Lecture starts at 9.05

Remind me to record!

Poll on moodle in week 5

**Simple functions** $(E, \mathcal{E}, \mu)$

$$f(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k} \qquad a_k \geq 0 \qquad A_k \in \mathcal{E}$$

"Functions that take finitely many values"

$$\mu(f) = \sum_{k=1}^{n} a_k \mu(A_k)$$

**Lemma** "Integration on simple functions works the way we want it to"

The integral of a simple is well defined (doesn't depend on how you write $f$)

- $\alpha > 0$ $\quad \mu(\alpha f) = \alpha \mu(f)$
- $\mu(f+g) = \mu(f) + \mu(g)$
- If $f \leq g$ then $\mu(f) \leq \mu(g)$
- $f = 0$ $\mu$-almost surely iff $\mu(f) = 0$

~~Pf~~ The first thing to notice is if $f(x) = \sum_k a_k \mathbb{1}_{A_k}$ then we can rewrite this expression so $a_k > 0$ for then we can rewrite this expression so $a_k > 0$ for each $k$ and the $(A_k)_{k=1}^n$ are pairwise disjoint, without changing the value of $\underline{\sum_k a_k \mu(A_k)}$.

First discard any term with $a_k = 0$

If $A_j \cap A_i \neq \emptyset$ then split this into $\underset{\hookrightarrow a_j}{A_j \setminus (A_i \cap A_j)}$

$\underset{\hookrightarrow a_i}{A_i \setminus (A_i \cap A_j)}$ and $\underset{\hookrightarrow a_i + a_j}{A_i \cap A_j}$.

So wlog $a_k > 0$ $k=1,n$ and all the $(A_k)_{k=1}^n$ are pairwise disjoint.

Well definedness

Suppose $f(x) = \sum_{k=1}^n a_k \mathbb{1}_{A_k} = \sum_{j=1}^m b_j \mathbb{1}_{B_j}$

wlog the coeff are $> 0$ and sets disjoint

$\text{supp}(f) = \bigcup_{k=1}^n A_k = \bigcup_{j=1}^m B_j$ so in particular

$\text{supp}(f) = \{x : f(x) \neq 0\}$

$$\bigcup_{k=1}^n A_k = \bigcup_{j=1}^m B_j$$

Also if $A_i \cap B_j \neq \emptyset$ then $a_i = b_j$

this is because if $x \in A_i \cap B_j$ then $f(x) = a_i$ and $f(x) = b_j$

so $a_i = b_j$

because $A_k \overset{\text{disjoint}}{=} \bigcup_{j=1}^m A_k \cap B_j$

Using this

$$\sum_{k=1}^n a_k \mu(A_k) = \sum_{k=1}^n a_k \sum_{j=1}^m \mu(A_k \cap B_j)$$

either $\mu(A_k \cap B_j) = 0$ or $a_k = b_j$

$$= \sum^n \sum^m \quad (A \cap B_j)$$

$k=1$ or $a_k = b_j$

$$= \sum_{k=1}^{n}\sum_{j=1}^{m} a_k \mu(A_k \cap B_j)$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{m} b_j \mu(A_k \cap B_j)$$

$$B_j = \overset{\text{disjoint}}{\bigcup_{k=1}^{n}} B_j \cap A_k$$

$$= \sum_{j=1}^{m} b_j \sum_{k=1}^{n} \mu(A_k \cap B_j)$$

$$= \sum_{j=1}^{m} b_j \mu(B_j)$$

This shows well definedness.

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Show

$\mu(\alpha f) = \alpha \mu(f)$.

$$f = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k} \quad \text{and} \quad \alpha f = \sum_{k=1}^{n} \alpha a_k \mathbb{1}_{A_k}$$

$$\mu(\alpha f) = \sum_{k=1}^{n} \alpha a_k \mu(A_k) = \alpha \sum_{k=1}^{n} a_k \mu(A_k) = \alpha \mu(f).$$

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$\mu(f+g) = \mu(f) + \mu(g)$

So write $f(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$ $g(x) = \sum_{j=1}^{m} c_j \mathbb{1}_{C_j}$

and wlog $a_k > 0$ $c_j > 0$ $A_k$ pairwise disjoint, $C_j$ disjoint

write $A_0 = E \setminus \bigcup_{k=1}^{n} A_k$ $a_0 = 0$

$C_0 = E \setminus \bigcup_{j=1}^{m} C_j$ $c_0 = 0$

$n$

$j=1$

$$f(x) = \sum_{k=0}^{n} a_k \mathbb{1}_{A_k} \quad \text{similarly with } g.$$

$$\bigcup_{k=0}^{n} A_k = E \quad \text{all } A_k \text{ are pairwise disjoint.}$$

