1 Introduction

Welcome to measure theory. This course introduces the modern theory of functions and integration which underpins most advanced analysis topics. In particular the theory of function spaces will be important in PDEs and the notion of measurable functions allows us to rigorously understand random variables.

The key example we will study is *Lebesgue measure* in $\mathbb{R}^d$. The goal of defining Lebesgue measure is to find a way of asigning length/area/volume/whatever its called if $d \geq 4$ to a subset of $\mathbb{R}^d$. It turns out that it is not possible to do this for every possible subset of $\mathbb{R}^d$, but it is possible to do this for every subset you are likely to come accross!

1.1 Integration

One of the most important results of measure theory is the ability to integrate 'against' the measures that we define. We want this new definition of the integral to agree with the Riemann integral on subsets of $\mathbb{R}^d$ and also allow us to integrate over sets that aren't subsets of $\mathbb{R}^d$ or with different weightings of the different parts of $\mathbb{R}^d$. This new theory of integration allows us to rigorously define expectation in probability theory and provides numerous convergence theorems which are some of the results you will use most from this course.

1.2 The most important things you will learn in this course

For your own knowledge of how measure and function 'really' work:

- How Lebesgue measure is constructed.

- How the Lebesgue integral is constructed.

- How product measures/spaces are constructed.

- How L^p spaces are defined.

For use in later courses:

- The fact that Lebesgue measure exists and does what you expect it to do.

- Why you can work with measures just by looking at how they behave on a π-system (find out what that is soon!).

- The different ways in which functions can converge.

- Equivalences between ways things converge.

- Convergence theorems: i.e. when convergence of functions implies convergence of their integrals.

- Important inequalities: Hölder/Cauchy-Schwartz, Minkowski, Jensen.

- When you can switch the order of integration.

2 σ-algebras, definition of a measure

2.1 Collections of subsets

We begin with some dry definitions of collections of sets and functions from collections of sets to $\mathbb{R}$. These will give us the key formal definition of measures.

We begin with the most basic defnition. An algebra is is collection of sets closed under finite set operations.

Definition 2.1 (Algebra). A collection of subsets of a space E, $\mathcal{A}$, is called and algebra if

- $\emptyset \in \mathcal{A}$, $E \in \mathcal{A}$.

- If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

- If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

- If $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$.

We next define a σ-algebra. This is the *key* defnition of a collection of sets for measure theory. The letter σ here denotes countability. A σ-algebra is a collection of subsets of a space E, which are closed under countable set operations.

Definition 2.2 (σ-algebra). A collection of subsets of a space E, $\mathcal{A}$ is a σ-algebra if

- $\emptyset \in \mathcal{A}$, $E \in \mathcal{A}$.

- If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

- If $A_1, A_2, \ldots$ is a countable collection of sets in $\mathcal{A}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

- If $A_1, A_2, \ldots$ is a countable collection of sets in $\mathcal{A}$ then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}$.

Lemma 2.3. *We can equivalently define a σ-algebra as a collection of sets which is contains $\emptyset$ and is closed under taking complements and countable unions.*

Proof. Suppose that $\mathcal{E}$ is closed under complements and taking countable unions and contains $\emptyset$, then it is clear that $E \in \mathcal{E}$. We need to show that if $(A_n)_n$ is a sequence in $\mathcal{E}$ then $\bigcap A_n \in \mathcal{E}$. We know that $\bigcap A_n = (\bigcup A_n^c)^c$ so this gives our result. $\qquad\square$

Example 2.4. In this course we really only deal with one 'concrete', non-trivial example of a σ-algebra. This is complicated to introduce and we will discuss bellow. However, in order to better understand the definition we give a few examples of things which are, and are not σ-algebras.

- The main example is the Borel σ-algebra which we will meet in week 2

- The power set of E is always a σ algebra.

- If $A \subset E$ then $\{\emptyset, A, A^c, E\}$ is a σ-algebra.

We now collect some results and further definitions about σ-algebras.

Lemma 2.5. *Suppose E is a space and $\mathcal{C}$ is a collection of σ algebras possibly uncountable. Then $\bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$ is also a σ-algebra.*

Proof. It is straightforward to check that every part of the definition of a σ-algebra holds for the intersection. $\qquad\square$

Corollary 2.6. *For any collection of subsets of a space E, $\mathcal{F}$ there is a smallest σ-algebra containing $\mathcal{F}$. We call this $\sigma(\mathcal{F})$ or the σ-algebra generated by $\mathcal{F}$.*

Proof. There exists at least one σ-algebra containing $\mathcal{F}$ since the set of all subsets of E is a σ algebra. Then we can consider the non-empty intersection $\bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$ where $\mathcal{C}$ is the collection of all σ-algebras which contain $\mathcal{F}$. We call this resulting σ- algebra the *σ-algebra generated by $\mathcal{F}$*. $\qquad\square$

Example 2.7 (Key example: Borel σ-algebra). If E is a topological space and $\mathcal{O}$ the family of open sets in E, then we write $\mathcal{B}(E)$ to be the σ-algebra generated by $\mathcal{O}$. This is called the Borel σ-algebra.

We are most interested in $\mathcal{B}(\mathbb{R}^d)$. We have the following result

Lemma 2.8. *$\mathcal{B}(\mathbb{R})$ is generated by the following sets.*

- *The collection of closed sets in $\mathbb{R}$.*

- *The collection of intervals of the form $(-\infty, b]$.*

- *The collection of intervals of the form $(a, b]$.*

Proof. Let us call $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ to be the σ-algebras generated by the sets above. We then want to show that $\mathcal{B}(\mathbb{R}) \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3$.

As $\mathcal{B}(\mathbb{R})$ contains all the open sets, it also contains all the closed sets (whose complements are open). Therefore, it also contains $\mathcal{B}_1$.

As $\mathcal{B}_1$ contains all the closed sets, and all the intervals $(-\infty, b]$ are closed then $\mathcal{B}_1$ contains the σ-algebra generated by these sets, namely $\mathcal{B}_2$.

As $\mathcal{B}_2$ contains $(-\infty, b]$ and $(-\infty, a]$ and is closed under complements it also contains, $(-\infty, b]$ and (a, ∞). As $\mathcal{B}_2$ is closed under intersection, this means it also contains $(a, b]$. This is true for all $a < b$ so $\mathcal{B}_2$ contains all sets of this form. Consequently, it contains $\mathcal{B}_3$.

Now we want to show that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_3$. This will conclude the proof. First we note, that we can make an open interval $(a, b) = \bigcup_n (a, b - 1/n]$ where the union is taken over all $n > (b - a)^{-1}$. Now we need to show that any open set in $\mathbb{R}$ is a countable union of open intervals. Let U be such an open set then let

$$O = \bigcup_{q \in \mathbb{Q} \cap U} \bigcup_{r \in \mathbb{Q} \text{ s.t } (q-r,q+r) \subseteq U} (q - r, q + r).$$

Then since O is a union of subsets of U then $O \subseteq U$. Suppose that $x \in U$ then there exists some ρ such that $(x - \rho, x + \rho) \subseteq U$. There is some rationals q, r such that $x \in (q - r, q + r) \subseteq (x - \rho, x + \rho)$ therefore $x \in O$. Consequently $U = O$. $\qquad\square$

We have two further defnitions of collections of sets which will be useful. These separate the two parts of the defnition of a σ-algebra.

Definition 2.9 (π-system). A collection of subsets of E, $\mathcal{A}$ is a π-system if

- $\emptyset \in \mathcal{A}$

- If $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$.

Definition 2.10 (D-system). A collection of subsets of E, $\mathcal{A}$ is a d-system if

- $E \in \mathcal{A}$.

- If $A, B \in \mathcal{A}$ with $A \subset B$ then $B \setminus A \in \mathcal{A}$.

- If $A_1 \subset A_2 \subset A_3 \subset \dots$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Lemma 2.11 (Dynkin's π-system lemma). *Let $\mathcal{A}$ be a π-system. Then any d-system containing $\mathcal{A}$ also contains the σ-algebra generated by $\mathcal{A}$.*

Proof. This is an exercise. $\qquad\square$

2.2 Set functions

Definition 2.12 (Set function). A set function ϕ is a function from a family of subsets of a space E, $\mathcal{A}$ to $\mathbb{R} \cup \{\infty\}$.

Definition 2.13 (Measure). A measure is a specific type of set function which satisfies certain axioms. A set function μ defined from a σ-algebra $\mathcal{A}$ is a measure if,

- $\mu(A) \geq 0$ for every $A \in \mathcal{A}$.

- $\mu(\emptyset) = 0$

- If $A_1, A_2, A_3, \ldots$ are all pairwise disjoint and in $\mathcal{A}$ then

$$\mu\left(\bigcup_n A_n\right) = \sum_n \mu(A_n).$$

We call this last property *countable additivity*.

Example 2.14 (Delta (function)). You've probably seen $\delta_{x_0}(x)$ used before; it is similar to the Kroeneker delta which appears in discrete spaces $\delta_{x,y} = 1$ if and only if $x = y$. This is the 'function' defined by $\int \delta_{x_0}(x) f(x) \mathrm{d}x = f(x_0)$. We can define a measure on $\mathbb{R}^d$ which will have this property by

$$\delta_{x_0}(A) = \left\{ \begin{array}{ll} 1 & x_0 \in A \\ 0 & x_0 \notin A \end{array} \right. .$$

Example 2.15 (Countable space). If $E = \{x_1, x_2, \ldots\}$ is a countable space and $F : E \to \mathbb{R}_{\geq 0}$ is a non-negative function then we can define a measure by $\mu(A) = \sum_n F(x_n) 1_{x_n \in A}$. In fact any measure on a countable set can be written this way by choosing $F(x_n) = \mu(\{x_n\})$.

Example 2.16 (Function (informally)). In the course we will define this rigorously later. However we can define a measure on $\mathbb{R}^d$ by integrating a function over subsets of $\mathbb{R}^d$. If f is a non-negative function then we define $\mu_f(A) = \int_A f(x) \mathrm{d}x$.

We also define two further possible properties of set functions

Definition 2.17 (Monotonicity). A set function ϕ is monotone if whenever $A \subseteq B$ we have $\phi(A) \leq \phi(B)$.

Definition 2.18 (Countable subadditivity). A set function ϕ is countably subadditive if for every sequence of sets $A_1, A_2, A_3, \ldots$ we have

$$\phi\left(\bigcup_n A_n\right) \leq \sum_n \phi(A_n).$$

Lemma 2.19. *If μ is a measure the μ is both monotone and countably subadditive.*

Proof. Suppose $A \subseteq B$ then $B = A \cup (B \setminus A)$ and this union is disjoint. Countable additivity then implies that $\mu(B) = \mu(A) + \mu(B \setminus A)$ and since $\mu(B \setminus A) \geq 0$ we have $\mu(A) \leq \mu(B)$.

Now take a sequence $A_1, A_2, A_3, \ldots$ and define $B_n = A_n \setminus \left(A_n \cap \bigcup_{k=1}^{n-1} A_k\right)$ then the B_n form a disjoint sequence with $\bigcup_n A_n = \bigcup_n B_n$. We also have, for every n, that $B_n \subseteq A_n$ so by monotonicity $\mu(B_n) \leq \mu(A_n)$. Then using countable additivity on the union of the B_n we have

$$\mu(\bigcup_n A_n) = \mu(\bigcup_n B_n) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n).$$

$\square$

Definition 2.20 (Measureable space). We call a pair $(E, \mathcal{A})$ of a space and a σ-algebra, a *measureable space*.

Definition 2.21 (Measure space). We call a triple $(E, \mathcal{A}, \mu)$ of a space, a σ-algebra and a measure a *measure space*.

Definition 2.22 (Finite measure space). We call a measure space $(E, \mathcal{A}, \mu)$ *finite* if $\mu(E) < \infty$.

Definition 2.23 (σ-finite measure space). We call a measure space, $(E, \mathcal{A}, \mu)$, σ-*finite* if there exists a countable collection $E_1, E_2, \dots \in \mathcal{A}$ such that

$$E = \bigcup_n E_n,$$

and

$$\mu(E_i) < \infty, \ \forall i.$$

Definition 2.24 (Borel measures and Radon measures). A measure μ on a subset of a topological space E is called a *Borel measure* if it is a measure with respect to the Borel σ-algebra.

A Borel measure is called a *Radon measure* if for every compact set $K \in \mathcal{B}(E)$ we have that $\mu(K) < \infty$.

Lemma 2.25 (Continuity of measure). *Let $(E, \mathcal{E}, \mu)$ be a measure space. Suppose that $(A_n)_n$ is a sequence of measurable sets with $A_1 \subseteq A_2 \subseteq \dots$ and $(B_n)_n$ is a sequence of measurable sets with $B_1 \supseteq B_2 \supseteq \dots$, and $\mu(B_1) < \infty$ then we have*

$$\mu\left(\bigcup_n A_n\right) = \lim_n \mu(A_n)$$

and

$$\mu\left(\bigcap_n B_n\right) = \lim_n \mu(B_n).$$

Proof. Let $\tilde{A}_n = A_n \setminus A_{n-1}$. We have that $\bigcup_n A_n = \bigcup_n \tilde{A}_n$. Furthermore, countable additivity gives us that

$$\mu\left(\bigcup_n \tilde{A}_n\right) = \sum_n \mu(\tilde{A}_n).$$

Therefore, we have $\sum_{n=1}^m \mu(\tilde{A}_n) \to \mu(\bigcup_n A_n)$. We also have $\sum_{n=1}^m \mu(\tilde{A}_n) = \mu\left(\bigcup_{n=1}^m \tilde{A}_n\right) = \mu(A_m)$.

