# Modes of convergence

$(X, \mathcal{A}, \mu)$

$f_n : X \to \mathbb{R}$ $\quad f : X \to \mathbb{R}$

## ① Pointwise convergence:

For each $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$

## ② Almost everywhere convergence:

For a.e. $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$.

## ③ Convergence in measure

$f_n$ converges to $f$ in measure

if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

Pointwise convergence a.e.
does not imply convergence
in measure and convergence
in measure does not imply
pointwise convergence. a.e.

**Prop. 3.1.2** If $(X, \mathcal{a}, \mu)$ is a <u>finite</u> measure space (if $\mu(X)<\infty$) then convergence a.e. implies convergence in measure.

Proof. Let $\varepsilon>0$. We need to show that

$$\lim_{n\to\infty} \mu\left(\{x\in X : |f_n(x)-f(x)|>\varepsilon\}\right)=0.$$

Let $A_k=\{x\in X : |f_k(x)-f(x)|>\varepsilon\}$

Let

$$B_n=\bigcup_{k=n}^{\infty} A_k.$$

This is the set of $x$ which are in some $A_k$ for $k\geq n$.

$$\boxed{B_n = \bigcup_{k=n}^{\infty} A_k.}$$

Observe that $B_n$ is a decreasing sequence of sets.

Consider the set $B_\infty = \bigcap_{n=1}^{\infty} B_n$. This is the set of $x$ for which are in infinitely many $A_k$.

Note that if $\lim_{n\to\infty} f_n(x) = f(x)$ then $x$ is only in finitely many sets $A_k$

$$A_k = \{x \in X : |f_k(x) - f(x)| > \varepsilon\}$$

so $x$ is not in $B_\infty$.

Since $f_n \to f$ a.e. we conclude that almost every point is not in $B_\infty$ so $B_\infty$ has measure 0.

Recall that we proved measure continuity theorems about increasing unions and decreasing intersections. In the decreasing case we needed a finiteness of measure hypothesis. Since our measure is finite we have:

$$\lim_{n\to\infty} \mu(B_n) = \mu\Big(\underbrace{\bigcap_{n=1}^{\infty} B_n}_{B_\infty}\Big) = 0.$$

Since $A_n \subset B_n$ it follows that

$$\lim_{n\to\infty} \mu(A_n) = 0$$

$A_n = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$

and that is what we wanted to show.

**Prop. 3.1.3** If $f_n$ converges to $f$ in measure then some subsequence of $f_n$ converges pointwise to $f$.

Proof. Say $f_n$ converges in measure to $f$. For $\varepsilon > 0$ we have that

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

We will construct a subsequence $f_{n_1}, f_{n_2} \dots$ that converges pointwise to $f$.

Choose $n_1$ so that:

$$\mu\left(\{x \in X : |f_{n_1}(x) - f(x)| > 1\}\right) \le \frac{1}{2}.$$

Choose $n_2 > n_1$ so that:

$$\mu\left(\{x \in X : |f_{n_2}(x) - f(x)| > \tfrac{1}{2}\}\right) \le \frac{1}{4}.$$

Choose $n_{k+1} > n_k$ inductively so that:

$$\mu\left(\{x \in X : |f_{n_k}(x) - f(x)| > \tfrac{1}{k}\}\right) \le \frac{1}{2^k}. \qquad (1)$$

We claim that the subsequence

$f_{n_k}$ converges pointwise a.e.

Define sets $A_k$ $k=1,2,\ldots$ by:

$$A_k = \left\{ x \in X : |f_{n_k}(x) - f(x)| > \frac{1}{k} \right\}.$$

Consider the set

$$A_\infty = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$$

As in the previous proof this is precisely the set of $x$ which lie in $A_k$ for $\infty$ many indices $k$.

If $x$ lies in only finitely many sets $A_k$ then for $k$ sufficiently large $|f_{n_k}(x) - f(x)| \leq \frac{1}{k}$ so $f_{n_k}(x) \to f(x)$ as $k \to \infty$.

