1 Measure Theory: Assignment - Constructing new measurable functions from old measurable functions

First let us have a definition

Definition We call ϕ a step function if we can write ϕ in the form $\phi(x) = \sum_{k=1}^{n} a_k 1_{(c_k,d_k]}$ where a_k, c_k and d_k are real numbers.

The goal of this sheet is to show we can approximate functions in $L^p(\mathbb{R})$ by step functions. We do this in three steps, one for each question.

Question 1.1. Let f be a non-negative measurable function in $L^p(\mathbb{R})$ for $p \in [1, \infty)$ by considering the functions $f_{n,m}(x) = f(x)1_{|x|\le n}1_{f(x)\le m}$, or otherwise, show that for every $\epsilon > 0$ there is a function g where g is a bounded, non-negative, measurable function that is 0 outside some closed bounded interval and $\|g - f\|_p \le \epsilon$. *7 marks*

Answer: Let us look at the sequence $f_{n,n}$ then we know that $f_{n,n} \to f$ pointwise and $|f_{n,n} - f|^p \le |f|^p$ therefore by dominated convergence $\|f_{n,n} - f\|_p \to 0$. Hence there exists some n such that $\|f_{n,n} - f\|_p \le \epsilon$, and for every n we have $f_{n,n}$ is bounded and has support in a bounded interval. □

Question 1.2. Let g be a non-negative, bounded, measurable function whose support is contained inside $[-M, M]$ for some $M < \infty$. Show that for any fixed $\epsilon > 0$ there is a *simple function*, h whose support is inside a closed bounded interval, such that $\|g - h\|_p < \epsilon$. *7 marks*

Answer: Take the standard approximation $g_n(x) = 2^{-n}\lfloor 2^n g(x)\rfloor$ then by the previous assignment g_n is a simple function and $|g_n(x) - g(x)| \le 2^{-n}$ then $\|g_n - g\|_p \le 2^{-np+1}M \to 0$ therefore we can find n large enough so that $\|g_n - g\|_p \le \epsilon$. We also have that $g_n(x) = 0$ whenever $g(x) = 0$ so our approximation is also 0 outside $[-M, M]$. □

Question 1.3. Suppose that A is a bounded Lebesgue measurable set, show that, given $\epsilon > 0$ there is a finite collection of disjoint, half open intervals I_k such that $\|1_A - \sum_{k=1}^{n} 1_{I_k}\|_p < \epsilon$ *Hint: look at the non credit exercise sheet from week 2.* Use this to show that if h is a simple function whose support is contained in $[-M, M]$ then there is a *step function*, ϕ, such that $\|h - \phi\|_p < \epsilon$. You may use Minkowski's inequality which says that $\|f_1 + f_2\|_p \le \|f_1\|_p + \|f_2\|_p$. *7 marks*

Answer: By the definition of Lebesgue outer measure and the fact that A is Lebesgue measureable we have that there exists a countable sequence of intervals with

$$\sum_n \lambda(I_n) \le \lambda(A) + \epsilon/2,$$

and $A \subseteq \bigcup_n I_n$. Also as $\sum_n \lambda(I_n) < \infty$ there exists an N such that $\sum_{n=N+1}^{\infty} \lambda(I_n) < \epsilon/2$. Then

$$\lambda(\left(\bigcup_{n=1}^{N} I_n\right) \Delta A) \le \lambda(\bigcup_n I_n \setminus A) + \lambda(\bigcup_{n=N+1}^{\infty} I_n) \le \epsilon.$$

This is because $\bigcup_{n=1}^{N} I_n \setminus A \subseteq \bigcup_n I_n \setminus A$ and $A \setminus \bigcup_{n=1}^{N} \subseteq \bigcup_n I_n \setminus \bigcup_{n=1}^{N} I_n$. Then

$$\|1_A - 1_{\bigcup_{n=1}^{N} I_n}\|_p \le \lambda(A\Delta \bigcup_{n=1}^{N} I_n)^{1/p} \le \epsilon^{1/p}$$

so we can make this arbitrarily small. Now if we have a simple function h whose support is contained in a bounded set we can write $h = \sum_{k=1}^{n} a_k 1_{A_k}$ and we will have $\lambda(A_k) < \infty$ as we are contained in a bounded set. Therefore we can find $I_{k,j}$ a finite collection of intervals for each k such that $\|1_{A_k} - 1_{\bigcup_j I_{k,j}}\|_p \leq \epsilon/a_k$ and wlog the $I_{k,j}$ are disjoint in j (for each k we have a disjoint finite collection). Then we can define

$$\phi = \sum_{k=1}^{n} a_k \sum_j 1_{I_{k,j}}.$$

This is a step function and using Minkowski's inequality we have

$$\|h - \phi\|_p \leq \sum_k a_k \|1_{A_k} - \sum_j I_{k,j}\|_p \leq \epsilon.$$

□

Question 1.4. Now show that for any (not necessarily non-negative) function $f \in L^p(\mathbb{R})$ with $p \in [1, \infty)$, and for any ϵ there exists a step function ϕ with $\|\phi - f\|_p \leq \epsilon$. (Note that step functions don't have to be positive). *4 marks*

Answer: Now suppose f is positive then there exists g which is bounded and with support contained in a bounded interval such that $\|f - g\|_p \leq \epsilon/3$ and we can find h simple such that $\|h - g\|_p \leq \epsilon/3$ and then ϕ a step function such that $\|h - \phi\|_p \leq \epsilon/3$. Therefore by Minkowski $\|f - \phi\| \leq \|f - g\|_p + \|g - h\|_p + \|h - \phi\|_p \leq \epsilon$.

Now suppose that f is not necessarily non-negative we can write $f = f_+ - f_-$ and there exist step functions ϕ_+ and ϕ_- such that $\|f_+ - \phi_+\|_p \leq \epsilon/2$ and $\|f_- - \phi_-\|_p \leq \epsilon/2$. Therefore, if we set $\phi = \phi_+ - \phi_-$ we have $\|f - \phi\|_p \leq \|f_+ - \phi_+\|_p + \|f_- - \phi_-\|_p \leq \epsilon$ so we are done. □