

# Calculating the International Geomagnetic Reference Field

Björn Sundin

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## 1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes  $\nabla \times \mathbf{B} = 0$  and the field can be written as the negative gradient of a potential  $V$ ;  $\mathbf{B} = -\nabla V$ . The IGRF model provides a list of spherical harmonic coefficients  $g_n^m$  and  $h_n^m$  which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^N \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) P_n^m(\cos \theta).$$

Here  $P_n^m(x)$  is the *Schmidt semi-normalized associated Legendre function* of  $n$ th degree and  $m$ th order,  $a = 6371.2$  km is Earth's mean radius,  $r$  is the distance from Earth's center to the point the magnetic field is to be calculated at,  $\theta$  is its co-latitude, and  $\phi$  is its longitude. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

Since we are using spherical coordinates, the magnetic field has the form

$$\mathbf{B} = -\nabla V = -\frac{dV}{dr} \hat{\mathbf{r}} - \frac{1}{r} \frac{dV}{d\theta} \hat{\boldsymbol{\theta}} - \frac{1}{r \sin \theta} \frac{dV}{d\phi} \hat{\boldsymbol{\phi}}$$

and the components can be expanded as

$$\begin{aligned} B_r &= \sum_{n=0}^N (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) P_n^m(\cos \theta) \\ B_\theta &= -\sum_{n=0}^N \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m \cos(m\phi) + h_n^m \sin(m\phi)) \frac{d}{d\theta} P_n^m(\cos \theta) \\ B_\phi &= \sum_{n=0}^N \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^n (g_n^m m \sin(m\phi) - h_n^m m \cos(m\phi)) P_n^m(\cos \theta) \end{aligned}$$

Here we see that we not only need  $P_n^m(\cos \theta)$ , we also need its derivative with respect to  $\theta$ .

## 2 Associated legendre functions

The  $n$ th degree,  $m$ th order associated legendre function with Schmidt semi-normalization is defined as (Winch et al., 2005)

$$P_n^m(x) = \alpha_n^m(x) D^{m+n} (x^2 - 1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!} \frac{1}{2^n n!}} (1 - x^2)^{m/2}, \quad (1)$$

$\delta_{m0}$  is the Kronecker delta (1 for  $m = 0$  and 0 otherwise) and  $D = d/dx$  is the differentiation operator with respect to  $x$ .

### 2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function  $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$  can be written as a polynomial by first expanding the binomial power:

$$(x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m} x^{2k} = \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m} (x^2 - 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \begin{cases} \frac{(2k)!}{(2k - n - m)!} x^{2k - n - m}, & \text{if } m + n \leq 2k \\ 0, & \text{if } m + n > 2k \end{cases}.$$

Since

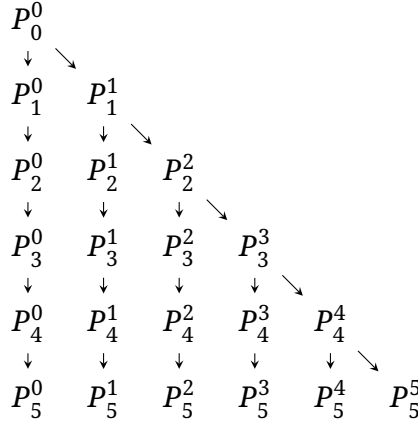
$$m + n > 2k \iff k < \frac{m + n}{2},$$

all terms with  $k < \lceil (m + n)/2 \rceil$  are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2 \rceil}^n \binom{n}{k} \frac{(2k)!}{(2k - n - m)!} (-1)^{n-k} x^{2k - n - m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that  $Q_n^m(x) = 0$  whenever  $m + n > 2n$ , i.e.  $m > n$ . A faster way to calculate the  $Q_n^m$ s is to use recurrence relations, and the fact that  $Q_n^m(x) = 0$  for  $m > n$  turns out to be important for doing that.

