$\phi : I \to \mathbb{R}$ is convex (where I is an interval)

if for every $t \in [0,1]$, $x, y \in I$ we have

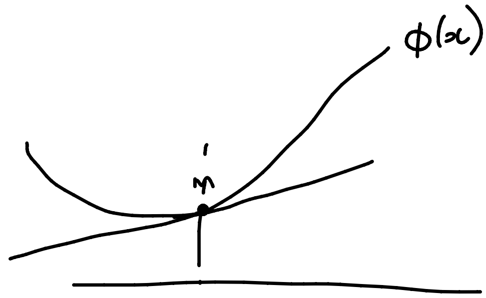
$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

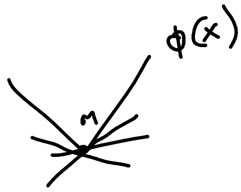
Lemma Let $\phi : I \to \mathbb{R}$ be convex and

$m \in \mathrm{Int}(I)$ then $\exists\ a, b$ real numbers

s.t. $\forall x \in I$ $\quad ax + b \leq \phi(x)$

and $\quad am + b = \phi(m)$





Proof take $x < m < y$ $\quad x, y \in I$

then by convexity as

$$m = \underbrace{\frac{y-m}{y-x}}_{\in [0,1]} x + \underbrace{\frac{m-x}{y-x}}_{\in [0,1]} y = \frac{x(y-m) + y(m-x)}{y-x} = \frac{m(y-x)}{y-x}$$

$$\phi(m) \leq \frac{y-m}{y-x}\phi(x) + \frac{m-x}{y-x}\phi(y)$$

$$(y-x)\phi(m) \leq (y-m)\phi(x) + (m-x)\phi(y)$$

$$(y-m + m-x)\phi(m) \leq (y-m)\phi(x) + (m-x)\phi(y)$$

$(y-m + m-x)\,\phi(m) =$

$$(y-m)(\phi(m)-\phi(x)) \leq (m-x)(\phi(y)-\phi(m))$$

$$\underbrace{\frac{\phi(m)-\phi(x)}{m-x}}_{\text{only depends on } x} \leq \underbrace{\frac{\phi(y)-\phi(m)}{y-m}}_{\text{only depends on } y}$$

This is true for any $x < m < y$

$$\sup_{x<m} \frac{\phi(m)-\phi(x)}{m-x} \leq \inf_{y>m} \frac{\phi(y)-\phi(m)}{y-m}$$

so there exists an a s.t. $\forall\, x < m < y$

$$\frac{\phi(m)-\phi(x)}{m-x} \leq a \leq \frac{\phi(y)-\phi(m)}{y-m}$$

for all $x < m$ $\quad \phi(m) - \phi(x) \leq am - ax$

$$\phi(x) \geq ax - am + \phi(m)$$

for all $y > m$ $\quad \phi(y) - \phi(m) \geq ay - am$

$$\phi(y) \geq ay - am + \phi(m)$$

so $\forall\, x \in I \quad \phi(x) \geq ax - am + \phi(m)$

when $x = m$ you have equality

Prop 1 **Jensen**

Suppose $(E, \mathcal{E}, \mu)$ is a measure space $f: E \to \mathbb{R}$ integrable measurable, $\mu(E) = 1$, ϕ a convex measurable function $\mathbb{R} \to \mathbb{R}$ then $\mu(\phi(f))$ makes sense and $\mu(\phi(f)) \geq \phi(\mu(f))$

<u>Example</u> $\phi(x) = x^2$ $E = [0,1]$ w/ Lebesgue + Borel σ-algebra

$$\frac{1}{4} = \left(\int_0^1 x\,dx\right)^2 \leq \int_0^1 x^2\,dx = \frac{1}{3}$$

Pf/ NB. As $\mu(E) = 1$ $\mu(f)$ is the average value of f.

In particular $\mu(f) \in$ interior (Range(f))

By our lemma with $m = \mu(f)$ $\exists\, a, b$ s.t.

$$ax + b \leq \phi(x) \quad \forall x, \qquad \underline{\underline{a\mu(f) + b = \phi(\mu(f))}}$$

$$af(x) + b \leq \phi(f(x))$$

integrating gives

$$a\mu(f) + b\underset{\overset{\parallel}{1}}{\mu(E)} \leq \mu(\phi(f))$$

$$\underset{\overset{\parallel}{\phi(\mu(f))}}{a\mu(f) + b} \leq \mu(\phi(f))$$

$$\phi(\mu(f)) \leq \mu(\phi(f))$$