Let $(\mathfrak{X}, \mathfrak{a})$ be a measurable space.

If $A \subset \mathfrak{X}$ and $A \in \mathfrak{a}$ we will call A an $\mathfrak{a}$-measurable set or, when there is no danger of confusion, we will call A a measurable set.

The example to keep in mind is the case $\mathbb{X} = \mathbb{R}$ where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable sets.

Another possible example would be

$\mathbb{X} = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$.

Proposition 2.1.1 Let $(X, \mathcal{A})$ be a measurable space and let $A \in \mathcal{A}$. For a function $f: A \to [-\infty, \infty]$ the following conditions are equivalent:

(a) for each $t \in \mathbb{R}$ $\{x \in A : f(x) \leq t\} \in \mathcal{A}$

(b) for each $t \in \mathbb{R}$ $\{x \in A : f(x) < t\} \in \mathcal{A}$

(c) for each $t \in \mathbb{R}$ $\{x \in A : f(x) \geq t\} \in \mathcal{A}$

(d) for each $t \in \mathbb{R}$ $\{x \in A : f(x) > t\} \in \mathcal{A}$.

Remark on the hypotheses in Prop. 2.1.1.

We are really interested in real valued functions on X. It is useful to add $+\infty$ and $-\infty$ as possible values when we are talking about order properties. This helps us deal with monotone increasing and decreasing sequences for example.

It is also convenient for technical reasons to allow our functions to be defined on subsets of X that is why we introduce the set A:

(a) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) \leq t\} \in \mathcal{A}$

(b) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) < t\} \in \mathcal{A}$

(c) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) \geq t\} \in \mathcal{A}$

(d) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) > t\} \in \mathcal{A}$.

Proof of Proposition.

(a) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) \le t\} \in \mathcal{A}$

(b) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) < t\} \in \mathcal{A}$

$$\underbrace{\{x \in A : f(x) < t\}}_{\text{want to show this is in } A} = \bigcup_n \underbrace{\{x \in A : f(x) \le t - \tfrac{1}{n}\}}_{\text{in } \mathcal{A} \text{ by hypothesis}}$$

so $(a) \Rightarrow (b)$.

(b) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) < t\} \in \mathcal{C}$

(c) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) \geq t\} \in \mathcal{C}$

$$\{x \in A : f(x) \geq t\} = \underbrace{A - \{x \in A : f(x) < t\}}_{\text{in } \mathcal{C} \text{ by hyp.}}$$

so (b) $\Rightarrow$ (c).

(c) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) \geq t\} \in \mathcal{C}$

(d) for each $t \in \mathbb{R}$ $\quad \{x \in A : f(x) > t\} \in \mathcal{C}$.

$$\{x \in A : f(x) > t\} = \bigcup_n \underbrace{\{x \in A : f(x) \geq t + \tfrac{1}{n}\}}_{\text{in } \mathcal{C} \text{ by hyp}}$$

so (c) $\Rightarrow$ (d).

(d) for each $t \in \mathbb{R}$ $\{x \in A : f(x) > t\} \in \mathcal{a}$.

(a) for each $t \in \mathbb{R}$ $\{x \in A : f(x) \le t\} \in \mathcal{a}$

$$\{x \in A : f(x) \le t\} = A - \{x \in A : f(x) > t\}$$

so (d) $\Rightarrow$ (a).

A function which satisfies one of these conditions is called an $\mathcal{A}$-measurable function. (When there is no danger of confusion we will call it a measurable function.)

When $\mathcal{A} = \mathcal{M}$ we say Lebesgue measurable.
$\mathcal{A} = \mathcal{B}(\mathbb{R})$ Borel

Example. Constant functions are $\mathcal{A}$-measurable for any measurable space $(X, \mathcal{A})$.

Prop. 2.1.6 Let f and g be $[0, +\infty]$ valued measurable functions on A and let $\alpha \geq 0$. Then αf and $f+g$ are measurable.

Proof.

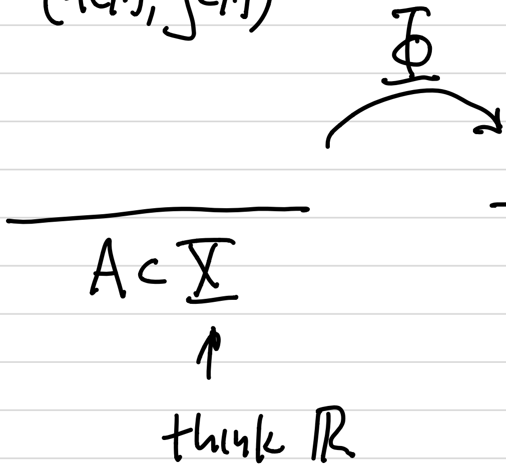
αf is easy.

