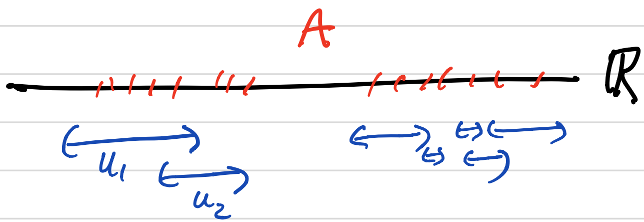
Recall that in lecture 1 we defined Lebesgue outer measure.

Our goal is to extend the definition of length to arbitrary subsets $A \subset \mathbb{R}$.



$\mathcal{U} = \{u_j\}$

Definition:

$$\lambda^*(A) = \inf_{\mathcal{U} = \{u_j\}} \left\{ \sum_{j=1}^{\infty} \text{length}(u_j) : A \subset \bigcup_{j=1}^{\infty} u_j \right\}$$

We did two calculations:

$\lambda^*(\mathbb{Q} \cap [0,1]) = 0$.

$\lambda^*([a,b]) = \text{length}([a,b]) = b-a.$

Now I want to describe some general properties of λ^*.

Prop.

(a) $\lambda^*(\emptyset) = 0$

(b) If $A \subset B \subset \mathbb{R}$

$\lambda^*(A) \leq \lambda^*(B)$

monotonicity

(c) If $\{A_j\}$ is a countably infinite sequence of sets then:

$$\lambda^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \lambda^*(A_j).$$

Countable sub-additivity

Proof of (a). $\lambda^*(\emptyset) = 0.$

Note that any set contains the null set so any set gives a cover of ϕ.

In particular the set $U_\varepsilon = (0, \varepsilon)$ covers the empty set so $\mathcal{U} = \{U_\varepsilon\}$

$$\lambda^*(\emptyset) \leq \text{length}((0,\varepsilon)) = \varepsilon$$

for all $\varepsilon > 0$ $\lambda^*(\emptyset) \leq 0$.

Thus $\lambda^*(\emptyset) = 0$.

Proof of (b) If $A \subset B \subset \mathbb{R}$ then $\lambda^*(A) \le \lambda^*(B)$.

If $\{U_j\}$ is a cover of B

then $B \subset \bigcup_j U_j$ so

$A \subset B \subset \bigcup_j U_j$.

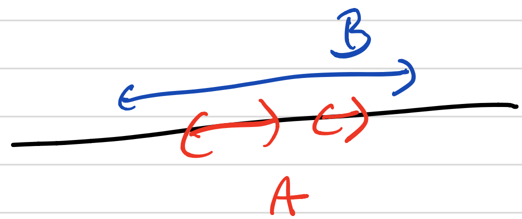
Thus $\{U_j\}$ is a cover of A.

$$\lambda^*(A) = \inf_{\mathcal{U} \text{ a cover of } A} \sum \text{lengths}(U_j)$$

$$\lambda^*(B) = \inf_{\mathcal{U} \text{ a cover of } B.} \sum$$

The set of covers of A is a subset of the set of covers of B.

"Taking an infimum over a smaller set gives a larger answer" so:



$$\lambda^*(B) = \inf_{\text{covers of } B} \ge \inf_{\text{covers of } A} = \lambda^*(A).$$

(c)

$$\boxed{\lambda^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \lambda^*(A_j)}$$

Let $\varepsilon > 0$.

We use the "$\varepsilon/2^n$ trick".

Find a cover $\mathcal{U}_j$ of A_j with

$$\sum_k \text{length}(U_{jk}) \leq \lambda^*(A_j) + \frac{\varepsilon}{2^j}$$

Let $\mathcal{U}_\infty$ be the union of all the covers $\mathcal{U}_j$ then $\mathcal{U}_\infty$ covers $\bigcup_n A_n$ and

