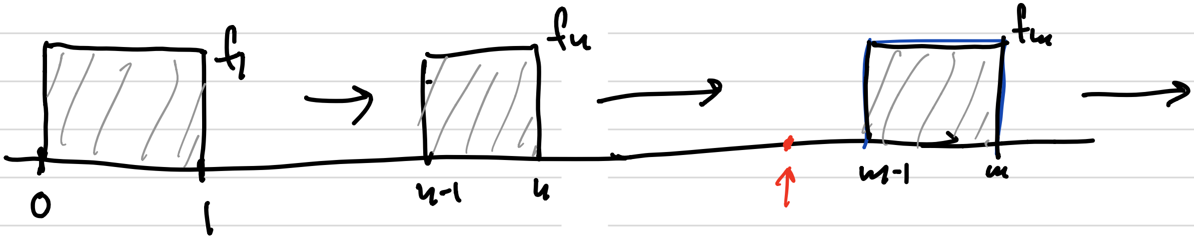
We want to prove some version of the monotone convergence theorem in the setting of general sequences of functions but we have to deal with the fact that there is a counterexample to the ~~simplest~~ formulation of such a result.

There is a sequence of functions f_n which converges pointwise to f so that

$$\lim_{n\to\infty} \int f_n \, d\mu \neq \int f \, d\mu.$$

Example. (Sliding bump phenomenon)

Let $f_n = \chi_{[n-1,\, n]}$ and $f = 0$. $\mu = \lambda =$ Lebesgue measure.



$$\int f_n \, d\lambda$$

$$\int f \, d\mu \neq \lim_{n\to\infty} \int f_n \, d\lambda$$

The next result says that even though equality may not hold it always fails in the same direction. Intuitively when you take the pointwise limit some "mass" may disappear.

In the case of monotone limits we could assume that the sequence f_n converged (perhaps to ∞).

The next result is more general in that it does not assume that the functions f_n converge. Instead it considers their limsup which exists whether or not they converge.

Theorem 2.4.4 (Fatou's Lemma) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of $[0, +\infty]$ valued $\mathcal{A}$-measurable functions on X.

Then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu.$$

Proof. Consider f_n.

Recall that $\liminf\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \underbrace{\inf_{k\geq n} f_k(x)}_{g_n(x)}$.

Note that $g_n = \inf\limits_{k\geq n} f_k$ is a non-decreasing sequence because, as n increases, we are taking infima over smaller sets:

$$m > n \Rightarrow \inf_{k\geq n} f_k \leq \inf_{k\geq m} f_k$$

Also observe that

$$g_n(x) = \inf_{k \geq n} f_k(x) \leq f_n(x)$$

so

$$\int g_n \, d\mu \leq \int f_n \, d\mu \quad .$$

For $k \geq n$ we have:

$$\int g_n d\mu \leq \int g_k \, d\mu \leq \int f_k d\mu$$

so

$$\int g_n d\mu \leq \inf_{k \geq 1} \int f_k \, d\mu \qquad \underline{(*)}$$

$$\int \liminf_{n\to\infty} f_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu$$

$$= \lim_{n\to\infty} \int g_n \, d\mu \qquad \text{by the Monotone Conv. Thm.}$$

$$\leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k \, d\mu \qquad \text{by } (\underline{*})$$

$$= \liminf_{n\to\infty} \int f_n \, d\mu$$

This is what we wanted to prove.