1 Assignment One - Lebesgue measure in $\mathbb{R}^d$

This assignment is about defining Lebesgue measure on $\mathbb{R}^d$ as opposed to $\mathbb{R}$. Later in the course we will also see product σ-algebras and product measures which give another way of doing this. We begin by defining Lebesgue outer measure, λ^* on $\mathbb{R}^d$ by first defining the measure of the rectangle $(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]$. We define

$$\lambda\left((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]\right) = (b_1 - a_1)(b_2 - a_2) \dots (b_d - a_d).$$

Then for any subset of $\mathbb{R}^d$, A, we define

$$\lambda^*(A) = \inf\{\Pi_{n=1}^{\infty} \lambda(R) \,:\, R_k \text{ are rectangles}, A \subseteq \bigcup_{n=1}^{\infty} R_n\}.$$

Question 1.1. Show that λ^* is indeed an outer measure.

Answer: We've got to check that λ^* gives measure 0 to the empty set, is monotone and countably subadditive. Firstly since $\emptyset \subseteq \emptyset$ and the $\emptyset$ is an (empty) union of intervals we have $\lambda^*(\emptyset) \leq 0$ and since λ^* is positive this gives $\lambda^*(\emptyset) = 0$.

Suppose $A \subseteq B$ then given $\epsilon > 0$ there exixt R a union of rectangles, such that $B \subseteq R$ and $\lambda(R) \leq \lambda^*(B) + \epsilon$. Then $A \subseteq I$ so $\lambda^*(A) \leq \lambda(I) \leq \lambda^*(B) + \epsilon$. As ϵ is arbitrary this gives $\lambda^*(A) \leq \lambda^*(B)$.

Let $A_1, A_2, \dots$ be a sequence of sets. Fixing $\epsilon > 0$ we can find R_n unions of rectangles, such that $\lambda(R_n) \leq \lambda^*(A_n) + 2^{-n}\epsilon$ and $A_n \subseteq R_n$. Therfore we have $A = \bigcup_n A_n \subseteq \bigcup_n (R_n)$ and $\lambda(\bigcup_n R_n) \leq \sum_n \lambda(R_n) \leq \sum_n \lambda^*(A_n) + \epsilon$. Therfore as ϵ is arbirtrary $\lambda(A) \leq \sum_n \lambda(A_n)$. This only works if $\lambda * (A_n) < \infty$ for each n but the other case is kind of obviously true.

$\square$

Question 1.2. Show that if R is a rectangle then $\lambda(R) = \lambda^*(R)$.

Answer: There are two possible proofs, both of which use compactness arguments and you can mix and match between them to some extent:

First via a compactness argument. As R is a rectangle this immediately gives $\lambda^*(R) \leq \lambda(R)$ so we want to prove the inequality in the other direction. Now suppose we have a sequence of rectangles $R_1, R_2, \dots$ such that $R \subseteq \bigcup_n R_n$ then there exists some M such that $R \subseteq (-M, M]^d$. Then without loss of generality we can assume $R_i \subseteq (-M-1, M+1]^d$ as we can just shrink the rectangles and still have a cover of R, and only loose mass (by monotonicity of λ). Now we can write $R = (a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]$ then for any $\epsilon > 0$ and small enough we can define $R_\epsilon = [a_1 + \epsilon, b_1 - \epsilon] \times \cdots \times [a_d + \epsilon, b_d - \epsilon]$. We can also write $R_k = (c_{1,k}, d_{1,k}] \times \dots (c_{d,k}, d_{d,k}]$ and define $R_{k,\delta} = (c_{1,k} - \delta 2^{-k}, d_{1,k} + \delta 2^{-k}) \times \dots (c_{d,k} - delta 2^{-k}, d_{d,k} + \delta 2^{-k})$. Now we have $R_\epsilon \subseteq \bigcup_n R_{n,\delta}$. Then we can use compactness to find some N such that $R_\epsilon \subseteq \bigcup_{n=1}^N R_{n,\delta}$. We can now compare the d-dimensional volume in the normal way to get.

