We defined the class of measurable functions, we defined simple functions and the integral of simple functions. It remains to define the integral of more general measurable functions (like functions we might care about).

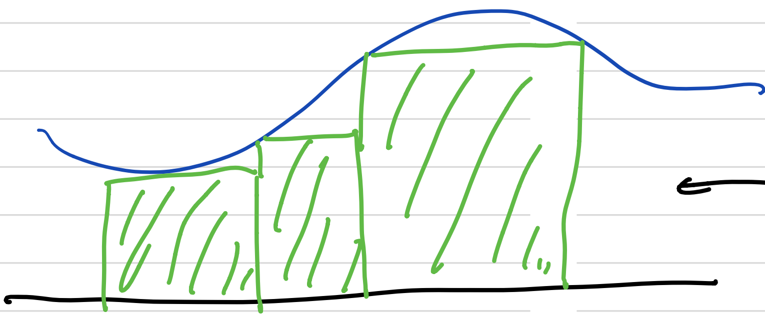
We start with the case of non-negative measurable functions.

This avoids a potential issue with cancelling $+\infty$ and $-\infty$.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

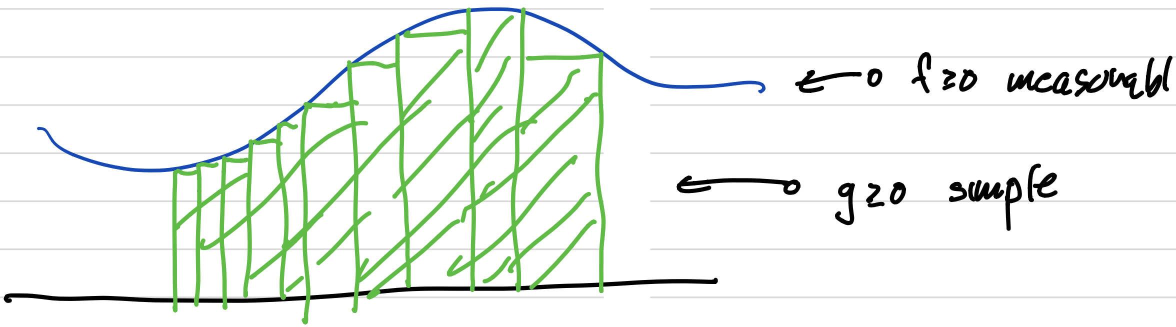
Definition: (p. 55) Let $f: X \to [0, \infty]$ be an $\mathcal{A}$ measurable function. Then

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu : g \in \mathcal{L}^+ \text{ and } g \leq f \right\}.$$



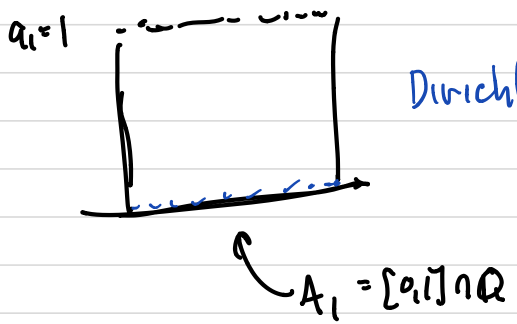
$\leftarrow$ $f \geq 0$ measurabl

$\leftarrow$ $g \geq 0$ simple



The answer is in $[0, \infty]$. $+\infty$ is a possible answer.

Recall that though the picture looks the Riemann integral picture it is different since the sets A_j in the sums $\sum a_j \chi_{A_j}$ are only measurable and not necessarily intervals.



Dirichlet function.

$f = \chi_{[0,1] \cap \mathbb{Q}}$

One good thing about this definition is that it seems natural.

One problem with this definition is that you are taking the supremum over a very large set.

