Last lecture!

10 December 2021 09:01

Good morning! Remind me to record!

Merry Christmas

Do the end of module feedback please

Radon - Nikodym Theorem

If $(E, \mathcal{E}, \mu)$ is a measure space and f is a non-neg measurable function then $\nu(A) = \mu(f \mathbb{1}_A)$ is a measure

R-N Thm says if ν is a measure s.t. $\mu(A) = 0 \Rightarrow \nu(A) = 0$ then $\exists$ a measurable function f s.t. $\nu(A) = \mu(f \mathbb{1}_A)$.

under some assumptions.

Defn Suppose $(E, \mathcal{E})$ is a measurable space and μ, ν are measures on $(E, \mathcal{E})$ the we say ν is absolutely continuous with respect to μ (and write $\nu \ll \mu$) if $\forall A \in \mathcal{E}$ with $\mu(A) = 0$ we have $\nu(A) = 0$.

For example δ_0 is not absolutely continuous with respect to Lebesgue measure as $\lambda(\{0\}) = 0$ but $\delta_0(\{0\}) = 1$.

If $\nu(A) = \mu(f \mathbb{1}_A)$, f measurable ≥ 0

then if $\mu(A) = 0$ then $\nu(A) = \mu(\underbrace{f \mathbb{1}_A}_{=0 \text{ a.e.}}) = \mu(0) = 0$.

$\nu \ll \mu$.

Lemma Alternative characterisation $\nu \ll \mu$ iff $\forall \varepsilon > 0$

$\exists\ \delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \nu(A) < \varepsilon$.

Theorem Let $(E, \mathcal{E})$ be a measurable space, μ, ν two finite measures with $\nu \ll \mu$. Then there exists $g : E \to [0, \infty]$ s.t. $\nu(A) = \mu(g \mathbb{1}_A)$

Remark The fact that we take μ, ν as finite measures isn't necessary. You can extend to the σ-finite case relatively straightforwardly. In Cohn's book.

The function g is unique up to identifying almost everywhere equal functions. We write $g = d\nu/d\mu$ and call it the Radon-Nikodym derivative of ν with respect to μ.

Proof Let us define the set $\mathcal{F}$ to be the set of all non-negative functions f with $\mu(f \mathbb{1}_A) \le \nu(A)$.

$\mathcal{F}$ is non-empty as $0 \in \mathcal{F}$.

We want to find $g \in \mathcal{F}$ s.t. $\mu(g) = \sup_{f \in \mathcal{F}} \mu(f)$, and show that g satisfies the conditions to be $d\nu/d\mu$.

As a first step we show that if $f_1, f_2 \in \mathcal{F}$ then $f_1 \vee f_2 \in \mathcal{F}$. So let us take any $A \in \mathcal{E}$ and let

$A_1 = A \cap \{f_1 \ge f_2\}$, $A_2 = A \cap \{f_1 < f_2\}$ these are both in $\mathcal{E}$

$$\mu(f_1 \vee f_2 \mathbb{1}_A) = \mu(f_1 \vee f_2 \mathbb{1}_{A_1}) + \mu(f_1 \vee f_2 \mathbb{1}_{A_2})$$

$$= \mu(f_1 \mathbb{1}_{A_1}) + \mu(f_2 \mathbb{1}_{A_2})$$

$$\leq \nu(A_1) + \nu(A_2) = \nu(A)$$

so $\mu(f_1 \vee f_2 \mathbb{1}_A) \leq \nu(A)$, works for every A so

$f_1 \vee f_2 \in \mathcal{F}$.

Now write $s = \sup_{f \in \mathcal{F}} \mu(f)$ so by definition of sup

there exists $f_n \in \mathcal{F}$ s.t. $\mu(f_n) \geq s - \frac{1}{n}$.

Then set $g_n = f_1 \vee f_2 \vee \dots \vee f_n$ then $g_n \geq f_n$ so

$\mu(g_n) \geq s - \frac{1}{n}$ and g_n is an increasing sequence.

So since its increasing we can define g by $g(x) = \lim_{n \to \infty} g_n(x)$

and by Monotone convergence $\mu(g \mathbb{1}_A) = \lim_{n \to \infty} \mu(g_n \mathbb{1}_A)$

Since $g_n \in \mathcal{F}$ for every n we have $\mu(g_n \mathbb{1}_A) \leq \nu(A)$

so $\mu(g \mathbb{1}_A) \leq \nu(A)$ so $g \in \mathcal{F}$.

and $\mu(g) = \lim_{n \to \infty} \mu(g_n) \geq \lim_{n \to \infty} (s - \frac{1}{n}) = s$

and as $g \in \mathcal{F}$ $\mu(g) \leq s$. So $\mu(g) = s$.

