(Record)

At this point, after much work we have constructed outer measures λ_d^* on $\mathbb{R}^d$ and σ-algebras of λ_d^* measurable sets $\mathcal{M}$ so that the restriction of λ_d^* to $\mathcal{M}$ is countably additive.

The last step in our program of constructing a useful theory of Lebesgue measure is to identify a large class of measurable sets.

Let's start with $d=1$.

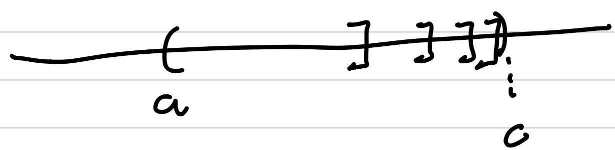
Theorem. Borel subsets of $\mathbb{R}$ are Lebesgue measurable.

Proof. The σ-algebra $\mathfrak{M}$ contains the sets $(-\infty, b]$. Since σ-algebras are closed under taking complements $\mathfrak{M}$ contains sets (b, ∞). Since σ algebras are closed under intersection $\mathfrak{M}$ contains

$$(-\infty, b] \cap (a, \infty) = (a, b].$$

Since $\mathfrak{M}$ is closed undertaking countable unions it contains

$$\bigcup_{n}^{\infty} (a, c - \tfrac{1}{n}] = (a, c).$$



Now every open set U in $\mathbb{R}$ is a countable union of open intervals so $\mathcal{M}$ contains all open sets. $\mathcal{M}$ contains all closed sets. Since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all ~~closed~~ open sets $\mathcal{M} \supset \mathcal{B}(\mathbb{R})$.

We have already established that $\mathcal{B}(\mathbb{R})$ contains your favourite sets.

We write λ for the restriction of the function λ^* to $\mathcal{M}$ and call it Lebesgue measure.

This theorem holds in $\mathbb{R}^d$.
The proof is similar but
not the same and
I will discuss it in a
posted video.

In general a measure μ is a function define on some σ-algebra $\mathcal{A}$ taking values in $[0,\infty]$ which satisfies

(1) $\mu(\emptyset)=0$.

(2) μ is countably additive:

$$\mu\Big(\bigcup_i A_i\Big)=\sum \mu(A_i)$$

when A_i are disjoint subsets of X.

We write this as:

$$(\mu, \mathcal{A}, X)$$

and call the triple a measure space.

$(\lambda_1, \mathcal{M}, \mathbb{R})$ or $(\lambda_1, \mathcal{B}(\mathbb{R}), \mathbb{R})$.

of subsets of X

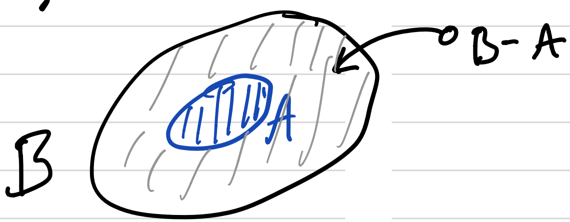
$(\lambda_d, \mathcal{M}, \mathbb{R}^d)$

Remark. Not every measure needs to come from an outer measure.

Prop. If $(\mu, \mathcal{a}, X)$ is a measure space and $A \subset B$ are in $\mathcal{a}$ then

① $\mu(A) \leq \mu(B)$ (monotonicity)

② $\mu(B-A) = \mu(B) - \mu(A)$ if $\mu(A) < \infty$.



Proof. A and $B-A$ are disjoint so

$$\mu(B) = \mu(A) + \mu(B-A)$$

by additivity.

Since $\mu(B-A) \geq 0$ we have $\mu(B) \geq \mu(A)$.

If $\mu(A) < \infty$ then $\mu(B) - \mu(A) = \mu(B-A)$.

(1.2.5)

Theorem (Measure continuity).

