Week 4 Video - Egoroff's Theorem

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Let $(E, \mathcal{E}, \mu)$ a finite measure space $(\mu(E) < \infty)$

let f_n be a sequence of real valued functions s.t. $f_n \to f$ almost everywhere. Then for each $\varepsilon > 0$ there exists a set A s.t. $\mu(A^c) < \varepsilon$ and $f_n \to f$ uniformly on A.

Pf/ For each n define $g_n(x) = \sup_{j \geq n} |f_j(x) - f(x)|$

so g_n is finite a.e (for n suff large)

$g_n \geq 0$ and $g_n \to 0$ almost everywhere.

$g_n \to 0$ almost everywhere $\Rightarrow$ $g_n \to 0$ in measure
$(\mu(E) < \infty)$.

We can find an n_k, for each k s.t.

$$\mu(\{x : g_{n_k} > \tfrac{1}{k}\}) < \varepsilon 2^{-k}$$

Define the sets $A_k = \{x : g_{n_k}(x) \leq \frac{1}{k}\}$

then let $A = \bigcap_k A_k$ $\quad \mu(A^c) = \mu(\bigcup_k A_k^c) \leq \sum_k \mu(A_k^c)$

$$\leq \sum_k \varepsilon 2^{-k} = \varepsilon$$

We want $f_n \to f$ uniformly on A.

For each δ $\exists$ k s.t. $1/k < \delta$ then as $A \subseteq A_k$

If $n \geq n_k$ $\quad |f_n - f| \leq g_{n_k} \leq \frac{1}{k} < \delta$

If $n \geq n_k$ $|f_n - f| \leq g_{n_k} \leq k$ -

as we're on A_k

So $f_n \to f$ uniformly on A.

This motivates a def^n:

Almost uniform convergence

So $f_n \to f$ almost uniformly on $(E, \mathcal{E}, \mu)$ if for every ε $\exists$ $A \in \mathcal{E}$ with $\mu(A^c) < \varepsilon$ s.t. $f_n \to f$ uniformly on A.