Thm. 2.4.5 (Dominated Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be a $[0,+\infty]$ integrable function on X and let $f_1, f_2 \ldots$ be $[-\infty, +\infty]$ valued $\mathcal{A}$-measurable functions such that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

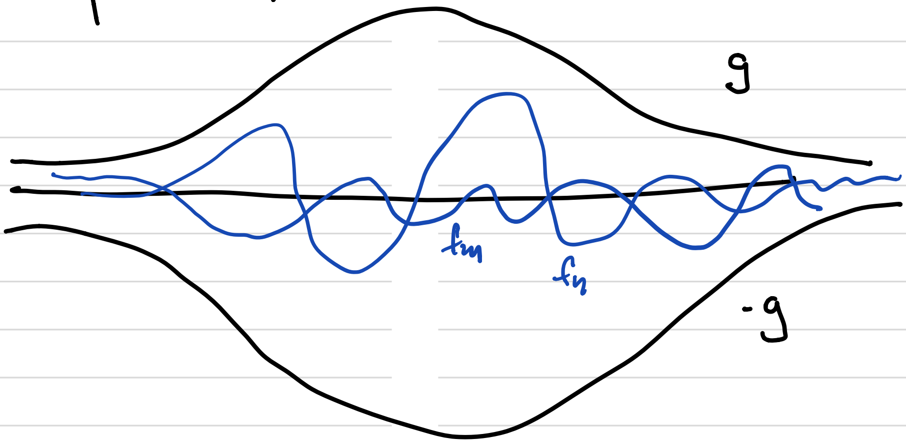
and

$$|f_n(x)| \leq g(x) \quad n = 1, 2, \ldots$$

Then f and f_n are integrable and

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

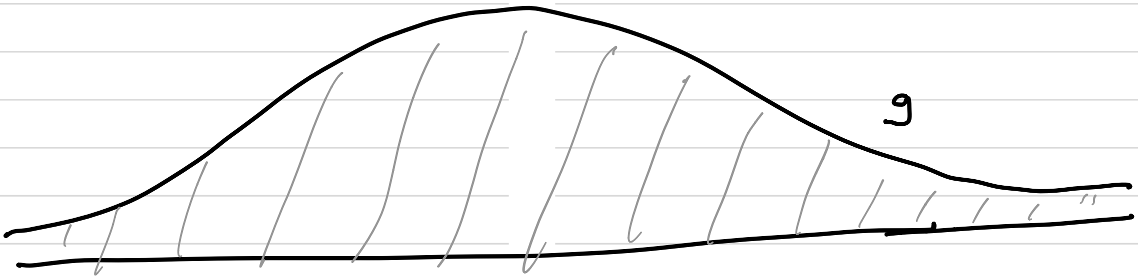
What is the picture?



$|f_n| \leq g$

$-g \leq f_n \leq g$

$\int g \, d\mu < \infty$

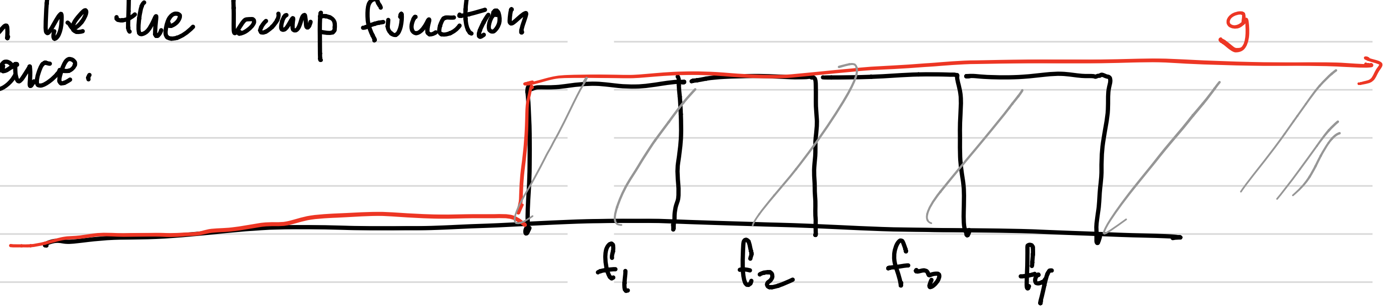


"area under the graph is finite"

What would happen if we consider the sliding bump functions where we know that the conclusion fails?

Let f_n be the bump function sequence.

What would g have to be?



Proof. Let me start with a general remark.

For any function f we have $|f| = f^+ + f^-$.

$f = f^+ - f^-$

We also have:

$$\underbrace{\int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu}_{\text{integrals of } [0,+\infty] \text{ valued functions.}}$$

Non-negative linearity for $[0,+\infty]$ valued functions

(Prop. 2.3.4) versus linearity for $\mathbb{R}$ valued $(-\infty,+\infty)$

functions (Prop. 2.3.6)

$$\int |f|\,d\mu = \int f^+ + f^- \,d\mu = \int f^+ d\mu + \int f^- d\mu$$

f integrable $\Rightarrow$ $|f|$ is integrable

$|f|$ integrable $\Rightarrow$ f is integrable

So if f is integrable iff $|f|$ is integrable.

