Recall the (efficient) definition of a σ-algebra:

A σ-algebra $\mathcal{A}$ is a collection of subsets of X such that

(a) $X \in \mathcal{A}$

(b) If $A \in \mathcal{A}$ then $A^c = X - A$ is in $\mathcal{A}$

(c) For each infinite sequence $\{A_i\}$ of sets in $\mathcal{A}$

$$\bigcup_{i=1}^{\infty} A_i \text{ is in } \mathcal{A}.$$

The next idea we want to capture is what it means for a given collection of sets to generate a σ-algebra.

Think about vectors $v_1 \dots v_k$ in $\mathbb{R}^n$. There are two ways to describe the subspace V of $\mathbb{R}^n$ generated by $v_1 \dots v_k$.

① V is the set of all linear combinations $a_1 v_1 + \dots a_k v_k$.

② V is the smallest subspace of $\mathbb{R}^n$ that contains $v_1 \dots v_k$. That is to say that V is the intersection of all subspaces of $\mathbb{R}^n$ containing $v_1 \dots v_k$.

In our setting the first approach is too complicated.

We use the second approach.

Proposition. Given a collection of subsets of X, $\mathcal{F}$ there is a unique smallest σ-algebra containing $\mathcal{F}$.

We start with a Lemma.

Lemma. Let X be a set.
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Proof. Let $\mathcal{C}$ be a non-empty collection of σ-algebras and let $\mathcal{A}$ be the intersection of these σ-algebras.

Since X is in each σ-algebra it is in $\mathcal{A}$. If $A \in \mathcal{A}$ then A is in each σ-algebra so A^c is in each σ-algebra.

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So A^c is in $\mathcal{A}$.

If $A_1, A_2 \ldots$ is in $\mathcal{A}$ then $\bigcap A_i$ is in each σ-algebra so $\bigcap A_i$ is in $\mathcal{A}$.

Cor. Given any collection $\mathcal{A}$ of sets of X there is a smallest σ-algebra containing $\mathcal{A}$.

Proof. Consider the intersection of all σ-algebras that contain $\mathcal{A}$. This is a σ-algebra and it is contained in any σ-algebra that contains $\mathcal{A}$.