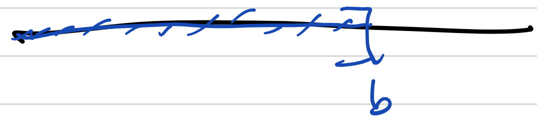
Thm. Every Borel set in $\mathbb{R}^d$ is λ_d^* measurable.

Proof. We start by showing that coordinate half-spaces are measurable:



$$H = H_{j,b} = \{ (x_1 \dots x_d) \in \mathbb{R}^d : x_j \le b \}.$$

This proof is similar to the proof for left infinite intervals in the video "A measurable set".

(I will drop the d in λ_d^*.)

Need to show that for any test set A:

$$\lambda^*(A) = \lambda^*(A \cap H) + \lambda^*(A \cap H^c).$$

The 1-dim argument shows that we can break up an open cover of A into open covers of $A \cap \mathring{H}$ and $A \cap H^c$ where $\mathring{H}$ is the interior of H. This gives

$$\lambda^*(A) = \lambda^*(A \cap \mathring{H}) + \lambda^*(A \cap H^c).$$

Let $Z = \{(x_1 \dots x_d) : x_j = b\}$ so that $H = \mathring{H} \cup Z$. Z has λ_d^* measure 0.

Now $\lambda^*(A \cap H) = \lambda^*((A \cap H^\circ) \cup (A \cap Z))$

Since $\lambda^*(A \cap Z) = 0$, by using the arguments from the homework (or proving it directly) we have:

$$\lambda^*(A \cap H) = \lambda^*(A \cap \overset{\circ}{H})$$

so

$$\lambda^*(A) = \lambda^*(A \cap H) + \lambda^*(A \cap H^c)$$

and H is measurable.

It follows that by taking intersections of $H_{j,a}^c$ and $H_{j,b}$ that

$$\{(x_1 \dots x_d) : a < x_j \leq b\}$$

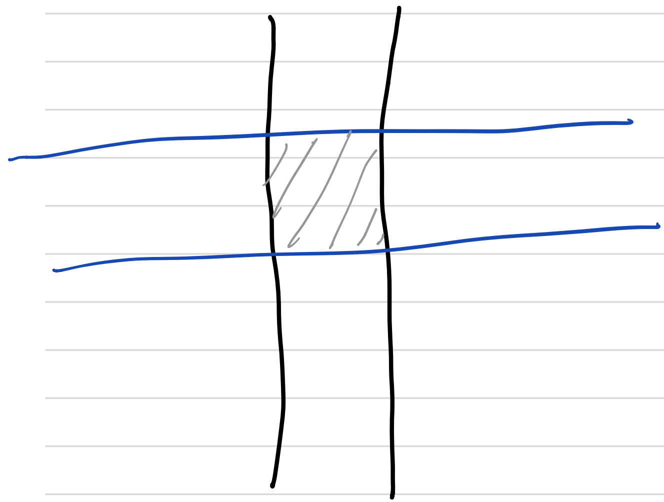
slab

are measurable. The homework shows that sets $\{(x_1 \dots x_d) : a < x_j < b\}$ are measurable using again that hyperplanes have measure 0.

By taking finite intersections we see that coordinate rectangles:

$$\{(x_1 \dots x_d) : a_1 < x_1 < b_1 , a_2 < x_2 < b_2 \dots a_d < x_d < b_d\}$$

are measurable.



We finished the 1-dim proof in the lecture 3 video by appealing to the fact that every open set in $\mathbb{R}$ is the union of countably many disjoint open intervals.

In fact we do not need the disjointness.

Lemma. Every open set in $\mathbb{R}^d$ is the union of a countable collection of open coordinate rectangles.

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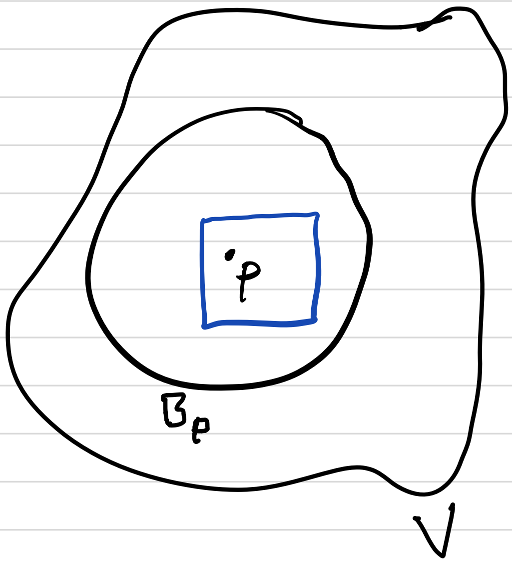
Proof. Consider the collection of coordinate rectangles: $\{(x_1 \dots x_d) : a_j < x_j < b_j\}$ where a_j and b_j are rational.

Let V be an open set and let $\mathcal{U}$ be the collection of rational coordinate rectangles contained in V. Clearly $\bigcup_{u \in \mathcal{U}} u \subset V$.

We just need to check that every $p \in V$ is contained in some rational coordinate rectangle.

This follows from the fact that there is a ball B containing p and contained in V.



Thus open sets are measurable.
Since the collection of measurable sets is a σ-algebra it follows that it contains the smallest σ-algebra containing the open sets and this is the σ-algebra of Borel sets.