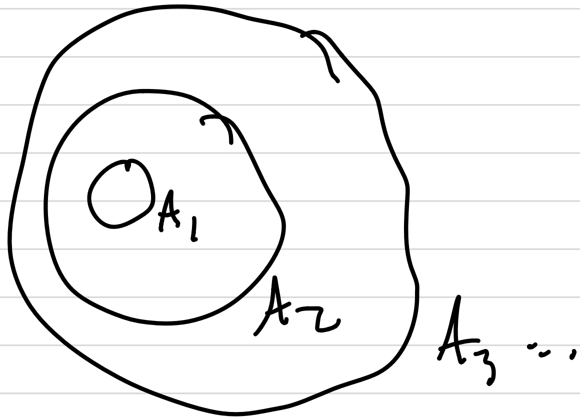
(a) If $\{A_k\}$ is an increasing sequence of sets $A_1 \subset A_2 \subset \cdots$ in $\mathcal{A}$ then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

(b) If $\{A_k\}$ is a decreasing sequence of sets in $\mathcal{A}$ and $\mu(A_1) < \infty$ then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

Proof. Suppose $\{A_k\}$ is increasing

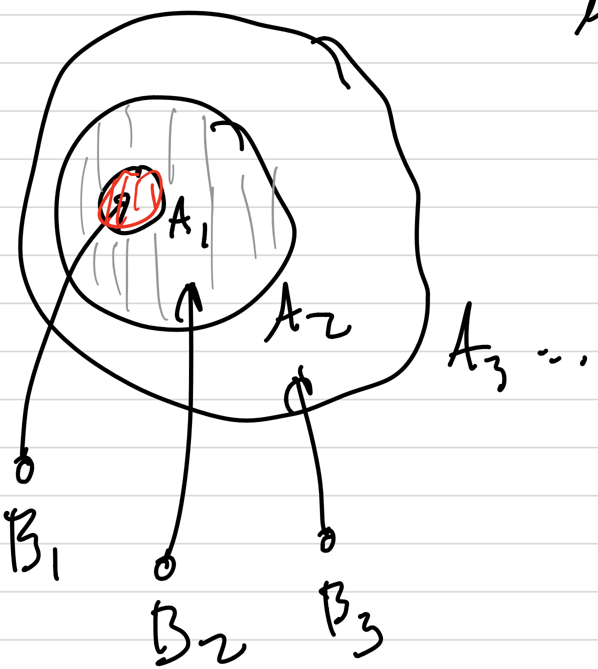


Write $B_1 = A_1$

$B_2 = A_2 - A_1$

$B_3 = A_3 - A_2$

$\vdots$



$B_1 \cup B_2 = A_2$

$B_1 \cup B_2 \cup B_3 = A_3$

$\vdots$

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right)$$

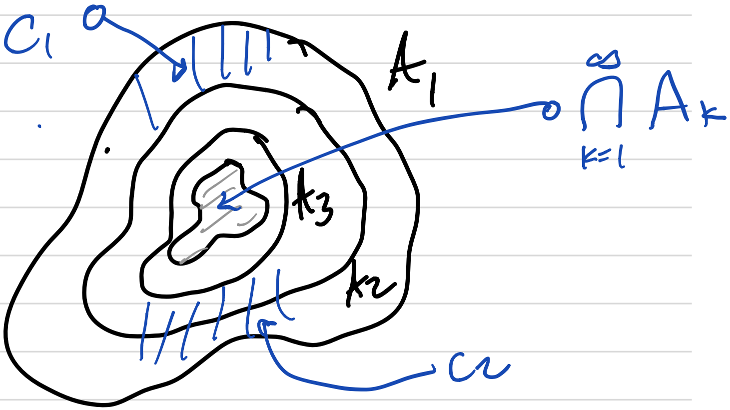
$$= \sum_{i=1}^{\infty} \mu(B_i) \quad \text{countable additivity}$$

$$= \lim_{k\to\infty} \sum_{i=1}^{k} \mu(B_i) \quad \text{def. of summation}$$

$$= \lim_{k\to\infty} \mu\left(\bigcup_{i=1}^{k} B_i\right) \quad \text{(finite) additivity}$$

$$= \lim_{k\to\infty} \mu(A_k) \quad \text{def. of } B_k.$$

(b) Now assume $\{A_k\}$ is decreasing.



Let $C_k = A_1 - A_k$ $\{C_k\}$ is increasing

and $\bigcup_{k=1}^{\infty} C_k = A_1 - \left(\bigcap_{k=1}^{\infty} A_k\right)$.

