Lecture 4 : Construction and uniqueness of measures

Remind me to record! Will start at 5 past 9.

$\mathcal{M}$ is a σ-algebra ← go through a bit of this

The Borel sets are contained in $\mathcal{M}$

Καραθεοδωρή

Carathéodory's extension theorem

Def^n A ring of sets, $\mathcal{A}$, of E is a collection of subsets satisfying

$$A, B \in \mathcal{A} \Rightarrow \underset{\underset{A \setminus (B \cap A)}{\parallel}}{A \setminus B} \in \mathcal{A} \quad \text{and} \quad A \cup B \in \mathcal{A}$$

Thm Let $\mathcal{A}$ be a ring of subsets of E and let $\mu : \mathcal{A} \to [0, \infty]$ be a countably additive set function $\mu(\emptyset) = 0$ (satisfies all the conditions to be a measure but its domain is not a σ-algebra)

Then μ extends to a measure on $\sigma(\mathcal{A})$

Example: We can construct Lebesgue measure using this theorem

$\mathcal{A}$ be the collection of finite disjoint unions of half open intervals $\lambda((a_1, b_1] \cup \dots (a_n, b_n]) = (b_1 - a_1) + \dots + (b_n - a_n)$

[N.B One can show that this collection of subsets is a ring and λ is countably additive on it]

Proof of Carathéodory's extension theorem

... an outer measure

~~Proof~~ of Caratheodory's extension theorem

We define an outer measure

$$\mu^*(B) = \inf \{ \sum_n \mu(A_n) : A_n \in \mathcal{A} \ \forall n, \ B \subset \bigcup_n A_n \}$$

$\mu^*(B) = \infty$ if there is no possible covering

Step 1: Show μ^* is an outer measure

$\emptyset \in A \setminus A$ for any $A \in \mathcal{A}$ so $\emptyset \in \mathcal{A}$ and $\mu(\emptyset) = 0$

and $\emptyset \subseteq \emptyset$ so $\mu^*(\emptyset) \leq \mu(\emptyset) = 0$ so $\mu^*(\emptyset) = 0$.

Monotonicity: If $B_1 \subseteq B_2$ then suppose $B_2 \subseteq \bigcup_n A_n$

then we have $B_1 \subseteq \bigcup_n A_n$

So consider the collection of sequences in $\mathcal{A}$ that cover B_1

this is a superset of the collection of sequences that cover B_2.

$$\inf \{ \sum_n \mu(A_n) : A_n \in \mathcal{A} \ \forall n, \ B_1 \subseteq \bigcup_n A_n \}$$

$$\leq \inf \{ \sum_n \mu(A_n) : A_n \in \mathcal{A} \ \forall n \ B_2 \subseteq \bigcup_n A_n \}$$

Countable subadditivity

let $B = \bigcup_n B_n$ then for each n $\exists$ $(A_{n,k})_{k \geq 1}$

s.t. $B_n \subseteq \bigcup_k A_{n,k}$ and $\sum_k \mu(A_{n,k}) \leq \mu^*(B_n) + \varepsilon 2^{-n}$

$A_{n,k} \in \mathcal{A} \ \forall n, k$

then $B \subseteq \bigcup_n \left(\bigcup_k A_{n,k} \right) = \bigcup_{n,k} A_{n,k}$ this is a countable sequence of $\mathcal{A}$ covering B

so $$\mu^*(B) \leq \sum_{n,k} \mu(A_{n,k})$$

$$= \sum_n \left(\sum_k \mu(A_{n,k}) \right)$$

$$\leq \sum_n \left(\mu^*(B_n) + \varepsilon 2^{-n} \right)$$

$$= \sum_n \mu^*(B_n) + \varepsilon$$

$$= \sum_n \mu^*(B_n) \quad + \varepsilon$$

As ε is arbitrary $\mu^*(B) \le \sum_n \mu^*(B_n)$

countable subadditivity.

Just like with Lebesgue measure

We call $\mathcal{M}$ the set of subsets A, of E satisfying

$\underline{\underline{\mu^*(B) = \mu^*(A \cap B) + \mu^*(A^c \cap B)}} \quad \forall B \subseteq E.$

<u>Step 2</u> Show μ^* agrees with μ on $\mathcal{A}$

(going to use the fact that μ is countably additive on $\mathcal{A}$)

[If we used this for Leb need to show μ is countably additive on finite disjoint unions of half open intervals $\mu(B_n) \to 0$ for a sequence $B_1 \subseteq B_2 \subseteq \dots$ with $\bigcap_n B_n = \emptyset$.]

