Lecture 6 - Intro to measurable functions

22 October 2021 08:50

Remind me to record!

Will start at 9.05

Feel free to vent about mistakes and typos

Please fill out the initial module feedback → Also feel free to tell me I'm doing okay :)

- Especially useful for me because its only my second time lecturing
- Pace of delivery — helpful if you can write in the comments if its too fast or too slow
- Quality of assignments — Is it a good length / difficulty
- Other exercise sheets — Length / difficulty

Second assignment is out today due 4th November 12

↳ Try and get marks from the first back soon

Next two wednesdays → Celebrating Ethnic minority mathematicians

→ Ada Lovelace day

Introducing measurable functions

How function interact with measure spaces

Dfn If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces

and $f : E \to F$ then I will call f measurable

if for every $A \in \mathcal{F}$ $\quad f^{-1}(A) \in \mathcal{E}$

Remark: This is very similar to the defn of continuity in topological spaces

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Lemma (2.2 in notes)

$A \subset \mathcal{F}$ and $\sigma(A) = \mathcal{F}$

Lemma (... in notes)

Suppose $\mathcal{A} \subseteq \mathcal{F}$ and $\sigma(\mathcal{A}) = \mathcal{F}$

If $f: E \to F$ is a function with

$f^{-1}(A) \in \mathcal{E}$ for every $\underline{\underline{A \in \mathcal{A}}}$ then f is measurable.

<u>Proof</u> $f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i)$

If $A \subseteq B$ $f^{-1}(B \setminus A) = f^{-1}(B) \setminus f^{-1}(A)$.

$\mathcal{B} = \{A \in \mathcal{F} : f^{-1}(A) \in \mathcal{E}\}$ so $\mathcal{A} \subseteq \mathcal{B}$

then $\mathcal{B}$ is a σ-algebra as

$\emptyset \in \mathcal{B}$ as $f^{-1}(\emptyset) = \emptyset$ $f^{-1}(F) = E$

and $\mathcal{B}$ is closed under complements and countable unions

so $f^{-1}(A^c) = f^{-1}(F) \setminus f^{-1}(A)$

so if $A \in \mathcal{B}$ then $f^{-1}(A) \in \mathcal{E}$ and $f^{-1}(F) = E$

so $f^{-1}(A^c) = (f^{-1}(A))^c \in \mathcal{E}$

so $A^c \in \mathcal{B}$

Then if $(A_n)_{n \geq 1} \subseteq \mathcal{B}$ then $\underbrace{f^{-1}(\bigcup_n A_n)}_{\in \mathcal{E}} = \bigcup_n \underbrace{f^{-1}(A_n)}_{\in \mathcal{E}}$

so $f^{-1}(\bigcup A_n) \in \mathcal{E}$ so $\bigcup_n A_n \in \mathcal{B}$

so $\mathcal{B}$ is a σ algebra $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{F}$

and so $\mathcal{B}$ must $\sigma(\mathcal{A}) =$ so $\mathcal{F} \subseteq \mathcal{B} \subseteq \mathcal{F}$

so $\mathcal{B} = \mathcal{F}$. so $f^{-1}(A) \in \mathcal{E}$ for every $A \in \mathcal{F}$

so $\mathcal{B} = \mathcal{F}$. so $f^{-1}(A) \in \mathcal{E}$ for every $A \in \mathcal{F}$
so f is measurable.

Example of how to use this fact.

So when f is real values and we have the Borel σ-algebra on $\mathbb{R}$.

$f: E \to \mathbb{R}$ $\{\mathcal{E}, \mathcal{B}(\mathbb{R})$

then to check f is measurable we only need to check

$f^{-1}((-\infty, b]) \in \mathcal{E}$ for every b

$f^{-1}((-\infty, b)) \in \mathcal{E}$ for every b

$f^{-1}((a, b]) \in \mathcal{E}$ for ever a, b

for your favorite subset generating $\mathcal{B}(\mathbb{R})$

<u>Lemma</u> If E, F are topological spaces equipped with their Borel σ-algebras and $f: E \to F$ is continuous then f is measurable wrt $\mathcal{B}(F)$ and $\mathcal{B}(E)$

Pf/ On the exercise sheet

Hint: Remember the open sets generate the Borel σ-algebra.

