Week 7 Video 2 -Transformations of the integral 1

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Defⁿ/Lemma **Restriction of measure**

Suppose $(E, \mathcal{E}, \mu)$ is a measure space and $A \in \mathcal{E}$ then the set of all measurable subsets of A

$\mathcal{E}_A = \{ B \in \mathcal{E} : B \subseteq A \}$ is a σ-algebra and $\mu|_{\mathcal{E}_A} = \mu|_A$ is a measure. Furthermore if f is measurable then

$\mu(f \mathbb{1}_A) = \mu_A(f)$ where $f \geq 0$ or integrable

Pf/ Exercise sheet.

We use this to understand Lebesgue measure on subset of $\mathbb{R}^d$ for example Intervals

if $I = [a,b]$ $\quad \int_a^b f(x)\,dx = \lambda(f \mathbb{1}_I)$

Propⁿ Let $(E, \mathcal{E}, \mu)$ be a measure space and f a non-negative, measurable, real valued function

Define $\nu(A) = \mu(f \mathbb{1}_A) = \int_A f \mu(dx)$ then ν is a measure and $\nu(g) = \mu(fg)$ for every non-negative g.

One of our first examples of how to construct a measure if $E = \mathbb{R}^d$ $\mu = \lambda$.

Pf/ $f \mathbb{1}_\emptyset = 0$ so $\nu(\emptyset) = \mu(0) = 0$

$\nu \geqslant 0$ as $f \geqslant 0$

If $A_1, A_2, A_3 \ldots$ are all disjoint then

$$\mathbb{1}_{\bigcup_n A_n} = \sum_n \mathbb{1}_{A_n}$$

Use the Beppo-Levi reformulation of monotone convergence

$$\mu\left(\sum_n f \mathbb{1}_{A_n}\right) \underset{\text{Beppo-Levi}}{=} \sum_n \mu(f \mathbb{1}_{A_n}) = \sum_n \nu(A_n)$$

$$\| \qquad\qquad\qquad\qquad \| \text{ — countable additivity}$$

$$\mu\left(f \sum_n \mathbb{1}_{A_n}\right) = \mu\left(f \mathbb{1}_{\bigcup_n A_n}\right) = \nu\left(\bigcup_n A_n\right)$$

This shows ν is a measure.

Now we want to show $\nu(g) = \mu(fg)$ $g \geq 0$ meas

If g is simple $g = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$

$$\nu(g) = \sum_{k=1}^{n} a_k \nu(A_k) = \sum_{k=1}^{n} a_k \mu(f \mathbb{1}_{A_k}) = \mu\left(f \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}\right)$$

$$= \mu(fg)$$

so $\nu(g) = \mu(fg)$ when g is simple

suppose g is not necessarilly simple

then $\exists$ a sequence g_n with $g_n \uparrow g$ g_n simple

and we will also have $fg_n \uparrow fg$ as $f \geqslant 0$

so $\nu(g) \underset{\text{MON}}{=} \lim_{n \to \infty} \nu(g_n) = \lim_{n \to \infty} \mu(fg_n) \underset{\text{MON}}{=} \mu(fg)$ ✓