# Calculating the International Geomagnetic Reference Field

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## 1 Introduction

The International Geomagnetic Reference Field (IGRF) is a three-dimensional model of Earth's internally generated magnetic field that is developed by the International Association of Geomagnetism and Aeronomy (IAGA) (Alken et al., 2021). A new version is released every five years, because the process by which Earth's magnetic field is generated (magnetohydrodynamic dynamo) causes Earth's magnetic field to change over time.

In a region of space without currents or electromagnetic waves, the Ampère-Maxwell equation becomes $\nabla \times \mathbf{B} = 0$ and the field can be written as the negative gradient of a potential $V$; $\mathbf{B} = -\nabla V$. The IGRF model provides a list of spherical harmonic coeffients $g_n^m$ and $h_n^m$ which are used in the following equation to calculate the potential:

$$V(r, \theta, \phi) = \sum_{n=0}^{N} \frac{a^{n+2}}{r^{n+1}} \sum_{m=0}^{n} \left( g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) P_n^m(\cos\theta).$$

Here $P_n^m(x)$ is the *Schmidt semi-normalized associated Legendre function* of $n$th degree and $m$th order, $a = 6371.2\,\mathrm{km}$ is Earth's mean radius, $r$ is the distance from Earth's center to the point the magnetic field is to be calculated at, $\theta$ is its co-latitude, and $\phi$ is its longitude. The coefficients are in reality functions of time. The IGRF provides them at five-year intervals, and between these points they can be estimated via linear interpolation.

Since we are using spherical coordinates, the magnetic field has the form

$$\mathbf{B} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\mathrm{d}V}{\mathrm{d}\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\mathrm{d}V}{\mathrm{d}\phi}\hat{\boldsymbol{\phi}}$$

and the components can be expanded as

$$B_r = \sum_{n=0}^{N} (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left( g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) P_n^m(\cos\theta)$$

$$B_\theta = -\sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left( g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta)$$

$$B_\phi = \sum_{n=0}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left( g_n^m m \sin(m\phi) - h_n^m m \cos(m\phi) \right) P_n^m(\cos\theta)$$

Here we see that we not only need $P_n^m(\cos\theta)$, we also need its derivative with respect to $\theta$.

# 2 Associated legendre functions

The $n$th degree, $m$th order associated legendre function with Schmidt semi-normalization is defined as (Winch et al., 2005)

$$P_n^m(x) = \alpha_n^m(x) D^{m+n}(x^2-1)^n$$

where

$$\alpha_n^m(x) = \sqrt{(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\frac{1}{2^n n!}}(1-x^2)^{m/2}, \tag{1}$$

$\delta_{m0}$ is the Kronecker delta (1 for $m = 0$ and 0 otherwise) and $D = \mathrm{d}/\mathrm{d}x$ is the differentiation operator with respect to $x$.

## 2.1 Expanded form

The definition given above involves repeated differentiation, which is not very efficient to implement directly on a computer. The function $Q_n^m(x) = P_n^m(x)/\alpha_n^m(x)$ can be written as a polynomial by first expanding the binomial power:

$$(x^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^{2k}$$

and then utilizing the following formula for repeated differentiation of a power function

$$D^{n+m}x^{2k} = \begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Both of these formulas can be proven by induction. Applying them, we get

$$Q_n^m(x) = D^{n+m}(x^2-1)^2 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \begin{cases} \dfrac{(2k)!}{(2k-n-m)!}x^{2k-n-m}, & \text{if } m+n \le 2k \\ 0, & \text{if } m+n > 2k \end{cases}.$$

Since

$$m+n > 2k \iff k < \frac{m+n}{2},$$

all terms with $k < \lceil (m+n)/2 \rceil$ are zero, and the equation can be simplified to

$$Q_n^m(x) = \sum_{k=\lceil (m+n)/2 \rceil}^{n} \binom{n}{k} \frac{(2k)!}{(2k-n-m)!} (-1)^{n-k} x^{2k-n-m}.$$

This is more readily calculated, although it is still very computationally expensive, with a sum of multiple terms and five factorials to calculate per term. Something we can deduce though is that $Q_n^m(x) = 0$ whenever $m+n > 2n$, i.e. $m > n$. A faster way to calculate the $Q_n^m$s is to use recurrence relations, and the fact that $Q_n^m(x) = 0$ for $m > n$ turns out to be important for doing that.

