Measures as operators on functions and the Riesz representation theorem

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Non-examinable

Measures as functions from σ-algebras $\to \mathbb{R}_+$

Now we've done integration we can consider measures as operators on functions.

$$\mu : \mathcal{E} \to \mathbb{R}$$

another view $\mu : C_c(X) \to \mathbb{R}$

Let X be a Hausdorff top space $\mathcal{B}(X)$ the Borel σ-algebra on X. ($X \subseteq \mathbb{R}^d$ for some d)

$C_c(X)$ is the vector space of all function on X which are continuous and have compact support with values in $\mathbb{R}$

From exercise sheet if $f \in C_c(X)$ then f is measurable wrt $\mathcal{B}(X)$ and $\mathcal{B}(\mathbb{R})$.

f will be bounded as its continuous of compact support.

μ a measure on $(X, \mathcal{B}(X))$ is a regular Radon measure if:

$\mu(K) < \infty$ for every K compact

$\mu(A) = \inf\{\mu(U) : U \text{ open } A \subseteq U\}$

$K \subseteq A\}$

$\mu(A) = \inf\{\mu(U) : U \text{ open } A \subseteq U\}$

$\mu(A) = \sup\{\mu(K) : K \text{ compact } K \subseteq A\}$

Consider a linear functional on $C_c(X)$, I_μ

$I_\mu(f) = \mu(f)$ the integral of f wrt μ.

we can see that f will be μ-integrable as $|f| \leq \sup_x |f(x)| \mathbb{1}_K$ where $K = \text{supp}(f)$ which is compact.

$$\mu(|f|) \leq \sup_x |f(x)| \mu(K) < \infty$$

Lemma Let U be open and μ a regular Radon measure on X then

$$\mu(U) = \sup\{\mu(f) : f \in C_c(X) \quad 0 \leq f \leq \mathbb{1}_U\}$$

~~Pf~~ Urysohm's Lemma if X is Hausdorff C closed, K_2 compact then $\exists$ $f : X \to [0,1]$ with $f|_{K_1} = 0$ and $f|_{K_2} = 1$.

By regularity $\mu(U) = \sup\{\mu(K) : K \subseteq U \ K \text{ compact}\}$

for any $K \subseteq U$ $\exists$ $f \in C_c(X)$ s.t. $f|_K = 1$ and $f|_{U^c} = 0$

and $f(x) \in [0,1]$ for every x.

so then ~~k~~ $\mathbb{1}_U \geq f \geq \mathbb{1}_K$

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so $\mu(U) \geq \mu(f) \geq \mu(K)$

$\sup\{\mu(f) : 0 \leq f \leq \mathbb{1}_U\} \leq \mu(U)$

and $\geq \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\} = \mu(U).$

We can then use this to find the measure of any $A \in \mathcal{B}(X)$ using regularity

<u>Riesz Representation Theorem</u>

Let X be a locally compact Hausdorff top space and let I be a linear functional on $C_c(X)$ with I positive if $f \geq 0$ then $I(f) \geq 0$ then $\exists!$ Radon measure μ s.t. $I(f) = \mu(f)$.

Pf First show uniqueness

If $\mu(f) = \nu(f)$ for every $f \in C_c(X)$ then by our lemma $\mu(U) = \nu(U)$ for every U open so by regularity $\mu(A) = \nu(A)$ for every $A \in \mathcal{B}(X)$ $\mu = \nu$.

Then define $\mu^*(U) = \sup\{I(f) : f \in C_c(X),\ 0 \leq f \leq \mathbb{1}_U\}$

when $\mathcal{U}$ is open

Then define $\mu^*(A) = \inf\{\mu^*(\mathcal{U}) : U \text{ open and } A \subseteq \mathcal{U}\}$

$\mathcal{U}$

We want to show $\mu^*|_{\mathcal{B}(X)}$ is a true measure.

Strategy

1. Check μ^* is well defined

2. Show μ^* is an outer measure

↪ Basically the same as in the construction of Lebesgue measure or Caratheodory's ext thm.

3. Once we've got an outer measure we define μ^* measurable sets as in Caratheodory then the proof that the measurable sets form a σ-algebra is identical

4. Show that the Borel sets are measurable by showing open sets are measurable

WTS $\mu^*(A) \geq \mu^*(A \cap \mathcal{U}) + \mu^*(A \cap \mathcal{U}^c)$

for $A \in \mathcal{P}(X)$ $\mathcal{U}$ open

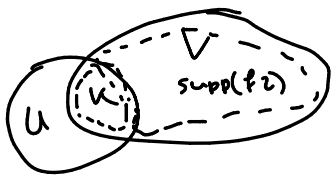
Find V open s.t. $A \subset V$ and

$\mu^*(V) \leq \mu^*(A) + \varepsilon$

then find f_1 s.t. $0 \le f_1 \le \mathbb{1}_{V\cap U}$ $f_1 \in C_c(X)$

and $I(f_1) \ge \mu^*(V\cap U) - \varepsilon$

Then define $K = \mathrm{supp}(f_1)$ so K^c is open



find f_2 s.t. $f_2 \in C_c(X)$

and $0 \le f_2 \le \mathbb{1}_{V\cap K^c}$

and $I(f_2) \ge \mu^*(V\cap K^c) - \varepsilon$

then $f_1+f_2 \in C_c(X)$ and $0 \le f_1+f_2 \le \mathbb{1}_V$

so $\mu^*(V) \ge I(f_1+f_2) = I(f_1) + I(f_2)$

$$\ge \mu^*(V\cap U) + \mu^*(V\cap K^c) - 2\varepsilon$$

then $K^c \supseteq U^c$ so by monotonicity

$$\mu^*(V) \ge \mu^*(V\cap U) + \mu^*(V\cap U^c) - 2\varepsilon$$

then $V \supseteq A$ so again by monotonicity

$$\mu^*(A)+\varepsilon \ge \mu^*(V) \ge \mu^*(A\cap U) + \mu^*(A\cap U^c) - 2\varepsilon$$

$$\mu^*(A) \ge \mu^*(A\cap U) + \mu^*(A\cap U^c) - 3\varepsilon$$

and ε is arbitrary so we have our conclusion.

Now we know $\mu^*|_{B(X)}$ is a true measure

Last step is to check $I_{\mu^*}(f) = \mu^*(f)$ agrees with I.

agrees wi