Since C_k is increasing by (1) we have:

$$\mu\left(\bigcup_{k=1}^{\infty} C_k\right) = \lim_{k\to\infty} \mu(C_k).$$

Now $\bigcup_{k=1}^{\infty} C_k = A_1 - \bigcap_{k=1}^{\infty} A_k$ so $\mu\left(\bigcup_{k=1}^{\infty} C_k\right) = \mu\left(A_1 - \bigcap_{k=1}^{\infty} A_k\right)$

$$= \mu(A_1) - \mu\left(\bigcap_{k=1}^{\infty} A_k\right).$$

Can subtract since $\mu(A_1) < \infty$.

Now $\lim_{k\to\infty} \mu(C_k) = \lim_k \mu(A_1 - A_k)$

$= \lim_k \mu(A_1) - \mu(A_k)$ $\mu(A_1) < \infty$

$= \mu(A_1) - \lim_k \mu(A_k)$

So we have

$$\mu(A_1) - \mu\left(\bigcap_{k=1} A_k\right) = \mu(A_1) - \lim_k \mu(A_k)$$

and $\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_k \mu(A_k)$

since $\mu(A_1) < \infty$.

Considering λ or λ_d.

Prop. 1.4.1 Let A be a Lebesgue measurable subset of $\mathbb{R}^d$. Then

(a) $\lambda(A) = \inf\{\lambda(U) : U \text{ is open and } A \subset U\}$

If A is bdd. then:

(b) $\lambda(A) = \sup\{\lambda(K) : K \text{ is compact and } K \subset A\}$.

Proof. Monotonicity of λ implies

$$\lambda(A) \le \inf\{\lambda(U)\}$$

$$\lambda(A) \ge \sup\{\lambda(K)\}$$

Proof that $\lambda(A) \ge \inf\{\lambda(U)\}$.

Let $\varepsilon>0$. There is a seq U_k of coord. rectangles $A \subset \bigcup_i U_i$ and $\sum_i \mathrm{vol}(U_i) \le \lambda(A)+\varepsilon$.

Let $U = \bigcup_i U_i$ then $\lambda(U) \le \sum_i \lambda(U_i) = \sum \mathrm{vol}(U_i)$

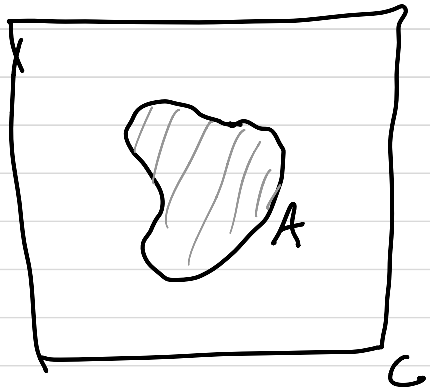
$\le \lambda(A)+\varepsilon$.

Since it is true for all ε we have:

$\lambda(U) \le \lambda(A)$.

Proof of (b). Say A is bdd.

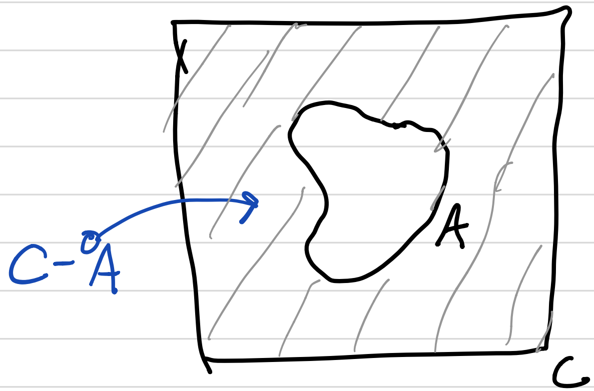
Let C be a closed coordinate rectangle that contains A.