$$\lambda^*\left(\bigcup_j A_j\right) \leq \sum_{j,k} \text{length}(U_{jk})$$

$$= \sum_j \sum_k \text{length}(U_{jk})$$

$$\leq \sum_j \lambda^*(A_j) + \frac{\varepsilon}{2^j}$$

$$\leq \left(\sum_j \lambda^*(A_j)\right) + \varepsilon$$

Since this holds for every $\varepsilon > 0$ we get

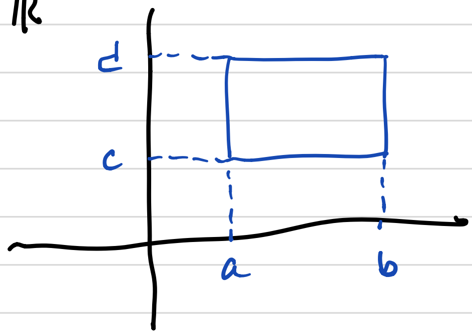
$$\lambda^*\left(\bigcup_j A_j\right) \leq \sum_j \lambda^*(A_j).$$

The construction of Lebesgue outer measure is so straightforward that we can use it in $\mathbb{R}^n$ for any n.

This should allow us to construct generalised area in $\mathbb{R}^2$, volume in $\mathbb{R}^3$ and

n-dimensional volume.

In this discussion the role of the role of the interval will be played by a coordinate rectangle in $\mathbb{R}^2$



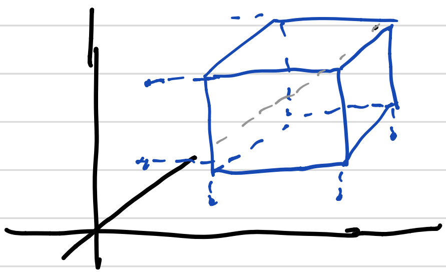
which we denote by:

$(a,b) \times (c,d)$ (open)

or $[a,b] \times [c,d]$. (closed)

area $= (b-a)(c-d)$.

In $\mathbb{R}^3$ we have a coordinate box:



Coordinate box:

$(a,b) \times (c,d) \times (e,f)$ in $\mathbb{R}^3$

with volume $= (b-a)(d-c)(f-e)$.

In $\mathbb{R}^n$ we have a set

$(a_1,b_1) \times \dots \times (a_n,b_n)$ in $\mathbb{R}^n$.

or

$\{x : a_j < x_j < b_j \text{ for } j=1\dots n\}$

with volume $\prod_{j=1}^{n} (b_j - a_j)$.

I will present the discussion for $n=2$. The general case differs only in that the notation is more complicated.

(Also the pictures are easier to draw in $\mathbb{R}^2$.)

Now let A be an arbitrary subset of $\mathbb{R}^2$.

Let $\{U_j\}$ be a covering of A ($A \subset \bigcup U_j$).

$\mathcal{U} = \{U_j\}$ be a covering of A by open rectangles (coordinate)

$$U_j = (a_j, b_j) \times (c_j, d_j)$$

For $A \subset \mathbb{R}^2$ define:

$$\lambda_2^*(A) = \inf_{\mathcal{U}} \sum \text{area}(u_i)$$

(For $A \subset \mathbb{R}^n$ we define

$$\lambda_n^*(A) = \inf_{\mathcal{U}} \sum \text{vol}(u_j).\ )$$

Prop.

(a) $\lambda_n^*(\emptyset) = 0$

(b) $\lambda_n^*(A) \leq \lambda_n^*(B)$.

(c) If $\{A_j\}$ is a countably infinite sequence of sets then:

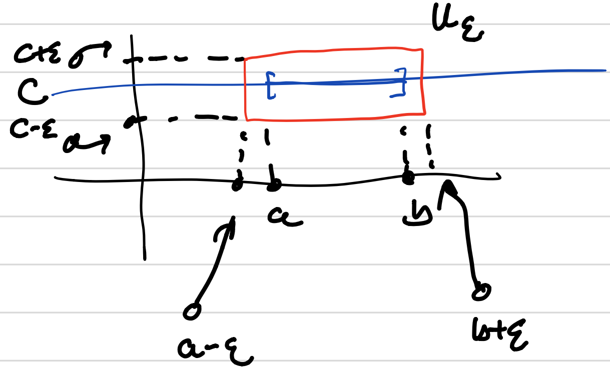
$$\lambda_n^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \lambda_n^*(A_j).$$

The proof is exactly the same as for the case of $\lambda = \lambda_1$.

