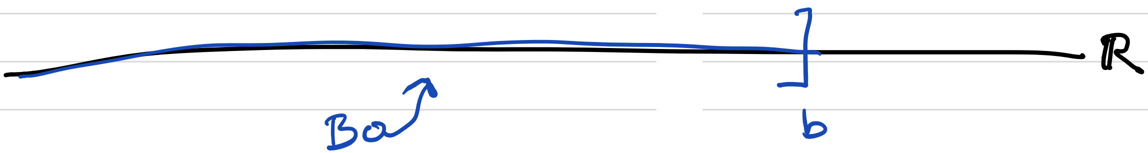
Second example of a measurable set.

Proposition. The half-infinite interval $B = (-\infty, b]$ is measurable.



The proof will follow the outline of our analysis of additivity for two sets contained in disjoint intervals.

Proof. Let A be a test set. We need to show

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) = \lambda^*(A).$$

(It might be helpful to remember this as:

"the pair $B | B^c$ divides A cleanly!")

Sub-additivity gives us:

$$\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \geq \lambda^*(A).$$

We need to show: $\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A).$

We need to show: $\lambda^*(A \cap B) + \lambda^*(A \cap B^c) \leq \lambda^*(A)$.

Let $\varepsilon > 0$. Let $\mathcal{U} = \{(a_n, b_n)\}$ be a cover of A with $\sum_{n=1}^{\infty} (b_n - a_n) < \lambda^*(A) + \varepsilon$.

We want to get a cover for A obtained by putting together covers for $A \cap B$ and $A \cap B^c$.

The cover for $A \cap B$ (respectively $A \cap B^c$) will give an upper bound for $\lambda^*(A \cap B)$ (respectively $\lambda^*(A \cap B^c)$).

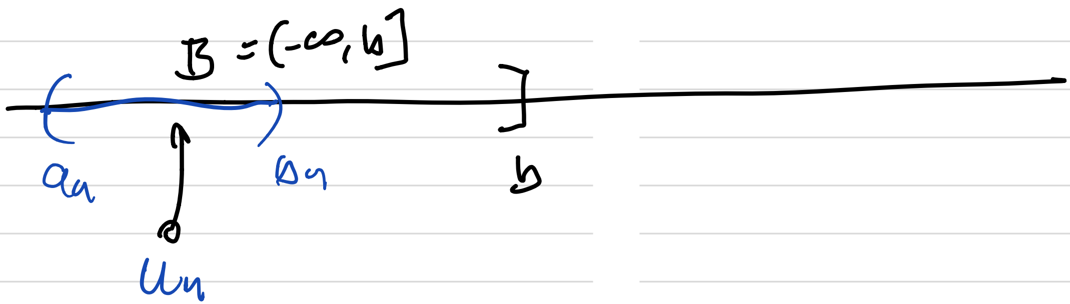
Recall $\mathcal{U} = \{(a_n, b_n)\}$

Consider $\mathcal{U}_1 = \{U_n \cap (-\infty, b)\}$ and

$\mathcal{U}_2 = \{U_n \cap (b, \infty)\}$.

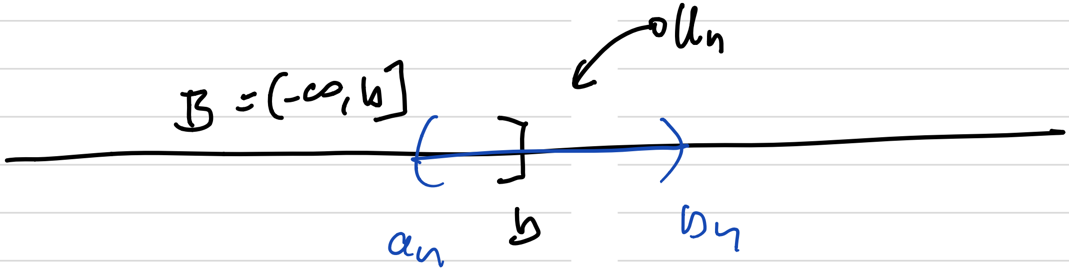
There are 3 configurations for (a_n, b_n):

①



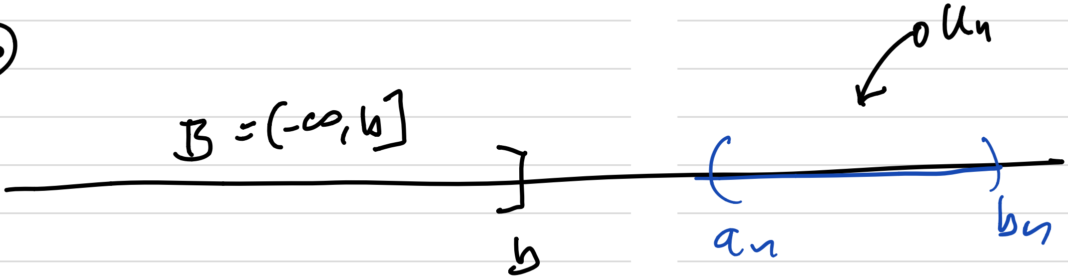
$U_n \in \mathcal{U}_1$

②



$(a_n, b) \in \mathcal{U}_1$ $\quad$ $(b, b_n) \in \mathcal{U}_2$

③



$(a_n, b_n) \in \mathcal{U}_2$.

$$\sum_{n=1}^{\infty} \text{length}(U_n) = \sum_{n=1}^{\infty} \text{length}(U_n \cap (-\infty, b)) + \text{length}(U_n \cap (b, \infty))$$

$$= \sum_{n=1}^{\infty} \text{length}(U_n \cap (-\infty, b)) + \sum_{n=1}^{\infty} \text{length}(U_n \cap (b, \infty))$$

$\mathcal{U}_1$ is almost a cover of $A \cap [-\infty, b]$. It fails to cover the point b. If we add the interval $U_+ = (b - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2})$ then it is a cover. Write $\mathcal{U}_1^+ = \mathcal{U} \cup \{U_+\}$.

We calculate:

$$\lambda^*(A\cap B)+\lambda^*(A\cap B^c)\le\sum_{u\in\mathcal{U}_1^+}\text{length}(u)+\sum_{u\in\mathcal{U}_2}\text{length}(u)$$

$$\le\varepsilon+\sum_{u\in\mathcal{U}_1}\text{length}(u)+\sum_{u\in\mathcal{U}_2}\text{length}(u)$$

$$\le\varepsilon+\sum_{n} b_n-a_n$$

$$\le\varepsilon+\lambda^*(A)+\varepsilon.$$

$$=\lambda^*(A)+2\varepsilon.$$

Letting $\varepsilon\to 0$ get we get:

$\lambda^*(A\cap B)+\lambda^*(A\cap B^c)\le\lambda^*(A)$ as desired.