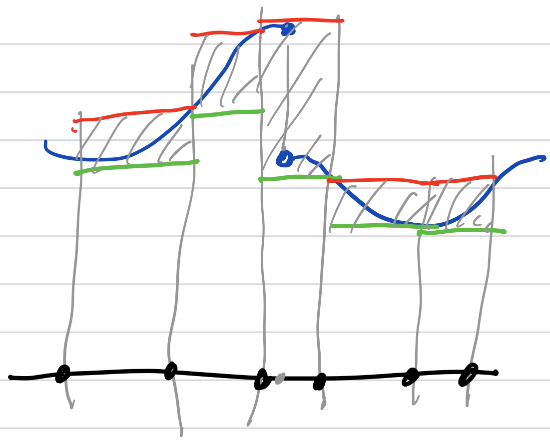
Riemann Integrability.

Let f be a bounded function defined on $[a,b]$. Let $P = \{a = a_0, \dots a_n = b\}$ be a partion of $[a,b]$.

To each partition P we associate a lower sum $L(P,f)$ and an upper sum $U(P,f)$.

We recall some basic facts about upper and lower sums taken from Cohn p.67.



Note that these are step functions rather than general simple functions.

(The pictures can be confusing since I draw the same picture for both.)

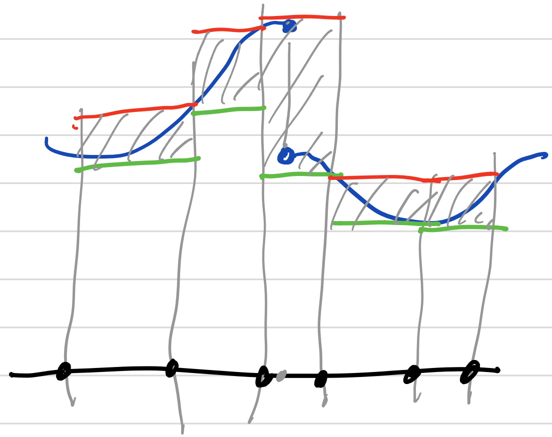
In particular these are Borel measurable.

(Measurable with respect to the σ-algebra $\mathcal{B}(\mathbb{R})$.)

Lemma 1.

For a partition P we have

$$L(P,f) \le U(P,f).$$



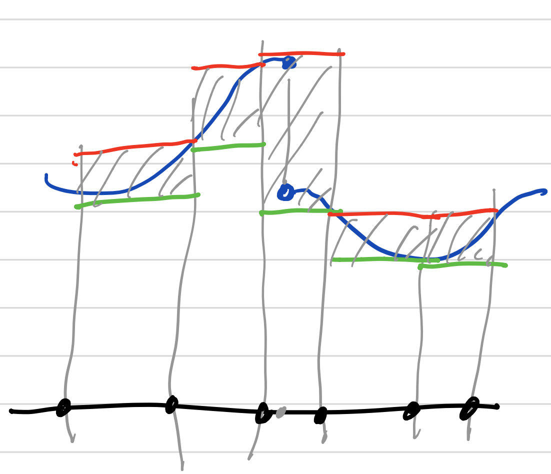
Lemma 2.

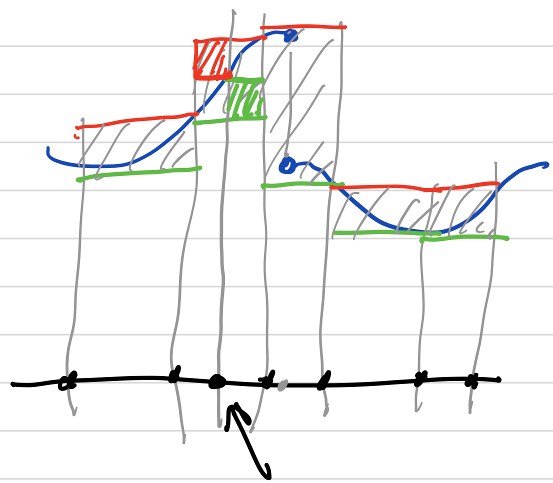
If P' is a refinement of P $(P \subset P')$ then

