Starting with a measure space $(X, \mathcal{A}, \mu)$

We have defined the integral for $\mathcal{A}$-measurable $[-\infty, +\infty]$ valued functions $f : X \to [-\infty, +\infty]$.

We have shown that the integral is linear:

$$\int \alpha f \, d\mu = \alpha \int f \, d\mu$$

$$\int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu$$

for $(-\infty, +\infty)$ valued functions

and $g \leq f \Rightarrow \int g \, d\mu \leq \int f \, d\mu$

We have gone from measures to integrals in stages.

We started by defining integrals of non-negative simple functions directly in terms of measures: $\mathcal{S}^+$

$$\int \sum a_i \chi_{A_i} \, d\mu = \sum a_i \mu(A_i).$$

We next defined integrals of non-negative measurable functions in terms of non-negative simple functions.

$$\int f \, d\mu = \sup_{\substack{g \le f \\ g \in \mathcal{S}^+}} \left\{ \int g \, d\mu \right\}$$

Lastly we defined the integral of a $[-\infty,+\infty]$ valued integrable function in terms of non-negative functions:

$$\int f\,d\mu = \int f^+ d\mu - \int f^- d\mu,$$

$$f = f^+ - f^-$$

f^+ non-neg
f^- non-neg

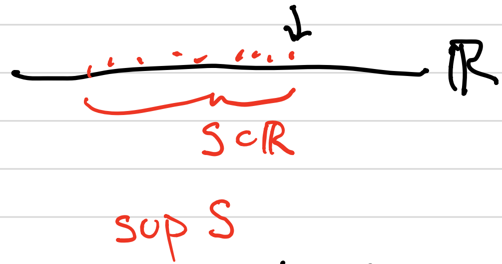
In order to prove linearity for real valued functions we proved some analog of linearity at each stage of our definition.

① If f, g are measurable then so are $f+g$ and αf.

② If f, g are integrable then so are $f+g$ and αf.

A technical issue arises in proving, say additivity for non-negative functions.

The definition of the integral for non-negative functions is in terms of a supremum. In general it is easy to get lower bounds for a supremum of a set of real numbers:



any element of S is a lower bound for $\sup S$.

It is not so easy to get an upper bound for sup S.

If f, g are non-negative measurable functions and $h_1 \in \mathcal{S}^+$ $h_1 \le f$

$h_2 \in \mathcal{S}^+$ $h_2 \le g$

then $h_1 + h_2 \in \mathcal{S}^+$ $h_1 + h_2 \le f+g$.

This shows $\int f+g \, d\mu \ge \int f \, d\mu + \int g \, d\mu$.

Thus we can prove half of what we want directly from the definition.

To prove the other half we need an alternative characterisation of the integral.

Proposition 2.3.3. Let $f : X \to [0, \infty]$ be measurable and $\{f_n\}$ a monotone sequence in $\mathcal{S}^+$ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

The important part of this result is that

$$\int f \, d\mu \leq \lim_{n \to \infty} \int f_n \, d\mu.$$

This gives us an <u>upper bound</u> for the integral.

Remark. This proposition is true in greater generality. We may remove the hypotheses on f and f_n.

Proposition 2.3.3. Let $f : X \to [0,\infty]$ be measurable and $\{f_n\}$ a monotone sequence in $\mathcal{S}^+$ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

This more general result is called the Monotone Convergence Theorem and we will prove it next week.

Comment on the proof of:

Proposition 2.3.3. Let $f : X \to [0,\infty]$ be measurable and $\{f_n\}$ a monotone sequence in $\mathcal{S}^+$ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

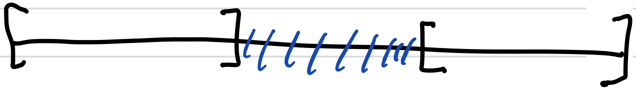
We prove this in two steps. First we assume $f \in \mathcal{S}^+$. In this case the proof comes down to considering the measure of an increasing union of sets.

For general f we use the fact that $\int f d\mu$ is approximated by $\int g d\mu$ where g is a simple function. We have $f_n \to f$ and we want $g_n \to g$ in order to apply version 1. We set $g_n = \min(f_n, g)$.

$$g_n \leq g_{n+1} \leq \dots$$

$$\lim_{n\to\infty} g_n(x) = g(x).$$

The measure of the Cantor set is 0.

$C =$ 

2^1 2 intervals length $1/3$

$\lambda(\text{cover}) = 2/3$

2^2 4 intervals length $1/9$

$4/9$

2^3 intervals length $\frac{1}{3^3}$

$\frac{2^3}{3^3} = \left(\frac{2}{3}\right)^3$

2^4 $\frac{1}{3^4}$ $\frac{2^4}{3^4} = \left(\frac{2}{3}\right)^4.$

$n \to \infty$ $\left(\frac{2}{3}\right)^n \to 0.$

$$0 \leq \lambda(C) \leq \lim_{n \to \infty} \frac{2^n}{3^n} = 0$$

$$\lambda(C) = 0.$$