Let me start by saying more about the last sentence of the proof of Thm. 5.2.1 (b).

$$\boxed{\text{(b)} \int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f_x \, d\nu \right) d\mu}$$

"Applying the Monotone Convergence theorem to the integrals of simple functions gives part (b)."

We proved that $f_n \to f$ where $f_n$ are simple functions and thus

$$\int_{X \times Y} f_n \, d(\mu \times \nu) = \int_X \left( \int_Y (f_n)_x \, d\nu \right) d\mu.$$

It follows that

$$\lim_{n \to \infty} \int_{X \times Y} f_n \, d(\mu \times \nu) = \lim_{n \to \infty} \int_X \left( \int_Y (f_n)_x \, d\nu(y) \right) d\mu(x)$$

The Monotone Convergence Thm. gives

$$\lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu\times\nu) = \int_{X\times Y} f \, d(\mu\times\nu).$$

$$\boxed{\int_X \left( \int_Y f_x \, d\nu \right) d\mu}$$

It also shows that for any fixed $x_0 \in X$

$$\int_Y (f_n)_{x_0} \, d\nu(y) \longrightarrow \int_Y f_{x_0} \, d\nu(y).$$

This says that the functions

$$x \mapsto \int_Y (f_n)_x \, d\nu(y) \quad \text{converge}$$

pointwise to the function:

$$x \mapsto \int_Y f_x \, d\nu(y).$$

This sequence of functions is non-decreasing so applying the MCT once more gives:

$$\int_X \left( \int_Y (f_n)_x \, d\nu(y) \right) d\mu(x) \to \int_X \left( \int_Y f_y \, d\nu(y) \right) d\mu(x).$$

So we get:

$$\boxed{(b) \int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left( \int_Y f_x \, d\nu \right) d\mu}$$

as claimed.

Thm. 5.2.2 (Fubini's Thm.) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be $\sigma$-finite measure spaces and let $f: X \times Y \to [-\infty, +\infty]$ be an $\mathcal{A} \times \mathcal{B}$ measurable function which is $\mu \times \nu$ integrable. Then:

(a) for $\mu$-a.e. $x \in X$ the section $f_x$ is $\nu$-integrable and for $\nu$ a.e. $y \in Y$ $f^y$ is $\mu$-integrable.

(b) Define

$$I_f(x) = \begin{cases} \int_Y f_x \, d\nu & \text{if } f_x \text{ is } \nu\text{-integrable} \\ 0 & \text{otherwise} \end{cases}$$

$$J_f(y) = \begin{cases} \int_X f^y \, d\mu & \text{if } f^y \text{ is } \mu\text{-integrable} \\ 0 & \text{otherwise} \end{cases}$$

then $I_f \in \mathcal{L}^1(X, \mu)$, $J_f \in \mathcal{L}^1(Y, \nu)$.

(c) $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X I_f \, d\mu = \int_Y J_f \, d\nu$.

Proof. Consider $f^+$ and $f^-$.

Prop. 5.1.2 implies that $f_x$, $(f^+)_x$ and $(f^-)_x$ are $\mathcal{B}$-measurable.

Prop. 5.2.1 implies that the functions

$$x \mapsto \int (f^+)_x \, d\nu \text{ and } x \mapsto \int (f^-)_x \, d\nu$$

are $\mathcal{A}$-measurable.

Since $f$ is $\mu \times \nu$ integrable we have

$$\int f^+ \, d(\mu \times \nu) < \infty \quad \text{and} \quad \int f^- \, d(\mu \times \nu) < \infty.$$

Thm. 5.2.1 applied to the non-negative functions $f^+$ and $f^-$ gives:

$$\int \left( \int (f^+)_x \, d\nu(y) \right) d\mu(x) = \int f^+ \, d(\mu \times \nu) < \infty \quad \text{and}$$

$$\int \left( \int (f^-)_x \, d\nu(y) \right) d\mu(x) = \int f^- \, d(\mu \times \nu) < \infty. \qquad (1)$$

So $x \mapsto \int (f^+)_x \, d\nu(y)$ and $x \mapsto \int (f^-)_x \, d\nu(y)$ are $\mu$-integrable. $\qquad (2)$

(2) implies that the functions

$$x \mapsto \int (f^+)_x \, d\nu(y) \quad \text{and} \quad x \mapsto \int (f^-)_x \, d\nu(y)$$

are finite for $\mu$ a.e. $x$. (Cor 2.3.14).

So for $\mu$ a.e. $x$ $\int (f_x)^+ d\nu < \infty$ <u>and</u> $\int (f_x)^- d\nu < \infty$

and thus $f_x$ is $\nu$-integrable.

This is assertion (a) for $f_x$.

Let $N$ be the set of $x$ for which

$\int (f_x)^+ d\nu < \infty$ and $\int (f_x)^- d\nu < \infty$. $\mu(N) = 0$.

Outside of $N$, $I_f(x) = \int f_x \, d\nu$

$$= \int f_x^+ - f_x^- \, d\nu$$

$$= \int f_x^+ d\nu - \int f_x^- d\nu$$

Since $I_f(x)$ the difference of two $\mu$-integrable functions, by (2), $I_f(x)$ is $\mu$-integrable.

Since $I_f(x)$ takes values in $(-\infty, \infty)$
$I_f \in \mathcal{L}^1(X, \mathfrak{A}, \mu)$.

This is assertion (b) for $I_f$.

As we have seen Prop. 5.2.1 applied to the non-negative functions $f^+$ and $f^-$ gives:

$$\int f^+ \, d(\mu\times\nu) = \int \left( \int (f^+)_x \, d\nu \right) d\mu$$
$$\int f^- \, d(\mu\times\nu) = \int \left( \int (f^-)_x \, d\nu \right) d\mu \qquad (3)$$

so...

$$\int f \, d(\mu \times \nu) = \int f^+ \, d(\mu \times \nu) - \int f^- \, d(\mu \times \nu)$$

using (3)

$$= \int \left( \int (f^+)_x \, d\nu \right) d\mu(x) - \int \left( \int (f^-)_x \, d\nu \right) d\mu(x)$$

$$= \int \left( \int (f^+)_x - (f^-)_x \, d\nu \right) d\mu(x)$$

$$= \int \left( \int f_x \, d\nu \right) d\mu(x)$$

$$= \int_X I_f \, d\mu.$$

Since $I_f = \int f_x \, d\nu$ outside of a set of measure 0.

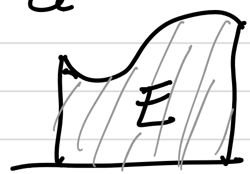
(Prop. 2.3.9)

This is assertion (c) for $I_f$.

Similar arguments give (a) for $f^Y$, (b) and (c) for $J_f$.

This completes the proof.

Example 5.3.1. Let $f: \mathbb{R} \to [0, +\infty]$ be a non-negative function.



$y = f(x)$

Let $E = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq f(x)\}$.

So $E$ is the region under the graph of $f$.

We claim that the area of $E$ is equal to the integral of $f$. This follows from Prop. 5.1.4 and the fact that $\lambda_2 = \lambda_1 \times \lambda_1$.

$$\lambda_2(E) = (\lambda_1 \times \lambda_1)(E) = \int_X \lambda_1(E_x)\, d\lambda(x) = \int_X f(x)\, d\lambda(x)$$