Measure Theory: Exercises (not for credit)

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Question 1 (The devil's staircase). In this question we construct a function which is continuous, flat almost everywhere and increases from 0 to 1 as x goes from 0 to 1 (This is quite a lot like doing research, you are only making progress at a measure 0 amount of time!). First we construct the Cantor set recursively. Let $C_0 = [0,1], C_1 = [0,1/3] \cup [2/3,1], C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1], \ldots$ where each C_n is constructed from C_{n-1} by removing the middle thirds of each of the closed intervals making up C_{n-1}. Let us write $C = \bigcap_n C_n$, so $C_n \downarrow C$ that is to say $C_1 \supset C_2 \supset \ldots$ and $C = \bigcap_n C_n$.

- Show that C is uncountable.
- Show that $\lambda(C) = 0$.
- Define $F_n(x) = \lambda(C_n \cap [0,x])/\lambda(C_n)$, show that $F(x) = \lim_n F_n(x)$ exists. *hint: try and find a reccurrence relationship for F_n in terms of F_{n-1} then use this to show F_n is a Cauchy sequence with the uniform norm on functions*
- Show that the function F is continuous for all x with $F(0) = 0$ and $F(1) = 1$.
- Show that for lebesgue almost every $x \in [0,1]$ we have that $F(x)$ is differentiable with $F'(x) = 0$.

Answer: First let us show that the Cantor set is uncountable. We can write every element of $[0,1]$ as an expansion in base 3. That is to say $x = k_1/3 + k_2/9 + k_3/27 + \cdots + k_n/3^n + \ldots$. Then $x \in C$ if and only if $k_n \in \{0,2\}$ for every n. Therefore the cardinality of the Cantor set is the same as the cardinality of all sequences or numbers which are either 0 or 2, which is uncountable by the standard Cantor diagonal argument to show uncountability of the reals.

Now let us show that $\lambda(C) = 0$. We have that $\lambda(C_0) = 1$ so we can apply our continuity of measure theorem to get that $\lambda(C) = \lim_n \lambda(C_n)$. Then we claim that $\lambda(C_n) = (2/3)\lambda(C_{n-1})$ since for each interval making up C_{n-1} we have removed a third of it. Working itteratively this gives that $\lambda(C_n) = (2/3)^n \lambda(C_0) = (2/3)^n$ therefore $\lambda(C) = \lim_n (2/3)^n = 0$. Notice here that we have shown the existence of a measure 0 set of uncountable cardinality.

This step is really tricky. We can define find a reccurence relationship for $F_n(x)$ using the fractal-like property of the Cantor set

$$F_n(x) = \begin{cases} F_{n-1}(3x)/2 & x \in [0,1/3] \\ 1/2 & x \in [1/3,2/3] \\ 1/2(1 + F_{n-1}(3x-2)) & x \in [2/3,1] \end{cases}$$

Now we want to look at $F_{n+1}(x) - F_n(x)$ we can split into the same 3 cases and get

$$F_{n+1}(x) - F_n(x) = \begin{cases} (F_n(3x) - F_{n-1}(3x))/2 & x \in [0,1/3] \\ 1/2 - 1/2 & x \in [1/3,2/3] \\ (F_n(3x-2) + 1 - (F_{n-1}(3x-2) + 1))/2 & x \in [2/3,1] \end{cases}$$

Then we have that $|F_n(x) - F_{n-1}(x)| \leq 2^{-n+1}|F_1(x) - F_0(x)| \leq 2^{-n+1}$ so $F_n(x)$ is a uniformly Cauchy sequence.

The next step relies on the previous one. We can check that $F_n(x)$ is continuous for each x. Then we know that the uniform limit of continuous functions is also continuous. $F_n(0) = \lambda(\emptyset)/\lambda(C_n) = 0$ so $F(0) = 0$ and $F_n(1) = \lambda(C_n)/\lambda(C_n) = 1$ so $F(1) = 1$.

Now suppose that $x \notin C$ then for n sufficiently large $x \notin C_n$ the interval $[0, 1] \setminus C_n$ is open so there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset [0, 1] \setminus C_n$. Then F_m will be constant on this set for any $m \geq n$ and the same value for each m. So $F(x)$ is constant on $(x - \delta, x + \delta)$ so $F'(x)$ exists and is 0. $\square$