This is an application of what we learnt in w10 about signed measures and the Radon-Nikodym Theorem.

__Definition__ If V is a Banach space (complete, normed vector space) then the dual space of V is written V' and is the space of all bounded linear operators $V \to \mathbb{R}$. We can put a norm on V' by

$$\|K\| = \sup_{v \neq 0} \frac{K(v)}{\|v\|} = \sup_{\|v\|=1} K(v).$$

=

Now if $g \in L^q(E, \mathcal{E}, \mu)$ then we can define a linear operator $K_g : L^p(E, \mathcal{E}, \mu) \to \mathbb{R}$ by $K_g(f) = \mu(fg)$.

This is bounded by Hölder's inequality

$$|\mu(fg)| \leq \mu(|fg|) \leq \|f\|_p \|g\|_q$$

so $\|K_g\| \leq \|g\|_q$.

__Theorem__ Let $(E, \mathcal{E}, \mu)$ be a __finite__ measure space and $p \in (1, \infty)$. Then the map $g \mapsto K_g$ defines a bijective isometry between $L^q(E, \mathcal{E}, \mu)$ and $(L^p(E, \mathcal{E}, \mu))'$. $(\|K_g\| = \|g\|_q)$.

__Remark__ We don't need the finite assumption its there to make the proof shorter.

Remark ... ...

just there to make the proof shorter.

We've already shown $g \mapsto K_g$ is linear and $\|K_g\| \leq \|g\|_q$.

Claim $\|K_g\| = \|g\|_q$.
We look at $f = \operatorname{sgn}(g(x))\, |g(x)|^{q-1}$ then

$f \in L^p$ $\quad \mu(|f|^p) = \mu(|g|^q)$ so $\|f\|_p = \|g\|_q^{q/p}$

so $$\frac{K_g(f)}{\|f\|_p} = \frac{\mu(|g|^q)}{\|g\|_q^{q/p}} = \|g\|_q^{q(1-\frac{1}{p})} = \|g\|_q$$

$\|K_g\| \geq \dfrac{K_g(f)}{\|f\|_p}$ and $\|K_g\| \leq \|g\|_q$ so $\|K_g\| = \|g\|_q$

Shows $g \longmapsto K_g$ is an isometry to some subset of $L^p(E)'$. We want to show its surjective.

Take K an arbitrary element of $L^p(E)'$ wts $\exists g$ s.t. $K = K_g$. $\quad$ wts $K(f) = \mu(fg)$ some g.

We can make a signed measure from K. Let $A \in \mathcal{E}$ then define a function $k : \mathcal{E} \to \mathbb{R}$ by $k(A) = K(\mathbb{1}_A)$.
$\mathbb{1}_A \in L^p$ since its a finite space. $\quad \|\mathbb{1}_A\|_p = \mu(A) < \infty$

Then define a function $K: \mathcal{L}$. & $\|\mathbb{1}_A\|_p = \mu(A) < \infty$

$\mathbb{1}_A \in L^p$ since its a finite space.

Claim: k is a signed measure

$$k(\emptyset) = K(0) = 0.$$

Now let $A_1, A_2, A_3, \dots$ be a sequence of disjoint sets in $\mathcal{E}$. Then

$$k\left(\bigcup_{j=1}^{n} A_j\right) = K\left(\mathbb{1}_{\bigcup_{j=1}^{n} A_j}\right) = K\left(\sum_{j=1}^{n} \mathbb{1}_{A_j}\right) = \sum_{j=1}^{n} K(\mathbb{1}_{A_j}) = \sum_{j=1}^{n} k(A_j)$$

so we know k is finitely additive

$$\left\| \mathbb{1}_{\bigcup_{j=1}^{\infty} A_j} - \mathbb{1}_{\bigcup_{j=1}^{n} A_j} \right\|_p = \left\| \mathbb{1}_{\bigcup_{j=n+1}^{\infty} A_j} \right\|_p = \mu\left(\bigcup_{j=n+1}^{\infty} A_j\right) \to 0$$

as we're in a finite measure space and the A_j are disjoint.

The fact that K is bounded means its continuous

$$\left| K\left(\mathbb{1}_{\bigcup_{j=1}^{n} A_j}\right) - K\left(\mathbb{1}_{\bigcup_{j=1}^{\infty} A_j}\right) \right| = \left| K\left(\mathbb{1}_{\bigcup_{j=n+1}^{\infty} A_j}\right) \right| \leq \|K\| \, \left\| \mathbb{1}_{\bigcup_{j=n+1}^{\infty} A_j} \right\|_p$$

$\to 0$ as $n \to \infty$

$$\text{so} \quad K\left(\mathbb{1}_{\bigcup_{j=1}^{\infty} A_j}\right) = \lim_{n\to\infty} K\left(\mathbb{1}_{\bigcup_{j=1}^{n} A_j}\right) = \lim_{n\to\infty} \sum_{j=1}^{n} K(\mathbb{1}_{A_j})$$

$|K(\mathbb{1}_{A_j})| \leq \|K\| \mu(A_j)$

$$\sum_{j=1}^{\infty} \mu(A_j) < \infty \qquad = \sum_{j=1}^{\infty} K(\mathbb{1}_{A_j})$$

$$k\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} k(A_j)$$

so k is countably additive.

