Our next project is to use the the theory of measures that we have developed to create a theory of integration.

In the construction of Lebesgue measure we had to decide which sets were eligible to have a measure.

Now we have to decide which functions are capable of having an integral. Once we have done that we will consider what the integral should be.

We have denoted a measure space by a triple $(X, \mathcal{A}, \mu)$ where X is a set, $\mathcal{A}$ is a σ-algebra and μ is a measure.

Examples: $(\mathbb{R}^d, \mathcal{M}, \lambda_d)$ and $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda_d)$.

A second example is counting measure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is counting measure

$$\mu(S) = \# S$$

and $\mathcal{P}(\mathbb{N})$ is the "power set" of $\mathbb{N}$ or the set of all subsets.

A family of examples:

We can "restrict" a measure to a measurable set.

Consider λ, Lebesgue measure on $\mathbb{R}$. Fix an interval $[a,b]$.

Define σ on $(\mathbb{R}, \mathcal{M})$ by

$$\sigma(A) = \lambda(A \cap [a,b])$$

Following Cohn we call a pair $(X, \mathcal{A})$ a measurable space.

Proposition 2.1.1 Let $(X, \mathcal{A})$ be a measurable space. Let $f: X \to [-\infty, +\infty]$ be a function.

The following conditions are equivalent:

1) The sets $\{x \in X : f(x) \leq t\}$ are in $\mathcal{A}$.
2) The sets $\{x \in X : f(x) < t\}$ are in $\mathcal{A}$.

If either condition holds we say that f is $\mathcal{A}$-measurable.

For each measure space $(X, \mathcal{A}, \mu)$ we will define an integral $\int f \, d\mu$ for $\mathcal{A}$-measurable functions.

Note:

$[a,b]$

$$\int f \, d\mu$$

no limits of integration

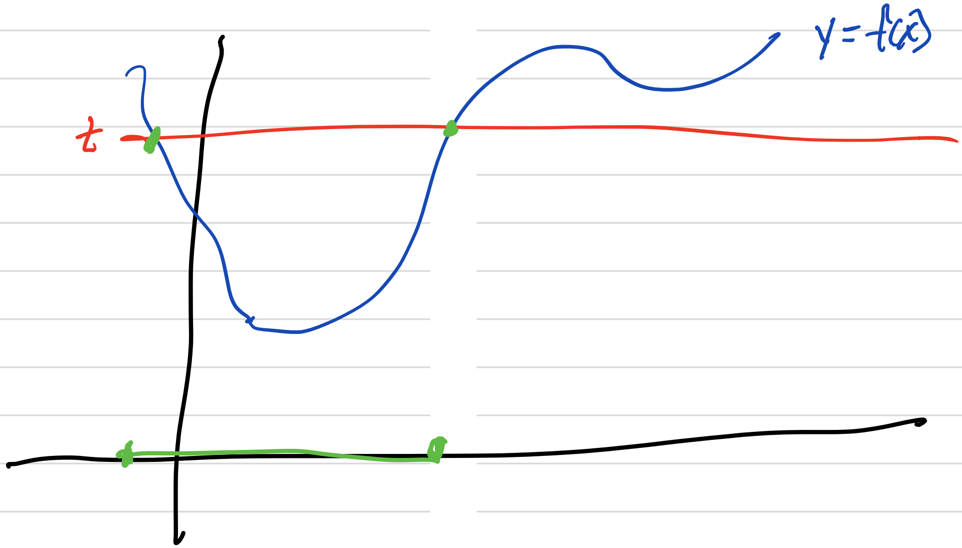
relevant measure

(Can deal with this by "restricting" our measure.)

The example to keep in mind is $(\mathbb{R}, \mathfrak{M}, \lambda)$ where $\mathfrak{M}$ is the σ-algebra of Lebesgue measurable sets. Here we talk about Lebesgue measurable functions.

