Lecture 16

26 November 2021 08:57

Good morning!

Lecture will start at 9.05

Remind me to record

Assignment 4 due on Thursday 2^{nd} of December at midday

More L^p spaces

Prove some facts about L^p spaces

- Define convergence in L^p
- Prove L^p is complete
- Talk about approximation in L^p

Def^n Convergence in L^p

If $(f_n)_{n\geq 1}$ and f are all in L^p then we say $f_n \to f$ in L^p if

$$\| f_n - f \|_p \to 0$$

A sequence $(f_n)_{n\geq 1}$ of L^p functions is a Cauchy sequence if $\forall\ \varepsilon > 0$ $\exists\ N$ s.t. $n, m \geq N$ implies $\|f_n - f_m\|_p < \varepsilon$.

Theorem $L^p(E)$ is complete. That is to say if $(f_n)_{n\geq 1}$ is a Cauchy sequence in $L^p(E)$ then $\exists$ an $f \in L^p(E)$

is a Cauchy sequence in $L^p(E)$ then $\exists$ [illegible]
s.t. $f_n \to f$ in $L^p(E)$.

Proof Find a canditate for f by finding a subsequence f_{n_k} that converges a.e. and taking its limit. Show that $f_n \to f$ in L^p using our integral convergence theorem (Fatou's lemma).

Let $n_1 = 1$ then we can define n_k recurssively by $\|f_{n_k} - f_{n_{k-1}}\|_p \leq 2^{-k}$

Then we will form a subsequence n_k s.t.

$$\sum \|f_{n_k} - f_{n_{k-1}}\|_p \leq 1 .$$

Then for any finite K by Minkowski's inequality

$$\Big\| \sum_{k=1}^{K} |f_{n_k} - f_{n_{k-1}}| \Big\|_p \leq \sum_{k=1}^{K} \|f_{n_k} - f_{n_{k-1}}\|_p \leq 1$$

By monotone convergence

$$\Big\| \sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k-1}}| \Big\|_p \leq 1 .$$

Therefore, $\sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k-1}}|$ is finite almost everywhere

so let $E' = \{ x : \sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k-1}}(x)| < \infty \}$

Then $\mu((E')^c) = 0$

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Then for $x \in E'$ so $(f_{n_k}(x))_{k \geq 1}$ is a Cauchy sequence so as Cauchy sequences in $\mathbb{R}$ converge we can define

$$f(x) = \begin{cases} \lim_{k \to \infty} f_{n_k}(x) & x \in E' \\ 0 & x \notin E' \end{cases}$$

Now we want to show $f_n \to f$ in L^p

Lets fix an $\varepsilon > 0$ (wtf N s.t. $n \geq N \Rightarrow \|f_n - f\| \leq \varepsilon$)

$\exists$ N s.t. $n, m \geq N \Rightarrow \|f_n - f_m\|_p \leq \varepsilon$

So for $n \geq N$ and k suff large

$$\|f_n - f_{n_k}\|_p \leq \varepsilon$$

$$\|f_n - f\|_p = \|f_n - \lim_{k \to \infty} f_{n_k}\| = \left(\mu\left(|f_n - \lim_{k \to \infty} f_{n_k}|^p\right)\right)^{1/p}$$

$$= \left(\mu\left(\lim_{k \to \infty}\left(|f_n - f_{n_k}|^p\right)\right)\right)^{1/p}$$

By Fatou's lemma $\leq \liminf_{k \to \infty} \left(\mu(|f_n - f_{n_k}|^p)\right)^{1/p}$

$\left(\mu(\liminf f_n) \leq \liminf \mu(f_n)\right)$ $= \liminf_{k} \|f_n - f_{n_k}\|_p$
$f_n \geq 0$

$\leq \varepsilon$

Def^n A family of functions $\mathcal{G}$ is dense in $L^p(E)$ if for every $f \in L^p(E)$, $\varepsilon > 0$ $\exists\, g \in \mathcal{G}$ s.t.

$$\|f-g\|_p \leq \varepsilon.$$

Prop^n Linear combinations of simple functions, step functions ($\phi(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{(c_k, d_k]}$) and continuous functions are all dense in $L^p(\mathbb{R})$ for $p \in [1, \infty)$.

Proof The first two results are proved in Assignment 4.

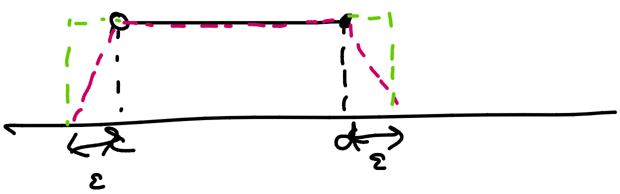
If $f \geq 0$ then $\exists\, g$ simple s.t. $\|f-g\|_p \leq \varepsilon$

If $f = f_+ - f_-$ then take g_+ and g_- approximating it and then

$$\|f-(g_+-g_-)\|_p \leq \|f_+-g_+\|_p + \|f_--g_-\|_p$$

Just going to prove that continuous functions are dense assuming that step functions are dense.

