MA359 Measure Theory

Lecture Wed. 10:05 – 10:50

Should be recorded.

Class Mon 10:05 – 10:50

Online material will be available on the course Moodle page.

Loosely following D. L. Cohn "Measure Theory" 2nd edition

Available online via Warwick Library.

Topics for week 1

xv – xxi

1.3.2, 1.3.3, 1.3.4

1.1.1, 1.1.2

You can ask questions by typing into chat or raising your hand. Ryan will read the questions out loud.

Lebesgue integration is a theory of integration which has some advantages over the Riemann integral.

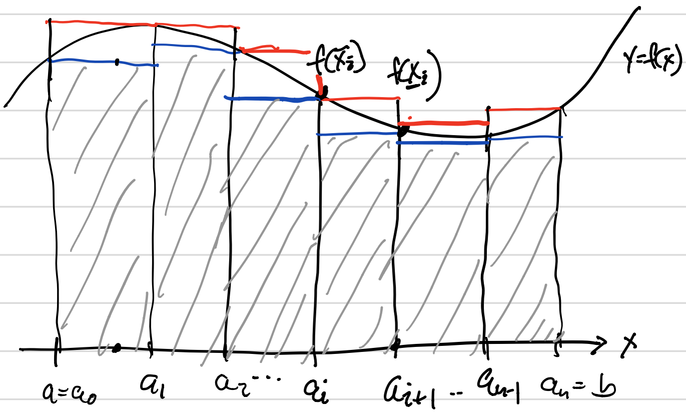
A theory of integration tells us two things:

① Which functions $f: \mathbb{R} \to \mathbb{R}$ can be integrated

② If f can be integrated what is

$$\int_a^b f\,dx.$$

In the case of the Riemann integral we answer question 1 in terms of upper and lower sums.



We say that f is Riemann integrable if, when you have a sequence of partitions of $[a,b]$ getting finer, the uppersums and lower sums converge to the same value.

We answer question 2 by saying that, when f is integrable, the integral

$$\int_a^b f(x)\,dx$$

is the common value of the upper sum limit and the lower sum limit.

In Lebesgue integration theory we will give a larger class of integrable functions.

If f is Riemann integrable then f will be Lebesgue integrable and the values of the two integrals will be the same.

The Lebesgue integral will have better properties with respect to limits than the Riemann integral.

With the Riemann integral if $f_n \to f$ uniformly then:

$$\int_a^b \lim_{n\to\infty} f_n \, dx = \lim_{n\to\infty} \int_a^b f_n \, dx$$

With the Lebesgue integral if $f_n \to f$ pointwise then:

$$\int_a^b \lim_{n\to\infty} f_n \, dx = \lim_{n\to\infty} \int_a^b f_n \, dx$$

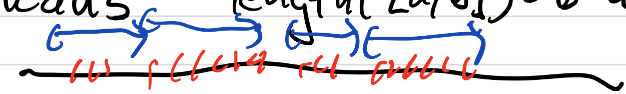
There are various situations where we want to construct new functions by taking limits of simpler functions.

The limit properties of the Lebesgue integral allow us to integrate these new functions.

If f is positive then $\int f\,dx$ is related to the area under the graph. In fact we can think of upper sums and lower sums geometrically as giving approximations to the area under the graph.

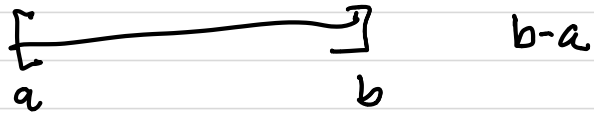
In this course we will describe a new theory of <u>area</u> but our starting point is a new theory of <u>length</u> which will lead us to the theory of the Lebesgue integral.

For intervals $[a,b] \subset \mathbb{R}$ we know what length means. $\text{length}([a,b]) = b-a$.



How do we make sense of the length of wilder sets $A \subset \mathbb{R}$?

Here is our first attempt to construct a generalized length function which we call Lebesgue outer measure.



Lebesgue outer measure

Given a set $A \subset \mathbb{R}$ we will cover A by sets of known measure. This should be an upper bound for the length of A so we will take the infimum over coverings.

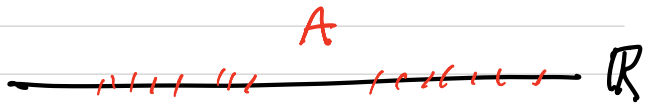
Formal definition:

The **Lebesgue outer measure** of $A \subset \mathbb{R}$ is defined to be

$$\lambda^*(A) = \inf_{\{U_j\}} \left\{ \sum_{j=1}^{\infty} \text{length}(U_j) : A \subset \bigcup_{j=1}^{\infty} U_j \right\}$$

where the U_j are open intervals (a,b) and $\text{length}((a,b)) = b - a$.

The key point in this definition is that we use countable coverings instead of finite coverings.



For our first example of a "wild set" consider

$$A = \mathbb{Q} \cap [0,1].$$

It is hard to draw this set because it is dense in the unit interval. Does this mean it is large?

The complement of A is <u>also</u> dense in the unit interval. Is it also large?

Claim: $\lambda^*(\mathbb{Q} \cap [0,1]) = 0$

The proof will depend on the fact that $\mathbb{Q} \cap [0,1]$ is countable:

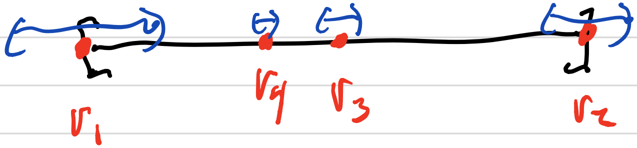
Let $r_1, r_2, r_3 \ldots$ be the elements of $\mathbb{Q} \cap [0,1]$.

