Week 9 - Friday lecture

03 December 2021 09:01

Good Morning! Lecture will start at 9.05

Remind me to record.

Hoping to get Assignment 3 marks back to you next week.

Before we start

For Fubini–Tonelli theorem you need to work in a σ-finite measure space (missed in videos!)

You use the previous thm about existence of product measure which is only for σ-finite measure spaces.

Q4 & 5 on the exercise sheet are examples of when Fubini–Tonelli fails.

Applications of Fubini–Tonelli

__Ex 1__: Let $(E, \mathcal{E}, \mu)$ be a measure space $f: E \to \mathbb{R}$ be measurable, non-negative

$$A = \{ (x,y) : 0 \leq y \leq f(x) \} \subseteq E \times \mathbb{R}$$

We can consider the space $(E \times \mathbb{R}, \mathcal{E} \times \mathcal{B}(\mathbb{R}), \mu \times \lambda)$

then we know

$$(\mu \times \lambda)(A) = \int \lambda(A_x) \mu(dx) = \int_{-\infty}^{\infty} \mu(A_y) \lambda(dy)$$

$$\lambda(A_x) = \lambda(\{ y : 0 \leq y \leq f(x) \}) = \lambda([0, f(x)]) = f(x)$$

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$\mu(A_y) = \mu(\{x : f(x) \geq y\})$

So we know that

$$\int_{-\infty}^{\infty} \mu(\{x : f(x) \geq y\})\, dy = \int_E f(x)\, \mu(dx)$$

so this gives an expression for the integral of a non-negative function in terms of its super-level sets

In prob often have estimates on $\mu(\{x : f(x) \geq y\})$ in prob context is $\mathbb{P}(X \geq y)$ and this allows you to turn them into estimates on $\mathbb{E}(X)$.

Example 2 Convolutions

Let $f, g \in L^1(\mathbb{R})$ then for every x the function $t \mapsto f(x-t)g(t)$ is measurable and in $L^1(\mathbb{R})$

Furthermore, $f*g(x)$ defined by

$$x \mapsto \begin{cases} \int_{\mathbb{R}} f(x-t)\, g(t)\, dt & \text{if } t \mapsto f(x-t)g(t) \text{ is integrable} \\ 0 & \text{o/w} \end{cases}$$

is in L^1 and satisfies $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$.

First lets check $t \mapsto f(x-t)g(t)$ is measurable

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As the product of measurable functions is measurable this will be true if $f(x-t)$ is measurable.

Define the function h by $h(t) = x-t$ then h is continuous so Borel measurable then $f(x-t) = f \circ h(t)$ and we know the composition of two measurable functions is measurable so $t \mapsto f(x-t)$ is measurable so $t \mapsto f(x-t)g(t)$ is measurable.

Now we want to check that its integrable for almost every x. We do this by integrating over x.

$$\int \left| \int f(x-t)g(t)\,dt \right| dx \leq \int\int |f(x-t)g(t)|\,dt\,dx$$

Using Fubini-Tonelli for the functions

$$= \int\int |f(x-t)|\,|g(t)|\,dx\,dt$$

$$= \int \left(\int |f(x-t)|\,dx \right) |g(t)|\,dt$$

$$= \int \left(\int |f(x)|\,dx \right) |g(t)|\,dt$$

$$= \|f\|_1 \, \|g\|_1$$

In particular, $\int \int |f(x-t)g(t)|\,dt\,dx$ is finite

so $\int |f(x-t)g(t)|\,dt$ is finite for almost every x.

Therefore we only altered our function on a set of measure 0.

$$\|f*g\|_1 \leq \|f\|_1 \|g\|_1$$

More convolution – Applications of L^p-theory

goal is if $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ $p \in (1,\infty)$ $\frac{1}{p} + \frac{1}{q} = 1$

then $f*g(x)$ is continuous function.

<u>Lemma</u> The shift map is continuous in L^p

$T_\tau : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R})$ by $(T_\tau f)(x) = f(x+\tau)$

then $\lim\limits_{\tau \to 0} \|T_\tau f - f\|_p = 0$

<u>Proof</u> We want to show given, $f, \varepsilon > 0$ there exists τ_* s.t. if $\tau < \tau_*$ then $\|T_\tau f - f\|_p \leq \varepsilon$

First, let us show the result for step functions

$$\phi = \sum_{k=1}^{n} a_k \mathbb{1}_{[c_k, d_k)}$$

$$\|T_\tau \phi - \phi\|_p \overset{\text{Minkowski}}{\leq} \sum_{k=1}^{n} |a_k| \, \|T_\tau \mathbb{1}_{[c_k,d_k)} - \mathbb{1}_{[c_k,d_k)}\|_p$$

$$= \sum_{k=1}^{n} |a_k| \, \|\mathbb{1}_{[c_k-\tau, d_k-\tau)} - \mathbb{1}_{[c_k,d_k)}\|_p$$

$$= \sum_{k=1}^{n} |a_k| \, \| \mathbb{1}_{([c_k-\tau, d_k-\tau)\Delta[c_k,d_k))} \|_p$$

$$= \sum_{k=1}^{n} |a_k| \, \lambda\left([c_k-\tau, d_k-\tau)\Delta[c_k,d_k)\right)^{1/p}$$

$$\leqslant \sum_{k=1}^{n} |a_k| (2|\tau|)^{1/p}$$

as $|\tau| \longrightarrow 0$ this is going to tend to 0.

Now we want to use the density of step functions

Given f we know $\exists \, \varphi$ (a step function) with $\|f-\varphi\|_p \leq \varepsilon/3$

$$\|T_\tau f - T_\tau \varphi\|_p = \left(\int |f(x+\tau) - \varphi(x+\tau)|^p dx \right)^{1/p}$$

$$\underset{\text{variables}}{\overset{\text{changing}}{=}} \left(\int |f(x) - \varphi(x)|^p dx \right)^{1/p} = \|f-\varphi\|_p$$

for this φ $\exists \, \tau_*$ s.t. $|\tau| \leq \tau_*$ implies $\|\varphi - T_\tau \varphi\|_p \leq \frac{\varepsilon}{3}$

$$\|T_\tau f - f\|_p \leqslant \|T_\tau f - T_\tau \varphi\|_p + \|T_\tau \varphi - \varphi\|_p + \|\varphi - f\|_p$$

$$= 2\|f-\varphi\|_p + \|T_\tau \varphi - \varphi\|_p$$

$$\leqslant \varepsilon.$$

if $|\tau| \leq \tau_*$

We're going to use this to show continuity of convolutions.

$$\ldots \; = \left| \int (f(x-t) - f(y-t)) g(t) \, dt \right|$$

convolutions.

$$|f*g(x) - f*g(y)| = \left|\int (f(x-t) - f(y-t))g(t)\,dt\right|$$

$$\leq \int |f(x-t) - f(y-t)||g(t)|\,dt$$

Hölder

$$\leq \|g\|_q \left(\int |f(x-t) - f(y-t)|^p\,dt\right)^{1/p}$$

changing variable

$$= \|g\|_q \left(\int |f(t-x+y) - f(t)|^p\,dt\right)^{1/p}$$

$$= \|g\|_q \|T_{-x+y}f - f\|_p$$

so this will go to 0 as $|x-y|$ goes to 0

$$|f*g(x) - f*g(y)| \leq \|g\|_q \|T_{-x+y}f - f\|_p$$

$$\to 0 \text{ as } |x-y| \to 0$$

so $f*g$ is continuous.

Also works if $f \in L^1$ and $g \in L^\infty$

and as $f*g(x) = g*f(x)$ it works if $f \in L^\infty, g \in L^1$.