Calculating the International Geomagnetic Reference Field

Björn Sundin

October 13, 2025

1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes it to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential V; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coeffients g_n^m and h_n^m which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^{N} \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) P_n^m(\cos\theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of nth degree and mth order, $a = 6371.2\,\mathrm{km}$ is Earth's mean radius, r is the distance from Earth's center to the point the magnetic field is to be calculated at, θ is its co-latitude, and ϕ is its longitude. The coefficients are functions of time, though this parameter has been omitted above. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

Since we are using spherical coordinates, the magnetic field has the form

$$\mathbf{B} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\mathrm{d}V}{\mathrm{d}\phi}\hat{\boldsymbol{\phi}}$$

and the components can be expanded as

$$B_r = \sum_{n=0}^{N} (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) P_n^m(\cos\theta)$$

$$B_\theta = -\sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta)$$

$$B_\phi = \sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left(g_n^m m \sin(m\phi) - h_n^m m \cos(m\phi) \right) P_n^m(\cos\theta)$$

Here we see that we not only need $P_n^m(\cos\theta)$, we also need its derivative with respect to θ.

2 Associated legendre functions

The nth degree, mth order associated legendre function with Schmidt semi-normalization is defined as (Winch et al., 2005)

$$P_n^m(x) = \alpha_n^m(x) D^{m+n}(x^2-1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}}\frac{1}{2^n n!}(1-x^2)^{m/2}, \tag{1}$$

δ_{m0} is the Kronecker delta (1 for $m=0$ and 0 otherwise) and $D = \mathrm{d}/\mathrm{d}x$ is the differentiation operator with respect to x.

2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m}x^{2k} = \begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m}(x^2-1)^2 = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Since

$$m+n > 2k \iff k < \frac{m+n}{2},$$

all terms with $k < \lceil (m+n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2\rceil}^{n}\binom{n}{k}\frac{(2k)!}{(2k-n-m)!}(-1)^{n-k}x^{2k-n-m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m + n > 2n$, i.e. $m > n$. A faster way to calculate the Q_n^ms is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

2.2 Recurrence relations

$$
\begin{tikzcd}
P_0^0 \arrow[d] \arrow[dr] & & & & & \\
P_1^0 \arrow[d] & P_1^1 \arrow[d] \arrow[dr] & & & & \\
P_2^0 \arrow[d] & P_2^1 \arrow[d] & P_2^2 \arrow[d] \arrow[dr] & & & \\
P_3^0 \arrow[d] & P_3^1 \arrow[d] & P_3^2 \arrow[d] & P_3^3 \arrow[d] \arrow[dr] & & \\
P_4^0 \arrow[d] & P_4^1 \arrow[d] & P_4^2 \arrow[d] & P_4^3 \arrow[d] & P_4^4 \arrow[d] \arrow[dr] & \\
P_5^0 & P_5^1 & P_5^2 & P_5^3 & P_5^4 & P_5^5
\end{tikzcd}
$$

Figure 1: Diagram showing how the recurrence relations are used to calculate P_n^m.

Instead of calculating each associated Legendre function by itself, a better idea is to calculate each P_n^m from previously calculated P_n^m values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single P_n^m up to $n = 13$, so we are effectively spreading the calculations out between P_n^ms and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each n is a row and each m is a column) leads to recurrence relations involving division by $\sin\theta$, which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. It all starts from P_0^0, which we can calculate to be

$$P_0^0(x) = \sqrt{(2-1)\frac{0!}{0!}}\frac{1}{2^0 \cdot 0!}(1-x^2)^0 = 1$$

with a derivative with respect to x (or θ, when $x = \cos\theta$) that is identically zero.

The recurrence relations are derived below and are illustrated individually in Figure 2; both use two previous values, but one of them might be zero.

2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^n fg = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$$

$$\begin{tikzcd} P^{m-1}_{n-1} \arrow[dr] & P^m_{n-1} \arrow[d] \\ & P^m_n \end{tikzcd}$$

(a) Diagonal

$$\begin{tikzcd} P^m_{n-2} \arrow[dd, bend right] \\ P^m_{n-1} \arrow[d] \\ P^m_n \end{tikzcd}$$

(b) Downwards

Figure 2: Diagrams illustrating the two recurrence relations used.

which can also be proven by induction.