Now we move onto the B_n, let $C_n = B_1 \setminus B_n$ then the C_n are an increasing sequence of measurable sets with $C_n \uparrow B_1 \setminus \bigcap_n B_n$. So by the first part we have $\mu(B_1 \setminus \bigcap_n B_n) = \lim_n \mu(C_n)$. Therefore

$$\mu(B_1) - \mu\left(\bigcap_n B_n\right) = \mu(B_1) - \lim_n \mu(B_n).$$

This gives the result as long as $\mu(B_1) < \infty$. If there exist an m such that $\mu(B_m) < \infty$ then we can renumber starting with m and repeat the argument above.

N.b. the fact that $\mu(B) < \infty$ implies $\mu(A) < \infty$ if $B \subset A$ follows from finite additivity. $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$. $\qquad \square$

3 Week 2 starts here - Outer measure, Lebesgue Measure

Definition 3.1 (Outer measure). We write $\mathscr{P}(E)$ to be the power set of E, that is to say the set of all subsets of E. An outer measure is a function, ν, from $\mathscr{P}(E) \to \mathbb{R}_+ \cup \{\infty\}$ such that

- $\nu(\emptyset) = 0$,

- If $A \subseteq B$ then $\nu(A) \leq \nu(B)$, (this is called *monotonicity*)

- If $A_1, A_2, \ldots$ is a sequence of subsets then $\nu\left(\bigcup_n A_n\right) \leq \sum_n \nu(A_n)$, (this is called *countable subadditivity*).

The key example of an outer measure is *Lebesgue outer measure*, defining this is our first step to defining Lebesgue measure.

Definition 3.2 (Lebesgue measure on unions of intervals). Let us call $\mathcal{I}$ the set of countable unions of half-open intervals $(a, b]$. That is to say I is the set of all sets of the form

$$\bigcup_n (a_1, b_1].$$

We also write $\mathcal{J}$ to be the set of finite unions of half open intervals. Then we define a set function λ from $\mathcal{J}$ to $\mathbb{R}$ by

$$\lambda\left((a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n]\right) = \sum_{i=1}^n (b_i - a_i).$$

We are most interested in λ defined on a single half open interval. Using this we can define Lebesgue outer measure.

Definition 3.3 (Lebesgue outer measure). We define Lebesgue outer measure on $\mathscr{P}(\mathbb{R})$ by

$$\lambda^*(A) = \inf\{\sum_n \lambda(I_n) : I_n \text{ are half open intervals}, A \subset \bigcup_n I_n\}.$$

Proposition 3.4. *Lebesgue outer measure is an outer measure and agrees with λ on any half open interval.*

Proof. We need to check each part of the definition of outer measure. First the fact that $\lambda^*(\emptyset) = 0$ follows from the fact that $\emptyset \in \mathcal{I}$ and $\lambda(\emptyset) = 0$. Now suppose that $A_1 \subset A_2$, then any set $B \in \mathcal{I}$ with $A_2 \subseteq B$ also has $A_1 \subseteq B$ so

$$\inf\{\sum_n \lambda(I_n) : I_n \text{ are intervals}, A_1 \subset B = \bigcup_n I_n\} \leq \inf\{\sum_n \lambda(I_n) : I_n \text{ are intervals}, A_2 \subset B = \bigcup_n I_n\},$$

as the infimum over a larger set will always be smaller. Now let us turn to the countable subadditivity. Let us take some sequence $A_1, A_2, \ldots$, if $\sum_n \lambda^*(A_n) = \infty$ then we are done. Therefore we can assume that $\sum_n \lambda^*(A_n) < \infty$. Now let us fix an arbitrary $\epsilon > 0$. Now by the definition of λ^* for each n there exists some $I_n \in \mathcal{I}$ such that $A_n \subseteq I_n$ and $I_n = \bigcup_k I_{n,k}$ where the $I_{n,k}$ are half open intervals, and $\sum_k \lambda(I_{n,k}) \leq \lambda^*(A_n) + \epsilon 2^{-n}$. Then the set $I = \bigcup_n I_n$ is in $\mathcal{I}$ and $\sum_{n,k} \lambda(I_{n,k}) = \sum_n \sum_k \lambda(I_{n,k}) \leq \sum_n \lambda^*(A_n) + \epsilon$. Therefore $\lambda^*(\bigcup_n A_n) \leq \sum_n \lambda^*(A_n) + \epsilon$.

Lastly if A is the interval $(a, b]$ then $(a, b] \in \mathcal{I}$ so $\lambda^*(A) \leq b - a$. Suppose that $(a, b] \subseteq (c_1, d_1] \cup (c_2, d_2] \cup \ldots$.

Suppose that $(a, b] \subseteq (c_1, d_1] \cup (c_2, d_2] \cup \ldots$. Then we have that for any ϵ, δ that

$$[a + \epsilon, b - \epsilon] \subseteq (c_1 - \delta/2, d_1 + \delta/2) \cup (c_2 - \delta/4, d_2 + \delta/4) \cup \ldots (c_k - 2^{-k}\delta, d_k + 2^{-k}\delta) \cup \ldots.$$

Then using compactness there exists some n such that

$$[a + \epsilon, b - \epsilon] \subseteq \bigcup_{k=1}^{n} (c_k - 2^{-k}\delta, d_k + 2^{-k}\delta).$$

Then we can compare the lengths of these two sets to get

$$b - a - 2\epsilon \leq \sum_{k=1}^{n} (d_k - c_k + 2^{-k+1}\delta) \leq \sum_{k=1}^{\infty} (d_k - c_k) + 2\delta.$$

Both ϵ and δ are arbitrary so we can let them go to 0 and get

$$b - a \leq \sum_{k} (d_k - c_k).$$

Now ranging over all possible covering sequences gives

$$b - a \leq \lambda^*((a, b]).$$

Note: When we are working in $\mathbb{R}^d$ as in the assignment the terms involving ϵ and δ will be multiplied by something involving the side lengths of rectangles. In order to run the proof you can say that wlog all the rectangles you are looking at are contained inside some fixed large rectangle. This will allow you to send ϵ and δ to zero without having to worry. $\qquad \square$

We want to turn this outer measure into a true measure. In order to do this we need to restrict λ^* to some subset of $\mathscr{P}(\mathbb{R})$.

Definition 3.5 (Lebesgue Measurable sets). We call a set $A \in \mathscr{P}(\mathbb{R})$ is *Lebesgue Measureable* if for any $B \in \mathscr{P}(\mathbb{R})$ we have
$$\lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(A^c \cap B).$$

Proposition 3.6. *The collection of Lebesgue measureable sets, $\mathscr{M}$, is a σ algebra.*

Proof. First let us notice that the definition of a Lebesgue measureable sets is symmetric in A and A^c, so $A \in \mathscr{M}$ implies that $A^c \in \mathscr{M}$.

Secondly we can see that $\emptyset \in \mathscr{M}$ as $\lambda^*(A \cap \emptyset) + \lambda^*(A \cap \emptyset^c) = \lambda^*(\emptyset) + \lambda^*(A \cap E) = 0 + \lambda^*(A)$. This also implies via the first point that $E \in \mathscr{M}$.

We then show that if $A_1, A_2 \in \mathscr{M}$ then $A_1 \cup A_2 \in \mathscr{M}$. Using the fact that $A_1 \in \mathscr{M}$ we have

$$\lambda^*(B \cap (A_1 \cup A_2)) = \lambda^*(B \cap (A_1 \cup A_2) \cap A_1) + \lambda^*(B \cap (A_1 \cup A_2) \cap A_1^c) = \lambda^*(B \cap A_1) + \lambda^*(B \cap A_2 \cap A_1^c).$$

We also have the identity $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$ therefore

$$\lambda^*(B \cap (A_1 \cup A_2)) + \lambda^*(B \cap (A_1 \cup A_2)^c) = \lambda^*(B \cap A_1) + \lambda^*(B \cap A_2 \cap A_1^c) + \lambda^*(B \cap A_1^c \cap A_2^c).$$

Then since $A_2 \in \mathscr{M}$ we have

$$\lambda^*(B \cap A_2 \cap A_1^c) + \lambda^*(B \cap A_1^c \cap A_2^c) = \lambda^*(B \cap A_1^c).$$

Therefore,

$$\lambda^*(B \cap (A_1 \cup A_2)) + \lambda^*(B \cap (A_1 \cup A_2)^c) = \lambda^*(B \cap A_1) + \lambda^*(B \cap A_1^c).$$

Then we use again the fact that $A_1 \in \mathscr{M}$ to get

$$\lambda^*(B \cap (A_1 \cup A_2)) + \lambda^*(B \cap (A_1 \cup A_2)^c) = \lambda^*(B).$$

This shows that $A_1 \cup A_2 \in \mathscr{M}$.

Now let us take an infinite sequence of disjoint sets $A_1, A_2, A_3, \ldots$ then we will show

$$\lambda^*(B) = \sum_{i=1}^{n} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{n} A_i^c\right)\right).$$

We can show this by induction. For the base case it just follows with $n = 1$ from the fact that $A_1 \in \mathscr{M}$. Then by induction suppose we know that

$$\lambda^*(B) = \sum_{i=1}^{n-1} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right).$$

Now since $A_n \in \mathscr{M}$ we have

$$\lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right) = \lambda^*\left(B \cap A_n \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right) + \lambda^*\left(B \cap A_n^c \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right).$$

Now since A_n is disjoint from $A_1, \ldots, A_{n-1}$ we have that $A_n \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right) = A_n$ so we have

$$\lambda^*\left(B \cap \left(\bigcap_{i=1}^{n-1} A_i^c\right)\right) = \lambda^*(B \cap A_n) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{n} A_i^c\right)\right).$$

This gives our induction step.

By monotonicity of the outer measure this gives that for any n we have

$$\lambda^*(B) \geq \sum_{i=1}^{n} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right).$$

Consequently we can let n tend to infinity to get

$$\lambda^*(B) \geq \sum_{i=1}^{\infty} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right).$$

Now we can use the countable subadditivity of λ^* to get

$$\lambda^*(B) \geq \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + \lambda^*\left((B \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right) = \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right).$$

Furthermore, the subadditivity of λ^* gives

$$\lambda^*(B) \leq \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right).$$

Therefore,

$$\lambda^*(B) = \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + \lambda^*\left(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right).$$

We have now shown that $\mathscr{M}$ is closed under complements and taking countable unions and contains $\emptyset$ which is sufficient to show that $\mathscr{M}$ is a σ-algebra. $\qquad \square$

Proposition 3.7. *The restriction of λ^* to $\mathscr{M}$ is a measure.*

Proof. We need to show that λ^* is countably additive on $\mathscr{M}$ so let $A_1, A_2, \ldots$ be a sequence of disjoint subsets in $\mathscr{M}$. In the proof that $\mathscr{M}$ is a σ-algebra we showed that

$$\lambda^*(B) \geq \sum_{i=1}^{\infty} \lambda^*(B \cap A_i) + \lambda^*\left(B \cap \left(\bigcap_{i=1}^{\infty} A_i^c\right)\right).$$

Now let us take the particular case where $B = \bigcup_{i=1}^{\infty} A_i$ this gives

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{n} \lambda^*(A_i) + \lambda^*\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) = \sum_{i=1}^{\infty} \lambda^*(A_i).$$

Countable subadditivity gives

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{n} \lambda^*(A_i),$$

so consequently

$$\lambda^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{n} \lambda^*(A_i).$$

$\square$

Remark 3.8. We now call the restriction of λ^* to $\mathscr{M}$, λ and call it *Lebesgue measure*.

We now want to know that there are some Lebesgue measureable sets. In order to do this we first show that all the intervals of the form $(-\infty, b]$ are Lebesgue measurable.

Lemma 3.9. *The intervals of the form $(-\infty, b]$ are Lebesgue measureable.*

Proof. Let B be a subset of $\mathbb{R}$ and let $I_1, I_2, \ldots$ be a sequence of half open intervals such that $B \subseteq I_1 \cup I_2 \cup \ldots$. Now let us define the (often empty) intervals $I_i^l = I_i \cap (-\infty, b]$ and $I_i^r = I_i \cap (b, \infty)$, these are also half open intervals. We have $B \cap (-\infty, b] \subseteq \bigcup_n I_n^l$ and $B \cap (b, \infty) \subseteq \bigcup_n I_n^r$. Therefore we have

$$\lambda^*(B \cap (-\infty, b]) \leq \sum_n \lambda(I_n^l), \quad \lambda^*(B \cap (b, \infty)) \leq \sum_n \lambda(I_n^r).$$

Using this we have

$$\lambda^*(B \cap (-\infty, b])) + \lambda^*(B \cap (b, \infty)) \leq \sum_n \lambda(I_n^l) + \sum_n \lambda(I_n^r) = \sum_n \lambda(I_n).$$

We can then take the infimum over all possible sequences of intervals covering B to get

$$\lambda^*(B \cap (-\infty, b])) + \lambda^*(B \cap (b, \infty)) \leq \lambda^*(B).$$

Combining this with countable subadditivity gives

$$\lambda^*(B) = \lambda^*(B \cap (-\infty, b])) + \lambda^*(B \cap (b, \infty)).$$

Therefore, $(-\infty, b]$ is Lebesgue measurable. $\square$

Corollary 3.10. *Every set in $\mathcal{B}(\mathbb{R})$ is Lebesgue measurable.*

Proof. The Borel σ algebra is the σ algebra generated by sets of the form $(-\infty, b]$ as shown last week. Therefore, as $\mathscr{M}$ is a σ-algebra and contains all the intervals of the form $(-\infty, b]$ then it contains the Borel σ-algebra. $\qquad\square$

The construction of Lebesgue measure via the outer measure can be generalised via Carathéodory's extension theorem. We briefly give the defintion of a ring of subsets.

