Measure Theory: Exercises (not for credit)

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Question 1. Use the uniqueness of extension theorem from the notes to show that Lebesgue measure in $\mathbb{R}^d$ is translation invariant on $\mathcal{B}(\mathbb{R})$. That is to say if we define the set $A+x = \{z \in \mathbb{R}^d : z = x+y, y \in A\}$. Show also that Lebesgue measure in $\mathbb{R}^d$ is rotationally invariant. That is to say if M is a rotation matrix in $\mathbb{R}^d$ then $\lambda(A) = \lambda(M^{-1}(A))$ where we understand M here to represent the map $x \mapsto Mx$.

Question 2. Let λ be Lebesgue measure on $\mathbb{R}^d$ let $L_{m,c} = \{(x,y) : y = mx + c\}$ for some $m, c \in \mathbb{R}$. Show that $\lambda L = 0$.

Question 3. Let f be the map on $\mathbb{R}^d$ given by $f(x) = 3x$. Write down an expression for $\lambda(f(A))$ and prove that it is correct.

Question 4. Prove that if A is Lebesgue measurable then $A = B \cup N$ where B is an F_σ set (a countable union of closed sets) and N is a null set. Show the converse, that if $A = B \cup N$ where B is F_σ and N is null then A is Lebesgue measurable.

Question 5. Show that there is a Lebesgue measurable subset of $\mathbb{R}^2$ whose projection under the map $(x,y) \mapsto x$ is not Lebesgue measurable.

Question 6. Let E, F be topological spaces and let $f : E \to F$ be a continuous function. Show that f is measurable with respect to the Borel σ algebras on E and F.

Question 7. Let $(E, \mathcal{A})$ be a measurable space and let $A \in \mathcal{A}$. Show that 1_A the indicator function of A is measurable with respect to any σ algebra on $\mathbb{R}$.