Measure Theory: Exercises (not for credit)

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Question 1. Suppose that $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Show that the set $\mathcal{A} \subseteq \mathcal{E} \times \mathcal{F}$ with $\mathcal{A} = \{A \times B \,:\, A \in \mathcal{E}, B \in \mathcal{F}\}$ is a π-system.

Question 2. Suppose that $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are measurable spaces. Let $\mathcal{A}_1 \subseteq \mathcal{E}$ and $\mathcal{A}_2 \subseteq \mathcal{F}$ be such that $\sigma(\mathcal{A}_1) = \mathcal{E}$ an $\sigma(\mathcal{A}_2) = \mathcal{F}$. Show that $\mathcal{E} \times \mathcal{F} = \sigma(\mathcal{A}_1 \times \mathcal{A}_2)$.

Question 3. Let $\mathcal{M}_1$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, and $\mathcal{M}_2$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^2$. Show that $\mathcal{M}_2 \neq \mathcal{M}_1 \times \mathcal{M}_1$.

Question 4. Let μ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (the measure that counts how many elements there are in a set) and let λ be Lebesgue measure on $\mathbb{R}$. Let f be the indicator function of the set $\{(x, x) \,:\, x \in \mathbb{R}\}$. Show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \mu(\mathrm{d}x) \lambda(\mathrm{d}y) \neq \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \lambda(\mathrm{d}x) \mu(\mathrm{d}x).$$

What part of the conditions of Fubini-Tonelli theorem doesn't hold to mean this can happen?

Question 5. Let $f(x, y) = 1_{x \geq 0}(1_{y \in [x, x+1)} - 1_{y \in [x+1, x+2)})$ Show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) \mathrm{d}x \mathrm{d}y \neq \int_{\mathbb{R}} f(x, y) \mathrm{d}y \mathrm{d}x.$$

What part of the conditions of Fubini-Tonelli theorem doesn't hold to allow this to happen?

Question 6. Let A be a bounded Borel subset of $\mathbb{R}$ with $\lambda(A) > 0$ show that the function $x \mapsto \lambda(A \cap (x + A))$ is continuous and is non-zero on some open interval containing 0. Define $diff(A) = \{z : z = x - y, x \in A, y \in A\}$ show that if A is a Borel subset of $\mathbb{R}$ with non-zero measure then $diff(A)$ contains some open interval around 0.