Applying it after performing the first derivative, we get

$$\begin{aligned} Q_n^m &= D^{n+m}(x^2-1)^n = D^{n+m-1}n(x^2-1)^{n-1}2x \\ &= 2n \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} \left(D^k x\right) \left(D^{n+m-1-k}(x^2-1)^{n-1}\right) \\ &= 2n \left(xD^{n+m-1}(x^2-1)^{n-1} + (n+m-1)D^{n+m-2}(x^2-1)^{n-1}\right) \end{aligned}$$

and since $Q_{n-1}^m = D^{n-1+m}(x^2-1)^{n-1}$ and $Q_{n-1}^{m-1} = D^{n-1+m-1}(x^2-1)^{n-1}$, we get

$$Q_n^m = 2nxQ_{n-1}^m + 2n(n+m-1)Q_{n-1}^{m-1}. \tag{2}$$

Since $Q_n^m = P_n^m/\alpha_n^m$, this can be rewritten

$$P_n^m = 2nx\frac{\alpha_n^m}{\alpha_{n-1}^m}P_{n-1}^m + 2n(n+m-1)\frac{\alpha_n^m}{\alpha_{n-1}^{m-1}}P_{n-1}^{m-1}.$$

Now we have the recurrence relation illustrated in Figure 2a. We will want to use this to calculate the topmost values in Figure 1, where $m = n$. But as we saw in the last section, $P_{m-1}^m = 0$. Hence, setting $m = n$ we get

$$P_m^m = 2m(2m-1)\frac{\alpha_m^m}{\alpha_{m-1}^{m-1}}P_{m-1}^{m-1}. \tag{3}$$

Expanding the alphas using equation (1), we get (assuming $m > 0$, since the first value to calculate using this recurrence relation is P_1^1)

$$\begin{aligned} \frac{\alpha_m^m}{\alpha_{m-1}^{m-1}} &= \sqrt{\frac{(2-0)(m-1+m-1)!}{(2-\delta_{m1})(m+m)!}}\frac{1}{2m}\frac{(1-x^2)^{m/2}}{(1-x^2)^{(m-1)/2}} \\ &= \sqrt{\frac{1+\delta_{m1}}{2m(2m-1)}}\frac{1}{2m}(1-x^2)^{1/2} \end{aligned}$$

and substituting this back into (3) results in

$$P_m^m(x) = \sqrt{(1+\delta_{m1})\frac{2m-1}{2m}}(1-x^2)^{1/2}P_{m-1}^{m-1}$$

or, with $x = \cos\theta$,

$$\boxed{P_m^m(\cos\theta) = A_m \sin(\theta) P_{m-1}^{m-1}(\cos\theta)}$$

where

$$\boxed{A_m = \sqrt{(1+\delta_{m1})\left(1 - \frac{1}{2m}\right)}}.$$

Differentiating this recurrence relation with respect to θ gives

$$\boxed{\frac{\mathrm{d}P_m^m(\cos\theta)}{\mathrm{d}\theta} = A_m \left(\cos(\theta) P_{m-1}^{m-1}(\cos\theta) + \sin(\theta)\frac{\mathrm{d}P_{m-1}^{m-1}(\cos\theta)}{\mathrm{d}\theta}\right)}.$$

2.2.2 Downwards

Finding the recurrence relation shown in Figure 2b was a bit more difficult, and the technique I found involves combining the recurrence relation in Figure 2a with the one illustrated in Figure 3.

$$\begin{tikzcd}
 & P_{n-2}^m \arrow[dd] \\
P_{n-1}^{m-1} \arrow[dr] & \\
 & P_n^m
\end{tikzcd}$$

Figure 3: Recurrence relation used together with the recurrence relation in Figure 2a to create the downward recurrence relation.