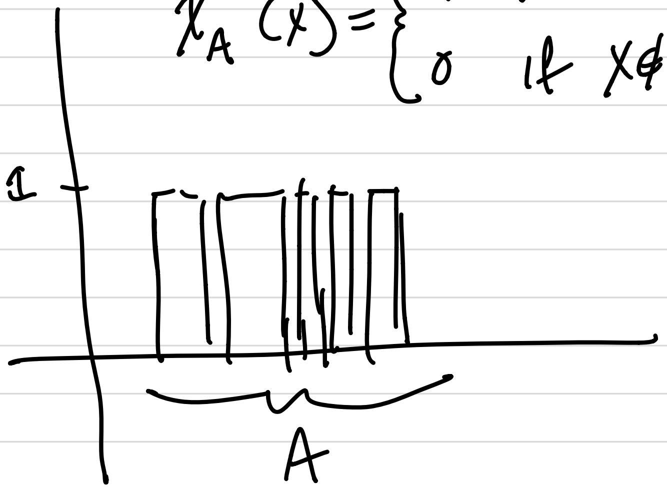
Example: If $f: \mathbb{R} \to \mathbb{R}$ is continuous it is Lebesgue measurable.

In this case $f^{-1}([-\infty, t))$ is open hence a Lebesgue measurable set.



Example. Characteristic function of a measurable set. Say $A \subset \mathbb{R}$ is Lebesgue measurable:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$



χ_A is Lebesgue measurable.

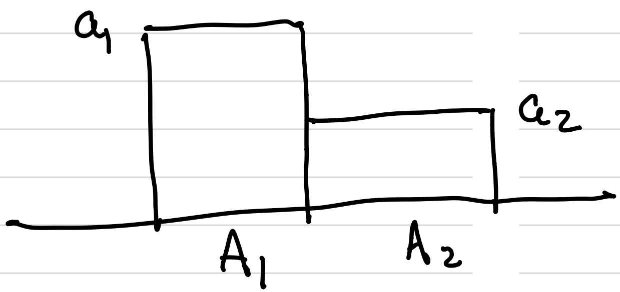
$f^{-1}([-\infty, t)) = \mathbb{R} - A$ or $\mathbb{R}$ or $\emptyset$

depending on the value of t.

The Dirichlet function

is χ_S where $S = [0,1] \cap \mathbb{Q}$.

Example. Linear combination of characteristic functions $\sum_{i=1}^{n} a_i \chi_{A_i}$.



$\mathcal{S}$

These functions are called simple functions. They are exactly the meas. fcns. taking finitely many values.

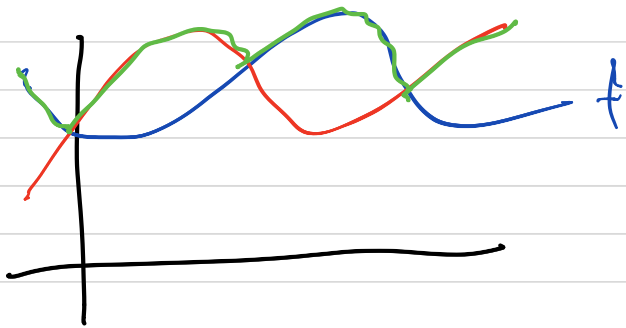
We want to show that the class of "$\mathcal{A}$-measurable" functions is closed under some natural operations.

These results will be useful when we define the integral.

Def. For $[-\infty,\infty]$ valued functions f, g define

$\max(f,g)$ by $x \mapsto \max(f(x), g(x))$

$\min(f,g)$ $\quad x \mapsto \min(f(x), g(x))$.



Prop. If f, g are $\mathcal{a}$-measurable then $\max(f,g)$ and $\min(f,g)$ are measurable.

