In the previous video (simple monotone) we proved:

Proposition 2.3.2. Let $f \in \mathcal{S}^+$ and let $\{f_n\}$ be a monotone sequence in $\mathcal{S}^+$ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

We will now prove Proposition 2.3.3 for which the conclusion is the same but the hypothesis is weakened to $f : X \to [0, \infty]$ is measurable and we don't require f to be a simple function.

Proposition 2.3.3. Let $f: X \to [0,\infty]$ be measurable and $\{f_n\}$ a monotone sequence in S^+ converging pointwise to to f then

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Proof. Recall that the definition of the integral of a non-negative measurable function is given in terms of non-negative simple functions.

That is to say:

$$\int f \, d\mu = \sup_{\substack{g \in \mathcal{L}^+ \\ g \leq f}} \int g \, d\mu$$

Since $f \geq f_n$ it is clear that

$$\int f \, d\mu \geq \int f_n \, d\mu$$

so

$$\int f \, d\mu \geq \lim_{n \to \infty} \int f_n \, d\mu.$$

so $$\int f\,d\mu \geq \lim_{n\to\infty} \int f_n\,d\mu.$$

We need to show the other inequality:

$$\lim_{n\to\infty} \int f_n\,d\mu \geq \int f\,d\mu.$$

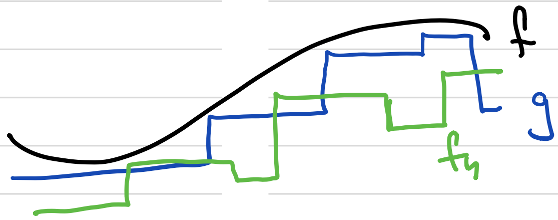
To show: $$\lim_{n\to\infty} \int f_n \, d\mu \geq \int f \, d\mu.$$

Since the right hand side is a supremum over $g \leq f$ we need to show that for any $g \in \mathcal{L}^+$ with $g \leq f$ that

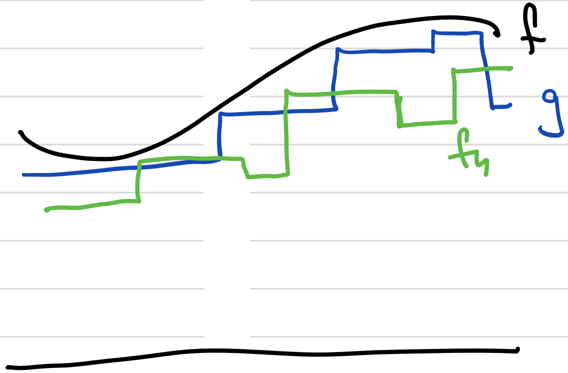
$$\lim_{n\to\infty} \int f_n \, d\mu \geq \int g \, d\mu.$$

So lets fix a $g \in \mathcal{L}^+$ with $g \leq f$.

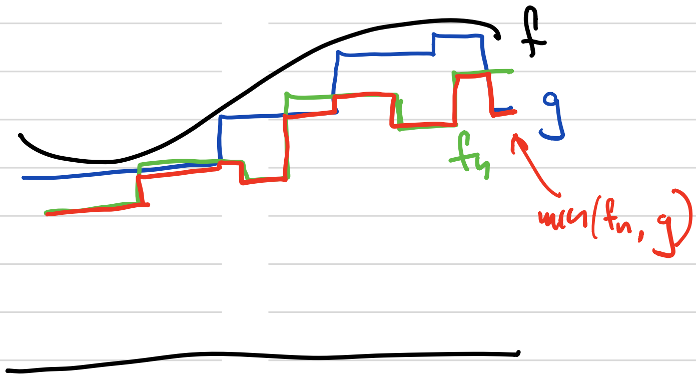
We have a monotone sequence f_n converging pointwise to f.



We want a monotone sequence converging pointwise to g.



Consider $\min(f_n, g)$.



Since the minimum of measurable functions is measurable and the minimum of simple functions is simple $\min(f_n, g) \in \mathcal{S}^+$.

Since $f_n \le f_m \Rightarrow \min(f_n, g) \le \min(f_m, g)$ this is a non-decreasing sequence of functions in $\mathcal{S}^+$.

Since for $x \in X$

$$\lim_{n\to\infty} \min(f_n, g)(x) = \min\left(\lim_{n\to\infty} f_n, g\right)(x) = \min(f, g)(x) = g(x)$$

the sequence $\min(f_n, g)$ converges pointwise to $g \in \mathcal{S}^+$.

By Proposition 2.3.2 $\lim_{n\to\infty} \int \min(f_n, g)\, d\mu = \int g\, d\mu$.

But $f_n \geq \min(f_n, g)$ so $\int f_n\, d\mu \geq \int \min(f_n, g)\, d\mu$ which gives:

$$\lim_{n\to\infty} \int f_n\, d\mu \geq \lim_{n\to\infty} \int \min(f_n, g)\, d\mu = \int g\, d\mu$$

as was to be shown.