This time we start by applying the first two derivatives:

$$\begin{aligned}
Q_n^m &= D^{n+m}(x^2-1)^n = D^{n+m-1} n (x^2-1)^{n-1} 2x \\
&= 2nD^{n+m-2}\left((x^2-1)^{n-1} + x(n-1)(x^2-1)^{n-2}2x\right) \\
&= 2nQ_{n-1}^{m-1} + 4n(n-1)D^{n+m-2}x^2(x^2-1)^{n-2}.
\end{aligned}$$

Applying the algebraic identity

$$x^2(x^2-1)^{n-2} = (x^2-1)(x^2-1)^{n-2} + 1(x^2-1)^{n-2} = (x^2-1)^{n-1} + (x^2-1)^{n-2}$$

and identifying $Q_{n-1}^{m-1} = D^{n+m-2}(x^2-1)^{n-1}$ and $Q_{n-2}^m = D^{n+m-2}(x^2-1)^{n-2}$,

$$\begin{aligned}
Q_n^m &= 2nQ_{n-1}^{m-1} + 4n(n-1)\left(Q_{n-1}^{m-1} + Q_{n-2}^m\right) \\
&= 2n(2n-1)Q_{n-1}^{m-1} + 4n(n-1)Q_{n-2}^m
\end{aligned}$$

which corresponds to the diagram in Figure 3. Now we multiply both sides by $(n+m-1)/(2n-1)$ and rearrange to get

$$2n(n+m-1)Q_{n-1}^{m-1} = \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m$$

which can be substituted in (2) to give

$$Q_n^m = 2nxQ_{n-1}^m + \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m$$

$$\iff \frac{2n-1-n-m+1}{2n-1}Q_n^m = \frac{n-m}{2n-1}Q_n^m = 2nxQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m$$

$$\iff Q_n^m = \frac{2n(2n-1)}{n-m}xQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}Q_{n-2}^m$$

or

$$P_n^m = \frac{2n(2n-1)}{n-m}x\frac{\alpha_n^m}{\alpha_{n-1}^m}P_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}\frac{\alpha_n^m}{\alpha_{n-2}^m}P_{n-2}^m. \tag{4}$$

Using (1) again, the alpha fractions are

$$\frac{\alpha_n^m}{\alpha_{n-1}^m} = \sqrt{\frac{n-m}{n+m}\frac{1}{2n}}$$

and

$$\frac{\alpha_n^m}{\alpha_{n-2}^m} = \sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}\frac{1}{4n(n-1)}}.$$

Back into (4),

$$P_n^m = \frac{2n(2n-1)}{n-m}x\sqrt{\frac{n-m}{n+m}\frac{1}{2n}}P_{n-1}^m$$
$$- \frac{4n(n-1)(n+m-1)}{n-m}\sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}\frac{1}{4n(n-1)}}P_{n-2}^m.$$

After some simplification and substituting $x = \cos\theta$,

$$\boxed{P_n^m(\cos\theta) = B_n^m\cos(\theta)P_{n-1}^m(\cos\theta) - C_n^m P_{n-2}^m(\cos\theta)}$$

where

$$\boxed{B_n^m = \frac{2n-1}{\sqrt{n^2-m^2}} \quad \text{and} \quad C_n^m = \sqrt{\frac{(n-1)^2-1}{n^2-m^2}}}$$

and the derivative becomes

$$\boxed{\frac{\mathrm{d}P_n^m(\cos\theta)}{\mathrm{d}\theta} = B_n^m\left(\cos(\theta)\frac{\mathrm{d}P_{n-1}^m(\cos\theta)}{\mathrm{d}\theta} - \sin(\theta)P_{n-1}^m(\cos\theta)\right) - C_n^m\frac{\mathrm{d}P_{n-2}^m(\cos\theta)}{\mathrm{d}\theta}}$$

References

Alken, P., Thébault, E., Beggan, C. D., Amit, H., Aubert, J., Baerenzung, J., Bondar, T. N., Brown, W. J., Califf, S., Chambodut, A., Chulliat, A., Cox, G. A., Finlay, C. C., Fournier, A., Gillet, N., Grayver, A., Hammer, M. D., Holschneider, M., Huder, L., … Zhou, B. (2021). International geomagnetic reference field: The thirteenth generation. *Earth, Planets and Space*, *73*(1). https://doi.org/10.1186/s40623-020-01288-x

Winch, D. E., Ivers, D. J., Turner, J. P. R., & Stening, R. J. (2005). Geomagnetism and schmidt quasi-normalization. *Geophysical Journal International*, *160*(2), 487–504. https://doi.org/10.1111/j.1365-246x.2004.02472.x