This Wed. Lecture on Fubini's Thm.

Th.-Fri. Post videos: Dynkin systems
Product σ-alg.
Product measures

Next Mon. Review this material.

Sat or Sun.? Post video: Tonelli's Thm, Fubini's Thm

Sun or Mon.? Post video: Lusin's Thm.

Next Wed. Review this additional material

Assignment 4 due Fri.

This week we introduced a family of normed linear spaces of functions and we showed that they are complete as metric spaces.

We introduced the L^p spaces and corresponding "modes of convergence" for functions: convergence with respect to the p-norm.

Unlike our previous discussion of "modes of convergence" where we ask, given $f_1, f_2 \ldots$ and f, when does $f_n \to f$. Now we can start with just the sequence f_n and determine from this sequence that there will be an f with $f_n \to f$.

We now have a tool to construct new measurable functions.

Mysterious conjugate exponents: $\frac{1}{p}+\frac{1}{q}=1.$

Prop. 3.3.2 (Holder's inequality). Say p, q are conjugate exponents and $p>1$.
If $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ then $fg \in \mathcal{L}^1$ and

$$\int |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q. \qquad (1)$$

Def. A continuous linear function on a normed such as L^p is a linear function $\ell : L^p \to \mathbb{R}$ such that

$$|\ell(f)| \leq C \cdot \|f\|_p \quad \text{for some } c > 0.$$

L^p

The Hölder inequality says that every $g \in L^q$ gives such a continuous linear function on L^p:

Define: $\ell(f) = \int f \cdot g \, d\mu$ then

$$|\ell(f)| = \left| \int f \cdot g \, d\mu \right| \leq \int |f \cdot g| \, d\mu \leq \|g\|_q \cdot \|f\|_p$$

$$= C \cdot \|f\|_p.$$

Prop. 3.3.3. (Minkowski's inequality.)

For $p \geq 1$ if $f, g \in \mathcal{L}^p$ then $f+g \in \mathcal{L}^p$ and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

$\|\cdot\|_p$ is a semi-norm on $\mathcal{L}^p$.

Fails to satisfy $\|f\|_p = 0 \Rightarrow f = 0$.

An element of L^p is an equivalence class of elements of $\mathcal{L}^p$.

Paradoxically it does not make sense to evaluate an element of $f \in L^p$ at a point. $f(x)$?

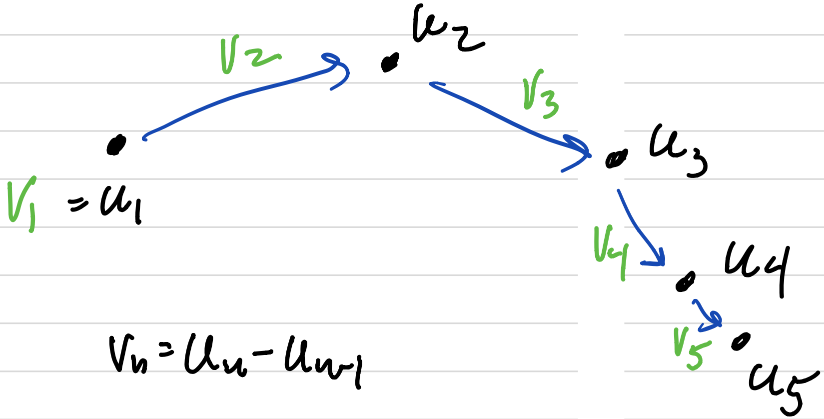
Typically in our arguments involving L^p we do not work with equivalence classes of functions but we choose concrete representative functions in $\mathcal{L}^p$.

These functions do have well defined values.

Prop. 3.2.5. Let V be a normed vector space. Then V is complete if and only if every absolutely convergent series in V is convergent.

The important implication for us is that

"If every abs. conv. series is convergent then V is complete."



$\in L^p$

$v_n = u_n - u_{n-1}$

We can turn a sequence into a series

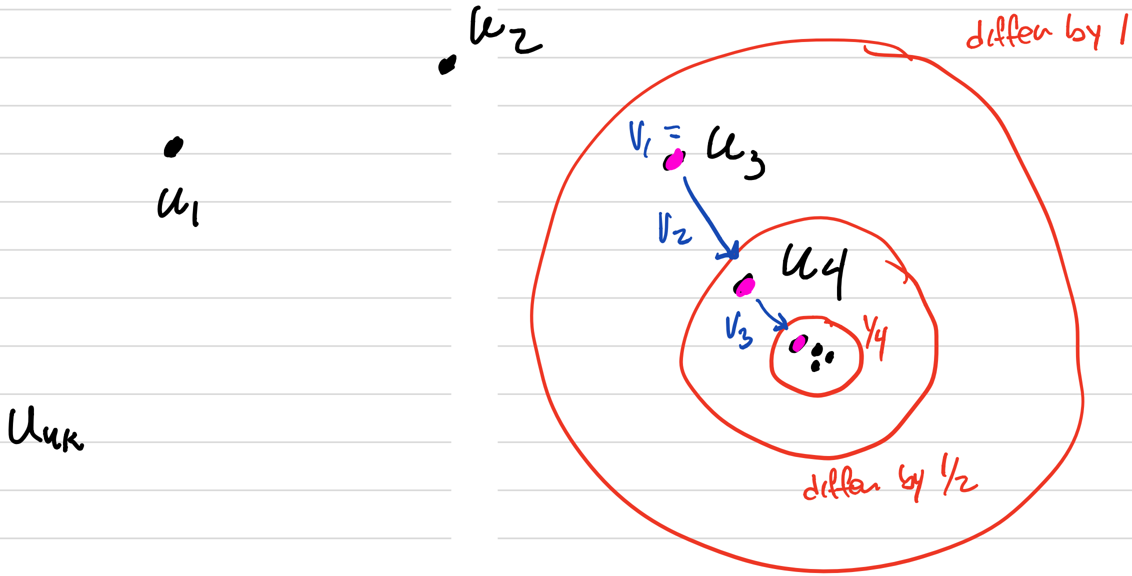
$$u_n = \sum_{k=1}^{n} v_k . \qquad \sum_{k=1}^{\infty} v_k$$

(Sequences and distance vs. norms and series.)

