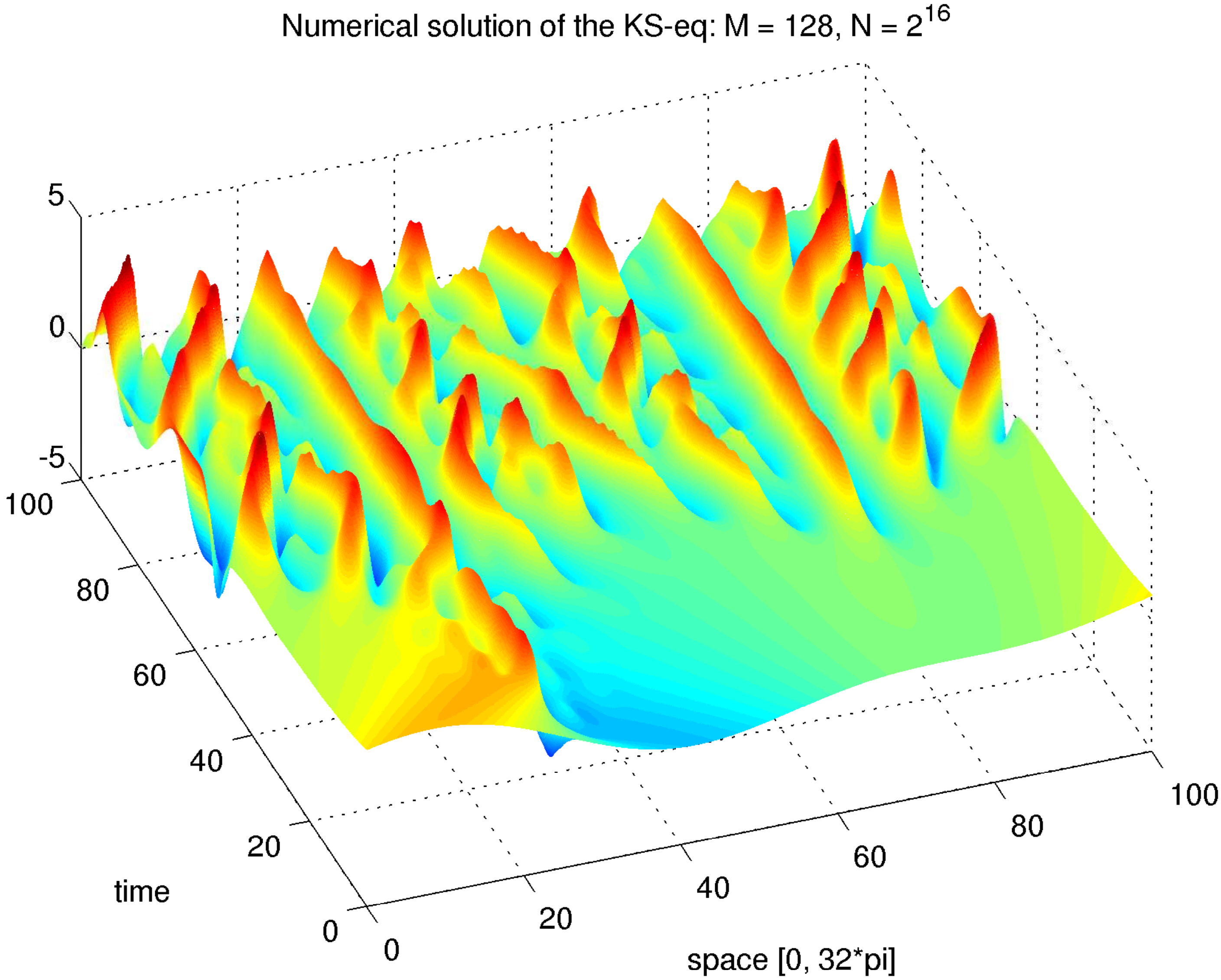