Now we want to show that the measure of

$$A_\infty = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k$$

is 0.

$$\boxed{A_\infty = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k}$$

$$\mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq \sum_{k=j}^{\infty} \mu(A_k) \qquad \text{(sub-additivity)}$$

$$\leq \sum_{k=j}^{\infty} \frac{1}{2^k} \qquad \mu(A_k) \leq \frac{1}{2^k} \text{ by (1)}$$

$$= \frac{1}{2^{j+1}} . \qquad \text{(summable errors)}$$

$$\boxed{\mu\left(\bigcup_{k=j}^{\infty} A_k\right) \leq \frac{1}{2^{j+1}}.}$$

So $\quad A_\infty = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k \subset \bigcup_{k=j}^{\infty} A_k \quad$ for all $j$

$$\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k\right) \leq \frac{1}{2^{j+1}}$$ for all $j$ and

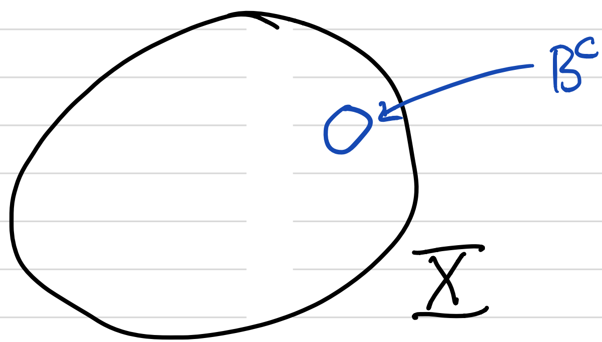
$$\mu\left(\bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k\right) = 0.$$ This finishes the proof.

Uniform convergence is a strong form of convergence.

We know for example that uniform convergence implies convergence of the Riemann integral.

It is interesting to know when we can recover some uniform convergence from the much weaker notion of pointwise convergence.

**Prop. 3.1.4. (Egoroff's Thm).** If $f_n$ converges pointwise to $f$ and $\mu$ is finite then for any $\varepsilon > 0$ there is a subset $B \subset X$ with $\mu(B^c) < \varepsilon$ so that $f_n$ restricted to $B$ converges uniformly to $f$ restricted to $B$.



Note that in this theorem we are not throwing away a set of measure 0.
We are throwing away a set of small but positive measure.

Proof of Egoroff's Theorem.

Say $f_n \to f$ a.e.

Let $g_n(x) = \sup_{j \geq n} |f_j(x) - f(x)|$.

If $f_n(x) \to f(x)$ then the quantity $|f_j(x) - f(x)|$ converges to $0$ so for some $n_0$ it is small for all $n \geq n_0$ thus $g_n(x) \to 0$.

This shows that $g_n \to 0$ a.e.

By Prop. 3.1.2 $g_n \to 0$ in measure (since $\mu$ is finite).

For each $k$ choose an $n_k$ such that

$$\mu\left(\left\{x \in X : g_{n_k}(x) > \frac{1}{k}\right\}\right) < \frac{\varepsilon}{2^k}.$$

Let $B_k = \left\{x \in X : g_{n_k}(x) \leq \frac{1}{k}\right\}$.

Let $B = \bigcap_{k=1}^{\infty} B_k$. This will be the set on which we have uniform convergence.

Let's check this:

Say $\delta > 0$. Let $k$ be such that $\frac{1}{k} < \delta$.
Let $x \in B$. Since $B \subset B_k$ we have

$$|f_n(x) - f(x)| \le g_{n_k}(x) \le \frac{1}{k} < \delta$$

for all $n \ge n_k$.

Now let's estimate the measure of $B^c$.
Since $B = \bigcap B_k$ $\quad B^c = \bigcup B_k^c$ so

$$\mu(B^c) = \mu\left(\bigcup_{k=1}^{\infty} B_k^c\right) \le \sum_{k=1}^{\infty} \mu(B_k^c) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

This completes the proof.