Example. For any horizontal or vertical line ℓ in $\mathbb{R}^2$ $\lambda_2^*(\ell) = 0$.

Proof. Let's start by considering a horizontal interval.

$$A = \{(x,y) : a \le x \le b,\ y = c\}$$



Consider the cover $\mathcal{U}_\varepsilon$ by a single set U_ε. $\mathcal{U}_\varepsilon = \{U_\varepsilon\}$.

$$U_\varepsilon = \{(x,y) : a-\varepsilon < x < b+\varepsilon,\ c-\varepsilon < y < c+\varepsilon\}$$

$$\lambda^*(A) = \inf_{\mathcal{U}} \sum \text{area}(U_j)$$

$$\leq \sum_{\mathcal{U}_\varepsilon} \text{area}(U_\varepsilon)$$

$$= \text{area}(U_\varepsilon)$$

$$= (b-a+2\varepsilon)(2\varepsilon)$$

As $\varepsilon \to 0$ this quantity tends to 0.

A horizontal line is a countable union of horizontal intervals.

$\lambda^*(A) = 0.$

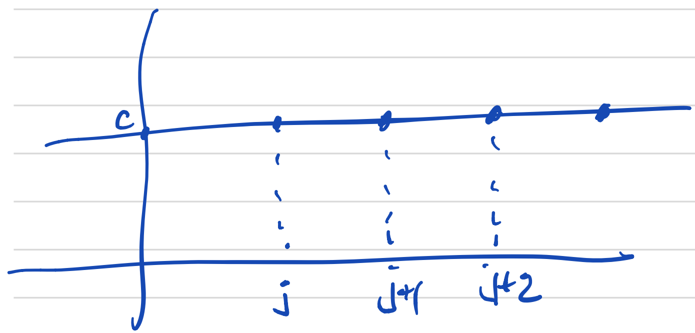
Now apply the countable subadditivity property for outermeasures.

This gives

$$\lambda^*(\text{line}) \leq \sum_{j=1}^{\infty} \lambda^*\big([j,j+1) \times \{c\}\big)$$

$$\leq \sum_{j=1}^{\infty} 0$$

$$= 0.$$



Now recall that an important point concerning the outer measure of the closed interval was that it agreed with the length of the interval.

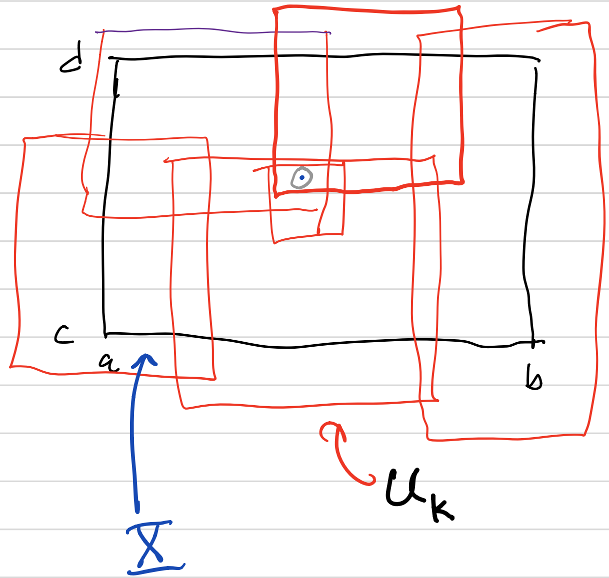
Thm. For a closed rectangle $R = I_1 \times I_2 \times \dots I_n \subset \mathbb{R}^n$

$$\lambda_n^*(R) = \mathrm{vol}(R) = \prod_{j=1}^{n} \mathrm{length}(I_j).$$

To prove this we will need a different property of open covers of compact sets in metric spaces. $\mathcal{U} = \{U_j\}$

Prop. If $\mathcal{U} = \{U_j\}$ is an open cover of a compact metric space X then there is a constant $\varepsilon > 0$ so that for any $x \in X$ the ball of radius ε around x is contained in a single set $U_j \in \mathcal{U}$.

