We consider the SDE:

$$\begin{cases} \mathrm{d}X_t & = V_t \,\mathrm{d}t \\ \mathrm{d}V_t & = -(\nabla_x \phi(X_t) + \lambda V_t)\,\mathrm{d}t + \mathrm{d}W_t \end{cases}$$

We aim to show:

$$\mathbb{E}\left(\exp\left(\phi(X_t) - \frac{1}{2}V_t^2\right)\right) < \infty$$

Consider $H(x,v) := \phi(x) + Ax^2 + 2Bxv + Cv^2$, for some A, B, C. By Ito's formula:

$$\mathrm{d}\left(e^{\alpha H(X_t,V_t)}\right) =$$
$$= \alpha(V_t\nabla_x\phi(X_t) + 2AX_tV_t + BV_t^2 - \lambda BX_tV_t - 2\lambda CV_t^2 - BX_t\nabla_x\phi(X_t) - 2CV_t\nabla_x\phi(X_t)$$
$$+ \frac{1}{2}\alpha(2CV_t + BX_t)^2 + 2C)e^{\alpha H(X_t,V_t)}\,\mathrm{d}t + (2BX_t + 2CV_t)e^{\alpha H(X_t,V_t)}\,\mathrm{d}W_t$$

Now pick $C = \frac{1}{2}$ so that the $V_t\nabla_x\phi(X_t)$ terms cancel. We also take expectations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(e^{\alpha H(X_t,V_t)}\right)$$
$$= \alpha\left(2AX_tV_t + BV_t^2 - \lambda BX_tV_t - \lambda V_t^2 - BX_t\nabla_x\phi(X_t) + \frac{\alpha}{2}(V_t + BX_t)^2 + 1\right)e^{\alpha H(X_t,V_t)}\,\mathrm{d}t$$

Now we pick $A = \frac{\lambda B}{2}$ so that the X_tV_t terms cancel. We are left with:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left(e^{\alpha H(X_t,V_t)}\right) &= \alpha\left(BV_t^2 - \lambda V_t^2 - BX_t\nabla_x\phi(X_t) + \frac{\alpha}{2}(V_t + BX_t)^2 + 1\right)e^{\alpha H(X_t,V_t)}\,\mathrm{d}t \\ &=: F(X_t,V_t)e^{\alpha H(X_t,V_t)}\end{aligned}$$

For X large enough, $X_t\nabla\phi(X_t)$ behaves approximately like X_t^2. Now pick $B = \frac{\lambda-1}{2}$. We then have (using Young's inequality):

$$\begin{aligned}F(X_t,V_t) &\sim -\alpha\left(\left(\lambda - \frac{\lambda-1}{2}\right)V_t^2 + \frac{\lambda-1}{2}X_t^2\right) + \frac{\alpha^2}{2}\left(V_t + \frac{\lambda-1}{2}X_t\right)^2 + \frac{\alpha}{2} \\ &= -\alpha\left(\frac{\lambda+1}{2}V_t^2 + \frac{\lambda-1}{2}X_t^2\right) + \frac{\alpha^2}{2}\left(V_t^2 + (\lambda-1)X_tV_t + \frac{(\lambda-1)^2}{4}X_t^2\right) + \frac{\alpha}{2} \\ &\leq -\alpha\left(\frac{\lambda+1}{2}V_t^2 + \frac{\lambda-1}{2}X_t^2\right) + \frac{\alpha^2}{2}\left(\left(1 + \frac{(\lambda-1)}{2}\right)V_t^2 + \frac{\lambda-1}{2}\left(1 + \frac{\lambda-1}{2}X_t^2\right)\right) + \frac{\alpha}{2} \\ &= -\alpha P(X_t,V_t) + \alpha^2 Q(X_t,V_t) + \frac{\alpha}{2}\end{aligned}$$

where:

$$P(X_t, V_t) = \frac{\lambda + 1}{2} V_t^2 + \frac{\lambda - 1}{2} X_t^2 \text{ and}$$
$$Q(X_t, V_t) = \frac{1}{2} \left(\frac{\lambda + 1}{2} V_t^2 + \left(\frac{\lambda - 1}{2} + \frac{(\lambda - 1)^2}{4} \right) X_t^2 \right)$$

We would like to show that there exists a constant C' such that $Q(X_t, V_t) \leq C' P(X_t, V_t)$. This is equivalent to showing that:

$$\left(\frac{\lambda - 1}{2} + \frac{(\lambda - 1)^2}{4} \right) X_t^2 \leq C'(\lambda - 1) X_t^2$$

i.e.:

$$\frac{(\lambda - 1)^2}{4} X_t^2 \leq (2C' - 1) \frac{\lambda - 1}{2} X_t^2$$

We can see that any $C' \geq \frac{\lambda+1}{4}$ would work as then:

$$(2C' - 1) \frac{(\lambda - 1)}{2} \geq \left(\frac{(\lambda + 1)}{2} - 1 \right) \frac{\lambda - 1}{2} = \frac{(\lambda - 1)^2}{4}$$

Then we obtain:

$$F(X_t, V_t) \leq (-\alpha + \alpha^2 C') P(X_t, V_t) + \frac{\alpha}{2} < 0$$

for $\alpha < \frac{1}{C'}$, when X and V are sufficiently large.