In the first video we showed that the spaces L^p are normed vector spaces with norms $\|f\|_p$.

A vector space V with a norm $\|\cdot\|_p$ is also a metric space where the metric is given by

$$d(f,g) = \|f-g\|_p.$$

Recall that a sequence of points x_n is a Cauchy sequence if for every $\varepsilon > 0$ there is an N_ε so that $m, n \geq N_\varepsilon$ implies that $d(x_n, x_m) \leq \varepsilon$.

A metric space is complete if every Cauchy sequence converges.

A complete normed vector space is also called a Banach space.

Given a vector space V with a norm $\|\cdot\|$ a <u>series</u> $\sum_{k=1}^{\infty} v_k$ of vectors v_k in V is <u>convergent</u>

if the limit of the sequence of partial sums, $\sum_{k=1}^{n} v_k$, exists.

A <u>series</u> is <u>absolutely convergent</u> if the series of real numbers

$$\sum_{k=1}^{\infty} \|v_k\|$$

converges.

Prop. 3.2.5. Let V be a normed vector space. Then V is complete if and only if every absolutely convergent series in V is convergent.

Proof. Assume V is complete and let $\sum_{k=1}^{\infty} v_k$ be an absolutely convergent series.

Let s_n be the partial sums of the series $\sum_{k=1}^{\infty} v_k$ and let t_n be the partial sums of the series $\sum_{k=1}^{\infty} \|v_k\|$.

If $m<n$ then

$$\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} v_k \right\| \le \sum_{k=m+1}^{n} \|v_k\| = t_n - t_m.$$

Since $\sum_{k=1}^{\infty} \|v_k\|$ is convergent the sequence t_n is convergent.

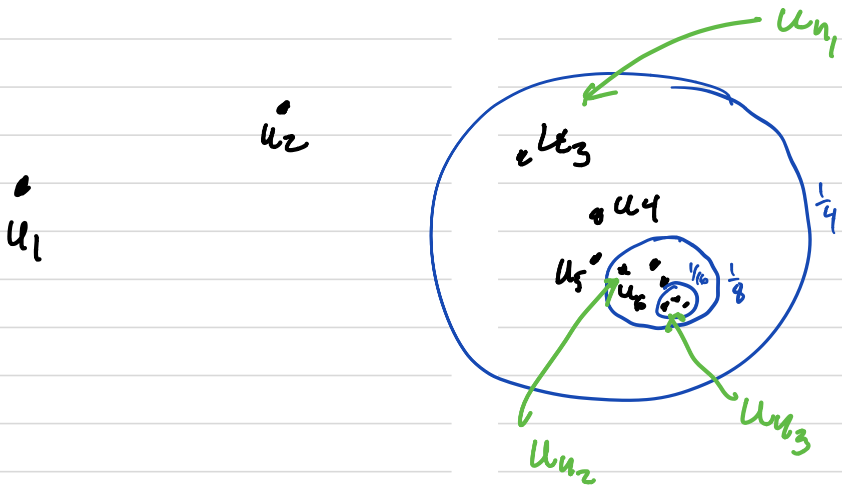
A convergent sequence of real numbers is a Cauchy sequence so t_n is a Cauchy sequence.

This implies that s_n is a Cauchy sequence. Since V is complete s_n converges so the series $\sum_{k=1}^{\infty} v_k$ converges since the sequence of partial sums converges.

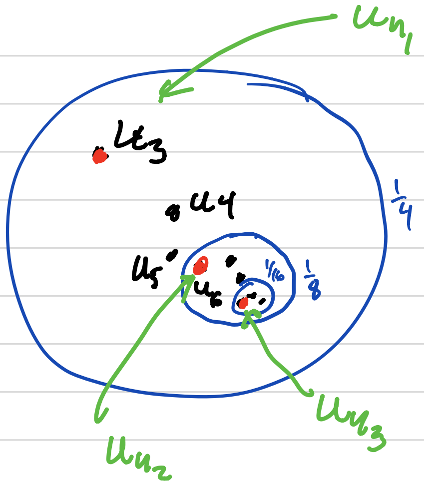
Now suppose that every absolutely convergent series in V is convergent.

