Corollary. (Beppo Levi's Theorem).

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\sum_{k=1}^{\infty} f_k$ be an infinite series whose terms are $[0, +\infty]$ valued $\mathcal{A}$-measurable functions. Then

$$\int \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Proof. If $g_n = \sum_{k=1}^{n} f_k$ and $g = \sum_{k=1}^{\infty} f_k$ then the non-negativity of the f_k imply that g_n is non-negative and it implies that the sequence

$$\dots g_n \leq g_{n+1} \dots$$

is non-decreasing so by Monotone Convergence

$$\lim_{n \to \infty} \int g_n \, d\mu = \int g \, d\mu.$$

$$\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu$$

$$\shortparallel$$

$$\lim_{n\to\infty} \int \sum_{k=1}^{n} f_k \, d\mu$$

$$\shortparallel$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, d\mu$$

$$\sum_{k=1}^{\infty} \int f_k \, d\mu$$

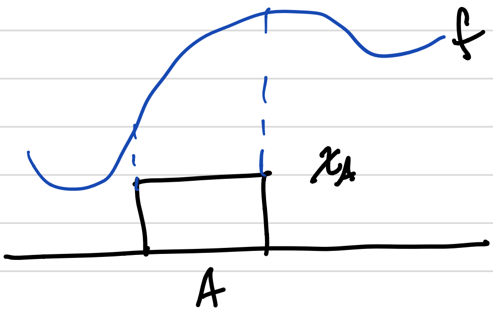
In the proper Riemann integral $\int_a^b f(x)\,dx$ we are implicitly considering integration over an interval $[a,b] \subset \mathbb{R}$.

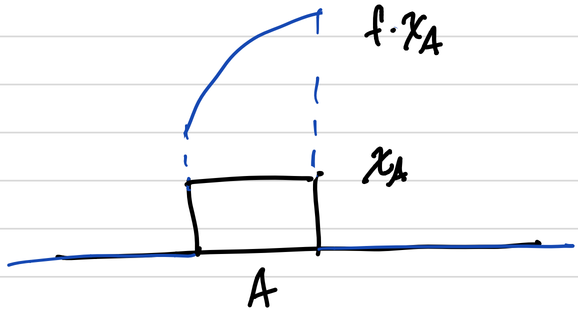
With the Lebesgue integral we are integrating over all of $\mathbb{R}$. In the general integral on a measure space $(X, \mathcal{A}, \mu)$ we are integrating over X.

Recall that we can restrict our integration to a measurable set A by defining

$$\int_A f \, d\mu = \int \chi_A \cdot f \, d\mu.$$

This has the effect of ignoring the values of f outside of the set A.





New measures from old.

Given a measure μ one simple way to create a new measure ν is to multiply μ by a positive constant c:

$$\nu(A) = c \cdot \mu(A).$$

We can think of ν as a rescaled version of μ.

It turns out that it is also possible to "rescale" a measure where the amount of "rescaling" varies from point to point.

Here is the construction.

Say that $(X, \mathcal{A}, \mu)$ is a measure space and that $f: X \to [0, +\infty]$ is $\mathcal{A}$-measurable.

We define a new measure

$$\nu : \mathcal{A} \to [0, +\infty] \text{ by setting}$$

$$\nu(A) = \int_A f \, d\mu.$$

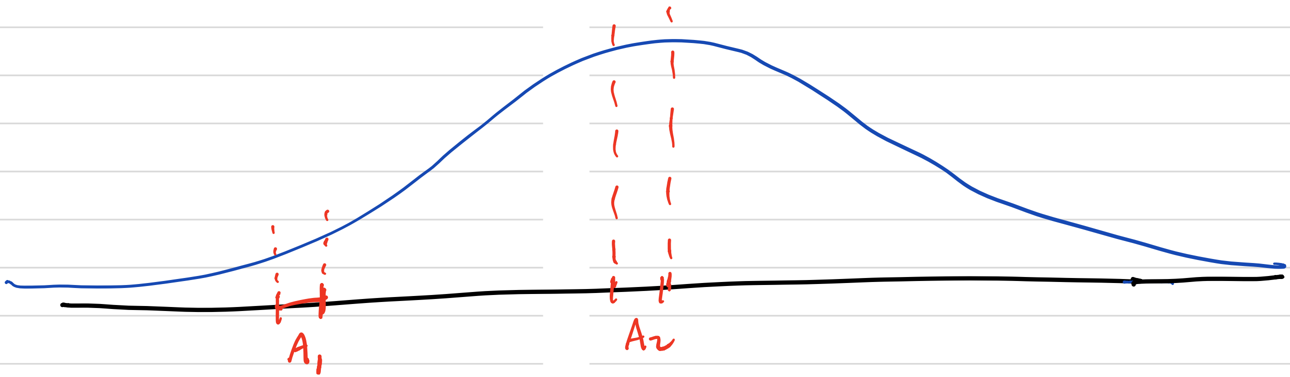
If $f(x) = c$ is a constant function then this is just rescaling as before. If f is not constant then we can think of f as describing a variable rescaling.

Example. μ is Lebesgue measure λ.

$$f(x) = \frac{1}{2\pi} e^{-x^2}.$$

$$\nu(A) = \frac{1}{2\pi} \int_A e^{-x^2} d\lambda.$$

This example has the property that $\nu(\mathbb{R}) = 1$.



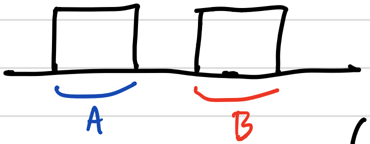
Let's check the properties of a measure.

① $\nu(\phi)=0$

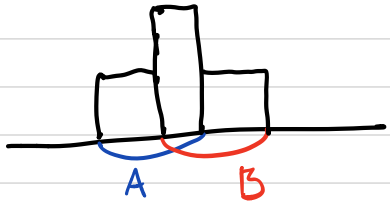
② If A_j is a countable sequence of disjoint sets then $\nu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \nu(A_j)$.

① $\nu(\phi) = \int_{\phi} f \, d\mu = \int f \cdot \chi_{\phi} \, d\mu = \int f \cdot 0 \, d\mu = 0.$

A key point in the proof of ② is that for disjoint sets A, B, $\chi_{A\cup B} = \chi_A + \chi_B$.



$\chi_A + \chi_B$
$= \chi_{A\cup B}$
(disjoint sets)



$\chi_A + \chi_B$

$\chi_A + \chi_B$
$\neq \chi_{A\cup B}$

(non-disjoint picture)

② $$\nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \int_{\cup A_j} f \, d\mu = \int f \cdot \chi_{\cup A_j} \, d\mu$$

disjointness of A_j

$$= \int f \cdot \left(\sum_j \chi_{A_j}\right) d\mu$$

$$= \int \sum_j f \cdot \chi_{A_j} \, d\mu$$

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$$= \sum_j \int f \cdot \chi_{A_j} \, d\mu$$

$$= \sum_j \nu(A_j).$$

$(X, \mathcal{A}, \nu)$ is a measure space.