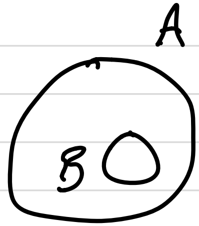
Want to discuss Fubini's theorem and iterated integrals for Lebesgue measure and for general measures.

At some point in this discussion we want to use a new tool: Dynkin classes.

These have some interest in their own right so I will begin by discussing them.

A collection $\mathcal{D}$ of subsets of X is a d-system on X if



(a) $X \in \mathcal{D}$

(b) $A - B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$ and $A \supset B$

(c) $\bigcup_n A_n \in \mathcal{D}$ whenever A_n is an increasing sequence of sets in $\mathcal{D}$: $A_1 \subset A_2 \subset A_3 \cdots$

(Cohn p. 37)

Lemma. Say that $(\mathfrak{X}, \mathcal{A})$ is a measurable space and μ and ν are measures defined on $\mathcal{A}$. Assume $\mu(\mathfrak{X}) = \nu(\mathfrak{X}) < \infty$.

Then the collection of sets A for which $\mu(A) = \nu(A)$ is a d-system.

Proof.

(a) $\mu(\mathfrak{X}) = \nu(\mathfrak{X})$ by assumption

(b) $\mu(A-B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A-B)$

(c) $\mu\left(\bigcup_n A_n\right) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_n A_n\right)$.

Just as for σ-algebras the intersection of two d-systems is a d-system.
If we are given a collection $\mathcal{C}$ of subsets of X we can form the smallest d-system containing $\mathcal{C}$ and we call this the d-system generated by $\mathcal{C}$. (Cohn p. 37)

In sum σ-algebras are good for defining measures while d-systems are good for comparing measures.

We say that a collection of sets $\mathcal{C}$ is a <u>π-system</u> if it is closed under taking finite unions. (p. 37)

As discussed in lecture 9 the collection of measurable rectangles is a π-system.

As we explained in Lecture 9 a fundamental property of d-systems that we need is the following:

Thm. 1.6.2. (Dynkin) Let X be a set and let $\mathcal{C}$ be a π-system on X. Then the <u>σ-algebra</u> generated by $\mathcal{C}$ coincides with the <u>d-system</u> generated by $\mathcal{C}$.

Proof. Let $\mathcal{D}$ be the d-system generated by $\mathcal{C}$.

Let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$.

Since every σ-algebra is a d-system $\sigma(\mathcal{C})$ contains the smallest d-system containing $\mathcal{C}$ so $\sigma(\mathcal{C}) \supset \mathcal{D}$.

To show the reverse inclusion we show that $\mathcal{D}$ is a σ-algebra.
This will show that $\mathcal{D}$ contains the smallest σ-algebra containing $\mathcal{C}$ so $\mathcal{D} \supset \sigma(\mathcal{C})$.

We do this in several steps.
First we show that $\mathcal{D}$ is closed under taking intersections $D \cap C$ with $D \in \mathcal{D}$ and $C \in \mathcal{C}$.

Let $\mathcal{D}_1 = \{ A \in \mathcal{D} : A \cap C \in \mathcal{D} \text{ for } C \in \mathcal{C} \}$.

We claim that $\mathcal{D}_1$ is a d-system.

We need to check 3 properties

① $X \in \mathcal{D}_1$: $X \in \mathcal{D}$ since $\mathcal{D}$ is a $\mathcal{D}$ system.

$X \cap C = C \in \mathcal{D}$ for $C \in \mathcal{C}$ since $\mathcal{C} \subset \mathcal{D}$.

② $\mathcal{D}_1$ is closed under proper differences:

Say $A \in \mathcal{D}_1$, $B \in \mathcal{D}_1$ and $B \subset A$.

$(A - B) \cap C = (A \cap C) - (B \cap C)$

Since $A, B \in \mathcal{D}_1$ $A \cap C \in \mathcal{D}$ and $B \cap C \in \mathcal{D}$

so $(A \cap C) - (B \cap C) \in \mathcal{D}$ hence $(A - B) \cap C \in \mathcal{D}$

hence $A - B \in \mathcal{D}_1$.

③ $\mathcal{D}_1$ is closed under taking increasing unions:

Say $A_1 \subset A_2 \subset \ldots$ and $A_n \in \mathcal{D}_1$.

Each $A_n \cap C$ is in $\mathcal{D}$.