## 2.2 Recurrence relations

$$
\begin{tikzcd}[row sep=small, column sep=small]
P_0^0 \arrow[d] \arrow[dr] & & & & & \\
P_1^0 \arrow[d] & P_1^1 \arrow[d] \arrow[dr] & & & & \\
P_2^0 \arrow[d] & P_2^1 \arrow[d] & P_2^2 \arrow[d] \arrow[dr] & & & \\
P_3^0 \arrow[d] & P_3^1 \arrow[d] & P_3^2 \arrow[d] & P_3^3 \arrow[d] \arrow[dr] & & \\
P_4^0 \arrow[d] & P_4^1 \arrow[d] & P_4^2 \arrow[d] & P_4^3 \arrow[d] & P_4^4 \arrow[d] \arrow[dr] & \\
P_5^0 & P_5^1 & P_5^2 & P_5^3 & P_5^4 & P_5^5
\end{tikzcd}
$$

**Figure 1:** Diagram showing how the recurrence relations are used to calculate $P_n^m$.

Instead of calculating each associated Legendre function by itself, a better idea is to calculate each $P_n^m$ from previously calculated $P_n^m$ values. This is especially fitting for the purposes of calculating the IGRF because we need to use every single $P_n^m$ up to $n = 13$, so we are effectively spreading the calculations out between $P_n^m$s and avoid repeating the same calculations many times.

The recurrence scheme can be done in many different ways. It turns out that doing it row-by-row (where each $n$ is a row and each $m$ is a column) leads to recurrence relations involving division by $\sin\theta$, which leads to the values exploding when near the poles. Instead we do it column-by-column, as illustrated in Figure 1. It all starts from $P_0^0$, which we can calculate to be

$$P_0^0(x) = \sqrt{(2-1)\frac{0!}{0!}}\frac{1}{2^0 \cdot 0!}(1-x^2)^0 = 1$$

with a derivative with respect to $x$ (or $\theta$, when $x = \cos\theta$) that is identically zero.

The recurrence relations are derived below and are illustrated individually in Figure 2; both use two previous values, but one of them might be zero.

$$
\begin{tikzcd}[row sep=small, column sep=tiny]
P_{n-1}^{m-1} \arrow[dr] & P_{n-1}^m \arrow[d] \\
& P_n^m
\end{tikzcd}
$$

**(a)** Diagonal

$$
\begin{tikzcd}[row sep=small]
P_{n-2}^m \arrow[dd, bend right] \\
P_{n-1}^m \arrow[d] \\
P_n^m
\end{tikzcd}
$$

**(b)** Downwards

**Figure 2:** Diagrams illustrating the two recurrence relations used.

### 2.2.1 Diagonal

The idea for finding the recurrence relations is to use the formula

$$D^n fg = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g)$$

which can also be proven by induction.

Applying it after performing the first derivative, we get

$$\begin{aligned}
Q_n^m &= D^{n+m}(x^2-1)^n = D^{n+m-1} n(x^2-1)^{n-1} 2x \\
&= 2n \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} \left(D^k x\right) \left(D^{n+m-1-k}(x^2-1)^{n-1}\right) \\
&= 2n \left(x D^{n+m-1}(x^2-1)^{n-1} + (n+m-1) D^{n+m-2}(x^2-1)^{n-1}\right)
\end{aligned}$$

and since $Q_{n-1}^m = D^{n-1+m}(x^2-1)^{n-1}$ and $Q_{n-1}^{m-1} = D^{n-1+m-1}(x^2-1)^{n-1}$, we get

$$Q_n^m = 2nx Q_{n-1}^m + 2n(n+m-1) Q_{n-1}^{m-1}. \tag{2}$$

Since $Q_n^m = P_n^m / \alpha_n^m$, we can rewrite this as

$$P_n^m = 2nx \frac{\alpha_n^m}{\alpha_{n-1}^m} P_{n-1}^m + 2n(n+m-1) \frac{\alpha_n^m}{\alpha_{n-1}^{m-1}} P_{n-1}^{m-1}.$$

Now we have the recurrence relation illustrated in Figure 2a. We will want to use this to calculate the topmost values in Figure 1, where $m = n$. But as we saw in the last section, $P_{m-1}^m = 0$. Hence, setting $m = n$ we get

$$P_m^m = 2m(2m-1) \frac{\alpha_m^m}{\alpha_{m-1}^{m-1}} P_{m-1}^{m-1}. \tag{3}$$

