In our discussion of the integral we have shown that any measure space $(X, \mathcal{A}, \mu)$ gives rise to an integration theory for measurable functions $f: X \to \mathbb{R}$.

Our prime example is Lebesgue measure.

What if we take a different example?

Consider the "Dirac mass" $(\mathbb{R}, P(\mathbb{R}), \delta_p)$ where

$$\delta_p(A) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{if } p \notin A. \end{cases}$$

What is $\int f \, d\mu$

when $\mu = \delta_p$?

(Start Recording]

We discussed the Riemann integral last week but I did not get to it. I will do it this week.

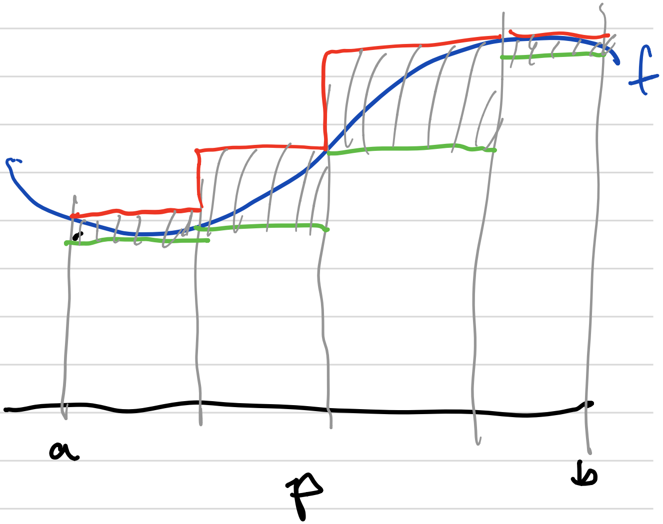
We make two important points. The first is the compatibility between the Riemann integral and the Lebesgue integral.

1. Every Riemann integrable function is Lebesgue integrable
2. When both integrals are defined they give the same value:

$$\int_a^b f(x)\,dx = \int f\,d\mu.$$

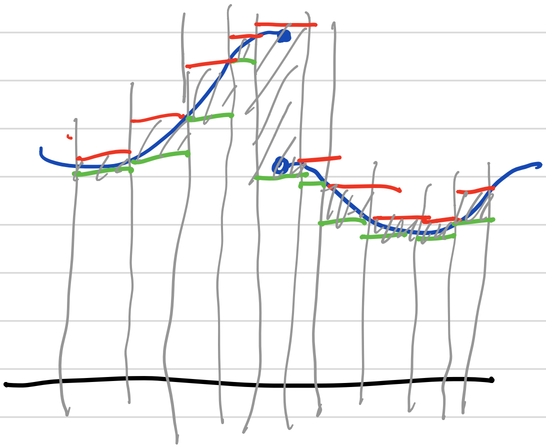
This fits in with the idea that the Lebesgue integration theory extends Riemann integration theory.

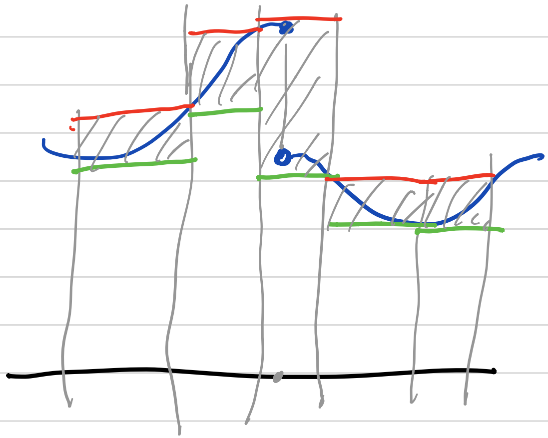
The second point is that we can use Lebesgue measure theory to give a precise description of which functions <u>are</u> Riemann integrable.



