The product σ-algebra

Given measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we want to define a product measure $\mu \times \nu$ on $X \times Y$.

A good example to keep in mind is

$X = \mathbb{R}$, $Y = \mathbb{R}$; $\mathcal{A} = \mathcal{B} = \mathcal{B}(\mathbb{R})$;

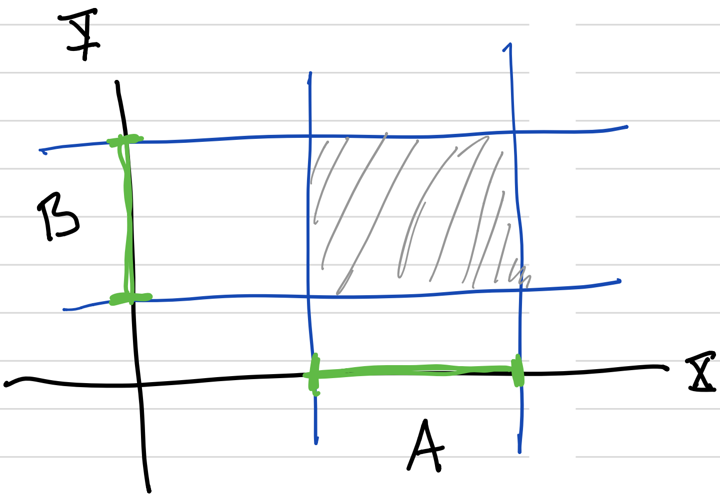
$\mu = \lambda$, $\nu = \lambda$ though as always we will work in greater generality.

As always when we are defining a measure we need to pay attention to the question of the appropriate σ-algebra on which to define it.

So we start by defining a product σ-algebra.

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces.

A measurable rectangle is a set $A \times B \subset X \times Y$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.



$A \in \mathcal{A}$, $B \in \mathcal{B}$

Definition. Given $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ the product σ-algebra is defined to be the σ-algebra generated by the collection of all measurable rectangles. We write it as: $\underline{\mathcal{A} \times \mathcal{B}}$. (Cohn p. 143)

Example 5.1.1. $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

$\mathcal{B}(\mathbb{R}^2)$ is generated by sets $(a,b] \times (c,d]$ (Prop. 1.1.5).

$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ is generated by the larger class of sets $A \times B$ where A, B are Borel and (despite my pictures) a Borel set in $\mathbb{R}$ can be much more complicated than an interval (think of a Cantor set).

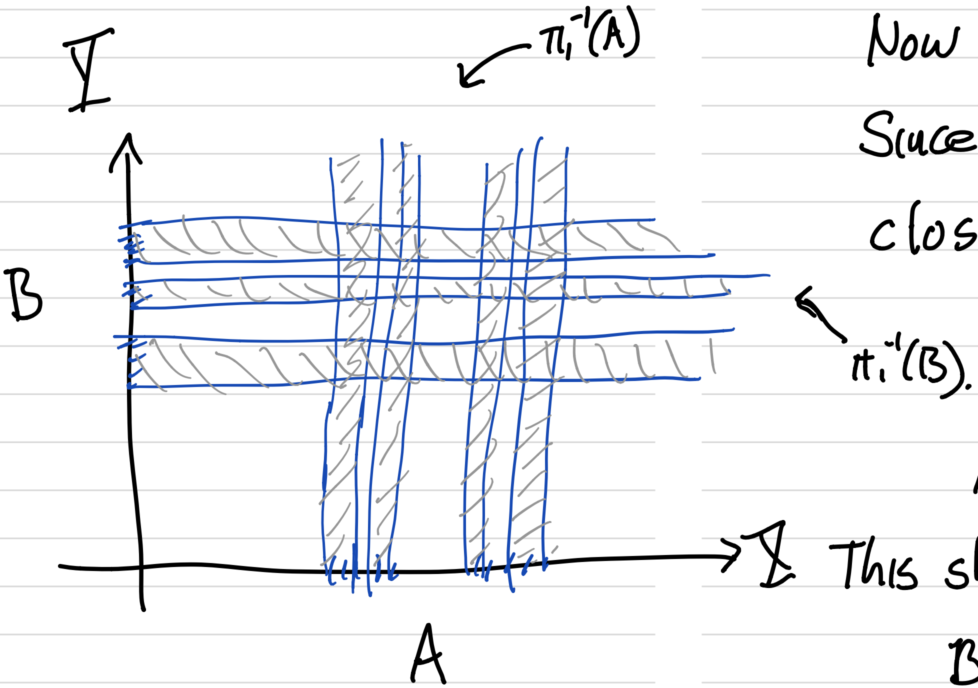
Since the generating class for $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ contains that for $\mathcal{B}(\mathbb{R}^2)$ we have $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \supset \mathcal{B}(\mathbb{R}^2)$.