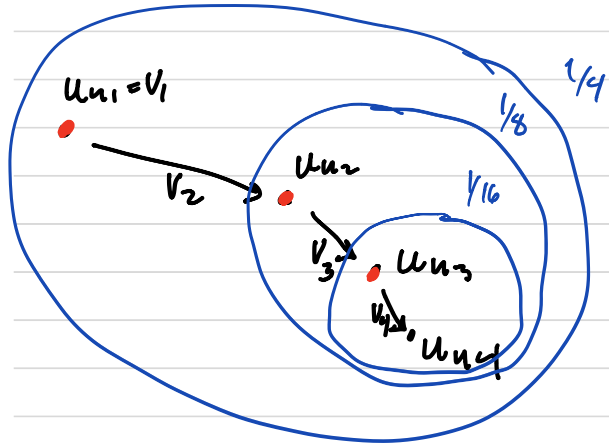
Let u_n be a Cauchy sequence.
We want to show that u_n converges.

We will construct a subsequence u_{n_k} of u_n that converges at a definite rate.



For each k we can find an n_k so that $n, m \ge n_k$ implies $\|u_n - u_m\| \le \frac{1}{2^{k+1}}$.
We may assume $n_1 < n_2 < n_3 < \dots$





Let $V_1 = u_{n_1}$

$V_2 = u_{n_2} - u_{n_1}$

$V_3 = u_{n_3} - u_{n_2}$

$\vdots$

$V_k = u_{n_k} - u_{n_{k-1}}$.

so $V_1 = u_{n_1}$

$V_1 + V_2 = u_{n_2}$

$V_1 + V_2 + V_3 = u_{n_3}$

$\vdots$

$$u_{n_k} = \sum_{j=1}^{k} V_k$$

We have $\|V_k\| = \|u_{n_k} - u_{n_{k-1}}\| \le \frac{1}{2^{k+1}}$.

By construction u_{n_k} is the sequence of partial sums of the series $\sum_{k=1}^{\infty} v_k$.

Since $\|v_k\| \le \frac{1}{2^k}$ the series $\sum_{k=1}^{\infty} v_k$ is absolutely convergent and hence convergent by assumption.

This tells us that the sequence of partial sums u_{n_k} converges to some u.

We show that the original Cauchy sequence $u_1, u_2, u_3 \ldots$ converges to u.

Given ε choose k so that $\|u - u_{n_k}\| < \frac{\varepsilon}{2}$ and $\|u_n - u_{n_k}\| \leq \frac{\varepsilon}{2}$ for $n \geq n_k$.

For $n \geq n_k$ we have

$$\|u - u_n\| \leq \|u - u_{n_k}\| + \|u_{n_k} - u_n\|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So u_n converges to u and $\sum_{k=1}^{\infty} v_k = u$.

We want to prove that L^p is complete. The argument makes essential use of extended real valued functions and we need a result from Ch. 2 which we did not prove at the time.
We prove it now.

Cor. 2.3.14. Let f be a $[-\infty, +\infty]$ valued integrable function on X. Then $|f(x)| < +\infty$ holds a.e.

Proof. This follows from Markov's inequality (2.3.10).

$$\mu(\{x \in X : |f(x)| \geq n\}) \leq \frac{1}{n} \int |f| \, d\mu.$$

So $\mu(\{x \in X : |f(x)| = +\infty\}) \leq \mu(\{x \in X : |f(x)| \geq n\})$

$$\leq \frac{1}{n} \int |f| \, d\mu.$$

Since $\int |f| \, d\mu < \infty$ and this equation holds for every n we have $\mu(\{x \in X : |f(x)| = +\infty\}) = 0$.

Comment on the Dominated Convergence Thm

There is a gap in the posted proof which Cor. 2.3.14 helps to resolve an issue pointed out by Scott Fowler-Wright.

The theorem deals with a $[-\infty, +\infty]$ valued function g and a sequence f_n with $|f_n| \leq g$.

In the proof we consider new sequences of functions $g - f_n$ and $f_n + g$ and their integrals.

We have not defined addition of $[-\infty, +\infty]$ valued functions since we don't assign a value to $(+\infty)+(-\infty)$ and we do not have a linearity theorem for the integral of extended real valued functions.

On the other hand since g is integrable we know that $\{x : |g(x)| = +\infty\}$ has measure 0. If we remove this set from X we do not change any of the integrals of g or f_n by Prop. 2.3.9.

