Measure Theory: Assignment Three - Simple functions and Lebesgue integration

Josephine Evans

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Let $(E, \mathcal{E}, \mu)$ be a measure space. Recall that we call a function $f : E \to \mathbb{R}$ simple if it is non-negative and can be written in the form $f(x) = \sum_{k=1}^n a_k 1_{A_k}(x)$ where the a_k are non-negative numbers and $A_k \in \mathcal{E}$ for each k.

Question 0.1. Show that if f is a simple function then f is measurable. *6 marks*

Question 0.2. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be measurable and non-negative. Recall our classic approximation

$$f_n(x) = \left(2^{-n} \lfloor 2^n f(x) \rfloor\right) \wedge n.$$

Show that the sequence $f_n(x)$ is non-decreasing for every x and has limit $f(x)$. Show that f_n is a simple function for each n. *7 marks*

Question 0.3. In this question let $f, f_1 f_2$ be measurable. Suppose $f : E \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is either non-negative or integrable. Let N be a null set. Define the function g by $g(x) = f(x) 1_N(x)$. Show that $\int g(x) \mu(\mathrm{d}x) = 0$. We say two function f_1 and f_2 are equal almost everywhere if $\mu\{x : f_1(x) \neq f_2(x)\} = 0$. Show that if f_1 and f_2 are integrable and equal almost everywhere then $\int f_1(x) \mu(\mathrm{d}x) = \int f_2(x) \mu(\mathrm{d}x)$. *12 marks*