Let me briefly recall why we are interested in integrating weird functions (motivated by a question from Oskar.)

In the 1800's mathematicians thought about new ways of defining functions.

Fourier described a method of writing an arbitrary function f on $[-\pi,\pi]$ as a series

$$f \sim a_0 + \sum_{k=1}^{\infty} b_k \cos(k\cdot x) + \sum_{k=1}^{\infty} c_k \sin(k\cdot x).$$

He gave a formula for calculating the coefficients in terms of integrals.

$$f \sim a_0 + \sum_{k=1}^{\infty} b_k \cos(k \cdot x) + \sum_{k=1}^{\infty} c_k \sin(k \cdot x).$$

How can you understand things like the integral of f?

Let $$f_n = a_0 + \sum_{k=1}^{n} b_k \cos(k \cdot x) + \sum_{k=1}^{n} c_k \sin(k \cdot x).$$

You know every thing about the integral of f_n. The only way to approach f is through a limit theorem.

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

The Dirichlet function example shows that the limits of nice functions can be messy.

Any theory of integration in which you can prove limit theorems will need to deal with weird functions.

It is reasonable to think of this weeks lectures as aiming towards the Dominated Convergence Thm.

This says that for a sequence of functions f_n converging pointwise to f

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

It has no unnecessary hypotheses about positivity or monotonicity it just has one absolutely necessary hypothesis that the family of functions f_n are "dominated" by an integrable function g.

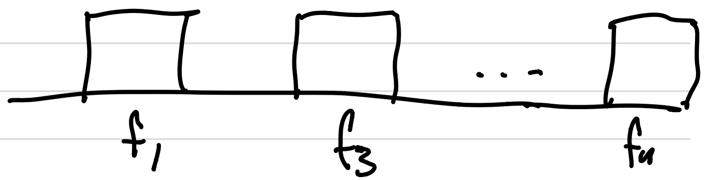
$$|f_n| \leq g.$$

The sliding bump example shows the necessity.

Having said that there are times when you want to use these other results since each has its own strengths.

Let's review the key theorems.

Monotone Convergence Theorem.



Theorem 2.4.1 (Monotone Convergence). Let f and $f_1, f_2 \ldots$ be $[0, +\infty]$ valued measurable functions on X. Suppose

$$f_1(x) \le f_2(x) \le \ldots$$

and

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Then $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$.

The idea is to reduce this to a previously proved result where the f_n are simple functions.

$$\begin{array}{ccccccc}
f_1 & \le & f_2 & \le & f_3 & \cdots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \\
\rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & &
\end{array}$$

$$
\begin{array}{ccccccc}
f_1 & \le & f_2 & \le & f_3 & \cdots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\rotatebox[origin=c]{90}{$\le$} & & \rotatebox[origin=c]{90}{$\le$} & & \rotatebox[origin=c]{90}{$\le$} & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \\
\rotatebox[origin=c]{90}{$\le$} & & \rotatebox[origin=c]{90}{$\le$} & & \rotatebox[origin=c]{90}{$\le$} & & \\
g_{1,1} & & g_{2,1} & & g_{3,1} & &
\end{array}
$$

$$\begin{array}{ccccccc}
f_1 & \le & f_2 & \le & f_3 & \cdots & \to f \\
\uparrow & & \uparrow & & \uparrow & & \nearrow \\
\vdots & & \vdots & & \vdots & & \\
g_{1,3} & & g_{2,3} & & g_{3,3} & & \\
\vee\!| & & \vee\!| & & \vee\!| & & \\
g_{1,2} & & g_{2,2} & & g_{3,2} & & \\
\vee\!| & & \vee\!| & & \vee\!| & & \\
\boxed{g_{1,1}} & & g_{2,1} & & g_{3,1} & &
\end{array}$$

$\underline{h_3} = \max\{g_{11} \ldots g_{3,3}\}.$

$n \times n$ box

$\underline{h_n} = \max\{ \underline{\qquad\qquad} \}$

Fatou's Lemma also considers a sequence of functions f_n. It assumes that the f_n are non-negative but makes no assumption about convergence.