To prove the opposite containment we need to show $A \times B \subset \mathcal{B}(\mathbb{R}^2)$ where $A, B \in \mathcal{B}(\mathbb{R})$.

Let π_1 and π_2 be the coordinate projections from $\mathbb{R}^2$ to $\mathbb{R}$: $\pi_1(x,y) = x$, $\pi_2(x,y) = y$.

Since π_1, π_2 are continuous they are Borel (2.1.2(a)).

It follows that $\pi_1^{-1}(A) \in \mathcal{B}(\mathbb{R}^2)$, $\pi_2^{-1}(B) \in \mathcal{B}(\mathbb{R}^2)$.



Now $A \times B = \pi_1^{-1}(A) \cap \pi_1^{-1}(B)$.

Since σ-algebras are closed under intersection

$$A \times B \in \mathcal{B}(\mathbb{R}^2).$$

This shows

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R}^2).$$

We conclude that:

$$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2).$$

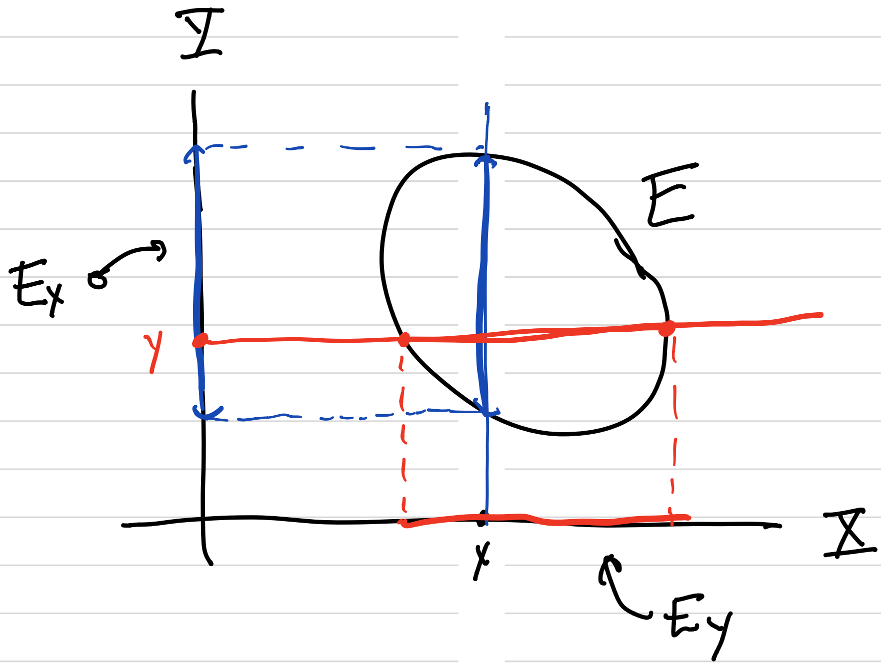
One of the tools we have for analysing a set $E \subset X \times Y$ is its collection of slices:

$$E_x = \{ y \in Y : (x,y) \in E \}$$

$$E^y = \{ x \in X : (x,y) \in E \}$$

We can think of these as "vertical" and "horizontal" slices.

