Week 7 Lecture 1
16 November 2021 09:01

Starting at 9.05

Remind me to record!

Assignment 1 marks are finally out!

Assignment 3 is in on Thursday

= Assignment 4 out Friday

Riemann integration vs Lebesgue integration

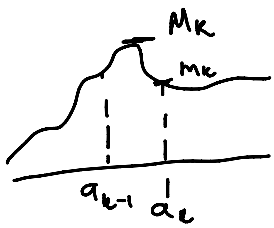
Def^n Let $[a,b]$ be an interval then a finite collection of real numbers $\{a_k\}_{k=1}^{n}$ is a partition of $[a,b]$ if $a = a_0 < a_1 < a_2 < \dots < a_n = b$.

I will denote partitions p or q but you might have seen $\underbrace{\mathcal{P} \text{ or } \pi}_{\text{not going to use this}}$.

Def^n Given a function f and a partition define

$$m_k = \inf\{f(x) : x \in [a_{k-1}, a_k]\}$$

$$M_k = \sup\{f(x) : x \in [a_{k-1}, a_k]\}$$



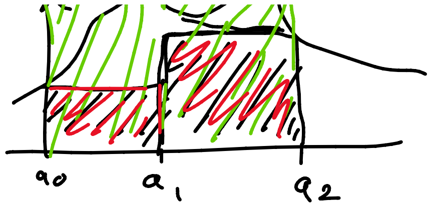
we use these to define

$$l(f,p) = \sum_{k=1}^{n} m_k (a_k - a_{k-1}) \leftarrow \text{lower sum}$$

$$u(f,p) = \sum_{k=1}^{n} M_k (a_k - a_{k-1}) \leftarrow \text{upper sum}$$

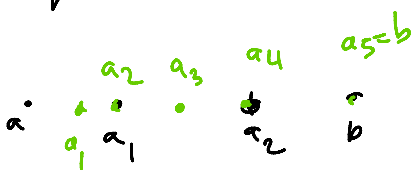
lower sum is sum of the red rectangles

upper sum is sum of



upper sum is sum of green rectangles

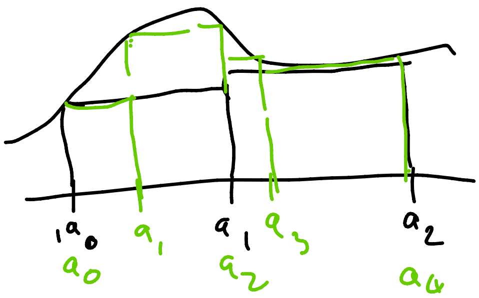
We say a partition q is a <u>refinement</u> of another partition p if every element of p is contained in q.



We call a sequence of partitions $\{p_n\}_{n\geq 1}$ <u>nested</u> if p_n is a refinement of p_{n-1}

If q is a refinement of p then

$$\ell(f,p) \leq \ell(f,q) \leq u(f,q) \leq u(f,p)$$



the lower sum will be larger on a refined partition

<u>Defn</u> A function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if $\sup_p \ell(f,p) = \inf_q u(f,q)$.

then we call this the Riemann integral of f.

[You might also have seen a definition looking ... whose max interval size

[You might also have seen a defi.....
nested sequences of partitions whose max interval size
goes to 0]

Before we prove our main theorem need 2 lemmas

<u>Lemma</u> Another characterisation of Riemann integrability:

$f: [a,b] \to \mathbb{R}$ is Riemann integrable iff for every $\varepsilon > 0$ $\exists$ p a partition with

$$u(f,p) - \ell(f,p) < \varepsilon.$$

Pf First suppose $\forall \varepsilon$ $\exists p$ s.t. $u(f,p) - \ell(f,p) < \varepsilon$

then as for this p

$$\ell(f,p) \leq \sup_q \ell(f,q) \leq \inf_q u(f,q) \leq u(f,p)$$

so $\inf_q u(f,q) - \sup_q \ell(f,q) < \varepsilon$

Since ε is arbitrary

$\inf_q u(f,q) = \sup_q \ell(f,q)$ so q is RI

Suppose f is Riemann integrable

$$\inf_q u(f,q) = \sup_q \ell(f,q) = \int_a^b f(x)dx$$

Therefore for any $\varepsilon > 0$ $\exists$ p_1 and p_2

$\ldots \int_a^b f(x)dx - \varepsilon/2$

Therefore for any $\varepsilon > 0$ $\exists$ P_1 and P_2

$u(f, P_1) \leq \int_a^b f(x)dx + \varepsilon/2$ and $\ell(f, P_2) \geq \int_a^b f(x)dx - \varepsilon/2$

now if we take $P = P_1 \cup P_2$

then P is a refinement of both P_1 and P_2

$$\int f dx - \varepsilon/2 \leq \ell(f, P_2) \leq \ell(f, P) \leq u(f, P) \leq u(f, P_1) \leq \int_a^b f(x)dx + \varepsilon/2$$

so $u(f, P) - \ell(f, P) \leq \varepsilon.$

<u>Lemma</u> Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable (measurable wrt $\mathcal{M}$ the σ-algebra of Lebesgue meas sets) and suppose $g : \mathbb{R} \to \mathbb{R}$ is equal to $g = f$ Lebesgue almost everywhere, then g is also Lebesgue measurable.

