Measure Theory: Exercises (not for credit)

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Question 1. Let C be a countable subset of $\mathbb{R}$. Show that $\lambda^*(C) = 0$.

Answer: We first show that $\lambda^*(\{b\}) = 0$. We have $\lambda^*(\{b\}) \leq \lambda((b-1/n, b]) = 1/n$. So letting $n \to \infty$ gives $\lambda(\{b\}) = 0$. Then we can write $C = \{b_1\} \cup \{b_2\} \cup \{b_3\} \cup \dots$ then, by countable subadditivity, we have $\lambda^*(C) \leq \sum_n \lambda^*(\{b_n\}) = 0$. □

Question 2. For each set $A \in \mathbb{R}^d$ show that there is a Borel subset, B, of $\mathbb{R}$ such that $\lambda(B) = \lambda^*(A)$, and $A \subseteq B$.

Answer: Let us define a sequence of intervals $I_{n,k}$ with two indices, by finding such a sequence with $A \subseteq \bigcup_k I_{n,k}$ and $\sum_k \lambda(I_{n,k}) \leq \lambda^*(A) + 2^{-n}$. Then write $J_n = \bigcup_k I_{n,k}$ for each k and $B_n = \bigcap_{i=1}^n J_n$. Since the $I_{n,k}$ are Borel sets it follows that the J_n are Borel sets and then that the B_n are Borel sets. We also have $A \subseteq B_n$ for each n and $\lambda(B_n) \leq \lambda(J_n) \leq \sum_k \lambda(I_{n,k}) \leq \lambda^*(A) + 2^{-n}$. We then let $n \to \infty$ and get $A \subseteq \bigcup_n B_n$ and $\lambda^*(\bigcup_n B_n) \leq \lambda^*(A)$. By monotonicity of λ^* we also have $\lambda^*(A) \leq \lambda(\bigcup_n B_n)$ therefore $\lambda^*(A) = \lambda(\bigcup_n B_n)$ and $\bigcup_n B_n$ is a Borel set as it is the countable intersection of Borel sets. □

Question 3. Let B be a Borel subset of $[0, 1]$ show that there exists a finite, disjoint sequence of half open intervals A such that $\lambda(A \triangle B) \leq \epsilon$. Here $A \triangle B = (A^c \cap B) \cup (A \cap B^c)$.

Answer: This is similar to question 1. Let us take a sequence I_n of half open intervals such that $B \subseteq \bigcup_n I_n$ and $\sum_n \lambda(I_n) \leq \lambda(B) + \epsilon/2$. Now since the sum $\sum_n \lambda(I_n)$ converges there exists an N such that $\sum_{n \geq N} \lambda(I_n) < \epsilon/2$. We then write $A = \bigcup_{n=1}^{N-1} I_n$ this is a finite union of half open intervals so can be expressed as a finite disjoint union of half open intervals. We also have that $B \cap A^c \subseteq \bigcup_{n \geq N} I_n$ therefore $\lambda(B \cap A^c) \leq \sum_{n \geq N} \lambda(I_n) \leq \epsilon/2$. We also have that $A \cap B^c \subseteq \bigcup_{n=1}^{\infty} I_n \setminus B$ so $\lambda(A \cap B^c) \leq \lambda\left(\bigcup_{n=1}^{\infty} I_n \setminus B\right) = \lambda(\bigcup_n I_n) - \lambda(B) \leq \epsilon/2$. Putting this together gives $\lambda(A \triangle B) \leq \epsilon$. □

Question 4. Let $(E, \mathcal{E}, \mu)$ be a finite measure space and let A_n be a sequence of measurable sets. Show that

$$\mu\left(\bigcup_n \bigcap_{m \geq n} A_m\right) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu\left(\bigcap_n \bigcup_{m \geq n} A_m\right).$$

Find an example to show that the last inequality is not necessarily true if μ is not finite.

Answer: The sequence $\bigcap_{m\geq n} A_m$ is increasing sequence. Therefore by continuity we have

$$\mu\left(\bigcup_n \bigcap_{m\geq n} A_m\right) = \lim_n \mu(\bigcap_{m\geq n} A_m).$$

By monotonocity of μ we have

$$\mu(\bigcap_{m\geq n} A_m) \leq \mu(A_m), \quad \forall\, m \geq n$$

therfore

$$\mu(\bigcap_{m\geq n} A_m) \leq \inf_{m\geq n} \mu(A_m).$$

Putting this all together gives the first inequality.

The second inequality is just the fact that $\liminf \leq \limsup$.

The sequence $\bigcup_{m\geq n} A_m$ is decreasing, and we are working in a finite measure space so all the sets have finite measure. By our continuity theorem this means that

$$\mu\left(\bigcap_n \bigcup_{m\geq n} A_m\right) = \lim_n \mu(\bigcup_{m\geq n} A_m).$$

By monotonicity we have

$$\mu(A_m) \leq \mu\left(\bigcup_{m\geq n} A_m\right) \quad \forall\, m \geq n.$$

Therefore we have

$$\sup_{m\geq n} \mu(A_m) \leq \mu\left(\bigcup_{m\geq n} A_m\right).$$

Putting all this together gives the last inequality.

For a counterexample let $\mu = \lambda$ on $\mathbb{R}$ and $A_n = [n, n+1]$ then $\bigcup_{m\geq n} A_m = [n, \infty)$ and $\bigcap_n \bigcup_{m\geq n} A_m = \bigcap_n [n, \infty) = \emptyset$. Therefore we have $\limsup_n \mu(A_n) = 1$ as $\mu(A_n) = 1$ for every n, but $\mu(\bigcap_n \bigcup_{m\geq n}) = \mu(\emptyset) = 0$. $\square$