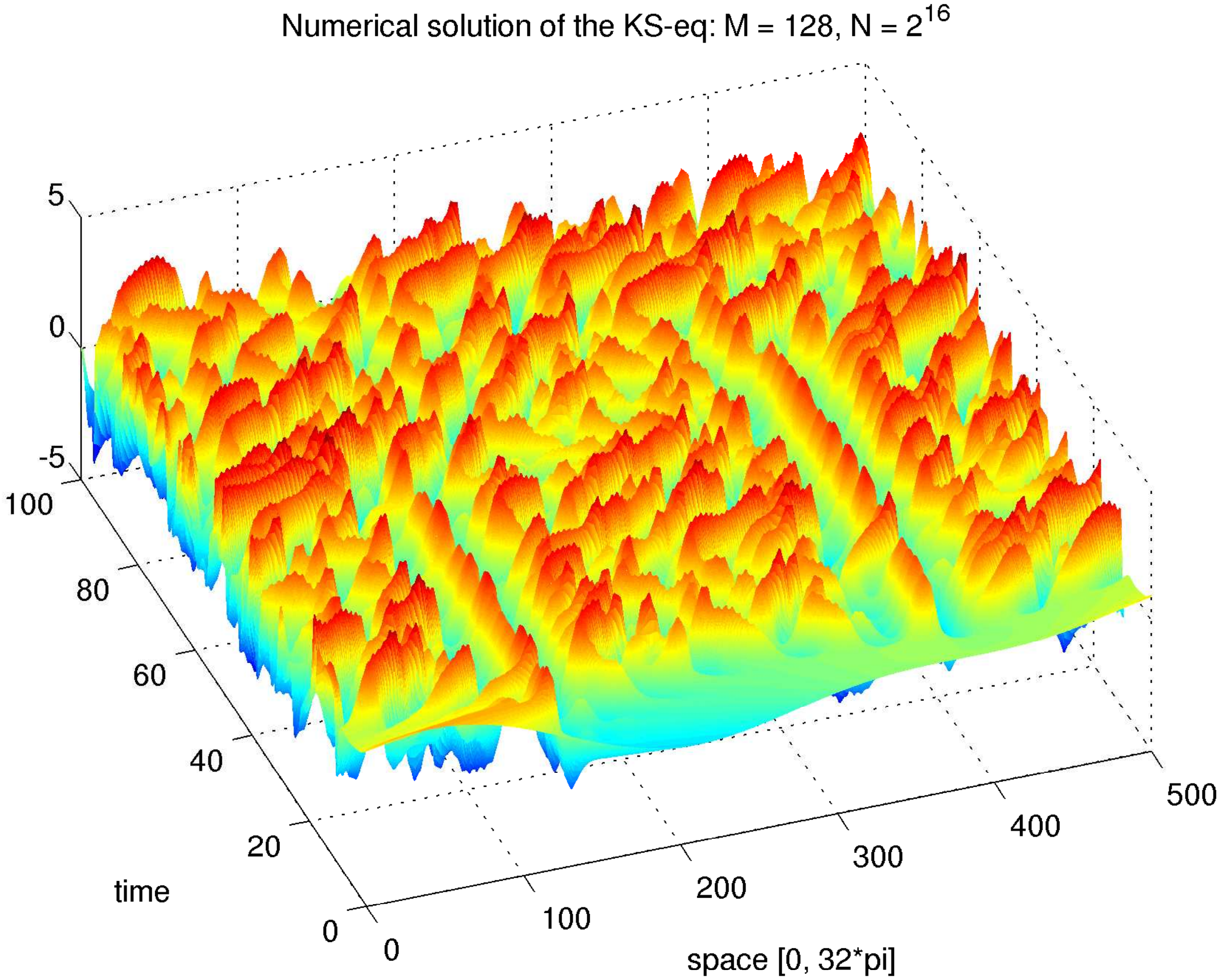