Idea here is to write $f$ and $g$ with the same sets

$$f(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_k \mathbb{1}_{A_k \cap C_j} \qquad \text{as } A_k = \underbrace{\bigcup_{j=1}^{m} A_k \cap C_j}_{\text{disjoint}}$$

$$g(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} c_j \mathbb{1}_{A_k \cap C_j}$$

we know the integral is well defined so we can use any representation

$$(f+g)(x) = \sum_{k=0}^{n} \sum_{j=0}^{m} (a_k + c_j) \mathbb{1}_{A_k \cap C_j}$$

$$\mu(f+g) = \sum_{k=0}^{n} \sum_{j=0}^{m} (a_k + c_j) \mu(A_k \cap C_j)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{m} a_k \mu(A_k \cap C_j) + \sum_{k=0}^{n} \sum_{j=0}^{m} c_j \mu(A_k \cap C_j)$$

$$= \mu(f) + \mu(g)$$

(can also rearrange using $\sum_{j=0}^{m} \mu(A_k \cap C_j) = \mu(A_k)$ to get origrinal for for $\mu(f)$ )

Now show monotonicity so $f \leq g$

so again we can write

so again we can write

$$f = \sum_{k=0}^{n} \sum_{j=0}^{m} a_k \mathbb{1}_{A_k \cap C_j} \qquad g = \sum_{k=0}^{n} \sum_{j=0}^{m} c_j \mathbb{1}_{A_k \cap C_j}$$

we want to get rid of the terms where $A_k \cap C_j = \emptyset$ so a convinient of writing this is to set $a_{kj} = a_k \mathbb{1}_{A_k \cap C_j \neq \emptyset}$

and $c_{k,j} = c_j \mathbb{1}_{A_k \cap C_j \neq \emptyset}$

It is now true that if $f(x) \leq g(x)$ for every $x$ the $a_{k,j} \leq c_{k,j}$ whenever $A_k \cap C_j \neq \emptyset$ if $x \in A_k \cap C_j$ then $f(x) = a_{kj}$ and $g(x) = c_{kj}$

$$\mu(f) = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{k,j}\, \mu(A_k \cap C_j) \leq \sum_{k=0}^{n} \sum_{j=0}^{m} c_{kj}\, \mu(A_k \cap C_j)$$

$$= \mu(g)$$

$\overline{\phantom{x}}$

$\mu(f) = 0$ iff $f = 0$ $\mu$ almost everywhere

$f(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$ is $0$ $\mu$ – a.e iff

for every $k$ $a_k = 0$ or $\mu(A_k) = 0$

iff $\sum_{k=1}^{n} a_k \mu(A_k) = 0$

Something happens $\mu$-almost everywhere if the set where it doesn't happen has $\mu$-measure 0.

**Defn:** If $f$ is non-negative, measurable and real valued then

$$\mu(f) = \sup\{\mu(g) : g \text{ simple } g \le f\}$$

N.B we can see it is monotone.
(we don't need to use monotone convergence to see this)

**Lemma** The two definitions agree on simple functions

if $f$ is simple $f(x) = \sum_k a_k \mathbb{1}_{A_k}$

$$\sum_k a_k \mu(A_k) = \sup\{\mu(g) : g \text{ simple } g \le f\}$$

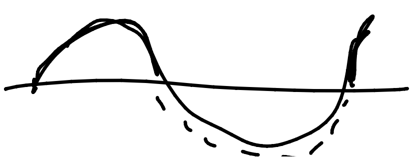
Pf/ On the exercise sheet.

Hint is use monotonicity of the second definition!

**Final defn**

If $f$ is measurable, real-valued

$$f = f_+ - f_- \quad \text{where} \quad f_+(x) = \max\{f(x), 0\}$$
$$f(x) = \max\{-f(x), 0\}$$



if $\mu(f_+), \mu(f_-) < \infty$

 if $\mu(f_+), \mu(f_-) < \infty$

we say $f$ is <u>integrable</u>

and define $\mu(f) = \mu(f_+) - \mu(f_-)$.