1 Assignment One - Lebesgue measure in $\mathbb{R}^d$

This assignment is about defining Lebesgue measure on $\mathbb{R}^d$ as opposed to $\mathbb{R}$. Later in the course we will also see product σ-algebras and product measures which give another way of doing this. We begin by defining Lebesgue outer measure, λ^* on $\mathbb{R}^d$ by first defining the measure of the rectangle $(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]$. We define

$$\lambda\left((a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]\right) = (b_1 - a_1)(b_2 - a_2) \ldots (b_d - a_d).$$

Then for any subset of $\mathbb{R}^d$, A, we define

$$\lambda^*(A) = \inf\{\sum_{n=1}^{\infty} \lambda(R) \,:\, R_k \text{ are rectangles}, A \subseteq \bigcup_{n=1}^{\infty} R_n\}.$$

Question 1.1. Show that λ^* is indeed an outer measure. *3 marks*

Question 1.2. Show that if R is a rectangle then $\lambda(R) = \lambda^*(R)$. *5 marks*

We now recall that a set, A, will be λ^*- measureable if for every set B

$$\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c).$$

As for the one dimensional case we write $\mathcal{M}$ for the set of Lebesgue measurable sets. We know from the proof of Carathéodory's extension theorem that $\mathcal{M}$ is a σ-algebra

Question 1.3. Explain why if $\lambda^*(A) = 0$ then $\lambda^*(B \cap A)$ will also be zero. Therefore show that if $\lambda^*(A) = 0$ or $\lambda^*(A^c) = 0$ then A is λ^*-measureable. *5 marks*

Question 1.4. In this question we will prove that every Borel subset of $\mathbb{R}^d$ is Lebesgue measurable.

- Show that every half space of the form $H_{j,b} = \{(x_1, \ldots, x_d) \,:\, x_j \leq b\}$ is Lebesgue measurebale. *5 marks*
- Show that every rectangle R is Lebesgue measurable. *2 marks*
- Show that every open set in $\mathbb{R}^d$ is a countable union of rectangles. *3 marks*
- Show that every Borel set is Lebesgue measureable. *2 marks*

Again we define Lebesgue measure on $\mathbb{R}^d$ to be the restriction λ^* to $\mathcal{M}$.