Week 7 Friday Lecture - Introducing L^p spaces
19 November 2021 08:58

Will start at 9.05

Remind me to record!

Assignment 4 is out later today. I'll post on the team when it's there.

About approximating functions in L^p

↳ All the material needed today + stuff that is restated on the sheet.

Due in 2^{nd} December.

We looked f a bounded, Riemann integrable function on $[a,b]$

we approximated f using sequences

$$g_n = \sum_{k=1}^{N_n} m_k^n \mathbb{1}_{[a_{k-1}^n, a_k^n)} \qquad h_n = \sum_{k=1}^{N_n} M_k^n \mathbb{1}_{[a_{k-1}^n, a_k^n)}$$

↳ increase to g ↳ decreases to h

Use dominated convergence → as f was bounded to show that

$$\lambda(g) = \lim_{n\to\infty} \lambda(g_n), \quad \lambda(h) = \lim_{n\to\infty} \lambda(h_n)$$

$$\lambda(g_n) = \ell(f, P_n) \qquad \lambda(h_n) = u(f, P_n)$$

$$\Rightarrow \lim_{n\to\infty} \ell(f,P_n) = \lim_{n\to\infty} u(f,P_n) = \text{Riemann integral of } f$$

we have that $g_n \leq f \leq h_n \quad \forall n$

so $h_n - f$ is a +ve $\qquad h_n - g_n$ is a +ve func

$h_n - f \leq h_n - g_n$ so taking limits

$0 \leq \lambda(h-f) \leq \lambda(h-g)$

$\lambda(h) = \lambda(g)$ so $\lambda(h-f) = 0$

so $h = f$ almost everywhere

so $\lambda(f) = \lambda(h)$ = Riemann integral of f

f is almost everywhere equal to a Lebesgue measurable functions so its Lebesgue measurable

g_n is an increasing sequence and we can see that $g_n \to f$ almost everywhere

so if f is measurable but not bounded then you can use Monotone convergence to see that $\lambda(f) = \lambda(g) = \lim_{n\to\infty} \lambda(g_n)$ = Riemann integral. $\blacksquare$

Classic example of a function which is Lebesgue integrable but not Riemann integrable

$f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ on $[0,1]$ with Lebesgue measure

then as $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense in $[0,1]$

For any partition P

$u(f,P) = 1$ and $\ell(f,P) = 0$

New topic L^p spaces

This is about looking at spaces of functions.

Why are we interested in looking at spaces of functions?

Main application is in PDE theory

↪ look for solutions to a PDE which lie in a particular function space

↪ Leads to looking at differential and integral operators on function spaces to lots of nice functional analysis.

L^p spaces leads naturally to Sobolev spaces most of Advanced PDEs.

Defn A normed space is a vector space V equipped with a norm $\|\cdot\|$ satisfying

- $\|v\| \in [0, \infty)$
- $\|\lambda v\| = |\lambda| \|v\|$ for $\lambda \in \mathbb{R}$
- $\|u + v\| \leq \|u\| + \|v\|$
- $\|v\| = 0$ iff $v = 0$

Defn Let $(E, \mathcal{E}, \mu)$ be a measure space then $L^p(E)$ is the space of measurable functions, $f: E \to \mathbb{R}$ with $\mu(|f|^p) < \infty$

equipped with norm $\|f\|_p = \mu(|f|^p)^{1/p}$

equipped with the norm $\|f\|_p = \mu(|f|^p)^{1/p}$

$p \in [1, \infty)$ we can't do $p<1$ as we wont have Δ-ineq.

Usually just write $L^p(E)$ but if we want to specify then σ-algebra, and the measure we write $L^p(E, \mathcal{E}, \mu)$.

If you see $L^p(\Omega)$ where $\Omega \subseteq \mathbb{R}^d$ you can assume the σ-algebra is the Borel σ-algebra and the measure is Lebesgue measure

If you are working in the space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \gamma)$ where γ is a measure which isn't Lebesgue measure then often write $L^p(\gamma)$ instead

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<u>Def^n</u> Suppose $(E, \mathcal{E}, \mu)$ is a measure space then we can look at the norm

$$\|f\|_\infty = \inf \{ c : |f| \le c \text{ almost everywhere} \}$$
$\hookrightarrow \inf(\emptyset) = \infty$

then $L^\infty(E)$ is the space of all f with $\|f\|_\infty < \infty$

<u>Important Remark</u>: Strictly speaking these are not norms on the given spaces. They are semi-norms.