Definition 3.11 (Ring). A collection of subsets, $\mathcal{A}$, of a space E is called a ring if for every $A, B \in \mathcal{A}$ we have $A \setminus B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.

Now we introduce Carathéodory's Extension theorem. We can see that the proof is in many ways very similar to the construction of Lebesgue measure.

Theorem 3.12 (Carathéodory's Extension Theorem). *Let $\mathcal{A}$ be a ring of subsets of E, and let $\mu : \mathcal{A} \to [0, \infty]$ be a countably additive set function. Then μ extends to a measure on $\sigma(\mathcal{A})$.*

Proof. We define the outer measure μ^* on $\mathscr{P}(E)$ by

$$\mu^*(B) = \inf\left\{\sum_n \mu(A_n) \ : \ A_n \in \mathcal{A} \forall n, B \subset \bigcup_n A_n\right\}.$$

$\mu^*(B) = \infty$ if there is not possible sequence of A_n so that B is contained in their union. We can see immediately that $\mu^*(\emptyset) = 0$ and μ^* is increasing.

As before we define $\mathscr{M}$ to be the set of μ^* measurable sets A that satisfy, for every $B \subseteq E$ that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

We want to show that $\mathscr{M}$ is a σ-algebra and that μ^* restricts to a measure on $\mathscr{M}$.

First we show that μ^* is countably subadditive. Suppose that we have a sequence B_n and want to show that

$$\mu^*\left(\bigcup_n B_n\right) \le \sum_n \mu^*(B_n).$$

Let us fix some $\epsilon > 0$ then for each n there is a sequence $A_{n,m} \in \mathcal{A}$ such that $B_n \subset \bigcup_m A_{n,m}$ and $\sum_m \mu(A_{n,m}) \le \mu^*(B_n) + \epsilon 2^{-n}$. Then $\bigcup_n B_n \subset \bigcup_{n,m} A_{n,m}$ and $\sum_{n,m} \mu(A_{n,m}) \le \sum_n \mu^*(B_n) + \epsilon$. Therefore $\mu^*\left(\bigcup_n B_n\right) \le \sum_n \mu(B_n) + \epsilon$. Since ϵ is arbitrary this gives the countable subadditivity.

Now we show that μ^* agrees with μ on $\mathcal{A}$. Let us take $A \in \mathcal{A}$ clearly $A \subseteq A$ so $\mu^*(A) \le \mu(A)$. Now suppose that there is a sequence $A_n \in \mathcal{A}$ such that $A \subseteq \bigcup_n A_n$. Then $A \cap A_n = A \setminus (A \setminus A_n) \in \mathcal{A}$. Therefore we use the countable subadditivity of μ on $\mathcal{A}$ to get

$$\mu(A) \le \sum_n \mu(A_n \cap A) \le \sum_n \mu(A_n).$$

Taking the infimum over such sequences gives $\mu(A) \le \mu^*(A)$. Therefore μ and μ^* agree on $\mathcal{A}$.

Now we show that $\mathscr{M}$ contains $\mathcal{A}$. That is to say we want to show that if $A \in \mathcal{A}$ then for every B

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

Using subadditivity of μ^* we have that $\mu^*(B) \le \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Therefore we want to show $\mu^*(B) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Let A_n be a sequence in $\mathcal{A}$ such that $\mu^*(B) \ge \sum_n \mu(A_n) - \epsilon$, $B \subseteq \bigcup_n A_n$, then we already know that $A \cap A_n$ will be in $\mathcal{A}$ we also have that $A^c \cap A_n = A_n \setminus (A \cap A_n) \in \mathcal{A}$. Therefore $\mu(B \cap A) \le \sum_n \mu(A_n \cap A)$ and $\mu(B \cap A^c) \le \sum_n \mu(A^c \cap A_n)$ and consequently

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \le \sum_n \left(\mu(A_n \cap A) + \mu(A_n \cap A^c)\right) = \sum_n \mu(A_n) \le \mu^*(B) + \epsilon.$$

As ϵ is arbitrary this gives the required result.

The next step is to show that $\mathscr{M}$ is a σ-algebra. We start with the algebra part. E and $\emptyset$ are in $\mathscr{M}$ as

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap \emptyset),$$

just because $B \cap E = B$ and $B \cap \emptyset = \emptyset$ and we know $\mu^*(\emptyset) = 0$. We also can see that

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

is symmetric in exchanging A and A^c so if $A \in \mathscr{M}$ then so is A^c. Now suppose $A_1, A_2 \in \mathscr{M}$. We notice that $(A_1 \cap A_2)^c \cap A_1 = (A_1^c \cup A_2^c) \cap A_1 = (A_1^c \cap A_1) \cup (A_2^c \cap A_1) = A_2^c \cap A^1$ and $A_1^c = A_1^c \cap (A_1 \cap A_2)^c$. Using this and the fact that A_1, A_2, A_1^c, A_2^c are in $\mathscr{M}$ we have

$$
\begin{aligned}
\text{Using that } A_1 \in \mathscr{M} \quad \mu^*(B) &= \mu^*(B \cap A_1) + \mu^*(B \cap A_1^c) \\
\text{Using that } A_2 \in \mathscr{M} \quad &= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap A_1 \cap A_2^c) + \mu^*(B \cap A_1^c) \\
\text{Using our first identity} \quad &= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap A_1^c) \\
\text{Using our second identiy} \quad &= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1) + \mu^*(B \cap (A_1 \cap A_2)^c \cap A_1^c) \\
\text{Using the fact that } A_1 \in \mathscr{M} \quad &= \mu^*(B \cap (A_1 \cap A_2)) + \mu^*(B \cap (A_1 \cap A_2)^c).
\end{aligned}
$$

Now that we have shown that $\mathscr{M}$ contains finite unions we want to show it countains countable unions. Let A_n be a sequence of disjoin sets in $\mathscr{M}$. Let us write $A = \bigcup_n A_n$. Then itterating our previous result we have for any B, n that

$$\mu^*(B) = \sum_{k=1}^{n} \mu^*(B \cap A_k) + \mu^*(B \cap A_1^c \cap \cdots \cap A_n^c).$$

Now as $A^c \subseteq A_1^c \cap A_2^c \cdots \cap A_n^c$ for each n we have $\mu^*(B \cap A^c) \leq \mu^*(B \cap A_1^c \cap \cdots \cap A_n^c)$. Therefore for each n

$$\mu^*(B) \geq \sum_{k=1}^{n} \mu^*(B \cap A_k) + \mu^*(B \cap A^c).$$

Letting $n \to \infty$ we have

$$\mu^*(B) \geq \sum_{n} \mu^*(B \cap A_n) + \mu^*(B \cap A^c).$$

Now we use the countable subadditivity of μ^* and the fact that $B \cap A = \bigcup_n (B \cap A_n)$ to get

$$\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

As the other inequality holds by subadditivity of μ^* we have that $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$ and hence $A \in \mathscr{M}$.

Lastly, we want to show that μ^* is a measure on $\mathscr{M}$. In order to do this we need to show that μ^* is countably additive on $\mathscr{M}$. In the last step we showed that for any B, and a sequence of disjoint sets A_n in $\mathscr{M}$ with $A = \bigcup_n A_n$, that

$$\mu^*(B) \geq \sum_{n} \mu^*(A_n \cap B) + \mu^*(B \cap A^c).$$

If we apply this identity with $B = A$ and use the fact that $A_n \cap A = A_n$ we get

$$\mu^*(A) \geq \sum_{n} \mu^*(A_n).$$

Since we already know that μ^* is countably subadditive this is sufficient to show that μ^* is countably additive and hence a measure on $\mathscr{M}$. $\qquad\square$

Theorem 3.13 (Uniqueness of Extension). *Let μ_1 and μ_2 be measures on $(E, \mathcal{E})$ with $\mu_1(E) = \mu_2(E) < \infty$. Suppose that $\mu_1 = \mu_2$ on $\mathcal{A}$ where $\mathcal{A}$ is a π-system generating $\mathcal{E}$, then $\mu_1 = \mu_2$ on $\mathcal{E}$.*

Proof. Let us consider $\mathcal{D} \subseteq \mathcal{E}$ defined as the measurable sets on which $\mu_1(A) = \mu_2(A)$. By hypothesis $E \in \mathcal{D}$ and $\mathcal{A} \subseteq \mathcal{D}$. We want to show that $\mathcal{D}$ is a σ-algebra and therefore $\mathcal{D} = \mathcal{E}$. Suppose that $A, B \in \mathcal{E}$ with $A \subseteq B$ then we have $\mu_i(A) + \mu_i(B \setminus A) = \mu_i(B) < \infty$. This means that if A and B are in $\mathcal{D}$ then so is $A \setminus B$. Therefore, $\mathcal{D}$ is a d-system containing the π-system $\mathcal{A}$ so by Dynkin's lemma is equal to $\mathcal{E}$. $\qquad\square$

4 Week 3 starts here - Outer measure and Lebesgue measure cont.

4.1 Properties of Lebesgue measure

This collection is a section of facts about Lebesgue measure and the set of Lebesgue measurable sets. We start with looking at $\mathcal{M}$ the σ-algebra of Lebesgue measurable sets.

Lemma 4.1 (Null sets are all Lebesgue measurable). *If A in $\mathscr{P}(\mathbb{R})$ and $\lambda^*(A) = 0$ then $A \in \mathcal{M}$.*

Proof. This is on the assignment. $\qquad\square$

We can actually characterise all Lebesgue measurable sets in terms of Null sets and Borel sets both of which we have shown are measurable. We have the following propersition which we wont prove in the course. We will show a similar result in the optional exercises.

Proposition 4.2. *A set $S \subseteq \mathbb{R}$ is Lebesgue measurable if and only there exists a Borel set B and a null set N such that $S = B \triangle N$.*

The most important thing we want to prove about $\mathcal{M}$ is that there exists a non-Lebesgue measurable set. Before we do this we need to explore a few more properties of Lebesgue measure itself.

Proposition 4.3. *Lebesgue measure is* regular *that is to say*

- $\lambda(A) = \inf\{\lambda(U) \,:\, U \text{ is open}, A \subseteq U\}$,

- $\lambda(A) = \sup\{\lambda(K) \,:\, K \text{ is compact}, K \subseteq A\}$.

Proof. By monotonicity we can see that $\lambda(A) \leq \inf\{\lambda(U) \,:\, U \text{ is open}, A \subseteq U\}$. Furthermore we can find a sequence of half open rectangles R_k such that $A \subseteq \bigcup_n R_n$ and $\sum_n \lambda(R_n) \leq \lambda(A) + \epsilon$. By slightly enlarging each of the half open rectangles we can produce another sequence of fully open rectangles $\tilde{R}_n$ such that $A \subseteq \bigcup_n \tilde{R}_n$ and $\lambda(A) \geq \sum_n \lambda(\tilde{R}_n) - 2\epsilon$. The set, $\bigcup_n \tilde{R}_n$ is open and ϵ can be made arbitrarily small so this shows $\lambda(A) \geq \inf\{\lambda(U) \,:\, U \text{ is open}, A \subseteq U\}$.

Monotonicity shows that $\lambda(A) \geq \sup\{\lambda(K) \,:\, K \text{ is compact}, K \subseteq A\}$. First let us assume that A is contained in some ball, B around 0. Now use the first part to find some open set U such that $B \setminus A \subseteq U$ and $\lambda(U) \leq \lambda(B \setminus A) + \epsilon$. Now let $K = B \setminus U$ then we have $K \subseteq A \subseteq B$ and $\lambda(K) \geq \lambda(B) - \lambda(U) \geq \lambda(B) - \lambda(B \setminus A) - \epsilon = \lambda(A) - \epsilon$ (here we use the fact that B, A, U, K will all have finite measure as they are inside B). As ϵ is arbitrary this concludes the proof when A is contained in a ball.

Now suppose that A is unbounded. Then let $A_n = A \cap B_n$ where B_n is the closed ball of radius n. We have that $\lambda(A_n) \to \lambda(A)$. If $\lambda(A) = \infty$ then we can find $K_n \subseteq A_n$ with $\lambda(K_n)$ arbitrarily close to $\lambda(A_n)$ therefore we can find such a sequence with $\lambda(K_n) \to \infty$. If $\lambda(A) \neq \infty$ then, given ϵ, there exists N such that $\lambda(A_n) \geq \lambda(A) - \epsilon$ for $n \geq N$. Then we can fine $K_N \subseteq A_N$ such that $\lambda(K_n) \geq \lambda(A_N) - \epsilon$ therefore $\lambda(K_N) \geq \lambda(A) - 2\epsilon$. This shows we can a compact set which is contained in A, with measure arbitrarily close to that of A. $\qquad\square$

We now want to show that Lebesgue measure is the only which assigns each interval the correct measure.

