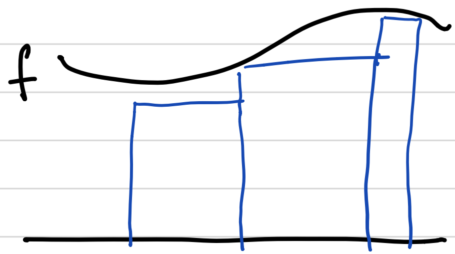
Let $(X, \mathcal{A}, \mu)$ be a measure space.

Definition: Let $f : X \to [0, +\infty]$ be an $\mathcal{A}$-measurable function. Then

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \in \mathcal{S}^{+} \text{ and } g \leq f \right\}. \quad \text{(p. 55)}$$



An important tool in establishing properties of the integral is the following:

Proposition 2.3.3. Let $f : X \to [0, \infty]$ be measurable and $\{f_n\}$ a non-decreasing sequence $f_n \in \mathcal{S}^+$ converging pointwise to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

We will prove this in 2 steps. We start by adding the strong hypothesis that f itself is a simple function.

Proposition 2.3.2. Let $f \in \mathcal{S}^+$ and let $\{f_n\}$ be a non-decreasing sequence in $\mathcal{S}^+$ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

Proof. We know that for functions in $\mathcal{S}^+$ $f \le g \Rightarrow \int f\,d\mu \le \int g\,d\mu$. (Properties of simple functions.)

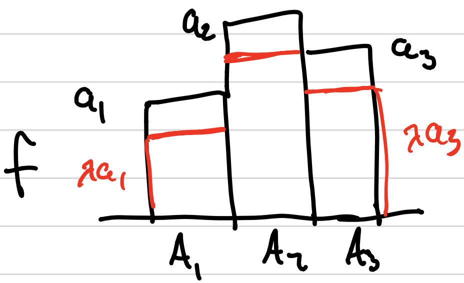
$\int f_1\,d\mu \le \int f_2\,d\mu \le \dots$ is a non-decreasing sequence bounded above by $\int f\,d\mu$. (from the def. of $\int$)

In particular $\lim\limits_{n\to\infty} \int f_n\,d\mu$ exists and

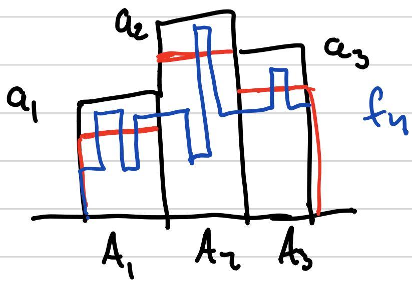
$$\lim_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu.$$

So we need to show $\lim_{n\to\infty} \int f_n \, d\mu \geq \int f \, d\mu$.

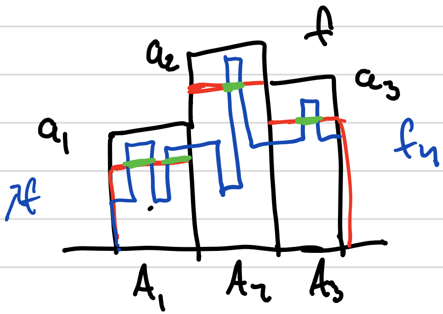
Let $\lambda < 1$. It suffices to show that $\lim_{n\to\infty} \int f_n \, d\mu \geq \lambda \cdot \int f \, d\mu$ and then let $\lambda \to 1$.

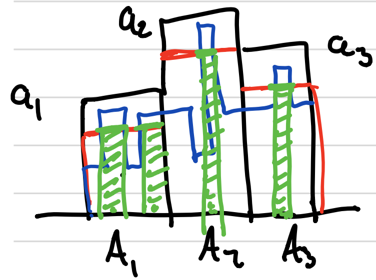


We want to look at the functions f_n with respect to λf.



We want to focus on those points x where $f_n(x) \geq \lambda f(x)$.





g_n is shown in green.

Let $g_n(x) = \begin{cases} \lambda f(x) & \text{if} \quad f_n(x) \geq \lambda f(x) \\ 0 & \text{if} \quad f_n(x) < \lambda f(x) \end{cases}$

So $g_n(x) \leq f_n(x)$.

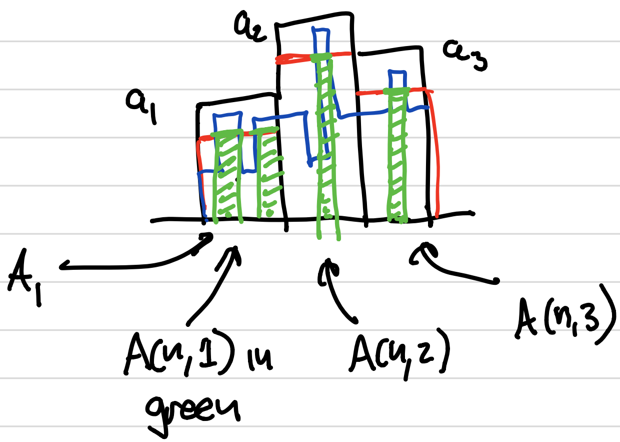
We want to write g_n explicitly as a simple function.

Write:

$$A(n,i) = \{x \in A_i : f_n(x) \geq \lambda f(x) = \lambda a_i\}.$$

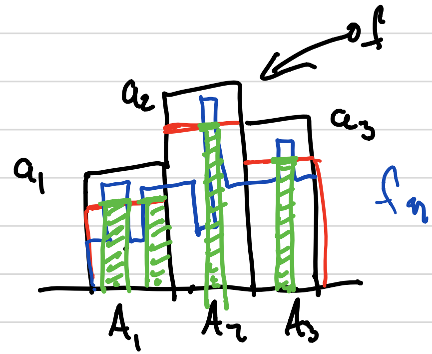
then

$$g_n = \sum_i \lambda a_i \cdot \chi_{A(n,i)}.$$



The fact that $f_n(x) \le f_m(x)$ for $n<m$ means that $A(n,i)$ is an non-decreasing sequence of sets for each fixed i.

$$A(n,i) \subset A(n+1,i) \subset \dots$$



The fact that $f_n(x) \to f(x) > \lambda f(x)$ means that for each $x \in A_i$ there is an n with $x \in A(n,i)$. Thus $\bigcup_{n\to\infty} A(n,i) = A_i$.

We know that for an increasing sequence of sets $\lim_{n\to\infty} \mu(A(n,i)) = \mu(\bigcup_n A(n,i)) = \mu(A_i)$.

This is one of our "measure continuity" results.

We use this to evaluate the limit of integrals of g_n.

$$\lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \sum_i \lambda \cdot a_i \, \mu(A(n,i))$$

$$= \sum_i \lambda \cdot a_i \lim_{n\to\infty} \mu(A(n,i))$$

$$= \sum \lambda \cdot a_i \, \mu(A_i)$$

$$= \lambda \sum a_i \, \mu(A_i)$$

$$= \lambda \int f \, d\mu.$$

Now $\quad g_n \leq f_n \quad$ so

$$\int g_n \, d\mu \leq \int f_n \, d\mu \leq \int f \, d\mu$$

and this gives

$$\lambda \int f \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu \leq \int f \, d\mu.$$

Since this is true for every $\lambda < 1$ we get

$$\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu$$

as was to be shown.