The following construction starts with an arbitrary $\mathcal{A}$-measurable function f and returns a sequence of simple functions f_n.

The construction is related to the construction of lower sums in the theory of the Riemann integral but in this case we are subdividing the range of our function rather than the domain.

Prop. Let f be a $[0,\infty]$-valued $\mathcal{A}$ measurable function.

There is a sequence of functions $\{f_n\}$ in $\mathcal{S}^+$ that satisfy

$$f_1(x) \le f_2(x) \le \ldots$$

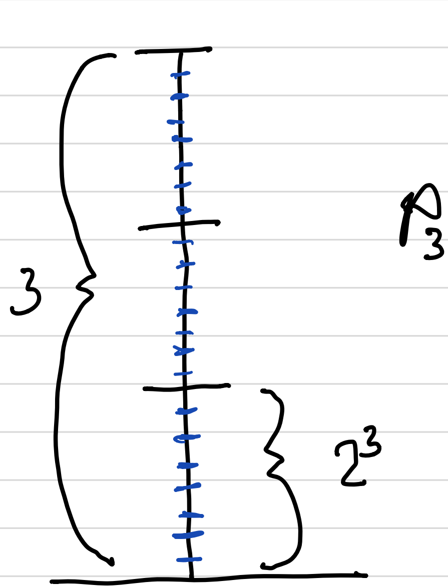
and $$f(x) = \lim_{n\to\infty} f_n(x) \text{ for each } x.$$

We start by constructing a sequence of partitions P_n of $\mathbb{R}$,

$$P_n = \{p_0 \dots\dots p_{n \cdot 2^n}\}$$

$$= \left\{0, \frac{1}{2^n}, \frac{2}{2^n} \dots \frac{n \cdot 2^n - 1}{2^n}, \frac{n \cdot 2^n}{2^n} = n\right\}$$

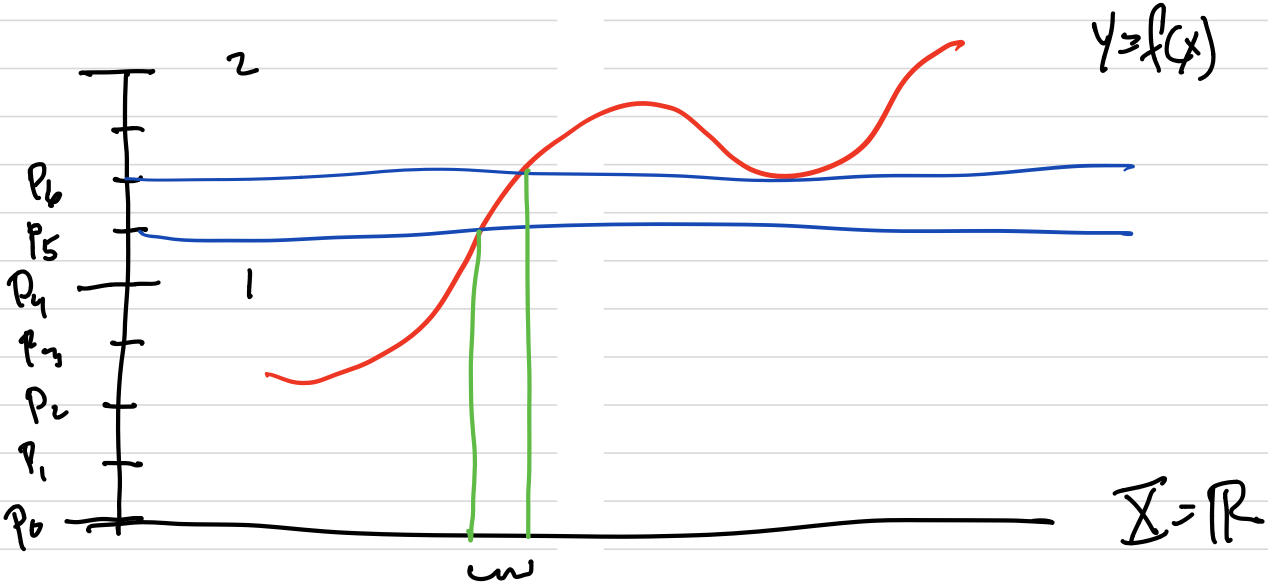
$$p_j = \frac{j}{2^n}.$$



We associate to the partition P_n the collection of $n \cdot 2^n$ intervals:

$$I_j = [p_j, p_{j+1}), \qquad j = 0, 1, \dots n2^n - 1.$$

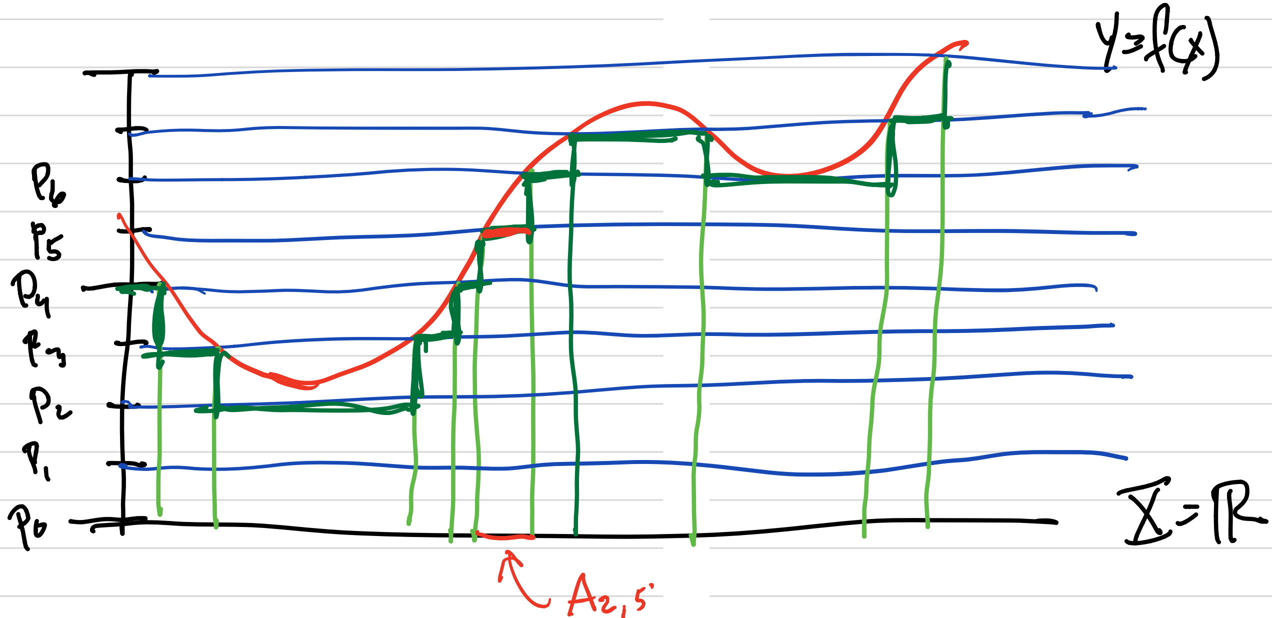
Let $A_{n,k} = \{ x \in \underline{X} : p_j \le f(x) < p_{j+1} \}$



$\mathcal{P}_2$

$A_{2,5} = \{x : p_5 \leq f(x) < p_6\}.$

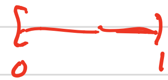
Define the function f_n to take the value p_k on the set $A_{n,k}$.



P_2 $\qquad$ f_2 is the green function

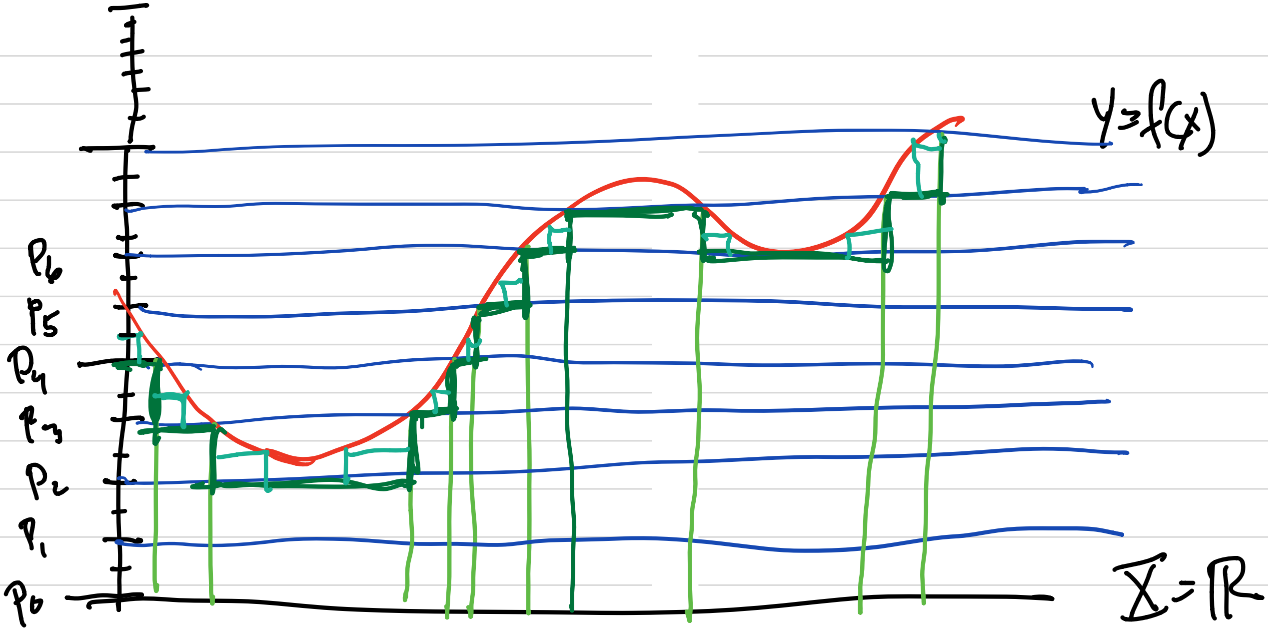
Since the partitions are nested:

$$P_1 \subset P_2 \subset P_3 \ldots$$



the corresponding intervals are nested,

It follows that $f_{n+1}(x) \geq f(x)$.



p_2

$f_2 \leq f_3$

Since the size of the intervals goes to 0 as $n \to \infty$ it follows that

$$f_n(x) \to f(x).$$

If we want to be more careful we would say that it is the quantities $\int f_n \, d\mu$ that correspond to the lower sums in the definition of the Riemann integral.

Note also that in my pictures I was taking the domain of

f to be $\mathbb{R}$ but in fact the domain can be any set X since the construction deals only with the range.