Now we want to show $\nu(A) = \mu(g \mathbb{1}_A)$ for every A.

$\mu(g \mathbb{1}_A) \leq \nu(A)$ $\forall A$

Now we want to

We know that $\mu(g \mathbb{1}_A) \leq \nu(A) \quad \forall A$

so we can define a new measure ν_0 by

$$\nu_0(A) = \nu(A) - \mu(g \mathbb{1}_A).$$

Then showing $g = d\nu/d\mu$ is equivalent to showing ν_0 is identically 0.

We're going to show this by contradiction. We're going to show that if $\nu_0 \neq 0$ then $\exists$ a measurable function $\tilde{g} \in \mathcal{F}$, s.t. $\mu(\tilde{g}) > s$.

Assume for contradiction that $\exists A$ with $\nu_0(A) > 0$ then by monotonicity $\nu_0(E) > 0$. Then since μ is finite there exists some $\varepsilon > 0$ s.t. $\nu_0(E) > \varepsilon \mu(E)$.

Now $\nu_0 - \varepsilon\mu$ is a finite signed measure.

So take a Hahn decomposition (P, N) for $\nu_0 - \varepsilon\mu$

then claim $\mu(P) > 0$ as if $\mu(P) = 0$ then

$\nu(P) = 0$ as $\nu \ll \mu$ so then

$$\nu_0(P) = \nu(P) - \mu(g \mathbb{1}_P) = 0 \quad \text{so then}$$

$$(\nu_0 - \varepsilon\mu)(E) = (\nu_0 - \varepsilon\mu)(P) + (\nu_0 - \varepsilon\mu)(N)$$

$$= \cancelto{0}{\nu_0(P)} - \cancelto{0}{\varepsilon\mu(P)} + (\nu_0 - \varepsilon\mu)(N)$$

$$\leq 0$$

... the choice of ε, so that $(\nu_0 - \varepsilon\mu)(E) > 0$

$= \nu$

this contradicts the choice of ε so that $(\nu_0 - \varepsilon\mu)(E) > 0$

therefore $\mu(P) > 0$.

$$= \nu(A) = \mu(g\mathbb{1}_A) + \nu_0(A) \overset{\text{by monotonicity}}{\geq} \mu(g\mathbb{1}_A) + \nu_0(A \cap P)$$

$(\nu_0 - \varepsilon\mu)(P) \geq 0$

so $\nu_0(P) \geq \varepsilon\mu(P)$

$$\geq \mu(g\mathbb{1}_A) + \varepsilon\mu(A \cap P)$$

$$= \mu(\mathbb{1}_A(g + \varepsilon\mathbb{1}_P))$$

so this works for any A so

$$\mu((g + \varepsilon\mathbb{1}_P)\mathbb{1}_A) \leq \nu(A)$$

so $g + \varepsilon\mathbb{1}_P \in \mathcal{F}$

and $\mu(g + \varepsilon\mathbb{1}_P) = \mu(g) + \varepsilon\mu(P) > \mu(g)$

This is a contradiction as $\mu(g) = \sup_{f \in \mathcal{F}} \mu(f)$.

Therefore ν_0 must be the zero measure so we must have

$$\nu(A) = \mu(g\mathbb{1}_A).$$

Uniqueness: Suppose we have another measurable function $h \geq 0$ with $\mu(g\mathbb{1}_A) = \mu(h\mathbb{1}_A) = \nu(A)$ $\forall A$

as $\nu(E) < \infty$ g, h are integrable so $g - h$ is integrable

then we know $\mu((g-h)\mathbb{1}_A) = 0$ $\forall$ $A \in \mathcal{E}$

so as g, h are measurable $\{x \in E : g-h \geq 0\}$ is measurable so $\mu((g-h)\mathbb{1}_{g \geq h}) = 0$

~~...~~ $(g-h)\mathbb{1}_{g \geq h}$ is non-negative

so $(g-h)\mathbb{1}_{g \geq h} = 0$ almost everywhere

similarly $(g-h)\mathbb{1}_{h \leq g} = 0$ almost everywhere

so $g = h$ almost everywhere.