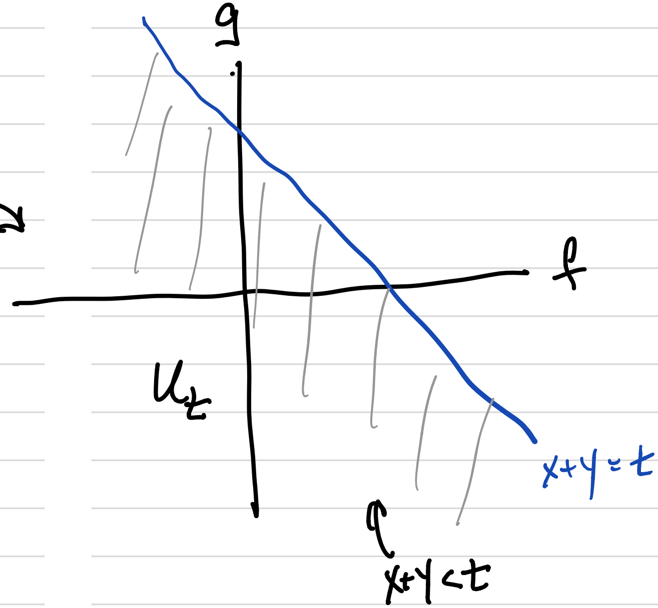
$$\underbrace{\{x \in A : \alpha f(x) < t\}}_{\text{want to prove that this set is measurable.}} = \underbrace{\{x \in A : f(x) < \tfrac{t}{\alpha}\}}_{\text{measurable by hypothesis}}$$

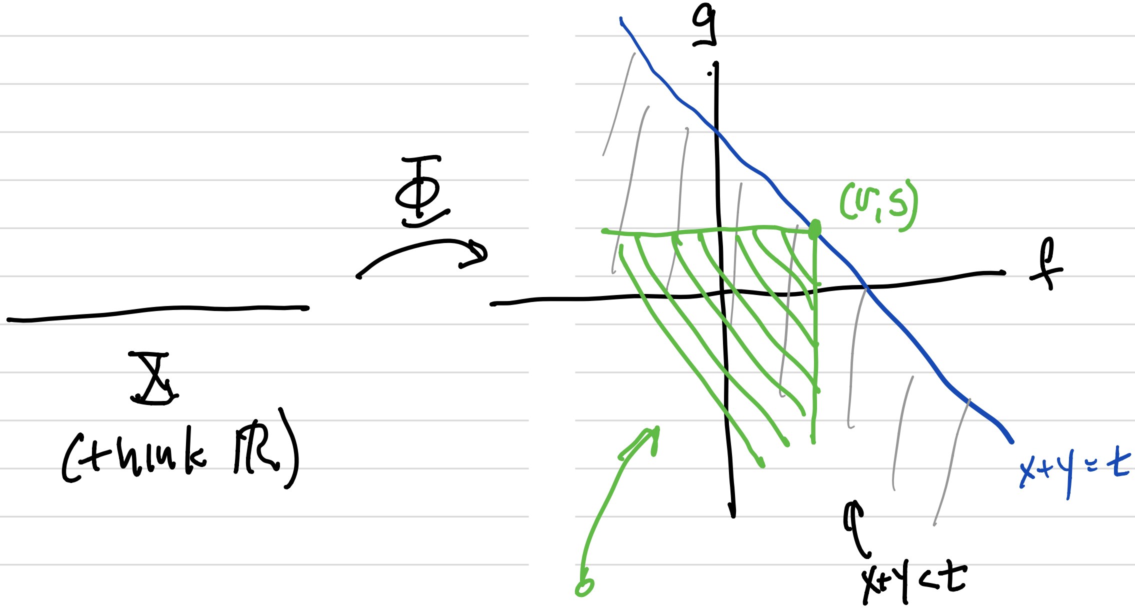
$f+g$: We want to show that the set $\{x \in A : f(x)+g(x) < t\}$ is in $\mathcal{A}$.

$$\{x \in A : f(x)+g(x) < t\} = \Phi^{-1}\big(\underbrace{(x,y) : x+y < t}_{U_t}\big)$$

$$\Phi(x) = (f(x), g(x))$$







$$\{(x,y): x<r\} \cap \{(x,y): y<s\}$$

Clearly $\{(x,y): x+y<t\}$ is the union of sets $\{(x,y): x<r\} \cap \{(x,y): y<s\}$ for different (r,s) with $r+s=t$.

It follows that $\{x \in A : f(x)+g(x)<t\}$ is the union of sets $\{x \in A : f(x)<r\} \cap \{x \in A : g(x)<s\}$ and each of these sets is $\mathcal{A}$-measurable.

