Calculating the International Geomagnetic Reference Field

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October 1, 2025

1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential V; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coeffients g_n^m and h_n^m which are used in the following equation to calculate the potential:

$$V(r,\theta,\phi) = \sum_{n=0}^{N} \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) P_n^m(\cos\theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of nth degree and mth order, $a = 6371.2\,\mathrm{km}$ is Earth's mean radius, r is the distance from Earth's center to the point the magnetic field is to be calculated at, θ is its co-latitude, and ϕ is its longitude. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

Since we are using spherical coordinates, the magnetic field has the form

$$\mathbf{B} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\mathrm{d}V}{\mathrm{d}\phi}\hat{\boldsymbol{\phi}}$$

and the components can be expanded as

$$B_r = \sum_{n=0}^{N} (n+1)\left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) P_n^m(\cos\theta)$$

$$B_\theta = -\sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta)$$

$$B_\phi = \sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m m \sin(m\phi) - h_n^m m \cos(m\phi)\right) P_n^m(\cos\theta)$$

Here we see that we not only need $P_n^m(\cos\theta)$, we also need its derivative with respect to θ.

2 Associated legendre functions

The nth degree, mth order associated legendre function with Schmidt semi-normalization is defined as (Winch et al., 2005)

$$P_n^m(x) = \alpha_n^m(x) D^{m+n}(x^2-1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\frac{1}{2^n n!}}(1-x^2)^{m/2}, \tag{1}$$

δ_{m0} is the Kronecker delta (1 for $m=0$ and 0 otherwise) and $D = \mathrm{d}/\mathrm{d}x$ is the differentiation operator with respect to x.

2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m}x^{2k} = \begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m}(x^2-1)^2 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Since

$$m+n > 2k \iff k < \frac{m+n}{2},$$

all terms with $k < \lceil (m+n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2\rceil}^{n}\binom{n}{k}\frac{(2k)!}{(2k-n-m)!}(-1)^{n-k}x^{2k-n-m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m+n > 2n$, i.e. $m > n$. A faster way to calculate the Q_n^ms is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

2.2 Recurrence relations

$$
\begin{array}{cccccc}
P_0^0 & & & & & \\
\downarrow & \searrow & & & & \\
P_1^0 & P_1^1 & & & & \\
\downarrow & \downarrow & \searrow & & & \\
P_2^0 & P_2^1 & P_2^2 & & & \\
\downarrow & \downarrow & \downarrow & \searrow & & \\
P_3^0 & P_3^1 & P_3^2 & P_3^3 & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \searrow & \\
P_4^0 & P_4^1 & P_4^2 & P_4^3 & P_4^4 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow \\
P_5^0 & P_5^1 & P_5^2 & P_5^3 & P_5^4 & P_5^5
\end{array}
$$

Figure 1: Diagram showing how the recurrence relations are used to calculate P_n^m.

Instead of calculating each associated Legendre function by itself, a better idea is to calculate each P_n^m from previously calculated P_n^m values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single P_n^m up to $n = 13$, so we are effectively spreading the calculations out between P_n^ms and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each n is a row and each m is a column) leads to recurrence relations involving division by $\sin\theta$, which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. It all starts from P_0^0, which we can calculate to be

$$P_0^0(x) = \sqrt{(2-1)\frac{0!}{0!}}\frac{1}{2^0 \cdot 0!}(1-x^2)^0 = 1$$

with a derivative with respect to x that is identically zero.

The recurrence relations are derived below and are illustrated individually in Figure 2; both use two previous values.

$$
\begin{array}{cc}
P_{n-1}^{m-1} & P_{n-1}^m \\
\searrow & \downarrow \\
 & P_n^m
\end{array}
\qquad\qquad
\begin{array}{c}
P_{n-2}^m \\
P_{n-1}^m \\
\downarrow \\
P_n^m
\end{array}
$$

(a) **(b)**

Figure 2: Diagrams illustrating the two recurrence relations used.

2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^n fg = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$$

which can also be proven by induction.

Applying it after first performing the first derivative, we get

$$\begin{aligned} Q_n^m &= D^{n+m}(x^2-1)^n = D^{n+m-1} n(x^2-1)^{n-1} 2x \\ &= 2n \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} \left(D^k x\right) \left(D^{n+m-1-k}(x^2-1)^{n-1}\right) \\ &= 2n \left(x D^{n+m-1}(x^2-1)^{n-1} + (n+m-1) D^{n+m-2}(x^2-1)^{n-1}\right) \end{aligned}$$

and since $Q_{n-1}^m = D^{n-1+m}(x^2-1)^{n-1}$ and $Q_{n-1}^{m-1} = D^{n-1+m-1}(x^2-1)^{n-1}$, we get

$$Q_n^m = 2nx Q_{n-1}^m + 2n(n+m-1) Q_{n-1}^{m-1}.$$

Since $Q_n^m = P_n^m / \alpha_n^m$, we can rewrite this as

$$P_n^m = 2nx \frac{\alpha_n^m}{\alpha_{n-1}^m} P_{n-1}^m + 2n(n+m-1) \frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} P_{n-1}^{m-1}.$$

Expanding the alphas using equation (1),

$$\begin{aligned} \frac{\alpha_n^m}{\alpha_{n-1}^m} &= \sqrt{\frac{(n-m)!(n-1+m)!}{(n-1-m)!(n+m)!}} \frac{2^{n-1}(n-1)!}{2^n n!} \\ &= \sqrt{\frac{n-1}{n+m}} \frac{1}{2n} \end{aligned}$$

and

$$\frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} = \sqrt{}$$

2.2.2 Downwards

References

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