Week 6 Lecture 2 - Mainly Egoroff and Lusin's theorem

12 November 2021 08:51

Will start at 9.05!

Remind me to record!

The big idea with these two theorems is that f, or a sequence $(f_n)_{n\geq 1}$ of measurable function will behave very nicely on a compact set of almost full measure.

<u>Egoroff's Theorem</u> Let $(E, \mathcal{E}, \mu)$ with a sequence of real valued measurable functions $(f_n)_{n\geq 1}$ and $f_n \longrightarrow f$ μ-almost everywhere

$$\mu(\{x : f_n(x) \not\to f(x)\}) = 0$$

Then for any $\varepsilon > 0$ $\exists$ a set A s.t. $\mu(A^c) < \varepsilon$ and $f_n \longrightarrow f$ uniformly on A.

<u>Example</u> Let $E = [0,1]$ $\mathcal{E} = \mathcal{B}([0,1])$ Lebesgue measure

$f_n(x) = x^n$ then we can take $A_\varepsilon = [0, 1-\varepsilon]$ then $f_n(x) \to 0$ uniformly on A_ε and $\lambda(A_\varepsilon^c) \leq \varepsilon$

<u>Proof</u> We take $g_n(x) = \sup_{j\geq n} |f_j(x) - f(x)|$

$g_n(x)$ is strictly decreasing $\to 0$ almost everywhere

also $g_n(x)$ is finite whenever $f_n(x) \to f(x)$

$g_n(x)$ is finite almost everywhere

We use our quasi-equivalence of convergence theorem

"In a finite measure space convergence a.e. $\Rightarrow$ convergence in measure"

$g_n(x) \to 0$ in measure.

Convergence in measure means for every $\delta > 0$

$\mu(\{x : g_n(x) > \delta\}) \to 0$ as $n \to \infty$

so in particular for every k

$\mu(\{x : g_n(x) > \frac{1}{k}\}) \to 0$ as $n \to \infty$

so $\exists\, n_k$ s.t. $\forall\, n \geq n_k$

$$\mu(\{x : g_n(x) > \tfrac{1}{k}\}) < 2^{-k}\varepsilon$$

$$\mu(\{x : g_{n_k} > \tfrac{1}{k}\}) < 2^{-k}\varepsilon$$

$A_k = \{x : g_{n_k}(x) \leq \frac{1}{k}\}$ let $A = \bigcap_k A_k$

$$\mu(A^c) = \mu\left(\left(\bigcap_k A_k\right)^c\right) = \mu\left(\bigcup_k A_k^c\right) \leq \sum_k \mu(A_k^c)$$

$$\leq \sum_k 2^{-k}\varepsilon = \varepsilon$$

Claim on A $f_n \to f$ uniformly.

We want that given $\delta > 0$ $\exists\, N(\delta)$ s.t.
$\forall\, x \in A$ $|f_n(x) - f(x)| < \delta$ when $n > N(\delta)$.

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$\forall\ x \in A \quad |f_n(x) - f(x)| < \delta$ when $n > N(\delta)$.

~~$\exists$~~ Lets fix $\delta > 0$ $\exists\ k$ with $\frac{1}{k} < \delta$

then for every $n \geq n_k$ as $A \subseteq A_k$ for $x \in A$

$$|f_n(x) - f(x)| \leq \sup_{j \geq n} |f_j(x) - f(x)| \leq \sup_{j \geq n_k} |f_j(x) - f(x)| = g_{n_k}(x)$$

then on A_k $g_{n_k}(x) \leq 1/k < \delta$

so on A $|f_n(x) - f(x)| < \delta$ whenever $n \geq n_k$

$f_n \to f$ uniformly on A.

Lusin's theorem is less important in this course than Egoroff's theorem.

In general should expect it as a consequence of Egoroff's theorem.

