

Integration model:

$$v(k+1) = v(k) + c_1 a(k) + c_2 a(k-1) \quad (1)$$

$$s(k+1) = s(k) + dt \ v(k) + c_3 a(k) + c_4 a(k-1) \quad (2)$$

We have to split each entry into its x- and y-components

$$v_x(k+1) = v_x(k) + c_1 a_x(k) + c_2 a_x(k-1) \quad (3)$$

$$v_y(k+1) = v_y(k) + c_1 a_y(k) + c_2 a_y(k-1) \quad (4)$$

$$s_x(k+1) = s_x(k) + dt \ v_x(k) + c_3 a_x(k) + c_4 a_x(k-1) \quad (5)$$

$$s_y(k+1) = s_y(k) + dt \ v_y(k) + c_3 a_y(k) + c_4 a_y(k-1) \quad (6)$$

The constants are defined as such:

$$\frac{1}{c_2} = \bar{c}_2 \quad (7)$$

$$\frac{c_1}{c_2} = \bar{c}_1 \quad (8)$$

$$\frac{1}{c_4} = \bar{c}_4 \quad (9)$$

$$\frac{c_3}{c_4} = \bar{c}_3 \quad (10)$$

Rearrange the constants:

$$c_2 = \frac{1}{\bar{c}_2} \quad (11)$$

$$c_1 = \bar{c}_1 \bar{c}_2 \quad (12)$$

$$c_4 = \frac{1}{\bar{c}_4} \quad (13)$$

$$c_3 = \bar{c}_3 \bar{c}_4 \quad (14)$$

Acceleration model with 4 parameters

$$a(k-1) = -\bar{c}_1 a(k) + \bar{c}_2 (v(k+1) - v(k)) \quad (15)$$

$$a(k-1) = -\bar{c}_3 a(k) + \bar{c}_4 (s(k+1) - s(k) - dt \ v(k)) \quad (16)$$

Model in matrix notation

$$\begin{bmatrix} a(k-1) \\ a(k-1) \end{bmatrix} = \begin{bmatrix} -a(k) & v(k+1) - v(k) & 0 & 0 \\ 0 & 0 & -a(k) & s(k+1) - s(k) - dt \ v(k) \end{bmatrix} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \end{bmatrix} \quad (17)$$

Add x and y components

$$\begin{bmatrix} a_x(k-1) \\ a_x(k-1) \\ a_y(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} -a_x(k) & v_x(k+1) - v_x(k) & 0 & 0 \\ 0 & 0 & -a_x(k) & s_x(k+1) - s_x(k) - dt \ v_x(k) \\ -a_y(k) & v_y(k+1) - v_y(k) & 0 & 0 \\ 0 & 0 & -a_y(k) & s_y(k+1) - s_y(k) - dt \ v_y(k) \end{bmatrix} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \end{bmatrix} \quad (18)$$

For easier implementation split into two models

$$\begin{bmatrix} a_x(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} -a_x(k) & v_x(k+1) - v_x(k) \\ -a_y(k) & v_y(k+1) - v_y(k) \end{bmatrix} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_3 \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} a_x(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} -a_x(k) & s_x(k+1) - s_x(k) - dt \ v_x(k) \\ -a_y(k) & s_y(k+1) - s_y(k) - dt \ v_y(k) \end{bmatrix} \begin{bmatrix} \bar{c}_2 \\ \bar{c}_4 \end{bmatrix} \quad (20)$$

Mind dump

$$\Rightarrow \quad (21)$$

$$a_x(k-1) = -\bar{c}_1 a_x(k) + \bar{c}_2 \left(v_x(k+1) - v_x(k) \right) \quad (22)$$

$$a_x(k-1) = -\bar{c}_3 a_x(k) + \bar{c}_4 \left(s_x(k+1) - s_x(k) - dt \ v_x(k) \right) \quad (23)$$

$$a_y(k-1) = -\bar{c}_1 a_y(k) + \bar{c}_2 \left(v_y(k+1) - v_y(k) \right) \quad (24)$$

$$a_y(k-1) = -\bar{c}_3 a_y(k) + \bar{c}_4 \left(s_y(k+1) - s_y(k) - dt \ v_y(k) \right) \quad (25)$$

