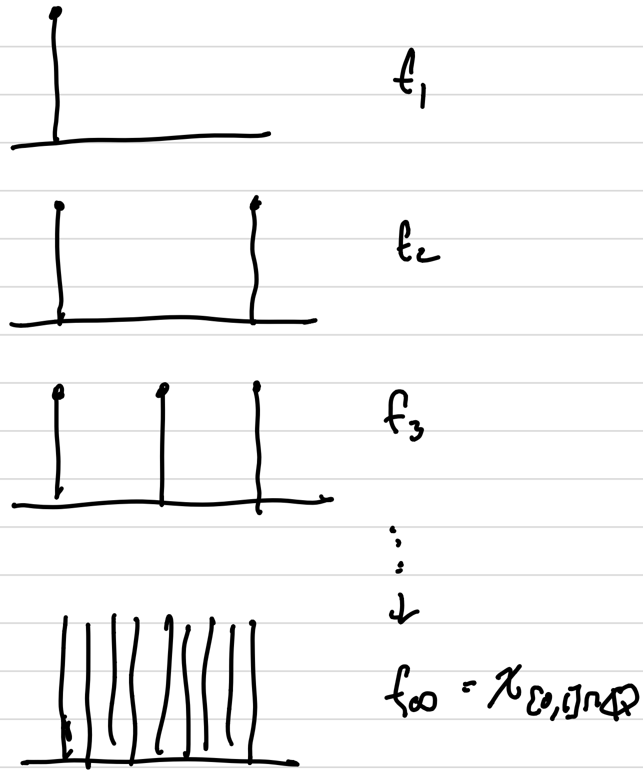
Question. Let $A \subset \mathbb{R}$ be a set of measure 0.

Let $f(x) = \begin{cases} \infty & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$

What is $\int f \, d\mu$?

Recall that the motivation for creating the Lebesgue integral was to create a theory of integration with good limit properties.

In week 1 we had gave an example of a sequence of Riemann integrable functions for which the pointwise limit was not Riemann integrable.



It will turn out that with this week's material we have completed our task.

We prove 4 limit theorems for the Lebesgue integral <u>and</u> we prove that the Lebesgue integral is compatible with the Riemann integral.

Riemann integrable functions f defined on $[a,b]$ are Lebesgue integrable and

$$\underbrace{\int f \, d\mu}_{\text{Lebesgue}} = \underbrace{\int_a^b f(x)\,dx}_{\text{Riemann}}.$$

Theorem 2.4.1 (Monotone Convergence). Let f and $f_1, f_2 \ldots$ be $[0, +\infty]$ valued measurable functions on X. Suppose

$$f_1(x) \le f_2(x) \le \ldots$$

and

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Then $$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

Remark. This is the 3rd time we have proven a version of this result. We started with f, f_n simple, then f_n simple and f measurable, now f_n measurable and f measurable.

We would like to "replace" the f_n by simple functions. We know that each f_n is a monotone limit of simple functions so introduce simple functions $g_{n,k}$ with

$$f_n = \lim_{k\to\infty} g_{n,k}.$$

$$\begin{array}{cccc}
f_1 & f_2 & \cdots\cdots & \to f \\
\uparrow & \uparrow & & \\
\vdots & \vdots & & \\
g_{1,3} & g_{2,3} & & \\
g_{1,2} & g_{2,2} & & \\
g_{1,1} & g_{2,1} & \cdots\cdots &
\end{array}$$

Corollary. (Beppo Levi's Theorem).

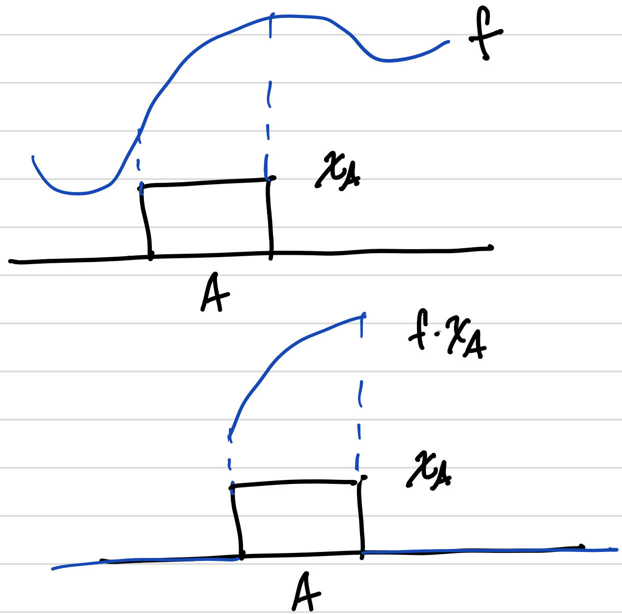
Let $(X, \mathcal{A}, \mu)$ be a measure space then

$$\int \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Method of attack. Apply the Monotone Convergence Theorem to the partial sums $\sum_{k=1}^{n} f_k$. Positivity of f_k implies that the sequence of partial sums is non-decreasing.

Recall $\int_A f\,d\mu = \int \chi_A \cdot f\,d\mu$.

Integrate the "restriction of f to A".



New measures from old.

Say that $(X, \mathcal{A}, \mu)$ is a measure space and that $f: X \to [0, +\infty]$ is $\mathcal{A}$-measurable.

We can define a new measure

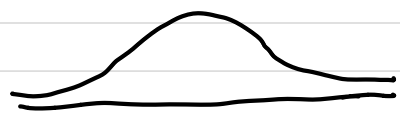
$$\nu : \mathcal{A} \longrightarrow [0, +\infty] \text{ by setting}$$

$$\nu(A) = \int_A f \, d\mu.$$

$$\nu(A) = \frac{1}{2\pi} \int_A e^{-x^2} dx$$

Gaussian distributions.

