Plan.

Today I will review the big picture and the videos "Product σ-algebras" and "Dynkin classes".

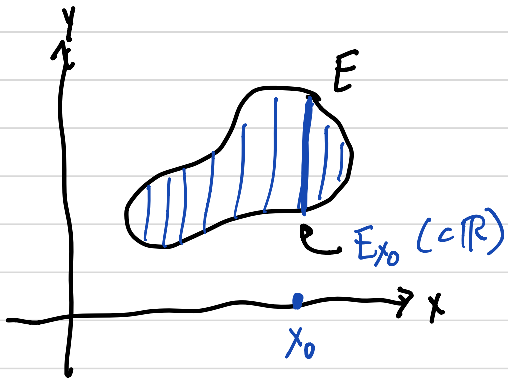
Wednesday I will review the remaining videos.

Ryan has an online support class today at 2. Dom has one Friday at 3.

Recond

We are aiming to prove two types of theorems. Let me state them loosely.

Given a measurable set $E \subset \mathbb{R}^2$




We define slices of a set $E \subset \mathbb{R}^2$.

$$E_{x_0} = \{ y : (x_0, y) \in E \}$$

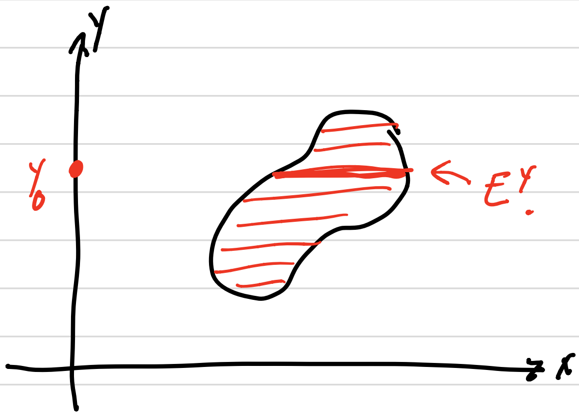
"Vertical Slice Theorem".

$$\lambda_2(E) = \int \lambda_1(E_x)\, d\lambda_1(x)$$

"Horizontal Slice Thm."

$$\lambda_2(E) = \int \lambda_1(E^y)\, d\lambda_1(y).$$

$E^{y_0} = \{x : (x, y_0) \in E\}$



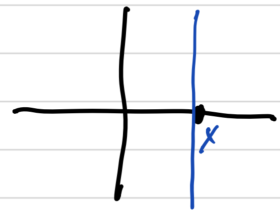
"Slice Theorem for $\mathbb{R}^2$"

$$\lambda_2(E) = \int \lambda_1(E_x)\, d\lambda_1(x) = \int \lambda_1(E^y)\, d\lambda_1(y).$$

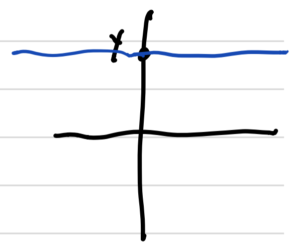
"Fubini Theorem for $\mathbb{R}^2$"

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be measurable then

$$\int f\, d\lambda_2 = \int \left( \int f(x,y)\, d\lambda_1(y) \right) d\lambda_1(x)$$




$$= \int \left( \int f(x,y)\, d\lambda_1(x) \right) d\lambda_1(y).$$




Remark: When $f = \chi_E$
then Fubini = Slice.

Our strategy is to start with a more general approach to the problem.

- Given any two measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ we construct a product $\sigma$-algebra $\mathcal{A} \times \mathcal{B}$ on $X \times Y$.

② We construct a product measure $\mu \times \nu$ defined on this product $\sigma$-algebra so that the Fubini and Slice theorems work for $\mu \times \nu$.

"Slice Theorem for product measure"

Say that $E \in \mathcal{A} \times \mathcal{B}$ then

$$(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

"Fubini Thm for product measure"

If $f$ is $\mathcal{A} \times \mathcal{B}$ measurable then

$$\int f \, d(\mu \times \nu) = \int \left( \int f(x,y)\, d\mu(x) \right) d\nu(y)$$

$$= \int \left( \int f(x,y)\, d\nu(y) \right) d\mu(x).$$

In order to recover our statements about $\mathbb{R}^2$ we need to:

① Identify the concrete Borel $\sigma$-algebra for $\mathbb{R}^2$: $\mathcal{B}(\mathbb{R}^2)$ with the abstract $\sigma$-algebra we have constructed: $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$

