Basic Properties of Simple Functions

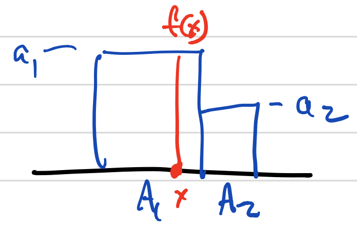
In the following discussion I want to fix a set X and a σ-algebra $\mathcal{A}$ of subsets of X.

$(X, \mathcal{A})$

In order to visualise the picture I suggest taking $X = \mathbb{R}$ and $\mathcal{A} = \mathcal{M}$.

$(\mathbb{R}, \mathcal{M})$

Definition. A simple function (with respect to a σ-algebra $\mathcal{A}$) is a function that can be written as $\sum_{i=1}^{n} a_i \chi_{A_i}$ with $A_i \in \mathcal{A}$.



Note that if $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ then we can always rewrite f so that the sets A_i are disjoint.

If the A_i are disjoint then the coefficients a_i are values of f namely:

$$a_i = f(x) \text{ for } x \in A_i.$$

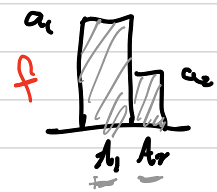
We write $\mathcal{S}$ for the family of simple functions $f : X \longrightarrow [-\infty, \infty]$.

We write $\mathcal{S}^+$ for the family of non-negative simple functions.

Our strategy in defining the integral is to define it for simple functions and then extend it to positive functions.

$$f = \sum_{i=1}^{n} a_i \chi_{A_i} \, d\mu$$

We define the integral of $f \in \mathcal{S}^+$ to be;



$$\int f \, d\mu = \int \sum_{i=1}^{n} a_i \chi_{A_i} \, d\mu = \sum_{i=1}^{n} a_i \cdot \mu(A_i).$$

The next result guarantees that if $f \in \mathcal{S}$ then the integral $\int f \, d\mu$ is well defined.

Prop. If f is a simple function

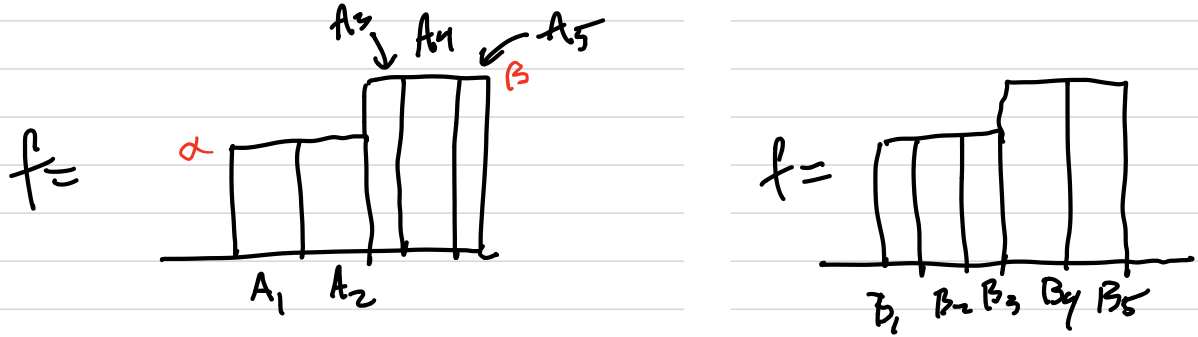
$$f = \sum_{i=1}^{m} a_i \chi_{A_i} \qquad (A_i \text{ disjoint})$$

and

$$f = \sum_{i=1}^{n} b_j \chi_{B_j} \qquad (B_j \text{ disjoint})$$

then

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{j=1}^{n} b_j \mu(B_j).$$



Proof. We may assume that no coefficients are equal to 0 so that

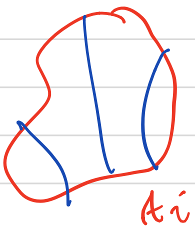
$$\bigcup A_i = \bigcup B_j = \{x ; f(x) > 0\}.$$

Now if $A_i \cap B_j \neq \emptyset$ then $c_i = b_j = f(x)$ for $x \in A_i \cap B_j$.

Lets check that the 2 potential expressions for $\int f \, d\mu$ give the same answer.

$$\sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \mu(A_i \cap B_j)$$



$$= \sum_{j=1}^{n} \sum_{i=1}^{m} b_j \mu(A_i \cap B_j)$$

$$= \sum_{j=1}^{n} b_j \mu(B_j).$$

Proposition. 2.3.1. Let f and g belong to $\mathcal{S}^+$ and let α be a non-negative real number. Then

(a) $\int \alpha f \, d\mu = \alpha \int f \, d\mu$

(b) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

(c) If $f(x) \leq g(x)$ holds for each $x \in X$ then $\int f \, d\mu \leq \int g \, d\mu$.

Proof. Suppose $f = \sum_{i=1}^{m} a_i \chi_{A_i}$ where

$a_1 \ldots a_m$ are non-neg. real numbers and

$A_1 \ldots A_m$ are disjoint measurable sets.

(a) $\int \alpha f \, d\mu = \sum_{i=1}^{m} \alpha a_i \mu(A_i) = \alpha \sum_{i=1}^{m} a_i \mu(A_i) = \alpha \int f \, d\mu.$

(b) $\int f+g \, d\mu = \sum_{i=1}^{m} \sum_{j=1}^{n} (a_i + b_j) \mu(A_i \cap B_j)$

$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \mu(A_i \cap B_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} b_j \mu(A_i \cap B_j)$

$$= \sum_{i=1}^{m} a_i \mu(A_i) + \sum_{j=1}^{n} b_j \mu(B_j)$$

$$= \int f \, d\mu + \int g \, d\mu.$$

(c) Now suppose that $f(x) \leq g(x)$ holds at each x in X. Then $g-f$ is in $\mathcal{A}^+$ so

$$\int g \, d\mu = \int (f + (g-f)) \, d\mu$$

$$= \int f \, d\mu + \int (g-f) \, d\mu \quad \text{(by (b))}$$

$$= \int f\,d\mu + \underbrace{\sum_i a_i\,\mu(A_i)}_{\geq 0} \qquad a_i \geq 0.$$

g-f

$$\geq \int f\,d\mu$$

The a_i are values of the function $g-f$ by the disjointness hypothesis and these values are non-negative since $f(x) \leq g(x)$ or $(g-f)(x) \geq 0$.