Note that if we define $\iota_x : Y \to X \times Y$ by $\iota_x(y) = (x,y)$ then $E_x = \iota_x^{-1}(E)$.



We will use these slices when we construct the product measure.

If we have a function $f: X \times Y \to [-\infty, +\infty]$, we can consider its restrictions to slices:

For $x \in X$ we define $f_x : Y \to [-\infty, +\infty]$ by $f_x(y) = f(x, y)$.

For $y \in Y$ we define $f^y : Y \to [-\infty, +\infty]$ by $f^y(x) = f(x, y)$. "Sections"

These play a role in the iterated integral.

Lemma 5.1.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces.

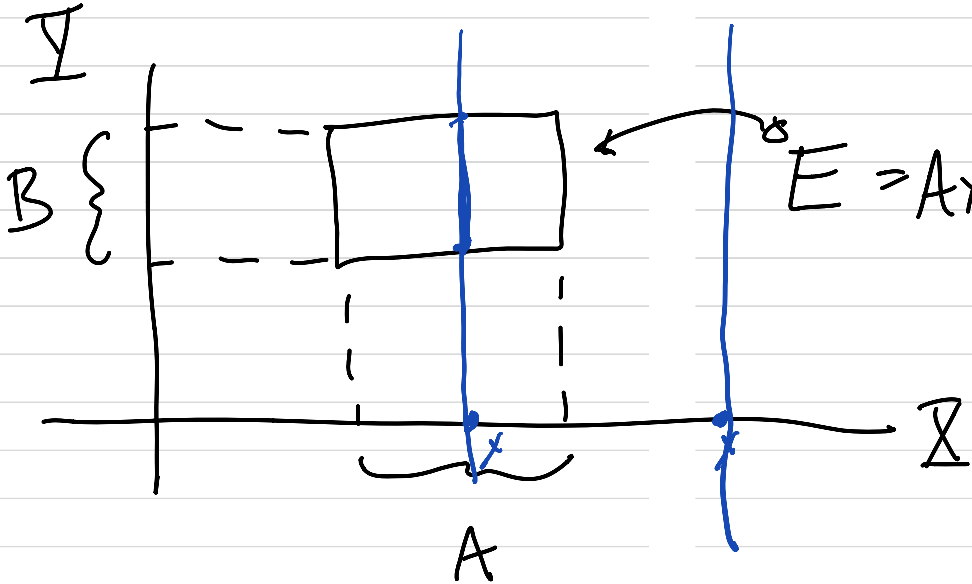
(a) Say $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $E_x \in \mathcal{B}$ for each $x \in X$.

(b) If f is an extended $\mathcal{A} \times \mathcal{B}$ measurable function then each section f_x is $\mathcal{B}$ measurable and each section f^y is $\mathcal{A}$ measurable.

Proof of (a). Say $x \in X$. Let $\mathcal{F}$ be the collection of all subsets E of $X \times Y$ for which E_x belongs to $\mathcal{B}$.

Let $\mathcal{R}$ be the collection of measurable rectangles.

$\mathcal{F}$ contains all measurable rectangles since $(A \times B)_x$ is either B (if $x \in A$) or $\emptyset$ (if $x \notin A$).



$E = A \times B$

$A \in \mathcal{A}$

$B \in \mathcal{B}$,

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Let's check that $\mathcal{F}$ is a σ-algebra.

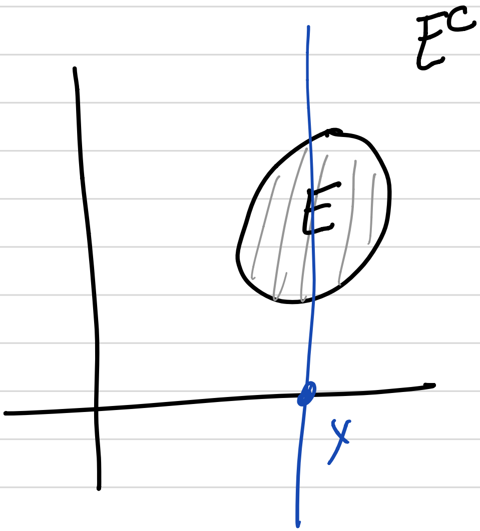
<u>General</u> <u>remark</u>. If f is a function then taking <u>inverse images</u> "commutes" with taking unions, intersections and complements. thus $f^{-1}(A^c) = (f^{-1}(A))^c$, $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$ etc.
(By contrast taking <u>images</u> of sets is not well behaved in this way.)