## 2.2 Recurrence relations



**Figure 1:** Diagram showing how the recurrence relations are used to calculate  $P_n^m$ .

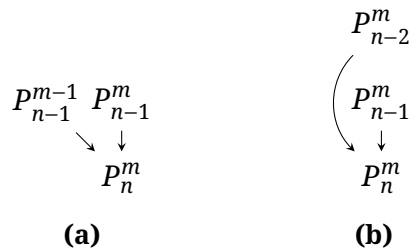
Instead of calculating each associated Legendre function by itself, a better idea is to calculate each  $P_n^m$  from previously calculated  $P_n^m$  values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single  $P_n^m$  up to  $n = 13$ , so we are effectively spreading the calculations out between  $P_n^m$ s and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each  $n$  is a row and each  $m$  is a column) leads to recurrence relations involving division by  $\sin \theta$ , which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. It all starts from  $P_0^0$ , which we can calculate to be

$$P_0^0(x) = \sqrt{(2-1) \frac{0!}{0!} \frac{1}{2^0 \cdot 0!}} (1-x^2)^0 = 1$$

with a derivative with respect to  $x$  that is identically zero.

The recurrence relations are derived below and are illustrated individually in Figure 2; both use two previous values.



**Figure 2:** Diagrams illustrating the two recurrence relations used.

### 2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^n f g = \sum_{k=0}^n \binom{n}{k} (D^k f)(D^{n-k} g)$$

which can also be proven by induction.

Applying it after first performing the first derivative, we get

$$\begin{aligned} Q_n^m &= D^{n+m}(x^2 - 1)^n = D^{n+m-1}n(x^2 - 1)^{n-1}2x \\ &= 2n \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} (D^k x) (D^{n+m-1-k}(x^2 - 1)^{n-1}) \\ &= 2n \left( x D^{n+m-1}(x^2 - 1)^{n-1} + (n+m-1) D^{n+m-2}(x^2 - 1)^{n-1} \right) \end{aligned}$$

and since  $Q_{n-1}^m = D^{n-1+m}(x^2 - 1)^{n-1}$  and  $Q_{n-1}^{m-1} = D^{n-1+m-1}(x^2 - 1)^{n-1}$ , we get

$$Q_n^m = 2nxQ_{n-1}^m + 2n(n+m-1)Q_{n-1}^{m-1}.$$

Since  $Q_n^m = P_n^m / \alpha_n^m$ , we can rewrite this as

$$P_n^m = 2nx \frac{\alpha_n^m}{\alpha_{n-1}^m} P_{n-1}^m + 2n(n+m-1) \frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} P_{n-1}^{m-1}.$$

Expanding the alphas using equation (1),

$$\begin{aligned} \frac{\alpha_n^m}{\alpha_{n-1}^m} &= \sqrt{\frac{(n-m)!(n-1+m)!}{(n-1-m)!(n+m)!} \frac{2^{n-1}(n-1)!}{2^n n!}} \\ &= \sqrt{\frac{n-1}{n+m} \frac{1}{2n}} \end{aligned}$$

and

$$\frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} = \sqrt{\frac{n-1}{n+m} \frac{1}{2n}}$$

### 2.2.2 Downwards

## References

- Alken, P., Thébault, E., Beggan, C. D., Amit, H., Aubert, J., Baerenzung, J., Bondar, T. N., Brown, W. J., Califf, S., Chambodut, A., Chulliat, A., Cox, G. A., Finlay, C. C., Fournier, A., Gillet, N., Grayver, A., Hammer, M. D., Holschneider, M., Huder, L., ... Zhou, B. (2021). International geomagnetic reference field: The thirteenth generation. *Earth, Planets and Space*, 73(1). <https://doi.org/10.1186/s40623-020-01288-x>
- Winch, D. E., Ivers, D. J., Turner, J. P. R., & Stening, R. J. (2005). Geomagnetism and schmidt quasi-normalization. *Geophysical Journal International*, 160(2), 487–504. <https://doi.org/10.1111/j.1365-246x.2004.02472.x>