Lecture 7 - Convergence of Measurable functions

26 October 2021 09:00

Remind me to record! Starts at 9.05

Please fill out the initial feedback questionnaire

$f_n \to f$ pointwise if $f_n(x) \to f(x)$ for every x

$f_n \to f$ uniformly if $\sup_x |f_n(x) - f(x)| \to 0$

Def^n Let $(E, \mathcal{E}, \mu)$ be a measure space a property holds almost everywhere if the set where it doesn't hold is null. (has measure 0).

Def^n so $(E, \mathcal{E}, \mu)$ a measure space $(f_n)_{n \geq 1}$, $f: E \to \mathbb{R}$

$f_n \to f$ almost everywhere if $\mu(\{x : f_n(x) \not\to f(x)\}) = 0$

Def^n $(E, \mathcal{E}, \mu)$ a measure space $(f_n)_{n \geq 1}$, $f : E \to \mathbb{R}$

We say $f_n \to f$ in measure if for every $\varepsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0 \text{ as } n \to \infty.$$

Examples:

1. $([0,1], \mathcal{B}([0,1]), \lambda)$

$f_n(x) = x^n$ so $f_n(x)$ converges to 0 almost everywhere and in measure but not pointwise or uniformly.

$f_n(1) \not\to 0$.

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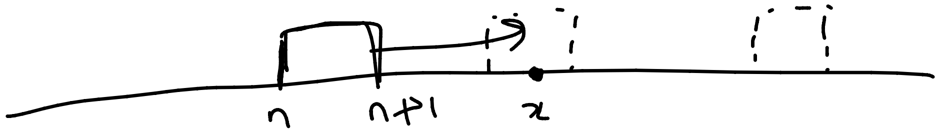
$$\{x : |f_n(x)| > \varepsilon\} = \{x : x^n > \varepsilon\} = \{x : x > \varepsilon^{1/n}\}$$

$$\mu(\{x : |f_n(x)| > \varepsilon\}) = (1 - \varepsilon^{1/n}) \to 0 \quad \text{as } n \to \infty$$

so it <u>does</u> converge in measure

2. $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ $\quad f_n(x) = \mathbb{1}_{[n,n+1]}(x)$

then $f_n(x) \to 0$ (everywhere) pointwise and almost everywhere but not uniformly and not in measure.



$$\mu(\{x : |f_n(x)| > \varepsilon\}) = \mu([n, n+1]) = 1 \not\to 0$$
as $n \to \infty$

take $(\varepsilon < 1)$

3. $f_1 = \mathbb{1}_{[0,\frac{1}{2})}$ $\quad f_2 = \mathbb{1}_{[1/2,1)}$ $\quad f_3 = \mathbb{1}_{[0,1/4)}$, $f_4 = \mathbb{1}_{[1/4,\frac{1}{2})}$

$f_5 = \mathbb{1}_{[1/2,3/4)}$ $\quad f_5 = \mathbb{1}_{[3/4,1)}$, $f_6 = \mathbb{1}_{[0,1/8)}$, $f_7 = \mathbb{1}_{[1/8,1/4)}$

$f_n \to 0$ in measure but not almost everywhere or uniformly.

$(\varepsilon < 1)$

if $n \geq 2 + 4 + \dots + 2^k$ then $\mu(\{x : |f_n(x)| > \varepsilon\}) \leq 2^{-k}$

so $\mu(\{x : |f_n(x)| > \varepsilon\}) \to 0$ as $n \to \infty$

f_n doesn't converge to 0 for any $x \in [0,1)$. Fix any x

then $f_n(x) = 1$ for infinitely many n so it can't converge to 0.

We have notions of stronger and weaker types of convergence. A notion of convergence is stronger than another if it implies the second type of convergence.

Unif convergence $\Rightarrow$ pointwise convergence

stronger   weaker.

Neither of convergence a.e. or convergence in measure implies the other.

uniform conv $\Rightarrow$ pointwise conv $\Rightarrow$ a.e. conv

unif conv $\Rightarrow$ conv in measure.

