Will start at 5past

Remind me to record!

Still time to do the initial feedback survey.

Egoroff's theorem + Lusin's theorem

Almost uniform convergence: $f_n \to f$ almost uniformly on $(E, \mathcal{E}, \mu)$ if $\varepsilon > 0$ $\exists A$ s.t. $f_n \to f$ uniformly on A and $\mu(A^c) < \varepsilon$.

Egoroff's thm if $\mu(E) < \infty$ ~~then~~ $(f_n)_{n \geq 1}$ f measurable and $f_n \to f$ a.e. then $f_n \to f$ almost uniformly.

Lusin's thm is a consequence If $f : (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu) \to \mathbb{R}$ is measurable, $A \in \mathcal{B}(\mathbb{R})$ $\mu(A) < \infty$ given $\varepsilon > 0$ $\exists K$ compact f is continuous on K and $\mu(A \setminus K) < \varepsilon$

Integration

Notation $f : (E, \mathcal{E}, \mu) \to \mathbb{R}$ three notations for the integral wrt to μ

$\mu(f)$, $\int_E f \, d\mu$ and $\int_E f(x) \, \mu(dx)$

I like this one because its v. compact.

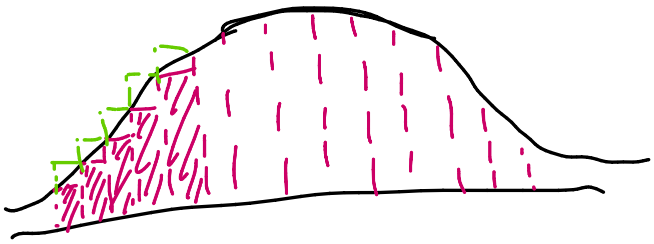
looks like an integral

useful to keep track of variables

Recap Riemann integration

Recap Riemann integration

1. Split the domain of f up into intervals
2. Estimate f above and below on these sub intervals to form little rectangles
3. Sum up the rectangles to estimate the area under the graph

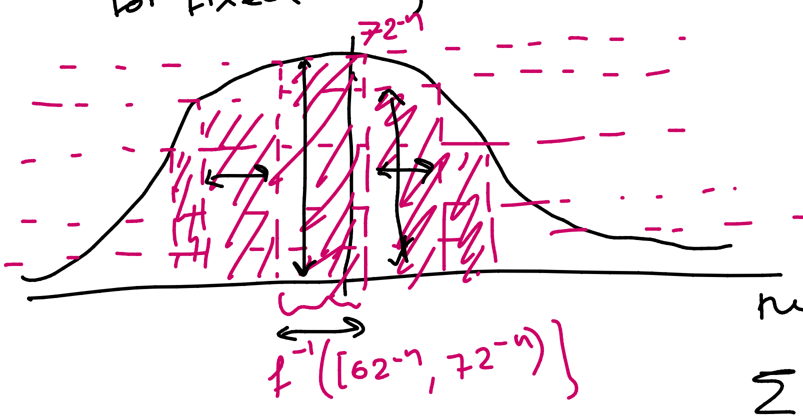


Lebesgue integration

Split the <u>range</u> of f up into little chunks (which could be intervals) $f: E \to \mathbb{R}$

we can split $\mathbb{R}$ into Intervals $[k2^{-n}, (k+1)2^{-n})$ for fixed n, $k \in \mathbb{Z}$ we estimate f by

$$\sum_k k2^{-n} \mathbb{1}_{\{f^{-1}([k2^{-n}, (k+1)2^{-n}))\}}$$



We estimate the area under the curve of f by

$$\sum_k \underbrace{k2^{-n}}_{\text{height of area}} \underbrace{\lambda(f^{-1}([k2^{-n}, (k+1)2^{-n})))}_{\text{"width" of area}}$$

First thing to note is if f is complicated then the set $f^{-1}([k2^{-n}, (k+1)2^{-n}))$ may be complicated

So we need measure theory in order to measure its length.

Immediate advantage: Using this we can extend the notion of integration to $f: E \to \mathbb{R}$ where E is not necessarilly a subset of $\mathbb{R}^d$

Useful for probability theory! Quite often want to integrate over function spaces.

Hidden advantages: It turns out Lebesgue integration is valid for a much wider set of functions.

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main intuitive reason Lebesgue integration wont get messed up by fast oscillation or functions jumping about infinitely often.

$f(x) = \mathbb{1}_{\mathbb{Q}}$ is Lebesgue integrable but not Riemann integrable

Stuff like $\sin(\frac{1}{x})$ is easier with Lebesgue

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Riemann integration doesn't behave well with respect to taking limits

We want to know when

$$\lim_{n\to\infty} \int f_n(x)\,dx = \int \lim_{n\to\infty} f_n(x)\,dx$$

This is true for Riemann integration if $f_n \to f$ uniformly for f.

But uniform convergence is very strong notion of convergence and its useful to have theorems that work for weaker types of convergence.

It turns out Lebesgue integration is much better for this.

almost everywhere convergence + something else allows us to switch orders of convergence and integration.

Switching orders of integration

With Lebesgue integration we can show

$$\int_{E_x}\int_{E_y} f(x,y)\,dy\,dx = \int_{E_y}\int_{E_x} f(x,y)\,dx\,dy$$

Fubini's theorem $\rightarrow$ week 9.

Both taking limits and switching the order of integration are useful for dealing with Fourier series.

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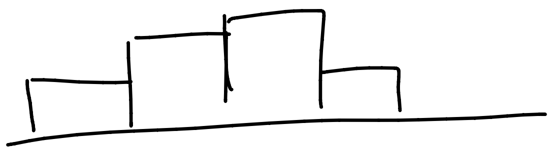
1. Simple functions a function f is simple

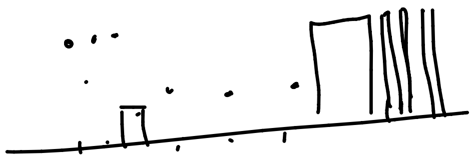
if $f(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$ where $a_k \geq 0$ and A_k is a meas set.

$\boxed{a_{k_1} = a_{k_2} \text{ then } f^{-1}(\{a_{k_1}\}) \text{ includes } A_{k_2}}$

These are measurable functions taking finitely many values.

(If you write f the right way $A_k = f^{-1}(\{a_k\})$)





Since simple functions take finitely many values it is easy to split up the range of

We define $\mu(f) = \sum_{k=1}^{n} a_k \mu(A_k) = \left(\sum_{k=1}^{n} a_k \mu(f^{-1}(\{a_k\})) \right)$

2. If f is positive and measurable we define $\mu(f)$ either by

$\rightarrow$ $\mu(f) = \lim_{n \to \infty} \mu(f_n)$ where $f_1 \leq f_2 \leq f_3 \dots$ $(f_n \uparrow f)$
$f_n \to f$

$\rightarrow$ $\mu(f) = \sup \{ \mu(g) : g \text{ simple } \; g \leq f \}$

3. Define it on functions which aren't non negative

$f_+(x) = \max \{ f(x), 0 \}$ and $f_-(x) = \max \{ -f(x), 0 \}$

$f = f_+ - f_-$

If $\mu(f_+) < \infty$ and $\mu(f_-) < \infty$ then

$$\mu(f) = \mu(f_+) - \mu(f_-)$$