Let $(X, \mathcal{A}, \mu)$ be a measure space.
Let $f : X \to \mathbb{R}$ be a measurable function.

For $1 \le p < \infty$ define the "p-norm" of f.

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}.$$

(At the moment it is not clear that this is a norm.)

In order for $\|f\|_p$ to be finite we need

$\int |f|^p d\mu < \infty$.

Let $\mathcal{L}^p = \mathcal{L}^p(X, \mathcal{A}, \mu)$ be the set of functions such that $\int |f|^p d\mu < \infty$.

This definition is compatible with our earlier definition of $\mathcal{L}^1 = \mathcal{L}^1(X, \mathcal{A}, \mu)$.

($f \in \mathcal{L}^1$ if f is $\mathbb{R}$ valued and $\int |f| d\mu < \infty$)

Lemma. For $1 \le p < \infty$ $\mathcal{L}^p$ is a vector space. (Cohn P. 92)

① Say $f \in \mathcal{L}^p$ then $\int |\alpha f|^p d\mu = |\alpha|^p \cdot \int |f|^p d\mu$
so $\alpha f \in \mathcal{L}^p$.

② Say $f, g \in \mathcal{L}^p$ then

$$|f(x) + g(x)|^p \le (|f(x)| + |g(x)|)^p$$

$$\le (2 \max\{|f(x)|, |g(x)|\})^p$$

$$\le 2^p |f(x)|^p + 2^p |g(x)|^p$$

If we integrate we get:

$$\int |f(x)+g(x)|^p \, d\mu \leq 2^p \int |f(x)|^p \, d\mu + 2^p \int |g(x)|^p \, d\mu$$

$< \infty$ since $f, g \in \mathcal{L}^p$.

$f+g \in \mathcal{L}^p$.

Remark. Typically $\mathcal{L}^p$ is an infinite dimensional vector space.

For example in $\mathcal{L}^2([-\pi,\pi],\lambda)$ the functions $\sin(kx)$ and $\cos(kx)$ for $k \geq 1$ are linearly independent.

Let V be a vector space. Recall that a function $v \mapsto \|v\|$ from V to $\mathbb{R}$ is a norm if:

a) $\|v\| \geq 0$

b) $\|v\| = 0$ if and only if $v = 0$

c) $\|\alpha v\| = |\alpha| \cdot \|v\|$

d) $\|v + w\| \leq \|v\| + \|w\|$.

We want to show that $\|\cdot\|_p$ is a norm on $\mathcal{L}^p$.

At the moment we can verify two norm properties.

(a) $\|f\|_p \geq 0$. $\quad |f|^p \geq 0 \Rightarrow \int |f|^p d\mu \geq 0 \Rightarrow \left(\int |f|^p d\mu\right)^{1/p} \geq 0$.

(c) $\|\alpha f\|_p = |\alpha| \, \|f\|_p$

$$\left(\int |\alpha f|^p d\mu\right)^{1/p} = \left(\int |\alpha|^p \cdot |f|^p d\mu\right)^{1/p}$$

$$= \left(|\alpha|^p \int |f|^p d\mu\right)^{1/p}$$

$$= |\alpha| \left(\int |f|^p d\mu\right)^{1/p}.$$

If $1 < p < \infty$ we define the conjugate exponent q so that $\frac{1}{p} + \frac{1}{q} = 1$.

It is often useful to use $\mathcal{L}^q$ in studying $\mathcal{L}^p$.

Example: If $p=2$ then $q=2$.

This relation shows up in various equivalent forms. For example:

If $\frac{1}{p}+\frac{1}{q}=1$ then $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$ or $q = \frac{p}{p-1}$

or $qp - q = p$ so $pq = p+q$.

We start by proving an inequality involving conjugate exponents p and q.

Lemma 3.3.1 Let x and y be non-negative numbers then

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}. \qquad (1)$$

Proof. Let $u = x^p$, $v = y^q$ so $x = u^{1/p}$, $y = u^{1/q}$.

