Let X be a finite set and let μ be the counting measure:

$$\mu(A) = \#A \quad \text{for } A \subset X.$$

If $f: X \to \mathbb{R}$ what is

$$\int f \, d\mu \, ?$$

$$\int f \, d\mu = \sum_{x \in X} f(x).$$

We have been talking about taking limits of functions.

Another place in mathematics where limits arise is in connection with metric spaces.

Recall that justifying "Fourier analysis" was part of the motivation for Lebesgue's theory of integration.

We will be talking about vector spaces of functions and the metrics we deal with come from norms on the vector spaces.

An example to keep in mind is $\mathbb{R}^n$.

$\mathbb{R}^n$ has a norm $v \mapsto \|v\|$. $\quad \|v\| = \left(\sum_{j=1}^{n} |v_j|^2 \right)^{1/2}$

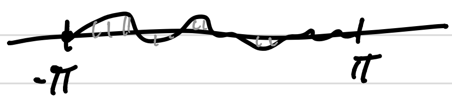
$\mathbb{R}^n$ has a distance $d(v,w) = \|v-w\|$ defined in terms of the norm.

An important distinction however is that the vector spaces we will be dealing with are infinite dimensional as is suggested by the expression: $f: [-\pi, \pi] \to \mathbb{R}$.

$$f(x) = a_0 + \sum_{j=1}^{\infty} b_j \sin(j \cdot x) + \sum_{j=1}^{\infty} c_j \cos(j \cdot x)$$

If we have some vector space of functions V then a norm $\|\cdot\|$ on V is a measure of the size of the function.

If f takes small values then $\|f\|$ should be small. If f takes large values then $\|f\|$ should be large.



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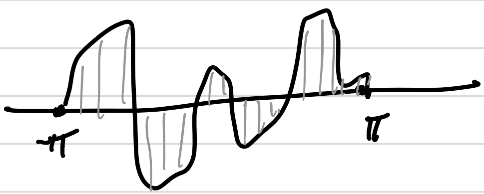

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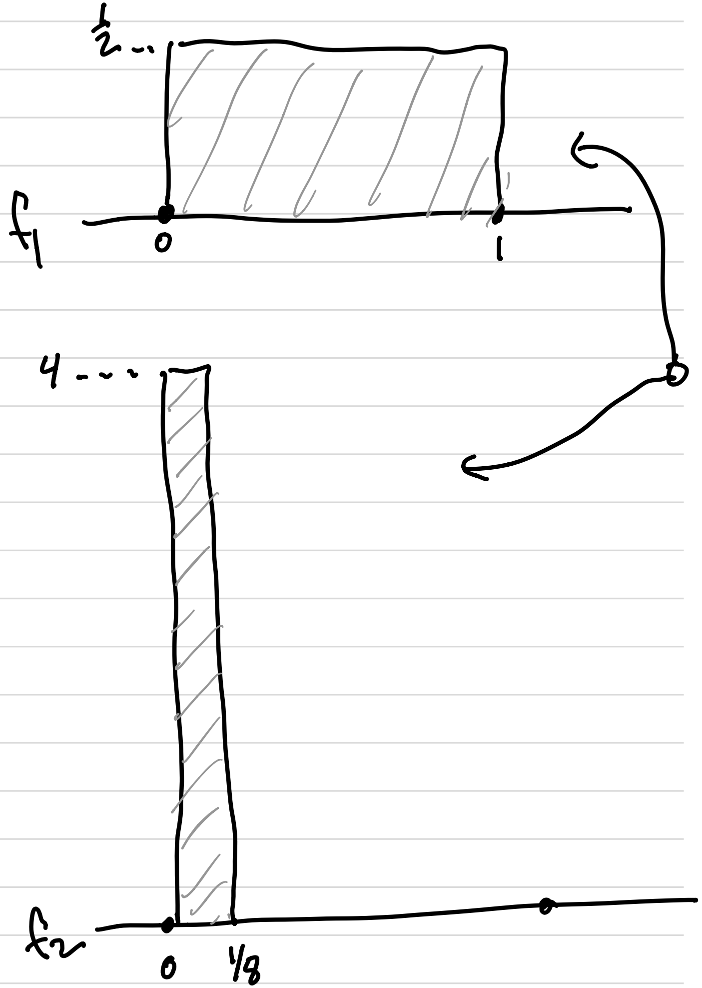
There are different ways for a function to be large.



same area

Which function should have the larger norm?