$$L(P', f) \geq L(P, f)$$

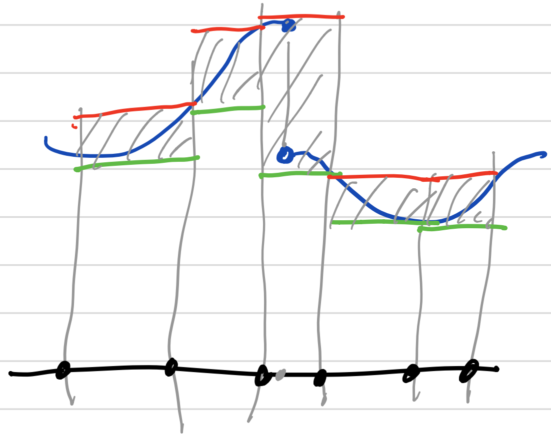
and

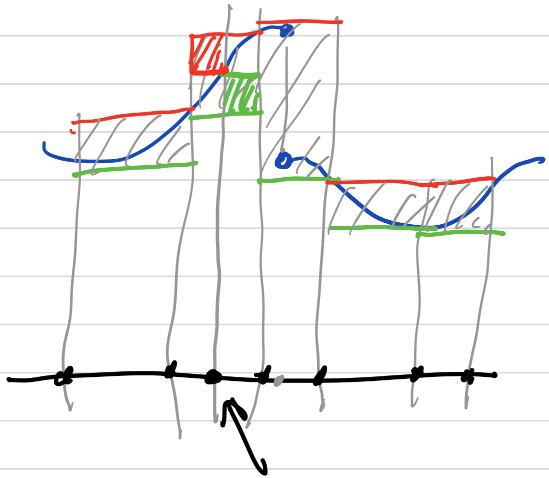
$$U(P', f) \leq U(P, f).$$





We can can see this by adding one point at a time to P.
When we do this the green curve can only increase and the red curve can only decrease.





Lemma 3.

Given two partitions P and P' we have

$$L(P,f) \leq U(P',f).$$

This follows by introducing P'' which is a common refinement of P and P'.

We have:

$$L(P,f) \leq L(P'',f) \leq U(P'',f) \leq U(P',f).$$

It follows from Lemma 3 that

$$\sup_P L(P,f) \le \inf_P U(P,f).$$

We say that f is Riemann integrable if

$$\sup_P L(P,f) = \inf_P U(P,f).$$

If f is Riemann integrable then $\int_a^b f\,dx$ is defined to be this common value.

Theorem. 2.5.4. Let $[a,b]$ be a closed bounded interval and let f be a bounded real valued function on $[a,b]$. Then

(a) if f is Riemann integrable then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.

(b) f is Riemann integrable if and only if it is continuous at almost every point of $[a,b]$.

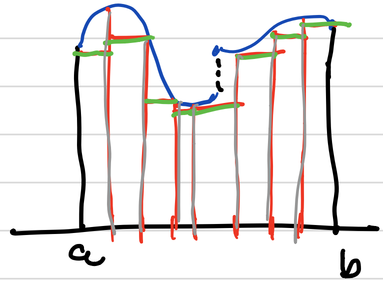
Proof.

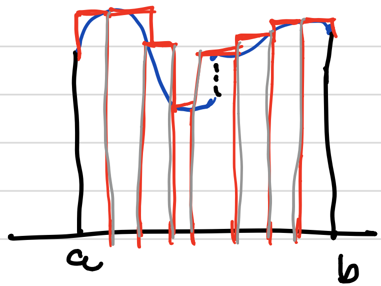
Choose an explicit sequence of partitions $\underline{P_n}$ with

and $P_1' \subset P_2' \subset P_3' \subset \dots$

and

$$U(P_n', f) - L(P_n', f) \leq \frac{1}{n}.$$





Let $P_n = \bigcup_{k=1}^{n} P_k'$. Then

P_n is a refinement of P_n'

so $\quad L(P_n', f) \leq L(P_n, f) \leq U(P_n, f) \leq U(P_n', f)$

and $\quad U(P_n, f) - L(P_n, f) \leq \frac{1}{n}$.