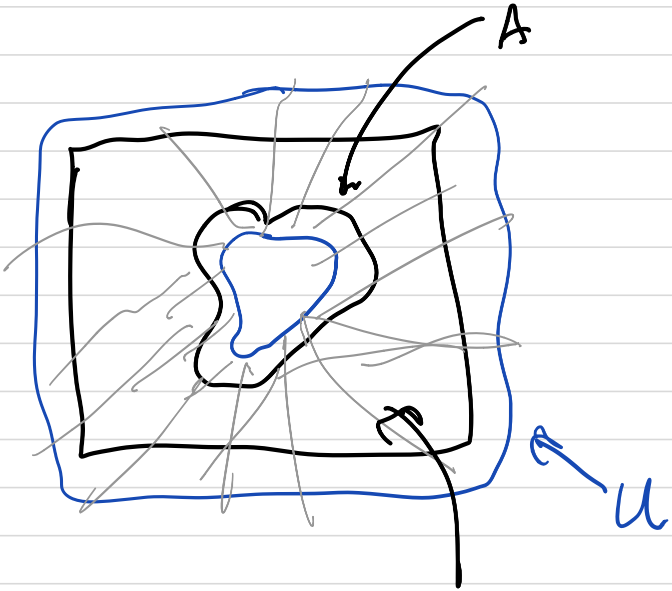


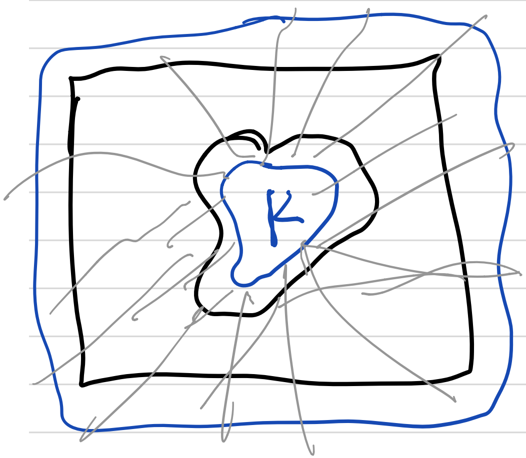
Remark. Measurability tells us that we can get an open outer approximation of $C-A$.

(Take C to be the test set.)

By taking the complement of an outer approx. of $C-A$ we get a closed inner approximation of A.







Using (a) there is an open set U containing $C-A$ with

$$\lambda(U) < \lambda(C-A) + \varepsilon \qquad (1)$$

Let $K = C - U$. $(K \subset A)$

$C \subset K \cup U$ so

$$\lambda(C) \leq \lambda(K \cup U) = \lambda(K) + \lambda(U) \quad (2).$$

We have

$$\lambda(U) < \lambda(C-A) + \varepsilon \qquad (1)$$

$$\lambda(C) \leq \lambda(K \cup U) = \lambda(K) + \lambda(U) \quad (2)$$

(1) gives: $\lambda(U) < \lambda(C) - \lambda(A) + \varepsilon$

Use (2) to replace $\lambda(C)$:

$$\lambda(U) < \lambda(K) + \lambda(U) - \lambda(A) + \varepsilon$$

So $$0 < \lambda(K) - \lambda(A) + \varepsilon$$

or $$\lambda(A) - \varepsilon < \lambda(K).$$

So A is arbitrarily well approximated from the inside by K.

Prop. If $A \subset \mathbb{R}^d$ is measurable and bounded then there are Borel sets E and F with

$$E \subset A \subset F \text{ and } \lambda(E) = \lambda(A) = \lambda(F).$$

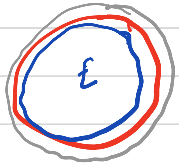
Conversely if A satisfies this "sandwich condition" and $\lambda(A) < \infty$ then A is measurable.

Proof. We can find a sequence of open sets U_n containing A with $\lambda(U_n) \longrightarrow \lambda(A)$.
We may assume that the U_n form a decreasing sequence. Let $F = \bigcap_n U_n$.

Then $\lambda(F) = \lim_{n \to \infty} \lambda(U_n) = \lambda(A)$.

Let K_n be the increasing union of a sequence of nested closed sets contained in A with $\lambda(K_n) \to \lambda(A)$. Let $E = \bigcup_{n=1}^{\infty} K_n$.

Then $\lambda(E) = \lim \lambda(K_n) = \lambda(A)$.



The converse follows from the fact that $\lambda(E) < \infty$ so $\lambda(F-E) = \lambda(F) - \lambda(E) = 0$.

In particular $A = A \cap E + A - E$

$A - E \subset F - E$ so $\lambda(A-E) = 0$.

Thus A is the union of a Borel set and a set of measure 0 so A is measurable.

Remark: An F_σ set is a countable union of closed sets.

A G_δ set is a countable intersection of open sets.

E is an F_σ and F is a G_δ.