If we have an arbitrary of real numbers we can get convergent sequences by considering liminf and limsup.

This is what Fatou's Lemma deals with.

It shows that:

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

$$\qquad\qquad 0 \qquad\qquad\qquad\qquad 1$$

On the left the lim inf forces $f_n(x)$ to yield a number. On the right lim inf forces the sequence $\int f_n \, d\mu$ to yield a number.

The point here is that equality may fail

(consider sliding bump functions) but it always fails in the same direction. Failure is related to "loss of mass" phenomenon.

$$f_n \to 0$$

$$\int f_n \, d\mu \to 1$$

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu .$$

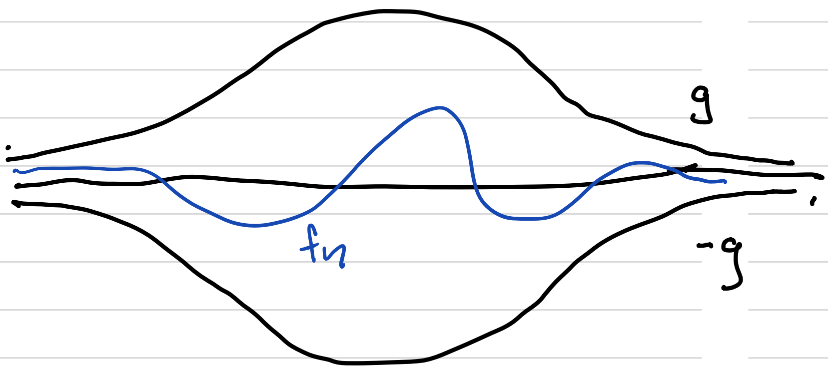
The proof is quick. It uses the fact that the liminf construction implicitly involves a non-decreasing function.

$$\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \underbrace{\inf_{k\ge n} f_k(x)}_{\text{non-decreasing}}$$

Dominated Convergence Thm.

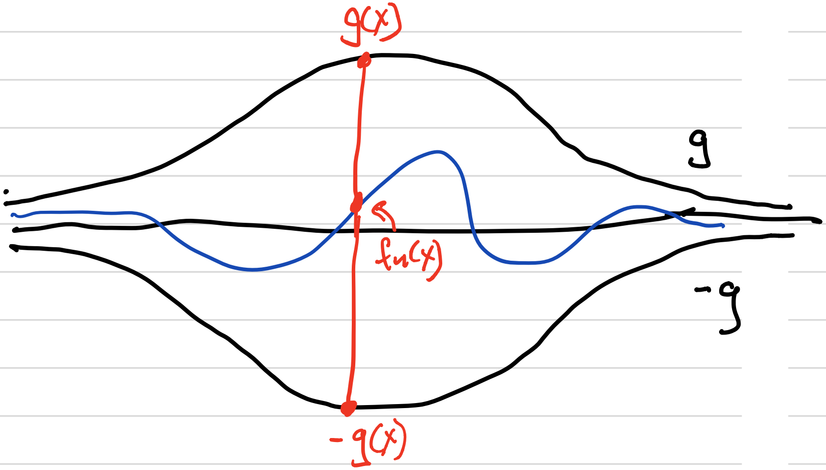
Hypothesis: The sequence f_n is "dominated" by a non-negative integrable function g. That is:

$$|f_n(x)| \le g(x).$$



Single function g dominates the family of functions f_n.

The technique of proof comes from observing that there are two ways to get non-negative functions out of this picture:



Consider: $x \rightsquigarrow g(x) - f_n(x)$

$x \longmapsto f_n(x) - (-g(x))$

$\qquad\qquad = f_n(x) + g(x)$

Applying Fatou to $f_n(x) + g(x)$ we get

$$\int \liminf f_n + g \, d\mu \leq \liminf \int f_n + g \, d\mu$$

or

$$\int g \, d\mu + \int \liminf f_n \, d\mu \leq \int g \, d\mu + \liminf \int f_n \, d\mu$$

