Theorem 1. Let μ^* be an outer measure on X. Then the collection $\mathcal{M}$ of μ^* measurable sets is a σ-algebra.

The proof involves a large number of steps none of which is too hard.

As a technique for managing these steps I want to break the proof into 3 Propositions.

The first involves algebras of sets which are like σ-algebras except the unions involved are finite unions.

Proposition 1. $\mathcal{M}$, the set of μ^* measurable sets, is an algebra.

Proof. We need to verify the axioms of an algebra:

(a) $X \in \mathcal{M}$

(b) If $B \in \mathcal{M}$ then $B^c \in \mathcal{M}$

(c) If $B_i \in \mathcal{M}$ for $i = 1 \dots n$ then $\bigcup_{i=1}^{n} B_i \in \mathcal{M}$.

(a) follows from the proposition proved in lecture 2 since $\mu^*(\emptyset) = 0$ and $X = \emptyset^c$.

$X^c = \emptyset$

(b) follows from the symmetry of the measurability condition:

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c)$$

$$(B^c)^c = B.$$

We now prove (c).

Say B_1 and B_2 are measurable we want to show that $B_1 \cup B_2$ is measurable which is to say:

$$\mu^*(A) = \mu^*(A \cap (B_1 \cup B_2)) + \underline{\mu^*(A \cap (B_1 \cup B_2)^c)}$$

We analyse the first term on the right.

$$\left(\text{To show: } \mu^*(A) = \mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c)\right)$$

Use the measurability of B_1 with test set $A \cap (B_1 \cup B_2)$ to get:

$$\mu^*(A \cap (B_1 \cup B_2)) = \underbrace{\mu^*(A \cap (B_1 \cup B_2) \cap B_1)}_{\mu^*(A \cap B_1)} + \underbrace{\mu^*(A \cap (B_1 \cup B_2) \cap B_1^c)}_{\mu^*(A \cap B_2 \cap B_1^c)}$$

(To show: $\mu^*(A) = \underline{\mu^*(A\cap(B_1\cup B_2))} + \underline{\mu^*(A\cap(B_1\cup B_2)^c)}$)

We rewrite the second term:

$$\underline{\mu^*(A\cap(B_1\cup B_2)^c)} = \mu^*(A\cap B_1^c \cap B_2^c)$$

Putting these together we get:

$$\underbrace{\mu^*(A\cap(B_1\cup B_2)) + \mu^*(A\cap(B_1\cup B_2)^c)} =$$

$$\mu^*(A\cap B_1) + \mu^*(\underline{A}\cap \underline{B_2}\cap \underline{B_1^c}) + \mu^*(\underline{A}\cap \underline{B_1^c}\cap \underline{B_2^c})$$

Now we use the measurability of $\underline{B_2^c}$ with test set $\underline{A\cap B_1^c}$.

$$\underbrace{\mu^*(A\cap(B_1\cup B_2)) + \mu^*(A\cap(B_1\cup B_2)^c)} = \mu^*(A\cap B_1) + \mu^*(A\cap B_1^c)$$

Now we use the measurability of B_1 with test set A:

$$\underbrace{\mu^*(A \cap (B_1 \cup B_2)) + \mu^*(A \cap (B_1 \cup B_2)^c)} = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c)$$

$$= \mu^*(A)$$

This proves $B_1 \cup B_2 \in \mathcal{M}$. The proof for finite unions follows.

This finishes the proof of Proposition 1.

The objective of Theorem 1 is to show that $\mathcal{M}$ is closed under taking countable unions.

Say $B_i \in \mathcal{M}$ for $i = 1 \ldots \infty$. We want to show that $\bigcup_{i=1}^{\infty} B_i \in \mathcal{M}$. That is

$$(*) \quad \mu^*(A) = \mu^*(A \cap (\cup B_i)) + \mu^*(A \cap (\cup B_i)^c)$$

This is equivalent to:

$$\mu^*(A) \geq \mu^*(A \cap (\cup B_i)) + \mu^*(A \cap (\cup B_i)^c)$$

by sub-additivity

$$(*) \qquad \mu^*(A) \geq \mu^*\left(A \cap \left(\bigcup B_i\right)\right) + \mu^*\left(A \cap \left(\bigcup B_i\right)^c\right)$$

$$\mu^*(A) \geq \mu^*\left(\bigcup_i A \cap B_i\right) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right)$$

Countable subadditivity implies

$$\sum_{i=1}^{\infty} \mu^*(A \cap B_i) \geq \mu^*\left(\bigcup_i A \cap B_i\right).$$

To establish $(*)$ it suffices to show:

$$(**) \qquad \mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right).$$

RHS of $(**)$ $\geq$ RHS of $(*)$.

Proposition 2. If B_i $i=1 \ldots \infty$ are disjoint then

$$(**) \quad \mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right).$$

(Summary: Prop. 2 establishes that μ satisfies the countable union property for disjoint sets.)