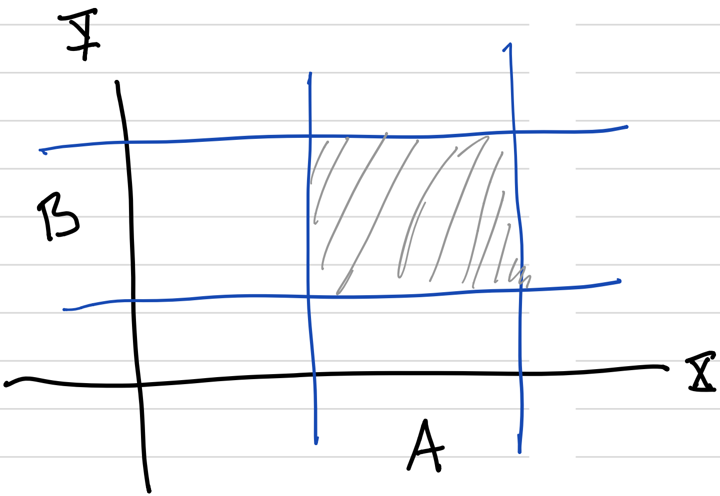
② Identify the concrete measure $\lambda_2$ with the abstract measure $\lambda_1 \times \lambda_1$ we have constructed.

We proceed with ①.

Let $(\mathfrak{X}, \mathcal{A})$ and $(\mathfrak{Y}, \mathcal{B})$ be measurable spaces.

A measurable rectangle is a set

$A \times B \subset \mathfrak{X} \times \mathfrak{Y}$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.



The product σ-algebra is generated by measurable rectangles is written: $\mathcal{A} \times \mathcal{B}$

(Cohn p. 143)

Example 5.1.1. $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

(From the Product σ-algebra video.)

Example 5.1.1. $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$

$\mathcal{B}(\mathbb{R}^2)$ is generated by sets $(a,b] \times (c,d]$ (Prop. 1.1.5).

$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ is generated by the larger class of sets $A \times B$ where $A, B$ are Borel and (despite my pictures) a Borel set in $\mathbb{R}$ can be much more complicated than an interval (think of a Cantor set).

Since the generating class for $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ contains that for $\mathcal{B}(\mathbb{R}^2)$ we have
$\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \supset \mathcal{B}(\mathbb{R}^2)$.

To prove the opposite containment we need to show $A \times B \subset \mathcal{B}(\mathbb{R}^2)$ where $A, B \in \mathcal{B}(\mathbb{R})$.

Let me refer you to the video for the details.

We want to connect the product $\sigma$-algebra $\mathcal{A}\times\mathcal{B}$ on $X\times Y$ to the $\sigma$-algebras $\mathcal{A}$ on $X$ and $\mathcal{B}$ on $Y$. In particular in order to make the "Slice Theorem" work we need slices $E_x$ to be in $\mathcal{B}$ and $E^y$ to be in $\mathcal{A}$.

"Slice Theorem for product measure"

Say that $E\in\mathcal{A}\times\mathcal{B}$ then

$$(\mu\times\nu)(E) = \int \nu(E_x)\,d\mu(x) = \int \mu(E^y)\,d\nu(y)$$

"Fubini Thm for product measure"

If $f$ is $\mathcal{A} \times \mathcal{B}$ measurable then

$$\int f \, d(\mu \times \nu) = \int \left( \boxed{\int f(x,y) \, d\mu(x)} \right) d\nu(y)$$

$$= \int \left( \int f(x,y) \, d\nu(y) \right) d\mu(x).$$

If we fix $x_0 \in X$ and consider

$$\boxed{\int f(x_0, y)\, d\mu(y)}$$

as a function of $y$ then

we want the function $y \mapsto f(x_0, y)$

to be $\mathcal{B}$-measurable.

Define $f_{x_0}(y) = f(x_0, y)$ and $f^{y_0}(x) = f(x, y_0)$.

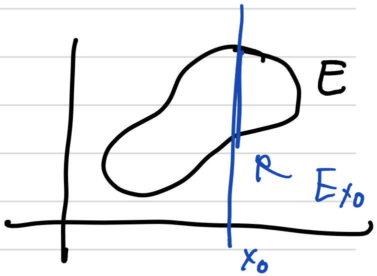
"sections"

**Lemma 5.1.2.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces.

(a) Say $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $E_x \in \mathcal{B}$ for each $x \in X$ and $E^y \in \mathcal{A}$ for each $y \in Y$. — for "slice"

