Week 5 Video 1 - Monotone convergence.

28 October 2021 11:34

Theorem Let f, $(f_n)_{n\geq 1}$ be non-negative, real valued measurable functions with $f_n \uparrow f$, ($f_1 \leq f_2 \leq f_3$ and $f_n \to f$)

on a measure space $(E, \mathcal{E}, \mu)$ then

$\mu(f_n) \uparrow \mu(f)$.

Recall : $\mu(f) = \int f(x)\mu(dx)$.

Recall $\mu(f) = \sup\{\mu(h) : h \text{ is simple } h \leq f\}$

Proof Key idea / what makes this work

Is our thm about continuity of measure

$A_1 \subseteq A_2 \subseteq A_3 \ldots \qquad \mu(\bigcup_n A_n) = \lim_{n\to\infty} \mu(A_n)$

Extending this result about measurable sets to measurable functions.

First case $f = \mathbb{1}_A$ and $f_n = \mathbb{1}_{A_n}$ so $f_n \uparrow f$

means $A_1 \subseteq A_2 \subseteq \ldots$ and ~~A_n~~ $A = \bigcup_n A_n$

$$\mu(f) = \mu(\mathbb{1}_A) = \mu(A) \underset{\substack{\text{continuity} \\ \text{of measure thm}}}{=} \lim_{n\to\infty} \mu(A_n) = \lim_n \mu(\mathbb{1}_{A_n}) = \lim_n \mu(f_n)$$

Second case $f = \mathbb{1}_A$ but the f_n are simple functions

Lets pick $\varepsilon \in (0,1)$ arbitrary and define

$A_n = \{x : f_n(x) > (1-\varepsilon)\}$ then as the f_n

$A_n = \{x : f_n(x) > (1-\varepsilon)\}$ then as the f_n are increasing $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$

and $\bigcup_n A_n = A$ because $f_n \uparrow \mathbb{1}_A$ so if $\underline{x \in A}$ then $f_n(x) \uparrow 1$ so $f_n(x) \geq (1-\varepsilon)$ for n suff large $\therefore \bigcup_n A_n = A$.

using our first case $\lim_{n\to\infty} \mu(\mathbb{1}_{A_n}) = \mu(\mathbb{1}_A)$

using linearity of the integral for simple functions

$$\mu((1-\varepsilon)\mathbb{1}_{A_n}) \uparrow (1-\varepsilon)\mu(\mathbb{1}_A)$$

and, $\underline{(1-\varepsilon)\mathbb{1}_{A_n} \leq f_n \leq f = \mathbb{1}_A}$

so $$\lim_{n\to\infty} \mu((1-\varepsilon)\mathbb{1}_{A_n}) \leq \lim_{n\to\infty} \mu(f_n) \leq \mu(A)$$

so $$(1-\varepsilon)\mu(A) \leq \lim_{n\to\infty} \underline{\mu(f_n) \leq \mu(A)}$$

since ε is arbitrary this gives

$$\lim_{n\to\infty} \mu(f_n) = \mu(A) = \mu(\mathbb{1}_A) = \mu(f)$$

By monotonicity of the integral $\mu(f_n)$ will be increasing so $\lim_{n\to\infty} \mu(f_n)$ always exists.

$f_n \leq f$ so $\lim_{n\to\infty} \mu(f_n) \leq \mu(f)$

<u>Third case</u> : $f, (f_n)_{n\geq 1}$ are all simple functions

$f = \sum a_k \mathbb{1}_{A_k}$ where wlog $a_k > 0$

$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$ where wlog $a_k > 0$

as f_n is a simple function so is

$a_k^{-1} f_n \mathbb{1}_{A_k}$ and $a_k^{-1} f_n \mathbb{1}_{A_k} \uparrow \mathbb{1}_{A_k}$

as $f_n \uparrow f$ so $\forall k$ if $x \in A_k$ then $f_n(x) \uparrow a_k \; {}^{= f(x)}$

so $a^{k} f_n(x) \uparrow 1$. if $x \notin A_k$ then $a_k^{-1} f_n \mathbb{1}_{A_k} = 0$.