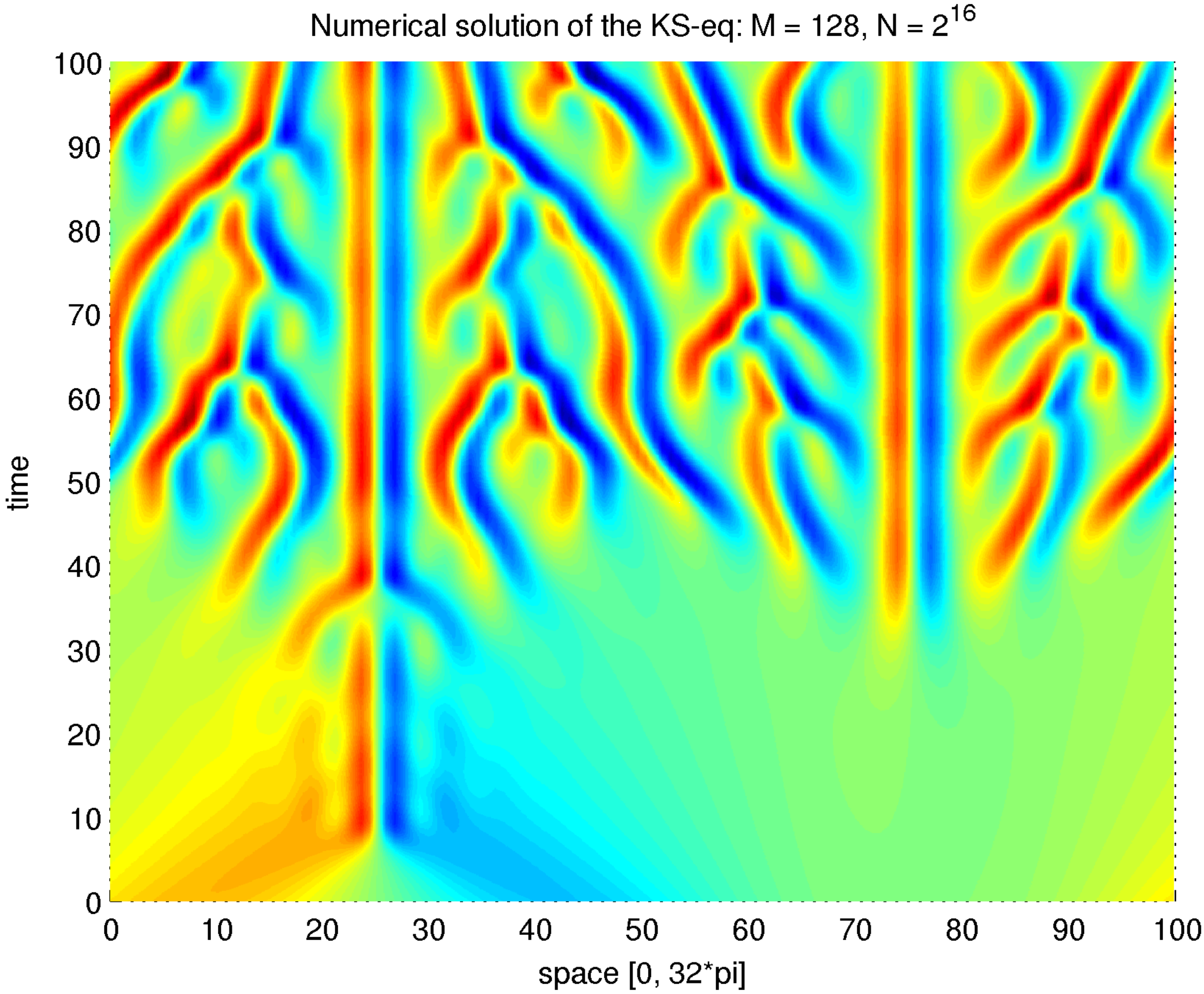