Of course $\mathcal{A}$ is only closed under countable unions. To deal with this we can require r to be rational and set $s = t - r$.

So

$$\bigcup_{r \in \mathbb{Q}} \{x \in A : f(x) < r\} \cap \{x \in A : g(x) < t - r\}$$

is in $\mathcal{A}$.

Prop. 2.1.6 Let f and g be $[0, +\infty]$ valued measurable functions on A and let $\alpha \geq 0$. Then αf and $f+g$ are measurable.

Prop. 2.1.7 Let f and g be real valued measurable functions on A and let $\alpha \in \mathbb{R}$. Then (a) αf, (b) $f+g$, (c) $f-g$, (d) $f \cdot g$ and (e) f/g are measurable (where the domain of f/g is $\{x \in A : g(x) \neq 0\}$).

Proof. ⓐ αf.

In analysing αf it is now possible that $\alpha < 0$. In this case:

switch the sign

$$\underbrace{\{x \in A : \alpha f(x) < t\}}_{\text{want to prove that this set is measurable.}} = \underbrace{\{x \in A : f(x) > \tfrac{t}{\alpha}\}}_{\text{measurable by hypothesis}}$$

ⓑ $f+g$

The proof of the case $f+g$ is the same as in the non-negative case.

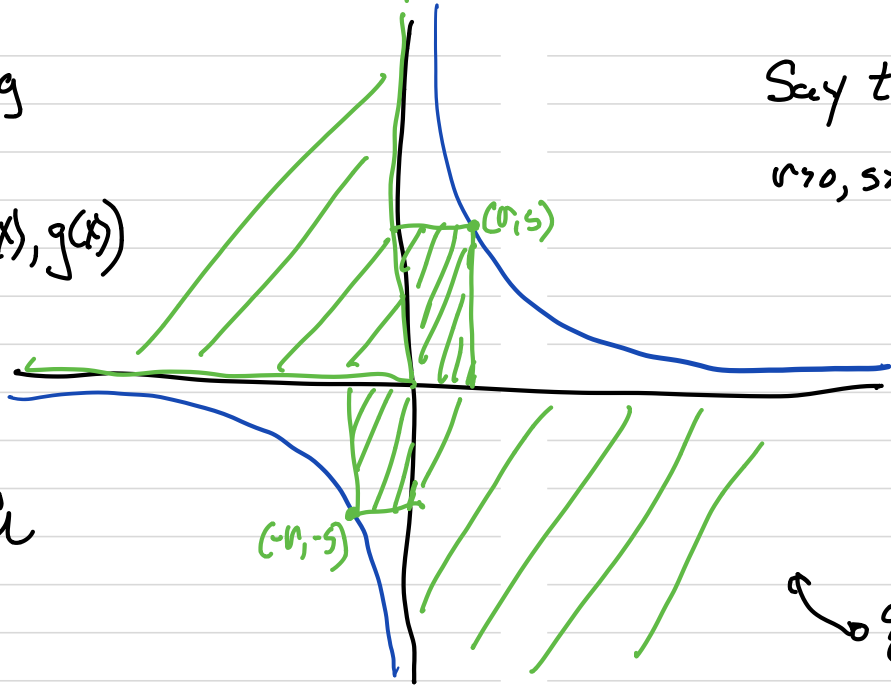
ⓒ $f-g$ The case of $f-g$ follows from ⓐ+ⓑ.

(d) $f \cdot g$

$\Phi(x) = (f(x), g(x))$

To show:

$\Phi^{-1}(U_t) \in \mathcal{A}$



Say $t > 0$.

$r > 0$, $s > 0$.

