Here is an issue: d-systems work in the context of finite measure spaces.

(We use facts like $\mu(B - A) = \mu(B) - \mu(A)$ for example.)

We would like to prove a Fubini theorem for $\mathbb{R}^2$.

The solution is to introduce <u>σ-finiteness</u>.

Given $(X, \mathcal{A}, \mu)$ we say μ is <u>σ-finite</u> if there is a sequence of sets $A_1, A_2 \ldots$ with $\mu(A_n) < \infty$ and $X = \bigcup_{n=1}^{\infty} A_n$. (Cohn p. 9)

$$\mathbb{R} = \bigcup_n [n, n+1) \longleftarrow$$ A_n are disjoint

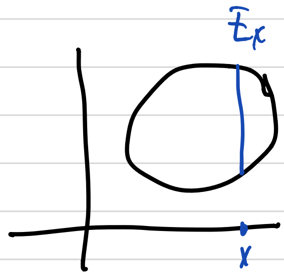
A_n are nested

$$\mathbb{R} = \bigcup_n [-n, n]$$

We need to upgrade our results to the σ-finite case.

We proved the following formula in the finite measure case:

Prop. 5.1.3(a) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function. and $y \mapsto \mu(E_y)$ is $\mathcal{B}$-measurable.



Here is a σ-finite version:

Prop. 5.1.3(b) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces.
If $E \subset X \times Y$, $E \in \mathcal{A} \times \mathcal{B}$ then $x \mapsto \nu(E_x)$ is an $\mathcal{A}$-measurable function and $y \mapsto \mu(E^y)$ is $\mathcal{B}$-measurable.

We have proved the following result in the finite measure case:

Cor. 1.6.3 (Agreement on generators). Let $(\mathbb{X}, \mathcal{A})$ be a measurable space and let $\mathcal{C}$ be a π-system s.t. $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν that satisfy $\mu(\mathbb{X}) = \nu(\mathbb{X}) < \infty$ and $\mu(C) = \nu(C)$ for $C \in \mathcal{C}$ then $\mu = \nu$.

Here is a version that works in a σ-finite setting:

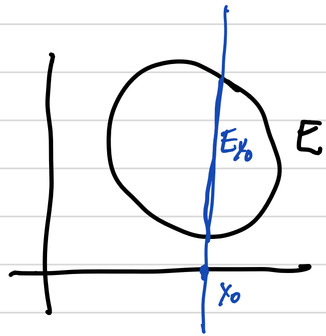
Cor. 1.6.4. Let $(\mathfrak{X}, \mathcal{A})$ be a measurable space, let $\mathcal{C}$ be a π-system on $\mathfrak{X}$ s.t. $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν are measures on $(\mathfrak{X}, \mathcal{A})$ that agree on $\mathcal{C}$ and if there is an increasing sequence of sets in $\mathcal{C}$ with $\mu(C_n) = \nu(C_n) < \infty$ so that $\bigcup_n C_n = \mathfrak{X}$ then $\mu = \nu$.

Note that the increasing sequence C_n is required to be in $\mathcal{C}$.

The method of proof is similar in both cases. If we have a measure μ and a sequence of sets D_n we introduce $\mu_n(E) = \mu(E \cap D_n)$. These are finite measures and we apply the finite measure result.

Depending on the circumstance it may be convenient to choose our sequence of sets to be <u>nested</u> or <u>disjoint</u>.

How does the possibility that our measures might be infinite affect the statements of our results?



