Recall the fundamental theorem of calculus

$f : [a,b] \to \mathbb{R}$ and is continuous

$F(t) = \int_a^t f(x)dx$ then $F(t)$ is differentiable

with derivative $f(t)$.

<u>Prop^n</u> [Change of Variables formula]

Let $\phi : [a,b] \to [\phi(a), \phi(b)]$ continuously differentiable

and strictly increasing (invertible)

then for all non-negative g on $[\phi(a), \phi(b)]$ we

have
$$\int_{\phi(a)}^{\phi(b)} g(y)dy = \int_a^b g(\phi(x))\, \phi'(x)\, dx$$

<u>Proof</u> The first idea is to approximate g by simple functions to reduce the problem to proving it for $g = \mathbb{1}_A$, $A \in \mathcal{B}(\mathbb{R})$

Then to show that it holds for $g = \mathbb{1}_{(c,d]}$ and use a Dynkin's lemma argument to go from intervals to any Borel set.

1. $g(y) = \mathbb{1}_{(c,d]}(y)$

$$\int_{\phi(a)}^{\phi(b)} \mathbb{1}_{(c,d]}(y)\, dy = \lambda\big((c,d] \cap [\phi(a), \phi(b)]\big)$$
$$= d \wedge \phi(b) - c \vee \phi(a)$$

$$\int_a^b \mathbb{1}_{(c,d]}(\phi(x))\, \phi'(x)\, dx$$

$$= \int_a^b \mathbb{1}_{(\phi^{-1}(c),\, \phi^{-1}(d)]}(x)\, \phi'(x)\, dx$$

$$= \int_{a \vee \phi^{-1}(c)}^{b \wedge \phi^{-1}(d)} \phi'(x)\, dx \overset{FTC}{=} \phi(b \wedge \phi^{-1}(d)) - \phi(a \vee \phi^{-1}(c))$$

$$= \phi(b) \wedge d - \phi(a) \vee c$$

so $\int_{\phi(a)}^{\phi(b)} g(y)\, dy = \int_a^b g(\phi(x))\, \phi'(x)\, dx$

when $g = \mathbb{1}_{(c,d]}$

$I = [\phi(a), \phi(b)]$

2. We want to extend this to $g = \mathbb{1}_A$, $A \in \mathcal{B}(I)$

$$\mathcal{D} = \left\{ A \in \mathcal{B}(I) : \int_{\phi(a)}^{\phi(b)} \mathbb{1}_A(y)\, dy = \int_a^b \mathbb{1}_A(\phi(x))\, \phi'(x)\, dx \right\}$$

so all half open intervals are in $\mathcal{D}$

We want to show $\mathcal{D}$ is a d - system

so wts if $A, B \in \mathcal{D}$ $A \subseteq B$ then $B \setminus A \in \mathcal{D}$

and if $A_1, A_2, A_3, \ldots \in \mathcal{D}$ with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ then $\bigcup_n A_n \in \mathcal{D}$

$\mathbb{1}_{B \setminus A} = \mathbb{1}_B - \mathbb{1}_A$ if $A \subseteq B$. If $A, B \in \mathcal{D}$

$$\int^{\phi(b)} \mathbb{1}_{B \setminus A}(y)\, dy = \int^{\phi(b)} \mathbb{1}_B(y)\, dy - \int^{\phi(b)} \mathbb{1}_A(y)\, dy$$

$$\int_{\phi(a)}^{\phi(b)} \mathbb{1}_{B\setminus A}(y)\,dy = \int_{\phi(a)}^{\phi(b)} \mathbb{1}_B(y)\,dy - \int_{\phi(a)}^{\phi(b)} \mathbb{1}_A(y)\,dy$$

$$= \int_a^b \mathbb{1}_B(\phi(x))\phi'(x)\,dx - \int_a^b \mathbb{1}_A(\phi(x))\phi'(x)\,dx$$

$$= \int_a^b \left(\mathbb{1}_B(\phi(x)) - \mathbb{1}_A(\phi(x))\right)\phi'(x)\,dx = \int_a^b \mathbb{1}_{B\setminus A}(\phi(x))\,\phi'(x)\,dx$$

so $B\setminus A \in \mathcal{D}$

Suppose $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup_n A_n$ and $A_n \in \mathcal{D}\ \forall n$

then let $g_n = \mathbb{1}_{A_n}$ then $g_n \uparrow \mathbb{1}_A = g$ and as

ϕ is increasing $g_n \circ \phi \uparrow g \circ \phi$

$$\int_{\phi(a)}^{\phi(b)} \mathbb{1}_A(y)\,dy \underset{MON}{=} \lim_n \int_{\phi(a)}^{\phi(b)} g_n(y)\,dy = \lim_n \int_a^b g_n(\phi(x))\,\phi'(x)\,dx$$

$$\underset{MON}{=} \int_a^b g(\phi(x))\phi'(x)\,dx = \int_a^b \mathbb{1}_A(\phi(x))\,\phi'(x)\,dx$$

$A \in \mathcal{D}$

So $\mathcal{D}$ is a d-system and the collection of half open intervals is a π-system contained in $\mathcal{D}$ so by Dynkin's lemma the σ-algebra generated by the half open intervals is contained in $\mathcal{D}$

so $\mathcal{D} = \mathcal{B}(I)$

$=$

3. Want to show our conclusion holds when g is simple $\quad g = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$

$$\int_{\phi(a)}^{\phi(b)} \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(y)\,dy = \sum_{k=1}^{n} a_k \int_{\phi(a)}^{\phi(b)} \mathbb{1}_{A_k}(y)\,dy$$

$$= \sum_{k=1}^{n} a_k \int_a^b \mathbb{1}_{A_k}(\phi(x))\,\phi'(x)\,dx$$

$$= \int_a^b \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(\phi(x))\,\phi'(x)\,dx$$

$$= \int_a^b g(\phi(x))\,\phi'(x)\,dx.$$

Now suppose g is +ve and measurable then there exists a sequence of simple functions with $g_n \uparrow g$ and we will also have $g_n \circ \phi \uparrow g \circ \phi$ so we can conclude using monotone convergence.