Lecture 1

05 October 2021 08:52

Will be recorded!

Intro to the course + σ-algebras

Starting at 9.05

Each week 2 lectures
+ Pre-recorded videos
+ lecture notes
+ exercise sheets

Tutorials which go through exercises

Also go through assignments

4 assignments best 3 make 15%

Due in ends of weeks 3, 5, 7, 9

1 live class Wed 11am → sign up for this

capacity of 48

2 online classes → separate tutorial channells

Email me

Q&A Chanel in the team

Underpins a lot further areas
→ functional analysis
→ Analysis of PDE
→ probability
→ Ergodic theory / dynamical systems

→ How to give a notion of size to potentiality complicated set.

→ A new theory of integration

Insides of the theory:

* Construction of Lebesgue measure
* Existence of non-Lebesgue measurable sets
* How the Leb integral is constructed
* Product measures and spaces
* Def^n of L^p (spaces of functions)

What you'll use in later courses:

* Fact that they exist / do what you expect
* How to work with measures by looking at "simple sets"
* Ways that functions can converge
* Convergence theorems
* Important functional inequalities
* When you can rigorously switch the order of integration.

Different collections of subsets.

Working in a space E (is just a set)

eg. $E = \mathbb{R}^d$

$E = [0,1)$

E is something completely abstract

Interested in "measuring the size" of different subsets of E.

We define some systems of subsets.

Similar to a topology

Def^n 2.1 An algebra of sets (Boolean algebra)

A collection of subsets, $\mathcal{A}$, of E is an algebra if it is "closed under <u>finite</u> set operations"

- $\emptyset \in \mathcal{A}$, $E \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ $\quad (A^c = E \setminus A)$
- $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

$$A_1 \cap A_2 \cap A_3 \dots \cap A_n \in \mathcal{A}$$

Def^n A σ-algebra is a collection, $\mathcal{A}$, of subsets of E "closed under countable set operations"

- $\emptyset \in \mathcal{A}$, $E \in \mathcal{A}$
- $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
- $A_1, A_2, A_3, \dots \in \mathcal{A}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Useful Lemma

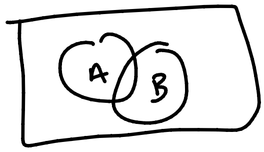
A σ-algebra is a collection of subsets containing $\emptyset$, closed under complements and countable unions.

~~Pf~~ $E = \emptyset^c$ so $E \in \mathcal{A}$

$$\bigcap_n A_n = \left(\bigcup_n A_n^c\right)^c$$

$$(A \cap B) = (A^c \cup B^c)^c$$

$\bigcap_n A_n = \left(\bigcup_n A_n^c \right)$



Examples of σ- algebras :

- Big example is Borel σ- algebra

→ In video 1 on the moodle

- The power set of E $\mathcal{P}(E)$
  the collection of all subsets of E.

- If E is not countable
  then the collection of all countable subsets of E is not a σ- algebra

- If $A \subseteq E$ $\{\emptyset, A, A^c, E\}$ is a σ-algebra

__Lemma__ Suppose E is a set and $\mathcal{C}$ is a collection of σ- algebras

$$\bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A} = \left\{ \begin{array}{c} \text{collection of subsets of } E \text{ which} \\ \text{are in every } \sigma\text{-algebra in } \mathcal{C} \end{array} \right\}$$

is a σ -algebra

~~Pf~~ $\emptyset, E$ are in every $\mathcal{A} \in \mathcal{C}$
so they are in the intersection

If $A \in \mathcal{A}$ for every $\mathcal{A} \in \mathcal{C}$ then $A^c \in \mathcal{A}$ for $\mathcal{A}$

so $A^c \in \bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$

Same with countable unions

if $A_1, A_2, A_3 \ldots$ in every σ-algebra then $\bigcup_{n=1}^{\infty} A_n$ in if $A_1, A_2, A_3 \dots$ in every σ-algebra then $\bigcup_{n=1}$ ... every σ-algebra.

Cor 2.6 Given a collection of subsets $\mathcal{F}$ then there exists a smallest σ-algebra containing $\mathcal{F}$.

contained in every σ-algebra containing $\mathcal{F}$.

We call this the σ-algebra generated by $\mathcal{F}$

$\sigma(\mathcal{F})$

~~Pf~~ $\mathcal{P}(E)$ contains $\mathcal{F}$ $\quad \mathcal{F} = \{A\}$

so let $\mathcal{C}$ be the collection of σ-algebras containing $\mathcal{F}$.

$$\mathcal{C} = \{\mathcal{A} : \mathcal{A} \subseteq \mathcal{P}(E), \mathcal{A} \text{ is a } \sigma\text{-algebra}, \mathcal{F} \subseteq \mathcal{A}\}$$

$\mathcal{P}(E) \in \mathcal{C}$ so $\mathcal{C}$ is non-empty

$$\sigma(\mathcal{F}) = \bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$$

← This is a σ-algebra by the lemma

$\mathcal{F} \subseteq \sigma(\mathcal{F})$ as $\mathcal{F} \subseteq \mathcal{A}$ for every $\mathcal{A} \in \mathcal{C}$

Its the smallest σ-algebra as if $\mathcal{A}$ is a σ-alg containing $\mathcal{F}$ the $\mathcal{A} \in \mathcal{C}$ so $\bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A} \subseteq \mathcal{A}$

$\boxed{A \cap B \subseteq A}$

Videos on the moodle:

→ Please look at Borel σ-algebra video before Friday

→ Vid on π-systems and d-system

q3 on the exercise