$\|f\|_p = 0$ iff $f = 0$ almost everywhere

it would be a norm if $\|f\|_p = 0$ when $f = 0$ everywhere.

We actually look at a quotient, so instead of considering elements of the space to be functions we look at the space of equivalence classes of functions

$f \sim g$ if $f = g$ almost everywhere

In this setting we call $\mathcal{L}^p(E)$ the space of functions where $\|\cdot\|_p$ is a seminorm

$L^p(E)$ the space of equivalence classes with $\|\cdot\|_p$ which is now a true norm.

We need to check these spaces are actually normed spaces

Theorem For $p \in [1, \infty]$ the space $L^p(E)$ is a vector space

Pf/ We need to show $f \in L^p(E)$ $\alpha \in \mathbb{R}$ then $\alpha f \in L^p(E)$

and $f, g \in L^p(E)$ then $f + g \in L^p(E)$

$p < \infty$ $\quad \mu(|\alpha f|^p) = \mu(|\alpha|^p |f|^p) = |\alpha|^p \mu(|f|^p) < \infty$

see from this that $\|\alpha f\|_p = |\alpha| \, \|f\|_p$

$$\mu(|f+g|^p) \leq \mu\left(\left(2 \max\{|f|, |g|\}\right)^p\right)$$

$$\leq \mu\left(2^p \left(|f| + |g|\right)^p\right)$$

$)) < \infty$

$$\leq \mu\left(2^p\,(|f|+|g|)\ \right)$$

$$\leq 2^p\left(\mu(|f|^p) + \mu(|g|^p)\right) < \infty$$

If $p = \infty$ then $\|\alpha f\|_\infty = |\alpha|\,\|f\|_\infty$

if $|f| \leq c$ a.e then $|\alpha f| \leq \alpha c$ a.e.

& if $f \leq c$ on N^c and $g \leq c'$ on M^c

then $f+g \leq c+c'$ on $(N \cup M)^c$

so if N, M are null sets $N \cup M$ is also a null set so $\|f+g\|_\infty \leq c+c' < \infty$

Can we show that $\|\cdot\|_p$ is a norm?

- $\|f\|_p = 0$ iff $f = 0$ ← we've already discusse

  change our notion of $=$ to $=$ a.e.

- $\|\alpha f\|_p = |\alpha|\,\|f\|_p$ we showed in the vector space proof.

- $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ ← This is harder to prove

It's called Minkowski's inequality $p \in [1,\infty)$

and we will prove it next week

- $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

First lets prove a useful Lemma

$\|f\|_\infty = C$ then $\exists$ a null set N s.t.

$|f| \leq C$ on N^c

$\|f\|_\infty \leq \inf \{c : |f| \leq C \text{ a.e.}\}$

want to show this inf is achieved

$\forall n \quad \exists \; N_n$ s.t. $|f| \leq C + \frac{1}{n}$ on N_n^c

then take $N = \bigcup_n N_n$

then $N^c = \bigcap_n N_n^c$ so on N^c $|f| \leq C + \frac{1}{n}$ $\forall n$

so $|f| \leq C$ on N^c and N is a null set as $\mu(N) \leq \sum \mu(N_n) = 0$

so if $\|f\|_\infty = C$ and N is a null set

s.t. $|f| \leq C$ on N^c

and $\|g\|_\infty = C'$ and M is a null set

s.t. $|g| \leq C'$ on M^c then look at

$|f(x) + g(x)|$ on $(N \cup M)^c = N^c \cap M^c$

then $|f(x) + g(x)| \leq |f| + |g| \leq C' + C$

so $\quad \|f+g\|_\infty = \sup_{(N\cup M)^c} |f+g| \leq C' + C = \|f\|_\infty + \|g\|_\infty$