Also P_{n+1} is a refinement of P_n.

For each n we can build functions g_n and h_n with

$$\int g_n \, dx = L(P_n, f)$$

$$\int h_n \, dx = U(P_n, f)$$

Then

$g_n(x) \le h_n(x)$ for $x \in [a,b]$

by Lemma 1 and

$g_1 \le g_2 \le g_3 \dots$

$h_1 \ge h_2 \ge h_3 \dots$

by Lemma 2 and the nestedness of the partitions.

Now let $g(x) = \lim_{n\to\infty} g_n(x)$ and $h(x) = \lim_{n\to\infty} h_n(x)$.

Since the sequences are monotone and uniformly bounded they converge at every point.

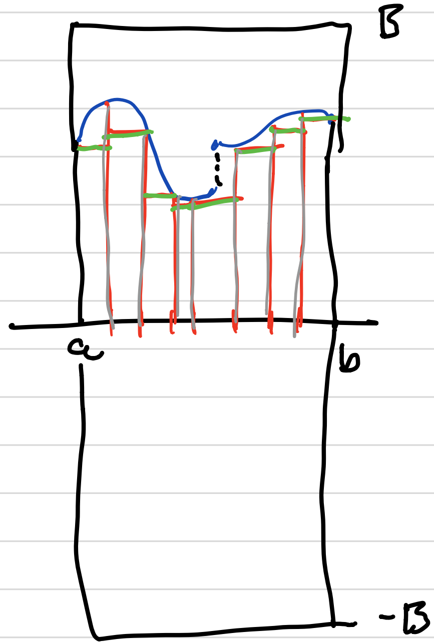
Then since limits of Borel measurable functions are Borel measurable (on the set where the limits exist) and the limit exists everywhere g and h are Borel measurable.

Furthermore by dominated convergence

$$\int g\,d\lambda = \lim_{n\to\infty} \int g_n\,d\lambda$$

$$\int h\,d\lambda = \lim_{n\to\infty} \int h_n\,d\lambda$$

and $\left| \int g_n\,d\lambda - \int h_n\,d\lambda \right| < \frac{1}{n}$



so $$\int g\,d\lambda = \lim \int g_n\,dy = \lim \int h_n\,dy = \int h\,d\lambda.$$

Now

$$h(x) = \lim_n h_n(x) \geq \lim_n g_n(x) = g(x).$$

So $h-g$ is non-negative and $\int h-g \, d\lambda = \int h \, d\lambda - \int g \, d\lambda = 0$.

We have seen in Corollary 2.3.12 that a non-negative function with integral 0 is equal to zero almost everywhere.

(This was phrased as $\int |f| = 0 \Rightarrow f = 0$ a.e.)

Since

$$g \le f \le h$$

we see that f is equal to a Borel function almost everywhere.

We need:

Prop. 2.2.2 If g is Lebesgue measurable and

$f = g$ a.e then f is Lebesgue measurable.

Proof of Prop. 2.2.2.

Say that $f(x) = g(x)$ for $x \notin N$ with $\lambda(N) = 0$.

We want to show that $\{x \in X : f(x) \le t\}$ is Lebesgue measurable.

$$\{x \in X : f(x) \le t\} = \left(\{x \in X : f(x) \le t\} \cap N^c\right) \cup \left(\{x \in X : f(x) \le t\} \cap N\right)$$

$$= \left(\{x \in X : g(x) \le t\} \cap N^c\right) \cup \underbrace{\left(\{x \in X : f(x) \le t\} \cap N\right)}$$

Measurable by hypothesis

Measurable because its complement has measure 0.

