How to construct measures using measurable functions

<u>Def^n</u> Image measure:

Suppose $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces

$f: E \to F$ is measurable

and μ is a measure on $(E, \mathcal{E})$

then ν defined by $\nu(A) = \mu(f^{-1}(A))$ is a measure on $(F, \mathcal{F})$ and we call it the image of μ under f.

Applying this to $\mathbb{R}$ – WARNING (this doesn't work in $\mathbb{R}^d$)

<u>Technical lemma</u>

Suppose $g: \mathbb{R} \to \mathbb{R}$ is non-constant, right-continuous and non-decreasing (implies it measurable)

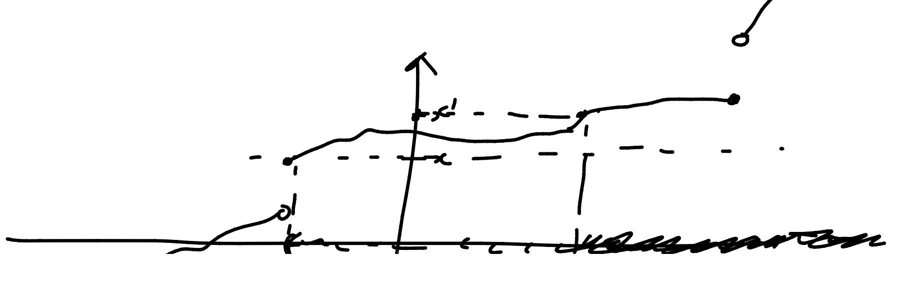
then define $g(-\infty) = \lim_{x \to -\infty} g(x)$ $\quad g(\infty) = \lim_{x \to \infty} g(x)$

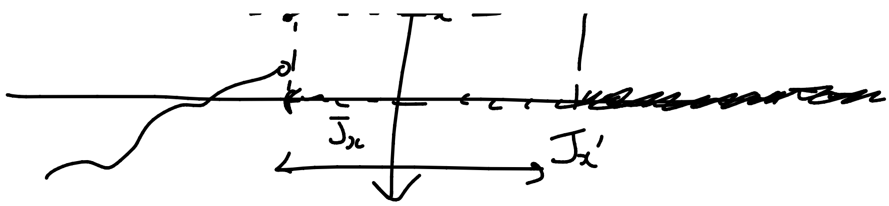
$I = (g(-\infty), g(\infty))$

then we define $f: I \to \mathbb{R}$ by

$f(x) = \inf_y \{y : g(y) \geq x\}$. The f is a partial inverse to g.

f is left continuous and non-decreasing.





~~Pf~~ Define $J_x = \{y : g(y) \geq x\}$ for $x \in I$

By the def^n of I we know J_x is non-empty (this shows f is well-defined $f(x) = \inf(J_x)$)

J_x is not the whole of $\mathbb{R}$ again by def^n of I

Since $g(y)$ is non increasing J_x is a half line.

If $y \in J_x$ and $y' \geq y$ then $y' \in J_x$

As g is right continuous if $y_n \downarrow y$ and $y_n \in J_x \; \forall n$ then $y \in J_x$ so J_x is a closed half line

$J_x = [y, \infty)$ for some y.

Because of this we can see (since g is non-decreasing)

If $x \leq x'$ then $J_x \supseteq J_{x'}$

so $f(x) \leq f(x')$

Also if $x_n \uparrow x$ then $J_x = \bigcap_n J_{x_n}$ so $f(x_n) \to f(x)$

so f is left-continuous.

<u>Theorem</u> Let g be a right-continuous, non-constant non-decreasing function from $\mathbb{R} \to \mathbb{R}$ then $\exists!$ Radon measure, dg, on $\mathbb{R}$ s.t $dg((a,b]) = g(b) - g(a)$ for each half open interval $(a,b]$. dg can give weight to the singleton $\{a\}$ if g has a jump

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We call this the Lebesgue-Stetjles measure associated with g. Furthermore every Radon measure on $\mathbb{R}$ can be represented this way.

Pf/ Define I and f as in the Lemma then set dg to be the image measure of Lebesgue measure on I under f.

$$dg((a,b]) = \lambda(\{ f^{-1}((a,b]) \}) = \lambda(\{x : a < f(x) \leq b\})$$

so then if $f(x) > a$ then $x > g(a)$
and if $f(x) \leq b$ then $x \leq g(b)$

so $$dg((a,b]) = \lambda((g(a), g(b)]) = g(b) - g(a)$$

Uniqueness follows either by the same proof as for Lebesgue measure.

If ν is a Radon measure

set $$g(y) = \nu((0,y]) \quad \text{for } y \geq 0$$
$$g(y) = \nu((y,0]) \quad \text{for } y < 0$$

then $\nu = dg$ by uniqueness of measures give the right measure to each half open interval.