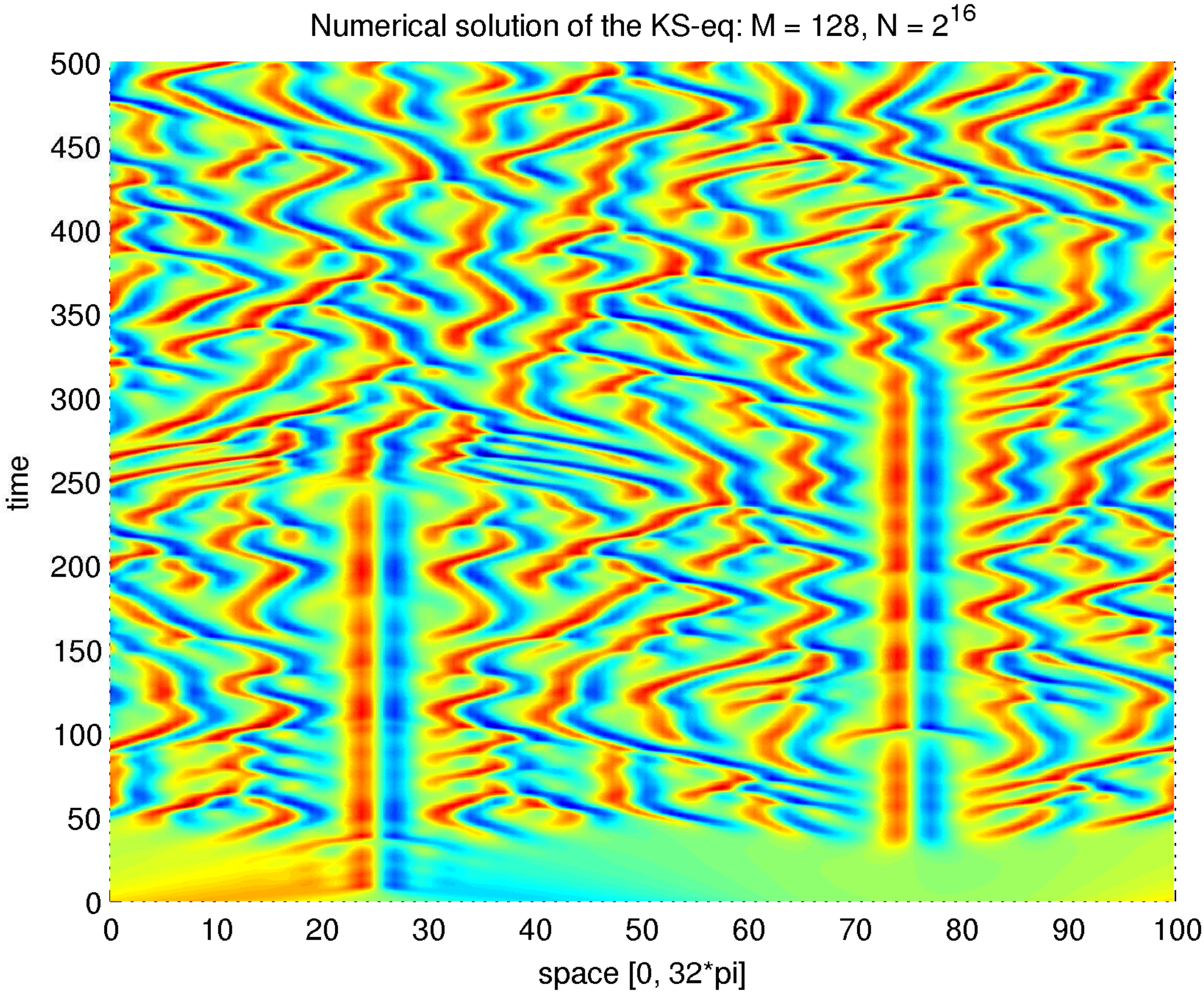
Numerical solution of the KS-eq: M = 128, N = 2^16
5
0
-5
100
80
60
40
20
0
time
0
20
40
60
80
100
space [0, 32*pi]