$$vol(R_\epsilon) \leq \sum_{n=1}^{N} vol(R_{k,\delta})$$

and we can bound the extra bits added to or taken away from the rectangles to get for $\delta < 1$

$$vol(R) - 2M^{d-1}\epsilon \leq \sum_{n=1}^{N} (vol(R_n) + (M+2)^{d-1}\delta 2^{-n}) \leq \sum_{n=1}^{\infty} vol(R_n) + 2\delta(M+2)^{d-1}.$$

We can now let both δ and ϵ tend to 0 to get,

$$vol(R) \leq \sum_{n=1}^{\infty} vol(R_n).$$

Therefore taking the infimum over all possible sequences we get

$$vol(R) \leq \lambda^*(R).$$

Therefore as $\lambda(R) = vol(R)$ we have $\lambda(R) = \lambda^*(R)$.

Second proof via the Carathéodory extension theorem part of the notes. Let us take R a rectangle and a sequence of rectangles $(R_n)_{n\geq 1}$ such that $R \subseteq R_n$. Then $\tilde{R}_n = R_n \cap R$ is also a rectangle and $R = \bigcup_n \tilde{R}_n$. We also have that λ is fairly clearly monotone on earch rectangle (remember its just normal d-dimensional volume in this case) so $\lambda(\tilde{R}_n) \leq \lambda(R_n)$. So we would like to show that $\lambda(R) \leq \sum_n \lambda(\tilde{R}_n)$ (i.e. a particular case of the fact that λ is countably subadditive on rectangles). We can write $R = \bigcup_{n=1}^N \tilde{R}_n \cup \left(\bigcup_{n=N+1}^\infty \tilde{R}_n \setminus \bigcup_{n=1}^N \tilde{R}_n\right)$. Let us write $B_n = \bigcup_{n=N+1}^\infty \tilde{R}_n \setminus \bigcup_{n=1}^N \tilde{R}_n$. We can see that $B_n \downarrow \emptyset$. As B_n is formed by taking away finitely many rectangles from a larger rectangle B_n can be written as a finite sequence of disjoint half open rectangles. We can therefore define $\lambda(B_n)$ to be the sum of the volumes of these rectangles and in the rest of this answer we consider λ to be well defined on disjoint unions of rectangles. We need to show $\lambda(B_n) \downarrow 0$. Let us argue by contradiction, suppose there exists some $\epsilon > 0$ so that $\lambda(B_n) \geq \epsilon$ for every n then by shrinking the rectangles making up B_n slightly we can find some C_n a disjoint sequence of rectangles with $\bar{C}_n \subseteq B_n$ and $\lambda(B_n \setminus C_n) \leq \epsilon 2^{-n-1}$. Then we have $\lambda(B_1 \setminus (C_1 \cap \cdots \cap C_n)) \leq \lambda((B_1 \setminus C_1) \cup \cdots \cup (B_n \setminus C_n)) \leq \sum_n \epsilon 2^{-n-1} = \epsilon/2$. Then we have that λ is additive on finite disjoint unions of rectangles so $\lambda((C_1 \cap \cdots \cap C_n)) = \lambda(B_n) - \lambda(B_n \setminus (C_1 \cap \cdots \cap C_n)) \geq \epsilon/2$. Now $K_n = \bar{C}_1 \cap \dots \bar{C}_n$ is non-empty and K_n is a decreasing sequence of bounded non-empty closed sets in $\mathbb{R}^d$ so $\bigcap_n K_n \neq \emptyset$ and $\bigcap_n K_n \subseteq \bigcap_n B_n$ which is a contradiction.

□

We now recall that a set, A, will be λ^*- measureable if for every set B

$$\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c).$$

As for the one dimensional case we write $\mathcal{M}$ for the set of Lebesgue measurable sets. We know from the proof of Carathéodory's extension theorem that $\mathcal{M}$ is a σ-algebra

Question 1.3. Explain why if $\lambda^*(A) = 0$ then $\lambda^*(B \cap A)$ will also be zero. Therefore show that if $\lambda^*(A) = 0$ or $\lambda^*(A^c) = 0$ then A is λ^*-measureable.

