Measure Theory: Exercises (not for credit)

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This is a shorter sheet because last week was longer and you have the assignment as well. I think the most useful question is 3.

Question 1. Let A_n be a sequence of measurable sets. Show that 1_{A_n} converges to 0 in measure if and only if $\mu(A_n) \to 0$. Furthermore show that 1_{A_n} converges almost everywhere if and only if $\mu(\bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k) = 0$.

Answer: This is mainly an exercise in understanding all the definitions. The first result is almost by definition. 1_{A_n} tends to 0 in measure if for every $\epsilon > 0$ the measure of $\{x : 1_{A_n} > \epsilon\}$ tends to 0. When $\epsilon > 1$ this set is empty so the result is true. When $\epsilon \leq 1$ we have $\{x : 1_{A_n} > \epsilon\} = A_n$ so convergence happens in measure iff $\mu(A_n) \to 0$.

The second part is a bit more subtle. 1_{A_n} tends to 0 at x if x does not appear in infinitel many A_n. The set of points which appears in infinitely many A_n is $\bigcap_n \bigcup_{m \geq n} A_m$ so the 1_{A_n} converges in measure if and only if $\mu(\bigcap_n \bigcup_{m\geq n} A_m) = 0$. □

Question 2. Let $(f_n)_{n\geq 1}, f, g$ be Borel measurable functions from $\mathbb{R} \to \mathbb{R}$. Suppose further that g is continuous. Show that if $f_n \to f$ almost everywhere then $g \circ f_n \to g \circ f$ almost everywhere. Can the conclusion fail if g is only continuous almost everywhere.

Answer: As f_n converges to f almost everywhere there is a set A with $f_n(x) \to f(x)$ for every $x \in A$ and $\lambda(A^c) = 0$. Now if $x \in A$, since g is continuous $g(f_n(x)) \to g(f(x))$ so $g \circ f_n \to g \circ f$ almost everywhere.

Now let $f_n(x) = x^n$ for $x \in (0, 1)$ and 1 everywhere else. Then f_n converges to 0 almost everywhere. Let $g(x) = 1$ when $x \neq 0$ but $g(0) = 0$. Then $g(f_n(x)) = 1$ for every $x \neq 0$ but $g(f(x)) = 0$ for every x so $g \circ f_n$ does not converge to $g \circ f$ almost everywhere. □

Question 3. Suppose that f is a measurable function from $(E, \mathcal{E}, \mu)$ to $(F, \mathcal{F})$ show that the image measure defined by $\nu(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{F}$ is indeed a measure.

Answer: We need to check that ν satisfies all the axioms to be a measure. $\nu(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$. We also have that $\nu \geq 0$ since $\mu \geq 0$. We now need to show that μ is countably additive. Suppose A_n is a sequence of disjoint sets in $\mathcal{F}$ then $\mu(\bigcup_n A_n) = \mu(f^{-1}(\bigcup_n A_n)) = \mu(\bigcup_n f^{-1}(A_n))$ now the fact that the A_n are disjoint imples that $f^{-1}(A_n)$ are disjoint (x can't end up in two disjoint sets). So by countable additivity of μ we have $\mu(\bigcup_n f^{-1}(A_n)) = \sum_n \mu(f^{-1}(A_n)) = \sum_n \nu(A_n)$. Therefore ν is countably additive. We have then shown that ν is a measure. □

Question 4. Suppose that $(E, \mathcal{E}, \mu)$ is a σ-finite measure space and f_n is a sequence of real valued measurable functions on E. Suppose that $f_n \to f$ almost everywhere. Show that there exists a sequence of sets A_n and a set B such that $E = \bigcup_n A_n \cup B$ and $\mu(B) = 0$ and f_n converges uniformly on each A_n.

Answer: I think this is a hard questions. We need to use Egoroff's theorem. First by σ-finiteness there exists a sequence of sets E_k such that $\mu(E_k) < \infty$ for every k and $E = \bigcup_k E_k$. Then let us fix ϵ we are going to show an intermediate claim: Given $\epsilon > 0$ there exists a sequence of sets $A_{k,\epsilon}$ such that f_n converges uniformly on each $A_{k,\epsilon}$ and $\mu((\bigcup_k A_{k,\epsilon})^c) < \epsilon$. In order to do this we apply Egoroff's theorem to each E_k to find a set $A_k \subseteq E_k$ with $\mu(E_k \setminus A_k) < \epsilon 2^{-k}$. Then we have that f_n converges uniformly on each $A_{k,\epsilon}$ and $\mu(E \setminus \bigcup_k A_{k,\epsilon}) = \mu(\bigcup_k (E_k \setminus A_{k,\epsilon}) \leq \sum_k \mu(E_k \setminus A_{k,\epsilon}) \leq \epsilon \sum_k 2^{-k} = \epsilon$. We have then proved the claim for each ϵ.

Now we take the sets $A_{k,2^{-m}}$ these form a countable collection what we really need to do is reorder them but lets show the theorem in an explicit way. Let $A_n = \bigcup_{k,m \leq n} A_{k,2^{-m}}$ then since f_n converges uniformly on each $A_{k,2^{-m}}$ it converges uniformly on any finite union of such sets so on each A_n. So then look at $\mu(E \setminus \bigcup_n A_n) \leq \mu(E \setminus \bigcup_k A_{k,2^{-m}})$ for any m by monotonicity of μ. Therefore $\mu(E \setminus \bigcup_n A_n) \leq 2^{-m}$ for any m, hence $\mu(E \setminus \bigcup_n A_n) = 0$. Therefore set $B = E \setminus \bigcup_n A_n)$ and we have proved the result. $\square$