Proof. Need to show

$$\{x \in X : \max(f,g)(x) \leq t\} \text{ is meas.}$$

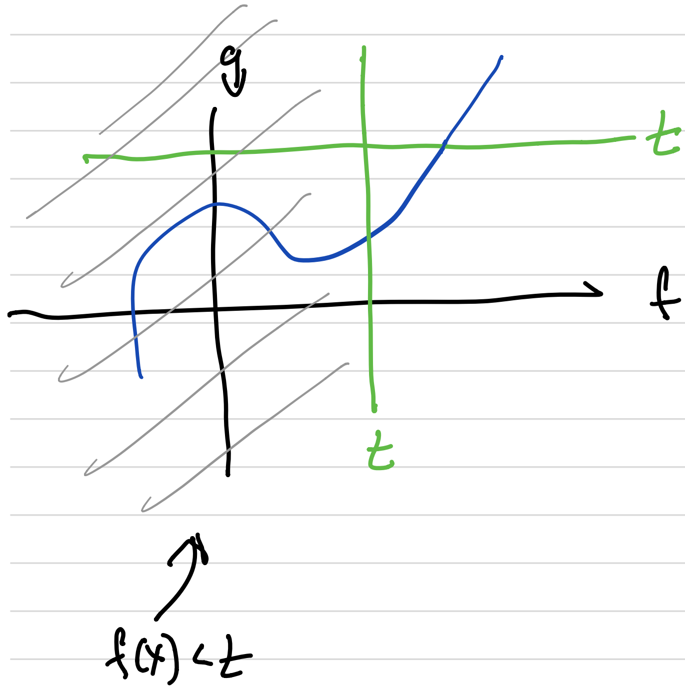
$$\{x \in X : \max(f,g)(x) \leq t\} =$$

$$\{x \in X : f(x) \leq t\} \cap \{x \in X : g(x) \leq t\}$$

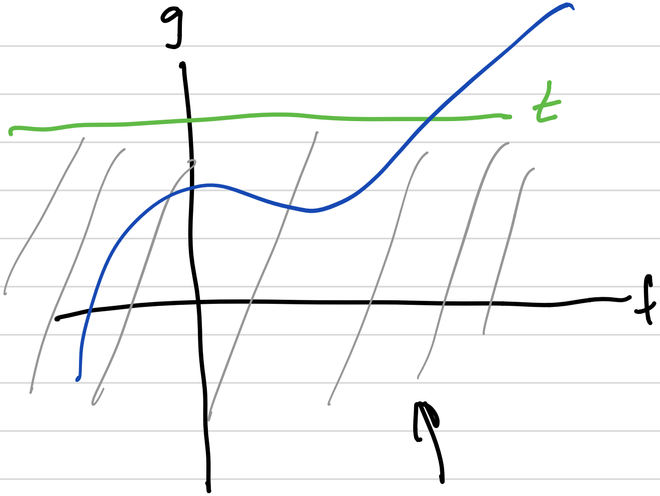
Let me draw a picture.

Say $X = \mathbb{R}$. A pair of functions f, g give a map from $\mathbb{R}$ to $\mathbb{R}^2$

