

Calculating the International Geomagnetic Reference Field

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1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential V ; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coefficients g_n^m and h_n^m , which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^N \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) P_n^m(\cos \theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of n th degree and m th order. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

2 Associated legendre functions

The n th degree, m th order associated legendre function is defined as

$$P_n^m(x) = \alpha_n^m(x) D^{m+n} (x^2 - 1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!} \frac{1}{2^n n!}} (1 - x^2)^{m/2},$$

δ_{m0} is the Kronecker delta (1 for $m = 0$ and 0 otherwise) and $D = d/dx$ is the differentiation operator with respect to x .

2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m} x^{2k} = \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m} (x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Since

$$m + n > 2k \iff k < \frac{m + n}{2},$$

all terms with $k < \lceil (m + n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2 \rceil}^n \binom{n}{k} \frac{(2k)!}{(2k - n - m)!} (-1)^{n-k} x^{2k - n - m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m + n > 2n$, i.e. $m > n$. A faster way to calculate the Q_n^m s is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

2.2 Recurrence relations

The idea is to

References

Alken, P., Thébault, E., Beggan, C. D., Amit, H., Aubert, J., Baerenzung, J., Bondar, T. N., Brown, W. J., Califf, S., Chambodut, A., Chulliat, A., Cox, G. A., Finlay, C. C., Fournier, A., Gillet, N., Grayver, A., Hammer, M. D., Holschneider, M., Huder, L., ... Zhou, B. (2021). International geomagnetic reference field: The thirteenth generation. *Earth, Planets and Space*, 73(1). <https://doi.org/10.1186/s40623-020-01288-x>

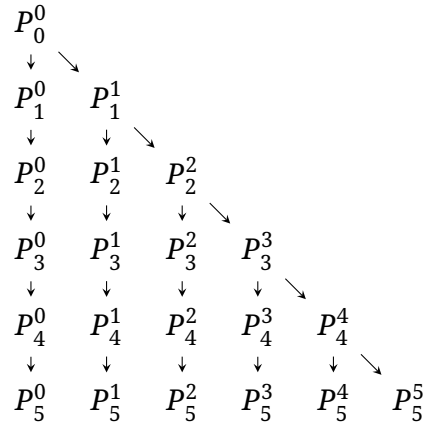


Figure 1: Diagram showing how the recurrence relations are used to calculate P_n^m .

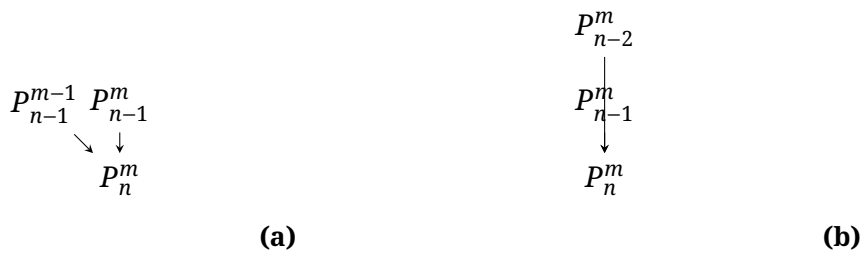


Figure 2: Diagrams illustrating the two recurrence relations used.