An additional remark:

In the DCT it suffices to assume $|f_n(x)| \leq g(x)$ for a.e. x again using Prop. 2.3.9.

Thm. 3.4.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let p satisfy $1 \le p < \infty$. Then $L^p(X, \mathcal{A}, \mu)$ is complete with respect to the norm $\|\cdot\|_p$.

Proof. Let f_k be a sequence of functions that belong to $\mathcal{L}^p$ and satisfy $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$ (which is to say that $\sum_{k=1}^{\infty} f_k$ is absolutely convergent).

We want to show that the series $\sum_{k=1}^{\infty} f_k$ converges.

Define $g: X \to [0, +\infty]$ by

$$g(x) = \left(\sum_{k=1}^{\infty} |f_k(x)| \right)^p.$$

We want to use g in an application of the DCT so we need to show g is integrable

In order to do that we will use the MCT.

Set $g_n(x) = \left(\sum_{k=1}^{n} |f_k(x)| \right)^p.$

$$g_n(x) = \left(\sum_{k=1}^{n} |f_k(x)| \right)^p \qquad g(x) = \left(\sum_{k=1}^{\infty} |f_k(x)| \right)^p.$$

g is the pointwise limit of the g_n.

The sequence g_n is non-decreasing.

By the Monotone Convergence Theorem:

$$\int g(x)\, dx = \lim_{n\to\infty} \int g_n(x)\, d\mu.$$

We want to estimate $\int g_n \, d\mu$

$$\left(\int g_n \, d\mu\right)^{1/p} = \left(\int \left(\sum_{k=1}^{n} |f_k|\right)^p d\mu\right)^{1/p}$$

$$= \left\| \sum_{k=1}^{n} |f_k| \right\|_p$$

$$\leq \sum_{k=1}^{n} \|f_k\|_p \qquad \text{Minkowski}$$

$$\leq \sum_{k=1}^{\infty} \|f_k\|_p < \infty. \qquad \text{Abs. conv. hyp.}$$

So $\left(\int g_n \, d\mu\right) \le \left(\sum_{k=1}^{\infty} \|f_k\|_p\right)^p < \infty.$

Thus $\int g \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu \le \left(\sum_{k=1}^{\infty} \|f_k\|_p\right)^p < \infty.$

It follows from Corollary 2.3.14 that $g(x)$ is finite a.e.

Define

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if } g(x) < +\infty \\ 0 & \text{otherwise} \end{cases}$$

on a set of measure 0

Since $|f(x)|^p \leq g(x)$ it follows that

$\int |f|^p d\mu \leq \int g \, d\mu < \infty$ and (since f is $\mathbb{R}$ valued) $f \in \mathcal{L}^p$.

f is a good candidate to be $\sum_{k=1}^{\infty} f_k$.

To show this we show that the sequence of partial sums $\sum_{k=1}^{n} f_k$ converges to f with respect to the L^p distance.

$$\left\| \sum_{k=1}^{n} f_k - f \right\|_p \longrightarrow 0$$

Specifically we need to show that

$$\lim_{n\to\infty}\left(\int\left(\sum_{k=1}^{n} f_k - f\right)^p d\mu\right)^{1/p} = 0$$

or

$$\lim_{n\to\infty}\int\left(\sum_{k=1}^{n} f_k - f\right)^p d\mu = 0$$

We want to do this by applying the Dominated convergence theorem to the functions:

$$\left(\sum_{k=1}^{n} f_k - f\right)^p$$

We know that these functions converge to 0 a.e.

We also want to check that they are dominated by the integrable function g:

$$\left|\sum_{k=1}^{n} f_k - \sum_{k=1}^{\infty} f_k\right|^p = \left|\sum_{k=n+1}^{\infty} f_k\right|^p \leq \left(\sum_{k=n+1}^{\infty} |f_k|\right)^p \leq \left(\sum_{k=1}^{\infty} |f_k|\right)^p = g.$$

By the DCT a sequence of functions dominated by an integrable function which converges pointwise to 0 a.e. has the property that its integrals converge to 0.

$$\lim_{n\to\infty} \int \left(\sum_{k=1}^{n} f_k - f \right)^p d\mu = \int 0 \, d\mu = 0$$

This completes the proof.