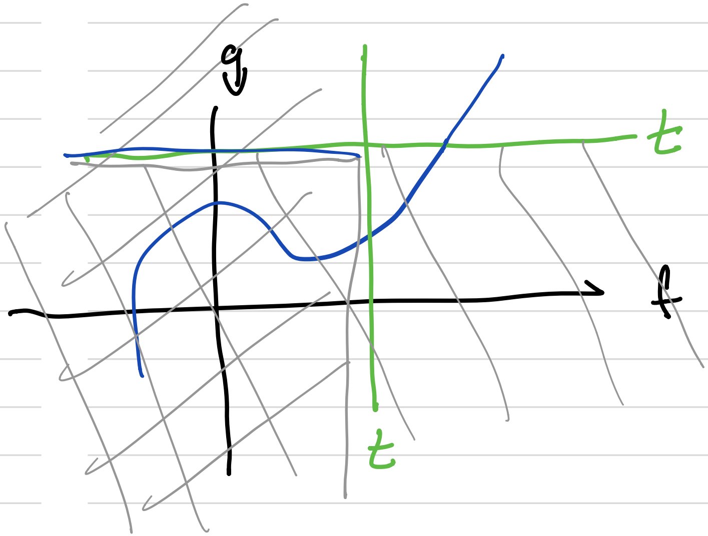
$x \mapsto (f(x), g(x))$



$\mathbb{R} \longrightarrow \mathbb{R}^2$



$g(x) < t$

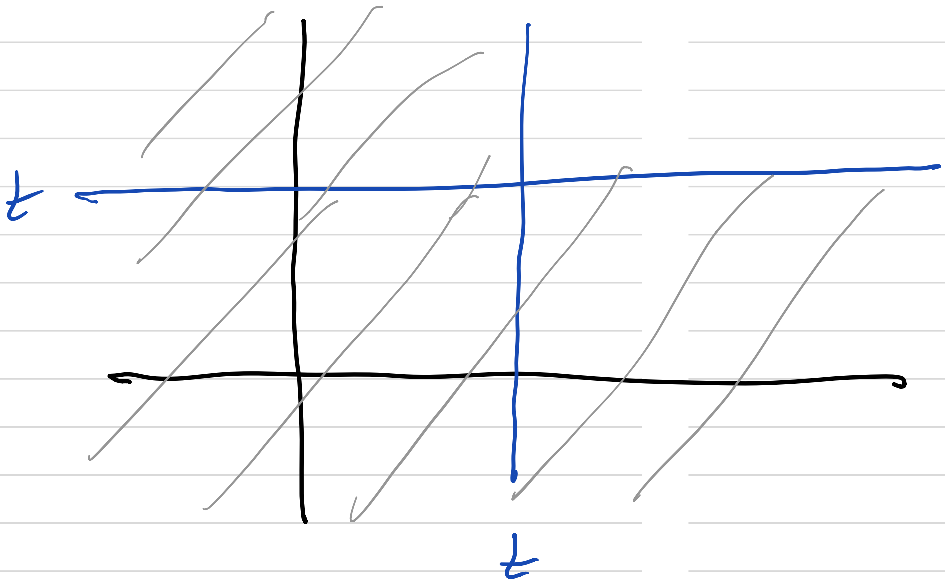


$f(x) < t$ and $g(x) < t$

Let's now consider

$$\{x \in X : \min(f,g)(x) \leq t\} =$$

$$\{x \in X : f(x) < t\} \cup \{x \in X : g(x) < t\}$$



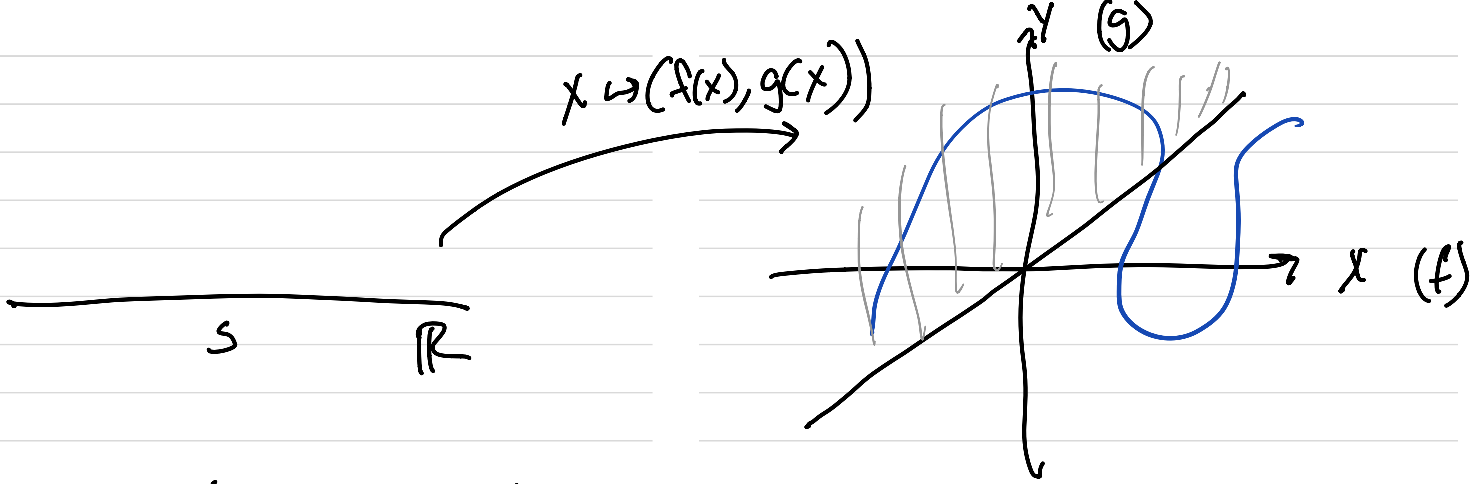
Prop. Let f, g be measurable functions then then the sets

