1 Measure Theory: Assignment Two - Constructing new measurable functions from old measurable functions

This sheet is all about building new measurable functions from existing measurable functions. Throughout this sheet we consider functions taking values in $\mathbb{R}$. We are interested in their measurability with respect to the Borel σ-algebra.

Question 1.1. Suppose that f is a measurable function. Show that $-f$ and λf are both measurable functions, where λ is a strictly positive contant.

Answer: $(-f)^{-1}((-\infty, b]) = f^{-1}([-b, \infty))$ which is measurable as $[-b, \infty)$ is a Borel set. Similarly, $(\lambda f)^{-1}((-\infty, b]) = f^{-1}((-\infty, b/\lambda])$ which is also measurable. *2 marks* $\square$

Question 1.2. Suppose that f_1 and f_2 are measurable. Show that $f_1 \vee f_2 = \min\{f_1, f_2\}$ is also measurable.

Answer: We look at the set $\{x : \min\{f_1(x), f_2(x)\} \leq b\}$ this is equal to $\{x : f_1(x) \leq b\} \cup \{x : f_2(x) \leq b\}$. Since both f_1 and f_2 are measurable this set will be measurable. Therefore $\min\{f_1, f_2\}$ is measurable. *3 marks* $\square$

Question 1.3. Suppose that $f_1, f_2, f_3, \ldots$ is a sequence of measurable functions. Show that $\inf_n f_n$ is also a measurable function. Use this to show that $\sup_n f_n$ is a measurable function as well.

Answer: let $f(x) = \inf\{f_n(x)\}$ then we have $f^{-1}((-\infty, b)) = \{x : f(x) < b\} = \{x : f_n(x) < b \,\text{for some}\, n\} = \bigcup_n \{x : f_n(x) < b\}$. Since f_n is measurable for every n $f^{-1}((-\infty, b))$ is the countable union of measurable sets so measurable. The corresponding result with supremums follows from the fact that $\sup_n\{f_n\} = -\inf_n\{-f_n\}$. Note that in order to use inf we had to work with $(-\infty, b)$. *5 Marks* $\square$

Question 1.4. Again let $f_1, f_2, f_3, \ldots$ be a sequence of measurbale functions. Show that $\limsup_n f_n$ and $\liminf_n f_n$ are both measurable functions. Why does this mean that $\lim_n f_n$ will be a measurble function if it exists.

Answer: Let us write $f(x) = \liminf_n f_n(x)$. Then $f^{-1}((-\infty, b)) = \{x : \liminf_n f_n(x) < b\} = \{x : \forall m \,\text{s.t}\, \inf_{n \geq m} f_n(x) < b\} = \bigcap_m \bigcup_{n \geq m} \{x : f_n(x) < b\}$. Therefore we have written this set as a countable union of measurable sets, so it is measurable. *5 marks* $\square$

Question 1.5. Suppose that f is a measurable function, show that f^2 is measurable.

Answer: The set $\{x : f(x)^2 \leq b\} = \{x : -\sqrt{b} \leq f(x) \leq \sqrt{b}\}$ which is measurable. *2 marks* $\square$

Question 1.6. Use the previous question to show that if f_1 an f_2 are measurable the $f_1 f_2$ is also measurable.

Answer: By the previous part f_1^2, f_2^2 and $(f_1+f_2)^2$ are all measurable. Therfore $f_1 f_2 = \frac{1}{2}((f_1+f_2)^2 - f_1^2 - f_2^2)$ is measurable. *3 marks* $\square$

Question 1.7. Let f be a measurable function with $f > 0$ everywhere. Show that $1/f$ is also measurable.

Answer: $(1/f)^{-1}((-\infty, b]) = \{x : 1/f(x) \leq b\} = \{x : f(x) \geq 1/b\} = f^{-1}([1/b, \infty))$ so this set is a measurable set therefore $1/f$ is measurable. *5 marks* $\square$