$rs = t$

$s = \frac{t}{r}$

$\{(x, y) : xy < t\} = U_t$

$\{(x, y) : 0 \le x < r, 0 \le y < s\}$ ✓

$\{(x, y) : x \le 0, y \ge 0\}$ ✓

$\{(x, y) : -r \le x \le 0, -s \le y \le 0\}$

$\{(x, y) : x \ge 0, y \le 0\}$

require $r \in \mathbb{Q}$

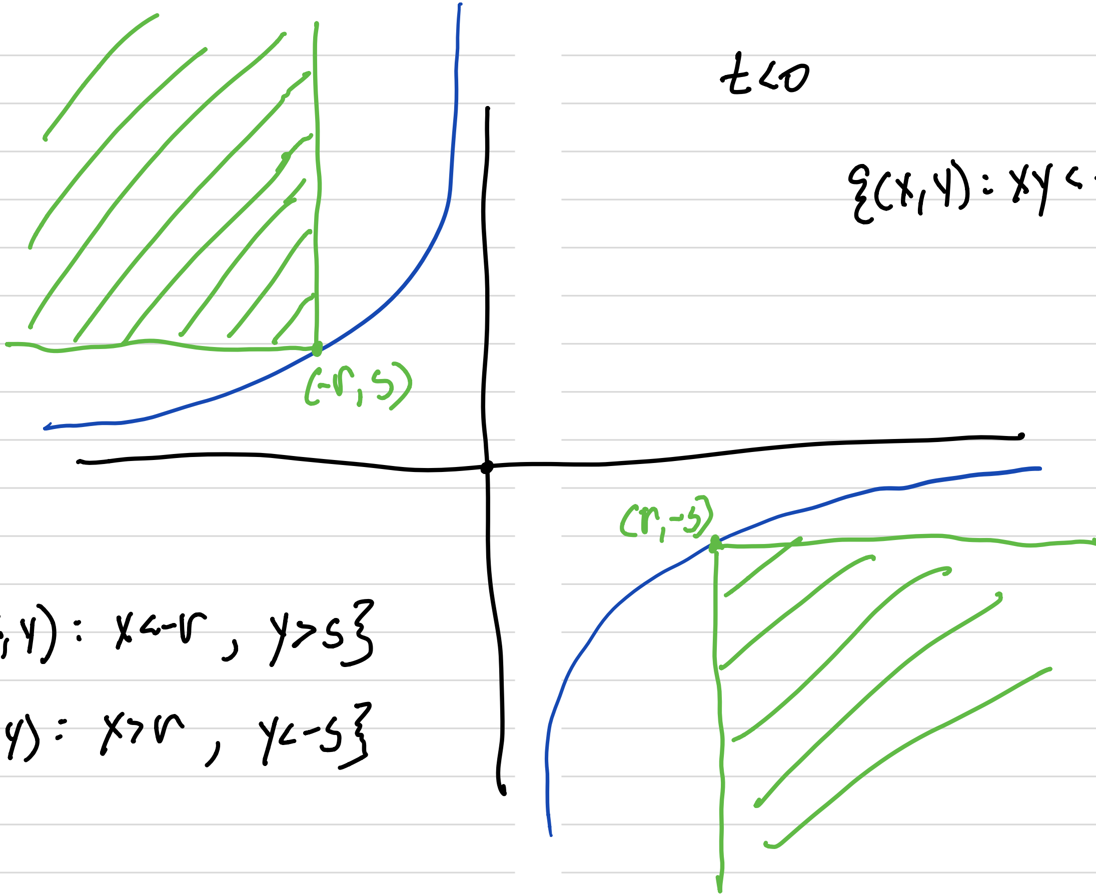
$\Phi^{-1}\left(\{(x, y) : 0 \le x < r, 0 \le y < \frac{t}{r}\}\right)$

$= \{x \in A : f(x) \ge 0\} \cap \{x \in A : f(x) < r\} \cap$

$\{x \in A : g(x) \ge 0\} \cap \{x \in A : g(x) < \frac{t}{r}\}$

$t < 0$

$\{(x,y) : xy < t\} = U_t$



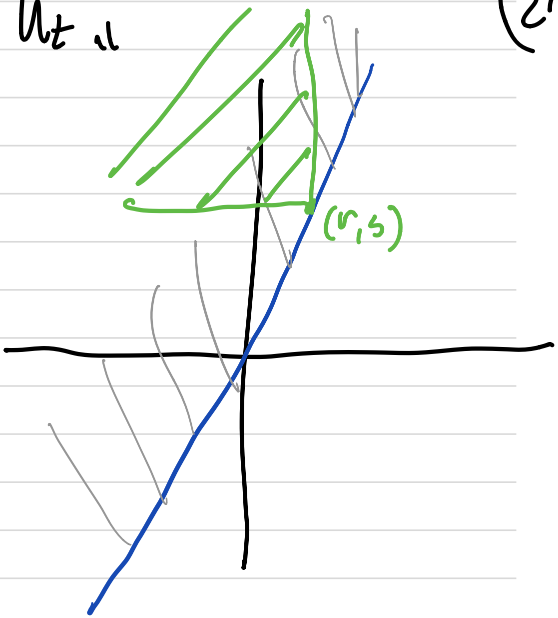
$\{(x,y) : x < -r,\ y > s\}$

$\{(x,y) : x > r,\ y < -s\}$

(e) f/g Need to check $\{x\in A : g(x)\neq 0\}\in \mathcal{A}$.

$$\left\{x\in A : \frac{f(x)}{g(x)} < t\right\} = \Big(\{x\in A: f(x)<t\cdot g(x)\}\cap\{x\in A: g(x)>0\}\Big) \cup \Big(\{x\in A: f(x)>t\cdot g(x)\}\cap\{x\in A: g(x)<0\}\Big)$$

U_t ,



$x < t\cdot y$

$\frac{x}{y} = t$

Again choose r rational.