Expanding the alphas using equation (1), we get (assuming $m > 0$, since the first value to calculate using this recurrence relation is $P_1^1$)

$$\begin{aligned}
\frac{\alpha_m^m}{\alpha_{m-1}^{m-1}} &= \sqrt{\frac{(2-0)(m-1+m-1)!}{(2-\delta_{m1})(m+m)!}} \frac{1}{2m} \frac{(1-x^2)^{m/2}}{(1-x^2)^{(m-1)/2}} \\
&= \sqrt{\frac{1+\delta_{m1}}{2m(2m-1)}} \frac{1}{2m} (1-x^2)^{1/2}
\end{aligned}$$

and substituting this back into (3) results in

$$P_m^m(x) = \sqrt{(1+\delta_{m1}) \frac{2m-1}{2m}} (1-x^2)^{1/2} P_{m-1}^{m-1}$$

or, with $x = \cos\theta$,

$$\boxed{P_m^m(\cos\theta) = \sqrt{(1+\delta_{m1}) \left(1 - \frac{1}{2m}\right)} \sin(\theta) P_{m-1}^{m-1}(\cos\theta)}$$

### 2.2.2 Downwards

Finding the recurrence relation shown in Figure 2b was a bit more difficult, and the technique I found involves combining the recurrence relation in Figure 2a with the one illustrated in Figure 3.

$$\begin{tikzcd} & P^m_{n-2} \arrow[dd] \\ P^{m-1}_{n-1} \arrow[dr] & \\ & P^m_n \end{tikzcd}$$

**Figure 3:** Recurrence relation used together with the recurrence relation in Figure 2a to create the downward recurrence relation.

This time we start by applying the first two derivatives:

$$\begin{aligned} Q_n^m &= D^{n+m}(x^2-1)^n = D^{n+m-1} n(x^2-1)^{n-1} 2x \\ &= 2nD^{n+m-2}\left((x^2-1)^{n-1} + x(n-1)(x^2-1)^{n-2} 2x\right) \\ &= 2nQ_{n-1}^{m-1} + 4n(n-1)D^{n+m-2}x^2(x^2-1)^{n-2}. \end{aligned}$$

Applying the algebraic identity

$$x^2(x^2-1)^{n-2} = (x^2-1)(x^2-1)^{n-2} + 1(x^2-1)^{n-2} = (x^2-1)^{n-1} + (x^2-1)^{n-2}$$

and identifying $Q_{n-1}^{m-1} = D^{n+m-2}(x^2-1)^{n-1}$ and $Q_{n-2}^m = D^{n+m-2}(x^2-1)^{n-2}$,

$$\begin{aligned} Q_n^m &= 2nQ_{n-1}^{m-1} + 4n(n-1)\left(Q_{n-1}^{m-1} + Q_{n-2}^m\right) \\ &= 2n(2n-1)Q_{n-1}^{m-1} + 4n(n-1)Q_{n-2}^m \end{aligned}$$

which corresponds to the diagram in Figure 3. Now we multiply both sides by $(n+m-1)/(2n-1)$ and rearrange to get

$$2n(n+m-1)Q_{n-1}^{m-1} = \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m$$

which can be substituted in (2) to give

$$\begin{aligned} Q_n^m &= 2nxQ_{n-1}^m + \frac{n+m-1}{2n-1}Q_n^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m \\ \iff \frac{2n-1-n-m+1}{2n-1}Q_n^m = \frac{n-m}{2n-1}Q_n^m &= 2nxQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{2n-1}Q_{n-2}^m \\ \iff Q_n^m &= \frac{2n(2n-1)}{n-m}xQ_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}Q_{n-2}^m \end{aligned}$$

or

$$P_n^m = \frac{2n(2n-1)}{n-m}x\frac{\alpha_n^m}{\alpha_{n-1}^m}P_{n-1}^m - \frac{4n(n-1)(n+m-1)}{n-m}\frac{\alpha_n^m}{\alpha_{n-2}^m}P_{n-2}^m. \tag{4}$$

Using (1) again, the alpha fractions are

$$\frac{\alpha_n^m}{\alpha_{n-1}^m} = \sqrt{\frac{n-m}{n+m}}\frac{1}{2n}$$

and

$$\frac{\alpha_n^m}{\alpha_{n-2}^m} = \sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}}\frac{1}{4n(n-1)}.$$

Back into (4),

$$\begin{aligned} P_n^m = {} & \frac{2n(2n-1)}{n-m}x\sqrt{\frac{n-m}{n+m}}\frac{1}{2n}P_{n-1}^m \\ & - \frac{4n(n-1)(n+m-1)}{n-m}\sqrt{\frac{(n-m)(n-m-1)}{(n+m)(n+m-1)}}\frac{1}{4n(n-1)}P_{n-2}^m. \end{aligned}$$

After some simplification,

$$\boxed{P_n^m = \frac{2n-1}{\sqrt{n^2-m^2}}xP_{n-1}^m - \sqrt{\frac{(n-1)^2-1}{n^2-m^2}}P_{n-2}^m}$$

# References


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