... signed measure and by our

So k is a signed measure and by our decomposition theorem $\exists$ (P, N) s.t. $E = P \cup N$
P is a +ve set for k, N is a negative set for k

$k_+(A) = k(A \cap P)$ $\qquad$ $k_-(A) = -k(A \cap N)$

$k = k_+ - k_-$.

k_+ and k_- are both finite measures.

We want to apply the Radon–Nikodym theorem so wts $k_+ \ll \mu$ and $k_- \ll \mu$.

So take some $A \in \mathcal{E}$ with $\mu(A) = 0$

$\mu(A \cap P) = 0 = \mu(A \cap N)$

so $k_+(A) = k(A \cap P) = K(\mathbb{1}_{A \cap P})$

then $\mathbb{1}_{A \cap P}$ is a.e. equal to the zero function since K is an operator on L^p it only sees functions up to almost everywhere equality so $K(\mathbb{1}_{A \cap P}) = K(0) = 0$

so $k_+(A) = 0$ and in exactly the same way $k_-(A) = 0$.

so we've shown $k_+, k_- \ll \mu$.

Therefore we can apply Radon-Nikodym to get functions g_+ and g_- s.t. $k_+(A) = \mu(g_+ \mathbb{1}_A)$ and $k_-(A) = \mu(g_- \mathbb{1}_A)$. Let $g = g_+ - g_-$.

now we know $K(\mathbb{1}_A) = \mu(g \mathbb{1}_A)$

we want to show $g \in L^q(E, \mathcal{E}, \mu)$ and $K(f) = \mu(fg)\ \forall f$.

Let us define $E_n = \{x : |g(x)| \leq n\}$

then $g\mathbb{1}_{E_n}$ is in L^q $\mu(|g\mathbb{1}_{E_n}|^q)$

$$= \mu(|g|^q \mathbb{1}_{E_n})$$
$$\leq n\,\mu(E_n)$$
$$\leq n\,\mu(E) < \infty.$$

Let us define two sequences of linear operators on L^p. K_n be define by $K_n(f) = K(f\mathbb{1}_{E_n})$

$\tilde{K}_n$ be defined by $\tilde{K}(f) = \mu(gf\mathbb{1}_{E_n})$

then $\|\tilde{K}_n\| = \|g\mathbb{1}_{E_n}\|$

$$\|K_n\| = \sup_{\|f\|=1} K(f\mathbb{1}_{E_n}) \leq \sup_{\|f\|_p=1} \frac{K(f\mathbb{1}_{E_n})}{\|f\mathbb{1}_{E_n}\|_p}$$

$\|f\mathbb{1}_{E_n}\|_p \leq \|f\|_p$

$$\leq \sup_{f\neq 0} \frac{K(f)}{\|f\|_p} = \|K\|$$

Claim: $K_n(\mathbb{1}_A) = \tilde{K}_n(\mathbb{1}_A)$

$$K_n(\mathbb{1}_A) = K(\mathbb{1}_{A\cap E_n}) = k(A\cap E_n) = \mu(g\mathbb{1}_{A\cap E_n})$$
$$= \mu(g\,\mathbb{1}_A\mathbb{1}_{E_n}) = \tilde{K}_n(\mathbb{1}_A).$$

So if h is a simple function the by linearity

$K_n(h) = \tilde{K}_n(h)$.

Now if $f \in L^p(E)$, $\varepsilon > 0$ then $\exists$ h simple with

$\|f-h\|_p < \varepsilon$.

$$|k_n(f)-\tilde{k}_n(f)| \leq |k_n(f)-k_n(h)| + \overbrace{|k_n(h)-\tilde{k}_n(h)|}^{0} + |\tilde{k}_n(h)-\tilde{k}_n(f)|$$

$$\leq \left(\|k_n\| + \|\tilde{k}_n\|\right)\|f-h\|_p$$

$$\leq \left(\|k_n\| + \|\tilde{k}_n\|\right)\varepsilon$$

since ε is arbitrary this means

$k_n(f) = \tilde{k}_n(f)$.

so $\|k_n\| = \|\tilde{k}_n\|$

we showed $\|k_n\| \leq \|k\|$

and $\|\tilde{k}_n\| = \|g\mathbb{1}_{E_n}\|_q$

so $\|g\mathbb{1}_{E_n}\|_q \leq \|k\| \quad \forall n$

$\mu(|g\mathbb{1}_{E_n}|^q) = \mu(|g|^q\mathbb{1}_{E_n})$ so $|g|^q\mathbb{1}_{E_n} \uparrow |g|^q$

so by monotone convergence $\|g\mathbb{1}_{E_n}\|_q \to \|g\|_q$

so $\|g\|_q \leq \|k\|$ so $g \in L^q$.

so we know $k = k_g$ on indicator functions, so we know it for simple function and since we can approximate any function in L^p by a simple $k = k_g$ on all $f \in L^p$.