

Calculating the International Geomagnetic Reference Field

Björn Sundin

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1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential V ; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coefficients g_n^m and h_n^m which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^N \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) P_n^m(\cos \theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of n th degree and m th order, $a = 6371.2$ km is Earth's mean radius, r is the distance from Earth's center to the point the magnetic field is to be calculated at, θ is its co-latitude, and ϕ is its longitude. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

Since we are using spherical coordinates, the magnetic field has the form

$$\mathbf{B} = -\nabla V = -\frac{dV}{dr} \hat{\mathbf{r}} - \frac{1}{r} \frac{dV}{d\theta} \hat{\boldsymbol{\theta}} - \frac{1}{r \sin \theta} \frac{dV}{d\phi} \hat{\boldsymbol{\phi}}$$

and the components can be expanded as

$$\begin{aligned} B_r &= \sum_{n=0}^N (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) P_n^m(\cos \theta) \\ B_\theta &= -\sum_{n=0}^N \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) \frac{d}{d\theta} P_n^m(\cos \theta) \\ B_\phi &= \sum_{n=0}^N \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m m \sin(m\phi) - h_n^m m \cos(m\phi)) P_n^m(\cos \theta) \end{aligned}$$

Here we see that we not only need $P_n^m(\cos \theta)$, we also need its derivative with respect to θ .

2 Associated legendre functions

The n th degree, m th order associated legendre function with Schmidt semi-normalization is defined as (Winch et al., 2005)

$$P_n^m(x) = \alpha_n^m(x) D^{m+n} (x^2 - 1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!} \frac{1}{2^n n!}} (1 - x^2)^{m/2}, \quad (1)$$

δ_{m0} is the Kronecker delta (1 for $m = 0$ and 0 otherwise) and $D = d/dx$ is the differentiation operator with respect to x .

2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m} x^{2k} = \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m} (x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Since

$$m + n > 2k \iff k < \frac{m + n}{2},$$

all terms with $k < \lceil (m + n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2 \rceil}^n \binom{n}{k} \frac{(2k)!}{(2k - n - m)!} (-1)^{n-k} x^{2k - n - m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m + n > 2n$, i.e. $m > n$. A faster way to calculate the Q_n^m s is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

2.2 Recurrence relations

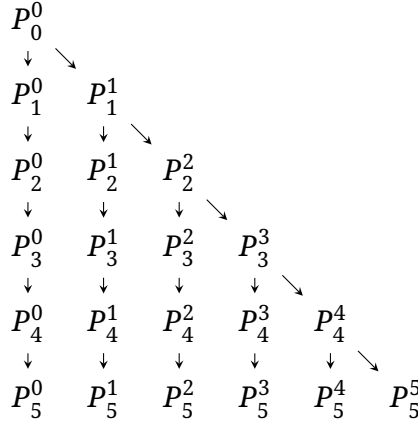


Figure 1: Diagram showing how the recurrence relations are used to calculate P_n^m .

Instead of calculating each associated Legendre function by itself, a better idea is to calculate each P_n^m from previously calculated P_n^m values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single P_n^m up to $n = 13$, so we are effectively spreading the calculations out between P_n^m s and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each n is a row and each m is a column) leads to recurrence relations involving division by $\sin \theta$, which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. It all starts from P_0^0 , which we can calculate to be

$$P_0^0(x) = \sqrt{(2-1) \frac{0!}{0!} \frac{1}{2^0 \cdot 0!}} (1-x^2)^0 = 1$$

with a derivative with respect to x (or θ , when $x = \cos \theta$) that is identically zero.

The recurrence relations are derived below and are illustrated individually in Figure 2; both use two previous values, but one of them might be zero.

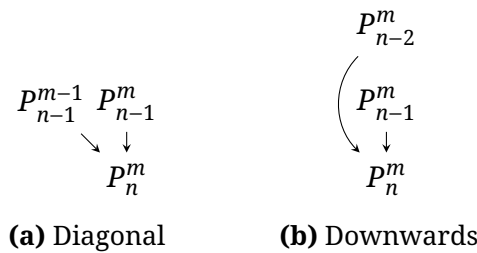


Figure 2: Diagrams illustrating the two recurrence relations used.

2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^n f g = \sum_{k=0}^n \binom{n}{k} (D^k f) (D^{n-k} g)$$

which can also be proven by induction.

Applying it after performing the first derivative, we get

$$\begin{aligned} Q_n^m &= D^{n+m} (x^2 - 1)^n = D^{n+m-1} n (x^2 - 1)^{n-1} 2x \\ &= 2n \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} (D^k x) (D^{n+m-1-k} (x^2 - 1)^{n-1}) \\ &= 2n \left(x D^{n+m-1} (x^2 - 1)^{n-1} + (n+m-1) D^{n+m-2} (x^2 - 1)^{n-1} \right) \end{aligned}$$

and since $Q_{n-1}^m = D^{n-1+m} (x^2 - 1)^{n-1}$ and $Q_{n-1}^{m-1} = D^{n-1+m-1} (x^2 - 1)^{n-1}$, we get

$$Q_n^m = 2nx Q_{n-1}^m + 2n(n+m-1) Q_{n-1}^{m-1}. \quad (2)$$

Since $Q_n^m = P_n^m / \alpha_n^m$, we can rewrite this as

$$P_n^m = 2nx \frac{\alpha_n^m}{\alpha_{n-1}^m} P_{n-1}^m + 2n(n+m-1) \frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} P_{n-1}^{m-1}.$$