$\|f_1\| = \|f_2\|$?

$\|f_1\| < \|f_2\|$?

$\|f_1\| > \|f_2\|$?

How do you take into account the effect of <u>larger</u> values on a <u>smaller</u> set in f_2 ?

There is no single best answer. Different problems may call for different answers.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let f be a measurable function on X.

For $1 \le p < \infty$ define the "p-norm"

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

(At the moment it is not clear that this is a norm.)

In order for $\|f\|_p$ to be finite we need

$$\int |f|^p d\mu < \infty.$$

Let $\mathcal{L}^p = \mathcal{L}^p(X, \mathcal{A}, \mu)$ be the set of functions such that $\int |f|^p d\mu < \infty$.

This definition is compatible with our earlier definition of $\mathcal{L}^1 = \mathcal{L}^1(X, \mathcal{A}, \mu)$.

$$\int |f| d\mu < \infty.$$

Lemma. For $1 \le p < \infty$ $\mathcal{L}_p$ is a vector space.
(Cohn p. 92)

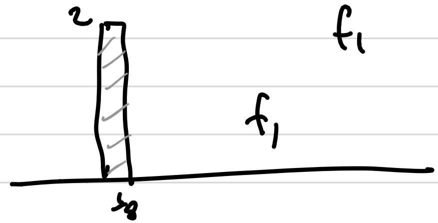
Remark. Typically $\mathcal{L}_p$ is an infinite dimensional vector space.

$\mathcal{L}_2([-\pi, \pi])$ contains functions $\sin(kx)$, $\cos(kx)$ for $k \ge 1$ and all of these are linearly independent.

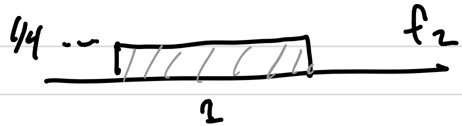
Examples.

$p=1$

L^1:



Same L^1 norm



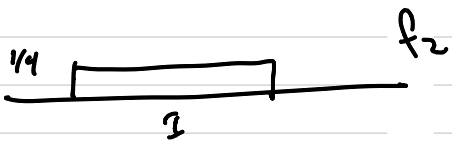
$$\|f_1\|_1 = 2 \cdot \frac{1}{8} = \frac{1}{4}$$

$$\|f_2\|_1 = \frac{1}{4} \cdot 1 = 1/4$$

L^2:

bigger

smaller



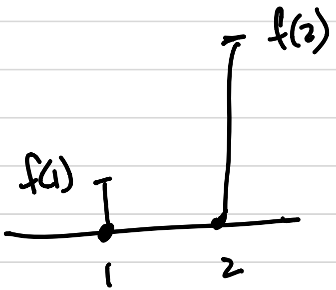
$$\|f_1\|_2 = \sqrt{4 \cdot 1/8} = \sqrt{1/2} = \frac{1}{\sqrt{2}} = .707$$

$$\|f_2\|_2 = \sqrt{\frac{1}{16} \cdot 1} = \sqrt{1/16} = \frac{1}{4} = .250$$

Intuition:

Larger values of p put more emphasis on the large values of the function.

Example. Consider the measure space $(X, \mathcal{A}, \mu)$ where $X = \{1, 2\}$. μ is counting measure.