## Convergence in mean.

We say that $f_n$ converges to $f$ in mean if

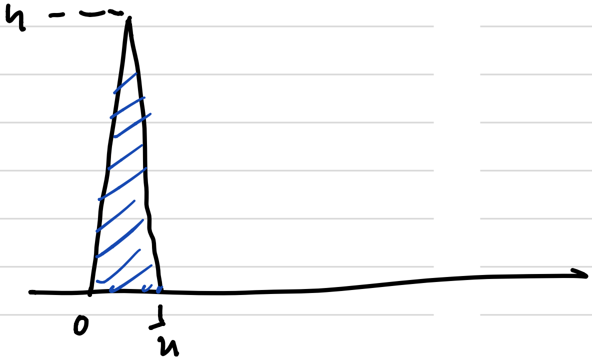
$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

Convergence in mean does not imply pointwise convergence.

Neither pointwise convergence nor convergence in measure imply convergence in mean.

## Example 3.

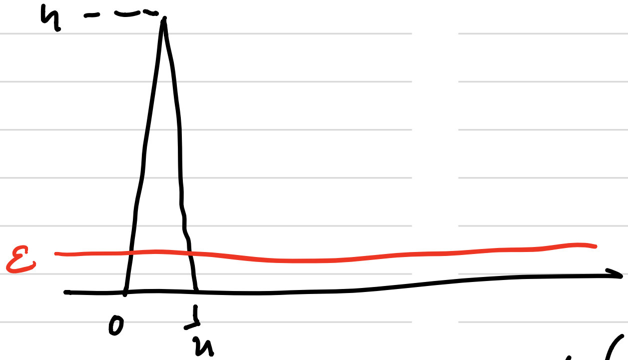
$f_n =$




$$\int |f_n - f|\, dx = \frac{1}{2}$$

This does not go to 0 so $f_n$ does not converge in mean.

$f_n =$




$$\mu(\{x \in X : |f_n(x) - f(x)|\}) \le \frac{1}{n}$$

so goes to 0.

Convergence in measure is only concerned with the size of the sets where small errors occur $\{x : |f_n(x) - f(x)| < \varepsilon\}$.

Convergence in mean involves not just where the errors occur but how big the errors are.

**Prop. 3.1.5** Let $(\mathfrak{X}, \mathcal{A}, \mu)$ be a measure space. Let $f_1, f_2 \ldots$ and $f$ belong to $\mathcal{L}^1(\mathfrak{X}, \mathcal{A}, \mu)$. If $f_n$ converges to $f$ in mean then $f_n$ converges in measure.

Proof. This follows from the Markov inequality (Prop. 2.3.10):

$$\mu(\{x \in \mathfrak{X} : |f_n(x) - f(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon} \int |f_n - f| \, d\mu.$$

Convergence a.e. or convergence in measure + domination hypothesis $\Rightarrow$ convergence in mean.

**Prop. 3.1.6.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_1, f_2 \dots$ and $f$ belong to $\mathcal{L}^1(X, \mathcal{A}, \mu)$. If $f_n$ converges to $f$ almost everywhere or in measure and there is an integable $g$ with

$$|f_n| \leq g \quad \text{and} \quad |f| \leq g \qquad (1)$$

then $f$ converges in mean.

Proof. Say $f_n \to f$ a.e. so $|f_n - f| \to 0$ a.e.

$$|f_n - f| \leq |f_n| + |f| \leq 2g$$

holds a.e. thus Dominated Convergence implies

$$\int |f_n - f| \, d\mu = 0.$$

Say $f_n \to f$ in measure and satisfies (1). Then every subsequence of $f_n$ has a sub-subsequence that converges to $f$ a.e. and hence in mean (by part 1).

If the original sequence did not converge in mean then then there would an $\varepsilon > 0$ and a subsequence $f_{n_k}$ with

$$\int |f_{n_k} - f| \, d\mu \geq \varepsilon$$

for all k. But this contradicts the fact that every sub-subsequence converges in mean.