This is a subset of a set of measure 0 so it is measurable.

Note: A measure μ with the property that a subset of a set of measure 0 is measurable is called <u>complete</u>. Lebesgue measure on $\mathcal{M}(\mathbb{R})$ is complete but not all measure spaces are complete. $\left((\mathbb{R}, \lambda, \mathcal{B}(\mathbb{R}))\right.$ is not complete.$\left.\right)$

Now that we know f is Lebesgue integrable it follows that $\int f\,d\mu$ exists and we have

$$\int f\,d\mu = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n). \quad \text{So } \int f\,d\mu = \int_a^b f\,dx.$$

So far we have proved (a) that a Riemann integrable function is Lebesgue measurable and the Riemann integral and Lebesgue integral agree. We have also shown that

$h = g$ a.e.

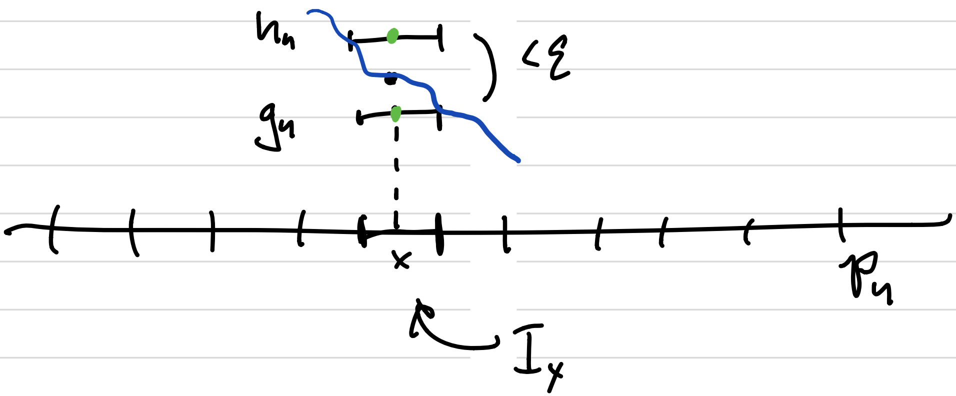
Want to show $h(x) = g(x)$

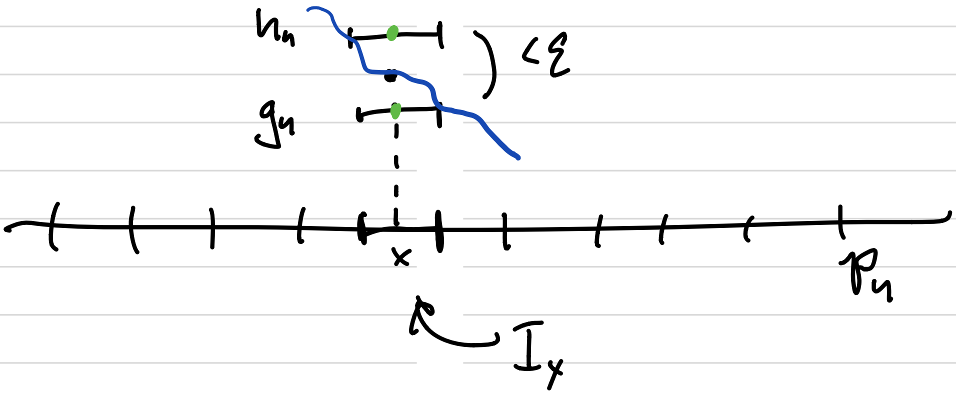
$\Rightarrow$ f is continuous at x

This shows that a Riemann integrable function is continuous a.e., which is half of (b).

Let x be a point where $h(x)=g(x)$ and assume x is not an endpoint of any P_n. Claim that f is continuous at x.

Let $\varepsilon>0$. We want to find a nbd. U of x so that for $x'\in U$ $|f(x)-f(x')|<\varepsilon$.





Choose an n so that $h_n(x) - g_n(x) < \varepsilon$.