Proposition 4.4. *Lebesgue measure is the only measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which assigns each half open interval its length. This is equally true with half open hyper-rectangles in $\mathbb{R}^d$.*

Proof. The collection of half open intervals is a π-system which generates the Borel σ-algebra. Therefore, if $\lambda(\mathbb{R})$ had been finite we could use Dynkin's uniqueness of extension Lemma to get that any other measure which agrees with Lebesgue measure on the half open intervals must agree with Lebesgue measure on the whole of the Borel σ-algebra. Instead let $E_n = [-n, n]^d$ then, by Dynkin's uniqueness of extension lemma we have that λ is the only measure on E_n, assigning each rectangle inside E_n its measure. Since every rectangle is bounded, eventually it is inside some E_n so if μ is a measure such that $\mu(R) = \lambda(R)$ for every rectangle then the restriction of μ to E_n must agree with the restriction of λ to E_n. We also have that, for any A, $\mu(A) = \lim_n \mu(A \cap E_n)$ by continuity of measure. So $\mu(A) = \lim_n \mu(A \cap E_n) = \lim_n \lambda(A \cap E_n) = \lambda(A)$. $\qquad\square$

Corollary 4.5. *Lebesgue measure is translation invariant. That is to say if we define the set $x + A = \{x + y, y \in A\}$ then $\lambda(x + A) = \lambda(A)$*

Proof. Define a new measure λ_x by $\lambda_x(A) = \lambda(x+A)$ then $\lambda_x((a, b]) = \lambda((a+x, b+x]) = b+x-(a+x) = b - a$. Therefore λ_x agrees with λ on the half open intervals and therefore agrees with λ on the whole of $\mathcal{B}(\mathbb{R})$. Again it is straightforward to extend this to $\mathbb{R}^d$. $\qquad\square$

Lastly, in the construction of Lebesgue measure we show that $\mathcal{M}$ is not the whole of $\mathcal{P}(\mathbb{R})$ and that there exist non-Lebesgue measureable sets.

Proposition 4.6. *There exists sets that are in $\mathcal{P}(\mathbb{R})$ which are not in $\mathcal{M}$.*

Proof. This proof involves the use of the axiom of choice. In fact it is known that it is necessary to use some form of the axiom of choice to prove the existence of a non-Lebesgue measurable set in $\mathbb{R}$.

We use an argument by contradiction, we begin by assuming every subset of $\mathbb{R}$ is Lebesgue measurable. We define an equivalence relation on $[0, 1)$ by saying $x \sim y$ exactly when $x - y \in \mathbb{Q}$. Using the axiom of choice we find a subset S of $[0, 1)$ which contains exactly one representative of each equivalence class. Next we define the set $S + q = \{s + q \,(\text{mod}\,1) : s \in S\}$ for each $q \in \mathbb{Q} \cap [0, 1)$. Then by our choice of S we have that
$$[0, 1) = \bigcup_{q \in \mathbb{Q} \cap [0,1)} (S + q),$$
where this union is disjoint. We can also see by translation invariance of λ that if S were Lebesgue measurable then we would have
$$\lambda(S) = \lambda(S + q)$$
for every q. Therefore, by countable additivity we would have
$$\lambda([0, 1)) = \sum_{q \in \mathbb{Q} \cap [0,1)} \lambda(S + q) = \sum_{q \in \mathbb{Q} \cap [0,1)} \lambda(S) = \infty.$$

$\qquad\square$

5 Measurable Functions

A big part of measure theory is the study of functions which are compatible with the measure spaces. We begin with a basic definition which will be satisfied by all the functions we are interested in.

Definition 5.1 (Mesasurable functions). If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are two measurable spaces and f is a function $E \to F$, then we say f is *measurable* if for every $A \in \mathcal{F}$ we have $f^{-1}(A) \in \mathcal{E}$.

Lemma 5.2. *Suppose that $\mathcal{A} \subset \mathcal{F}$ is such that $\sigma(\mathcal{A}) = \mathcal{F}$. If f is a function such that for every $A \in \mathcal{A}$ we have $f^{-1}(A) \in \mathcal{E}$ then f is measurable.*

Proof. First we note that

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i),$$

and

$$f^{-1}(B \setminus A) = f^{-1}(B) \setminus f^{-1}(A).$$

Now if we consider $\{A \in \mathcal{F} : f^{-1}(A) \in \mathcal{E}\}$ then this is a σ-algebra, as $\mathcal{E}$ is a σ-algebra and f^{-1} preserves set operations. Therefore, $\{A \in \mathcal{F} : f^{-1}(A) \in \mathcal{E}\}$ is a σ-algebra containing $\mathcal{A}$ therefore $\mathcal{F} \subseteq \{A \in \mathcal{F} : f^{-1}(A) \in \mathcal{F}\}$ so f is measurable. $\square$

Remark 5.3. In particular note that the above lemma means that whenever we have $f : E \to \mathbb{R}$ and $\mathbb{R}$ is equipped with the Borel σ algebra, we know that f is measurable if $f^{-1}((-\infty, b])$ is a measurable set for every b.

Lemma 5.4. *If E, F are topological spaces, equipped with their Borel σ-algebras, and we have $f : E \to F$ is a continuous map then f is measurable.*

Proof. This is on the exercise sheet. $\square$

Lemma 5.5. *If f_n is a sequence of measurable function taking values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then the following functions are all measurable:*

- $-f_1$

- λf_1 *for $\lambda > 0$ a fixed contant.*

- $f_1 \wedge f_2$

- $f_1 \vee f_2$

- $f_1 + f_2$,

- $f_1 f_2$,

- $\sup_n f_n$,

- $\inf_n f_n$,

- $\limsup_n f_n$,

- $\liminf_n f_n$.

Proof. We only show two result. The rest are on the assignment.

In order to show that any of these functions are measureable we want to look at $f^{-1}((-\infty, b])$ or a similar set. $(f_1 \vee f_2)^{-1}((-\infty, b]) = \{x : \max\{f_1(x), f_2(x)\} \leq b\} = \{x : f_1(x) \leq b \text{ and } f_2(x) \leq b\} = \{x : f_1(x) \leq b\} \cap \{x : f_2(x) \leq b\} = f_1^{-1}((-\infty, b]) \cap f_2^{-1}((-\infty, b])$. Now since f_1 and f_2 are both measureable the sets $f_1^{-1}((-\infty, b])$ and $f_2^{-1}((-\infty, b])$ are both measurable. We also know that the intersection of two measurable sets is measurable.

$(f_1 + f_2)^{-1}((b, \infty)) = \{x : f_1(x) + f_2(x) > b\}$. Now if $f_1(x) > b - f_2(x)$ then there exists a $q \in \mathbb{Q}$ such that $f_1(x) > q > b - f_2(x)$. Let us define the set $A = \bigcup_{q \in \mathbb{Q}} \{x : f_1(x) > q\} \cap \{x : f_2(x) > b - q\}$. Since f_1, f_2 are both measurable A is a countable union of measurable sets so measurable. We can also see that if $x \in A$ then $f_1(x) + f_2(x) > b$ and our observation shows that in fact $A = \{x : f_1(x) + f_2(x) > b\}$. Therefore, $f_1 + f_2$ is measurable. $\qquad\square$

Definition 5.6 (Image measure)**.** We can use a measurable function f to define an image measure. Suppose μ is a measure on $(E, \mathcal{E})$ and f is a measurable function $(E, \mathcal{E}) \to (F, \mathcal{F})$ then we can define a new measure ν by saying that

$$\nu(A) = \mu(f^{-1}(A)),$$

for every $A \in \mathcal{F}$. We write $\nu = \mu \circ f^{-1}$.

We can use the notion of image measure to construct further measures from Lebesgue measure.

Lemma 5.7. *Suppose $g : \mathbb{R} \to \mathbb{R}$ and that g is non-constant, right-continuous and non-decreasing. Let us define $g(-\infty) = \lim_{x \to -\infty} g(x)$ and $g(\infty) = \lim_{x \to \infty} g(x)$ and let us call the interval $I := (g(-\infty), g(\infty))$ (this might be the whole of $\mathbb{R}$. Define a partial inverse to g by $f : I \to \mathbb{R}$ by*

$$f(x) = \inf_y \{x \leq g(y)\}.$$

Then f is left-continuous and non-decreasing and $f(x) \leq y$ if and only if $x \leq g(y)$.

Proof. Fix $x \in I$ and consider the set $J_x = \{y \in \mathbb{R} : x \leq g(y)\}$ by definition of I we know that J_x is non empty and is not the whole of $\mathbb{R}$ (this shows that f is well defined). As g is non-decreasing, if $y \in J_x$ and $y' \geq y$, then $y' \in J_x$. As g is right-continuous, if $y_n \in J_x$ and $y_n \downarrow y$ then $y \in J_x$ (noting the $\leq$ sign in J_x). Now using this we have that if $x \leq x'$ then $J_x \supseteq J_{x'}$ so $f(x) \leq f(x')$. We also have that if $x_n \uparrow x$ then $J_x = \bigcap_n J_{x_n}$, so $f(x_n) \to f(x)$. $\qquad\square$

Theorem 5.8. *Let g be a non-constant, right-continuous and non-decreasing function from $\mathbb{R} \to \mathbb{R}$. There exists a unique Radon measure on $\mathbb{R}$ such that for all $a, b \in \mathbb{R}$ with $a < b$*

$$\mathrm{d}g((a, b]) = g(b) - g(a).$$

We call this measure the Lebesgue Steitjles *measure associated with g. Furthermore, every Radon measure on $\mathbb{R}$ can be represented this way.*

Proof. Define I and f as in the Lemma above. Then we can construct $\mathrm{d}g$ as the image measure of Lebesgue measure on I. That is to say we can let $\mathrm{d}g = \lambda \circ f^{-1}$. If this is the case then

$$\mathrm{d}g((a, b]) = \lambda(\{x : f(x) > a, f(x) \leq b\}) = \lambda((g(a), g(b)]) = g(b) - g(a).$$

The standard argument for uniqueness of measures (as for that of Lebesgue measure) gives uniqueness of this measure.

Finally, if ν is a Radon measure on $\mathbb{R}$ then we can define a function g, by

$$g(y) = \nu((0, y]), \quad y \geq 0, \quad g(y) = -\nu((y, 0]), \quad y < 0.$$

Then $\nu = \mathrm{d}g$ by uniqueness. $\qquad\square$

5.1 Random variables and the measure theoretic formulation of probability - in brief

The structure of measure theory allows us to put probability theory on a firm footing.

Definition 5.9. We call a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ a *probability space* (and use different letters for the different bits) if $\mathbb{P}(\Omega) = 1$. In this setting we still have Ω is a set, $\mathcal{F}$ is a σ-algebra and $\mathbb{P}$ is a measure. We call $\mathbb{P}$ a probability measure.

The set Ω is not really described and we view it as the *space of all possible events*. We call $A \subset \Omega$ and event, and $\mathbb{P}(A)$ the probability of an event happening.

Definition 5.10. A *random variable*, X is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to another measurable space $(E, \mathcal{A})$.

Under this way of writing things we have $\mathbb{P}(X \in B) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in B\}) = \mathbb{P}(X^{-1}(B))$, where $X^{-1}(B) \in \mathcal{F}$ as X is measurable. We call $X^{-1}(B)$ the event that $X \in B$. When working with probability people usually suppress the argument ω.

Definition 5.11. The *law of a random variable*, X is the image measure of $\mathbb{P}$ under the measurable function X. I.e. if $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{A})$ then we define a measure μ_X on $(E, \mathcal{A})$ by $\mu_X(B) = \mathbb{P}(X \in B)$.

Remark 5.12. The law of a random variable is an object which allows us to understand both probability densities and discrete probability distributions in the same way.

If X takes values in $\mathbb{R}$ then the distribution function of X, $F_X(x) = \mu_X((-\infty, x])$. If X has density $f(x)$ then μ_X is equal to the measure given by $\mu_X(A) = \int_A f(x)\mathrm{d}x$, though we still don't know how to integrate over complicated sets.

6 Week 4 starts here - Measurable function cont.

6.1 Convergence of measurable functions

Definition 6.1 (Almost everywhere / Almost surely)**.** We use the short hand *almost everywhere* (or *almost surely* in a probability space) to discuss properties that are true everywhere except a measure zero set.

Definition 6.2 (Convergence almost everywhere)**.** Let $(E, \mathcal{E}, \mu)$ be a measureable space. A sequence of measureable functions, $(f_n)_{n \geq 1} : E \to F$, *converges almost everywhere* to f if

$$\mu\left(\{x \in E : f_n(x) \not\to f(x)\}\right) = 0$$

Definition 6.3 (Convergence in measure)**.** Let $(E, \mathcal{E}, \mu)$ be a measureable space. A sequence of real valued measureable functions, $(f_n)_{n \geq 1} : E \to F$, *converges in measure* to f if for every $\epsilon > 0$

$$\mu\left(\{x : |f(x) - f_n(x)| > \epsilon\}\right) \to 0, \quad \text{as } n \to \infty.$$

Example 6.4. The sequence of functions $f_n(x) = x^n$ converges to 0 Lebesgue almost everywhere on $[0, 1]$, and in measure, but it doesn't converge pointwise as it doesn't converge at $x = 1$.

