~~Markov's inequality~~

Let $(E, \mathcal{E}, \mu)$ be a measure space f non-negative, measurable, real valued function and $\lambda > 0$

then $$\mu(\{x : f(x) > \lambda\}) \leq \frac{\mu(f)}{\lambda}$$

~~Pf~~

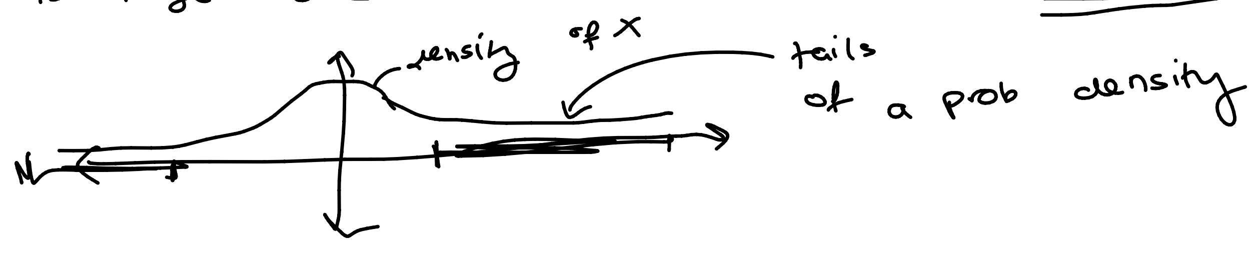
$$\lambda \, \mathbb{1}_{\{f(x) > \lambda\}} \leq f$$

Using monotonicity of the integral

$$\mu(\lambda \mathbb{1}_{\{f > \lambda\}}) \leq \mu(f)$$

$$\mu(\{x : f(x) > \lambda\}) \leq \mu(f)/\lambda$$

↳ Bounds on the size of the set where f is large are sometimes called <u>tail estimates</u>



If $f \in L^p$ we can get more refined tail estimates out of Markov's inequality

$$\mu(\{x : |f(x)| > t\}) = \mu(\{x : |f(x)|^p > t^p\})$$

$$\leq \frac{\mu(|f|^p)}{\;}$$

$$\leqslant \frac{\mu(|f|^p)}{t^p}$$

If $f \in L^p$ then we get a stronger control over how fast $\mu(\{x : |f(x)| > t\}) \to 0$ as $t \to \infty$.

In probability $\mathbb{P}(X > x) \leq x^{-p}\, \mathbb{E}(|X|^p)$

In particular get Chebychev's ineq

$$\mathbb{P}(\,|X - \underset{\substack{\uparrow \\ \text{mean of } X}}{\mu}| > t\,) \leqslant \frac{\mathrm{Var}(X)}{t^2}.$$

We can do something better when $e^{\alpha f}$ is integrable for every α. Sometimes called a Tchernoff bound

General principle

$$\mu(\{f(x) > t\}) = \mu(\{e^{\alpha f} > e^{\alpha t}\}) \leq \underbrace{\mu(e^{\alpha f})\, e^{-\alpha t}}_{\text{minimize this side in } \alpha.}$$

Example If $X \sim N(0, \sigma^2)$

$$\mathbb{E}(e^{\alpha X}) = e^{\alpha^2\sigma^2/2}$$

$$\mathbb{P}(X > t) \leqslant e^{\alpha^2\sigma^2/2 - \alpha t}$$

$$\frac{\alpha^2\sigma^2}{2} - \alpha t = \frac{1}{2}\left(\alpha\sigma - \frac{t}{\sigma}\right)^2 - \frac{t^2}{2\sigma^2}$$

so if we choose $\alpha = t/\sigma^2$ we get so if we choose $\alpha = t/\sigma^2$ we got

$$\mathbb{P}(X > t) \leq e^{-t^2/2\sigma^2}$$