<u>Lemma 2.5</u> Constructing measurable functions from others

Let $(f_n)_{n \geq 1}$ be a sequence of real valued measurable function $f_n : (E, \mathcal{E}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Then the following are all measurable functions

- $-f_1$
- λf_1 for $\lambda > 0$ a fixed const
- $f_1 \wedge f_2$ ($\min\{f_1, f_2\}$)
- $f_1 \vee f_2$ ($\max\{f_1, f_2\}$)
- $f_1 + f_2$
- $f_1 f_2$
- $\sup_n f_n$
- $\inf_n f_n$
- $\limsup_n f_n$
- $\liminf_n f_n$

<u>Proof</u> $f_1 \vee f_2$ and $f_1 + f_2$ harder than the ~~ex~~ assignment.

By the previous lemma we can just check $(f_1 \vee f_2)^{-1}((-\infty, b]) \in \mathcal{E}$ for every $b \in \mathbb{R}$

$$(f_1 \vee f_2)^{-1}((-\infty, b]) = \{x : \max\{f_1(x), f_2(x)\} \leq b\}$$
$$= \{x : f_1(x) \leq b \text{ and } f_2(x) \leq b\}$$
$$= \{x : f_1(x) \leq b\} \cap \{x : f_2(x) \leq b\}$$
$$= f_1^{-1}((-\infty, b]) \cap f_2^{-1}((-\infty, b])$$

since f_1 and f_2 are measurable

$$f_1^{-1}((-\infty, b]) \in \mathcal{E} \text{ and } f_2^{-1}((-\infty, b]) \in \mathcal{E}$$

so $$f_1^{-1}((-\infty, b]) \cap f_2^{-1}((-\infty, b]) \in \mathcal{E}$$

is measurable

so $f_1((-\infty, b]) \cap f_2((-\infty, b]) \in \mathcal{E}$

so $(f_1 \vee f_2)^{-1}((-\infty, b]) \in \mathcal{E}$ so $(f_1 \vee f_2)$ is measurable

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$(f_1+f_2)^{-1}((b,\infty)) = \{x : f_1(x) + f_2(x) > b\}$

the same

Now if $f_1(x) > b - f_2(x)$ then

$\exists\ q \in \mathbb{Q}$ s.t. $f_1(x) > q > b - f_2(x)$

$A = \bigcup_{q \in \mathbb{Q}} \left(\{x : f_1(x) > q\} \cap \{x : f_2(x) > b - q\} \right)$

if $x \in A$ then $\exists\ q$ s.t.

$f_1(x) > q$ and $f_2(x) > b - q$

so $f_1(x) + f_2(x) > b$

Now if $x \in (f_1+f_2)^{-1}((b,\infty))$ then $\exists\ q$ s.t.

$f_1(x) > q$ and $f_2(x) > b-q$

so $x \in \{x : f_1(x) > q\} \cap \{f_2(x) > b-q\}$ so $x \in A$

so $A = (f_1+f_2)^{-1}((b,\infty))$

and $A = \underbrace{\bigcup_{q \in \mathbb{Q}}}_{\text{countable union}} \underbrace{\left(\underbrace{f_1^{-1}((q,\infty))}_{\in \mathcal{E}} \cap \underbrace{f_2^{-1}((b-q,\infty))}_{\in \mathcal{E}} \right)}_{\mathcal{E}}$,

so $A \in \mathcal{E}$ so $f_1 + f_2$ is measurable.

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week 4: material including a video about image measure

from w3 notes

+ assignment 2.

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