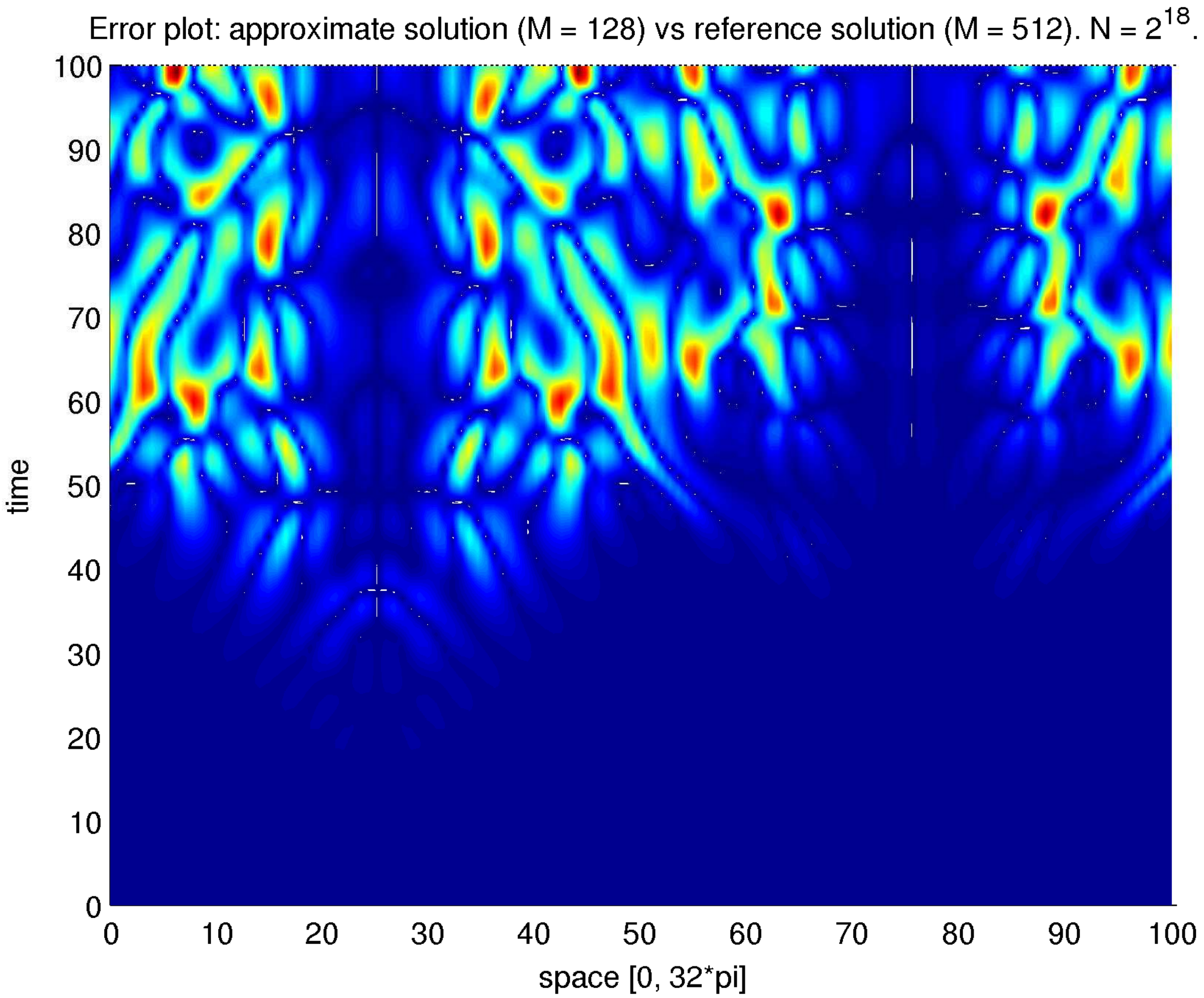
Numerical solution of the KS-eq: M = 128, N = 2^16
5
0
-5
100
80
60
40
20
0
time
0
100
200
300
400
500
space [0, 32*pi]



