Measure Theory: Exercises (not for credit)

Josephine Evans

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Question 1 (The devil's staircase). In this question we construct a function which is continuous, flat almost everywhere and increases from 0 to 1 as x goes from 0 to 1 (This is quite a lot like doing research, you are only making progress at a measure 0 amount of time!). First we construct the Cantor set recursively. Let $C_0 = [0,1], C_1 = [0,1/3] \cup [2/3,1], C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1], \dots$ where each C_n is constructed from C_{n-1} by removing the middle thirds of each of the closed intervals making up C_{n-1}. Let us write $C = \bigcap_n C_n$, so $C_n \downarrow C$ that is to say $C_1 \supset C_2 \supset \dots$ and $C = \bigcap_n C_n$.

- Show that C is uncountable.
- Show that $\lambda(C) = 0$.
- Define $F_n(x) = \lambda(C_n \cap [0,x])/\lambda(C_n)$, show that $F(x) = \lim_n F_n(x)$ exists. *hint: try and find a reccurrence relationship for F_n in terms of F_{n-1} then use this to show F_n is a Cauchy sequence with the uniform norm on functions*
- Show that the function F is continuous for all x with $F(0) = 0$ and $F(1) = 1$.
- Show that for lebesgue almost every $x \in [0,1]$ we have that $F(x)$ is differentiable with $F'(x) = 0$.