Good morning! Will start at 9.05

Remind me to record!

Last reminder of the assignment which is due on Thursday at midday.

Product measure

When we've got two measure spaces $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ we want to produce a sensible σ-algebra and measure on $E \times F$.

Def^n Cartesian product

$$E \times F = \{(x,y) : x \in E, y \in F\}$$

Ex: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

Def^n The product σ-algebra which we write $\mathcal{E} \times \mathcal{F}$ is the σ-algebra generated by the collection of sets of the form $\Pi = \{A \times B,\ A \in \mathcal{E},\ B \in \mathcal{F}\}$, Π is a collection of subsets of $E \times F$, and $\mathcal{E} \times \mathcal{F} = \sigma(\Pi)$

Ex: $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

we can kind of already see this as $\mathcal{B}(\mathbb{R}^2)$ is generated by sets of the form $(a,b] \times (c,d]$

Q2: on the exercise sheet this week says that if $\mathcal{E} = \sigma(\mathcal{A}_1)$ and $\mathcal{F} = \sigma(\mathcal{A}_2)$ then $\mathcal{E} \times \mathcal{F} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$

Remark $\mathcal{M}_d$ is the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^d$ then $\mathcal{M}_2 \neq \mathcal{M}_1 \times \mathcal{M}_1$. This is also on the example sheet.

We define two maps

$\pi_E : E \times F \to E \quad \pi_E(x,y) = x, \quad \pi_F : E \times F \to F, \quad \pi_F(x,y) = y.$

<u>Lemma</u> 1. π_E and π_F are both measurable function wrt to $\mathcal{E} \times \mathcal{F}$ and $\mathcal{E}$ resp $\mathcal{F}$.

2. If C is in $\mathcal{E} \times \mathcal{F}$ define $C_x = \{ y \in F : (x,y) \in C \}$

$C_y = \{ x \in E : (x,y) \in C \}$ These are often called sections

we have $C_x \in \mathcal{F}$ and $C_y \in \mathcal{E}$

3. If $f : E \times F \to G$ is measurable $E \times F$ with $\mathcal{E} \times \mathcal{F}$ σ-algebra and G has some σ-algebra $\mathcal{G}$

then $f_x : F \to G$ defined by $f_x(y) = f(x,y)$

and $f_y : E \to G$ defined by $f_y(x) = f(x,y)$

are both measurable.

<u>Proof</u> 1. Let $A \in \mathcal{E}$ then $\pi_E^{-1}(A) = A \times F \in \Pi \subseteq \mathcal{E} \times \mathcal{F}$

so $\pi_E^{-1}(A)$ is measurable so π_E is a measurable function.

2. $\mathcal{C}$ be the collection of sets, C, in $\mathcal{E} \times \mathcal{F}$ with $C_x \in \mathcal{F} \ \forall x$ and $C_y \in \mathcal{E} \ \forall y$.

Then $\Pi \subseteq \mathcal{C}$ as $(A \times B)_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}$

We want to show $\mathcal{C}$ is a σ-algebra as then we would have $\mathcal{C} \supseteq \sigma(\Pi) = \mathcal{E} \times \mathcal{F}$ so would have

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$\mathcal{C} = \mathcal{E} \times \mathcal{F}$.

$\emptyset \in \mathcal{C}$ as $\emptyset = \emptyset \times \emptyset$, $E \times F \in \mathcal{C}$ as $E \times F \in \Pi$

If $C \in \mathcal{C}$ then

$$(C^c)_x = \{y : (x,y) \in C^c\} = \{y : (x,y) \notin C\} = \{y : (x,y) \in C\}^c$$
$$= (C_x)^c$$

so $C_x \in \mathcal{E}$ as $C \in \mathcal{C}$ so $(C^c)_x = (C_x)^c \in \mathcal{E}$

so this works for any x and any y

so $C^c \in \mathcal{C}$ so $\mathcal{C}$ is closed under complements

If $C_1, C_2, \dots$ are all in $\mathcal{C}$ then

$$\left(\bigcup_n C_n\right)_x = \{y : (x,y) \in \bigcup_n C_n\} = \bigcup_n \{y : (x,y) \in C_n\}$$
$$= \bigcup_n (C_n)_x \qquad \text{so as } (C_n)_x \in \mathcal{E} \ \forall n$$

$\left(\bigcup_n C_n\right)_x \in \mathcal{E}$. This works for any x, y

so $\bigcup_n C_n \in \mathcal{C}$. This show $\mathcal{C}$ is closed under

countable unions.

