Measure Theory: Exercises (not for credit)

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Question 1. Show that the definition of the integral for a non-negative function is consistent with the defintion of the integral for simple functions. That is to say if $f = \sum_k a_k 1_{A_k}$ is a simple function then

$$\sum_k a_k \mu(A_k) = \sup\{\mu(h) \,:\, h\,\text{simple},\, h \leq f\}$$

Question 2. Let f be an integrable, real valued function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that $\mu(f 1_A) = 0$ for every $A \in \mathcal{E}$ show that this implies that $f = 0$ almost everywhere. Let $\mathcal{A}$ be a π-system generating $\mathcal{E}$ and containing E. Suppose that $\int f 1_A \mu(\mathrm{d}x)\, (= \mu(f 1_A)) = 0$ for every $A \in \mathcal{A}$ show that then $f = 0$ almost everywhere.

Question 3. Find a three sequences of real valued integrable functions, $(f_n)_{n\geq 1}, (g_n)_{n \geq 1}, (h_n)_{n\geq 1}$, all of which converge to 0 almost everywhere and where

- $\lim_n \int f_n(x)\mathrm{d}x = \infty$
- $\lim_n \int g_n(x)\mathrm{d}x = 1$
- $\limsup_n \int h_n(x)\mathrm{d}x = -\liminf_n \int h_n(x)\mathrm{d}x = 1.$

Question 4. Let $(f_n)_{n\geq 1}$ be a sequence of real valued, measurable functions (not necessarily non-negative) on $(E, \mathcal{E}, \mu)$. Suppose that f_1 is integrable and $f_1(x) \leq f_2(x) \leq f_3(x) \leq \dots$ for every x and $f_n(x) \to f(x)$. Show that $\lim_n \int f_n(x)\mu(\mathrm{d}x) = \int f(x)\mu(\mathrm{d}x)$.

Question 5. In lectures we proved Beppo-Levi as a consequence of the monotone convergence theorem. Show that if we assume the result in Beppo-Levi then we can prove the monotone convergence theorem as a consequence.