By our second case $\mu(a_k^{-1} f_n \mathbb{1}_{A_k}) \uparrow \mu(A_k)$ as $n \to \infty$

so by lineary of the integral for simple functions

$$\mu(a_k a_k^{-1} f_n \mathbb{1}_{A_k}) \uparrow \mu(a_k \mathbb{1}_{A_k})$$
$$\shortparallel$$
$$\mu(f_n \mathbb{1}_{A_k}) \uparrow \mu(a_k \mathbb{1}_{A_k})$$

Again using linearity

$$\mu(f_n \sum_k \mathbb{1}_{A_k}) \uparrow \mu(\sum_k a_k \mathbb{1}_{A_k}) = \mu(f)$$

$$\mu(f_n \sum_k \mathbb{1}_{A_k}) \uparrow \mu(f)$$

supp(f_n) must be contained in supp(f) as

$\underline{0 \leq f_n \leq f}$ and supp$(f) = \bigcup_k A_k$

$$f_n \sum_k \mathbb{1}_{A_k} = f_n \sum_k \mathbb{1}_{\bigcup_k A_k} = f_n$$

so $\mu(f_n) \uparrow \mu(f)$

Fourth case: f is +ve and measurable

then $(f_n)_{n \geq 1}$ are all simple.

Recall $\mu(f) = \sup\{\mu(g) : g \text{ simple } g \leq f\}$

so fix some g with g simple $g \leq f$.

Set $g_n = f_n \wedge g = \min\{f_n, g\}$

then g_n is also simple.

So as $f_n \uparrow f$ and $g \leq f$ we have $g_n \uparrow g$

So by our third case $\mu(g_n) \uparrow \mu(g)$

We also have $g_n \leq f_n$ by monotonicity of integration $\mu(g_n) \leq \mu(f_n) \leq \mu(f)$.

So taking limits $\mu(g) \leq \lim_{n\to\infty} \mu(f_n) \leq \mu(f)$

We can take the supremum over all g simple $g \leq f$ in this inequality.

$$\sup\{\mu(g) : g \text{ simple } g \leq f\} \leq \lim_n \mu(f_n) \leq \mu(f)$$

$$\mu(f) \leq \lim_n \mu(f_n) \leq \mu(f)$$

so $\lim_n \mu(f_n) = \mu(f)$.

Fifth and final case:

f is +ve measurable, f_n are +ve and measurable

Idea: create a se~~z~~quence of simple functions g_n

s.t. $g_n \uparrow f$ and $g_n \leq f_n$ so the $\lim_n \mu(g_n) \leq \lim_n \mu(f_n)$

So we set $g_n = \left(2^{-n} \lfloor 2^n f_n \rfloor\right) \wedge n$ $k2^{-n}$

biggest number of the form $k2^{-n}$ smaller than $f_n(x)$

so $g_n(x) \leq f_n(x)$

and if $f_n(x) \leq n$ then $g_n(x) \geq f_n(x) - 2^{-n}$

so $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x)$

$$g_n(x) = \left(2^{-n} \lfloor 2^n f_n(x) \rfloor\right) \wedge n \leq \left(2^{-(n+1)} \lfloor 2^{n+1} f_n(x) \rfloor\right) \wedge (n+1)$$

$$\leq \left(2^{-(n+1)} \lfloor 2^{n+1} f_{n+1}(x) \rfloor\right) \wedge (n+1)$$

using that f_n is increasing.

By the fourth case $\mu(g_n) \uparrow \mu(f)$

and $\mu(g_n) \leq \mu(f_n)$

so $\lim_n \mu(g_n) \leq \lim_{n\to\infty} \mu(f_n) \leq \mu(f)$

$\lim_n \mu(g_n) = \mu(f)$

so

$\mu(f)$

so

$$\lim_{n\to\infty} \mu(f_n) = \mu(f).$$