So $\mathcal{C}$ is a σ-algebra so contains $\sigma(\Pi)$

so $\mathcal{C} = \mathcal{E} \times \mathcal{F}$.

3. Now $f : E \times F \to G$

take $A \in \mathcal{G}$ $\quad f_x^{-1}(A) = \{y \in F : f(x,y) \in A\} = (f^{-1}(A))_x$

$f^{-1}(A) = \{(x,y) : f(x,y) \in A\}$

$f^{-1}(A) \in \mathcal{E} \times \mathcal{F}$ as f is measurable

$f^{-1}(A) = \{(x,y) : f(x,y) \in A\}$

We know that $f^{-1}(A) \in \mathcal{E} \times \mathcal{F}$ as f is measurable and so we know by part 2. that $(f^{-1}(A))_x \in \mathcal{F}$.

<u>Theorem</u> 11.6 (Product measure)

Given two <u>σ-finite</u> $(E, \mathcal{E}, \mu)$ and $(F, \mathcal{F}, \nu)$ there exists a unique measure $\mu \times \nu$ on $\mathcal{E} \times \mathcal{F}$ s.t.

$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B)$ when $A \in \mathcal{E}$ and $B \in \mathcal{F}$.

Furthermore the maps $x \mapsto \nu(C_x)$ and $y \mapsto \mu(C_y)$ for $C \in \mathcal{E} \times \mathcal{F}$ are measurable and

$$\mu \times \nu(C) = \int_E \nu(C_x)\,\mu(dx) = \int_F \mu(C_y)\,\nu(dy)$$

Pf/ First in order to show existence we want to show $C \mapsto \int_E \nu(C_x)\mu(dx)$ is a measure on $\mathcal{E} \times \mathcal{F}$ with right properties. For uniqueness we use a Dynkin's uniqueness of extension argument (which why we need σ-finiteness).

First we check $x \mapsto \nu(C_x)$ is measurable.

Let us work first in the case where μ, ν ~~is~~ are finite.

Let $\tilde{\mathcal{C}}$ be the collection of all sets in $\mathcal{E} \times \mathcal{F}$ s.t. $x \mapsto \nu(C_x)$ is measurable and $y \mapsto \mu(C_y)$ is measurable.

$\Pi \subseteq \tilde{\mathcal{C}}$ as $x \mapsto \nu((A\times B)_x) = \mathbb{1}_A(x)\,\nu(B)$ which is measurable.

Q1 on the exercise sheet asks you to show Π is a π-system.

<u>Recall</u> <u>Dynkin's lemma</u> If $\mathcal{A}$ is a π-system, $\mathcal{D}$ is a d-system with $\mathcal{A} \subseteq \mathcal{D}$ then $\sigma(\mathcal{A}) \subseteq \mathcal{D}$

So if we can show that $\tilde{\mathcal{C}}$ is a d-system then

$\sigma(\Pi) = \mathcal{E}\times\mathcal{F} \subseteq \tilde{\mathcal{C}} = \mathcal{E}\times\mathcal{F}$

WTS: $\tilde{\mathcal{C}}$ is a d-system.

If $C_1 \subseteq C_2$ both in $\tilde{\mathcal{C}}$ then $\nu((C^2\setminus C^1)_x) = \underbrace{\nu(C^2_x) - \nu(C^1_x)}_{\text{use the fact that } \nu \text{ is finite.}}$

this is measurable.

If $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ all in $\tilde{\mathcal{C}}$ then

$$\nu((\bigcup_n C_n)_x) = \nu(\bigcup_n (C_n)_x) \underset{\text{continuity}}{=} \lim_{n\to\infty} \nu((C_n)_x)$$

$\nu((\bigcup_n C_n)_x)$ is the limit of measurable functions

so measurable.