Sequence converging is equivalent to the series converging.

Now we would like the series to converge quickly. (Control on norms of terms.)

We can find a subsequence that converges as fast as we like.



$$u_{n_1} = u_3 \qquad v_1 = u_{n_1} \qquad \|v_2\| \le 1$$

$$u_{n_2} = u_4 \qquad v_2 = u_{n_2} - u_{n_1} \qquad \|v_3\| \le \tfrac{1}{2}$$

$$v_3 \ldots\ldots \qquad \|v_4\| \le \tfrac{1}{4}$$

$\sum v_j$ is an absolutely convergent series since $\sum \|v_j\|$ converges.

The following result plays an essential role in the proof of the completeness theorem:

Cor. 2.3.14. Let f be a $[-\infty, +\infty]$ valued integrable function on X. Then $|f(x)| < +\infty$ holds a.e.

Proof. This follows from Markov's inequality (2.3.10).

Comment on the Dominated Convergence Thm.

There is a gap in the posted proof which Cor. 2.3.14 helps to resolve an issue pointed out by Scott Fowler-Wright.

The theorem deals with a $[-\infty, +\infty]$ valued function g and a sequence f_n with $|f_n| \leq g$.

In the proof we consider new sequences of functions $g - f_n$ and $f_n + g$ and their integrals.

We have not defined addition of $[-\infty, +\infty]$ valued functions since we don't assign a value to $(+\infty)+(-\infty)$ and we do not have a linearity theorem for the integral of extended real valued functions.

On the other hand since g is integrable we know that $\{x : |g(x)| = +\infty\}$ has measure 0. If we remove this set from X we do not change any of the integrals of g or f_n by Prop. 2.3.9.

Thm. 3.4.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let p satisfy $1 \le p < \infty$. Then $L^p(X, \mathcal{A}, \mu)$ is complete with respect to the norm $\|\cdot\|_p$.

Proof. Let f_k be a sequence of functions that belong to $\mathcal{L}^p$ and satisfy $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$ (which is to say that $\sum_{k=1}^{\infty} f_k$ is absolutely convergent).

We want to show that the series $\sum_{k=1}^{\infty} f_k$ converges.

Fix an $x \in X$.

Now $\sum_{k=1}^{\infty} f_k(x)$ converges if it converges absolutely. That is if $\sum_{k=1}^{\infty} |f_k(x)| < \infty$.

We want to control the set of x for which

$$\sum_{k=1}^{\infty} |f_k(x)| = \infty.$$

Define $g: X \to [0, +\infty]$ by

$$g(x) = \left(\sum_{k=1}^{\infty} |f_k(x)| \right)^p.$$

We use the monotone convergence theorem together with the Minkowski inequality to show that g is integrable.

We use Cor. 2.3.14 to show that

$|g(x)| < \infty$ a.e.

This allows us to define a "target function".
(to which the original series should converge.)

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } g(x) < +\infty \\ 0 & \text{otherwise} \quad (\text{on a set of measure } 0) \end{cases}$$

Since we are talking about convergence in L^p we have freedom to define our functions arbitrarily on sets of measure 0.

f is constructed as the result of "pointwise convergence a.e. of a subsequence."

It is not true that the original sequence converges a.e. to f but it is true that the original sequence converges in p-norm to f:

$$\left\| \sum_{k=1}^{n} f_k - f \right\|_p \to 0$$

We prove this using the Dominated Convergence Theorem using the function g in our "domination condition".

This week we gave the ultimate answer to the question of "why integrate weird functions?"

If we want our function spaces to be closed under taking limits of Cauchy sequences then these limits might be "funny functions".

Instead of asking that the individual functions in our function spaces L^p be well behaved we are asking that the function spaces L^p be well behaved.

You cannot have both!

Historically we have moved from 19th century mathematics to 20th century mathematics.

We have introduced the tools necessary for giving a precise answer to Fourier's question:

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} b_k \sin(kx) + \sum_{j=1}^{\infty} a_j \cos(jx).$$

Say $f \sim a_0 + \sum_{k=1}^{\infty} b_k \sin(kx) + \sum_{j=1}^{\infty} a_j \cos(jx)$.

It might seem most logical to use pointwise convergence a.e. In fact pointwise convergence is hard to understand.

The most straightforward thing to do is to use the geometry of the Hilbert space L^2.

We look at the sequence:

$$f_n(x) = a_0 + \sum_{k=1}^{n} b_k \sin(kx) + \sum_{j=1}^{n} a_j \cos(jx).$$

We find a subsequence f_{n_k} for which the sums $f_n(x)$ converge pointwise to some f_∞.

We show that $f_n \to f_\infty$ in the L^2-norm.