(1) is equivalent to:

$$u^{1/p} \cdot v^{1/q} \le \frac{u}{p} + \frac{v}{q}. \qquad (2)$$

$$\boxed{u^{1/p} \cdot v^{1/q} \le \frac{u}{p} + \frac{v}{q}. \qquad (2)}$$

Divide both sides of (2) by v (which is positive):

$$u^{1/p} v^{1/q - 1} \le \frac{u/v}{p} + \frac{1}{q}$$

$1 = \frac{1}{p} + \frac{1}{q}$

$\frac{1}{q} - 1 = -\frac{1}{p}$

$$u^{1/p} v^{-1/p} \le \frac{u/v}{p} + \frac{1}{q}$$

$$\left(\frac{u}{v}\right)^{1/p} \le \frac{u/v}{p} + \frac{1}{q}$$

Write $t = \frac{u}{v}$, (2) is equivalent to

$$t^{1/p} \le \frac{t}{p} + \frac{1}{q} \qquad (3)$$

for $t > 0$.

$$\boxed{t^{1/p} \le \frac{t}{p} + \frac{1}{q}} \tag{3}$$

Write $$g(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p}.$$

To prove (3) we use calculus to show that $g(t) \ge 0$.

$$\boxed{g(t) = \frac{t}{p} + \frac{1}{q} - t^{1/p}}$$

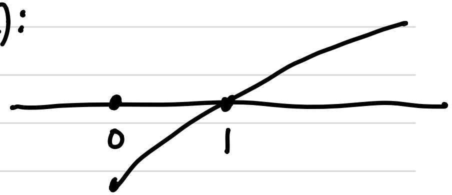
$$g'(t) = \frac{1}{p} - \left(\frac{1}{p}\right) t^{1/p - 1}$$

$$= \frac{1}{p}\left(1 - t^{-1/q}\right) \qquad \left(\text{since } -\frac{1}{q} = \frac{1}{p} - 1\right)$$

So for $t<1$ $\quad t^q<1$ and $t^{-1/q}>1$ $\quad (1-t^{-1/q})<0$

For $t>1$ $\quad t^q>1$, $t^{-1/q}<1$ $\quad$ so $g'(t)>0$.

$g'(t)$:



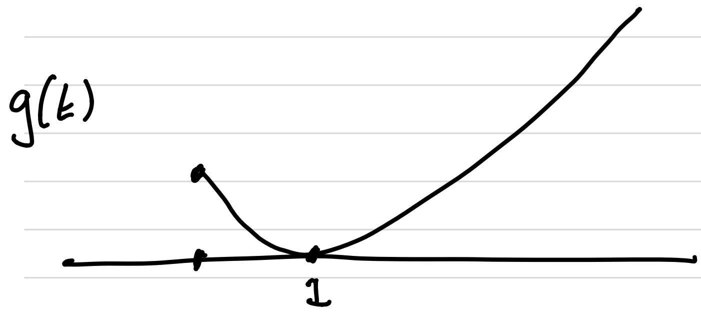
$$-\frac{1}{q} = \frac{1}{p} - 1$$

Thus

$g(t)$ has a unique minimum at $t=0$.

Since $g(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$

we are done.



Prop. 3.3.2 (Holder's inequality). Say p, q are conjugate exponts and $p>1$.

If $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ then $fg \in \mathcal{L}^1$ and

$$\int |fg| \, d\mu \le \|f\|_p \cdot \|g\|_q. \qquad (1)$$

If $p=q=2$ this is a version of Cauchy's inequality.

Proof. $1 < p, q < \infty$.

Using Lemma 3.3.1. gives:

$$|f(x) \cdot g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \quad (2).$$

We begin by proving (1) in the special case when $\|f\|_p = \|g_p\| = 1$.

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} \quad \text{so} \quad \|f\|_p = 1 \Leftrightarrow \int |f|^p d\mu = 1.$$

Similarly $\int |g|^q d\mu = 1$. Integrating (2) gives:

$$\int |fg| \, d\mu \leq \frac{1}{p} \int |f|^p d\mu + \frac{1}{q} \int |g|^q d\mu = \frac{1}{p} + \frac{1}{q} = 1. \quad (3)$$

$$\boxed{\int |fg|\, d\mu \le 1 \qquad (3)}$$

If $\|f\|_p$ or $\|f\|_q$ is zero then the $f \cdot g$ is zero ae and the inequality holds.

If both are non-zero we apply (3) to $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$ which both have norm 1.

This gives: $\displaystyle\int \frac{|f \cdot g|}{\|f\|_p \cdot \|g\|_q}\, d\mu \le 1$

or $\displaystyle\int |f \cdot g|\, d\mu \le \|f\|_p \cdot \|g\|_q$ as was to be shown.

Prop. 3.3.3. (Minkowski's inequality.)