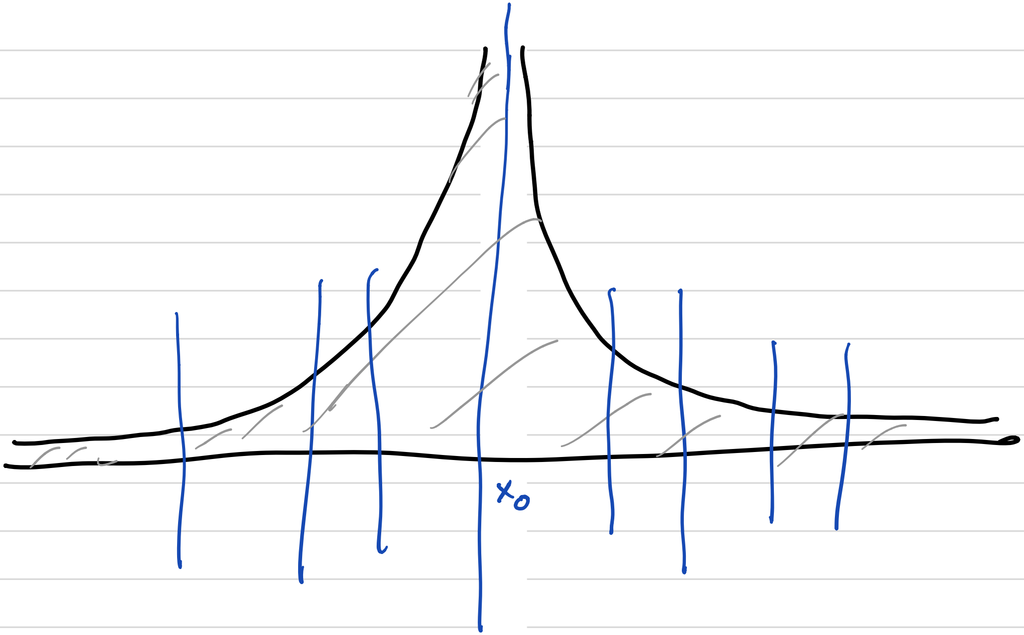
"Slice Theorem for product measure"

Say that $E \in \mathcal{A} \times \mathcal{B}$ then

$$(\mu \times \nu)(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

$(\mu \times \nu)(E)$ might be $+\infty$. Even if $\mu \times \nu(E) < \infty$,

for a given x, $\nu(E_x)$ might be $+\infty$.



$$\int f(x_0, y)\, d\nu(y) = \infty.$$

"Fubini Thm for product measure"

If f is $\mathcal{A}\times\mathcal{B}$ measurable then

$$\int f \, d(\mu\times\nu) = \int\left(\int f(x,y)\, d\mu(x)\right) d\nu(y)$$

$$= \int\left(\int f(x,y)\, d\nu(y)\right) d\mu(x).$$

Recall that $\int f$ has a value if $\int f^+ < \infty$ or $\int f^- < \infty$.

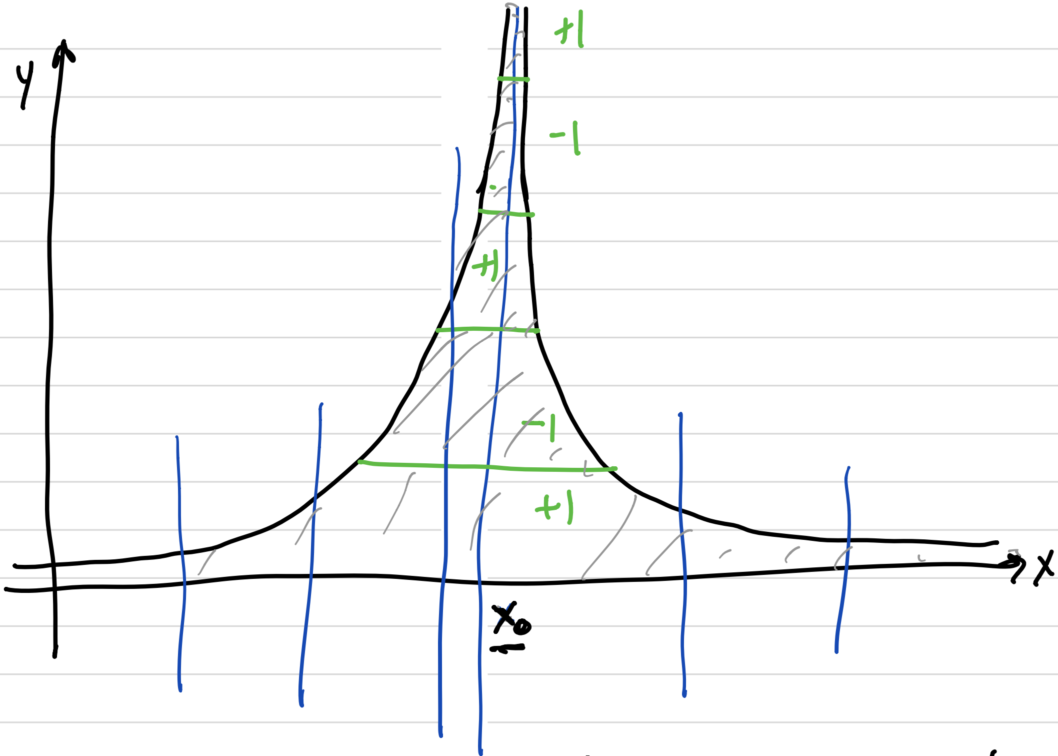
Otherwise we are trying to assign a meaning to $(+\infty)-(+\infty)$ which we don't know how to do.

