Borel sigma-algebra

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E be a topological space

E is a set $\quad\quad \mathcal{O}$ = collection of open sets (closed under finite intersection, arbitrary unions)

The Borel σ-algebra on E is written $\mathcal{B}(E)$
and is $\sigma(\mathcal{O})$

$\sigma(\mathcal{O})$ is the smallest σ-algebra containing $\mathcal{O}$

We're most interested in $\mathcal{B}(\mathbb{R})$ or $\mathcal{B}(\mathbb{R}^d)$

Lemma (2.8 in notes)

$\mathcal{B}(\mathbb{R})$ is also generated by

1. the collection of closed subset of $\mathbb{R}$
2. the collection of intevals $(-\infty, b]$ $\quad b \in \mathbb{R}$
3. the collection of half open intervals $(a, b]$, $a, b \in \mathbb{R}$

"4". the collection of half open intervals

Proof Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ be the σ-algebras generated by the sets in 1. 2. 3.

$$\mathcal{B}(\mathbb{R}) \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \mathcal{B}_3 \supseteq \mathcal{B}(\mathbb{R})$$

$\mathcal{B}(\mathbb{R})$ contains all the open sets, closed under complements
contains all the closed sets.

$\mathcal{B}_1$ contains $(-\infty, b]$ for every b as this is closed
so $\mathcal{B}_1$ contains $\mathcal{B}_2$.

$\mathcal{B}_2$ contains $(-\infty, a]$ and $(-\infty, b]$ for any $a < b$

$\mathcal{B}_2$ contains $(-\infty, a]$ and

so it contains $(a, \infty) = (-\infty, a]^c$

and hence also $(a, b] = (a, \infty) \cap (-\infty, b]$

So $\mathcal{B}_2$ contains $\mathcal{B}_3$

Now we see that $\mathcal{B}_3$ also contains all open intervals (a,b) as $(a,b) = \bigcup_n (a, b-\frac{1}{n}]$ unions taken over n large enough $a < b - \frac{1}{n}$

(Aside $\Rightarrow$ $\mathcal{B}_3$ contains $\mathcal{B}_4$ σ-algebra generated by all open intervals)

We want to show that any open set U is in $\mathcal{B}_3$

Take U an arbitrary open set

$$O = \bigcup_{q \in \mathbb{Q} \cap U} \; \bigcup_{\substack{r \in \mathbb{Q} \text{ s.t.} \\ (q-r, q+r) \subseteq U}} (q-r, q+r)$$

$O \subseteq U$ as its the union of subset of U

WTS: $U \subseteq O$ take $x \in U$

by def^n of open set in Euclidean topology $\exists$ some interval $(x-p, x+p) \subseteq U$

Then we can shrink the interval a bit to find q, r s.t. $x \in (q-r, q+r) \subseteq (x-p, x+p) \subseteq U$

so $x \in O$. Therefore $O = U$

O is a countable union of intervals so $O \in \mathcal{B}_3$

so all open sets are in $\mathcal{B}_3$

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}_3$.

Aside: The key points about $\mathbb{R}$ and its top we use countable base for the topology (Intervals with rational points) and separability of $\mathbb{R}$.