Lebesgue measure is regular so

$\lambda(A) = \inf\{\lambda(U) : U \text{ open } A \subseteq U\}$

and $\lambda(A) = \sup\{\lambda(K) : K \text{ compact}, K \subseteq A\}$

<u>Lusin's Theorem</u>: Suppose f is a measurable function $f: \mathbb{R}^d \to \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$ is a Borel set with $\lambda(A) < \infty$ then for any $\varepsilon > 0$ there exists K, compact, $K \subseteq A$ $\lambda(A \setminus K) < \varepsilon$ s.t. $f|_K$ is continuous.

<u>Remark</u>: More detailed version of this theorem in Cohn's book.

Proof/ First we do a special case

Suppose f can only take countably many values $a_1, a_2, a_3, \ldots$. Let us define $A_k = f^{-1}(\{a_k\}) \cap A$

$A = \bigcup_n A_n$ so by continuity of measure $\lambda(A) = \lim_{n\to\infty} \lambda\left(\bigcup_{k=1}^{n} A_k\right)$

Given $\varepsilon > 0$ there exists n s.t.

$\lambda(A) - \lambda\left(\bigcup_{k=1}^{n} A_k\right) \to 0$

$\lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right) < \varepsilon/2$

By the regularity of Lebesgue measure $\exists K_1, \ldots, K_n$ compacts with $K_k \subseteq A_k$ and $\lambda(A_k \setminus K_k) < \varepsilon/2n$

Let $K = \bigcup_{k=1}^{n} K_k$ This is compact.

$\lambda(A \setminus K) = \lambda\left(A \setminus \bigcup_{k=1}^{n} A_k\right) + \lambda\left(\bigcup_{k=1}^{n} A_k \setminus \bigcup_{k=1}^{n} K_k\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

and $f|_K$ is continuous because the A_k are all disjoint so the K_k are all disjoint and $f|_{K_i}$ is conts. for each i. So we've proved Lusin's thm in this case!

Now take a general f and let $f_n = 2^{-n}\lfloor 2^n f \rfloor$

so f_n can only take countably many values

$2^{-n} \geq f(x) - f_n(x) \geq 0$ so $f_n(x) \to f(x)$ for every x.

$f_n \to f$ almost everywhere.

So by Egoroff's theorem $\exists$ B, $B \subseteq A$, s.t. $\lambda(A \setminus B) < \varepsilon/4$ and $f_n \to f$ uniformly on B by regularity $\exists$ $K \subseteq B$ compact s.t. $\lambda(B \setminus K) < \varepsilon/4$ so $K \subseteq A$ $\lambda(A \setminus K) < \varepsilon/2$

Using our special case of Lusin's theorem $\exists$ $K_n \subseteq K$ s.t. $\lambda(K \setminus K_n) \leq \varepsilon 2^{-n-1}$ and $f_n|_{K_n}$ is continuous

Then set $K_\infty = \bigcap_n K_n$ then K_∞ is compact

and $\lambda(A \setminus K_\infty) = \lambda(A \setminus K) + \lambda(K \setminus K_\infty) = \lambda(A \setminus K) + \lambda\left(\bigcup_n (K \setminus K_n)\right)$

$$\leq \frac{\varepsilon}{2} + \varepsilon \sum_n 2^{-n-1} = \varepsilon$$

Now $K_\infty \subseteq K$ and $K_\infty \subseteq K_n$ for each n

so $f_n|_{K_\infty}$ is continuous for every n

and $f_n \to f$ uniformly on K_∞

... continuous functions is continuous

The uniform limit of continuous functions is continuous so $f|_{K_\infty}$ is continuous.