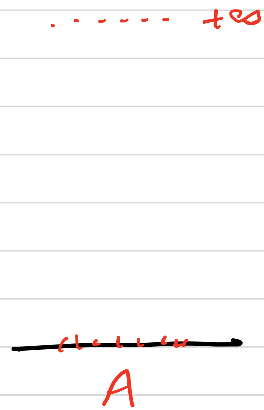
Given a measure space $(X, \mathcal{A}, \mu)$ we say that a property holds <u>μ- almost everywhere</u> if the set of points for which the property does not hold has μ-measure 0.

μ a.e.

a.e.

Proposition 2.3.9. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and g be $[-\infty, \infty]$ valued $\mathcal{A}$ measurable functions on X that agree almost everywhere. If $\int f\,d\mu$ exists then $\int g\,d\mu$ exists and $\int f\,d\mu = \int g\,d\mu$.



Proof. Assume f and g are non-negative.
Let $A = \{x \in X : f(x) \neq g(x)\}$ and let

$$h(x) = \begin{cases} +\infty & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We claim that $\int h\, d\mu = 0$. Note that h is not itself a simple function. It is non-negative so its integral is the sup of integrals of simple functions $g \in \mathcal{S}^+$ with $g \le f$.

A snappy way to prove this is to observe that h is the pointwise limit of functions $n \cdot \chi_A$ and apply Prop. 2.3.3. We have $f \le g + h$ so $\int f\, d\mu \le \int g\, d\mu + \int h\, d\mu = \int g\, d\mu$.

Similarly $\int g\, d\mu \le \int f\, d\mu$. We can reduce the general case to the non-negative case.

Remark on the Monotone Convergence Thm.

We can prove a (slightly) stronger version of the Theorem by weakening the hypothesis so that we only assume monotonicity and convergence a.e.

Given a sequence f_n we let N be the set of points for which some hypothesis fails. So $\mu(N)=0$.

Consider the sequence $f_n \cdot \chi_{N^c}$.

$f_n \cdot \chi_{N^c} \to f \cdot \chi_{N^c}$ pointwise.

The Thm. as proved shows that

$$\int f \cdot \chi_{N^c} \, d\mu = \lim_{n\to\infty} \int f_n \cdot \chi_{N^c} \, d\mu.$$

Since $f_n \cdot \chi_{N^c} = f_n$ a.e. and $f \cdot \chi_{N^c} = f$ a.e.,

Prop. 2.3.9 gives $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$

Markov inequality.

Prop. Let f be a $[0,+\infty]$ valued $\mathcal{A}$-measurable function on X. If t is a positive real number and if A_t is defined by $A_t = \{x \in X : f(x) > t\}$ then

$$\mu(A_t) \le \frac{1}{t}\int_{A_t} f\,d\mu \le \frac{1}{t}\int f\,d\mu.$$

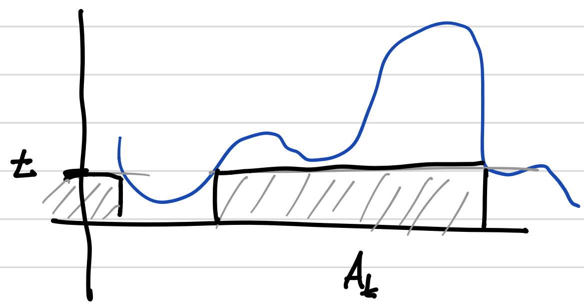
Proof.

$$0 \le t\cdot\chi_{A_t} \le f\cdot\chi_{A_t} \le f$$

and Prop 2.3.4(c) imply that

$$\int t\cdot\chi_{A_t}\,d\mu \le \int_{A_t} f\,d\mu \le \int f\,d\mu$$

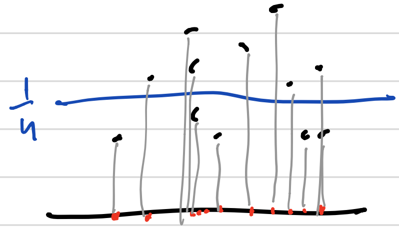
$$\Rightarrow \mu(A_t) \le \frac{1}{t}\int_{A_t} f\,d\mu \le \frac{1}{t}\int f\,d\mu.$$



Cor. 2.3.12 Let f be a $[-\infty, \infty]$ valued $\mathcal{A}$-measurable function on X that satisfies

$$\int |f| \, d\mu = 0$$

then $f = 0$ μ-almost everywhere.



Proof. $\mu\left(\{x \in X : |f(x)| \geq \frac{1}{n}\}\right) \leq n \cdot \int |f| \, d\mu = 0.$

$$\{x \in X : f(x) \neq 0\} = \bigcup_n \{x \in X : |f(x)| \geq \frac{1}{n}\}.$$

Countable sub-additivity implies

$$\mu(\{x \in X : f(x) \neq 0\}) = 0.$$