Proof:

As a first step towards proving Proposition 2 we consider a finite collection of sets $B_1 \dots B_n$.

We show: $$(I_n) \quad \mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right)$$

by induction on n.

For $n=1$ we have

$$(I_1) \quad \mu^*(A) = \mu^*(A \cap B_1) + \mu^*(A \cap B_1^c)$$

which is just the measurability of B_1.

Now assume the statement holds for n.

That is we assume:

$$(I_n) \qquad \mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right)$$

Let's analyse the right hand term.

Measurability of B_{n+1} with respect to the test set $A \cap \left(\bigcap_{n=1}^{n} B_i^c\right)$ gives:

$$\mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right)\right) = \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}\right) + \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}^c\right)$$

$$\mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right)\right) = \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}\right) + \mu^*\left(A \cap \left(\bigcap_{i=1}^{n} B_i^c\right) \cap B_{n+1}^c\right)$$

Disjointness implies that $B_{n+1} \subset B_i^c$

$$\mu^*(A \cap B_{n+1}) \qquad \mu^*\left(A \cap \bigcap_{i=1}^{n+1} B_i^c\right)$$

(I_n) $\qquad \mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \underline{\mu^*(A \cap \bigcap_{i=1}^{n} B_i^c)}$

So $\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \underline{\mu^*(A \cap B_{n+1}) + \mu^*(A \cap \bigcap_{i=1}^{n+1} B_i^c)}$

$$= \sum_{i=1}^{n+1} \mu^*(A \cap B_i) + \mu^*(A \cap \bigcap_{i=1}^{n+1} B_i^c).$$

This is statement (I_{n+1}). Thus $(I_n) \Rightarrow (I_{n+1})$ and this completes the induction and proves (I_n) for all n.

Now we need to go from a finite collection of sets B_i to a countably infinite collection.

We have:

$$(I_n) \qquad \mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right)$$

As n increases $\bigcap_{i=1}^{n} B_i^c$ decreases. Monotonicity of μ^* gives:

$$\mu^*(A) = \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{n} B_i^c\right)$$

$$\geq \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right)$$

$$\mu^*(A) \geq \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \bigcap_{i=1}^{\infty} B_i^c\right)$$

Replace intersection by union.

$$\mu^*(A) \geq \sum_{i=1}^{n} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Letting $n \to \infty$ we get:

$$(**) \quad \mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

Thus we have finished the proof of Proposition 2.

Proposition 3. If $B_i \in \mathfrak{M}$ for $i = 1 \ldots \infty$ then $\bigcup_{i=1}^{\infty} B_i \in \mathfrak{M}$.

This proof of Prop. 3 will complete the proof of the Theorem.

Proof of Prop. 3.

Now say we have an arbitrary sequence of sets $\{B_i\}$.

$B_1, \; B_2, \; B_3, \; B_4, \; B_5 \dots$

Consider the following sequence of disjoint sets:

$B_1, \; B_1^c \cap B_2, \; B_1^c \cap B_2^c \cap B_3 \dots\dots \; B_1^c \cap B_2^c \cap \dots B_{n-1}^c \cap B_n \dots$

Each of these sets is measurable (since $\mathcal{M}$ is an algebra by Prop 1.)

$B_1, \quad B_1^c \cap B_2, \quad B_1^c \cap B_2^c \cap B_3 \; \dots\dots$

$B_1^c \cap B_2^c \cap \dots B_{n-1}^c \cap B_n$

These sets are pairwise disjoint so their union is in $\mathcal{M}$ by Prop. 2.

The union of this sequence of sets is

$\bigcup_{j=1}^{\infty} B_j$ so $\bigcup_{j=1}^{\infty} B_j \in \mathcal{M}$ and the proof of Prop3 is complete.

Proving Prop 3 completes the proof of Thm 1.

Theorem 2. The restriction of μ^* to the σ-algebra of μ^* measurable sets is countably additive.

Proof. Let B_j $j=1\ldots\infty$ be a collection of disjoint sets.

To see that μ^* is countably additive consider the equation:

$$(**) \qquad \mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap B_i) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

from Prop. 2

and let the test set A be $\bigcup_{i=1}^{\infty} B_i$.

We get

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \geq \sum_{i=1}^{\infty} \mu^*\left(\bigcup_{i=1}^{\infty} B_i \cap B_i\right) + \mu^*\left(\bigcup_{i=1}^{\infty} B_i \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c\right)$$

$$= \sum_{i=1}^{\infty} \mu^*(B_i) + 0$$

So we have:

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \geq \sum_{i=1}^{\infty} \mu^*(B_i).$$

Subadditivity gives: $\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \sum_{i=1}^{\infty} \mu^*(B_i)$

so we conclude that

$$\mu^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu^*(B_i)$$

when the B_i are disjoint.