Example 6.5. The- sequence of functions $f_n(x) = 1_{[n,n+1]}(x)$ converges to 0 Lebesgue almost everywhere (in fact everywhere) but not in measure.

Example 6.6. Consider the sequence of functions $f_1 = 1_{[0,1/2)}, f_2 = 1_{[1/2,1)}, f_3 = 1_{[0,1/4)}, f_4 = 1_{[1/4,1/2)}, f_5 = 1_{[1/2,3/4)}, f_6 = 1_{[3/4,1)}, f_7 = 1_{[0,1/8)}, f_8 = 1_{[1/8,1/4)} \ldots$ then f_n converges to 0 in measure, but $f_n(x)$ does not converge for any x.

We can prove a quasi-equivalence between these two notions of measure. Before we do this we need to introduce a very useful lemma, the Borel-Cantelli Lemma. We introduce it here as it is used to prove the following theorem but it is a useful tool to have whilst doing measure theory, particularly probability theory. First let us also introduce some more notation

Definition 6.7. Let $(A_n)_n$ be a sequence of measurable sets then we have the following names

$$\limsup_n A_n = \bigcap_n \bigcup_{m \geq n} A_m = \{A_n \text{ infinitely often }\},$$

and

$$\liminf_n A_n = \bigcup_n \bigcap_{m \geq n} A_m = \{A_n \text{ eventually }\}.$$

The last names are more comon when the A_n are events in a probability space.

Lemma 6.8 (First Borel-Cantelli Lemma). *Let $(E, \mathcal{E}, \mu)$ be a measure space. Then if $\sum_n \mu(A_n) < \infty$ it follows that $\mu(\limsup_n A_n) = 0$.*

Proof. For any n we have

$$\mu(\limsup_n A_n) \leq \mu \left(\bigcup_{m \geq n} A_m \right) \leq \sum_{m \geq n} \mu(A_m).$$

Then the right hand side goes to zero as $n \to \infty$, so $\mu(\limsup_n A_n) = 0$. $\qquad\square$

Lemma 6.9 (Second Borel-Cantelli Lemma). *Let $(E, \mathcal{E}, \mu)$ be a probability space ($\mu(E) = 1$). Then suppose that $\mu(A_i \cap A_j) = \mu(A_i)\mu(A_j)$ (the events are pairwise independent) for every i, j and that $\sum_n \mu(A_n) = \infty$ then it will follow that $\mu(\limsup_n A_n) = 1$.*

Proof. First we note that $\mu(A_i^c \cap A_j^c) = \mu((A_i \cup A_j)^c) = 1 - \mu(A_i \cup A_j) = 1 - \mu(A_i) - \mu(A_j) + \mu(A_i)\mu(A_j) = (1 - \mu(A_i))(1 - \mu(A_j))$. We use the inequality $1 - a \leq e^{-a}$. Let $a_n = \mu(A_n)$ then

$$\mu \left(\bigcap_{m=n}^N A_m^c \right) = \Pi_{m=n}^N (1 - a_m) \leq \exp \left(- \sum_{m=n}^N a_m \right) \to 0, \quad \text{as } N \to 0.$$

Therefore, $\mu \left(\bigcap_{m \geq n} A_m^c \right) = 0$ for every n. So $\mu(\limsup_n A_n) = 1 - \mu(\bigcup_n \bigcap_{m \geq n} A_m^c) = 1$. $\qquad\square$

Theorem 6.10. *Let $(E, \mathcal{E}, \mu)$ be a measure space and $(f_n)_n$ be a sequence of measurable functions. Then we have the following:*

- *Suppose that $\mu(E) < \infty$ and that $f_n \to 0$ almost everywhere, then $f_n \to 0$ in measure.*

- *If $f_n \to 0$ in measure then there exists some subsequence $(n_k)_k$ such that $f_{n_k} \to 0$ almost everywhere.*

Proof. Suppose that $f_n \to 0$ almost everywhere. Then

$$\mu(\{x : |f_n(x)| \leq \epsilon\}) \geq \mu \left(\bigcap_{m \geq n} \{x : |f_m(x)| \leq \epsilon\} \right) \uparrow \mu(|f_n| \leq \epsilon \text{ eventually}) \geq \mu(\{x : f_n(x) \to 0\}) = \mu(E),$$

therefore,

$$\mu(\{x : |f_n(x)| > \epsilon\}) = \mu(E) - \mu(\{x : |f_n(x)| \leq \epsilon\}) \to 0.$$

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Now suppose that $f_n \to 0$ in measure. We can find a subsequence n_k such that

$$\mu(\{x \,:\, |f_{n_k}(x)| > 1/k\}) \leq 2^{-k}.$$

Therefore

$$\sum_k \mu(\{x \,:\, |f_{n_k}(x)| > 1/k\}) < \infty.$$

Therefore by the first Borel-Canteli lemma we have that

$$\mu\left(\{x \,:\, |f_{n_k}(x)| > 1/k \text{ infinitely often}\}\right) = 0.$$

Therefore $f_{n_k} \to 0$ almost everywhere. $\qquad \square$

6.2 Egoroff's Theorem and Lusin's Theorem

Theorem 6.11 (Egoroff's Theorem). *Let $(E, \mathcal{E}, \mu)$ be a finite measure space and $(f_n)_n$ be a sequence of real valued measurable functions on E. If μ is finite and if f_n converges μ-almost everywhere to f then for each positive ϵ there is a set A with $\mu(A^c) < \epsilon$, such that f_n converges uniformly on A to f.*

Proof. For each n let $g_n(x) = \sup_{j \geq n} |f_j(x) - f(x)|$. Then g_n is a positive function which is finite almost everywhere. The sequence $(g_n)_n$ converges to 0 almost everywhere and so in measure. Therefore, for each positive integer k we can find n_k such that

$$\mu\left(\{x \,:\, g_{n_k} > 1/k\}\right) < \epsilon 2^{-k}.$$

Define sets $A_k = \{x \,:\, g_{n_k}(x) \leq 1/k\}$ and let $A = \bigcap_k A_k$. The set A has

$$\mu(A^c) = \mu\left(\bigcup_k A_k^c\right) \leq \sum_k \mu(A_k^c) \leq \sum_k \epsilon 2^{-k} = \epsilon.$$

We want to show that f_n converges uniformly to f on A. For each δ there exists a k such athat $1/k < \delta$, then as $A \subseteq A_k$, we have that for every $n \geq n_k$,

$$|f_n - f| \leq g_{n_k} \leq 1/k < \delta,$$

uniformly on all $x \in A_k$ and hence in A. $\qquad \square$

This proof motivates the following definition of convergence of functions.

Definition 6.12 (Almost uniform convergence). We say a sequence of functions $(f_n)_{n \geq 1}$ converges *almost uniformly* on a measure space $(E, \mathcal{E}, \mu)$ if for every $\epsilon > 0$ there exists a set A with $\mu(A^c) < \epsilon$ with $f_n \to f$ uniformly on A.

We can use Egoroff's theorem to prove a result called Lusin's theorem. First let us recall the definition of regularity

Definition 6.13. Let E be a topological space and μ be a measure on $(E, \mathcal{B}(E))$ then say μ is *regular* if for every $A \in \mathcal{B}(E)$ we have

- $\mu(A) = \inf\{\mu(U) \,:\, A \subseteq U, U \text{ is open}\}$,

- $\mu(A) = \sup\{\mu(K) \,:\, K \subseteq A, K \text{ is compact}\}$.

Theorem 6.14 (Lusin's Theorem). *Suppose that f is a measurable function and $A \subseteq \mathbb{R}^d$ is a Borel set and $\lambda(A) < \infty$ then for any $\epsilon > 0$ there is a compact subset K of A with $\lambda(A \setminus K) < \epsilon$ such that the restriction of f to K is continuous.*

Remark 6.15. This theorem can be generalised to locally compact Hausdorff spaces, see Cohn's book.

Proof. Suppose first that f only takes countably many values, $a_1, a_2, a_3, \dots$ on A the let $A_k = \{x \in A : f(x) = a_k\}$, by measurablility of f we can see that $A_k = f^{-1}(\{a_k\})$ is measurable. We know that $A = \bigcup_n A_n$ so by continuity of measure $\lambda(\bigcup_{k=1}^n A_k) \uparrow \lambda(A)$. Since $\lambda(A) < \infty$ we have that for any $\epsilon > 0$ there exists n such that $\lambda(A \setminus \bigcup_{k=1}^n A_k) < \epsilon/2$. By the regularity of Lebesgue measure we can find compact subsets $K_1, \dots, K_n$ such that $\lambda(A_n \setminus K_n) \leq \epsilon/2n$. Then let $K = \bigcup_{k=1}^n K_k$. This is a compact subset of A and

$$\lambda(A \setminus K) \leq \lambda(A \setminus \bigcup_{k=1}^n A_k) + \lambda(\bigcup_{k=1}^n A_k \setminus \bigcup_{k=1}^n K_k) < \epsilon/2 + \epsilon/2.$$

Now f restricted to K is continuous since the K_i are disjoint and f is constant on each K_i.

Now we have proved the special case where f takes countably many values we can use this to prove the theorem for general f. Let $f_n = 2^{-n} \lfloor 2^n f \rfloor$ then $2^{-n} \geq f(x) - f_n(x) \geq 0$ so $f_n(x) \to f(x)$, uniformly. Now, f_n can only take countably many values, so by our special case of Lusin's theorem there exists a $K_n \subseteq K$, compact, such that $\lambda(A \setminus K_n) \leq \epsilon 2^{-n-1}$, and f_n is continuous on K_n. Now let $K_\infty = \bigcap_n K_n$, then K_∞ is compact and $\lambda(A \setminus K_\infty) = \lambda(A \setminus K_\infty) = \lambda(\bigcup_n (A \setminus K_n)) \leq \epsilon/2 + \sum_n \epsilon 2^{-n-1} = \epsilon$. Now we have that f_n converges uniformly to f on K_∞ and f_n is continuous on K_∞ for each n. As the uniform limit of continuous functions is continuous this shows that f is continuous on K. □

7 Integration

We now get to the definition of the Lebesgue integral which is the second important object that we construct in this course. There are several different notations for the integral of a function f with respect to a measure μ. We have

$$\mu(f) = \int_E f \mathrm{d}\mu = \int_E f(x) \mu(\mathrm{d}x).$$

When you are integrating with respect to Lebesgue measure the most common notation is

$$\int_E f(x) \mathrm{d}x.$$

Before we start constructing the integral we'll briefly discuss the motivations for how to construct it. Firstly, you've already seen the Riemann integral. We can describe the strategy of Riemann integration - very loosly - as splitting the *domain* of the function into equal sized chunks, estimating the height of the function on each chunk then adding them together. Broadly what happens with Lebesgue integration is that we split the *range* of the function into equal sized chunks, estimate the size of the part of the domain which will end up in that chunk of range then sum everything up. We need the theory of measure in order to do this because the bit of the domain corresponding to chuncks of the range can be quite weird sets whose size it wouldn't be possible to measure. The first motivation for this is that whilst Riemann integration only works for functions from subsets of $\mathbb{R}^d$ to $\mathbb{R}$, Lebesgue integration allows the domain on the function to be quite weird, (as long as it is a measure space). As an example, this is helpful for taking expectations rigorously because expectations are integral of random variables and the domain of a random variable is a probability space which may not be explicit.

The second big motivation for introducing a new theory of integration is the issue of convergence. It is important in many practical applications of integration theory to know when $\lim_n \int f_n(x)\mathrm{d}x = \int \lim_n f_n(x)\mathrm{d}x$ or when $\int_{E_x} \int_{E_y} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{E_y} \int_{E_x} f(x,y)\mathrm{d}y\mathrm{d}x$. Lebesgue integration allows us to rigorously find conditions on f under which these statements will be true. This is often not possible in a satisfactory way with the Riemann theory of integration. We will see some of these convergence theorems next week and then switching the order of integration towards the end of the course (currently planned for week 9). The most important motivation for developing good convergence theorems was the development of Fourier series. We want to know when it is possible to integrate a Fourier series term by term.

The strategy for constructing the integral is to begin by defining $\mu(f)$ when f belongs to a special class of measurable functions that we call *simple functions*. We then define the integral to progressively larger classes of functions.

Definition 7.1 (Simple functions). Let $(E, \mathcal{E}, \mu)$ be a measure space. The set of simple functions on this space taking values in $\mathbb{R}$ are functions of the form

$$f(x) = \sum_{k=1}^{n} a_k 1_{A_k}(x).$$

Here, the A_k are disjoint sets in $\mathcal{E}$, 1_A represents the indicator function of the set, and the a_k are non-negative real numbers. We note that this representation of f is not unique.

