Measure Theory: Exercises (not for credit)

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Questions 5 and 6 are long and not super important so only do them if you are enjoying doing exercise sheets!

Question 1. Suppose that f is a measurable function from $\mathbb{R} \to \mathbb{R}$, show that g defined by $g(x) = f(x+\tau)$ is also measurable, where τ is a fixed real number.

Question 2. Show the restriction of measure makes sense. That is to say if $(E, \mathcal{E}, \mu)$ is a measure space and $A \in \mathcal{E}$ then show that $\mathcal{E}_A = \{B \in \mathcal{E} : B \subset A\}$ is a σ-algebra and $\mu_A = \mu|_{\mathcal{E}_A}$ is a measure. Show further that if g is a non-negative measurable function then $\mu_A(g) = \mu(g1_A)$.

Question 3. Show that the function $sin(x)/x$ is not Lebesgue integrable over $[1, \infty]$ but the limit as $n \to \infty$ of

$$\int_1^n \sin(x)/x \mathrm{d}x.$$

Question 4. Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measureable spaces. We call K a kernel on if for each $x \in E$ the function $A \mapsto K(x, A)$ is a measure on $(F, \mathcal{F})$, and for every $A \in \mathcal{F}$ the function $x \mapsto K(x, A)$ is a measurable function on $(E, \mathcal{E})$. Suppose that K is a kernel, μ is a measure on $(E, \mathcal{E})$ and f is a $[0, \infty]$ valued measurable function on $(F, \mathcal{F})$. Show that

- $A \mapsto \int K(x, A)\mu(\mathrm{d}x)$ is a measure on $(F, \mathcal{F})$,
- $x \mapsto \int f(y)K(x, \mathrm{d}y)$ is a measurable function on $(E, \mathcal{E})$
- If ν is the measure defined by $\nu(A) = \int K(x, A)\mu(\mathrm{d}x)$ then $\nu(f) = \int \int f(y)K(x, \mathrm{d}y)\mu(\mathrm{d}x)$.

Question 5. Recall in the last exercise sheet we constructed the devils staircase function $f : [0, 1] \to [0, 1]$ which is continuous and non-decreasing and flat everywhere except the Cantor set. Define a function by

$$g(y) = \inf\{x \in [0, 1] : f(x) = y\},$$

such a function is well defined since as f is continuous the intermediate value theorem shows there must be at least one x such that $f(x) = y$.

- Show that the range of g is contained inside the Cantor set.
- Show that $f(g(y)) = y$ and that hence g is injective.
- Let A be a non-Lebesgue measurable subset of $[0, 1]$ (such as was constructed by Vitali) show that $B = g(A)$ is Lebesgue measurable.
- Show that B is not Borel measurable.
- Show that $h(x) = 1_B(x)$ is Riemann integrable (note this shows the existence of a function that is Riemann integrable and not Borel measurable).

Question 6.

- Construct a sequence of set by $K_1 = [0,1]$ and then $K_2 = [0,1/3] \cup [2/3,1]$ then instead of removing the middle thirds as in the construction of the Cantor set to construct K_n we remove the middle α_n proportion of every interval so that $\lambda(K_n) = (1-\alpha_n)\lambda(K_{n-1})$. Then for $\beta \in (0,1)$ choose $\alpha_n = 1 - \beta^{2^{-n}}$, and show that $\bigcap_n K_n$ is a closed set with no interior points and $\lambda(\bigcup_n K_n) = \beta$.

- Now assume (you can prove this fairly easily but its not the point of the question) that you can find a function f_n which is cotinuous and takes values in $[0,1]$ such that $f_n = 0$ on K_n and 1 on K_{n-1}^c. Show that f_n is an increasing sequence of functions which converges to 1_U where $U = (\bigcap_n K_n)^c$.

- Show that 1_U is not Riemann integrable.