First show that if $f = \mathbb{1}_{(c,d]}$ then $\exists\, g$ continuous s.t. $\|f-g\|_p \leq \varepsilon$.



$$g_{\varepsilon,c,d}(x) = \begin{cases} 0 & x \notin (c-\varepsilon, d+\varepsilon) \\ (x-c+\varepsilon)/\varepsilon & x \in [c-\varepsilon, c) \\ 1 & x \in [c, d) \\ -(x-d-\varepsilon)/\varepsilon & x \in [d, d+\varepsilon) \end{cases}$$

$$|\mathbb{1}_{(c,d]} - g_{\varepsilon,c,d}| \leq \mathbb{1}_{[c-\varepsilon, c]} + \mathbb{1}_{[d, d+\varepsilon]}$$

$$\| \mathbb{1}_{[c,d]} - g_{\varepsilon,c,d} \|_p \leq \| \mathbb{1}_{[c-\varepsilon, c)} + \mathbb{1}_{[d, d+\varepsilon]} \|_p$$

$$\leq 2\varepsilon$$

If we have a step function

$\varphi(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{(c_k, d_k]}$ and let $g(x) = \sum_{k=1}^{n} a_k \, g_{\varepsilon/2|a_k|n, c_k, d_k}$

$$\| \varphi - g \|_p \leq \sum_{k=1}^{n} |a_k| \, \| \mathbb{1}_{(c_k, d_k]} - g_{\varepsilon/2|a_k|n, c_k, d_k} \|_p$$

$$\leq \sum_{k=1}^{n} \cancel{|a_k|} \frac{\cancel{2}\varepsilon}{\cancel{2|a_k|} n} = \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon$$

So given φ a step function and $\varepsilon > 0$

we can find g a continuous function s.t.

$\| \varphi - g \|_p \leq \varepsilon$

Then given any $f \in \mathcal{L}^p(E)$, $\varepsilon > 0$ $\exists$ φ a step function st. $\| f - \varphi \|_p \leq \varepsilon/2$

and there will then exist g continuous s.t.

$\| \varphi - g \|_p \leq \varepsilon/2$ so

$$\| f - g \|_p = \| f - \varphi + \varphi - g \|_p \leq \| f - \varphi \|_p + \| \varphi - g \|_p$$

$$\leq \varepsilon .$$

I missed some lemmas showing some things were measurable

<u>Lemma</u>

The composition of two measurable functions is measurable

If $(E, \mathcal{E})$, $(F, \mathcal{F})$ and $(G, \mathcal{G})$ are all measurable spaces and $f: E \to F$, $g: F \to G$ are both measurable then $g \circ f : E \to G$ is also measurable

Pf/ Take any $A \in \mathcal{G}$ wts

$$(g \circ f)^{-1}(A) \in \mathcal{E}$$

$$(g \circ f)^{-1}(A) = \{ x \in E : g(f(x)) \in A \}$$

Let $B = g^{-1}(A) = \{ y \in F : g(y) \in A \}$

then $B \in \mathcal{F}$ as g is measurable

$$(g \circ f)^{-1}(A) = \{ x \in E : g(f(x)) \in A \} = \{ x \in E : g(y) \in A, y = f(x) \}$$

$$= \{ x \in E : y \in B, y = f(x) \}$$

$$= \{ x \in E : f(x) \in B \}$$

$$= f^{-1}(B)$$

so $f^{-1}(B) \in \mathcal{E}$ as f is measurable

Therefore $(g \circ f)$ is measurable.

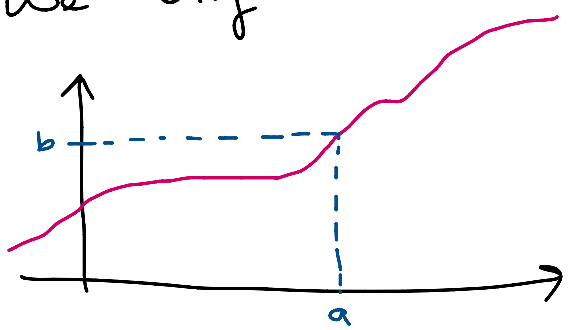
Lemma

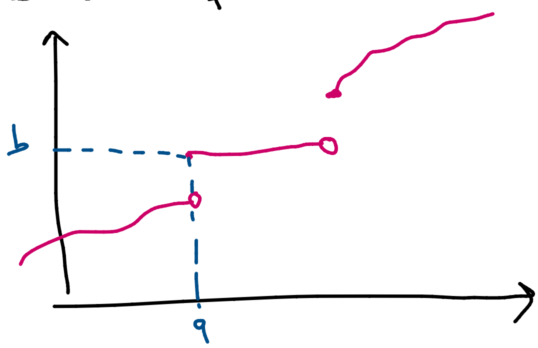
If $f: \mathbb{R} \to \mathbb{R}$ and f is monotone

then f is measurable wrt $\mathcal{B}(\mathbb{R})$

Proof Suppose f is non-decreasing

we only need to check $f^{-1}((-\infty, b])$





$f^{-1}((-\infty, b])$ is going to be of the form

$(-\infty, a]$, $(-\infty, a)$ or $\emptyset$ or $(-\infty, \infty)$

all of these are Borel measurable so

f is Borel measurable.