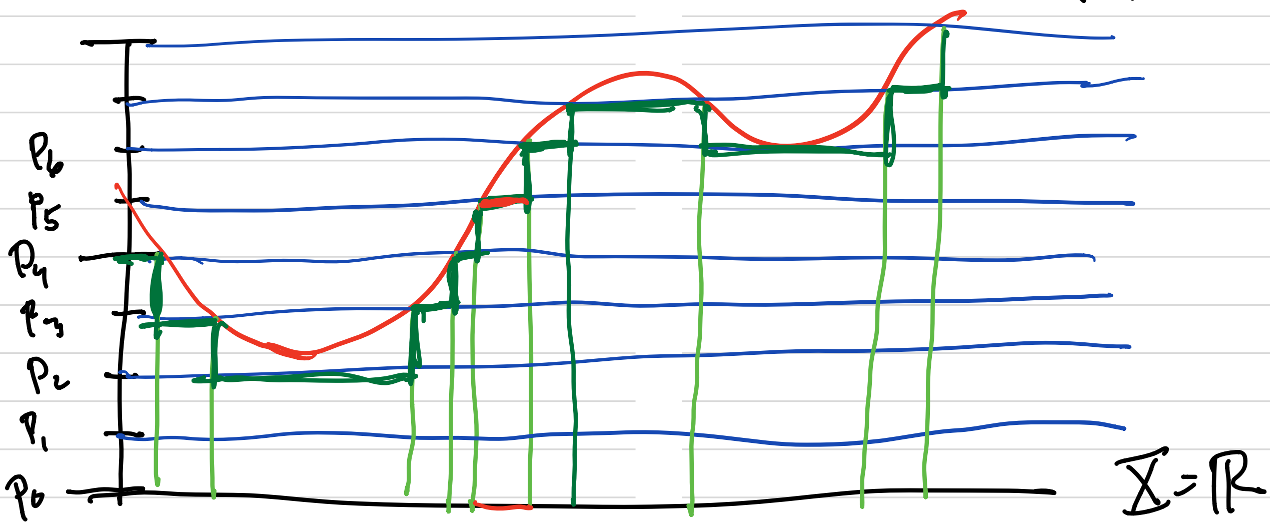
A second problem is that the supremum definition is good for getting lower bounds on $\int f \, d\mu$ but not good for getting upper bounds on $\int f \, d\mu$.

Is there some more checkable way to verify that you have calculated the integral?

Can you simultaneously get lower bounds, upperbounds and reduce the collection of step functions you have to consider?

Recall our "lower sum construction" from last week:

f in red.



f_2 in green.

P_2

We proved that for any measurable $f \geq 0$ there is a sequence of functions $f_n \in \mathcal{S}^+$

$$f_1(x) \leq f_2(x) \leq f_3(x) \dots \quad \text{(monotone sequence)}$$

with

$$\lim_{n \to \infty} f_n(x) = f(x). \quad \text{(pointwise convergence)}$$

Is it true that

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu \, ?$$

Yes.

Proposition 2.3.3. Let $f : X \to [0, \infty]$ be measurable and $\{f_n\}$ a monotone sequence converging pointwise to to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

In fact f_n can be any sequence satisfying these two properties. It need not come from the lower sums construction.

I will give an application of this result.

Proposition 2.3.4. Let f, g be $[0, +\infty]$ valued measurable functions then

$$\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu.$$

Proof that $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$.

We have $\{f_n\} \in \mathcal{E}^+$ with f_n monotone, $f_n \to f$ pointwise.
We have $\{g_n\} \in \mathcal{E}^+$ with g_n monotone, $g_n \to g$ pointwise.
So $\int f_n\,d\mu \to \int f\,d\mu$ and $\int g_n\,d\mu \to \int g\,d\mu$ by Prop. 2.3.3.

Now $\{f_n + g_n\} \in \mathcal{E}^+$, $f_n + g_n$ monotone and $f_n + g_n \to f+g$ pointwise. So again by Prop. 2.3.3

$$\int f_n + g_n\,d\mu \to \int f+g\,d\mu$$

Putting this together:

$$\int f+g \, d\mu = \lim_{n\to\infty} \int f_n+g_n \, d\mu = \lim_{n\to\infty} \left(\int f_n \, d\mu + \int g_n \, d\mu \right) = \int f \, d\mu + \int g \, d\mu.$$

Prop. 2.3.3

simple fcns property

Prop. 2.3.3

We now deal with $[-\infty, \infty]$ valued functions. Given such an f we can construct two $[0, \infty]$ valued functions.

