At this point we have defined the integral of non-negative measurable functions.

We want to

① establish some basic properties of this integral for non-negative functions

② extend the integral to real valued functions. This is the last step in our step by step definition of the integral

③ establish some basic properties of the integral for real valued functions.

In order to do ① we will take advantage of Prop. 2.3.3 about realizing the value of an integral as a limit of values of integrals of simple functions.

Proposition 2.3.4. Let f, g be $[0, +\infty]$ valued measurable functions then

(a) If $f(x) \le g(x)$ for $x \in X$ then $\int f \, d\mu \le \int g \, d\mu$.

(b) $\int \alpha f \, d\mu = \alpha \int d\mu$ for $\alpha \ge 0$.

(c) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Proof of

(a) If $f(x) \le g(x)$ for $x \in X$ then $\int f\,d\mu \le \int g\,d\mu$.

This follows from the definition of the integral as a supremum. If $f \le g$ and $h \le f$ then $h \le g$

so

$$\int f\,d\mu = \sup_{\substack{h \in \mathcal{L}^+ \\ h \le f}} \int h\,d\mu \le \sup_{\substack{h \in \mathcal{L}^+ \\ h \le g}} \int h\,d\mu = \int g\,d\mu.$$

sup over a smaller set

sup over a larger set

Proof of (b) $\int \alpha f \, d\mu = \alpha \int d\mu$ for $\alpha \geq 0$.

According to Prop. 2.1.8 (lower sum construction) there is a monotone sequence of $f_n \in \mathcal{S}^+$ with $f_n \to f$ pointwise. Then $\alpha f_n \to \alpha f$ pointwise

so

$$\int \alpha f \, d\mu \underset{\text{Prop. 2.3.3}}{=} \lim_{n \to \infty} \int \alpha f_n \underset{\substack{\text{Linearity of} \\ \text{the integral} \\ \text{for simple} \\ \text{functions.}}}{=} \lim_{n \to \infty} \alpha \int f_n \, d\mu \underset{\text{Prop 2.3.3}}{=} \alpha \int f \, d\mu$$

Proof of (c) $\int (f+g)\,d\mu = \int f\,d\mu + \int g\,d\mu$

Let $f_n \in \mathcal{S}^+$ with f_n monotone, $f_n \to f$ pointwise and

$g_n \in \mathcal{S}^+$ with g_n monotone, $g_n \to g$ pointwise then

$\{f_n + g_n\}$ is monotone and converges to $f+g$ pointwise

so

$$\int f+g\,d\mu \underset{\uparrow\ 2.3.3}{=} \lim_{n\to\infty} \int f_n+g_n\,d\mu \underset{\uparrow\ \text{simple fcns}}{=} \lim_{n\to\infty}\left(\int f_n\,d\mu + \int g_n\,d\mu\right) \underset{\uparrow\ 2.3.3}{=} \int f\,d\mu + \int g\,d\mu.$$

Now we extend the definition of the integral to general measurable functions.

In particular we reduce the general case to the case of non-negative functions.

Say that $f: X \to [-\infty, \infty]$ is $\mathcal{A}$-measurable.

f can be written as the difference of two non-negative functions.

We now deal with $[-\infty, \infty]$ valued functions.
Given such an f we can construct two $[0, \infty]$ valued functions.

Say f is measurable. Define

$f : X \to [-\infty, \infty]$

$f^+ = \max(f, 0)$

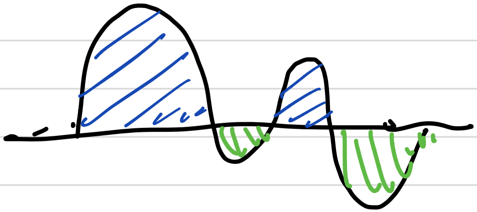
$f^+ : X \to [0, \infty]$

$f^- = \max(-f, 0) = -\min(f, 0)$

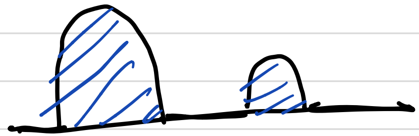
$f^- : X \to [0, \infty]$

We have shown that the max of two measurable functions is measurable and constant functions are always measurable so f^+, f^- are measurable.

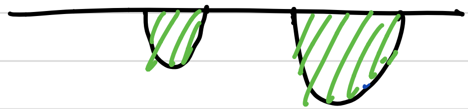
f



$f^+ = \max(f, 0)$



$\min(f, 0)$



$\min(f, 0) = -\max(-f, 0)$

$f^- = -\min$

$f^- = \max(-f, 0)$

Say f is measurable.

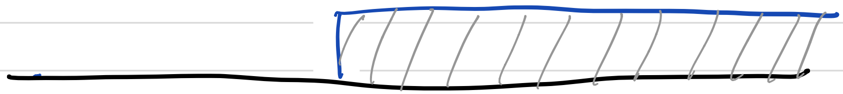
Def. f has an integral if $\int f^+ d\mu < \infty$ or $\int f^- d\mu < \infty$.