<u>Defn</u> Let $(A_n)_{n \geq 1}$ be a sequence of meas sets

then

$$\limsup_n A_n = \bigcap_n \bigcup_{m \geq n} A_m = \{A_m \text{ infintely often}\}$$

$x \in$ ... x in infinitely many

$$\liminf_n A_n = \bigcup_n \bigcap_{m \geq n} A_m = \{A_m \text{ eventually}\}$$

$x \in$ ... $x \in$ every A_m for m suff large

Borel Cantelli I $(E, \mathcal{E}, \mu)$ be a measure space

Then $(A_n)_{n \geq 1} \subseteq \mathcal{E}$ if $\sum_n \mu(A_n) < \infty$

then $\mu(A_n \text{ i.o.}) = 0$

inf often

inf often

Pf/ $\mu(A_n \text{ i.o}) = \mu\left(\bigcap_n \bigcup_{m\geq n} A_m\right) \leq \mu\left(\bigcup_{m\geq n} A_m\right) \leq \sum_{m\geq n} \mu(A_m)$ for any n

$\to 0$ as $n \to \infty$

Borel- Cantelli II

So let $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob space

$(A_n)_{n\geq 1} \subseteq \mathcal{F}$ and $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$ (pairwise indep.)

then if $\sum_n \mathbb{P}(A_n) = \infty$ then

$$\mathbb{P}(A_n \text{ i.o}) = 1.$$

Pf/ $\mathbb{P}(A_i^c \cap A_j^c) = \mathbb{P}(A_i^c)\mathbb{P}(A_j^c)$ (check this)

$\underline{\underline{1-a \leq e^{-a}}}$ let $a_n = \mathbb{P}(A_n)$

$$\mathbb{P}\left(\bigcap_{m=n}^{N} A_m^c\right) = \prod_{m=n}^{N}(1-a_m) \leq \exp\left(-\sum_{m=n}^{N} a_m\right) \to 0$$

as $N \to \infty$

so $\mathbb{P}\left(\bigcap_{m\geq n} A_m^c\right) = 0$ for every n

and $\mathbb{P}(A_n \text{ i.o}) = 1 - \mathbb{P}\Big(\underbrace{\bigcup_n \bigcap_{m\geq n} A_m^c}_{\substack{\text{countable}\\ \text{union of}\\ \text{null sets so null.}}}\Big) = 1$

<u>Theorem</u> (Quasi - equivalence of convergence)

Let $(E, \mathcal{E}, \mu)$ be a measure space

functions

Let $(E, \mathcal{E}, \mu)$ be a measure space

f_n a sequence of real valued meas functions

1. If $\mu(E) < \infty$ and $f_n \to 0$ a.e. then

$f_n \to 0$ in measure

2. If $f_n \to 0$ in measure then $\exists$ a subseq.

f_{n_k} s.t. $f_{n_k} \to 0$ a.e.

Pf/ 1. Fix $\varepsilon > 0$

$$\mu(\{x : |f_n(x)| \leq \varepsilon\}) \geq \mu\left(\bigcap_{m \geq n} \{x : |f_m(x)| \leq \varepsilon\}\right)$$

$$\mu\left(\bigcap_{m \geq n} \{x : |f_m(x)| \leq \varepsilon\}\right) \uparrow \mu(\{x : |f_n(x)| \leq \varepsilon \text{ eventually}\})$$

$$\geq \mu(\{x : f_n(x) \to 0\}) = \mu(E)$$

$$\mu(\{x : |f_n(x)| \leq \varepsilon\}) \to \mu(E)$$

taking complements

$$\mu(\{x : |f_n(x)| > \varepsilon) \to 0$$

2. $f_n \to 0$ in measure so for every k

$\exists$ n_k s.t. $\mu(\{x : |f_{n_k}(x)| > \frac{1}{k}\}) \leq 2^{-k}$

then $\sum_k \mu(\{x : |f_{n_k}(x)| > \frac{1}{k}\}) < \infty$

so by BC I we have

$$\mu\left(\{x : |f_{n_k}(x)| > \tfrac{1}{k} \quad \text{i.o.}\}\right) = 0$$

Therefore $f_{n_k} \to 0$ almost everywhere.