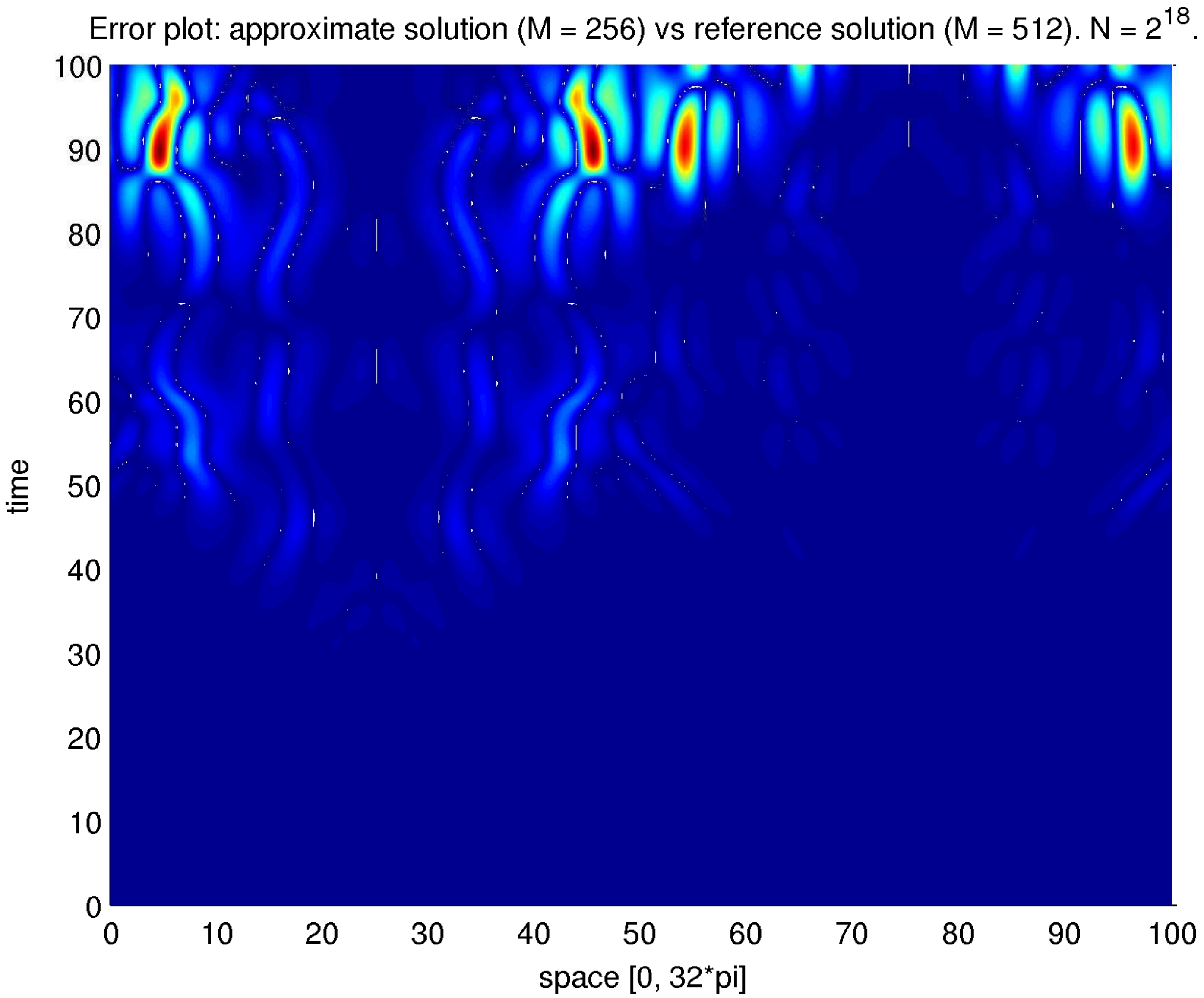
Numerical solution of the KS-eq: M = 128, N = 2^16
time
space [0, 32*pi]
0
10
20
30
40
50
60
70
80
90
100



