Measure Theory: Exercises (not for credit)

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Question 1. Show that the definition of the integral for a non-negative function is consistent with the defintion of the integral for simple functions. That is to say if $f = \sum_k a_k 1_{A_k}$ is a simple function then

$$\sum_k a_k \mu(A_k) = \sup\{\mu(h) \,:\, h\,\text{simple},\, h \le f\}$$

Answer: To write the answer it is convinient to write $\mu(h)$ for our integral on simple functions defined by $\mu(\sum_{k=1}^n a_k 1_{A_k}) = \sum_k a_k \mu(A_k)$ and $\mu_*(f) = \sup\{\mu(h) \,:\, h\,\text{simple},\, h \le f\}$. We want to show that μ and μ_* agree on simple functions. As f is a simple function with $f \le f$ we have that $\mu_*(f) \ge \mu(f)$. We also know that integration with respect to μ is monotone on simple functions so if $h \le f$ and h simple the $\mu(h) \le \mu(f)$ so taking supremums over all possible such h gives $\mu_*(f) \le \mu(f)$. Therefore $\mu_*(f) = \mu(f)$. □

Question 2. Let f be an integrable, real valued function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that $\mu(f 1_A) = 0$ for every $A \in \mathcal{E}$ show that this implies that $f = 0$ almost everywhere. Let $\mathcal{A}$ be a π-system generating $\mathcal{E}$ and containing E. Suppose that $\int f 1_A \mu(\mathrm{d}x) \,(= \mu(f 1_A)) = 0$ for every $A \in \mathcal{A}$ show that then $f = 0$ almost everywhere.

Answer: For the first part we know that $f^{-1}([0,\infty)) \in \mathcal{E}$ therefore $\int f 1_{f \ge 0} \mu(\mathrm{d}x) = 0$ since $f 1_{f \ge 0}$ is non-negative the results from lectures imply that $f 1_{f \ge 0} = 0$ almost everywhere. We can argue similarly to see that $f 1_{f \le 0} = 0$ almost everywhere.

For the second part consider $\mathcal{D} = \{A \in \mathcal{E} \,:\, \mu(f 1_A) = 0\}$. Then suppose $A, B \in \mathcal{D}$ and $A \subseteq B$ then $\mu(f 1_{B \setminus A}) = \mu(f 1_B) - \mu(f 1_A) = 0$ so $B \setminus A \in \mathcal{D}$. Suppose also that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ then $|f 1_{A_n}| \le |f|$ and $f 1_{A_n} \to f 1_A$ where $A = \bigcup_n A_n$. So by dominated convergence $\mu(f 1_{A_n}) = \mu(f 1_A)$ therefore if the $A_n \in \mathcal{D}$ for every n then $\mu(f 1_A) = \lim_n \mu(f 1_{A_n}) = 0$, so $A \in \mathcal{D}$. We also have that $\mathcal{A} \subseteq \mathcal{D}$ and $E \in \mathcal{A}$ so $\mathcal{D}$ is a d-system and so by Dynkin's lemma it contains $\sigma(\mathcal{A}) = \mathcal{E}$. □

Question 3. Find a three sequences of real valued integrable functions, $(f_n)_{n \ge 1}, (g_n)_{n \ge 1}, (h_n)_{n \ge 1}$, all of which converge to 0 almost everywhere and where

- $\lim_n \int f_n(x) \mathrm{d}x = \infty$
- $\lim_n \int g_n(x) \mathrm{d}x = 1$
- $\limsup_n \int h_n(x) \mathrm{d}x = -\liminf_n \int h_n(x) \mathrm{d}x = 1$.

Answer: We can take $f_n = n^2 1_{[0,1/n)}$ then $\lambda(f_n) = n$ for every n, but $f_n(x)$ converges to 0 everywhere except $x = 0$.

We can take $g_n = n 1_{[0,1/n)}$.

We can take $h_n = (-1)^n n 1_{[0,1/n)}$ $\square$

Question 4. Let $(f_n)_{n\geq 1}$ be a sequence of real valued measurable functions on $(E, \mathcal{E}, \mu)$. Suppose that f_1 is integrable and $f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$ for every x and $f_n(x) \to f(x)$. Show that $\lim_n \int f_n(x)\mu(\mathrm{d}x) = \int f(x)\mu(\mathrm{d}x)$.

Answer: So define $g_n = f_n - f_1$ then g is an increasing sequence of non-negative real valued measurable functions and $g_n \to f - f_1$ so by monotone convergence $\mu(g_n) \to \mu(f - f_1)$ therefore $\mu(f_n) \to \mu(f)$ by adding $\mu(f_1)$ which is finite, to each side. $\square$

Question 5. In lectures we proved Beppo-Levi as a consequence of the monotone convergence theorem. Show that if we assume the result in Beppo-Levi then we can prove the monotone convergence theorem as a consequence.

Answer: Suppose that f_n is an increasing sequence of non-negative, real vauled, measurable functions. Define $g_n = f_n - f_{n-1}$ for $n \geq 2$ and $g_1 = f_1$, then the g_n are all non-negative and measurable. Then we have that $f_n = \sum_{k=1}^n g_n$ and $f = \sum_{n=1}^\infty g_n$. Then by the conclusion of Beppo-Levi we have $\mu(f) = \mu(\sum_n g_n) = \sum_n \mu(g_n) = \lim_n \sum_{k=1}^n \mu(g_n) = \lim_n \mu(\sum_{k=1}^n g_n) = \lim_n \mu(f_n)$ so the conclusion of the monotone convergence theorem holds. $\square$