This shows $\tilde{\mathcal{C}}$ is a d-system.

By Dynkin's lemma $\mathcal{E}\times\mathcal{F} = \sigma(\Pi) \subseteq \tilde{\mathcal{C}}$ so

$\tilde{\mathcal{C}} = \mathcal{E}\times\mathcal{F}$ so $x \mapsto \nu(C_x)$ is measurable for every $C \in \mathcal{E} \times \mathcal{F}$.

So this allows us to define the ~~m~~measure

$$(\mu \times \nu)_1 (C) = \int_E \nu(C_x) \, \mu(dx).$$

We can also define

$$(\mu \times \nu)_2 (C) = \int_F \mu(C_y) \, \nu(dy)$$

If $C = A \times B$ then

$$(\mu \times \nu)_1 (A \times B) = \int_E \mathbb{1}_A(x) \, \nu(B) \, \mu(dx) = \mu(A) \nu(B)$$

$$(\mu \times \nu)_2 (A \times B) = \int_F \mathbb{1}_B(y) \, \mu(A) \, \nu(dy) = \mu(A) \nu(B).$$

Now ~~th~~ as μ, ν are finite we can apply

Dynkin's uniqueness of ext theorem

If you have two finite measures which agree on the whole space, and agree on a π-system generating the σ-algebra then they are the same.

$(\mu \times \nu)_1 = (\mu \times \nu)_2$ and will also coincide with any other measure which has

$$(\mu \times \nu)(A \times B) = \mu(A) \nu(B).$$

Extension to the σ-finite case

$\exists$ set $(E_n)_{n\geq 1} \in \mathcal{E}$ $\quad (F_n)_{n\geq 1} \in \mathcal{F}$

with $\mu(E_n) < \infty \ \forall n$ $\quad \nu(F_n) < \infty \ \forall n$

$E = \bigcup_n E_n$, $\quad F = \bigcup_n F_n$ $\quad$ wlog $\quad E_1 \subseteq E_2 \subseteq E_3 \cdots$
$F_1 \subseteq F_2 \subseteq F_3 \cdots$

We know that if $C \subseteq E_n \times F_n$ then by working with the restrictions of $\mathcal{E}, \mu, \mathcal{F}, \nu$ to E_n, F_n we have

$x \mapsto \nu(C_x)$ is measurable and $y \mapsto \mu(C_y)$ is measurable

Therefore for any C

$x \mapsto \nu((C \cap (E_n \times F_n))_x)$ is measurable for any n

as $n \to \infty$ $\quad (C \cap (E_n \times F_n))_x \uparrow$ to C_x

so $\quad \nu((C \cap (E_n \times F_n))_x) \to \nu(C_x)$

by continuity of measure

so $\quad x \mapsto \nu(C_x)$ is the limit of measurable functions so measurable, the same proof works for $y \mapsto \mu(C_y)$.

So the measures

$$(\mu \times \nu)_1(C) = \int_E \nu(C_x)\,\mu(dx)$$

and $$(\mu \times \nu)_2(C) = \int_F \mu(C_y)\,\nu(dy)$$ make sense

So we've done existence in the σ-finite case

Now uniqueness let $(\mu \times \nu)_2$ be any other measure with

Now uniqueness let $(\mu\times\nu)_3$ be any other measure with

$(\mu\times\nu)_3(A\times B) = \mu(A)\nu(B)$.

By our finite measure result

$(\mu\times\nu)_1(C\cap(E_n\times F_n)) = (\mu\times\nu)_3(C\cap(E_n\times F_n))$

and $C = \bigcup_n (C\cap(E_n\times F_n))$ and this is an increasing union so by continuity of measure

$$(\mu\times\nu)_1(C) = \lim_{n\to\infty}(\mu\times\nu)_1(C\cap(E_n\times F_n)) =$$
$$\lim_{n\to\infty}(\mu\times\nu)_3(C\cap(E_n\times F_n)) = (\mu\times\nu)_3(C).$$

So $(\mu\times\nu)_1 = (\mu\times\nu)_3$ and so the product measure is unique.