Now we have the recurrence relation illustrated in Figure 2a. We will want to use this to calculate the topmost values in Figure 1, where $m = n$. But as we saw in the last section, $P_{m-1}^m = 0$. Hence, setting $m = n$ we get

$$P_m^m = 2m(2m-1) \frac{\alpha_m^m}{\alpha_{m-1}^{m-1}} P_{m-1}^{m-1}. \quad (3)$$

Expanding the alphas using equation (1), we get (assuming $m > 0$, since the first value to calculate using this recurrence relation is P_1^1)

$$\begin{aligned} \frac{\alpha_m^m}{\alpha_{m-1}^{m-1}} &= \sqrt{\frac{(2-0)(m-1+m-1)!}{(2-\delta_{m1})(m+m)!}} \frac{1}{2m} \frac{(1-x^2)^{m/2}}{(1-x^2)^{(m-1)/2}} \\ &= \sqrt{\frac{1+\delta_{m1}}{2m(2m-1)}} \frac{1}{2m} (1-x^2)^{1/2} \end{aligned}$$

and substituting this back into (3) results in

$$P_m^m(x) = \sqrt{(1+\delta_{m1})} \frac{2m-1}{2m} (1-x^2)^{1/2} P_{m-1}^{m-1}$$

or, with $x = \cos \theta$,

$$P_m^m(\cos \theta) = \sqrt{(1+\delta_{m1})} \left(1 - \frac{1}{2m}\right) \sin(\theta) P_{m-1}^{m-1}(\cos \theta)$$

2.2.2 Downwards

Finding the recurrence relation shown in Figure 2b was a bit more difficult, and the technique I found involves combining the recurrence relation in Figure 2a with the one illustrated in Figure 3.

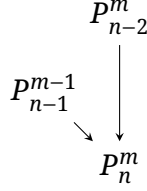


Figure 3: Recurrence relation used together with the recurrence relation in Figure 2a to create the downward recurrence relation.

This time we start by applying the first two derivatives:

$$\begin{aligned} Q_n^m &= D^{n+m}(x^2 - 1)^n = D^{n+m-1}n(x^2 - 1)^{n-1}2x \\ &= 2nD^{n+m-2}\left((x^2 - 1)^{n-1} + x(n-1)(x^2 - 1)^{n-2}2x\right) \\ &= 2nQ_{n-1}^{m-1} + 4n(n-1)D^{n+m-2}x^2(x^2 - 1)^{n-2}. \end{aligned}$$

Applying the algebraic identity

$$x^2(x^2 - 1)^{n-2} = (x^2 - 1)(x^2 - 1)^{n-2} + 1(x^2 - 1)^{n-2} = (x^2 - 1)^{n-1} + (x^2 - 1)^{n-2}$$

and identifying $Q_{n-1}^{m-1} = D^{n+m-2}(x^2 - 1)^{n-1}$ and $Q_{n-2}^m = D^{n+m-2}(x^2 - 1)^{n-2}$,

$$\begin{aligned} Q_n^m &= 2nQ_{n-1}^{m-1} + 4n(n-1)\left(Q_{n-1}^{m-1} + Q_{n-2}^m\right) \\ &= 2n(2n-1)Q_{n-1}^{m-1} + 4n(n-1)Q_{n-2}^m \end{aligned}$$

which corresponds to the diagram in Figure 3. Now we multiply both sides by $(n+m-1)/(2n-1)$ and rearrange to get

$$2n(n+m-1)Q_{n-1}^{m-1} = \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m$$

which can be substituted in (2) to give

$$\begin{aligned} Q_n^m &= 2nxQ_{n-1}^m + \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m \\ \iff \frac{2n-1-n-m+1}{2n-1}Q_n^m &= \frac{n-m}{2n-1}Q_n^m = 2nxQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m \\ \iff Q_n^m &= \frac{2n(2n-1)}{n-m}xQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}Q_{n-2}^m \end{aligned}$$

or

$$P_n^m = \frac{2n(2n-1)}{n-m}x\frac{\alpha_n^m}{\alpha_{n-1}^m}P_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}\frac{\alpha_n^m}{\alpha_{n-2}^m}P_{n-2}^m. \quad (4)$$

Using (1) again, the alpha fractions are

$$\frac{\alpha_n^m}{\alpha_{n-1}^m} = \sqrt{\frac{n-m}{n+m}} \frac{1}{2n}$$

and

$$\frac{\alpha_n^m}{\alpha_{n-2}^m} = \sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}} \frac{1}{4n(n-1)}.$$

Back into (4),

$$\begin{aligned} P_n^m &= \frac{2n(2n-1)}{n-m} x \sqrt{\frac{n-m}{n+m}} \frac{1}{2n} P_{n-1}^m \\ &\quad - \frac{4n(n-1)(n+m-1)}{n-m} \sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}} \frac{1}{4n(n-1)} P_{n-2}^m. \end{aligned}$$

After some simplification,

$$P_n^m = \frac{2n-1}{\sqrt{n^2-m^2}} x P_{n-1}^m - \sqrt{\frac{(n-1)^2-1}{n^2-m^2}} P_{n-2}^m$$

References

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