We can give $\int f \, d(\mu \times \nu)$ meaning by

① assuming $f \geq 0$ or $\left(\int f^- < \infty \right)$

② assuming f is integrable. $\left(\int f^+ < \infty \text{ and } \int f^- < \infty \right)$

Even if f is integrable it may still be the case that $y \mapsto \int f(x_0, y) \, d\nu(y)$ may fail to be integrable for some values of x_0. In order to state our theorem carefully we need to throw out these values.



$f(x,y)$ is $\mu\times\nu$ integrable but $\int f(x_0, y)\, d\nu(y)$ does <u>not</u> exist.

In the end we wind up with 1 version of the "Slice Theorem" and 2 versions of "Fubini's Theorem": f non-negative, f integrable.

"Slice Theorem":

Thm. 5.1.4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Then there is a unique measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$ such that

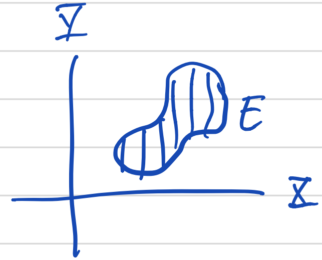
$$(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$$

for each measurable rectangle $A \times B$.

Furthermore ...

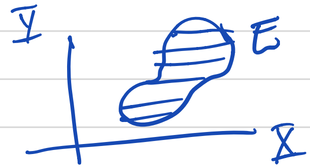
Furthermore for $E \in \mathcal{A} \times \mathcal{B}$

$$\mu \times \nu (E) = \int_X \nu(E_x)\, d\mu(x)$$



"vertical slice formula"

and $$\mu \times \nu (E) = \int_Y \mu(E^y)\, d\nu(y).$$



"horizontal slice formula"

Def.

The measure $\mu \times \nu$ is the product of μ and ν.

Note that the statement gives us two potential formulas for $\mu \times \nu$.

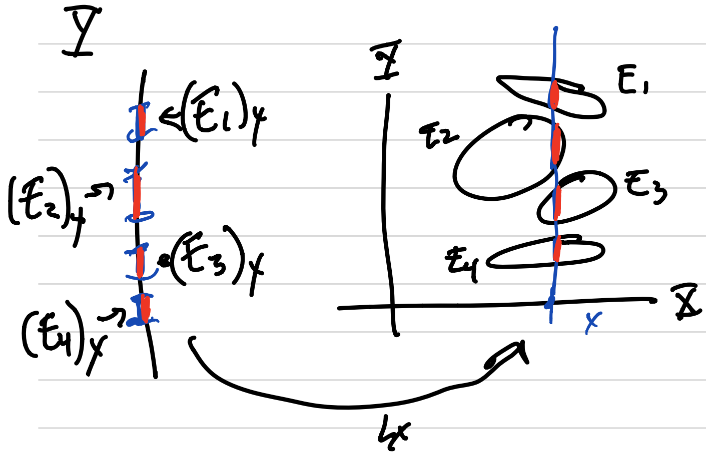
For example:

Say $(\mu \times \nu)_1(E) = \int \nu(E_x)\, d\mu(x)$. (vertical slice formula)

We check that $(\mu \times \nu)_1$ is a measure.

Countable additivity.

Let E_n be disjoint sets.



$$(\mu\times\nu)\left(\bigcup_n E_n\right) = \int \nu\left(\underbrace{\left(\bigcup_n E_n\right)_x}\right) d\mu(x)$$

$$= \int \nu\left(\underbrace{\bigcup_n (E_n)_x}\right) d\mu(x)$$

$$L_x(y) = (x,y)$$

$$L_x^{-1}(E) = E_x.$$

$$L_x^{-1}\left(\bigcup_n E_n\right) = \bigcup_n L_x^{-1}(E_n)$$

$$\underbrace{\left(\bigcup_n E_n\right)_x} = \underbrace{\bigcup_n (E_n)_x}.$$

$$\boxed{= \int \nu\Big(\bigcup_n (E_n)_x\Big)\, d\mu(x)}$$

ν is a measure on Y. $(E_n)_x$ are disjoint subsets of Y. We use the countable additivity of ν.

$$= \int \sum_n \nu\big((E_n)_x\big)\, d\mu(x)$$

Beppo-Levy Thm. 2.4.2

For a sequence of non-neg. fcns. can interchange order of summation & integration.