$$\{x \in A : f(x) < g(x)\}$$
$$\{x \in A : f(x) \leq g(x)\}$$
$$\{x \in A : f(x) = g(x)\}$$

are measurable.

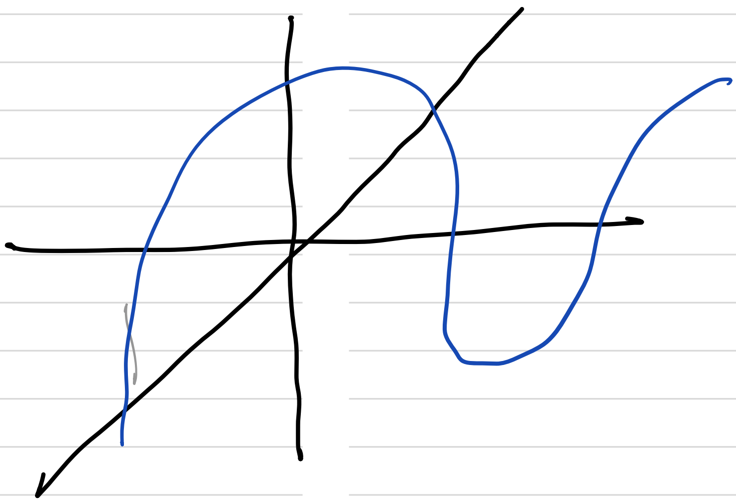
Say f, g are Lebesgue meas. fcns. defined on $\mathbb{R}$. Then they give us a map from