<u>Lusin's</u> Let $(E, \mathcal{E}, \mu)$ be a measure space where E is a subset of $\mathbb{R}^d$, $\mathcal{E} = \mathcal{B}(E)$ and μ is a regular measure

$$\mu(A) = \inf\{\mu(U) : A \subseteq U,\ U \text{ open}\} = \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}$$

If f is a measurable function $E \to \mathbb{R}$ then if A has $\mu(A) < \infty$ then given $\varepsilon > 0$ ($\exists$ K compact $K \subseteq A$ s.t. $\mu(A \setminus K) < \varepsilon$ s.t.

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Reason you might expect it as a consequence of Egoroff's theorem is that we can approximate measurable functions by continuous functions (we'll do this later) then we find a set where they are uniformly converging the uniform limit of continuous functions is continuous so this should give use the result.

<u>Example</u> $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ as a function on $[0,1]$ is measurable. Then f is continuous of $[0,1] \setminus \mathbb{Q}$ which is a set of measure 1 and by regularity of Lebesgue measure $\exists\ K \subseteq [0,1] \setminus \mathbb{Q}$ with measure at $1-\varepsilon$.

We can actually construct such a set. If we enumerate the rationals in $[0,1]$ $q_1, q_2, q_3, \dots$ then set $U = \bigcup_n (q_n - \varepsilon 2^{-n-1}, q_n + \varepsilon 2^{-n-1})$

then U is an open set and $K = [0,1] \cap U^c$ is a closed and bounded set so compact and $\mathbb{Q} \cap K = \emptyset$ so $f|_K = 0$ and constant functions are continuous

$$\lambda(K) \geq 1 - \lambda(U)$$
$$\geq 1 - \sum_n \varepsilon 2^{-n}$$
$$= 1 - \varepsilon$$

Proof Lusin's Theorem

Two steps 1. Show the conclusion when f takes countably many values

2. Extend to all functions by finding a sequence f_n of functions only taking countably many values s.t. $f_n \to f$ uniformly.

Suppose f takes values $a_1, a_2, a_3, \dots$

define the set $A_k = f^{-1}(\{a_k\}) \cap A$

then $A = \bigcup_k A_k$ then A_k are all disjoint

if $x \in A_k \cap A_j$ then $f(x) = a_k$ and $f(x) = a_j$ #

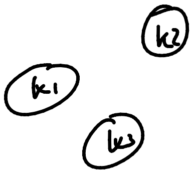
By continuity of measure $\mu(A) = \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} A_k\right)$

so there exists n s.t. $\mu\left(\bigcup_{k=1}^{n} A_k\right) \geq \mu(A) - \varepsilon/2$

then using regularity we can find K_k compact with $K_k \subseteq A_k$ $\mu(A_k \setminus K_k) \leq \frac{\varepsilon}{2n}$

then let $K = \bigcup_{k=1}^{n} K_k$ this is a finite union of compact sets. so K is compact.

Then the K_k are disjoint as the A_k are disjoint so by the Hausdorff property they are separated so by the Hausdorff property they are separated



so since $f|_{K_k}$ is constant for each k

$f|_K$ is continuous.

Now we want to extend to general f we approximate f by $f_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$

$f_n(x)$ takes values in the set $\{k2^{-n} : k \in \mathbb{Z}\}$

and $2^{-n} \geq f(x) - f_n(x) \geq 0$ so $f_n \to f$ uniformly

For each n $\exists$ K_n with $\mu(A \setminus K_n) < \varepsilon 2^{-n}$ with $f_n|_{K_n}$ being continuous then let

$K = \bigcap_n K_n$ then K is compact

$$\mu(A \setminus K) \leq \sum_n \mu(A \setminus K_n) \leq \sum_n \varepsilon 2^{n} = \varepsilon$$

and $f_n|_K$ is continuous as $K \subseteq K_n$ for each n

then f is the ~~cont~~ uniform limit of cts function on K so $f|_K$ is continuous.