For example:

$r_1 = 0$, $r_2 = 1$, $r_3 = \frac{1}{2}$,

$r_4 = \frac{1}{3}$, $r_5 = \frac{2}{3}$, $r_6 = \frac{1}{4}$ ...

Let $\varepsilon > 0$ be given. Let U_j^ε be the open interval centered at r_j of length $\varepsilon/2^j$.



(These are not actually disjoint.)

$\mathcal{U}_\varepsilon = \{U_j^\varepsilon\}$.

$$\sum_{j=1}^{\infty} \operatorname{length}(U_j) = \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j}$$

$$= \varepsilon.$$

$$\lambda^*(\mathbb{Q} \cap [0,1]) = \inf_{\varepsilon > 0} \varepsilon = 0.$$

Remark: If we only considered finite coverings we would get a different answer.

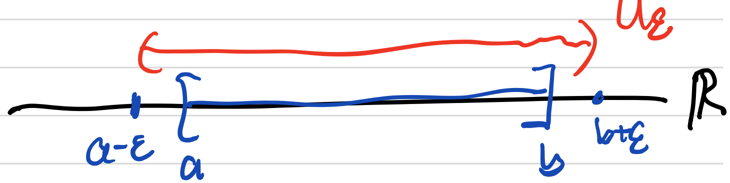
In defining $\lambda^*([a,b])$ we used the length of the interval as one ingredient in our definition but in the process of defining λ^* we we may have redefined "length"$([a,b])$. Let us check that we haven't changed the length of the interval.

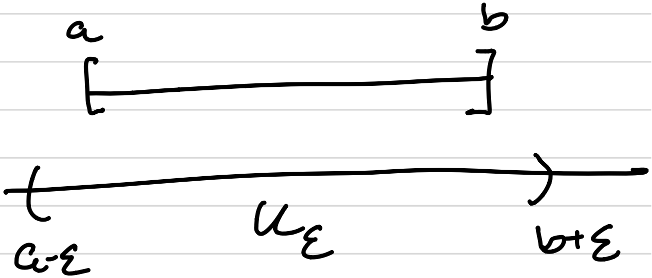
Prop. $\lambda^*(\underset{I}{[a,b]}) = b-a.$

Proof. We first show

$$\lambda^*(I) \leq \text{length}(I)$$

It suffices to cover $[a,b]$ with a single interval. Say $\varepsilon > 0$. Let





So $\lambda^*([a,b]) \le \overbrace{(b+\varepsilon)-(a-\varepsilon)}^{\text{length}(U_\varepsilon)}$

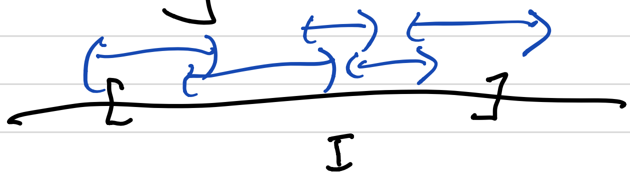
$= b-a+2\varepsilon.$

Let $\varepsilon \to 0$ get $\lambda^*([a,b]) \le b-a.$

Now we show

$\lambda^*([a,b]) \ge b-a.$

Let $\mathcal{U}=\{U_i\}$ be a covering of $I=[a,b]$



I is compact so there is a finite subcover

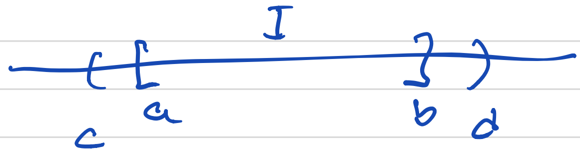
$U_1 \ldots U_n$

Clearly

$$\sum_{k=1}^{N} \text{length}(u_{j_k}) \le \sum_{k=1}^{\infty} \text{length}(u_k)$$

Now we prove by induction on the number of intervals that for a finite cover $\{u_1 \dots u_n\}$ of an interval $[a,b]$ we have $\sum \text{length}(u_i) \ge b-a$.

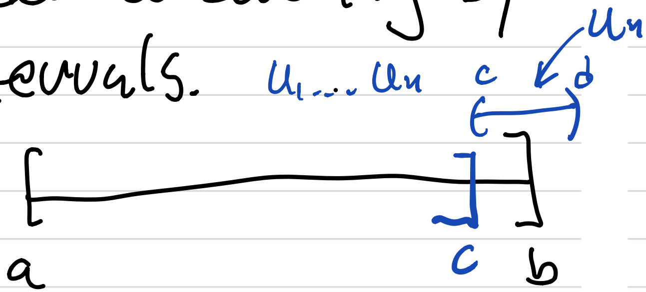
I interval



$b-a < \underline{d-c}$.

Say that our claim holds for covers with $n-1$ intervals.

Consider a covering by n intervals. $U_1 \dots U_n$



Some interval contains b, call it $U_n = (c,d)$.

We may assume $c > a$.

The remaining $n-1$ intervals $U_1 \dots U_{n-1}$ cover $[a,c]$ so

$$\sum_{j=1}^{n} \text{length}(U_j) = \sum_{j=1}^{n-1} \text{length}(U_j) + \text{length}(U_n)$$

$$\geq c-a + d-c$$

$$= d-a$$

$$> b-a.$$

Cor. The open cover $\mathcal{U}^\varepsilon$ of the rationals leaves out some points when $\varepsilon < 1$.