Let I_x be the partition piece of P_n containing x.

By hypothesis $x \in \mathrm{int}(I_x)$. Set $U = \mathrm{int}(I_x)$.

Since g_n and f_n are constant on I_x we have

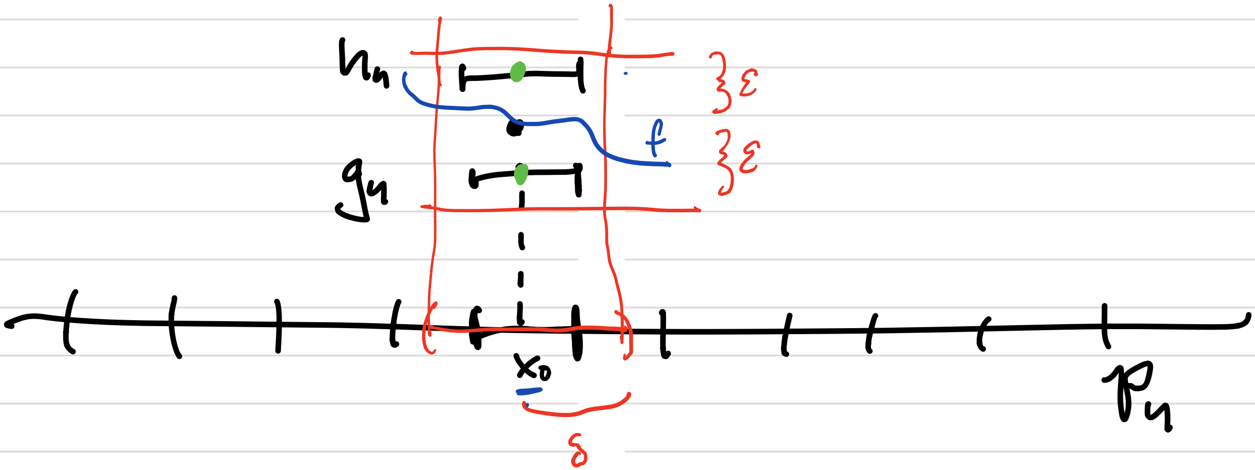
$$g_n(x) \le f(x') \le h_n(x) \text{ and } g_n(x) \le f(x) \le h_n(x)$$

for $x' \in I_x$ so $|f(x) - f(x')| < \varepsilon$.

Remark. The converse is also true.

If f is continuous at x and the size of the partition pieces goes to 0 then

$h(x) = g(x)$.



Given $\varepsilon > 0$ there is a δ so that $|x_0 - x| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. Choose an n so that the partition of $\mathcal{P}_n$ piece containing x has diameter less than δ.

Then $|h_n(x) - g_n(x)| < \varepsilon$.

Now suppose f is a bounded and continuous a.e.
We want to show that f is Riemann integrable.
This will complete the proof of the Theorem.

Choose a nested sequence of partitions P_n of $[a,b]$ where the size of the maximal interval goes to 0.

Repeat the construction of g_n, h_n, g and h.

By the remark $g(x) = h(x)$ at when f is continuous at x but $x \notin P_n$.

Thus $g(x) = h(x)$ a.e.

Two functions which agree a.e. have

the same Lebesgue integral. (Prop. 2.3.9).

So $\int g \, d\mu = \int h \, d\mu$.

By dominated convergence

$$\lim_{n\to\infty} \int g_n \, d\mu = \int g \, d\mu = \int h \, d\mu = \lim_{n\to\infty} \int h_n \, d\mu$$

so given any $\varepsilon > 0$ we can find an n with

$$\int h_n \, d\mu - \int g_n \, d\mu < \varepsilon \quad \text{or}$$

$$L(P_n, f) - U(P_n, f) < \varepsilon.$$

This shows that f is Riemann integrable.

Remark. In the Wednesday lecture I gave a different definition of Riemann integrability. It follows from this proof that the two definitions are equivalent.