~~Since~~ WTS: $\mu(A) = \mu^*(A)$ for all $A \in \mathcal{A}$

as $A \in \mathcal{A}$ $A \subseteq A$ $\mu^*(A) \le \mu(A)$

Then suppose $(A_n)_{n \ge 1}$ is a sequence in $\mathcal{A}$ with $A \subseteq \bigcup_n A_n$

then $A \cap A_n = (A \setminus (A \setminus A_n)) \in \mathcal{A}$

$$A = \bigcup_n (A \cap A_n) \quad \left(= A \cap \bigcup_n A_n = A \right)$$

by <u>countable subadditivity</u> [only need to prove countable subadditivity for λ]

$$\mu(A) \le \sum_n \mu(A_n \cap A)$$

By monotonicity $\mu(A_n \cap A) \le \mu(A_n)$ for each n

so $\mu(A) \le \sum_n \mu(A_n \cap A) \le \sum_n \mu(A_n)$

Therefore taking infimum over all possible ~~sets~~ sequences

$\mu(A) \leq \mu^*(A)$.

Step 3 Want to show $\mathcal{A} \subseteq \mathcal{M}$

i.e. If $A \in \mathcal{A}$ then $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$ $\forall B \subseteq E$.

By subadditivity $\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c)$

Let $(A_n)_{n \geq 1}$ be a sequence in $\mathcal{A}$ s.t.

$$\mu^*(B) \geq \sum_n \mu(A_n) - \varepsilon$$ and $B \subseteq \bigcup_n A_n$

Then $\underline{A \cap A_n} \in \mathcal{A}$ and $\underline{A^c \cap A_n} = A_n \setminus (A \cap A_n) \in \mathcal{A}$

$A \cap B \subseteq \bigcup_n (A \cap A_n)$ and $A^c \cap B \subseteq \bigcup_n (A^c \cap A_n)$

By defn of μ^*

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_n \mu(A_n \cap A) + \sum_n \mu(A^c \cap A_n)$$

$$= \sum_n \left(\mu(A_n \cap A) + \mu(A^c \cap A_n) \right)$$

$$= \sum_n \mu(A_n)$$ By countable (finite) additivity

$$\leq \mu^*(B) + \varepsilon$$

and ε is arbitrary so

$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \mu^*(B)$

We already showed the other inequality so

$\mathcal{A} \subseteq \mathcal{M}$.

Step 4 $\mathcal{M}$ is a σ-algebra $\underline{\text{and}}$ $\mu^*|_{\mathcal{M}}$ is a measure

Pf / ... exactly the same as video 1 this week

Pf/ Is exactly the same as video 1 this week

Show that $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra so $\sigma(\mathcal{A}) \subseteq \mathcal{M}$ and $\mu^*|_{\mathcal{M}}$ is a measure so $\mu^*|_{\sigma(\mathcal{A})}$ is just a restriction of this measure so also a measure.

<u>Uniqueness of extension</u> C.E.T shows there exists an extension but there could be others

<u>Theorem</u> Dynkin's uniqueness of extension theorem

Let μ_1 and μ_2 be measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$.

Suppose that $\mu_1 = \mu_2$ on $\mathcal{A}$ where $\mathcal{A}$ is a π-system generating $\mathcal{E}$ $(\sigma(\mathcal{A}) = \mathcal{E})$, then $\mu_1 = \mu_2$ on $\mathcal{E}$.

<u>Example</u>: (almost) half open intervals are a π-system generating $\mathcal{B}(\mathbb{R})$

Pf/ Basically done all the work with Dynkin's lemma

$\mathcal{D} \subseteq \mathcal{E}$

$\mathcal{D} = \{A \in \mathcal{E} \mid \mu_1(A) = \mu_2(A)\}$ so $\mathcal{A} \subseteq \mathcal{D}$

then $\mathcal{D}$ is a d-system as

If $A, B \in \mathcal{D}$ with $A \subseteq B$

$\mu_1(A) = \mu_2(A)$ $\quad \mu_1(B) = \mu_2(B)$

if

$\mu_1(A) = \mu_2(A)$ $\quad\quad$ $\mu_1(B) = \mu_2(B)$

$\mu_i(B) = \mu_i(A) + \mu_i(B \setminus A) \Rightarrow \mu_1(B \setminus A) = \mu_2(B \setminus A)$

so $B \setminus A \in \mathcal{D}$

so $\mathcal{D}$ is a d-system

Dynkins Lemma says if $\mathcal{A}$ is a π-system and $\mathcal{D}$ is a d-system ~~σ(A)~~ w/ $\mathcal{A} \subset \mathcal{D}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{D}$ as well

$\mathcal{D} \subseteq \mathcal{E}$ $\quad\quad$ $\sigma(\mathcal{A}) = \mathcal{E} \subseteq \mathcal{D}$ so $\mathcal{D} = \mathcal{E}$.