Calculating the International Geomagnetic Reference Field

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1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential V; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coeffients g_n^m and h_n^m, which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^{N} \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) P_n^m(\cos\theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of nth degree and mth order, $a = 6371.2$ is Earth's mean radius, r is the distance from Earth's center, θ is the co-latitude, and ϕ is the longitude. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

2 Associated legendre functions

The nth degree, mth order associated legendre function is defined as

$$P_n^m(x) = \alpha_n^m(x) D^{m+n} (x^2 - 1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!}} \frac{1}{2^n n!} (1 - x^2)^{m/2},$$

δ_{m0} is the Kronecker delta (1 for $m = 0$ and 0 otherwise) and $D = \mathrm{d}/\mathrm{d}x$ is the differentiation operator with respect to x.

2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m} x^{2k} = \begin{cases} \dfrac{(2k)!}{(2k-n-m)!} x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m}(x^2-1)^2 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \begin{cases} \dfrac{(2k)!}{(2k-n-m)!} x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Since

$$m+n > 2k \iff k < \frac{m+n}{2},$$

all terms with $k < \lceil (m+n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2 \rceil}^{n} \binom{n}{k} \frac{(2k)!}{(2k-n-m)!} (-1)^{n-k} x^{2k-n-m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m+n > 2n$, i.e. $m > n$. A faster way to calculate the Q_n^ms is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

2.2 Recurrence relations

The idea is to instead calculate the associated Legendre functions using *recurrence relations*, where each P_n^m is calculated from previously calculated P_n^m values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single P_n^m up to $n = 13$, so we are effectively spreading the calculations out between P_n^ms and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each n is a row and each m is a column) leads to recurrence relations involving division by $\sin\theta$, which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. The recurrence relations are derived below.

$$
\begin{tikzcd}
P_0^0 \arrow[d] \arrow[dr] & & & & & \\
P_1^0 \arrow[d] & P_1^1 \arrow[d] \arrow[dr] & & & & \\
P_2^0 \arrow[d] & P_2^1 \arrow[d] & P_2^2 \arrow[d] \arrow[dr] & & & \\
P_3^0 \arrow[d] & P_3^1 \arrow[d] & P_3^2 \arrow[d] & P_3^3 \arrow[d] \arrow[dr] & & \\
P_4^0 \arrow[d] & P_4^1 \arrow[d] & P_4^2 \arrow[d] & P_4^3 \arrow[d] & P_4^4 \arrow[d] \arrow[dr] & \\
P_5^0 & P_5^1 & P_5^2 & P_5^3 & P_5^4 & P_5^5
\end{tikzcd}
$$

Figure 1: Diagram showing how the recurrence relations are used to calculate P_n^m.

$$
\begin{tikzcd}
P_{n-1}^{m-1} \arrow[dr] & P_{n-1}^{m} \arrow[d] \\
& P_n^m
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
P_{n-2}^{m} \arrow[d, dash] \\
P_{n-1}^{m} \arrow[d] \\
P_n^m
\end{tikzcd}
$$

(b)

Figure 2: Diagrams illustrating the two recurrence relations used.

2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^{n+m} fg = \sum_{k=0}^{n+m} \binom{n+m}{k} (D^k f)(D^{n+m-k} g)$$

which can also be proven by induction.

2.2.2 Downwards

References

Alken, P., Thébault, E., Beggan, C. D., Amit, H., Aubert, J., Baerenzung, J., Bondar, T. N., Brown, W. J., Califf, S., Chambodut, A., Chulliat, A., Cox, G. A., Finlay, C. C., Fournier, A., Gillet, N., Grayver, A., Hammer, M. D., Holschneider, M., Huder, L., … Zhou, B. (2021). International geomagnetic reference field: The thirteenth generation. *Earth, Planets and Space*, *73*(1). https://doi.org/10.1186/s40623-020-01288-x