Answer: λ^* is monotone and $B \cap A \subseteq A$ so $\lambda^*(B \cap A) \leq \lambda^*(A) = 0$. The measurablility condition is A is measurable if for every B

$$\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c).$$

We know that $\lambda^*(B) \leq \lambda^*(B \cap A) + \lambda^*(B \cap A^c)$ by countable subadditivity. Furthermore by monotonicity $\lambda^*(B \cap A^c) \leq \lambda^*(B)$, therefore if $\lambda^*(A) = 0$ then $\lambda^*(A \cap B) = 0$ so $\lambda^*(B \cap A) + \lambda^*(B \cap A^c) \leq \lambda^*(B) \leq \lambda^*(B \cap A) + \lambda^*(B \cap A^c)$. Therefore we have equality in all the inequalities and A is measureable.

5 marks □

Question 1.4. In this question we will prove that every Borel subset of $\mathbb{R}^d$ is Lebesgue measurable.

- Show that every half space of the form $H_{j,b} = \{(x_1, \dots, x_d) : x_j \leq b\}$ is Lebesgue measurebale.
- Show that every rectangle R is Lebesgue measurable.
- Show that every open set in $\mathbb{R}^d$ is a countable union of rectangles.
- Show that every Borel set is Lebesgue measureable.

Answer: We want to show that for every set B we have

$$\lambda^*(B) = \lambda^*(H_{j,b} \cap B) + \lambda^*(H_{j,b}^c \cap B).$$

Let us take some sequence of rectangles $R_1, R_2, \dots$ such that $B \subseteq \bigcup_n R_n$. Then we can construct the new rectangles $R_i^l = R_i \cap H_{j,b}$ and $R_i^r = R_i \cap H_{j,b}^c$. This produces another sequence of rectangles with $B \cup H_{j,b} \subseteq \bigcup_n R_n^l$ and $B \cup H_{j,b}^c \subseteq \bigcup_n R_n^r$. We also have $\sum_n \lambda(R_n^l) + \sum_n \lambda(R_n^r) = \sum_n \lambda(R_n)$. Therefore we have

$$\lambda^*(H_{j,b} \cap B) + \lambda^*(H_{j,b}^c \cap B) \leq \sum_n \lambda(R_n^l) + \sum_n \lambda(R_n^r) = \sum_n (R_n).$$

We can do this with any collection of rectangles containing B so taking the infimum gives

$$\lambda^*(H_{j,b} \cap B) + \lambda^*(H_{j,b}^c \cap B) \leq \lambda^(B).$$

By countable subadditivity we have

$$\lambda^*(B) \leq \lambda^*(H_{j,b} \cap B)^*_\lambda(H_{j,b}^c \cap B).$$

5 marks

Now for the second point, we know that the set of Lebesgue measurable sets is a σ-algebra. Therefore we know that $H_{j,a}^c$ is also Lebesgue measurable. Then the rectangle $(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d] = (H_{1,a_1}^c \cap H_{1,b_1}) \cap \cdots \cap (H_{d,a_d}^c \cap H_{d,a_d})$. Therefore this rectangle is measurable. *2 marks*

We can fit a small d-dimensional cube, with a rational side length, into $B(x, r)$ with centre at the same place. Let us write $Q(q, r)$ for the d-cube centred at $q \in \mathbb{R}^d$ with side length r. Therefore, given an open set U let

$$O = \bigcup_{q \in \mathbb{Q}^d \cap U} \bigcup_{r \in \mathbb{Q} \text{ s.t.} Q(q,r) \subseteq U} Q(q, r).$$

Then we have $O \subseteq U$ as it is the union of subsets of U. We also have that every point in U will be contained in a small ball inside U with a rational centre and therfore also one of the $Q(q, r)$ therefore $O = U$. *3 marks*

Every open set is the countable union of cubes. Therefore as the cubes are Lebesgue measurable and the collection of Lebesgue measurable sets is a σ-algebra all the open sets are Lebesgue measurable. Therefore the collection of Lebesgue measurable sets is a σ-algebra containing all the open sets it must contain the σ-algebra generated by the open sets which is the Borel σ-algebra. *2 marks* ☐ Again we define Lebesgue measure on $\mathbb{R}^d$ to be the restriction λ^* to $\mathcal{M}$.