Definition 7.2 (The integral of a simple function). Still working in the setting above, let $f(x) = \sum_{k=1}^{n} a_k 1_{A_k}(x)$, then we can define

$$\mu(f) = \sum_{k=1}^{n} a_k \mu(A_k).$$

8 Week 5 starts here -Integration cont.

Definition 8.1 (Simple functions). Let $(E, \mathcal{E}, \mu)$ be a measure space. The set of simple functions on this space taking values in $\mathbb{R}$ are functions of the form

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Definition 8.2 (The integral of a simple function). Still working in the setting above, let $f(x) = \sum_{k=1}^{n} a_k 1_{A_k}(x)$, then we can define

$$\mu(f) = \sum_{k=1}^{n} a_k \mu(A_k).$$

Lemma 8.3. *The integral of a simple function is well defined (it doesn't depend on the choice of representation of the simple function) and satisfies the following properties.*

- *For $\alpha > 0$ we have $\mu(\alpha f) = \alpha \mu(f)$*

- *$\mu(f + g) = \mu(f) + \mu(g)$.*

- *If $f(x) \leq g(x)$ for every $x \in E$ then $\mu(f) \leq \mu(g)$.*

- $f = 0$ μ-*almost everywhere if and only if* $\mu(f) = 0$

Proof. Let us first look at the well definedness, without loss of generality lets assume the a_k, b_j are strictly positive and the sets A_k are disjoint and similar for the B_j. Let us suppose that $f = \sum_{k=1}^{n} a_k 1_{A_k} = \sum_{k=1}^{m} b_k 1_{B_k}$ which are both simple function representations. Then we can see that $\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{m} B_k$ and that $a_i = b_j$ if $A_i \cap B_j \neq \emptyset$. Using this we can write

$$\mu(f) = \sum_{k=1}^{n} a_k \mu(A_k)$$

$$\left(\text{as } A_k = \bigcup_j (A_k \cap B_j) \text{ as } \bigcup_{k=1}^{n} A_k = \bigcup_{j=1}^{m} B_j \right) \quad = \sum_{k=1}^{n} \sum_{j=1}^{m} a_k \mu(A_k \cap B_j)$$

$$(\text{as } a_k = b_j \text{ if } A_k \cap B_j \neq \emptyset \text{ so } a_k = b_j \text{ or } \mu(A_k \cap B_j) = 0) \quad = \sum_{k=1}^{n} \sum_{j=1}^{m} b_j \mu(A_k \cap B_j) = \sum_{j=1}^{m} b_j \sum_{k=1}^{n} \mu(A_k \cap B_j)$$

$$= \sum_{j=1}^{m} b_j \mu(B_j).$$

Now we move on to the linearity properties. These come naturally from the defintion,

$$\mu(\alpha f) = \sum_{k=1}^{n} \alpha a_k \mu(A_k) = \alpha \sum_{k=1}^{n} a_k \mu(A_k) = \alpha \mu(A_k).$$

When we are dealing with two simple functions simultaneously it is useful to write them both in a representation where the measurable sets appearing are the same for both functions. If we let $g = \sum_{j=1}^{m} c_j 1_{C_j}$ then let us write $A_0 = E \setminus \bigcup_{k=1}^{n} A_k, a_0 = 0$ and define C_0, c_0 similarly then we can write $f + g$ as a simple function via

$$f(x) + g(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} (a_k + c_j) 1_{A_k \cap C_j}$$

and we have

$$\mu(f + g) = \sum_{k=0}^{n} \sum_{j=0}^{m} (a_k + c_j) \mu(A_k \cap C_j).$$

We note that $\bigcup_{k=0}^{n} A_k = \bigcup_{j=0}^{m} C_j = E$ and the A_k are mutually disjont, and the C_j are mutually disjoint. Therefore

$$\mu(f + g) = \sum_{k=0}^{n} \sum_{j=0}^{m} (a_k + c_j) \mu(A_k \cap C_j)$$

$$= \sum_{k=0}^{n} a_k \sum_{j=0}^{m} \mu(A_k \cap C_j) + \sum_{j=0}^{m} c_j \sum_{k=0}^{n} \mu(A_k \cap C_j)$$

$$\text{as the unions of the } A_k \text{ or } C_j \text{ fill the space} \quad = \sum_{k=0}^{n} a_k \mu(A_k) + \sum_{j=0}^{m} c_j \mu(C_j)$$

$$\text{as } a_0 = c_0 = 0 \quad = \sum_{k=1}^{n} a_k \mu(A_k) + \sum_{j=1}^{m} c_j \mu(C_j)$$

$$= \mu(f) + \mu(g).$$

Now we move onto the monotonicity of the integral. Let us express f and g as before, again the goal is to represent the two functions using the same measurable sets. We can rewrite as

$$f = \sum_{k=0}^{n} \sum_{j=0}^{m} a_k 1_{A_k \cap C_j} = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{k,j} 1_{A_k \cap C_j},$$

where $a_{k,j} = a_k 1_{A_k \cap C_j \neq \emptyset}$. Here again we are using the fact that $\bigcup_{j=0}^{m} C_j$ fill the space. In the same way we can write

$$g = \sum_{k=1}^{n} \sum_{j=1}^{m} c_{k,j} 1_{A_k \cap C_j},$$

where $c_{k,j} = c_j 1_{A_k \cap C_j \neq \emptyset}$. Then if $f(x) \leq g(x)$ for every x we know that this means $a_{k,j} \leq c_{k,j}$ for every k, j. Then by the well definedness of the integral we have

$$\mu(f) = \sum_{k=1}^{n} \sum_{j=1}^{m} a_{k,j} \mu(A_k \cap C_j) \leq \sum_{k=1}^{n} \sum_{j=1}^{m} c_{k,j} \mu(A_k \cap C_j) = \mu(g).$$

Lastly, we look at when $\mu(f) = 0$. First if $f = 0$ μ-almost everywhere then $a_k = 0$ or $\mu(A_k) = 0$ for each k, therefore $\mu(f) = 0$. If $\mu(f) = 0$ then since all the terms $a_k \mu(A_k) \geq 0$ we have that $a_k = 0$ or $\mu(A_k) = 0$ for each k. $\qquad \square$

We are now going to extend the definition of the integral to a larger class of functions.

Definition 8.4 (Lebesgue integral for positive functions). Let f be a positive, measurable function, we define

$$\mu(f) = \sup\{\mu(g) \ : \ g \text{ is a simple function}, g \leq f\}$$

Lemma 8.5. *The above definition of Lebesgue integral for positive functions is consistent with the defintion for simple functions. That is to say that if $f = \sum_{k=1}^{n} a_k 1_{A_k}$ where $a_k \geq 0$ and the A_k are disjoint measurable sets then*

$$\sum_{k=1}^{n} a_k \mu(A_k) = \sup\{\mu(g) \ : \ g \text{ is a simple function}, g \leq f\}.$$

Proof. This is on the exercise sheet. $\qquad \square$

Definition 8.6 (Final definition of Lebesgue integral). Suppose that f is a measurable function which is not necessarily positive. Then we call f, *μ-integrable* or *Lebesgue integrable* if $\mu(|f|) < \infty$. In this case we can write $f = f_+ - f_-$ where f_+ and f_- are both positive and measureable ($f_+ = \max\{f, 0\}$). We then define the integral of f by

$$\mu(f) = \mu(f_+) - \mu(f_-).$$

Remark 8.7. Notice that we haven't yet proved that these definitions of the integral behave the way we hope (e.g. are linear, monotone, etc). In order to do this we need to prove some convergence results.

8.1 Convergence theorems for integrals of functions

This is one of the most useful parts of the course. In follow on courses in PDE and probability you will use these theorems again and again.

Lemma 8.8. *If f and g are non-negative, measurable, real valued functions on $(E, \mathcal{E}, \mu)$ and $f \leq g$ then $\mu(f) \leq \mu(f)$.*

Proof. If $f \le g$ then we recall that by definition $\mu(f) = \sup\{\mu(\tilde{f}) \ : \ \tilde{f} \text{ is simple, } \tilde{f} \le f\}$. So if h is a simple function with $h \le f$ then we also have that $h \le g$. Therefore, we can write

$$\begin{aligned} \mu(g) &= \sup\{\mu(h) \ : \ h \text{ is simple, } h \le g\} \\ &= \max\{\sup\{\mu(h) \ : \ h \text{ simple, } h \le f\}, \sup\{\mu(h) \ : \ h \text{ simple, } h \le g, h \text{ is not } \le \text{f}\}\} \\ &= \max\{\mu(f), \sup\{\mu(h) \ : \ h \text{ simple, } h \le f, h \text{ is not } \le \text{g}\}\} \ge \mu(f). \end{aligned}$$

$\square$

Theorem 8.9 (Monotone Convergence Theorem). *Let f be a non-negative, measurable real-valued function and let f_n be a sequence of such functions. Then if $f_n \uparrow f$ we will have that $\mu(f_n) \uparrow \mu(f)$.*

Proof. We will break this proof down into progressively more complicated cases. First we note that by monotonicity $\lim_n \mu(f_n) \le \mu(f)$ and therefore it is sufficient to prove $\mu(f) \le \lim_n \mu(f_n)$.

First let us look at the case where $f_n = 1_{A_n}$ and $f = 1_A$, then the assumptions imply that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ and $A = \bigcup_n A_n$. Then this result is the same as the continuity of a measure as proved before.

Now let us keep $f = 1_A$ and let f_n be a sequence of simple functions. Pick $\epsilon \in (0,1)$ arbitrary. Then let $A_n = \{x \ : \ f_n(x) > 1 - \epsilon\}$ then we have that $A_n \uparrow A$, and $(1-\epsilon)1_{A_n} \le f_n$. Therefore by the first case we have that $\lim_n \mu(f_n) \ge \lim_n \mu((1-\epsilon)1_{A_n}) = \lim_n(1-\epsilon)\mu(1_{A_n}) = (1-\epsilon)\mu(f)$. Since ϵ is arbitrary this gives the result.

Now we look at the case where both f and f_n are simple functions. We can write $f = \sum_{k=1}^n a_k 1_{A_k}$, where wlog each a_k is strictly positive, then we have that $a_k^{-1} f_n 1_{A_k} \uparrow 1_{A_k}$ so we can apply the previous case to each part of f. Specifically

$$\mu(f_n) = \mu(\sum_{k=1}^n 1_{A_k} f_n) = \sum_{k=1}^n a_k \mu(a_k^{-1} 1_{A_k} f_n) \uparrow \sum_{k=1}^n a_k \mu(A_k) = \mu(f).$$

Here the first equality follows from the fact that the support of f_n must be included in the support of f since $0 \le f_n \le f$.

Our next case is when f is positive and measurable and f_n are all simple. Let us pick g a simple function with $g \le f$ then $g_n = f_n \wedge g = \min\{f_n, g\}$ is a sequence of simple functions increasing to g. Therefore, by our previous case $\mu(g_n) \uparrow \mu(g)$. Furthermore $g_n \le f_n$ so by monotonicity

$$\mu(g) = \lim_n \mu(g_n) \le \lim_n \mu(f_n).$$

As g is an arbitrary this means that $\sup\{\mu(g) \ : \ g \text{ is a simple function}, g \le f\} \le \lim_n \mu(f_n)$.

The last case is the most general where both f_n and f are positive and measureable. In this case we introduce our favorite kind of approximation (which is very similar to what we used in Lusin's theorem)

$$g_n = \left(2^{-n} \lfloor 2^n f_n \rfloor\right) \wedge n,$$

then g_n is a sequence of simple functions with $g_n \uparrow f$ and $g_n \le f_n$, proving this is an exercise on the assignment. Therefore we have

$$\mu(f) = \lim_n \mu(g_n) \le \lim_n \mu(f_n) \le \mu(f).$$

Hence we have the required result. $\square$

We have a corollary of this result which can be thought of as another definition of $\mu(f)$ for non-negative, measurable f. This makes concrete our hand-waving definition of how Lebesgue measure works as splitting the range of f up into equal chunks and using this to approximate the area under the curve. This is often a more practically useful definition of the integral of a positive function.

Corollary 8.10. *Let f be a non-negative, measurable, real valued function on $(E, \mathcal{E}, \mu)$ and define*

$$f_n = \left(2^{-n}\lfloor 2^n f \rfloor\right) \wedge n,$$

then $\mu(f) = \lim_n \mu(f_n)$.

Proof. $f_n \uparrow f$ so by monotone convergence $\lim_n \mu(f_n) = \mu(f)$. $\qquad\square$

We can now use this to prove that the integral of positive measurable functions has the desired properties.

Proposition 8.11. *Suppose f and g are non-negative, real valued, measurable function on a space $(E, \mathcal{E}, \mu)$ then*

- *For every $\alpha > 0$ $\mu(\alpha f) = \alpha \mu(f)$,*

- *$\mu(f + g) = \mu(f) + \mu(g)$*

- *If $f \leq g$ then $\mu(f) \leq \mu(g)$*

- *$\mu(f) = 0$ if and only if f is 0 almost everywhere.*

Proof. Suppose that f_n is a sequence of simple functions with $f_n \uparrow f$ then αf_n is a sequence of simple functions with $\alpha f_n \uparrow \alpha f$. Monotone convergence tells us that $\mu(\alpha f) = \lim_n \mu(\alpha f_n)$. We can use our previous results for simple functions to get that $\mu(\alpha f_n) = \alpha \mu(f_n)$. Therefore $\mu(\alpha f) = \lim_n \mu(\alpha f_n) = \lim_n \alpha \mu(f_n) = \alpha \mu(f)$.

For the sum let f_n and g_n be sequences of simple functions with $f_n \uparrow f$ and $g_n \uparrow g$. Then we have $(f_n + g_n) \uparrow f + g$ and by monotone convergence and the results for simple functions we have $\mu(f + g) = \lim_n \mu(f_n + g_n) = \lim_n \mu(f_n) + \lim_n \mu(g_n) = \mu(f) + \mu(g)$.