We can put E_x in this framework since $E_x = \iota_x^{-1}(E)$, the operation of "taking slices" commutes with unions, intersections and complements.

$\mathcal{F}$ is closed under taking complements:

If $E \in \mathcal{F}$ then $E_x \in \mathcal{B}$ so $(E_x)^c \in \mathcal{B}$

but $(E_x)^c = (E^c)_x$ so $(E^c)_x \in \mathcal{B}$ and $E^c \in \mathcal{F}$.

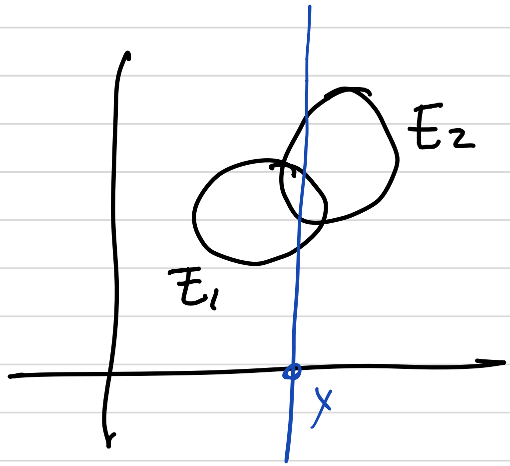


$\mathfrak{F}$ is closed under taking countable unions:

If $E_n \in \mathfrak{F}$ then $(E_n)_x \in \mathcal{B}$ so $\bigcup_n (E_n)_x \in \mathcal{B}$

but $\bigcup_n (E_n)_x = \left(\bigcup_n E_n\right)_x$ so $\left(\bigcup_n E_n\right)_x \in \mathcal{B}$ and

$\bigcup_n E_n \in \mathfrak{F}$.



Let us write $\mathcal{R}$ for the collection of all measurable rectangles. Since $\mathcal{F}$ is a σ-algebra that contains $\mathcal{R}$ $\mathcal{F}$ contains the smallest σ-algebra containing $\mathcal{R}$ which is exactly $\mathcal{A} \times \mathcal{B}$.

This means that if $E \in \mathcal{A} \times \mathcal{B}$ then $E \in \mathcal{F}$ so E_x is in $\mathcal{B}$.

