

Difference Scheme

The Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

Difference scheme:

$$\begin{aligned} u_t &\approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k} \\ u_{xx} &\approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} = \frac{1}{h^2} Au \\ u_{xxxx} &\approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} = \frac{1}{h^4} AAu \\ (u^2)_x &\approx \frac{\mu \delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} = \frac{1}{2h} D \end{aligned}$$

$$U^{n+1} = U^n - \frac{k}{h^2} AU^n - \frac{k}{h^4} AAU^n - \frac{k}{4h} D(U^n \odot U^n)$$

\odot = Element-wise multiplication

Consistency

Consistency

$$\begin{aligned}u_t &= \frac{\Delta u}{k} + O(k) \\u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\\mu \delta[u(x)^2] &= \mu[u(x + \frac{h^2}{2}) - u(x - \frac{h}{2})^2] \\&= 2huu_x + O(h^3) \\uu_x &= \frac{\mu \delta u^2}{4h} + O(h^2)\end{aligned}$$

Local Truncation Error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k,h \rightarrow 0} 0 \tag{1}$$

\Rightarrow Consistent

Stability

Stability analysis of the KS-equation

Using Von Neumann stability analysis on a linearized version of the equation where the $\frac{1}{2}(u^2)_x$ term is replaced by $\frac{1}{2}(\rho(x)u)_x$. $\rho(x) = U^0$ is the initial condition $f(x, 0)$. We will look at the scheme

$$U^{n+1} = (I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}D(RU^n))$$

$$U_m^{n+1} = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n) - \frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n)$$

$$\text{Let } U_m^n = \xi^n e^{i\beta x_m} \text{ and } U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m} \text{ s.t.}$$

$$\begin{aligned} \xi &= 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\ &\quad - \frac{k}{4h}(e^{i\beta h} e^{i\beta h} - e^{-i\beta h} e^{-i\beta h}) \\ &= 1 - \frac{2k}{h^2}(\cos(\beta h) - 1) - \frac{k}{h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h)) - \frac{k}{4h}2i\sin(2\beta h) \end{aligned}$$

The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$. Let $r = r/k^4$.

$$\begin{aligned} |\xi|^2 &= (1 - 4rh^2(\cos(\beta h) - 1) - 2r(3 - 4\cos(\beta h) + \cos(2\beta h)))^2 + k\frac{rh^2}{4}\sin^2(2\beta h) \\ &= \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right) \end{aligned}$$

We want $|\xi| \leq 1 + \mu k$, so we need $\psi = |1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)| \leq 1 + \tilde{\mu}k$. Let $q = \sin^2\left(\frac{\beta h}{2}\right)$. For the case $(1 \leq 16rq^2 \leq 2)$ we achieve

$$\psi = |1 + 4rh^2q - 16rq^2| \leq 1 \xRightarrow{0 \leq x \leq 1} (1/16 \leq r \leq 1/8)$$

For the case $(0 \leq 16rq^2 \leq 1)$ we get no further result. Let us now assume that $(1 \leq 16rq^2 \leq 2)$ and $r = 1/8$.

$$|\xi|^2 = \left(1 + \frac{4}{8}qh^2 - \frac{16}{8}q^2\right)^2 + k\frac{h^2}{4 \cdot 8} \sin(2\beta h)$$

$$\max \left(1 + \frac{xh^2}{2} - 2x^2\right) | (0 \leq x \leq 1) \text{ gives } x = \frac{h^2}{8}$$

$$=$$