Def. If f has an integral then the integral of f is:

$$\int f\,d\mu = \int f^+ d\mu - \int f^- d\mu.$$

(This integral can take the value $+\infty$ or $-\infty$.)

Def. f is integrable if $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$. (We also say $f \in \mathcal{L}^1$ or $f \in \mathcal{L}^1(\mu)$)



$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

f has an integral since $\int f^+ d\mu = \infty$, $\int f^- d\mu = 0$.

$$\int f \, d\mu = +\infty.$$

If f is integrable then $\int f \, d\mu \neq \pm\infty$.

Our next objective is to prove linearity for real valued functions. We start with a Lemma.

Lemma 2.3.5 Let f_1, f_2, g_1, g_2 be non-neg real valued integrable functions s.t. $f_1 - f_2 = g_1 - g_2$. Then

$$\int f_1 \, d\mu - \int f_2 \, d\mu = \int g_1 \, d\mu - \int g_2 \, d\mu.$$

To show $\int f_1\,d\mu - \int f_2\,d\mu = \int g_1\,d\mu - \int g_2\,d\mu$.

Proof. $f_1 - f_2 = g_1 - g_2$ ← Hypothesis.

$$f_1 + g_2 = g_1 + f_2$$

$$\int f_1 + g_2\,d\mu = \int g_1 + f_2\,d\mu$$

↙ pos. linearity ↘ pos. linearity

$$\int f_1\,d\mu + \int g_2\,d\mu = \int g_1\,d\mu + \int f_2\,d\mu$$

$$\int f_1\,d\mu - \int f_2\,d\mu = \int g_1\,d\mu - \int g_2\,d\mu$$

Prop. 2.3.6 Let f, g be real valued integrable functions on X and let α be a real number.

Then:

(a) αf is integrable and $\int \alpha f \, d\mu = a \int f \, d\mu$

(b) $f+g$ are integrable and

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$$

(c) if $f(x) \leq g(x)$ for $x \in X$ then $\int f \, d\mu \leq \int g \, d\mu$.

Proof. (a) Say f is integrable so $\int f^+ d\mu < \infty$ and $\int f^- d\mu < \infty$.

If $\alpha > 0$ then

$$(\alpha f)^+ = \alpha f^+$$

$$(\alpha f)^- = \alpha f^-$$

so $\int (\alpha f)^+ = \alpha \int f^+ < \infty$

$\int (\alpha f)^- = \alpha \int f^- < \infty$

Thus αf is integrable.

$$\int \alpha f \, d\mu = \int \alpha f^+ \, d\mu - \int \alpha f^- \, d\mu$$

$$= \alpha \int f^+ d\mu - \alpha \int f^- d\mu$$

$$= \alpha \left(\int f^+ d\mu - \int f^- d\mu \right)$$

$$= \alpha \int f \, d\mu.$$

If $\alpha < 0$ then

$$(\alpha f)^+ = \alpha f^-$$

$$(\alpha f)^- = \alpha f^+$$

but otherwise the argument is the same.

If f, g integrable then $f+g$ is integrable.

(b) Now $(f+g)^+ \le f^+ + g^+$ so

$$\int (f+g)^+ \, d\mu \le \int f^+ \, d\mu + \int g^+ \, d\mu < \infty.$$

Also $(f+g)^- \le f^- + g^-$ so

$$\int (f+g)^- \, d\mu \le \int f^- \, d\mu + \int g^- \, d\mu < \infty$$

Thus $f+g$ is integrable.

Now

$$f+g = (f+g)^+ - (f+g)^-$$

$$f+g = (f^+ - f^-) + (g^+ - g^-) = (f^+ + g^+) - (f^- + g^-)$$

so

$$(f+g)^+ - (f+g)^- = (f^+ + g^+) - (f^- + g^-)$$

and by Lemma 2.3.5 we have

$$\int (f+g)^+ d\mu - \int (f+g)^- d\mu = \int f^+ + g^+ d\mu - \int f^- + g^- d\mu$$

$$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu = \int f^+ + g^+ \, d\mu - \int f^- + g^- \, d\mu$$

$$\swarrow \qquad\qquad \downarrow \text{ pos. lin.} \qquad\qquad \downarrow \text{ pos. lin.}$$

$$\int f+g \, d\mu \qquad\qquad \int f^+ d\mu + \int g^+ d\mu - \int f^- d\mu - \int g^- d\mu$$

$$= \int f^+ d\mu - \int f^- d\mu + \int g^+ d\mu - \int g^- d\mu$$

$$= \int f \, d\mu + \int g \, d\mu.$$

(c) If $f(x) \le g(x)$ for all $x \in X$ then $g(x) - f(x) \ge 0$. Using the inequality for non-negative meas. fcns. we have $\int g(x) - f(x) \, d\mu \ge 0$.

Using (a) and (b) we have

$$0 \le \int g(x) - f(x) \, d\mu = \int g(x) - \int f(x) \, d\mu$$

So $\int f(x) \, d\mu \le \int g(x) \, d\mu$.