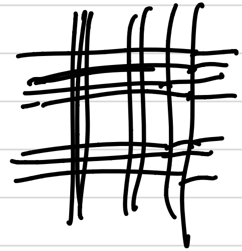
Example: As measures on $\mathbb{R}^2$: $\lambda_2 = \lambda_1 \times \lambda_1$.

$\lambda_1 \times \lambda_1$ has the property that for measurable rectangles $A \times B \subset \mathbb{R}^2$ $(\lambda_1 \times \lambda_1)(A \times B) = \lambda_1(A) \cdot \lambda_1(B)$.
In particular for actual rectangles $[a,b] \times [c,d]$ $\lambda_1 \times \lambda_1$ gives the same answer as $\lambda_2$.

In Prop 1.4.3 we showed that any measure which agrees with Lebesgue measure on coordinate rectangles <u>is</u> Lebesgue measure. So $\lambda_1 \times \lambda_1$ is equal to Lebesgue measure $\lambda_2$.

A consequence of this is that $\lambda_2(A\times B)=\lambda_1(A)\times\lambda_1(B)$ for measurable rectangles.



The next result is a version of Fubini's Thm for non-negative functions.

Prop. 5.2.1 Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces and let $f : X \times Y \longrightarrow [0, +\infty]$ be an $\mathcal{A} \times \mathcal{B}$ measurable function. Then

(a) $x \mapsto \int_Y f_x \, d\nu$ is $\mathcal{A}$-measurable and

$y \mapsto \int_X f^y \, d\mu$ is $\mathcal{B}$-measurable.

(b) $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f_x \, d\nu \right) d\mu$

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \left( \int_X f^y \, d\mu \right) d\nu.$$

Can write: $\int_X \left( \int_Y f_x \, d\nu \right) d\mu = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x)$

Proof. Consider first the case when $f$ is a characteristic function.

$$f = \chi_E \quad \text{for } E \in \mathcal{A}\times\mathcal{B}.$$

Now $f_x(y) = f(x,y) = \begin{cases} 1 & (x,y)\in E \\ 0 & \text{otherwise} \end{cases}$

So $f_x(y) = \begin{cases} 1 & (x,y)\in E \\ 0 & \text{otherwise} \end{cases} = \chi_{E_x}$

The function $x \mapsto \int f_x \, d\nu = \int \chi_{E_x} \, d\nu = \nu(E_x)$.

is the (vertical) slice measure function for E.

We proved that this was $\mathcal{A}$-measurable in Prop. 5.1.3 b. This is (a).

The equality of the integral:

$$\int f \, d(\mu \times \nu) = \int \chi_E \, d(\mu \times \nu) = (\mu \times \nu)(E)$$

and the integrals of the horizontal and vertical slice functions is proved in the Product Measure Theorem 5.1.4.

Now consider the case when $f = \sum a_i \chi_{A_i}$ is a non-neg. simple function.

The functions in (a) are linear combinations of measurable functions so they are measurable.

The linearity of the integral shows that the equations in (b) continue to hold.

When $f$ is a non-negative $\mathcal{A} \times \mathcal{B}$ measurable function then Prop. 2.1.8 tells us that $f$ is a pointwise monotone limit of simple functions.

Part (a) follows from the fact that the pointwise limit of measurable functions is measurable (Prop. 2.1.5)

Applying the Monotone Convergence theorem to the integrals of simple functions gives part (b).