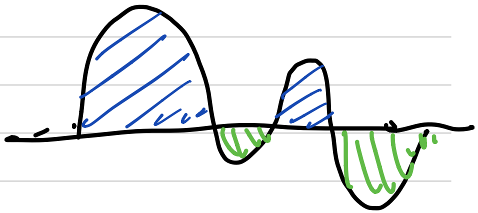
Say f is measurable. Define

$$f^+ = \max(f, 0)$$

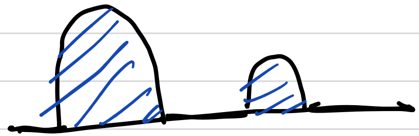
$$f^- = \max(-f, 0) = -\min(f, 0)$$

We have shown that the max of two measurable functions is measurable and constant functions are always measurable.

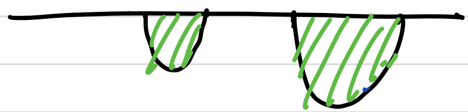
f



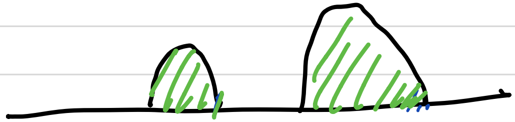
$f^+ = \max(f,0)$



$-f^- = \min(f,0)$



$f^- = \max(-f,0)$



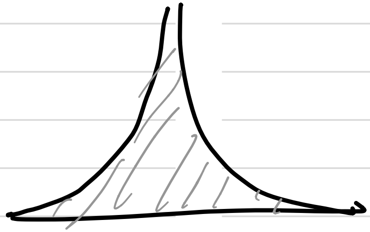
Def. f is <u>integrable</u>

if $\int f^+ d\mu < \infty$ and

$\int f^- d\mu < \infty$.

$f(x) = \frac{1}{|x|}$ is not integrable.

$$\int \frac{1}{|x|} d\mu$$

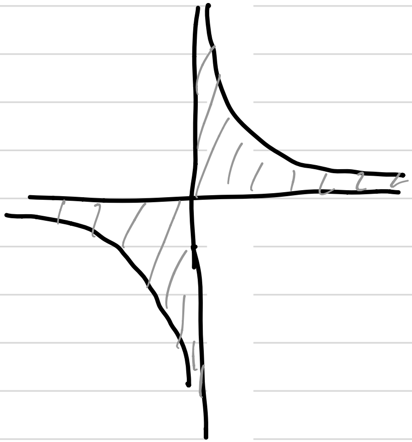


$\int f^+ d\mu = \infty$

$\int f^- d\mu = 0$

$f(x) = \frac{1}{x}$

is not integrable



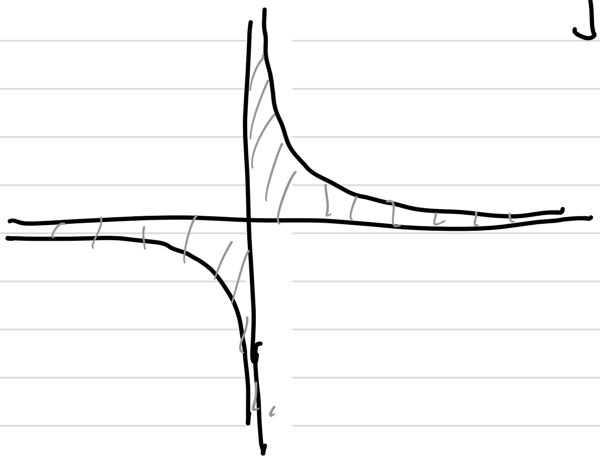
$\int f^+ d\mu = \infty$

$\int f^- d\mu = \infty$

$f(x) = \frac{1}{x^3}$ is integrable

$\int f^+ d\mu < \infty$

$\int f^- d\mu < \infty$



If f is integrable then

Def. $\int f \, d\mu = \int f^+ d\mu - \int f^- d\mu.$

We also say that an integrable function is $\mathcal{L}^1$.

I will show that the integral on $\mathcal{L}^1$ functions has all the expected properties.