(Prop. 2.3.8 in Cohn.)

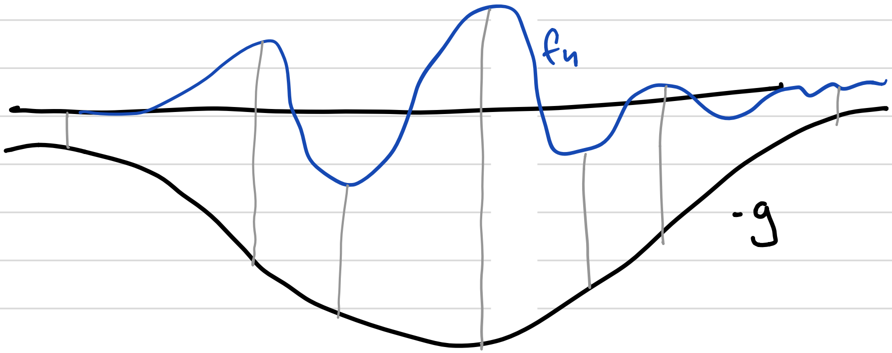
Returning to the proof –

Now g is integrable and $|f_n| \leq g$ so

If $|f_n| \leq g$ and g is integrable

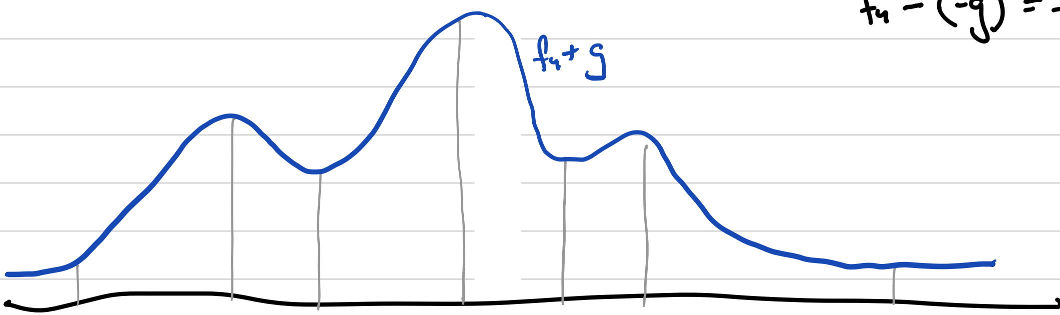
then f_n is integrable.

We prove the equality by proving two inequalities.



Add to f_n:

$f_n - (-g) = f_n + g$.



Since $f_n \geq -g$ we have $f_n + g \geq 0$.

Since $f_n+g \geq 0$ we can apply Fatou's Lemma.

This gives:

$$\int \liminf_{n\to\infty}(f_n+g)\,d\mu \leq \liminf_{n\to\infty} \int f_n+g\,d\mu$$

$$\int f\,d\mu + \int g\,d\mu = \int f+g\,d\mu \qquad\qquad \liminf \int f_n\,d\mu + \int g\,d\mu$$

Since $f_n \to f$ pointwise we have $\liminf(f_n+g) = f+g$

$$\int f_n+g\,d\mu = \int f_n\,d\mu + \int g\,d\mu$$

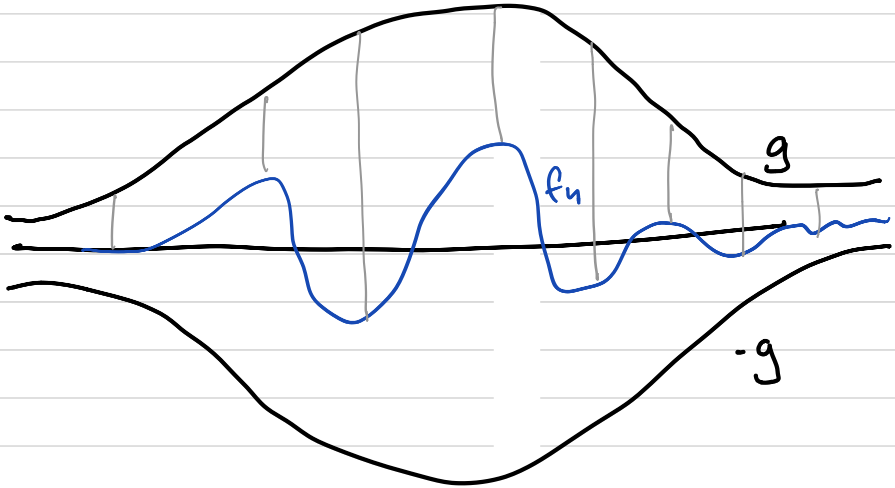
so

$$\int f\,d\mu + \int g\,d\mu = \int f+g\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu + \int g\,d\mu$$