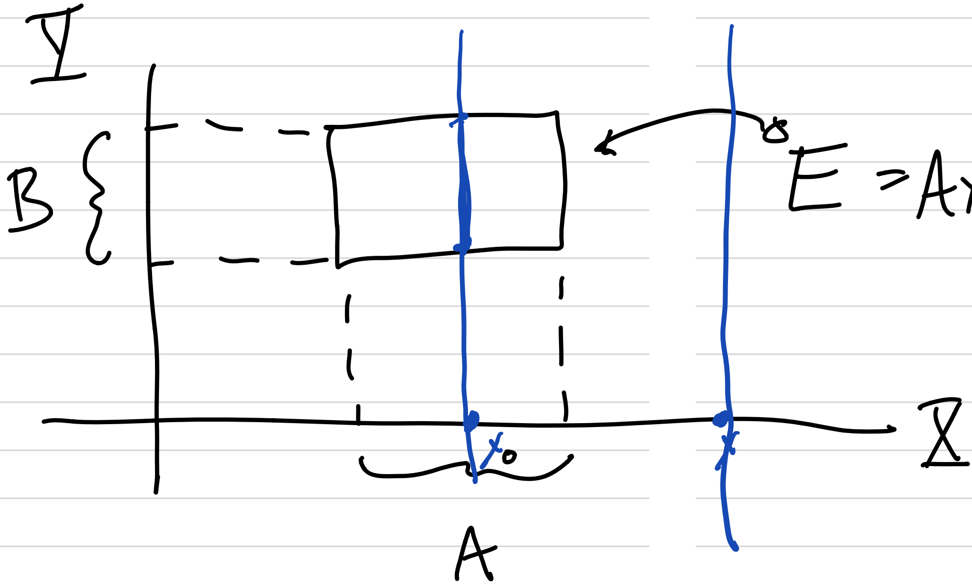
(b) If $f$ is real valued then $f_x$ is $\mathcal{B}$-measurable and $f^y$ is $\mathcal{A}$-measurable. — for "Fubini"

Proof of (a). Say $x_0 \in X$. Let $\mathcal{F}$ be the collection of all subsets $E$ of $X \times Y$ for which $E_{x_0}$ belongs to $\mathcal{B}$.




Let $\mathcal{R}$ be the collection of measurable rectangles.

$\mathcal{F}$ contains all measurable rectangles since $(A \times B)_{x_0}$ is either $B$ (if $x_0 \in A$) or $\emptyset$ (if $x_0 \notin A$).




$E = A \times B$ $\quad A \in \mathcal{A}$

$B \in \mathcal{B}$,

$$E_x = \begin{cases} B & \text{if } x_0 \in A \\ \emptyset & \text{if } x_0 \notin A. \end{cases}$$

$\mathcal{F}$ contains meas. rectangles.

Let's check that $\mathcal{F}$ is a $\sigma$-algebra.

General remark. If $f$ is a function then taking inverse images "commutes" with taking unions, intersections and complements. thus $f^{-1}(A^c) = (f^{-1}(A))^c$, $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$ etc.

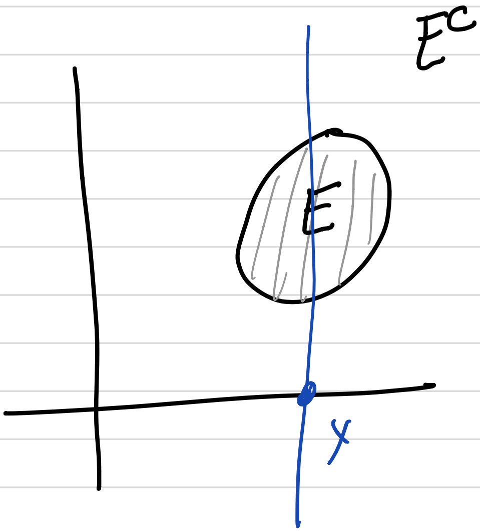
(By contrast taking images of sets is not well behaved in this way.) $\quad i_x(y) = (x, y)$

We can put $E_x$ in this framework since $E_x = i_x^{-1}(E)$, the operation of "taking slices" commutes with unions, intersections and complements.

$\mathcal{F}$ is closed under taking complements:

If $E \in \mathcal{F}$ then $E_x \in \mathcal{B}$ so $(E_x)^c \in \mathcal{B}$

but $(E_{x_0})^c = (E^c)_{x_0}$ so $(E^c)_{x_0} \in \mathcal{B}$ and $E^c \in \mathcal{F}$.

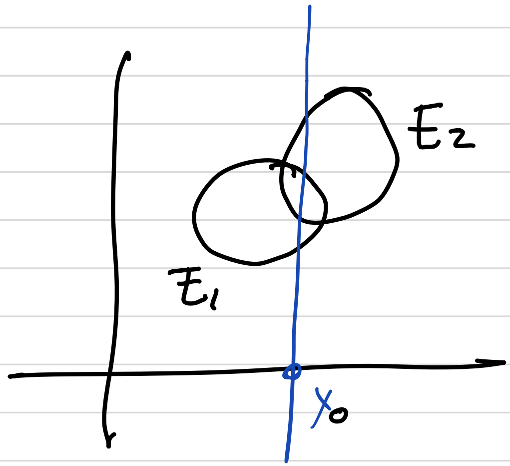


$\hat{\mathcal{F}}$ is closed under taking countable unions:

If $E_n \in \hat{\mathcal{F}}$ then $(E_n)_{x_0} \in \mathcal{B}$ so $\bigcup_n (E_n)_{x_0} \in \mathcal{B}$

but $\bigcup_n (E_n)_{x_0} = \left(\bigcup_n E_n\right)_{x_0}$ so $\left(\bigcup_n E_n\right)_{x_0} \in \mathcal{B}$ and

$\bigcup_n E_n \in \hat{\mathcal{F}}$.



