Measure Theory: Assignment Three - Simple functions and Lebesgue integration

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Let $(E, \mathcal{E}, \mu)$ be a measure space. Recall that we call a function $f : E \to \mathbb{R}$ simple if it is non-negative and can be written in the form $f(x) = \sum_{k=1}^n a_k 1_{A_k}(x)$ where the a_k are non-negative numbers and $A_k \in \mathcal{E}$ for each k.

Question 0.1. Show that if f is a simple function then f is measurable. *6 marks*

Answer: As f is simple we can write it as $f(x) = \sum_{k=1}^n a_k 1_{A_k}$ then if B is a Borel set in $\mathbb{R}$ we have $f^{-1}(B) = \bigcup_{ks.t.a_k \in B} A_k$ this is a finite union of measurable sets so also measurable. Therefore f is measurable. □

Question 0.2. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be measurable and non-negative. Recall our classic approximation

$$f_n(x) = \left(2^{-n} \lfloor 2^n f(x) \rfloor\right) \wedge n.$$

Show that the sequence $f_n(x)$ is increasing for every x and has limit $f(x)$. Show that f_n is a simple function for each n. *7 marks*

Answer: Firstly, to show f is simple, we can write it as $a_{k,n} = k2^{-n}$ for $k = 1, 2, \dots, n2^n$, and let $A_{k,n} = f^{-1}([a_{k,n}, a_{(k+1),n}))$ then we have that $f_n(x) = \sum_{k=0}^{n2^n} a_{k,n} 1_{A_{k,n}}$, then by defintion $a_{k,n} \geq 0$ and as f is measurable and $[a_{k,n}, a_{(k+1),n}) \in \mathcal{B}(\mathbb{R})$ we have that $A_{k,n} \in \mathcal{E}$.

To show f_n converges to f, we note that $f_n(x)$ is the largest number of the for $k2^{-n}$ which is smaller than $f(x)$ therefore $|f_n(x) - f(x)| \leq 2^{-n}$ which shows that $f_n(x) \to f(x)$ (and also that it converges uniformly but that isn't part of the questions).

To show that $f_n(x)$ is an increasing sequence, suppose that $f_n(x) = k2^{-n}$ then this means that $f(x) \in [k2^{-n}, (k+1)2^{-n})$. Now we know that $f_{n+1}(x)$ is the smallest number of the form $j2^{-n-1}$ with $j \in \mathbb{N}$ that is less than or equal to $f(x)$ so as $f(x) \in [k2^{-n}, (k+1)2^{-n})$, we have that $f_{n+1}(x)$ is either $(2k)2^{-n-1}$ or $(2k+1)2^{-n-1}$ both of which are greater than or equal to $k2^{-n} = f_n(x)$ so $f_{n+1}(x) \geq f_n(x)$. □

Question 0.3. Suppose $f : E \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ is either non-negative or integrable. Let N be a null set. Define the function g by $g(x) = f(x) 1_N(x)$. Show that $\mu(g) = 0$. We say two function f_1 and f_2 are equal almost everywhere if $\mu\{x : f_1(x) \neq f_2(x)\} = 0$. Show that this implies $\int f_1(x) \mu(\mathrm{d}x) = \int f_2(x) \mu(\mathrm{d}x)$. *12 marks*

Answer: First let us work in the case that $f \geq 0$. Then we use the fact that if f_n is an increasing sequence of measurable functions then $\mu(f) = \lim_n \mu(f_n)$. We can construct such a sequence as in the previous questions. Now let $g_n = f_n 1_N$, then we will have that $g_n \uparrow g$. If $f_n = \sum_{k=1}^{N_n} a_{k,n} 1_{A_{k,n}}$ then we can express g_n by $g_n = \sum_{k=1}^{N_n} a_{k,n} 1_{A_{k,n} \cap N}$ and we have that $\mu(g_n) = \sum_{k=1}^{N_n} a_{k,n} \mu(A_{k,n} \cap N)$. As $\mu(N) = 0$ by monotonicity we have that $\mu(A_{k,n} \cap N) = 0$, so $\mu(g_n) = 0$. Therefore $\mu(g) = \lim_n \mu(g_n) = 0$.

Now in the case that f is integrable, we have that $g = f_+ 1_N - f_- 1_N$ and so we have $\mu(g) = \mu(f_+ 1_N) - \mu(f_- 1_N)$ and using the result for positive f we have that $\mu(g) = 0$.

Now suppose that $f_1 = f_2$ almost everywhere, let us define $N = \{x : f_1(x) \neq f_2(x)\}$ then we have that $\mu(N) = 0$. We have that $\mu(f_1) = \mu(f_1 1_N) + \mu(f_1 1_{N^c}) = \mu(f_1 1_N) + \mu(f_2 1_{N^c} = \mu(f_2 1_{N^c})$ as $f_1 = f_2$ on N^c and using the first part of the questions $\mu(f_1 1_N) = 0$. We also have that $\mu(f_1 1_N) = 0$ so $\mu(f_2) = \mu(f_2 1_N) + \mu(f_2 1_{N^c}) = \mu(f_2 1_{N^c})$ therefore $\mu(f_1) = \mu(f_2)$. $\square$