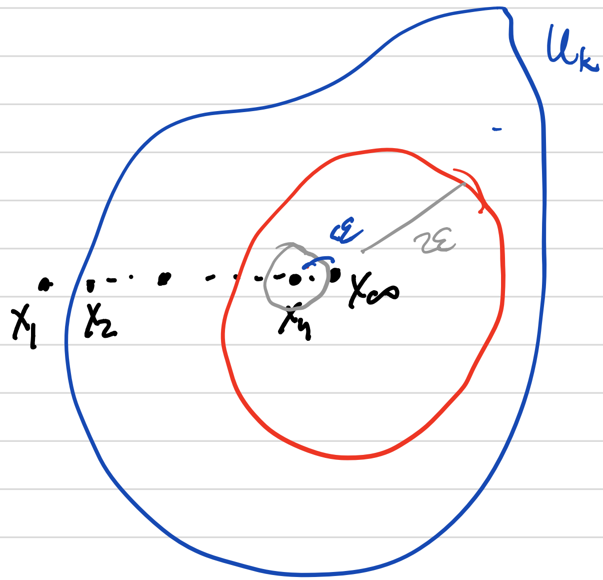
ε is called the <u>Lebesgue constant</u> of the cover.



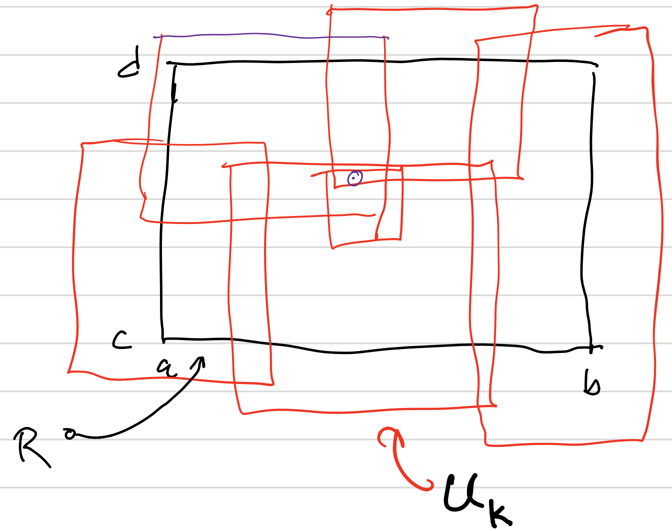
Proof. Say we have $\mathbb{X}$ and $\mathcal{U}$ but the proposition fails. Then for every n there is an $x_n \in \mathbb{X}$ where $B(x_n, \frac{1}{n})$ is not contained in a single open set.

Using compactness there is a subsequence $x_{n_j} \to x_\infty$. But $x_\infty \in U_k$ and U_k is open so there is some ball $B(2\varepsilon, x_\infty) \subset U_k$.

Choose n so that $d(x_n, x_\infty) < \varepsilon$ and $\frac{1}{n} < \varepsilon$ then $B(\varepsilon, x_n) \subset B(2\varepsilon, x_\infty) \subset U_k$ contradicting our assumption.



Thm. λ_2^* satisfies $\lambda_2^*(R) = \text{area}(R)$ when R is a rectangle.



Proof. As before it is easy to see that $\lambda_2^*(R) \leq \text{area}(R)$ by constructing a covering with a single rectangle.

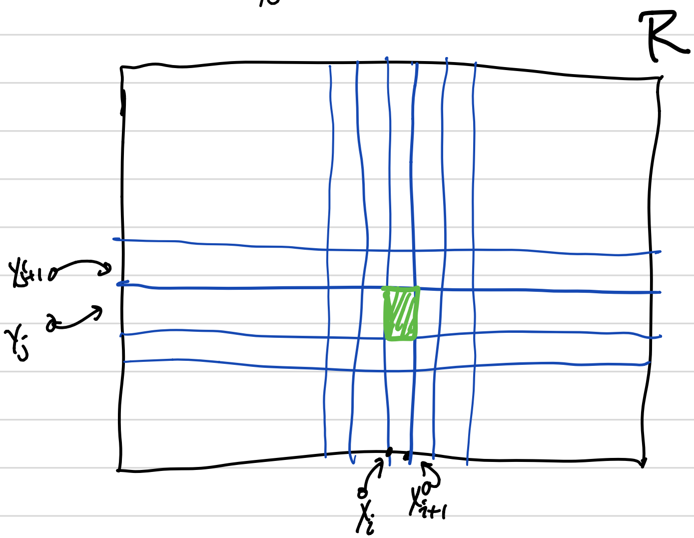
Now we prove the reverse inequality that $\lambda_2^*(R) \geq \text{area}(R)$.