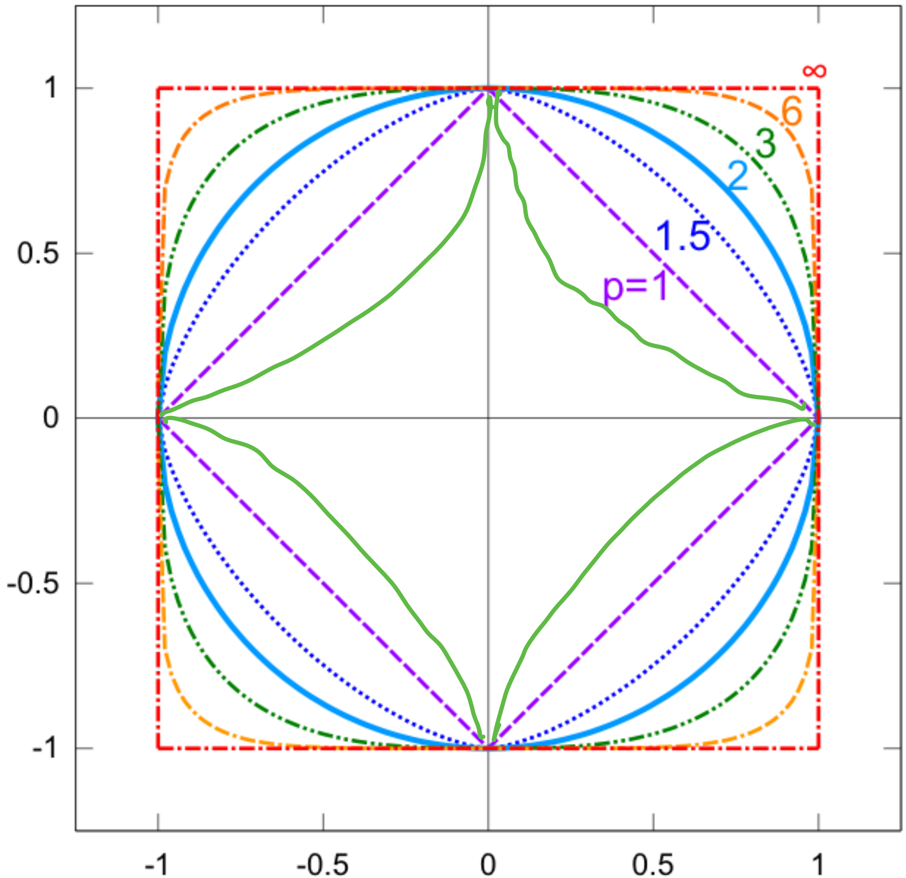


$(X, \mathcal{A}, \mu) = \mathbb{R}^2$. $\int f \, d\mu = f(1) + f(2)$.

$\|f\|_p = \left(|f(1)|^p + |f(2)|^p \right)^{1/p}$

What does the L^p unit ball look like?

Unit circles for p-norms in $\mathbb{R}^2$



$p=1$

L^1 norm

$\int |f| d\mu$

$\|f\|_1 = |f(1)| + |f(2)|$

L^∞ norm

$\|f\|_\infty = \max\{|f(1)|, |f(2)|\}$

$p=1$

$p=\infty$

As $p \to \infty$ the $\|f\|_p$ norm becomes more or less the sup norm.

We can define:

$$\|f\|_\infty = \max\{|f(1)|, |f(2)|\}.$$

The unit ball is convex if $p \geq 1$.

The limiting cases $p=1$ and "$p=\infty$" or the sup norm are somewhat exceptional.

The case $p=2$ gives the classic Euclidean norm on $\mathbb{R}^2$.

Extreme cases are the L^1 norm and the L^∞ norm. L^p interpolates between these.

Let V be a vector space. Recall that a function $v \mapsto \|v\|$ from V to $\mathbb{R}$ is a norm if:

a) $\|v\| \geq 0$

b) $\|v\| = 0$ if and only if $v = 0$

c) $\|\alpha v\| = |\alpha| \cdot \|v\|$

d) $\|v+w\| \leq \|v\| + \|w\|$.

Is $\|\cdot\|_p$ really a norm on $\mathcal{L}^p$?

Want to show that $\|\cdot\|_p$ is a norm on $\mathcal{L}_p$.

(a) $\|f\|_p \geq 0$. $\quad |f|^p \geq 0 \Rightarrow \left(\int |f|^p d\mu \geq 0.\right)^{1/p} \geq 0.$

(c) $\|\alpha f\|_p = |\alpha| \, \|f\|_p$.

$$\left(\int |\alpha f|^p d\mu\right)^{1/p} = \left(\int |\alpha|^p \cdot |f|^p d\mu\right)^{1/p}$$

$$= \left(|\alpha|^p \int |f|^p d\mu\right)^{1/p}$$

$$= |\alpha| \left(\int |f|^p d\mu\right)^{1/p}.$$

(d) is equivalent to the convexity of the unit ball.

If $1<p<\infty$ we define the conjugate exponent q so that $\frac{1}{p}+\frac{1}{q}=1$.

Example: If $p=2$ then $q=2$.

