Lecture 3 - Lebesgue outer measure

12 October 2021 08:52

Remind me to record!

Will start at 5 past

- Assigment was out on last Friday due on 21^{st}
- Support – Two online Monday 11am
  Friday 11am

  One F2F Wednesday 11am ← Need to sign up on tabula

- Support classes – Not recorded
  ↳ Latexed solution sheets

Outer measure

$\nu : \underline{\mathcal{P}(E)} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is an outer measure IF:

↑ power set of E

Domain has to be the power set

- $\nu(\emptyset) = 0$
- It is <u>monotone</u> $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$
- It is <u>countably</u> <u>subadditive</u>

  $A_1, A_2, A_3 \dots$ $\nu\left(\bigcup_n A_n\right) \leq \sum_n \nu(A_n)$

<u>Key example</u> Lebesgue outer measure

First we define Lebesgue measure on an interval

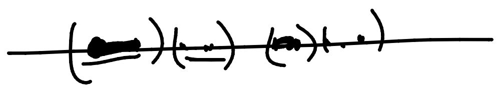
$\lambda((a,b]) = b-a$

↑ half open interval.

$\tilde{I}$ to be the set of countable unions of half open intervals.

<u>Defn</u> Lebesgue outer measure for $A \in \mathcal{P}(\mathbb{R})$

$$\lambda^*(A) = \inf \left\{ \sum_n \lambda(I_n) \;\middle|\; I_n \text{ are all intervals and } A \subseteq \bigcup_n I_n \right\}$$



<u>Remark</u>: We could do this with open intervals or closed intervals or half open the other way.

Book Cohn: he does it with open intervals.

<u>Propn</u> (1.4 in the notes)

Lebesgue outer measure is an outer measure and agrees with λ on half open intervals.

<u>Proof</u> 1. $\lambda^*(\phi) = 0$ so ϕ is a countable collection of half open intervals

so $\lambda^*(\phi) \leq \lambda(\phi) = 0$

2. Monotonicity: $A_1 \subseteq A_2$ then for any $B \subset I$ with $A_2 \subseteq B$ then $A_1 \subseteq B$ therefore

$$\inf \Big\{ \sum_n \lambda(I_n) : I_n \text{ intervals},\ A_1 \subseteq \underbrace{\bigcup_n I_n}_{B} \Big\} \leq \inf \Big\{ \sum_n \lambda(I_n) : I_n \text{ intervals } A_2 \subseteq \underbrace{\bigcup_n I_n}_{B} \Big\}$$

as inf on LHS is taken over a bigger set.

3. <u>Countable subadditivity</u> : Let $A_1, A_2, A_3, \dots$ be a sequence

If $\sum_n \lambda^*(A_n) = \infty$ the we're done

$\lambda^*(\bigcup_n A_n) \leq \infty$

We look at the case where $\sum_n \lambda^*(A_n) < \infty$

Then $\exists$ a collection $I_{n,k}$ of intervals s.t.

$\sum_k \lambda(I_{n,k}) \leq \lambda^*(A_n) + 2^{-n}\varepsilon$ and $A_n \subseteq \bigcup_k I_{n,k}$

$$\bigcup_n A_n \subseteq \bigcup_{n,k} I_{n,k}$$

and
$$\sum_n \sum_k \lambda(I_{n,k}) \leq \sum_n (\lambda^*(A_n) + 2^{-n}\varepsilon)$$
$$= \sum_n \lambda^*(A_n) + \varepsilon$$

and

$$\lambda^*(\bigcup_n A_n) \leq \sum_n \lambda^*(A_n) + \varepsilon$$

ε is arbitrary so $\lambda^*(\bigcup_n A_n) \leq \sum_n \lambda^*(A_n)$

We want to show: $\lambda^*((a,b]) = b-a$

$(a,b] \subseteq (a,b]$ and $(a,b]$ is a countable union of half open intervals

$$\lambda^*((a,b]) \leq \lambda((a,b]) = b-a$$

$(a,b] \subseteq (c_1,d_1] \cup (c_2,d_2] \cup \dots$

Suppose $(a,b] \subseteq (c_1,d_1] \cup (c_2,d_2] \cup \dots$

then $[a+\varepsilon, b-\varepsilon] \subseteq (c_1,d_1) \cup (c_2,d_2) \cup \dots$

Every open cover of a compact set has a finite sub-cover

There exists n s.t.

$[a+\varepsilon, b-\varepsilon] \subseteq (c_1,d_1) \cup \dots \cup (c_n,d_n)$

by taking lengths of both sides

$$b-a-2\varepsilon \leq \sum_{k=1}^{n} (d_k-c_k) \leq \sum_{k=1}^{\infty} (d_k-c_k)$$

so as ε was arbitrary

$$b-a \leq \sum_{k=1}^{\infty} (d_k-c_k)$$

and $\lambda^*((a,b]) = \inf \left\{ \sum_{k=1}^{\infty} (d_k-c_k) \;\middle|\; (a,b] \subseteq \bigcup_k (c_k,d_k] \right\}$

take the inf over all possible covering with half open intervals

$$b-a \leq \lambda^*((a,b])$$

We want to turn λ^* into a proper measure

We want to find a smaller collection of subsets of $\mathbb{R}$, $\mathcal{M}$, s.t. $\lambda^*|_{\mathcal{M}}$ is a measure.

Call these the Lebesgue measurable subsets

Defn $A \in \mathcal{P}(\mathbb{R})$ is Lebesgue measurable

<u>Def^n</u> $A \in \mathcal{P}(\mathbb{R})$ is Lebesgue measurable iff for every $B \in \mathcal{P}(\mathbb{R})$

$$\lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(A^c \cap B)$$

Call $\mathcal{M}$ the set of all Lebesgue measurable sets

<u>Video 1</u> Show $\mathcal{M}$ is a σ-algebra

λ^* restricted to $\mathcal{M}$ is a measure.

<u>Video 2</u> All Borel sets are in $\mathcal{M}$.

To do this we show $(-\infty, b] \in \mathcal{M}$ for every b.