The third point is proved before.

Now $\mu(f) = 0$ if and only if $\sup\{\mu(h) : h \text{ simple}, h \leq f\} = 0$ which is if and only if $\mu(h) = 0$ for every $h \leq f$ and h simple. Now we know that if h is simple $\mu(h) = 0$ if and only if h is zero almost everywhere. This tells us that $\mu(f) = 0$ if and only if $h \leq f$ and h simple, implies $h = 0$ almost everywhere. If we look at the set on which f is positive we have $\{x : f(x) > 0\} = \bigcup_n \{x : f(x) > 1/n\}$ so by continuity of measure if $\mu(f > 0) > 0$ then there exists some n such that $\mu(f > 1/n) > 0$ therefore f is zero almost everywhere if and only if we can't fit any simple function underneath f which is not zero almost everywhere. $\qquad\square$

Using this we can give another writing of the monotone convergence theorem.

Proposition 8.12 (Beppo-Levi). *Suppose that $(f_n)_{n \geq 0}$ is a sequence of real-valued measurable functions. Then $\mu(\sum_n f_n) = \sum_n \mu(f_n)$.*

Proof. Write $g_n = \sum_{k=1}^n f_k$ then $g_n \uparrow \sum_n f_n$

$$\sum_n \mu(f_n) = \lim_n \sum_{k=1}^n \mu(f_k),$$

then by linearity

$$\lim_n \sum_{k=1}^n \mu(f_k) = \lim_n \mu(\sum_{k=1}^n f_k) = \lim_n \mu(g_n),$$

then using monotone convergence we have

$$\lim_n \mu(g_n) = \mu(\sum_n f_n).$$

$\qquad\square$

We can also prove that our notion of convergence for integrable functions behaves in the way we expect. First we prove a helpful Lemma

Lemma 8.13. *Let f_1, f_2, g_1, g_2 all be non-negative, integrable, real valued functions such that $f_1 - f_2 = g_1 - g_2$ then we have $\mu(f_1) - \mu(f_2) = \mu(g_1) - \mu(g_2)$.*

Proof. We have that $f_1 + g_2 = g_1 + f_2$ so $\mu(f_1 + g_2) = \mu(g_1 + f_2)$, using linearity we have

$$\mu(f_1) + \mu(g_2) = \mu(g_1) + \mu(f_2)$$

since all the integrals involved are finite we can rearrange this to give

$$\mu(f_1) - \mu(f_2) = \mu(g_1) - \mu(g_2).$$

$\square$

Proposition 8.14. *Suppose that f and g are integrable, real valued function on $(E, \mathcal{E}, \mu)$ then*

- *For every $\alpha > 0$ we have $\mu(\alpha f) = \alpha \mu(f)$, we also have $\mu(-f) = -\mu(f)$*

- *The function $f + g$ is integrable and $\mu(f + g) = \mu(f) + \mu(g)$*

- *If $f \leq g$ then $\mu(f) \leq \mu(g)$*

Proof. Let us write $f = f_+ - f_-$ and $g = g_+ - g_-$ where these are split into the positive and negative parts of f and g. Then $\alpha f = \alpha f_+ - \alpha f_-$ so $\mu(\alpha f) = \mu(\alpha f_+) - \mu(\alpha f_-) = \alpha(\mu(f_+) - \mu(f_-))$. Similarly $-f = f_- - f_+$ so $\mu(-f) = \mu(f_-) - \mu(f_+) = -\mu(f)$.

First we have that $|f + g| \leq |f| + |g|$ so $f + g$ is integrable. For the second point we need to use the Lemma above. We know that $(f_+ + g_+) - (f_- + g_-) = (f + g)_+ - (f + g)_-$ and all of $(f_+ + g_+), (f_- + g_-), (f + g)_+, (f + g)_-$ are non-negative and integrable so using the lemma we have

$$\mu(f+g) = \mu((f+g)_+) - \mu((f+g)_-) = \mu(f_+ + g_+) - \mu(f_- + g_-) = \mu(f_+) - \mu(f_-) + \mu(g_+) - \mu(g_-) = \mu(f) + \mu(g).$$

For the last point if $f \leq g$ then $g - f$ is a non-negative measurable function so $\mu(g - f) \geq 0$ and $\mu(g) = \mu(f + (g - f))$. We then use the linearity from the last point to get $\mu(f + (g - f)) = \mu(f) + \mu(g - f) \geq \mu(f)$. $\square$

Theorem 8.15 (Fatou's Lemma). *Let f_n be a sequence of non-negative measurable function then we have the following result*

$$\mu\left(\liminf_n f_n\right) \leq \liminf_n \mu(f_n)$$

Remark 8.16. I always have trouble remembering which way around the inequality goes in this lemma. A helpful example is if $f_n = 1_{[n,n+1)}$ and μ is Lebesgue measure. Then $\lambda(f_n) = 1$ for every n and $\liminf_n f_n = 0$. This is also an instructive example for why the limits can fail to be the same. Essentially here the mass we are trying to integrate escapes to infinity.

Proof. This is essentially a consequence of monotone convergence. Let $g_n = \inf_{k \geq n} f_k$, then g_n is a non-decreasing sequence of measurable functions and $g_n \leq f_n$ for each n. By definition of the g_n we also know that $\liminf f_n = \liminf_n g_n = \lim_n g_n$. Using Monotone convergence we then have

$$\mu(\liminf_n f_n) = \mu(\lim_n g_n) = \lim_n \mu(g_n).$$

Then using monotonicity we have

$$\mu(g_n) \leq \mu(f_n)$$

for each n, so consequently

$$\lim_n \mu(g_n) = \liminf_n \mu(g_n) \leq \liminf_n \mu(f_n).$$

Putting these all together gives the result. $\square$

Fatou's lemma is key to proving our next important convergence theorem.

Theorem 8.17 (Dominated convergence theorem). *Let f_n be a sequence of functions and f another function such that $f_n \to f$ almost everywhere. Suppose further that there exists a positive function g such that $|f| \leq g, |f_n| \leq g$ for every n and $\mu(g) < \infty$, then $\lim_n \mu(f_n) = \mu(f)$. The function g is called the* dominating function.

Proof. Let us first suppose that $f_n \to f$ and the domination conditions hold everywhere. Then we have that $g + f_n$ is a sequence of non-negative measurable functions whose limit is $g + f$. Applying Fatou's lemma gives

$$\mu(g + f) \leq \liminf_n \mu(g + f_n) = \mu(g) + \liminf_n \mu(f_n),$$

subtracting $\mu(g)$ from each side (which we can do as it is finite) gives

$$\mu(f) \leq \liminf_n \mu(f_n).$$

Similarly $g - f_n$ is a sequence of non-negative measurable functions whose limit is $g - f$. Applying Fatou's lemma again gives

$$\mu(g) - \mu(f) \leq \mu(g) + \liminf_n (-\mu(f_n)) = \mu(g) - \limsup_n \mu(f_n).$$

Rearranging this since all the quantities are finite gives

$$\limsup_n \mu(f_n) \leq \mu(f).$$

Putting both parts together gives

$$\mu(f) \leq \liminf_n \mu(f_n) \leq \limsup_n \mu(f_n) \leq \mu(f).$$

Therefore the limit of the sequence $\mu(f_n)$ exists and is equal to $\mu(f)$.

The extension of this result to when the conditions only hold almost everywhere is due to the fact that the integrals of any function is unchanged by modifying that function on a measure zero set. This type of result will be discussed in more detail when we introduce *Lebesgue spaces*. It isn't really the point of this particular theorem, we just give the full version here so we are able to apply it. $\square$

9 More integration: Week 7 starts here

The following is a useful criteria for when we can differentiate under the integral sign which also serves as a good example of how to use the dominated convergence theorem.

Theorem 9.1 (Differentiation under the integral sign). *Let $(E, \mathcal{E}, \mu)$ be a measure space and $f : U \times E \to \mathbb{R}$ be a function such that $x \mapsto f(t, x)$ is integrable for every t, and $t \mapsto f(t, x)$ is differentiable for every x, and suppose further that there exists an integrable function $g(x)$ such that*

$$\left| \frac{\partial f(t, x)}{\partial t} \right| \leq g(x), \quad \forall t \in U$$

then the function $x \mapsto \partial f(t, x)/\partial t$ is integrable and the function $F(t) = \int_E f(t, x)\mu(\mathrm{d}x)$ is differentiable with

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \int_E \frac{\partial f}{\partial t}(t, x)\mu(\mathrm{d}x).$$

Notice here we are using a different notation for the integral with respect to μ. We do this because it is helpful to be able to emphasise that we integrate in x but not t.

Proof. Let ϵ_n be an arbitrary sequence which tends towards 0. Let

$$g_n(t, x) = \frac{f(t + \epsilon_n, x) - f(t, x)}{\epsilon_n} - \frac{\partial f}{\partial t}(t, x).$$

First we notice that $g_n \to 0$ and $g_n + \partial f / \partial t$ is measurable so $\partial f / \partial t$ is the limit of measurable functions so measurable. By the mean value theorem we have $|g_n| \leq 2g$ for each n. Therefore, by dominated convergence we have

$$\int g_n(t, x)\mu(\mathrm{d}x) \to 0.$$

This gives the required result. $\qquad\square$

We have a couple of useful facts about integration which don't fit into a big section.

Definition 9.2 (Restriction of a Measure)**.** Suppose that $(E, \mathcal{E}, \mu)$ is a measure space and $A \in \mathcal{E}$ then the set of measurable subsets of A is a σ algebra we call $\mathcal{A}$ and the restriction of μ to $\mathcal{E}_A$ is a measure we call μ_A. Furthermore we have that if f is a measurable function on E then $\mu(f 1_A) = \mu_A(f)$.

Remark 9.3. The last part is actually a lemma whose proof is an exercise.

We can use this defintion to make sense of Lebesgue integrals on intervals (for example). If $I = [a, b]$ then $\int_a^b f(x)\mathrm{d}x = \lambda(f 1_I)$.

We also notice that we can define a measure using a positive function f.

Proposition 9.4. *Let $(E, \mathcal{E}, \mu)$ be a measure space and let f be a non-negative measurable function. Define $\nu(A) = \mu(f 1_A)$ for each $A \in \mathcal{E}$. Then ν is a measure on E and for all non-negative g we have*

$$\nu(g) = \mu(fg)$$

Proof. First we need to show that ν is indeed a measure. $f 1_\emptyset = 0$ so we have $\nu(\emptyset) = 0$ as required. We will also have that $\nu(A) \geq 0$ since f is non-negative. We show countable additivity, we note that if A an B are disjoin then $1_{A \cup B} = 1_A + 1_B$ and furthermore if $A_1, A_2, \ldots$ is a sequence of disjoint sets then $1_{\bigcup_n A_n} = \sum_n 1_{A_n}$. With this reformulation $\nu(\bigcup_n A_n) = \mu(f 1_{\bigcup_n A_n}) = \mu(\sum_n f 1_{A_n})$ using the Beppo-Levi reformulation of monotone convergence we have

$$\mu(\sum_n f 1_{A_n}) = \sum_n \mu(f 1_{A_n}) = \sum_n \nu(A_n),$$

which is our desired result.

Now we want to show that if $g \geq 0$ then $\nu(g) = \mu(fg)$. Let us begin with the case where $g = 1_A$ for some measurable A, then $\nu(g) = \nu(A) = \mu(f 1_A) = \mu(fg)$ so the result follows by definition. Then using the linearity of μ we can see that if g is a simple function then $\nu(g) = \mu(fg)$. Now suppose that g is not necessarilly simple, we can constuct (in our standard way) a sequence of simple functions, g_n, which increase to g then by monotone convergence we have that $\nu(g) = \lim_n \nu(g_n) = \lim_n \mu(fg_n)$. Now fg_n is a sequence of function which increases to fg so using monotone convergence we have that $\lim_n \mu(fg_n) = \mu(fg)$ so we have that $\nu(g) = \mu(fg)$. $\qquad\square$

There are also a few facts about Riemann integration which work in pretty much exactly the same way for Lebesgue integration. For example the fundamental theorem of calculus holds equally well in this case. We will see in general that when something is Riemann integrable it is also Lebesgue integrable which will prove all these in general.

Theorem 9.5 (Fundamental theorem of calculus)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and set $F(t) = \int_a^t f(x)\mathrm{d}x = \lambda(1_{[a,t]}f)$, then F is differentiable with $F'(t) = f(t)$. Furthermore, let $F : [a, b] \to \mathbb{R}$ have continuous derivative f, then $F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$.*

Proof. Given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - t| \leq \delta$ implies that $|f(x) - f(t)| \leq \epsilon$ therefore if we take $|h| \leq \delta$ then

$$\left| \frac{1}{h}(F(t+h) - F(t)) - f(t) \right| = \frac{1}{|h|} \left| \int_t^{t+h} (f(x) - f(t))\mathrm{d}x \right| \leq \frac{1}{|h|} \int_{t \wedge (t+h)}^{t \vee (t+h)} |f(x) - f(t)|\mathrm{d}x.$$

Now we can use the fact that inside the integral $|x - t| \leq \delta$ so we have

$$[\left| \frac{1}{h}(F(t+h) - F(t)) - f(t) \right| \leq \frac{1}{|h|} \int_{t \wedge (t+h)}^{t \vee (t+h)} \epsilon \mathrm{d}x = \epsilon.$$

Therefore $\lim_{h \to 0}(F(t+h) - F(t))/h = f(t)$.