Numerical solution of the KS-eq: M = 128, N = 2^16
time
500
450
400
350
300
250
200
150
100
50
0
0
10
20
30
40
50
60
70
80
90
100
space [0, 32*pi]



Error plot: approximate solution (M = 128) vs reference solution (M = 512). N = 2^18.
time
0
10
20
30
40
50
60
70
80
90
100
space [0, 32*pi]



Error plot: approximate solution (M = 256) vs reference solution (M = 512). N = 2^18.
time
space [0, 32*pi]
0
10
20
30
40
50
60
70
80
90
100

Difference Scheme

The Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$
$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

Difference scheme:

$$u_t \approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k}$$
$$u_{xx} \approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} = \frac{1}{h^2}Au$$
$$u_{xxxx} \approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} = \frac{1}{h^4}AAu$$
$$(u^2)_x \approx \frac{\mu\delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} = \frac{1}{2h}D$$

$$U^{n+1} = U^n - \frac{k}{h^2}AU^n - \frac{k}{h^4}AAU^n - \frac{k}{4h}D(U^n \odot U^n)$$

$$U^{n+1} = (I - A - B)U^n - \frac{1}{2}D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

Implicit scheme:

Crank-Nicholson on u_{xx} and u_{xxxx}

$$(I + \frac{k}{2h^2}A + \frac{k}{2h^4}AA)U^{n+1} = (I - \frac{k}{2h^2}A - \frac{k}{2h^4}AA)U^n - \frac{k}{4h}D(U^n \odot U^n)$$

$$(I + A + B)U^{n+1} = (I - A - B)U^n - \frac{1}{2}D(U^n \odot U^n)$$

Consistency

Consistency

$$\begin{aligned} u_t &= \frac{\Delta u}{k} + O(k) \\ u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\ u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\ \mu\delta[u(x)^2] &= \mu[u(x + \frac{h}{2}^2) - u(x - \frac{h}{2})^2] \\ &= 2huu_x + O(h^3) \\ uu_x &= \frac{\mu\delta u^2}{4h} + O(h^2) \end{aligned}$$

Local Truncation Error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k,h\to 0} 0 \tag{1}$$

$\Rightarrow$ Consistent

Stability

Von Neumann stability analysis: $U_m^n = \xi^n e^{i\beta x_m}$

Nonlinear term: $\frac{1}{2}(u^2)_x \approx \frac{1}{2}(\rho(x)u)_x$

where $\rho(x) = U^0$

$U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$.

Implicit scheme:

$$\left(I + \frac{k}{h^2}A + \frac{k}{h^4}A^2\right)U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n$$

$$|\xi|^2 \leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + k\frac{rh^2}{4}$$

Not stable unless $r = 0$.

Explicit scheme:

$$U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n$$

$$|\xi|^2 = \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

For the case $(1 \leq 16r\alpha \leq 2)$ where $\alpha = \sin^2(\frac{\beta h}{2})$:

$$\left|1 + 4rh^2\alpha - 16r\alpha^2\right| \leq 1 \underset{0\leq\alpha\leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

Using $r = 1/8$

$$|\xi|^2 \leq \left(1 + \frac{h^4}{32}\right)^2 + k\frac{h^2}{32} \qquad\qquad (\alpha = \frac{h^2}{8})$$

Not stable, which was expected. But parallels can be drawn to the nonlinear scheme, which numerically gives an upper bound

$$r = \frac{k}{h^2} = \frac{1}{8}$$