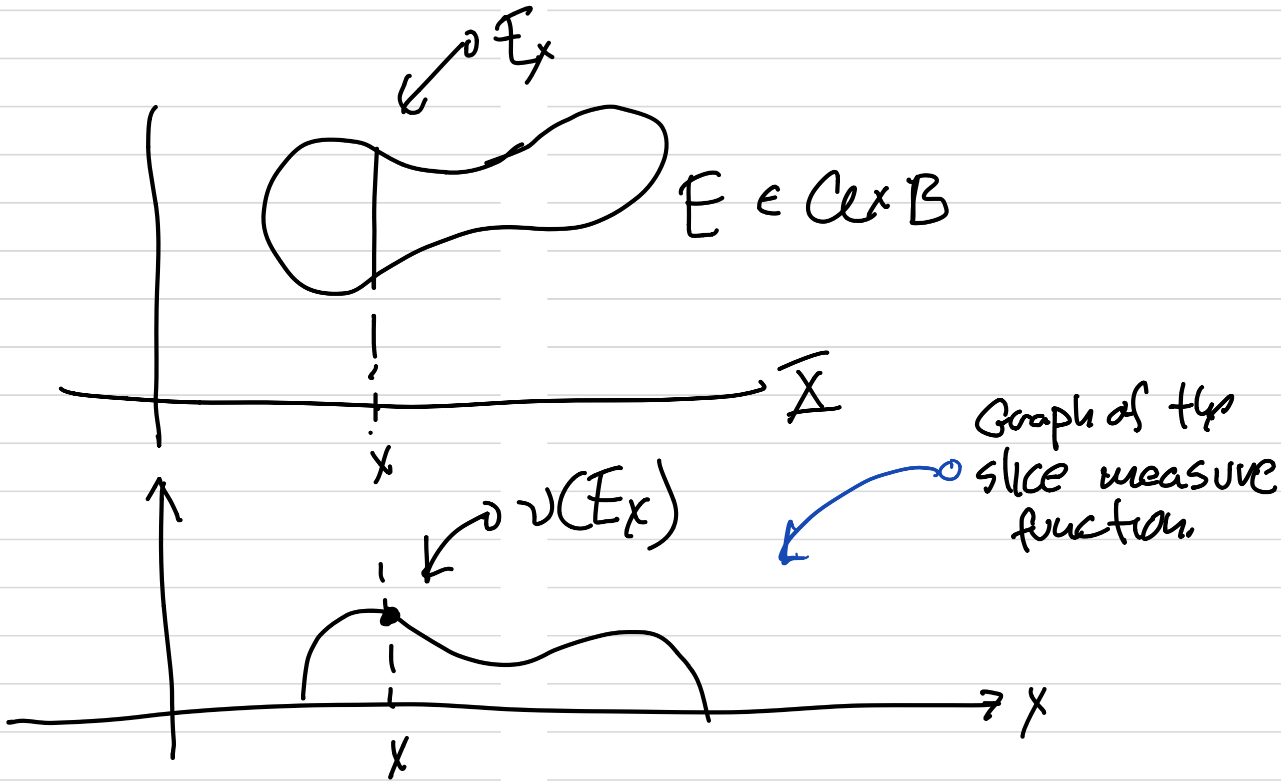
The proof of (b) follows from (a) since, for D a Borel set in $\mathbb{R}$ if f is $\mathcal{A}\times\mathcal{B}$ measurable then $f^{-1}(D) \in \mathcal{A}\times\mathcal{B}$ and by (a) $(f^{-1}(D))_x \in \mathcal{B}$.
But $(f^{-1}(D))_x = (f_x)^{-1}(D)$ so $(f_x)^{-1}(D) \in \mathcal{B}$.
Hence f_x is measurable.

By the same reasoning f^y is measurable.
This completes the proof of the Lemma.

Now say that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces.

According to the Lemma, E_x is in $\mathcal{B}$ so the function $x \mapsto \nu(E_x)$ is well defined.

Call $x \mapsto \nu(E_x)$ the "slice measure function".

