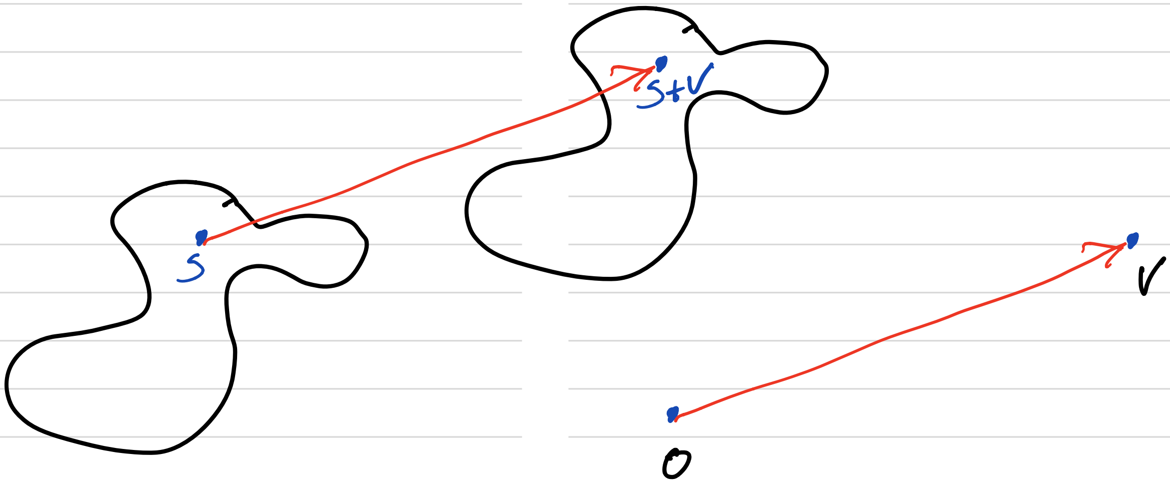
Proposition. Lebesgue measure λ_d in $\mathbb{R}^d$ is translation invariant.

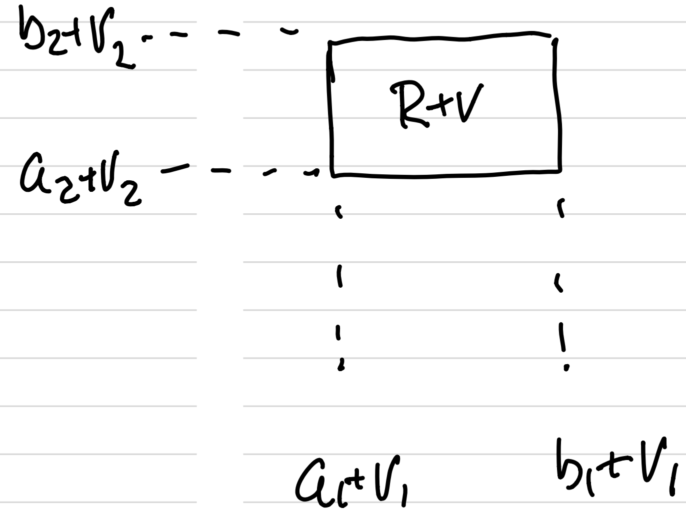
Proof. Let $v \in \mathbb{R}^d$ and let $F_v : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be given by $F(x) = x+v$. We want to show two things: If B is measurable then ① $F(B)$ is measurable and ② $\lambda(B) = \lambda(F(B))$.

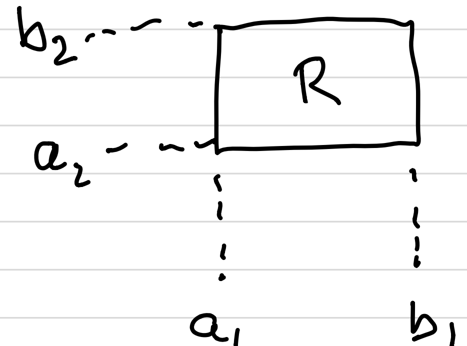
Let $S \subset \mathbb{R}^d$ be a set and v a vector. Write $F_v(S) = S+v = \{s+v : s \in S\}$.

This is the translate of S by v.



Note that if R is a coordinate rectangle then R and $R+v$ have the same dimensions so $\mathrm{vol}(R+v) = \mathrm{vol}(R)$.





$\mathrm{Area}(R) = (b_1-a_1)(b_2-a_2)$

$\mathrm{Area}(R+v) = (b_1+v_1-a_1-v_1)(b_2+v_2-b_1-v_2)$
$= (b_1-a_1)(b_2-a_2)$

If $\mathcal{U}$ is a collection of coordinate rects. $\mathcal{U} = \{R_j\}$ then write $\mathcal{U}+v$ for $\{R_j + v\}$. The correspondence $\mathcal{U} \to \mathcal{U}+v$ maps covers of S to covers of $S+v$ while $\mathcal{U}' \longrightarrow \mathcal{U}'-v$ maps covers of $S+v$ to covers of S. Each correspondence preserves sums of volumes. We conclude that

$\lambda^*(S) = \lambda^*(S+v)$.

If B is λ^*-measurable then

$$\lambda^*(A) = \lambda^*(A \cap B) + \lambda^*(A \cap B^c) \quad \text{so}$$

$$\begin{array}{ccccc}
\lambda^*(A+v) = & \lambda^*((A+v) \cap (B+v)) & + & \lambda^*((A+v) \cap (B+v)^c) \\
 & \| & & \| \\
\| & \lambda^*((A\cap B)+v) & + & \lambda^*((A \cap B^c)+v) \\
 & \| & & \| \\
\lambda^*(A) & \lambda^*(A \cap B) & + & \lambda^*(A \cap B^c)
\end{array}$$

and thus $B+v$ is λ^* measurable.

It also follows that

$$\lambda(B+V) = \lambda^*(B+V) = \lambda^*(B) = \lambda(B).$$

This completes the proof.