Recall that bounded function f on $[a,b]$ is Riemann integrable if the upper sums $u(f,P)$ and the lower sums $l(f,P)$ converge as $\operatorname{mesh}(P) \to 0$.

We know that continuous functions are Riemann integrable and some discontinuous functions.

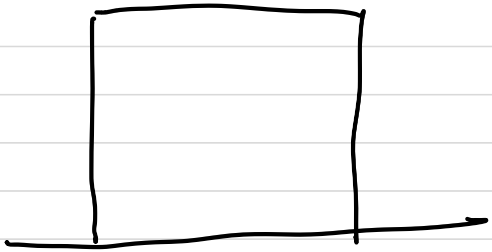




How bad can the set of discontinuities be before f is no longer Riemann integrable?

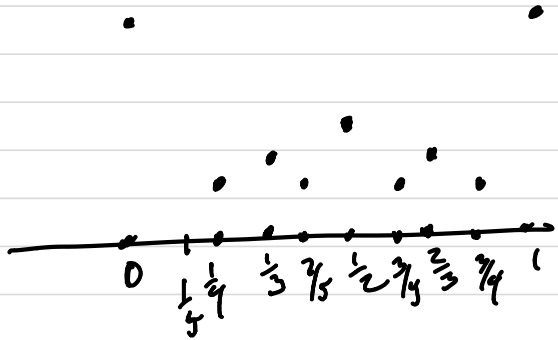
Examples.

Dirichlet function:



$$f = \chi_{[0,1]\cap\mathbb{Q}}$$

Discontinuous at every point in $[0,1]$.



$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Discontinuous at the rational points in $[0,1]$ s.

Theorem 2.5.4. Let $[a,b]$ be a closed bounded interval and let f be a bounded real valued function on $[a,b]$. Then

(a) f is Riemann integrable if and only if it is continuous at almost every point of $[a,b]$

(b) if f is Riemann integrable then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.

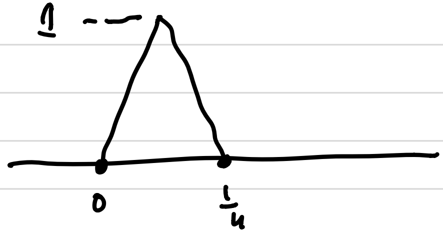
I have been making the point that we have been developing Lebesgue integration theory because we want an integration theory with good limit properties.

I have not mentioned that there are different types of limits we could consider.

Limits. $(X, \mathcal{A}, \mu)$

$f_n : X \to \mathbb{R}$ $\quad$ $f : X \to \mathbb{R}$

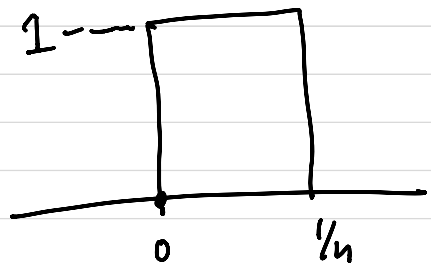
① Pointwise convergence:



For each $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$

$f_n \to 0$

② Pointwise convergence a.e.:



For a.e. $x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$.

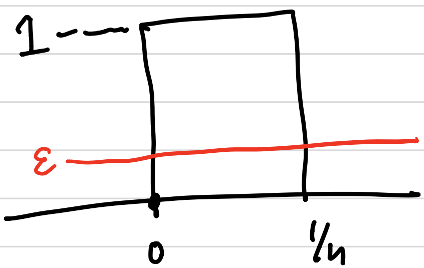
$f_n \to 0$ a.e.

③ Convergence in measure

f_n converges to f in measure

if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$



$f_n \to 0$ a.e.