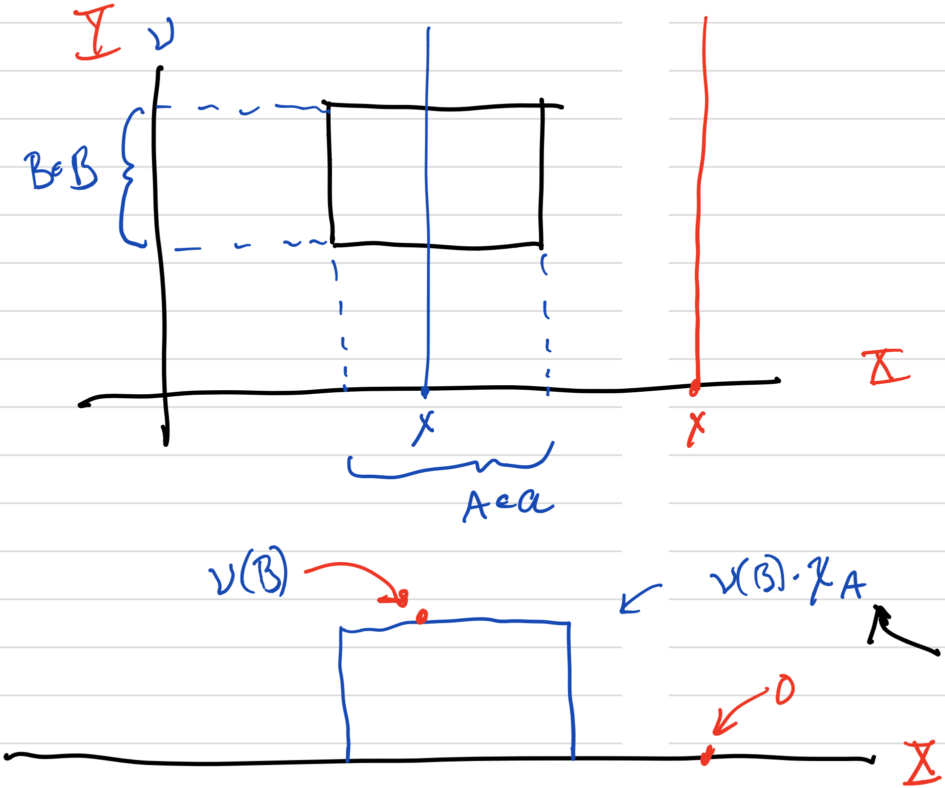


What can we say about this function?

Prop. 5.1.3(a) Let $(\mathbb{X}, \mathcal{A}, \mu)$ and $(\mathbb{Y}, \mathcal{B}, \nu)$ be finite measure spaces.
If $E \subset \mathbb{X} \times \mathbb{Y}$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function.

Proof strategy: Let $\mathcal{F}$ be the collection of sets E for which the slice measure function is $\mathcal{A}$-measurable. We show that $\mathcal{F}$ contains $\mathcal{R}$ and $\mathcal{F}$ is a d-system.
We will finish by invoking Dynkin's Thm.

$\mathcal{F}$ contains the collection of measurable rectangles.



slice measure function.

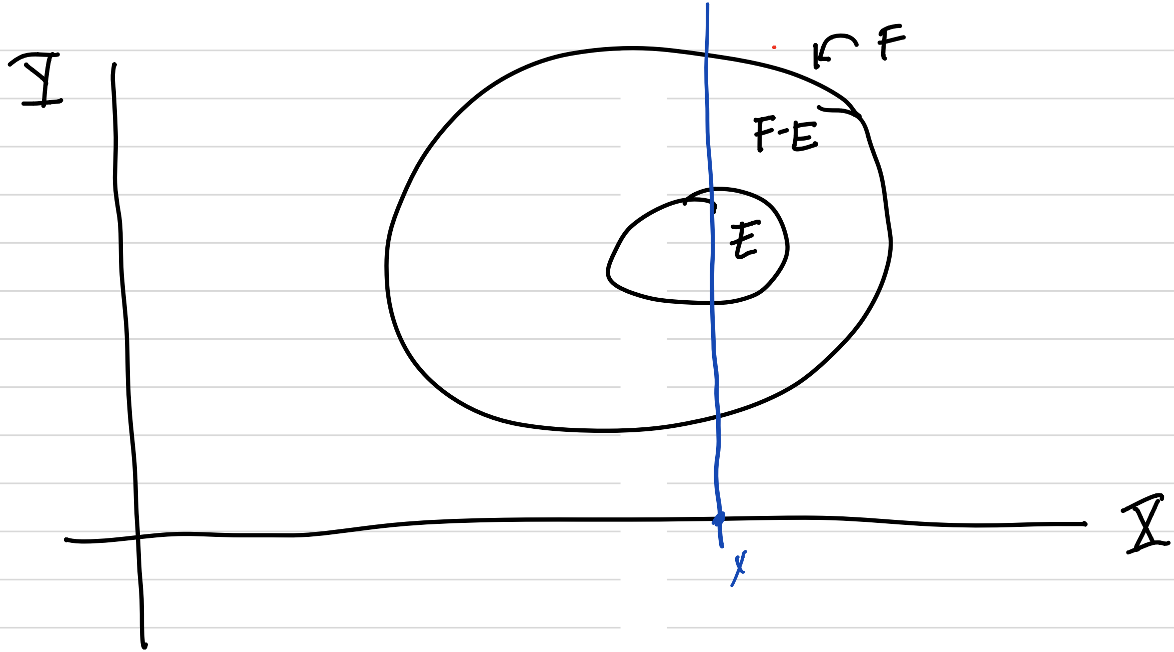
So the slice measure function is a measurable: $A \in \mathcal{A}$ so χ_A is $\mathcal{A}$-measurable and $\nu(B) \cdot \chi_A$ is $\mathcal{A}$-measurable.