The fact that ν is countably additive follows from ① for disjoint sets A, B $\quad \chi_{A \cup B} = \chi_A + \chi_B$.



② Beppo Levi's Theorem.

Theorem 2.4.4 (Fatou's Lemma) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of $[0, +\infty]$ valued $\mathcal{A}$-measurable functions on X.

Then

$$\int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu.$$

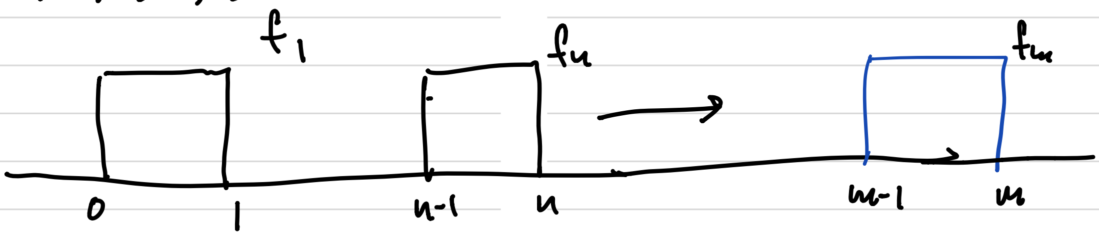
When is the inequality strict?

Fatou's Lemma: $$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu .$$

Non-strict inequality example: "Sliding bump functions"

$\mu = \lambda =$ Lebesgue measure

$f_n = \chi_{[n-1,n]}$



$$\int f_n \, d\lambda = \int \chi_{[n,n+1]} \, d\lambda = \lambda([n, n+1]) = 1.$$

$\liminf_{n\to\infty} f_n = 0 \qquad \liminf_{n\to\infty} \int f_n \, d\lambda = 1.$

The sliding bump functions show more generally that pointwise convergence does not imply convergence of the integral (without some extra hypothesis).

$$f_n \to 0 \quad \text{pointwise}$$

$$\int f_n \, d\mu \to 1$$

Thm. 2.4.5 (Dominated Convergence Theorem)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let g be a $[0, +\infty]$ integrable function on X and let $f_1, f_2 \ldots$ be $[-\infty, +\infty]$ valued $\mathcal{A}$-measurable functions such that

$$f(x) = \lim_{n \to \infty} f_n(x)$$

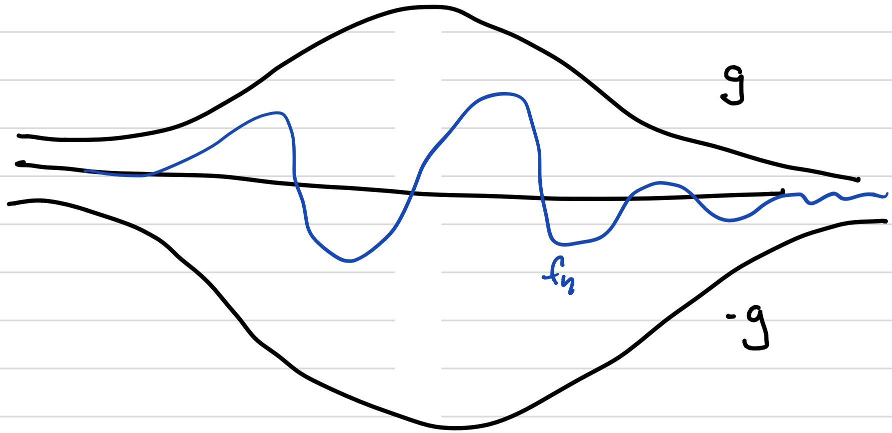
and

$$|f_n(x)| \le g(x) \quad n = 1, 2, \ldots$$

Then f and f_n are integrable and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

Picture:



$|f_n| \leq g.$

What would happen if we consider the sliding bump functions?

What would g be?

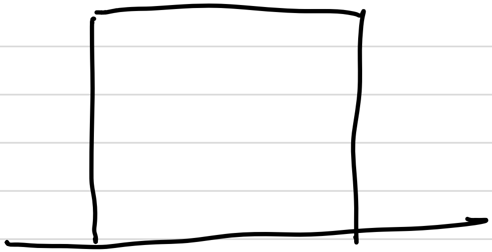
Theorem 2.5.4. Let $[a,b]$ be a closed bounded interval and let f be a bounded real valued function on $[a,b]$. Then

(a) f is Riemann integrable if and only if it is continuous at almost every point of $[a,b]$

(b) if f is Riemann integrable then f is Lebesgue integrable and the Riemann and Lebesgue integrals of f coincide.

Examples.

Dirichlet function:

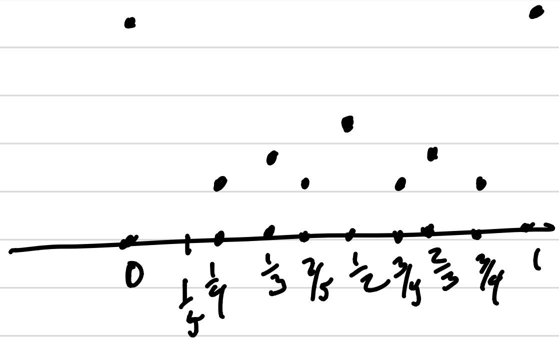


$$f = \chi_{[0,1]\cap\mathbb{Q}}$$

Discontinuous on $[0,1]$

$\lambda([0,1]) = 1$

Not Riemann integrable



$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Discontinuous on $\mathbb{Q}\cap[0,1]$.

$\lambda(\mathbb{Q}\cap[0,1]) = 0$.

Riemann integrable.