Remind me to record! Will start at 9.05

Next week Breather week

* No new notes
* Lectures will be recap + questions
  ↳ Egoroff & Lusin's theorem
  Modes of convergence
  Construction of Lebesgue
* One video which is on Riesz represetation theorem
  – non-examinable.
* List of other interesting things to read
* 3rd Assignment is out today 18th November
  Thursday week 7. at 12

Integration using our final definition

Def^n We say a real valued measurable function f is integrable if $\mu(|f|) < \infty$

Recall $f = f_+ - f_-$ where $f_+ = \max\{f, 0\}$

$\mu(|f|) < \infty$ iff $\mu(f_+) < \infty$ and $\mu(f_-) < \infty$

Def^n If f is integrable then we define
$\mu(f) = \mu(f_+) - \mu(f_-)$

Lemma If f_1, f_2, g_1, g_2 are all non-negative integrable functions s.t. $f_1 - f_2 = g_1 - g_2$

then $\mu(f_1) - \mu(f_2) = \mu(g_1) - \mu(g_2)$

Pf/ $f_1 + g_2 = g_1 + f_2$

so $\mu(f_1 + g_2) = \mu(g_1 + f_2)$ linearity for integral of +ve functions

$\mu(f_1) + \mu(g_2) = \mu(g_1) + \mu(f_2)$ since all the func^n are integrable

$\mu(f_1), \mu(f_2), \mu(g_1), \mu(g_2) < \infty$

rearrange $\mu(f_1) - \mu(f_2) = \mu(g_1) - \mu(g_2)$

Remark If f is integrable and $f = g - h$

where g, h are both non-negative and integrable

then $\mu(f) = \mu(g) - \mu(h)$

Prop^n Suppose f, g are integrable real-valued

functions on $(E, \mathcal{E}, \mu)$ then:

- for all $\alpha > 0$ $\mu(\alpha f) = \alpha \mu(f)$ and $\mu(-f) = -\mu(f)$

- $f + g$ is also integrable and $\mu(f+g) = \mu(f) + \mu(g)$

- If $f \leq g$ then $\mu(f) \leq \mu(g)$.

Pf/ Let us write $f = f_+ - f_-$ and $g = g_+ - g_-$

- $\alpha f = \alpha f_+ - \alpha f_-$ so $\mu(\alpha f) = \mu(\alpha f_+) - \mu(\alpha f_-)$

$$= \alpha\mu(f_+) - \alpha\mu(f_-) = \alpha\mu(f)$$

$$\mu(-f) = \mu(f_- - f_+) = \mu(f_-) - \mu(f_+) = -\mu(f).$$

- $|f+g| \leq |f| + |g|$ so $\mu(|f+g|) \leq \underbrace{\mu(|f|) + \mu(|g|)}_{<\infty}$

so $f+g$ is integrable

and $f+g = (f_+ + g_+) - (f_- + g_-)$

$f_+ + g_+ \neq (f+g)_+$ always so here we really need the lemma

$\mu(f+g) = \mu(f_+ + g_+) - \mu(f_- + g_-)$ using linearity for the func^n

$$= \mu(f_+) - \mu(f_-) + \mu(g_+) - \mu(g_-)$$

$$= \mu(f) + \mu(g)$$

- If $f \leq g$ then $g - f \geq 0$

using last •

$$\mu(g) = \mu(f + (g-f)) = \mu(f) + \underbrace{\mu(g-f)}_{\geq 0}$$

$$\geq \mu(f).$$

□

Back to convergence theorem's

key step proving dominated conv.

<u>Fatou's Lemma</u> Let $(f_n)_{n\geq 1}$ be a sequence of

non-negative measurable functions then

$$\mu\left(\liminf_n f_n\right) \leq \liminf_n \mu(f_n)$$

$$\boxed{\liminf_n f_n = \lim_{n\to\infty} \inf_{k\geq n} f_k}$$

<u>Remark / Classic example</u>

take $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $f_n = \mathbb{1}_{[n,n+1]}$

then $\lambda(f_n) = 1$ for every n but $f_n \to 0$ everywhere

so $\liminf_n f_n = 0$

$$\underset{\overset{\|}{0}}{\mu\left(\liminf_n f_n\right)} \leq \liminf_n \underset{\overset{\|}{1}}{\mu(f_n)}$$

~~Pf~~ We'll use monotone convergence

Let $g_n = \inf_{k\geq n} f_k$ then g_n is a non-decreasing sequence of non-negative measurable functions.