Pf/ Lets us take any $B \in \mathcal{M}$ we want to show $g^{-1}(B) \in \mathcal{M}$.

We can write $g^{-1}(B) = (f^{-1}(B) \cup \{x : g(x) \in B, f(x) \notin B\})$
$\setminus \{x : g(x) \notin B, f(x) \in B\}$

$\{x : g(x) \in B, f(x) \notin B\}, \{x : g(x) \notin, f(x) \in B\} \subseteq \{x : f(x) \neq g(x)\}$

we said $f(x) = g(x)$ λ - almost everywhere

so $\lambda(\{x : f(x) \neq g(x)\}) = 0$

$\{x : g(x) \in B, f(x) \notin B\}, \{x : g(x) \notin B, f \in B\}$ are both so $\lambda(\{x: f(x) \neq g(x)\})$

So $\{x : g(x) \in B,\ f(x) \in B\}$, $\{x : g(x) \notin B, f \in B\}$ are both null sets. This means they are in $\mathcal{M}$.

$$g^{-1}(B) = \underbrace{f^{-1}(B)}_{\substack{\in \mathcal{M} \\ \text{as } f \text{ is} \\ \text{measurable}}} \cup \underbrace{\{x : g(x) \in B,\ f(x) \notin B\}}_{\in \mathcal{M}} \setminus \underbrace{\{x : g(x) \notin B, f(x) \in B\}}_{\in \mathcal{M}}$$

So we can for $g^{-1}(B)$ by a countable (2) number of set operations on sets in $\mathcal{M}$

so $g^{-1}(B) \in \mathcal{M}$.

Theorem Let $[a,b]$ be an interval. Suppose f is positive bounded and Riemann integrable, then it is also Lebesgue measurable, Lebesgue integrable and the two integrals agree.

Pf/ The important part of proving integrability is proving measurability.

Key idea is to approximate by simple functions related to the upper and lower sum.

By the first lemma we can produce a nested sequence of partitions P_n s.t.

$$u(f, P_n) - l(f, P_n) \leq \frac{1}{n}$$

($\exists$ q_n s.t. $u(f,q_n) - \ell(f,q_n) \leq \frac{1}{n}$ then take

$p_n = q_1 \cup q_2 \cup \dots \cup q_n$ then as p_n is a refinement

of q_n $u(f,p_n) - \ell(f,p_n) \leq \frac{1}{n}$ and we've constructed
the p_n to be nested)

Lets write $p_n = \{a_k^n\}_{k=0}^{N_n}$

Two sequences of simple functions

$$g_n = \sum_{k=1}^{N_n} m_k^n \mathbb{1}_{[a_{k-1}^n, a_k^n)}(x) \qquad h_n = \sum_{k=1}^{N_n} M_k^n \mathbb{1}_{[a_{k-1}^n, a_k^n)}(x)$$

we can see that $\lambda(g_n) = \ell(f,p_n)$

$\lambda(h_n) = u(f,p_n)$

Since ~~the~~ p_ns are nested g_n is non-decreasing

and h_n is non-increasing.

So $g(x) = \lim_{n\to\infty} g_n(x)$ exists as does $h(x) = \lim_{n\to\infty} h_n(x)$

we also have $g_n(x) \leq h_n(x) \leq \sup_{y \in [a,b]} f(y)$

so g_n and h_n are measurable for each n

so g and h are also Lebesgue measurable as
they are the limit of Lebesgue measurable
functions and they are bounded by $\sup_{[a,b]} f$

so they are integrable.

So by dominated convergence with $\sup_{[a,b]} f$ as

So by dominated convergence with $\sup_{[a,b]} f$ as a dominating function

$$\lambda(g) \overset{DOM}{=} \lim_{n\to\infty} \lambda(g_n) = \lim_{n\to\infty} \ell(f, p_n) =$$

$$\lim_{n\to\infty} u(f, p_n) = \lim_{n\to\infty} \lambda(h_n) \overset{DOM}{=} \lambda(h)$$

so $\lambda(g) = \lambda(h)$ we also have $h_n \geq g_n$ for every n and $h_n \geq f \geq g_n$ for every n

so $\lambda(h-g) = 0$ and $h-g \geq 0$

so $h-g = 0$ almost everywhere

$h-f \leq h-g$ and $h-f \geq 0$

so $h-f = 0$ almost everywhere

so $h=f$ a.e. so f is Lebesgue measurable

$\lambda(f) = \lambda(h) = \lim_n u(f, p_n) =$ Riemann integral of f.