For $\varepsilon < 1$:

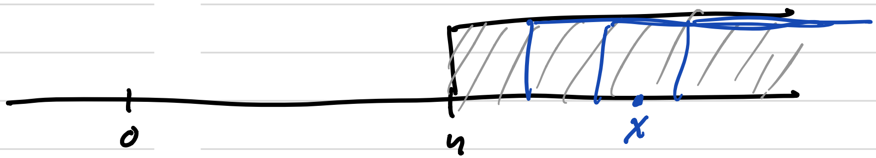
$$\mu\left(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}\right) = \frac{1}{n}$$

$f_n \to 0$ in measure

Pointwise convergence does not imply convergence in measure and convergence in measure does not imply pointwise convergence.

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = \mu([n, +\infty)) = \infty.$$

Example 1.

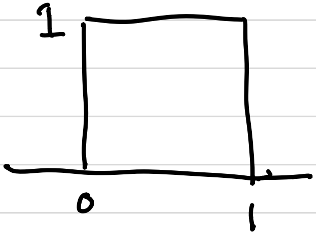


$f_n = \chi_{[n,+\infty)}$.

$f_n \to 0$ pointwise but not in measure.

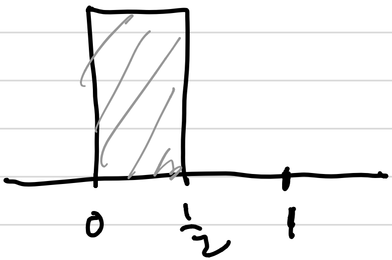
Example 2.

$f_1 =$

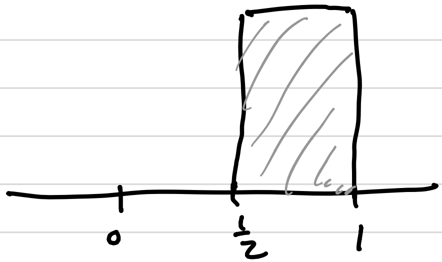


$= \chi_{[0,1]}$

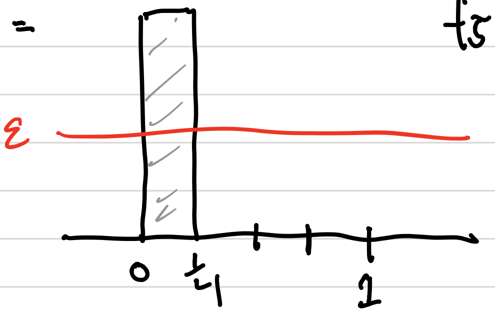
$f_2 =$



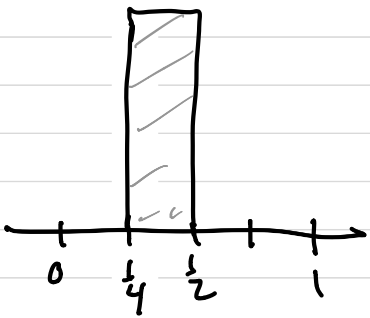
$f_3 =$



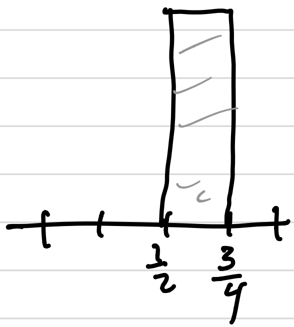
$f_4 =$



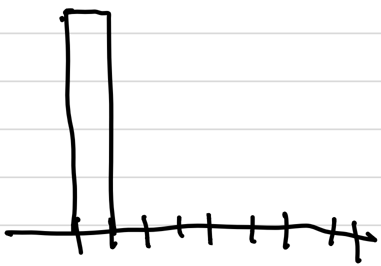
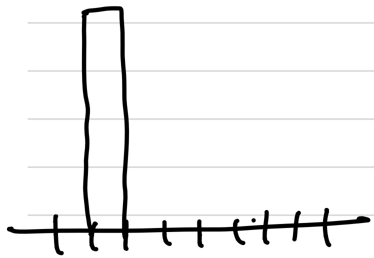
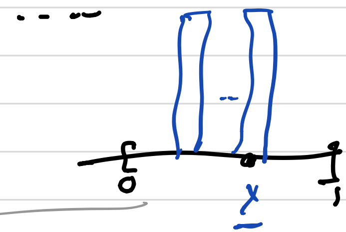
$f_5 =$



$f_6 =$

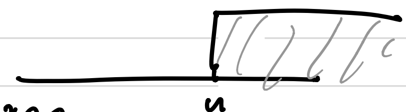


f_7

$f_8 =$  $f_9 =$  ...... 