Since $\int g\,d\mu < \infty$ we can subtract it from both sides to get:

$$\int f\,d\mu \leq \liminf \int f_n\,d\mu \qquad (1)$$

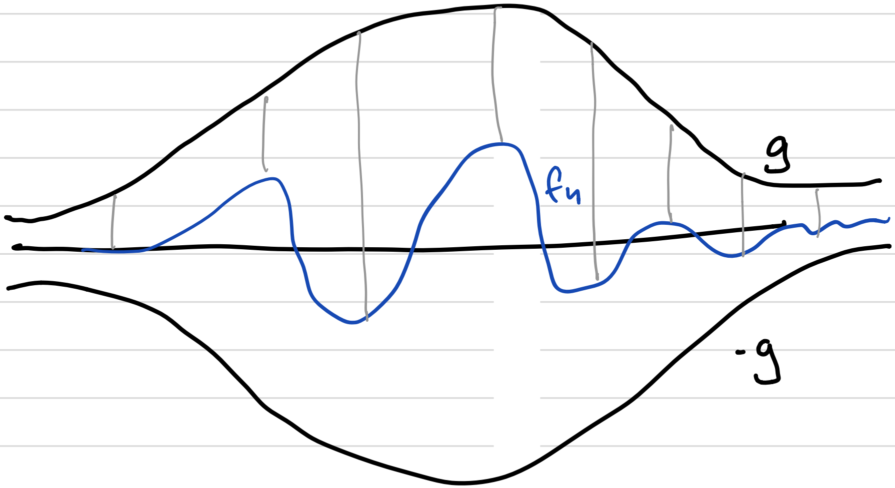
This is the first inequality.



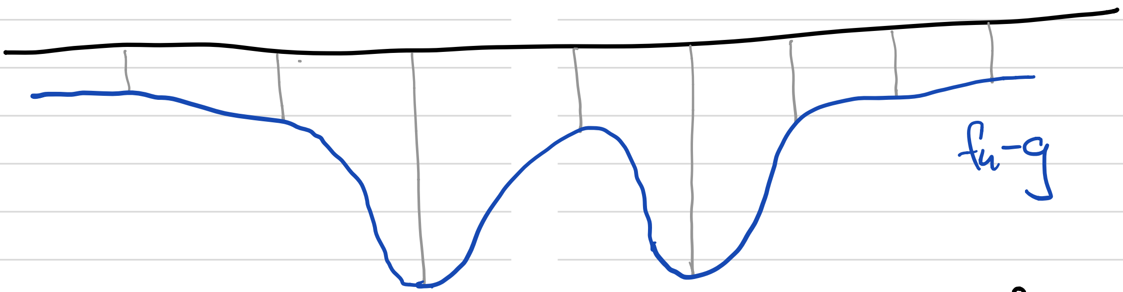
$g \geq f_n$
$0 \geq f_n - g$
$g - f_n \geq 0.$

Now we consider the other inequality $g \geq f_n$ or $0 \geq f_n - g$ or $g - f_n \geq 0$.

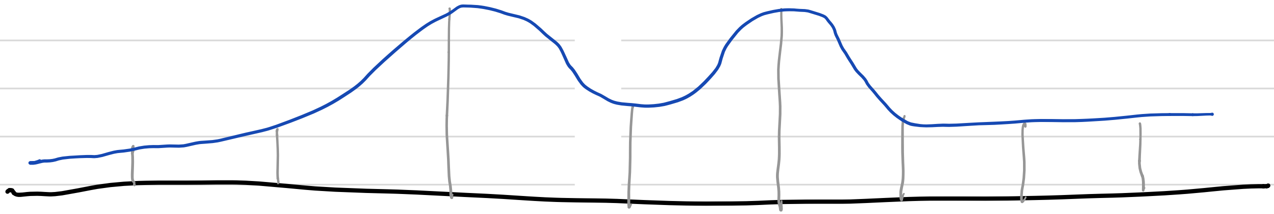
So the functions $f_n - g$ are non-positive and the functions $g - f_n$ are non-negative



$g \geq f_n$



$0 \geq f_n - g$



$g - f_n \geq 0.$

Applying Fatou's Lemma we get:

$$\int \liminf_n \, g - f_n \, d\mu \leq \liminf_n \int g - f_n \, d\mu$$

$$\int g \, d\mu - \int f \, d\mu \leq \int g \, d\mu + \liminf_n \left(- \int f_n \, d\mu \right)$$

$$= \int g \, d\mu - \limsup_n \int f_n \, d\mu$$

$$- \int f \, d\mu \leq - \limsup_n \int f_n \, d\mu$$

$$-\int f \, d\mu \leq -\limsup_n \int f_n \, d\mu$$

$$\int f \, d\mu \geq \limsup_n \int f_n \, d\mu \qquad (2)$$

Putting the 2 inequalities together we have:

$$\int f\,d\mu \overset{(2)}{\geq} \limsup_{n} \int f_n\,d\mu \geq \liminf_{n} \int f_n\,d\mu \overset{(1)}{\geq} \int f\,d\mu$$

Thus we see that $\lim_{n\to\infty} \int f_n\,d\mu$ exists and is equal to $\int f\,d\mu$. This is what we wanted to show.