Material from last week.

The point is to lay the foundations for the definition and basic properties of the Lebesgue integral. Formally the proofs do not depend on working with the Lebesgue integral, they work for a measure space $(X, \mathcal{a}, \mu)$.

Topics:

① What are $\mathcal{A}$-measurable functions?

$\mathcal{A}$ is a σ-algebra on a set X.

Think $X = \mathbb{R}$, $\mathcal{A} = \mathcal{M}(\mathbb{R})$.

4 equivalent definitions

Discussed in lecture and in "measurable functions".

Primarily interested in real valued functions on X ($\mathbb{R}$).

Allow our functions to have a domain smaller than all of X. Introduce $A \subset X$. Useful in considering $\frac{1}{f}$ for example.

Done in posted video but not lecture.

Allow our functions to take on values $+\infty, -\infty$. Useful when considering limsup for example.

Collection of $\mathcal{A}$-measurable functions is closed under max, min, sum, product, taking limsups, taking limits.

Some of these properties are discussed in the lecture and some, more clearly in the "measurable functions video".

For properties involving pairs of functions f, g this in terms of pictures of $\mathbb{R}^2$.

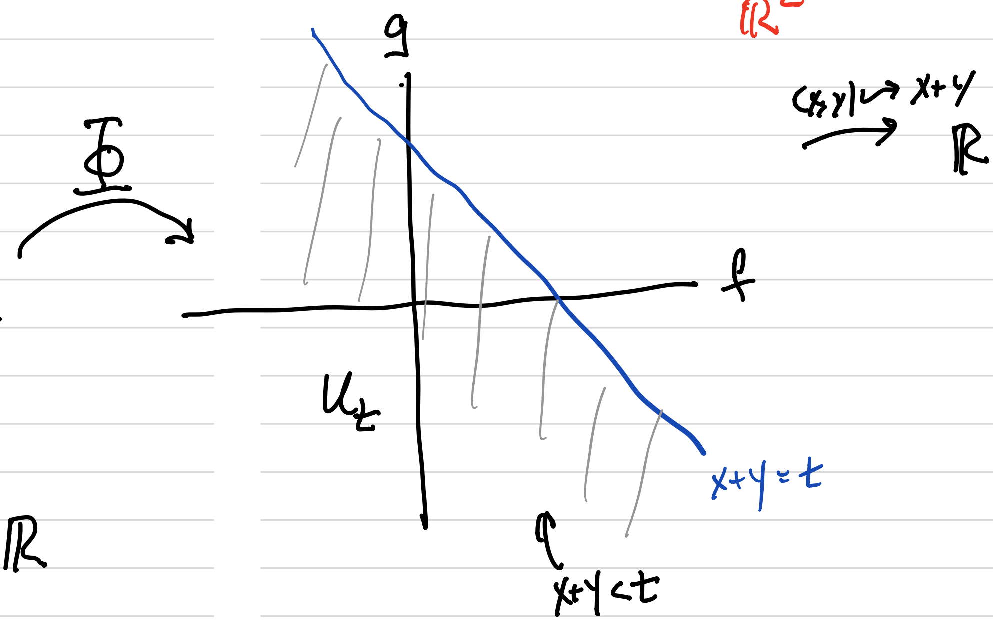
$$\Phi : X \to \mathbb{R}^2 \qquad \Phi(x) = (f(x), g(x)).$$

$$\begin{array}{ccccc} x & \longmapsto & (f(x), g(x)) & & f(x)+g(x) \\ X & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \end{array}$$

$f+g$: We want to show that the set $\{x \in A : f(x)+g(x) < t\}$ is in $\mathcal{A}$.

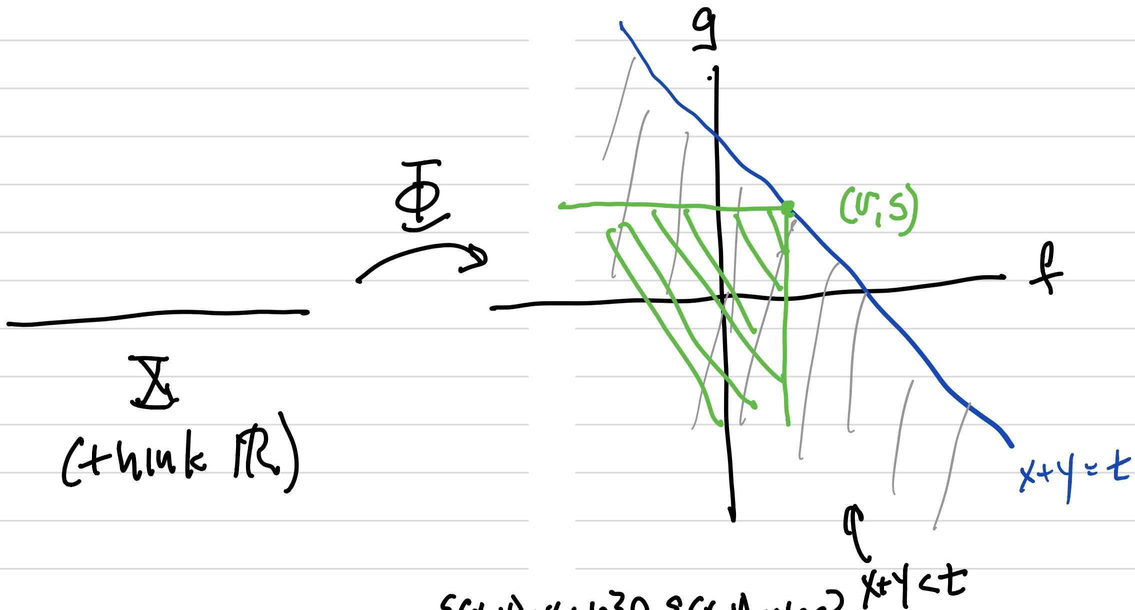
$$\{x \in A ; f(x)+g(x) < t\} = \Phi^{-1}\big(\underbrace{(x,y) : x+y < t}_{U_t}\big)$$