$f_n \to 0$ in measure but not pointwise.

Prop. 3.1.2 If $(X, \mathcal{A}, \mu)$ is a finite measure space (if $\mu(X) < \infty$) then convergence a.e. implies convergence in measure.

The example  uses the infinite measure.

Prop. 3.1.3 If f_n converges to f in measure then some subsequence of f_n converges pointwise to f.

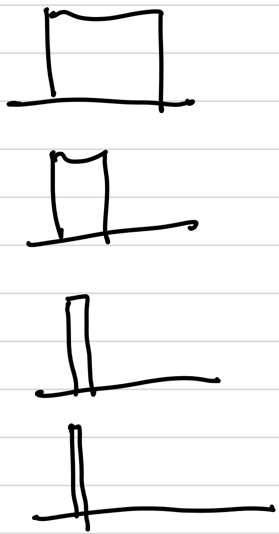
In the previous example

$f_1 =$

$f_2 =$

$f_4 =$

$f_8 =$



$f_{2^n} \to 0$ pointwise.

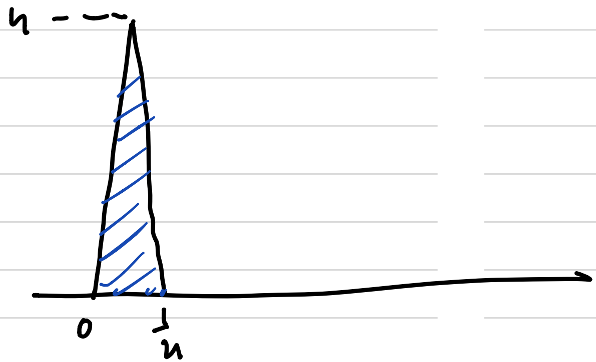
Convergence in mean.

We say that f_n converges to f in mean if

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0.$$

Convergence in measure does not imply convergence in mean:

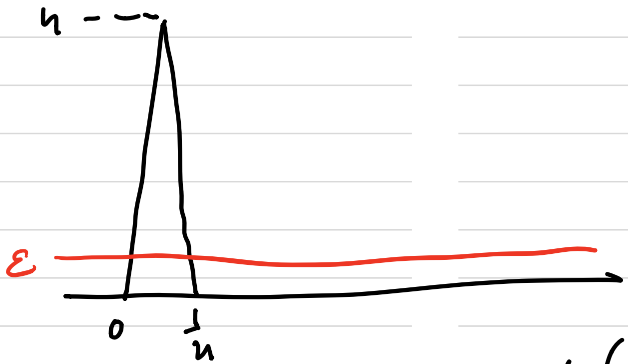
$f_n =$



$$\int |f_n - \overset{0}{f}|\, dx = \frac{1}{2}$$

This does not go to 0 so f_n <u>does</u> not converge in mean.

$f_n =$



$$\mu(\{x \in X : |f_n(x) - f(x)|\}) \leq \frac{1}{n}$$

so goes to 0.

Uniform convergence is a strong form of convergence.

We know for example that uniform convergence implies convergence of the Riemann integral.

It is interesting to know when we can recover some uniform convergence.

Prop. 3.1.4. (Egoroff's Thm). If f_n converges pointwise to f then for any $\varepsilon > 0$ there is a subset $B \subset X$ with $\mu(B^c) < \varepsilon$ so that f_n restricted to B converges uniformly to f restricted to B.