For $p \geq 1$ if $f, g \in \mathcal{L}^p$ then $f+g \in \mathcal{L}^p$ and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. First consider the case $p=1$.

$$\|f+g\|_1 = \int |f+g| \, d\mu \leq \int |f| \, d\mu + \int |g| \, d\mu = \|f\|_1 + \|g\|_1$$

Now assume $1<p<\infty$ and let q be the conjugate exponent to p.

We have shown that if $f, g \in \mathcal{L}^p$ then $f+g \in \mathcal{L}^p$.

Consider $\left(|f+g|^{p-1}\right)^q = |f+g|^{(p-1)q}$ (using $pq = p+q$)

$$= |f+g|^p.$$

Now $f+g \in \mathcal{L}^p \Rightarrow |f+g| \in \mathcal{L}^p \Rightarrow |f+g|^p \in \mathcal{L}^1 \Rightarrow \left(|f+g|^{p-1}\right)^q \in \mathcal{L}^1$

$\Rightarrow |f+g|^{p-1} \in \mathcal{L}^q.$

$$\int |f+g|^p \, d\mu < \infty$$

$$\int \left(|f+g|^p\right)^1 d\mu < \infty$$

We have

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leq |f| \cdot |f+g|^{p-1} + |g||f+g|^{p-1} \quad (1)$$

Integrating gives:

$$\int |f+g|^p d\mu \leq \int |f| \cdot |f+g|^{p-1} d\mu + \int |g| \, |f+g|^{p-1} d\mu$$

We consider the terms on the right.

$$\boxed{\int |f| \cdot |f+g|^{p-1} \, d\mu} \qquad \boxed{\int |g| \, |f+g|^{p-1} \, d\mu}$$

Applying the Holder inequality using $f \in \mathcal{L}^p$, $|f+g|^{p-1} \in \mathcal{L}^q$

$$\int |f| \cdot |f+g|^{p-1} \, d\mu \leq \|f\|_p \cdot \| |f+g|^{p-1} \|_q$$

$$= \|f\|_p \cdot \left(\int |f+g|^{(p-1)q} \, d\mu \right)^{1/q}$$

$$= \|f\|_p \cdot \left(\int |f+g|^p \, d\mu \right)^{1/q} \qquad (p-1)q = p.$$

Similarly $$\int |g|\,|f+g|^{p-1}\,d\mu \le \|g\|_p \left(\int |f+g|^p\,d\mu\right)^{1/q}$$

so $$\int |f+g|^p\,d\mu \le \left(\|f\|_p + \|g\|_p\right)\left(\int |f+g|^p\,d\mu\right)^{1/q}$$

If $\int |f+g|^p\,d\mu = 0$ then Minkowski's inequality holds and we are done.

If $\int |f+g|^p \, d\mu \neq 0$ then divide both sides of the inequality by this quantity to get:

$$\frac{\int |f+g|^p \, d\mu}{\left(\int |f+g|^p \, d\mu\right)^{1/q}} \leq \|f\|_p + \|g\|_p$$

exponent is $1 - \frac{1}{q}$.

The left hand side is:

$$\left(\int |f+g|^p d\mu\right)^{1-1/q} = \left(\int |f+g|^p d\mu\right)^{1/p} = \|f+g\|_p.$$

so we get $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

and we are done.

Let's recall the properties of a norm.

a) $\|v\| \geq 0$

b) $\|v\| = 0$ if and only if $v = 0$

c) $\|\alpha v\| = |\alpha| \cdot \|v\|$

d) $\|v+w\| \leq \|v\| + \|w\|$.

$\|\cdot\|_p$ satisfies a c and d.

What about b?

If $\|f\|_p = 0$ then $\left(\int |f|^p \, d\mu \right)^{1/p} = 0$

so $\int |f|^p \, d\mu = 0 \Rightarrow |f|^p = 0$ a.e.

$\Rightarrow f = 0$ a.e.

This shows that $f=0$ a.e, but it does not show that $f=0$.

We would like to modify $\mathcal{L}^p$ so that two functions which are equal a.e. become equal.

Define $\mathcal{N}^p \subset \mathcal{L}^p$ to be the vector space of functions that are equal 0 a.e.

Set $L^p = \mathcal{L}^p / \mathcal{N}^p$.

We can think of an element of L^p as an equivalence class of functions.

$\|\cdot\|_p$ gives a norm on L^p but typically it gives only a semi-norm on $\mathcal{L}^p$.