$A_1 \cap C \subset A_2 \cap C \subset \ldots$

Since $\mathcal{D}$ is a d-system $\bigcup_n (A_n \cap C)$ is in $\mathcal{D}$.

Now $\left(\bigcup_n A_n\right) \cap C = \bigcup_n (A_n \cap C)$ so

$\left(\bigcup_n A_n\right) \cap C$ is in $\mathcal{D}$. So $\bigcup_n A_n \in \mathcal{D}_1$.

We conclude that $\mathcal{D}_1$ is a d-system.

$\mathcal{C}$ is closed under taking finite intersections.
so for $C_0 \in \mathcal{C}$ and any $C \in \mathcal{C}$ $C_0 \cap C \in \mathcal{C}$.
Thus $C_0 \in \mathcal{D}_1$. So $\mathcal{C} \subset \mathcal{D}_1$.

Thus $\mathcal{D}_1$ contains the smallest d-system containing $\mathcal{C}$ so $\mathcal{D}_1 \supset \mathcal{D}$.

Thus $\mathcal{D}$ is closed under the operation of taking intersections with elements of $\mathcal{C}$.

We have not quite proved what we wanted to so we do this trick again.

Define $\mathcal{D}_2 = \{ A \in \mathcal{D} : A \cap E \in \mathcal{D}$ for each $E \in \mathcal{D} \}$.

We claim that $\mathcal{D}_2$ is a d-system.

① $X \in \mathcal{D}_2$

② Say $A \in \mathcal{D}_2$, $B \in \mathcal{D}_2$ and $B \subset A$.

For $E \in \mathcal{D}$

$A \cap E \in \mathcal{D}$ $\quad B \cap E \in \mathcal{D}$

But $(A-B)\cap E = (A\cap E)-(B\cap E)$ so
$(A-B)\cap E \in \mathcal{D}$.
Thus $A-B \in \mathcal{D}_2$.

③ Say $A_1 \subset A_2 \subset \ldots$ and $A_n \in \mathcal{D}_2$.
For $E \in \mathcal{D}$
$(A_1\cap E) \subset (A_2 \cap E) \subset \ldots$ with $A_n \cap E \in \mathcal{D}$

Thus $\bigcap_n (A_n \cap E) \subset \mathcal{D}$.
$\bigcap_n (A_n \cap E) = (\bigcap_n A_n) \cap E \in \mathcal{D}$ and
$\bigcap_n A_n \in \mathcal{D}_2$.

$$\mathcal{D}_2 = \{A \in \mathcal{D} : A \cap E \in \mathcal{D} \text{ for each } E \in \mathcal{D}\}.$$

Now our disussion of $\mathcal{D}_1$ shows that $\mathcal{C} \subset \mathcal{D}_2$,

As before $\mathcal{D}_2 \supset \mathcal{D}$.

This shows that $\mathcal{D}$ is closed under taking finite intersections.

Since $\mathcal{D}$ contains X for any $A \in \mathcal{D}$
$A \subset X$ so $X - A = A^c \in \mathcal{D}$.

Since $\mathcal{D}$ is closed under taking complements $\mathcal{D}$ is also closed under taking finite unions.

We finish the proof of the theorem by showing that a d-system closed under taking finite unions is a σ-algebra.

We need to show it is closed under taking <u>arbitrary</u> unions.

Let A_n be an arbitrary countable collection of sets in $\mathcal{D}$.

Let $B_n = \bigcup_{i=1}^{n} A_n$. B_n is an increasing union so $\bigcup_n B_n \in \mathcal{D}$. But $\bigcup_n B_n = \bigcup_n A_n$ so $\bigcup_n A_n \in \mathcal{D}$.

This completes the proof of the theorem.

Cor. 1.6.3 (Agreement on generators).
Let $(\mathbb{X}, \mathcal{A})$ be a measurable space and let $\mathcal{C}$ be a π-system s.t. $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν that satisfy $\mu(\mathbb{X}) = \nu(\mathbb{X}) < \infty$ and $\mu(C) = \nu(C)$ for $C \in \mathcal{C}$ then $\mu = \nu$.

Proof. Let $\mathcal{D} = \{ A \in \mathcal{A} : \mu(A) = \nu(A) \}$.
$\mathcal{D}$ is a d-system. $\mathcal{C} \subset \mathcal{D}$ is a π-system.
Thm. 1.6.2 implies that $\mathcal{D} \supset \sigma(\mathcal{C}) = \mathcal{A}$.
Thus $\mu(A) = \nu(A)$ for each $A \in \mathcal{A}$.