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu$$

or

$$\int f \, d\mu \leq \liminf \int f_n \, d\mu$$

Applying Fatou to

$g(x)-f_n(x)$ we get

$$\int \liminf g-f_n \, d\mu \le \liminf \int g-f_n \, d\mu$$

or

$$\int g \, d\mu + \int \liminf(-f_n) d\mu \le \int g \, d\mu + \liminf \int (-f_n) d\mu$$

$$\int \liminf(-f_n) d\mu \le \liminf \int (-f_n) d\mu$$

$$-\int \limsup f_n \, d\mu \le -\limsup \int f_n \, d\mu$$

$$\int \limsup f_n \, d\mu \ge \limsup \int f_n \, d\mu$$

or $\int f \, d\mu \ge \limsup \int f_n \, d\mu.$

So

$$\int f \, d\mu \ge \limsup \int f_n \, d\mu \ge \liminf \int f_n \, d\mu \ge \int f \, d\mu$$

or

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

QED

The next result is one of many that say that you can "ignore sets of measure 0".

There are however different arguments in different settings that are required to do this.

You get no credit on the exam for saying that "you can ignore sets of measure 0."

Proposition 2.3.9. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and g be $[-\infty, \infty]$ valued $\mathcal{A}$ measurable functions on X that agree almost everywhere. If $\int f d\mu$ exists then $\int g d\mu$ exists and $\int f d\mu = \int g d\mu$.

Proof is based on the idea that if A is a set of measure 0 and

$$h(x) = \begin{cases} \infty & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

then $\int h \, d\mu = 0$. We discussed this

last Monday.

If $f(x) = g(x)$ for $x \notin A$ and $\mu(A) = 0$ then

$f \leq g + h$ so

$$\int f \, d\mu \leq \int g + h \, d\mu \leq \int g \, d\mu + \int h \, d\mu = \int g \, d\mu.$$

Theorem 2.4.1 (Monotone Convergence). Let f and $f_1, f_2 \dots$ be $[0, +\infty]$ valued measurable functions on X. Suppose

$$f_1(x) \leq f_2(x) \leq \dots$$

and

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for a.e. x.

Then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.

Markov inequality.

Prop. Let f be a $[0,+\infty]$ valued $\mathcal{A}$-measurable function on X. If t is a positive real number and if A_t is defined by

$$A_t = \{x \in X : f(x) > t\}$$

then

$$\mu(A_t) \le \frac{1}{t}\int_{A_t} f\,d\mu \le \frac{1}{t}\int f\,d\mu.$$

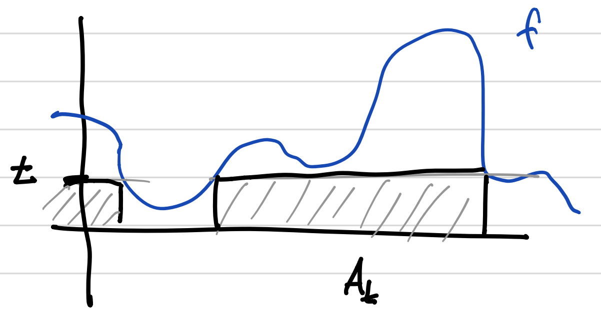
Proof.

$$0 \le t\cdot\chi_{A_t} \le f\cdot\chi_{A_t} \le f$$

and Prop 2.3.4(c) imply that

$$\int t\cdot\chi_{A_t}\,d\mu \le \int_{A_t} f\cdot d\mu \le \int f\,d\mu$$

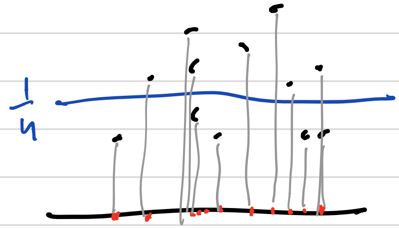
$$\Rightarrow \mu(A_t) \le \frac{1}{t}\int_{A_t} f\,d\mu \le \frac{1}{t}\int f\,d\mu.$$



Cor. 2.3.12 Let f be a $[-\infty, \infty]$ valued $\mathcal{A}$-measurable function on X that satisfies

$$\int |f| \, d\mu = 0$$

then $f = 0$ μ-almost everywhere.



We will use this in the discussion of the Riemann integral which is delayed until next week.