In the other direction $\mathrm{d}/\mathrm{d}t(F(t) - \int_a^t f(x)\mathrm{d}x) = 0$ so $F(t) - \int_a^t f(x)\mathrm{d}x$ is constant in t (by the mean value theorem), so $F(t) = F(a) + \int_a^t f(x)\mathrm{d}x$. This gives us the result. $\square$

We can use the fundamental theorem of calculus to prove the standard result about change of variables. This time we can exploit our new machinary.

Proposition 9.6. *Let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be continuously differentiable and stricly increasing then for all non-negative g on $[\phi(a), \phi(b)]$ we have*

$$\int_{\phi(a)}^{\phi(b)} g(y)\mathrm{d}y = \int_a^b g(\phi(x))\phi'(x)\mathrm{d}x.$$

Proof. First suppose that g is the indicator function of an interval $(c, d]$ then we want to prove that

$$\int_{\phi(a)}^{\phi(b)} 1_{(c,d]}(x)\mathrm{d}y = \int_a^b 1_{(c,d]}(\phi(x))\phi'(x)\mathrm{d}x.$$

Here the left hand side is equal to $[\phi(a), \phi(b)] \cap (c, d]$ and the right hand side is

$$\int_{a \vee c}^{b \wedge d} \phi'(x)\mathrm{d}x,$$

using the fundamental theorem of calculus this is

$$\phi(b \wedge d) - \phi(a \vee c) = \phi(b) \wedge \phi(d) - \phi(a) \vee \phi(c) = [\phi(a), \phi(b)] \cap (c, d]$$

where here we used the fact that ϕ was increasing to commute it with min and max.

Now we have shown our proposition holds when g is the indicator function of a half open interval. By linearity of the integral it will hold when g is the indicator function of a finite disjoint union of half open intervals. Now let $\mathcal{D}$ be the set of all Borel sets A such that 1_A satisfies our proposition. As the name suggests we want to show that $\mathcal{D}$ is a d-system. If $A \subseteq B$ and $A, B \in \mathcal{D}$ then $1_{B \setminus A} = 1_B - 1_A$ so the proposition will hold for $B \setminus A$ by linearity of the integral. Suppose that $A_1 \subseteq A_2 \subseteq A_3 \ldots$ then let $g_n = 1_{A_n}$ then $g_n \uparrow 1_A = g$ and $g_n \circ \phi \uparrow g \circ \phi$, as ϕ is increasing so if

$$\int_{\phi(a)}^{\phi(b)} g_n(y)\mathrm{d}y = \int_a^b g_n(\phi(x))\phi'(x)\mathrm{d}x$$

for each n applying monotone convergence to each side gives the result for $g = 1_A$. This shows that $\mathcal{D}$ is a d-system. Applying Dynkin's lemma then shows that for every $A \in \mathcal{B}(R)$ we have that the proposition holds with $g = 1_A$.

Linearity of the integral allows us to extend this result to any simple function g. We can then use monotone convergence in exactly the same way as for the last part to extend it to any non-negative measurable g. $\square$

9.1 Agreement with Riemann Integral

We now turn our attention to the case wher μ is Lebesgue measure, λ. We want to show that our new defintion of the integral will agree with the Riemann integral when they are both defined. Let us first recall the definitions associated with the Riemann integral.

Definition 9.7. Let $[a, b]$ be an interval in $\mathbb{R}$ then a finite sequence of real numbers $\{a_k\}_{k=0}^n$ is called a *partition* of the interval if $a = a_0 < a_1 < \cdots < a_n = b$. I usually denote a partition with a lower case p or q. You also often see the notation $\mathscr{P}$.

Definition 9.8. If we have two partitions $p = \{a_k\}_{k=0}^n$ and $q = \{b_j\}_{j=0}^m$ then we say q is a *refinement* of p if every element of p appears in q.

Definition 9.9. We call a sequence of partitions $(p_n)_{n \geq 1}$ *nested* if for every n we have that p_{n+1} is a refinement of p_n.

Definition 9.10. If we have a partition $p = \{a_k\}_{k=0}^n$ and a function f then we can define

$$m_k = \inf\{f(x) \,:\, x \in [a_{k-1}, a_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \,:\, x \in [a_{k-1}, a_k]\}.$$

Then we have the upper sum and lower sum associated to the partition which are defined as

$$l(f, p) = \sum_{k=1}^n m_k(a_k - a_{k-1}), \quad u(f, p) = \sum_{k=1}^n M_k(a_k - a_{k-1}).$$

Remark 9.11. We can check that if q is a refinement of p then

$$l(f, p) \leq l(f, q) \leq u(f, q) \leq u(f, p).$$

Definition 9.12. We call a function f, *Riemann integrable* on $[a, b]$ if for every sequence of nested partitions $(p_n)_{n \geq 1}$ we have that $\lim_n l(f, p_n) = \lim_n u(f, p_n)$ and we call this limit the Riemann integral of f or $\int_a^b f(x)\mathrm{d}x$.

Lemma 9.13. *A function, f, is Riemann integrable if and only if there exists a partition p such that*

$$u(f, p) - l(f, p) < \epsilon.$$

Proof. First suppose that f satisfies that for every ϵ there exists a p such that $u(f, p) - l(f, p) < \epsilon$ then

$$\inf_q u(f, q) - \sup_q l(f, q) \leq u(f, p) - l(f, p) < \epsilon.$$

Since, ϵ is arbitrary this implies

$$\inf_q u(f, q) = \sup_q l(f, q)$$

and therefore f is Riemann integrable.

Suppose that f is Riemann integrable then we know that $\inf_q u(f, q) = \sup_q l(f, q) = \int f\mathrm{d}x$. Therefore, given ϵ there exists p_1 and p_2 so that $u(f, p_1) \leq \int f\mathrm{d}x + \epsilon/2$ and $l(f, p_2) \geq \int f\mathrm{d}x - \epsilon/2$. Now let $p = p_1 \cup p_2$, that is to say the partition made up of all the point in both p_1 and p_2. In this case p_1 and p_2 are both refinements of p, so we have

$$\int f\mathrm{d}x - \epsilon/2 \leq l(f, p_2) \leq l(f, p) \leq u(f, p) \leq u(f, p_1) \leq \int f\mathrm{d}x + \epsilon/2,$$

so

$$u(f, p) - l(f, p) < \epsilon.$$

$\square$

Theorem 9.14. *Let $[a, b]$ be an interval. Suppose that f is a bounded, function which is Riemann integrable on $[a, b]$, then it is Lebesgue integrable and the Riemann integral agrees witht the Lebesgue integral.*

Proof. As f is bounded we only need to show that it is Lebesgue measurable in order for it to be integrable. Using the lemma above there exists a nested sequence of partitions p_n such that $u(f, p_n) - l(f, p_n) < 1/n$ for each n. Let us define two sequences of functions g_n and h_n. We write $p_n = \{a_k^n\}_{k=0}^{N_n}$, and recall the definition of m_k^n and M_k^n associated to the partition. Then we define

$$g_n := \sum_{k=1}^{N_n} m_k^n 1_{[a_{k-1}^n, a_k^n)}, \quad h_n := \sum_{k=1}^{N_n} M_k^n 1_{[a_{k-1}^n, a_k^n)}.$$

Here we can see that g_n and h_n are both sequences of simple functions. We also have that g_n is a monotonically increasing sequence and h_n is a monotonically decreasing sequence. As f is bounded so are the sequences $g_n(x)$ and $h_n(x)$ for each x so we define $g(x) = \lim_n g_n(x)$ and $h(x) = \lim_n h_n(x)$, these are both bounded, Borel measurable functions. We also have that $g_n(x) \le f(x) \le h_n(x) \le \sup_{[a,b]} f$ so consequently $g(x) \le f(x) \le h(x)$. We can see that $\lambda(g_n) = l(f, p_n)$ and $\lambda(h_n) = u(f, p_n)$. We can use $\sup_{[a,b]} f$ as a dominating function, so we have by dominated convergence that

$$\lambda(g) = \lim_n \lambda(g_n) = \lim_n l(f, p_n) = \lim_n u(f, p_n) = \lim_n \lambda(h_n) = \lambda(h).$$

We also have that $h - g \ge 0$ and $\lambda(h - g) = 0$ so we know that $h = g$ Lebesgue almost everywhere and as $h - f \le h - g$ we know that $f = h = g$ almost everywhere. Therefore, f is almost everywhere equal to a measurable function and it is bounded so is Lebesgue integrable $\qquad \square$

We finish this section with some examples of functions which are Lebesgue integrable but are not Riemann integrable. The most classic example of this is

Example 9.15. Let $f(x) = 1_{\mathbb{Q}}$ then f is Lebesgue integrable but not Riemann integrable on $[0, 1]$ (or any other interval). In order to see that this function is not Riemann integrable we can see that for any partition p as the rationals an the irrationals are dense in $[0, 1]$ then $l(f, p) = 0$ and $u(f, p) = 1$ therefore if we take a seqence of nested partitions p_n then we wont have the limits of $l(f, p_n)$ and $u(f, p_n)$ meeting.

10 Norms and inequalities

Let us remember what a normed space is with an emphasis on how this fits in with functions spaces.

Definition 10.1 (Normed space)**.** A normed space is a vector space $\mathcal{V}$ equipped with a norm, $\| \cdot \|$, which should satisfy

- $\|v\| \in \mathbb{R}_+$

- $\|\lambda v\| = |\lambda| \|v\|$

- $\|v + u\| \le \|v\| + \|u\|$

- $\|v\| = 0$ if and only if $v = 0$.

We are interested in normed spaces of functions, where the norms come from integrating quantites.

Definition 10.2. $L^p(E)$ Suppose that $(E, \mathcal{E}, \mu)$ is a measure space, and $p \geq 1$, then we have the associated L^p space, which is the space of bounded measureable functions equipped with the norm

$$\|f\|_p = (\mu(|f|^p))^{1/p}.$$

When we are working on $\Omega \subseteq \mathbb{R}^d$ with Lebesgue measure we often write the space $L^p(\Omega)$ to be the set of functions with $\int_\Omega |f|^p \mathrm{d}x < \infty$. We then often write the norm $\|\cdot\|_{L^p(\Omega)}$ if we have not previously specified which space we are working in. If we are working with the measure $\gamma(x)\mathrm{d}x$ for some positive function γ on $\mathbb{R}^d$ then we write $L^p(\gamma), \|\cdot\|_{L^p(\gamma)}$. We also calle $L^p(E)$ or $L^p(\Omega)$ or $L^p(\gamma)$ the space of measurable functions where the associated norm is finite.

We also have the supremum norm.

Definition 10.3. Suppose $(E, \mathcal{E}, \mu)$ is a measure space. We have the following norm on measurable functions

$$\|f\|_\infty = \inf\{\lambda : |f| \leq \lambda \,\text{almost everywhere}\}.$$

We also call the space $L^\infty(E)$ the space of all measurable functions on E with $\|f\|_\infty < \infty$.

Remark 10.4. Strictly speaking the norms defined above are seminorms. This is because all these norms will vanish for a function f, where f is non-zero but is equal to zero almost everywhere. When working in L^p spaces we consider two functions the same if they are equal almost everywhere.

Strictly speaking we no longer consider functions f when we work in L^p spaces we instead consider equivalence classes of functions with the equivalence relation $f \sim g$ if $f = g$ almost everywhere. When working in this setting we write $\mathcal{L}^p$ for the space of measurable function equipped with the p-seminorm and L^p for the space of equivalence classes of functions equipped with the p-norm. Most of the time we wont really think about an element of L^p as an equivalence class and *hopefully* it quickly becomes natural to think about functions as defined up to alteration on a null set.

Theorem 10.5. *For $p \in [1, \infty]$ the space $L^p(E)$ is indeed a vector space.*

Proof. We need to show $f \in L^p(E)$ implies that $\alpha f \in L^p(E)$ for $\alpha \in \mathbb{R}$ (or $\mathbb{C}$) and $f, g \in L^p(E)$ implies that $f + g \in L^p(E)$, specifically we need to show that $\|\alpha f\|_p < \infty$ and $\|f + g\|_p < \infty$.

For $p < \infty$, the first point we can use the linearity of the integral to get

$$\mu(|\alpha|^p |f|^p) = |\alpha|^p \mu(|f|^p) < \infty.$$

Now, in a slightly more complex way we have

$$\int |f(x)+g(x)|^p \mu(\mathrm{d}x) \leq \int (2\max\{|f(x)|, |g(x)|\})^p \mu(\mathrm{d}x) \leq \int 2^p (|f(x)|^p + |g(x)|^p) \mu(\mathrm{d}x) \leq 2^p (\|f\|_p^p + \|g\|_p^p) < \infty.$$

For $p = \infty$ for the first point it follows immediately from the definition that $\|\alpha f\|_\infty = |\alpha| \|f\|_\infty < \infty$. For the second point since the union of two null sets is null we have that $f + g$ is equivalent to a function which is uniformly bounded. Therefore it is clear that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. $\qquad\square$

In order to progress further with normed spaces of functions we need to be able to prove the triangle inequality for the p-norms. This inequality is called *Minkowski's inequality*. In the next section we prove it as well as several other inequalities which are very useful when working with function spaces.