Let $\mathcal{U} = \{U_k\}$ be a countable covering of R by open rectangles.

Chop up R into rectangles $S^{i,j}$ with diameter less than the Lebesgue constant of the covering.

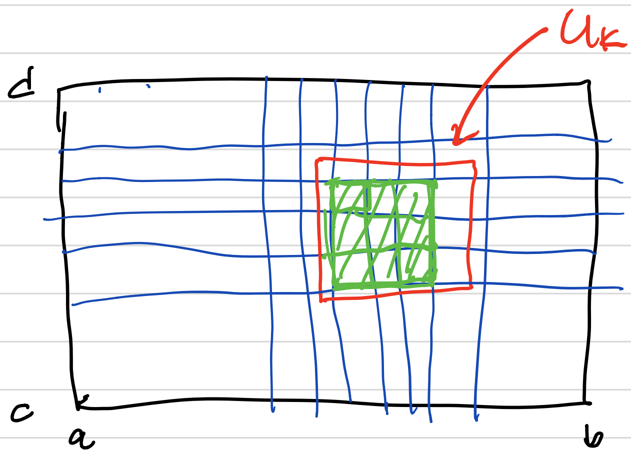
$$S^{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

$$\text{area}(R) = \sum_{i,j} \text{area}(S^{i,j}).$$



Now for any U_k we can consider the squares $S_{k,l}$ completely contained in U_k. The union of $S_{k,l}$ is a rectangle contained in U_k so

$$\sum_{S_{k,l} \subset R_k} \text{area}(S_{k,l}) \le \text{area}(U_k)$$

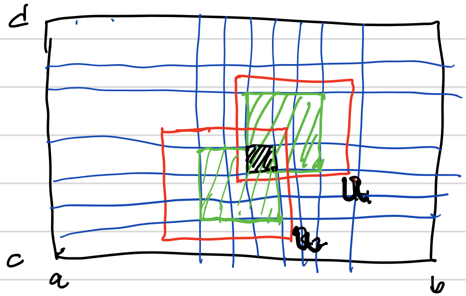


I want to label the rectangles $S_{i,j}$ in a different way.

Now for any U_k we can consider the squares $S_{k,l}$ completely contained in R_k. The union of $S_{k,l}$ is a rectangle contained in R_k so

$$\sum_{S_{k,l} \subset U_k} \text{area}(S_{k,l}) \le \text{area}(U_k)$$

A given square can show up in more than one collection $S_{k,j}$:



but every square shows up in at least one $S_{k,l}$ by the property of the Lebesgue number.

We have

$$\text{area}(R) = \sum_{i,j} \text{area}(S^{i,j}) \leq \sum_{k,l} \text{area}(S_{k,l})$$

$$\leq \sum_{k} \sum_{l} \text{area}(S_{k,l})$$

$$\leq \sum_{k} \text{area}(U_k)$$

$$\text{area}(R) \leq \lambda_2^*(R) \text{ as desired.}$$

Let us formalise this discussion a little.

Let μ^* be a function from the set of subsets of a set X to $[0,\infty]$.

Def. We say that μ^* is an outer measure on a set X if $\mu^*(A) \in [0,\infty]$ for each $A \subset X$ and

(a) $\mu^*(\emptyset) = 0$

(b) if $A \subset B \subset X$ then

$$\mu^*(A) \leq \mu^*(B)$$

(c)

$$\mu^*\Big(\bigcup_n A_n\Big) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

Restatement of our result:

Thm. λ_n^* is an outer measure on $\mathbb{R}^n$.