Measure Theory: Exercises (not for credit)

Josephine Evans

November 22, 2021

Question 1. Use the inequality $(x-y)^2 \geq 0$ to show that $xy \leq (x^2+y^2)/2$.

Answer: $(x-y)^2 \geq 0$ turns into $x^2+y^2-2xy \geq 0$ so rearranging gives the inequality. □

Question 2. Draw a graph of the function $t = s^{p-1}$. Let L be the are underneath the graph and above the s-axis and U be the area above the graph and to the right of the t-axis. Compute the areas of $L \cap \{s : 0 \leq s \leq x\}$ and $U \cap \{t : 0 \leq t \leq x\}$ and draw these two sets on your picture. Use this to show that $xy \leq x^p/p + y^q/q$.

Answer: It is hard to type an answer to this but the area under the graph will have size x^p/p and the area above the graph will have size y^q/q and the box of width x and height y is contained inside the union of the two areas. □

Question 3. Let $(E, \mathcal{E}, \mu)$ be a finite measure space and let $p_1 \leq p_2$ show that if $f \in L^{p_2}$ then $f \in L^{p_1}$.

Answer: You can do this using either Holder or Jensen's inequality. The method via Jensen is

$$\left(\int_E |f(x)|^{p_1} \mu(\mathrm{dx})\right)^{p_2/p_1} = \mu(E)^{p_1/p_2} \left(\frac{1}{\mu(E)} \int_E |f(x)|^{p_1} \mu(\mathrm{d}x)\right)^{p_2/p_1}.$$

Then the measure $\mu/\mu(E)$ gives the space E measure 1, and $x \mapsto x^{p_2/p_1}$ is a convex function so we can apply Jensen to get

$$\mu(E)^{p_1/p_2} \left(\frac{1}{\mu(E)} \int_E |f(x)|^{p_1} \mu(\mathrm{d}x)\right)^{p_2/p_1} \leq \mu(E)^{p_1/p_2} \frac{1}{\mu(E)} \int |f(x)|^{p_1 * p_2/p_1} \mu(\mathrm{d}x) = \mu(E)^{p_1/p_2-1} \|f\|_{p_2}^{p_2}.$$

Putting this all together we have

$$\|f\|_{p_1} \leq \mu(E)^{p_1/p_2-1} \|f\|_{p_2}.$$

□

Question 4. Let X be a random variable. Prove the identity

$$\mathbb{E}(|X|^p) = \int_0^\infty p x^{p-1} \mathbb{P}(|X| > x) \mathrm{d}x$$

and hence show that if for all $q > p$ we have $\mathbb{P}(|X| > x) = O(x^{-q})$ as $x \to \infty$ then $X \in L^p$. (Recall that X is a measurable function from a probability space to $\mathbb{R}$, $\mathbb{P}$ is the measure on this space and $\mathbb{E}$ is the notation for integrating with respect to $\mathbb{P}$.

Answer:

$$\begin{aligned}
\mathbb{E}(|X|^p) &= \int_\Omega |x|^p \mathbb{P}(\mathrm{d}x) = \int_\Omega \left(\int_0^{|x|} p y^{p-1} \mathrm{d}y \right) \mathbb{P}(\mathrm{d}x) \\
&= \int_\Omega \int_0^\infty p y^{p-1} 1_{y \le |x|} \mathrm{d}y \mathbb{P}(\mathrm{d}x) \\
&= \int_0^\infty \int_\Omega 1_{y \le |x|} \mathbb{P}(\mathrm{d}x) p y^{p-1} \mathrm{d}y \\
&= \int_0^\infty \mathbb{P}(|X| \ge y) p y^{p-1} \mathrm{d}y.
\end{aligned}$$

Now if $\mathbb{P}(|X| > x) = O(x^{-q})$ for every $q > p$ then there will exist a C such that $\mathbb{P}(|X| > x) \le C x^{-p-1}$ then

$$\mathbb{P}(|X|^p) \le \int_1^\infty \min\{p y^{p-1}, C y^{-2}\} \mathrm{d}y < \infty.$$

$\square$