$\mathbb{R}$ to $\mathbb{R}^2$ by $\mapsto (f(x), g(x))$

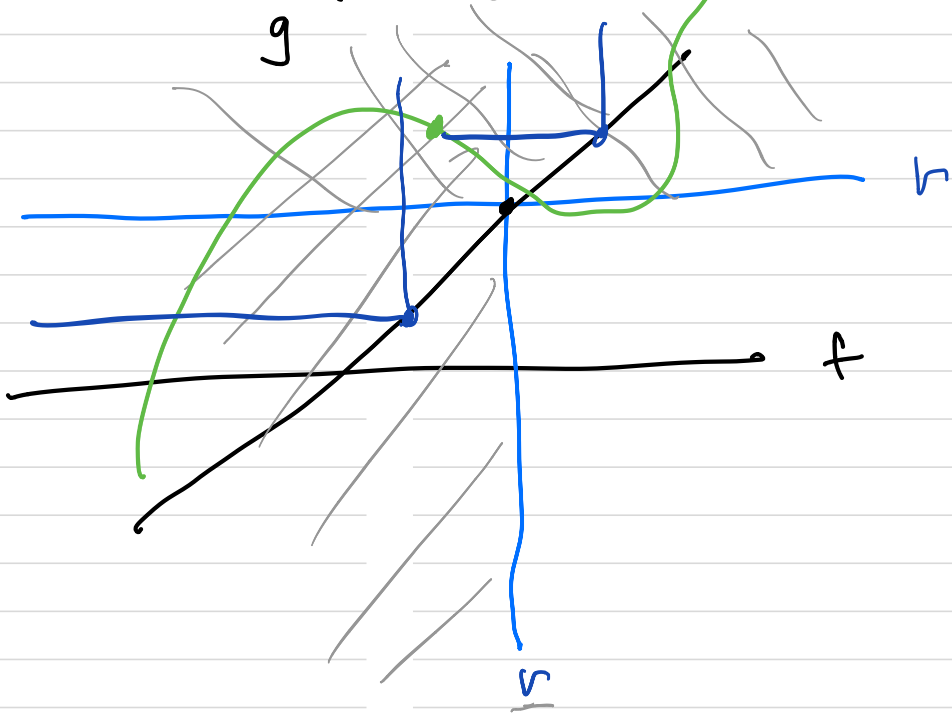


The set where $f(x) < g(x)$ is the inverse image of the set where $y<x$.

This is the gray region.



We want to show that this set is in $\mathcal{A}$.



Consider $\{f < r\} \cap \{r < g\}$.

Say $f(x) < g(x)$. There is some r s.t.

$f(x) < r < g(x)$ so

$$x \in f^{-1}([-\infty, r)) \cap f^{-1}((r, \infty])$$

and $\{x \in A : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} f^{-1}([-\infty, r)) \cap f^{-1}((r, \infty])$.

The only problem with this representation is that the union is not countable.

We have dealt with this issue before. Let us restrict to rational r. We can do that because rationals are dense so the open interval $(f(x), g(x))$ is guaranteed to contain a rational.

The set $\{x \in X : g(x) < f(x)\}$ is also measurable as is $\{x \in X : g(x) \geq f(x)\} = \{x \in X : g(x) < f(x)\}^c$.

The set $\{x \in X : g(x) = f(x)\}$ is the difference so is also measurable.

Prop. Let f_n be a sequence of meas. functions.

(a) $\sup_n f_n$ and $\inf_n f_n$ are measurable

(b) $\limsup_n f_n$ and $\liminf_n f_n$ are measurable

(c) $\lim_n f_n$ is meas. on the meas. set where it is defined. (In part. if we know the lim exists everywhere it is measurable.)

Proof. (a) Fix t.

Consider $\{x \in X : \sup_n f_n < t\} = \bigcap_n \{x \in A : f_n(x) < t\}$

(b) $\{x \in X : \limsup_n f_n < t\} = \bigcup_n \bigcap_{m=n}^{\infty} \{x \in A : f_m(x) < t\}$

(c) To say $\lim_n f_n(x)$ exists is to say that

$\limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$

We have already seen that the set

where two meas. fcns. are equal is meas.

$\lim_{n\to\infty} f_n(x)$ is the restriction of the meas.

function $\limsup f_n$ to this set.

Our strategy in defining the integral is to define it for simple functions and then extend it to positive functions.

We would like

$$\int \sum_{i=1}^{n} a_i \chi_{A_i} \, d\mu = \sum_{i=1}^{n} a_i \cdot \mu(A_i).$$

Potential problem is that we might get positive ∞ and negative ∞ terms.

No good way to cancel such terms.

We deal with this by restricting to non-negative functions.

non-negative simple functions

Call this class of functions $\mathcal{S}^+$.

Answer might be finite or $+\infty$.

No cancellation issues.

