(N) \leq L$ and $L = \inf\{\nu(A) : A \text{ negative}\}$

so $\nu(N) = L$.

Now lets take $P = N^c$ we want to show P is a positive set. Argue by contradiction: suppose $A \subseteq P$ and $\nu(A) < 0$ then by our lemma $\exists$ $B \subseteq A$, B negative and $\nu(B) \leq \nu(A)$. Then $B \cap N = \emptyset$ as $B \subseteq P = N^c$ so $B \cup N$ is a negative set an $\nu(B \cup N) = \nu(B) + \nu(N) < \nu(N) = L$ this the contradicts the definition of L. So no such A can exist so P is a positive set for ν.

This decomposition is in general not unique, its often

This decomposition is in general not unique, its often unique "up to null sets"

<u>Theorem</u> Jordan decomposition theorem

Let $(E, \mathcal{E})$ be a measurable space, ν a finite signed measure then $\exists$ measure ν_+ and ν_- s.t. $\forall$ $A \in \mathcal{E}$ $\nu(A) = \nu_+(A) - \nu_-(A)$.

~~Pf~~ Take some Hahn-decomposition (P, N) then

define $\nu_+(A) = \nu(A \cap P)$ and $\nu_-(A) = -\nu(A \cap N)$

then as $A = (A \cap P) \cup (A \cap N)$ this is a disjoint union

$\nu(A) = \nu(A \cap P) + (A \cap N) = \nu_+(A) - \nu_-(A)$.

$\nu_+(\emptyset) = \nu(\emptyset \cap P) = \nu(\emptyset) = 0$ similarly for $\nu_-(\emptyset)$.

and if $(A_n)_{n \geq 1}$ is a disjoint sequence then

$$\nu_+\left(\bigcup_n A_n\right) = \nu\left(\left(\bigcup_n A_n\right) \cap P\right) = \nu\left(\bigcup_n (A_n \cap P)\right) = \sum_n \nu(A_n \cap P)$$
$$= \sum_n \nu_+(A_n)$$

similarly for ν_-. So ν_+ and ν_- are indeed measures $\nu_+(A) \geq 0$ as $A \cap P \subseteq P$ and positive set.

So we have our decomposition.

<u>Remark</u>: If ν_+ and ν_- are constructed this way then we have $\nu_+(A) = \sup\{\nu(B) : B \subseteq A, B \in \mathcal{E}\}$

then we have $\nu_+(A) = \sup\{\nu(B) : B \subseteq A, B \in \Sigma\}$

so $\nu_+(A) = \nu(A \cap P) \leq \sup\{\nu(B) : B \subseteq A, B \in \Sigma\}$

and if there exists $B \subseteq A$ with

$\nu(B) > \nu(A \cap P)$ then

$$
\begin{aligned}
\nu(B) &= \nu(B \cap P) + \nu(B \cap N) \\
&\leq \nu(B \cap P) \\
&= \nu_+(B) \leq \nu_+(A)
\end{aligned}
$$
by monotonicity of ν_+

which is a contradiction.

So the values of ν_+ and ν_- don't depend on the particular Hahn decomposition used.