$$\Phi(x) = (f(x), g(x))$$



$A \subset X$

think $\mathbb{R}$



$\{(x,y): x<r\} \cap \{(x,y): y<s\}$

$\{x \in A : f(x) < r\}$
$\cap \{x \in A : g(x) < s\}$

Clearly $\{(x,y): x+y<t\}$ is the union of sets $\{(x,y): x<r\} \cap \{(x,y): y<s\}$ for different (r,s) with $r+s=t$.

Consider two settings. ① $[0,+\infty]$ valued functions. Here we define sums and mult. by positive scalars.

$x+\infty = \infty$
$\infty+\infty = \infty$

② $(-\infty,\infty)$ valued ($\mathbb{R}$-valued) functions. Here we discuss mult. by arbitrary scalars, $f+g$, $f-g$, $f\cdot g$, f/g.

"Basic properties of simple functions", Lecture.

Definition of simple functions

$f = \sum a_i \chi_{A_i}$. $\quad \mathcal{S}, \mathcal{S}^+$

$$\int f \, d\mu = \sum a_i \cdot \mu(A_i).$$

Definition of the integral of a simple function: $\int f \, d\mu$

To start with we only work with non-negative functions to avoid the issue of cancelling $+\infty$s and $-\infty$s.

Consider $\quad f(x) = 1 \quad x \geq 0 \quad f(x) = -1 \quad x < 0.$

Basic properties of the integral

① Well defined.

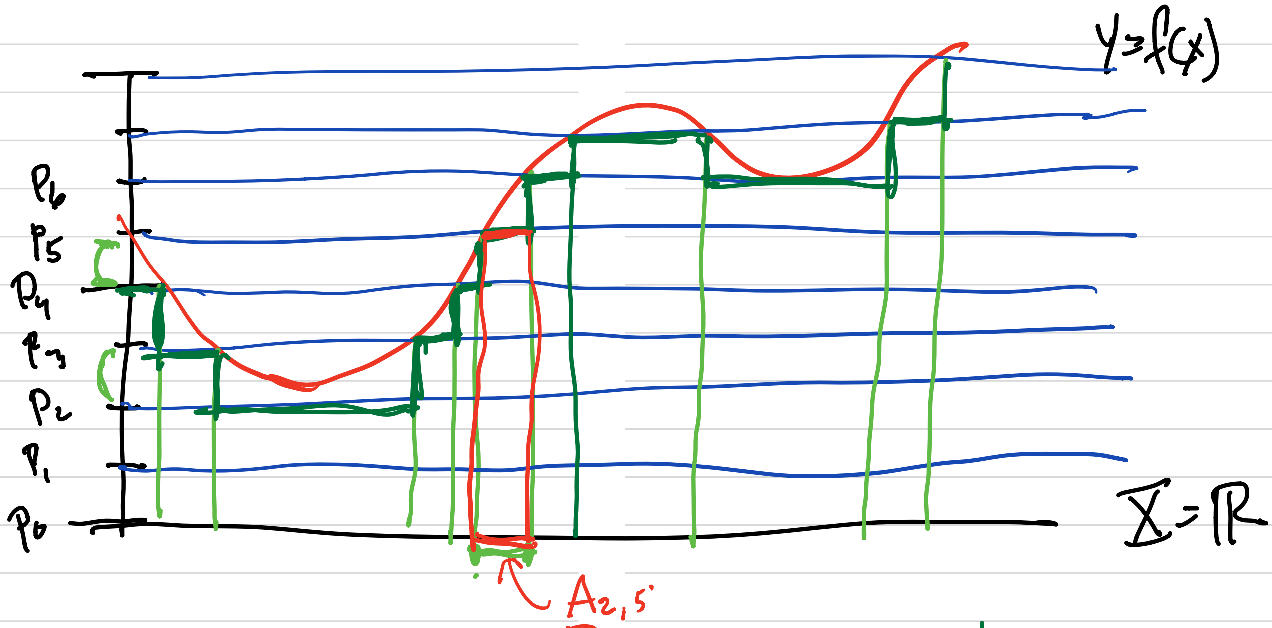
② Linear.

③ Monotone $\quad f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu.$

"Lower sum construction"

Given a measurable f

This gives us a way to get a sequence of simple functions f_n from a measurable function by constructing a sequence of partitions of the range of the function.



P_2

f_2 is the green function

f_3

f_4

$\vdots$

The sequence f_n has 3 properties

1. $f_n \in \mathcal{S}^+$ (simple)

2. $f_n(x) \le f_{n+1}(x) \le \dots$ (monotonicity)

3. $\lim_{n\to\infty} f_n(x) = f(x)$ (pointwise convergence)