Prop. Let f be a $[0,\infty]$-valued meas. fcn. then there is a sequence of functions $\{f_n\}$ of fcns in $\mathcal{S}^+$ that satisfy

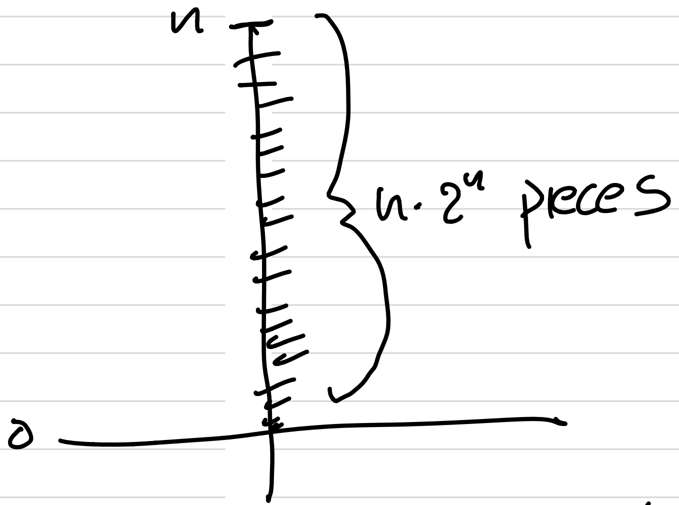
$$f_1(x) \le f_2(x) \le \dots$$

and

$$f(x) = \lim_{n\to\infty} f_n(x) \text{ for each } x.$$

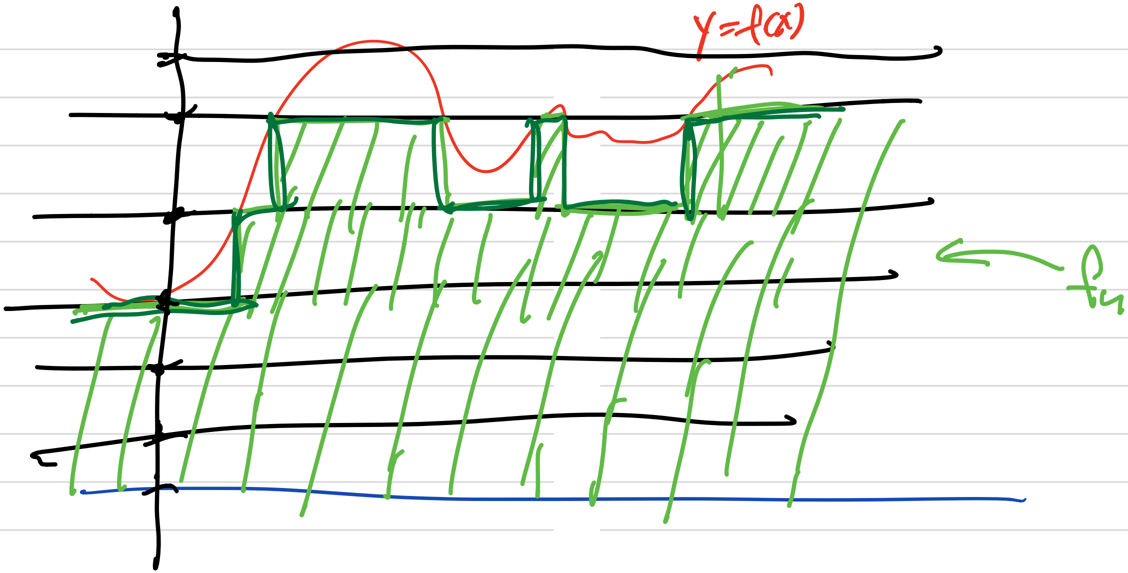
$$\lim_{n\to\infty} \int f_n(x)\,d\mu.$$

Proof. We subdivide the domain



$k = 1 \dots n2^n$ $\quad A_{n,k} = \{x \in A : \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n}\}$

Define the function f_n to take the value $\frac{k-1}{2^n}$ in the set $A_{n,k}$ for $k = 1 \dots n2^n$.



Note that the (n+1)st collection of range points contains the n-th collection.