Let us write $\mathcal{R}$ for the collection of all measurable rectangles. Since $\mathcal{F}$ is a $\sigma$-algebra that contains $\mathcal{R}$, $\mathcal{F}$ contains the smallest $\sigma$-algebra containing $\mathcal{R}$ which is exactly $\mathcal{A} \times \mathcal{B}$.

This means that if $E \in \mathcal{A} \times \mathcal{B}$ then $E \in \mathcal{F}$ so $E_{x_0}$ is in $\mathcal{B}$.

This finishes the proof of (a). (b) follows from (a).

"Slice Theorem for product measure"

Say that $E \in \mathcal{A} \times \mathcal{B}$ then

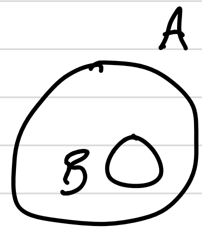
$$(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

In order for the "Slice Thm." to make sense we need $\nu(E_x)$ to make sense and we need $x \mapsto \nu(E_x)$ to be $\mathcal{A}$-measurable because we want to integrate this function.

Prop. 5.1.3(a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function.

Proof strategy: Let $\mathcal{F}$ be the collection of sets $E$ for which the slice measure function is $\mathcal{A}$-measurable. We show that $\mathcal{F}$ contains $\mathcal{R}$ and $\mathcal{F}$ is a d-system.
We will finish by invoking Dynkin's Thm.

A collection $\mathcal{D}$ of subsets of $X$ is a **d-system** on $X$ if




(a) $X \in \mathcal{D}$

(b) $A - B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$ and $A \supset B$

(c) $\bigcup_n A_n \in \mathcal{D}$ whenever $A_n$ is an increasing sequence of sets in $\mathcal{D}$: $A_1 \subset A_2 \subset A_3 \cdots$

(Cohn p. 37)

Thm. 1.6.2. (Dynkin) Let $X$ be a set and let $\mathcal{C}$ be a $\pi$-system on $X$. Then the $\sigma$-algebra generated by $\mathcal{C}$ coincides with the d-system generated by $\mathcal{C}$.

(Take $\mathcal{C} = \mathcal{R}$.)

The key difference between a d-system and a $\sigma$-algebra is that $\sigma$-algebras are closed under taking intersections.

Only one technique to use:

Construct new d-system where sets have property X.

Show that this new d-system contains the d-system generated by $\mathcal{C}$.

Conclude that sets in the d-system generated by $\mathcal{C}$ have property X.

We apply this trick twice.

Let $\mathcal{D}_1 = \{ A \in \mathcal{D} : A \cap C \in \mathcal{D} \text{ for } C \in \mathcal{C} \}$

We show $\mathcal{D}_1$ is a d-system that contains $\mathcal{C}$ so $\mathcal{D}_1$ contains $\mathcal{D}$.

Thus $\mathcal{D}$ is closed under taking intersections with sets <u>in $\mathcal{C}$</u>.

Define $\mathcal{D}_2 = \{ A \in \mathcal{D} : A \cap E \in \mathcal{D}$ for each $E \in \mathcal{D} \}$.

We show $\mathcal{D}_2$ is a d-system.

Using part 1 we show $\mathcal{D}_2$ contains $\mathcal{C}$.

Thus $\mathcal{D}$ is closed under taking arbitrary finite intersections.

Now we show that $\mathcal{D}$ is a $\sigma$-algebra.

Since $\mathcal{D}$ contains $X$ for any $A \in \mathcal{D}$
$A \in \mathcal{D}$ so $X - A = A^c \in \mathcal{D}$.

Since $\mathcal{D}$ is closed under taking complements $\mathcal{D}$ is also closed under taking finite unions.

We finish the proof of the theorem by showing that a d-system closed under taking finite unions is a $\sigma$-algebra.

We need to show it is closed under taking arbitrary unions.

Let $A_n$ be an arbitrary countable collection of sets in $\mathcal{D}$.

Let $B_n = \bigcup_{i=1}^{n} A_n$. $B_n$ is an increasing union so $\bigcup_n B_n \in \mathcal{D}$. But $\bigcup_n B_n = \bigcup_n A_n$ so $\bigcup_n A_n \in \mathcal{D}$.

This completes the proof of the theorem.