Using monotone convergence

$$\mu\left(\lim_{n\to\infty} g_n\right) = \lim_{n\to\infty} \mu(g_n)$$

$$= \mu\left(\liminf_n f_n\right)$$

We also have that $g_n \leq f_n$ for each n

using this

so $\mu(g_n) \leq \mu(f_n)$

$\lim \mu(g_n) = \liminf \mu(g_n) \leq \liminf \mu(f_n)$

so $\quad \mu(g_n) \le \mu(f_n)$

so $\quad \lim_{n\to\infty} \mu(g_n) = \liminf_n \mu(g_n) \le \liminf_n \mu(f_n)$

$$\mu(\liminf_n f_n) = \lim_{n\to\infty} \mu(g_n) = \liminf_n \mu(g_n) \le \liminf_n \mu(f_n)$$

Theorem **Dominated Convergence**

Let $(f_n)_{n\ge1}$ be a sequence of measurable functions and f another function so that $f_n \to f$ everywhere. Suppose further that there exists a $g \ge 0$ s.t. $|f_n| \le g$ $\forall n$ and $|f| \le g$ ~~then~~ and $\mu(g) < \infty$

$$\lim_{n\to\infty} \mu(f_n) = \mu(f)$$

Remarks * We call g a dominating function

* The presence of g stops the mass in f_n "escaping to infinity"

* If $\mu(E) < \infty$ then we can use constant functions as dominating functions and we sometimes call this bounded convergence.

Pf The idea is to use g in order to construct sequences of non-negative functions to construct sequences of non-negative functions which we can apply Fatou's lemma to

$g + f_n$ is non-negative and $g + f_n \to g + f$

$f_n \geq -|f_n| \geq -g$

so applying Fatou's lemma

$$\mu(\liminf_n (g + f_n)) \leq \liminf_n (\mu(g + f_n))$$

$$\mu(g+f) \leq \mu(g) + \liminf_n \mu(f_n)$$

$$\mu(g) + \mu(f) \leq \mu(g) + \liminf_n \mu(f_n)$$

as $\mu(g) < \infty$ we can subtract it from both sides

$$\mu(f) \leq \liminf_n \mu(f_n).$$

Similarly, $g - f_n$ is a sequence of non-negative measurable functions with $g - f_n \to g - f$

non-negative as $g - f_n \geq g - |f_n| \geq g - g = 0$

Applying Fatou again gives

$$\mu(\liminf_n (g - f_n)) \leq \liminf_n \mu(g - f_n)$$

$$\mu(g - f) \leq \liminf \mu(g - f_n)$$

$$\mu(g-f) \leq \liminf_n \mu(g-f_n)$$

$$\mu(g) - \mu(f) \leq \mu(g) + \liminf_n (-\mu(f_n))$$

$$\mu(g) - \mu(f) \leq \mu(g) - \limsup_n \mu(f_n)$$

then as f, f_n, g are all integrable all of $\mu(g), \mu(f)$

$\boxed{|f| \leq g \text{ and } \mu(g) < \infty}$

$\mu(f_n) \leq \mu(|f_n|) \leq \mu(g)$ for every n

so $\limsup_n \mu(f_n) \leq \mu(g) < \infty$ so all the numbers in the inequality are finite

so we can safely rearrange

$$\mu(g) - \mu(f) \leq \mu(g) - \limsup_n \mu(f_n)$$

$$\Rightarrow \quad \limsup_n \mu(f_n) \leq \mu(f)$$

We also $\quad \mu(f) \leq \liminf_n \mu(f_n)$

putting them together gives

$$\mu(f) \leq \liminf_n \mu(f_n) \leq \limsup_n \mu(f_n) \leq \mu(f)$$

therefore we have $\liminf_n \mu(f_n) = \limsup_n \mu(f_n) = \mu(f)$