Thus $A \times B \in \mathcal{A}$. In particolar $X \times Y \in \mathcal{A}$.

This is the first property of a d-system.

The second property is that if $E, F \in \mathcal{A}$ and $E \subset F$ then $F - E \in \mathcal{A}$.

Say $E, F \in \mathcal{A}$. The slice measure function for $F - E$ is $\nu((F-E)_x) = \nu(F_x - E_x) = \nu(F_x) - \nu(E_x)$. Thus it is the difference of functions which are $\mathcal{a}$-measurable so it is $\mathcal{a}$-measurable.



Y
F
F-E
E
X
x

If E_n is an increasing seq. of sets in $\mathcal{A}$ then

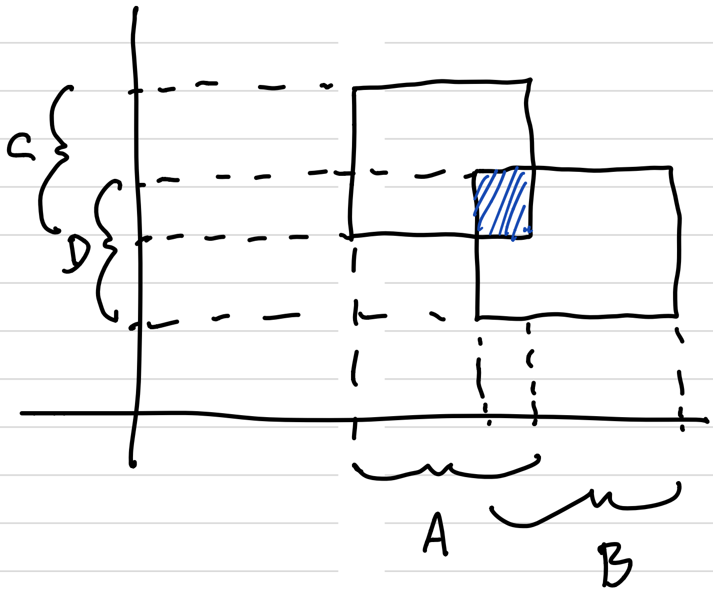
$$\nu\left(\left(\bigcup_n E_n\right)_x\right) = \nu\left(\bigcup_n (E_n)_x\right)$$

$$\lim_n \nu\left((E_n)_x\right).$$

Since the limit of $\mathcal{A}$-measurable functions is a measurable $\bigcup_n E_n \in \mathcal{A}$.

Thus $\mathcal{A}$ is a d-system containing $\mathcal{R}$.

$\mathcal{R}$ is a π-system:



The collect. of meas. rects. is a π system

$$(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D).$$

Recall:

Thm. 1.6.2. (Dynkin) Let X be a set and let $\mathcal{R}$ be a π-system on X. Then the σ-algebra generated by $\mathcal{R}$ coincides with the d-system generated by $\mathcal{R}$.

We complete the proof of the Proposition. $\mathcal{F}$ is a d-system containing the π-system $\mathcal{R}$ so $\mathcal{F}$ contains the σ-algebra generated by $\mathcal{R}$ which is $\mathcal{A} \times \mathcal{B}$. Thus the slice measure function is measurable for every $E \in \mathcal{A} \times \mathcal{B}$.