Extended forumal with added features (7 parameters)

$$v(k+1) = v(k) + c_0 v(k-1) + c_1 a(k) + c_2 a(k-1) \quad (26)$$

$$s(k+1) = s(k) + c_3 s(k-1) + dt \ v(k) + c_4 v(k-1) + c_5 a(k) + c_6 a(k-1) \quad (27)$$

$$(28)$$

$$c_2 a(k-1) = v(k+1) - v(k) - c_0 v(k-1) - c_1 a(k) \quad (29)$$

$$c_6 a(k-1) = s(k+1) - s(k) - c_3 s(k-1) - dt \ v(k) - c_4 v(k-1) - c_5 a(k) \quad (30)$$

$$(31)$$

$$a(k-1) = \frac{v(k+1) - v(k)}{c_2} - \frac{c_0}{c_2} v(k-1) - \frac{c_1}{c_2} a(k) \quad (32)$$

$$a(k-1) = \frac{s(k+1) - s(k)}{c_6} - \frac{c_3 s(k-1)}{c_6} - \frac{dt}{c_6} v(k) - \frac{c_4}{c_6} v(k-1) - \frac{c_5}{c_6} a(k) \quad (33)$$

$$(34)$$

Rearrange the coefficients

$$c_2 = \frac{1}{\bar{c}_2} c_0 = \frac{\bar{c}_0}{\bar{c}_2} \quad (35)$$

$$c_1 = \frac{\bar{c}_1}{\bar{c}_2} \quad (36)$$

$$c_6 = \frac{1}{\bar{c}_6} \quad (37)$$

$$c_3 = \frac{\bar{c}_3}{\bar{c}_6} \quad (38)$$

$$c_4 = \frac{\bar{c}_4}{\bar{c}_6} \quad (39)$$

$$c_5 = \frac{\bar{c}_5}{\bar{c}_6} \quad (40)$$

$$(41)$$

Imagine all coefficient have dashes

$$a(k-1) = c_2(v(k+1) - v(k)) - c_0v(k-1) - c_1a(k) \quad (42)$$

$$a(k-1) = c_6(s(k+1) - s(k) - dt v(k)) - c_3s(k-1) - c_4v(k-1) - c_5a(k) \quad (43)$$

$$(44)$$

Equation for the model

$$a(k-1) = -c_0v(k-1) - c_1a(k) + c_2(v(k+1) - v(k)) \quad (45)$$

$$a(k-1) = -c_3s(k-1) - c_4v(k-1) - c_5a(k) + c_6(s(k+1) - s(k) - dt v(k)) \quad (46)$$

$$(47)$$

Here we solve the following system:

$$\begin{bmatrix} a(k-1) \\ a(k-1) \end{bmatrix} = \begin{bmatrix} v(k-1) & -a(k) & v(k+1) - v(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s(k-1) & -v(k-1) & -a(k) & + (s(k+1) - s(k) - dt v(k)) \end{bmatrix} \begin{bmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \\ \bar{c}_5 \\ \bar{c}_6 \end{bmatrix} \quad (48)$$

We add the x and y components (imagine the matrix also contains the indices)

$$\begin{bmatrix} a_x(k-1) \\ a_x(k-1) \\ a_y(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} v(k-1) & -a(k) & v(k+1) - v(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s(k-1) & -v(k-1) & -a(k) & + (s(k+1) - s(k) - dt v(k)) \\ v(k-1) & -a(k) & v(k+1) - v(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -s(k-1) & -v(k-1) & -a(k) & + (s(k+1) - s(k) - dt v(k)) \end{bmatrix} \begin{bmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \\ \bar{c}_4 \\ \bar{c}_5 \\ \bar{c}_6 \end{bmatrix} \quad (49)$$

We split the model into two models for easier implementation

$$\begin{bmatrix} a_x(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} v_x(k-1) & -a_x(k) & v_x(k+1) - v_x(k) \\ v_y(k-1) & -a_y(k) & v_y(k+1) - v_y(k) \end{bmatrix} \begin{bmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \bar{c}_2 \end{bmatrix} \quad (50)$$

$$\begin{bmatrix} a_x(k-1) \\ a_y(k-1) \end{bmatrix} = \begin{bmatrix} -s_x(k-1) & -v_x(k-1) & -a_x(k) & s_x(k+1) - s_x(k) - dt \, v_x(k) \\ -s_y(k-1) & -v_y(k-1) & -a_y(k) & s_y(k+1) - s_y(k) - dt \, v_y(k) \end{bmatrix} \begin{bmatrix} \bar{c}_3 \\ \bar{c}_4 \\ \bar{c}_5 \\ \bar{c}_6 \end{bmatrix} \quad (51)$$