Prop. 3.3.2 (Holder's inequality).
If $f\in \mathcal{L}^p$ and $g\in \mathcal{L}^q$ then $fg\in \mathcal{L}^1$ and

$$\int |fg|\, d\mu \le \|f\|_p \cdot \|g\|_q.$$

If $p=q=2$ this is Cauchy's inequality.

Minkowski's inequality.

Prop. If $f, g \in \mathcal{L}^p$ then $f+g \in \mathcal{L}^p$ and

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Let V be a vector space. Recall that a function $v \mapsto \|v\|$ from V to $\mathbb{R}$ is a norm if:

a) $\|v\| \geq 0$

b) $\|v\| = 0$ if and only if $v = 0$

c) $\|\alpha v\| = |\alpha| \cdot \|v\|$

d) $\|v+w\| \leq \|v\| + \|w\|$.

Minkowski's inequality gives property d for $\|\cdot\|_p$. We have a), c) and d). What about b)?

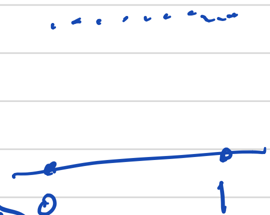
If $\|f\|_p = 0$ then $\left(\int |f|^p d\mu\right)^{1/p} = 0$

so $\int |f|^p d\mu = 0 \Rightarrow \underline{\int |f| d\mu = 0.}$

This shows that $f=0$ a.e, but it does not show that $f=0$.

Consider the Dirichlet function $\chi_{\mathbb{Q}\cap[0,1]}$ in $\mathcal{L}^p(\mathbb{R}, \mathcal{M}, \lambda)$.

$\int \chi_{\mathbb{Q}\cap[0,1]} = 0$ but $\chi_{\mathbb{Q}\cap[0,1]} \neq 0$



We would like to modify $\mathcal{L}^p$ so that two functions which are equal a.e. become equal.

Define $\mathcal{N}^p \subset \mathcal{L}^p$ to be the vector space of functions that are equal a.e.

Set $L^p = \mathcal{L}^p / \mathcal{N}^p$.

We can think of an element of L^p as an equivalence class of functions.

$\|\cdot\|_p$ gives a <u>norm</u> on L^p.

not on $\mathcal{L}^p$.

A vector space V with a norm $\|\cdot\|_p$ is also a metric space where the metric is given by

$$d_p(f,g) = \|f-g\|_p.$$

We can talk about convergence in a metric space.

Convergence in L^1 is the same as convergence in mean.

$f_n \to f$ in mean $\quad \int |f_n - f|^1 d\mu \to 0$

We have created new "modes of convergence" one for every $p \in [1,\infty)$.

$f_n \to f$ in L^p if

convergence of p-th moment.

$$\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0.$$

An important property of metric spaces is completeness which is to say that every Cauchy sequence converges.

Finite dimensional normed vector spaces are always complete but for ∞ dimensional vector spaces this is a subtle property.

Thm. 3.4.1. Let $(X, \mathcal{A}, \mu)$ be a meas. space and let p satisfy $1 \le p < \infty$. Then $L^p(X, \mathcal{A}, \mu)$ is complete under the norm $\|\cdot\|_p$.

You may recall that historically the Lebesgue integral was motivated by Fourier's claim that every function f on $[-\pi, \pi]$ can be written as:

$$f(x) = a_0 + \sum_{j=1}^{\infty} b_j \sin(j \cdot x) + \sum_{j=1}^{\infty} c_j \cos(j \cdot x)$$

For which f can we write:

$$f(x) = a_0 + \sum_{j=1}^{\infty} b_j \sin(j \cdot x) + \sum_{j=1}^{\infty} c_j \cos(j \cdot x) \; ?$$

This holds for $f \in L^2$.

Write

$$f_n(x) = a_0 + \sum_{j=1}^{n} b_j \sin(j \cdot x) + \sum_{j=1}^{n} c_j \cos(j \cdot x)$$

In what sense does $f_n \to f$? Convergence will take place in the L^2 norm.

For which collections of a_0, b_j, c_k does this expression define a function.

$$a_0^2 + \sum_j b_j^2 + \sum_k c_k^2 < \infty.$$

(We have proved the hard part which is the completeness of L^2. I will leave the rest of the proof for a Fourier Analysis course.)

Completeness of L^2 requires that we use the Lebesgue integral in order to define L^2.

It is not true if we define L^2 with respect to the Riemann integral.