Lecture 2

08 October 2021 08:49

Measures

Remind me to record!

→ Support classes: Separate sign up for each week

→ Assignment out today due in 21st at noon

Put on moodle

Based on Lecture 3 material and the videos after

Put out next weeks lecture notes

Extra q null sets:

Measures

What is a measure?

Def^n Set function $\Phi : \mathcal{A} \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$\mathcal{A}$ is a collection of subsets of some space E.

Def^n μ is a measure if its a set function

$\mu : \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$

$\mathcal{A}$: σ-algebra

- $\mu(A) \geq 0$ for every $A \in \mathcal{A}$
- $\mu(\emptyset) = 0$
- If $A_1, A_2, A_3 \dots$ of pairwise disjoint elements of $\mathcal{A}$

then $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$

"countable additivity"

Remark Why does countability come into def^n of σ-alg?

You'll see in week 2 constructing measures constructing

You'll see in week 2 constructing measures constructing σ-alg and meas at the same time.

<u>Examples</u> δ "function" $\delta_{x,y} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

1. $\delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{cases}$ on $\mathcal{P}(E)$

2. Let E be countable $E = \{x_1, x_2, x_3, \dots\}$

$F : E \to \mathbb{R}_{\geq 0}$

$\mu_F(A) = \sum_n F(x_n) \mathbb{1}_{x_n \in A}$ is a measure on $\mathcal{P}(E)$

in fact any measure on $(E, \mathcal{P}(E))$ can be constructed this way $F(x_n) = \mu(\{x_n\})$

3. "Non-rigorous example"

If $f : \Omega \to \mathbb{R}_{\geq 0}$ is a function $\Omega \subseteq \mathbb{R}^d$

then $\mu_f(A) = \int_A f(x)\,dx$ is a measure

We don't know how to define $\int_A f(x)\,dx$ when A is complicated.

=

<u>Lemma</u>

If μ is a measure on $(E, \mathcal{A})$

then μ is <u>monotone</u> [if $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$

and μ is <u>countably subadditive</u> [if $A_1, A_2, \dots$ is a sequence in $\mathcal{A}$ then $\mu(\bigcup_n A_n) \le \sum_n \mu(A_n)$

Pf ~~A~~ $B = A \cup (B \setminus A)$ this is a disjoint union

$$\mu(B) = \mu(A) + \underbrace{\mu(B \setminus A)}_{+ve} \qquad \mu(B) \geq \mu(A)$$

$$B_n = A_n \setminus \left(\left(\bigcup_{i=1}^{n-1} A_i \right) \cap A_n \right)$$

then the B_n form a disjoint seq. by monotonicity

$$\mu(B_n) \le \mu(A_n)$$

by countable <u>add</u> $\mu(\bigcup_n A_n) = \mu(\bigcup_n B_n) = \sum_n \mu(B_n)$

$$\le \sum_n \mu(A_n)$$

<u>Def^n</u> $(E, \mathcal{A})$ where E is a set

$\mathcal{A}$ is a σ-alg on E

call this a <u>measurable space</u>

<u>Def^n</u> $(E, \mathcal{A}, \mu)$ where E set, $\mathcal{A}$ σ-alg on E, μ meas on $\mathcal{A}$

is a <u>measure space</u>

<u>Def^n</u> $(E, \mathcal{A}, \mu)$ is a <u>finite</u> measure space if

$\mu(E) < \infty$ (n.b. by monotonicity $\mu(A) < \infty$ $\forall A \in \mathcal{A}$)

Def^n $(E, \mathcal{A}, \mu)$ is a <u>σ-finite</u> measure space if $\exists$ $E_1, E_2, E_3, \ldots \in \mathcal{A}$

s.t. $\bigcup_n E_n = E$ and $\mu(E_n) < \infty$ $\forall n$

$\mathbb{R}$ is σ-finite $\qquad \mathbb{R} = \bigcup_n \left([n, n+1] \cup [-n, -n-1] \right)$

<u>Lemma</u> 2.22 <u>Continuity of measure</u>

Let $(E, \mathcal{A}, \mu)$ be a measure space

then let $A_n \in \mathcal{A}$ for each n $\quad A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$

and $B_n \in \mathcal{A}$ $\forall n$ $\quad B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ <u>and</u> $\underbrace{\mu(B_1) < \infty}$

$$\mu\left(\bigcup_n A_n\right) = \lim_{n \to \infty} \mu(A_n)$$

$$\mu\left(\bigcap_n B_n\right) = \lim_{n \to \infty} \mu(B_n)$$

Pf/ the sequences $\mu(A_n)$ is increasing and $\mu(B_n)$ is decreasing

$\therefore$ they both have limits

Let $\tilde{A}_n = A_n \setminus A_{n-1}$ then $(\tilde{A}_n)_{n \geq 1}$ is a sequence of disjoint sets

and $\bigcup_n A_n = \bigcup_n \tilde{A}_n$

Countable additivity gives $\mu\left(\bigcup_n A_n\right) = \sum_n \mu(\tilde{A}_n)$

This implies $\sum_{k=1}^{N} \mu(\tilde{A}_k) \to \mu\left(\bigcup_n A_n\right)$

as $N \to \infty$

also by countable additivity, as $A_N = \bigcup_{k=1}^{N} \tilde{A}_k$

$$\sum_{k=1}^{N} \mu(\tilde{A}_k) = \mu(A_N)$$

$$\therefore \quad \mu(A_N) \to \mu(\bigcup_n A_n).$$

The result with (B_n)s is taking setminus ...

$C_n = B_1 \setminus B_n$ then the C_n are an increasing sequence

by the first result C_n is increasing $C_n \uparrow (B_1 \setminus \bigcap_n B_n)$

$$\mu(B_1 \setminus B_n) \to \mu(B_1 \setminus \bigcap_n B_n)$$

$=$ ~~$\mu(B)$~~ if $B \subseteq A$ $\quad \mu(A) = \mu(B) + \mu(A \setminus B)$

$$\mu(A \setminus B) = \mu(A) - \mu(B)$$

$$\mu(B_1) - \mu(B_n) \xrightarrow[n \to \infty]{} \mu(B_1) - \mu(\bigcap_n B_n)$$

so as $\mu(B_1) < \infty$ we can rearrange $\begin{cases} \mu(B_n) < \infty \\ \mu(\bigcap_n B_n) < \infty \end{cases}$

$$\mu(B_n) \longrightarrow \mu(\bigcap_n B_n) \quad \square$$

Lemma 2.22 doesn't hold if $\mu(B_1) = \infty$

$B_n = [n, \infty)$ for each n

μ be Lebesgue measure $\quad B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$

$\mu(B_n) = \infty$ for every n

but $\mu(\bigcap_n B_n) = \mu(\phi) = 0$

$E = \{1, 2, 3, \dots\}$

$\mu(\{n\}) = 1 \quad \forall n$ $\qquad B_n = \{n, n+1, n+2, \ldots\}$

$\mu(\{n, n+1, \ldots\}) = \infty$

but $\bigcap_n (\{n, n+1, n+2, \ldots\}) = \emptyset$

& $\mu(\bigcap_n B_n) = 0$ but $\mu(B_n) = \infty$ for every n.