Remind me to record!

Will start at 9.05

Remember: Assignments are due on Thursday

Dynkin's uniqueness of extension theorem

If $\mathcal{A}$ is a π-system and $\mathcal{E} = \sigma(\mathcal{A})$ and μ_1, μ_2 are two measures on $(E, \mathcal{E})$ with $\mu_1 = \mu_2$ on $\mathcal{A}$ and $\mu_1(E) = \mu_2(E) < \infty$ then $\mu_1 = \mu_2$ on all of $\mathcal{E}$

<u>Key Prop^n</u> Lebesgue measure is the only measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which assigns each Interval / rectangle its length (also works in $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

~~Pf~~ The half open intervals are a π-system generating $\mathcal{B}(\mathbb{R})$

Equally, the half open intervals inside $(-n, n]$ generate $\mathcal{B}((-n,n])$ and so suppose μ is a measure which gives every half open interval its length then by Dynkin's theorem $\mu = \lambda$ on $\mathcal{B}((-n,n])$

$$\mu((-n,n]) = \lambda((-n,n]) = 2n$$

Let $E_n = (-n, n]$ and fix some $A \in \mathcal{B}(\mathbb{R})$ and set $A_n = A \cap E_n$ then by continuity of and set $A_n = A \cap E_n$ then by continuity of measure $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ and $\lambda(A) = \lim_{n\to\infty} \lambda(A_n)$

and we know $\mu(A_n) = \lambda(A_n)$ for each n

so $\mu(A) = \lambda(A)$. So $\mu = \lambda$ on $\mathcal{B}(\mathbb{R})$

<u>Corollary</u> Lebesgue is translation invariant

$(x+A) = \{ x+y : y \in A \}$ $\quad \lambda(x+A) = \lambda(A)$

~~Pf~~ Define a measure λ_x on $\mathcal{B}(\mathbb{R})$

by $\lambda_x(A) = \lambda(x+A)$

Then for any half open interval $(a,b]$

$\lambda_x((a,b]) = \lambda((a+x, b+x]) = (b+x) - (a+x)$
$= b-a$

so λ_x gives half open intervals their length

Therefore $\lambda_x = \lambda$ on $\mathcal{B}(\mathbb{R})$

$\lambda(x+A) = \lambda(A)$ for every $A \in \mathcal{B}(\mathbb{R})$

Similar way: Rotation Invariance in $\mathbb{R}^d$

Reflection Invariance not on ex sheet

Dilations work the way you would expect

<u>Prop^n</u> (Vitali) There exists sets that are $\mathcal{P}(\mathbb{R})$ but are not in $\mathcal{M}$ (Lebesgue Measurable)

In video $\mathcal{M}$ = "the sets that differ from a Borel set by a null set"

This proof we use the axiom of choice. It was

In this proof we use the axiom of choice. It was shown much later that the existence of a non-Leb measurable set is only provable if you use the axiom of choice.

We argue by contradiction: Suppose all of $\mathcal{P}(\mathbb{R})$ is in $\mathcal{M}$.

Define an equivalence relationship on $(0,1]$ by

$x \sim y$ if $x - y \in \mathbb{Q}$

Uncountable number of equivalence classes.

Using the axiom of choice we form a set S which contains exactly one element of each equivalence class.

$\rightarrow$ If $w, z \in S$ then $w - z \notin \mathbb{Q}$.

Then for every $q \in \mathbb{Q}$ we define a new set $(q+S)_{\text{mod } 1} = \{ s+q \pmod 1 : \forall s \in S \}$

$$(0,1] = \bigcup_{q \in \mathbb{Q}} (S+q) \pmod 1$$

$x \in (0,1]$ then $\exists$ an element of $[x]_\sim$ in S

the equiv class of x

call this y. Then as $y \sim x$

$\exists$ a $q \in \mathbb{Q}$ s.t. $x - y = q$ or $y - x = q$

so then $x \in (S+q)$ for this particular q. $(y+1+q) \bmod 1$

$$(0,1] = \bigcup_{q \in \mathbb{Q} \cap (0,1]} (S+q)_{(\text{mod } 1)}$$

$$(0,1] = \bigcup_{q \in \mathbb{Q} \cap (0,1]} (S+q) \pmod 1$$

this is a disjoint union

if $x \in S+q$ and $S+q'$

then $\exists\ y, y' \in S$ s.t.

$$x = y + q = y' + q'$$

then $y - y' = q' - q \in \mathbb{Q}$ which isn't possible.

So the union is disjoint.

$$(0,1] = \bigcup_{q \in \mathbb{Q} \cap (0,1]} (S+q) \bmod 1$$

So as we've assumed everything is measurable
by countable additivity

$$\lambda((0,1]) = \sum_{q \in \mathbb{Q} \cap (0,1]} \lambda((S+q) \bmod 1) = 1$$

But by translation invariance of λ

$$\lambda((S+q) \bmod 1) = \lambda(S)$$

So $$\sum_{q \in \mathbb{Q} \cap (0,1]} \lambda((S+q)_{\bmod 1}) = \sum_{q \in \mathbb{Q} \cap (0,1]} \lambda(S) = \begin{cases} 0 & \text{if } \lambda(S) = 0 \\ \infty & \text{if } \lambda(S) > 0 \end{cases}$$

So we've got a contradiction.

So S, $S+q$ for any q are not Lebesgue measurable.

Non-examinable

Banach - Tarski paradox

You can break a ball in $\mathbb{R}^d$ $d \geq 3$ up into finitely many pieces and move those pieces round under translation, rotation and reflection to create two balls of the same size.

Whenever you have a set $A \in \mathcal{M}$ you can move it around by reflection rotation and translation and keep the same Lebesgue measure.

But if $A \in \mathcal{M}$ we can't give it any notion of size.

Idea of the proof: Is split the Ball up into non-leb measurable sets then move them around to form two balls.