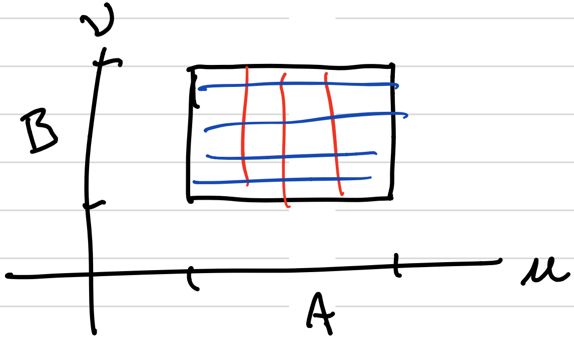
$$= \sum_n \int \nu\big((E_n)_x\big)\, d\mu(x)$$

Def of $(\mu\times\nu)_1$

$$= \sum_n (\mu\times\nu)_1(E_n).$$

So $(\mu\times\nu)_1\Big(\bigcup_n E_n\Big) = \sum_n (\mu\times\nu)_1(E_n)$.

Why is $(\mu\times\nu)_1 = (\mu\times\nu)_2$? Check that $(\mu\times\nu)_1 = (\mu\times\nu)_2 = (\mu\times\nu)$ on measurable rectangles. Apply Cor. 1.6.4 where $\mathcal{C}$ is the collection of measurable rectangles.



$\underline{\mu(A)\cdot\nu(B)}$

The non-negative Fubini Thm:

Prop. 5.2.1 Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f : X \times Y \longrightarrow [0, +\infty]$ be an $\mathcal{A} \times \mathcal{B}$ measurable function. Then

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x)$$

$$= \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y).$$

Proof. When $f = \chi_E$ this is just the slice theorem.

Extend to $f = \sum_j a_j \chi_{E_j}$ by linearity.

Extend to measurable f by observing f is the pointwise limit of increasing simple functions f_n and take limits appropriately.

Thm. 5.2.2 (Fubini's Thm.) Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $f: X \times Y \to [-\infty, +\infty]$ be an $\mathcal{A} \times \mathcal{B}$ measurable function which is $\mu \times \nu$ integrable. Then:

(a) Define

$$I_f(x) = \begin{cases} \int_Y f_x \, d\nu & \text{if } f_x \text{ is } \nu\text{-integrable} \\ 0 & \text{otherwise} \end{cases}$$

$$J_f(y) = \begin{cases} \int_X f^y \, d\mu & \text{if } f^y \text{ is } \mu\text{-integrable} \\ 0 & \text{otherwise} \end{cases}$$

then $I_f \in \mathcal{L}^1(X, \mu)$, $J_f \in \mathcal{L}^1(Y, \nu)$.

(b) $\int_{X \times Y} f \, d(\mu \times \nu) = \int_X I_f \, d\mu = \int_Y J_f \, d\nu$.

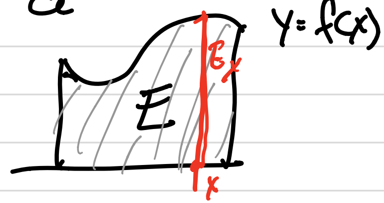
To prove 5.2.2 we apply 5.2.1 to the non-negative functions f^+ and f^-.

Example: As measures on $\mathbb{R}^2$: $\lambda_2 = \lambda_1 \times \lambda_1$.

$\lambda_1 \times \lambda_1$ has the property that for measurable rectangles $A \times B \subset \mathbb{R}^2$ $\lambda_2(A \times B) = \lambda_1(A) \cdot \lambda_1(B)$.
In particular for actual rectangles $[a,b] \times [c,d]$ $\lambda_1 \times \lambda_1$ gives the same answer as λ^2.

In 1.4.3 we showed that any measure which agrees with Lebesgue measure on coordinate rectangles is Lebesgue measure. So $\lambda_1 \times \lambda_1$ is equal to Lebesgue measure.

Example 5.3.1. Let $f: \mathbb{R} \to [0, +\infty]$ be a non-negative function.



Let $E = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le f(x)\}$.

So E is the region under the graph of f.

We claim that the area of E is equal to the integral of f. This follows from Prop. 5.1.4 and the fact that $\lambda_2 = \lambda_1 \times \lambda_1$.

$$\lambda_2(E) = (\lambda_1 \times \lambda_1)(E) = \int_X \lambda_1(E_x)\, d\lambda(x) = \int_X f(x)\, d\lambda(x)$$