Proofs of Fubini–Tonelli Theorem

Recall a theorem from Tuesday

Thm (11.6) Given two measure spaces $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ then $\exists$ a unique measure $(\mu \times \nu)$ on $\mathcal{E} \times \mathcal{F}$

s.t. $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$

Furthermore if $C \in \mathcal{E} \times \mathcal{F}$

$$(\mu \times \nu)(C) = \int_E \nu(C_x)\,\mu(dx) = \int_F \mu(C_y)\,\nu(dy)$$

and the functions $x \mapsto \nu(C_x)$ and $y \mapsto \mu(C_y)$ are measurable

<u>Theorem</u> (Fubini–Tonelli)

Let $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ be measure spaces

f a non-negative $\mathcal{E} \times \mathcal{F}$ measurable function

then $x \mapsto \int_F f(x,y)\,\nu(dy)$ and

$y \mapsto \int_E f(x,y)\,\mu(dx)$ are both measurable

and
$$(\mu \times \nu)(f) = \int_E \left(\int_F f(x,y)\,\nu(dy) \right) \mu(dx)$$
$$= \int_F \left(\int_E f(x,y)\,\mu(dx) \right) \nu(dy)$$

Pf/ The situation is symmetric so we only prove $x \longmapsto \int_F f(x,y)\,\nu(dy)$ is measurable

$$(\mu\times\nu)(f) = \int_E \int_F f(x,y)\,\nu(dy)\,\mu(dx)$$

We are going to build up progressively more complicated fs

First if $f = \mathbb{1}_C$ for some $C \in \mathcal{E}\times\mathcal{F}$

$$\int_F f(x,y)\,\nu(dy) = \int_F \mathbb{1}_C(x,y)\,\nu(dy) = \int_F \mathbb{1}_{C_x}(y)\,\nu(dy)$$

$= \nu(C_x)$ we know this is measurable by the first thm in this video

we also know that $(\mu\times\nu)(\mathbb{1}_C) = (\mu\times\nu)(C) = \int \left(\nu(C_x)\right)\mu(dx)$

So this proves our result in the case f is an indicator function.

Next step: f is a simple function

$$f = \sum_{k=1}^{n} c_k \mathbb{1}_{C_k}$$

then $\int_F f(x,y)\,\nu(dy) = \sum_{k=1}^{n} c_k \int \mathbb{1}_{C_k}(x,y)\,\nu(dy)$

$$= \sum_{k=1}^{n} c_k\,\nu((C_k)_x)$$

This is the sum of measurable functions so measurable.

$= \sum^{n} c_k \int \nu((C_k)_x)\,\mu(dx)$

measurable.

$$\int_E \left(\int_F f(x,y)\, \nu(dy) \right) \nu(dx) = \int_E \sum_{k=1}^{n} c_k \nu((C_k)_x) = \sum_{k=1}^{n} c_k \int_E \nu((C_k)_x) \mu(dx)$$

$$= \sum_{k=1}^{n} c_k (\mu\times\nu)(C_k) = (\mu\times\nu)(f).$$

Now we want to look at general non-negative f. Then there exists a sequence of simple functions f_n with $f_n \uparrow f$

so the function $(y \mapsto f_n(x,y))$ are simple functions which increase to the function $(y \mapsto f(x,y))$ for each x so by monotone convergence

$$\underbrace{\int_F f_n(x,y)\, \nu(dy)}_{\text{these are all measurable functions}} \uparrow \underbrace{\int f(x,y)\, \nu(dy)}_{}$$

these are all measurable functions of x for each n

Is the limit of measurable functions so measurable.

by our result for simple functions

$$g_n = \int_F f_n(x,y)\, \nu(dy) \quad \text{and} \quad g = \int f(x,y)\, \nu(dy)$$

then $g_n \uparrow g$ so $\mu(g_n) \uparrow \mu(g)$ by MON

$$\int_E \int_F f_n(x,y)\, \nu(dy)\, \mu(dx) \to \int_E \int_F f(x,y)\, \nu(dy)\, \mu(dx)$$

|| by our result for simple func^n

$$(\mu\times\nu)(f_n) \xrightarrow[\text{MON}]{} (\mu\times\nu)(f)$$

so $$\int_E \int_F f(x,y)\, \nu(dy)\, \mu(dx) = (\mu \times \nu)(f).$$

Thm $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ measure spaces

f is $\mathcal{E} \times \mathcal{F}$ measurable and is also integrable

wrt $(\mu \times \nu)$

Define the functions

$$g(x) = \begin{cases} \int_F f(x,y)\, \nu(dy) & \int_F |f(x,y)|\, \nu(dy) < \infty \\ 0 & \text{o/w} \end{cases}$$

$$h(y) = \begin{cases} \int_E f(x,y)\, \mu(dx) & \int_E |f(x,y)|\, \mu(dx) < \infty \\ 0 & \text{o/w} \end{cases}$$

these are both measurable and integrable and

$(\mu \times \nu)(f) = \mu(g) = \nu(h)$.

Proof Basically by writing $f = f_+ - f_-$ then apply the result for +ve functions to f_+ and f_-

But its a bit more fiddly than this because of checking everything is finite.

$(\mu \times \nu)(|f|) < \infty$ and by our result for +ve func^n $(\mu \times \nu)(|f|) = \int_E \int_F |f(x,y)|\, \nu(dy)\, \mu(dx)$

Therefore $\int_F |f(x,y)|\, \nu(dy)$ is finite almost everywhere and integrable

$$\left| \int_F f(x,y)\, \nu(dy) \right| \leq \int_F |f(x,y)|\, \nu(dy)$$

so is finite a.e. so a.e $=$ to $g(x)$.

so a.e =

so $\searrow$ is finite a.e.

Let $A = \{x : \int_F |f(x,y)|\,\nu(dy) < \infty\}$

the function $y \mapsto f(x,y)$ is integrable for $x \in A$

then $\int f(x,y)\,\nu(dy)\,\mathbb{1}_{x\in A} = \left(\int f_+(x,y)\,\nu(dy) - \int f_-(x,y)\,\nu(dy)\right)\mathbb{1}_{x\in A}$

then using the fact that $\mu(A^c) = 0$

$$(\mu\times\nu)(f) = (\mu\times\nu)(f_+) - (\mu\times\nu)(f_-)$$

$$= \int_E\int_F f_+(x,y)\,\nu(dy)\,\mu(dx) - \int_E\int_F f_-(x,y)\,\nu(dy)\,\mu(dx)$$

$$= \int_E \underbrace{\left(\int_F f_+(x,y)\,\nu(dy)\right)}_{<\infty} \mathbb{1}_A\,\mu(dx) - \int_E \underbrace{\left(\int_F f_-(x,y)\,\nu(dy)\right)}_{<\infty} \mathbb{1}_A\,\mu(dx)$$

$$= \int_E \left(\int_F f_+(x,y)\,\nu(dy) - \int_F f_-(x,y)\,\nu(dy)\right)\mathbb{1}_A(x)\,\mu(dx)$$

$$= \int_E \int_F f(x,y)\,\nu(dy)\,\mathbb{1}_{x\in A}\,\mu(dx)$$

$$= \int_E\int_F f(x,y)\,\nu(dy)\,\mu(dx)$$