

Kuramoto-Sivashinsky equation

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$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

About Kuramoto-Sivashinsky equation

- ▶ Stiff
- ▶ Firefronts

Forward and central difference

$$u_t \approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k}$$

$$u_{xx} \approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} = \frac{1}{h^2} A u$$

$$u_{xxxx} \approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} = \frac{1}{h^4} A^2 u$$

$$(u^2)_x \approx \frac{\mu \delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} = \frac{1}{2h} D$$

Explicit scheme

$$U^{n+1} = U^n - \frac{k}{h^2} A U^n - \frac{k}{h^4} A^2 U^n - \frac{k}{4h} D(U^n \odot U^n)$$

\odot = Element-wise multiplication

- Unstable for $k > r \cdot h^4$

Crank-Nicolson

- ▶ Trapezoidal rule
- ▶ Not applied to non-linear term

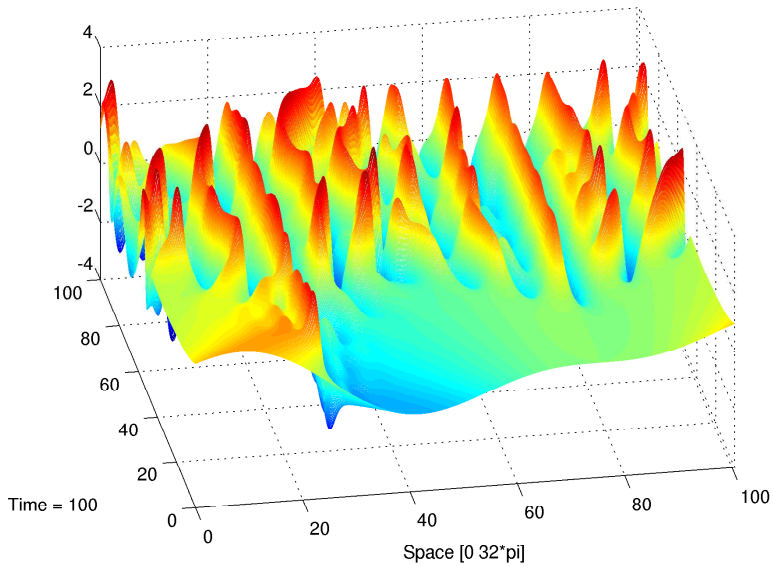
$$\left[\frac{U^{n+1} - U^n}{k} = -\frac{1}{2h^2}A(U^{n+1} + U^n) - \frac{1}{2h^4}A^2(U^{n+1} + U^n) \right] - \frac{1}{4h}D(U^n \odot U^n)$$

Implicit scheme

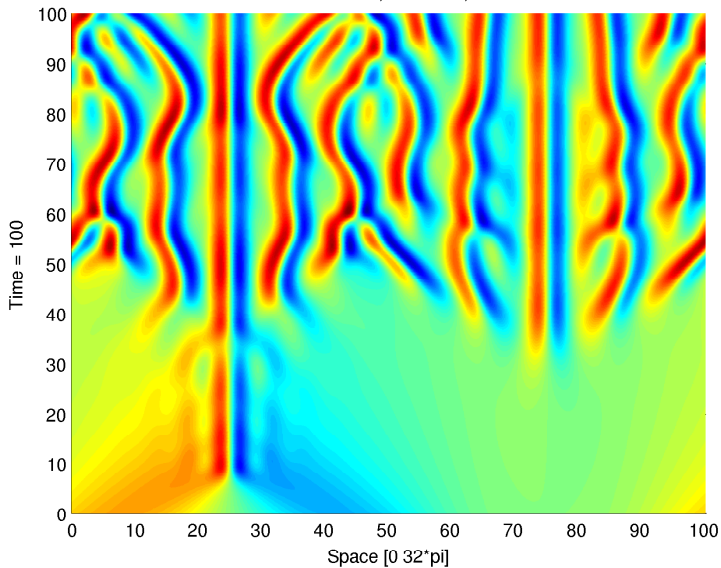
$$(I + \frac{k}{2h^2}A + \frac{k}{2h^4}A^2)U^{n+1} = (I - \frac{k}{2h^2}A - \frac{k}{2h^4}A^2)U^n - \frac{k}{4h}D(U^n \odot U^n)$$

- ▶ Crank-Nicholson
- ▶ Explicit non-linear term
- ▶ Stable

Numerical solution, $h = 0.196$, $k = 0.024$



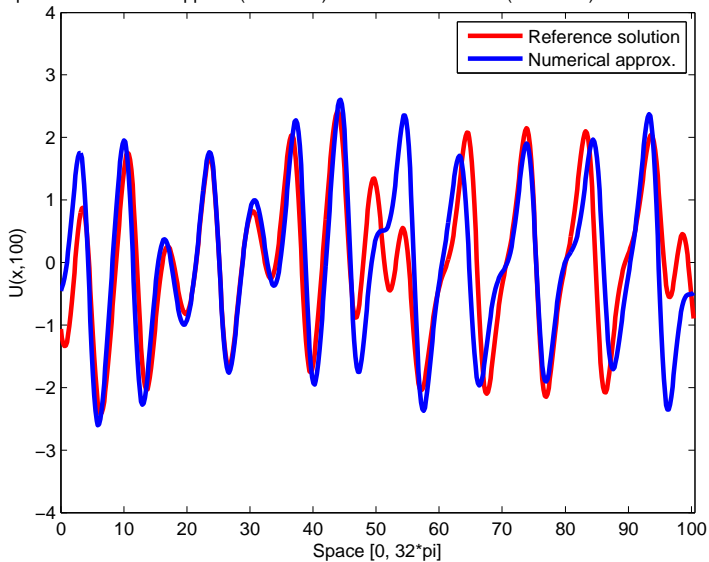
Numerical solution, $h = 0.196$, $k = 0.024$



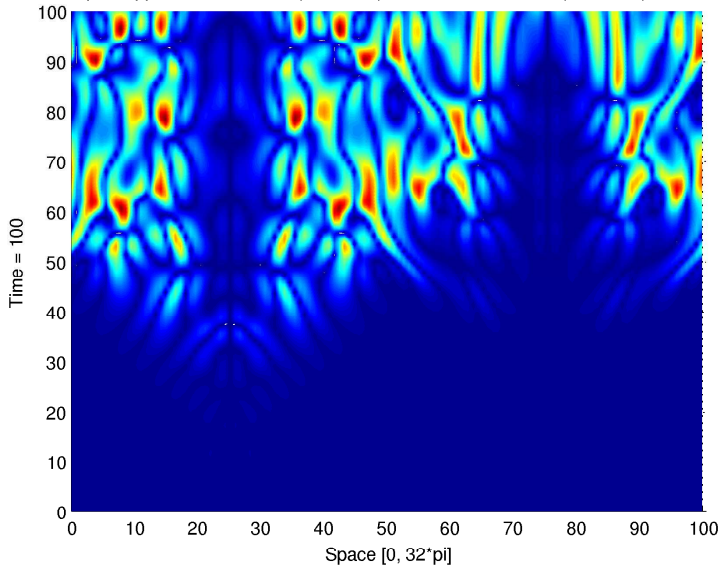
Reference solution

- ▶ No analytic solution
- ▶ MATLAB ode15s
- ▶ semi discretization and stiff system

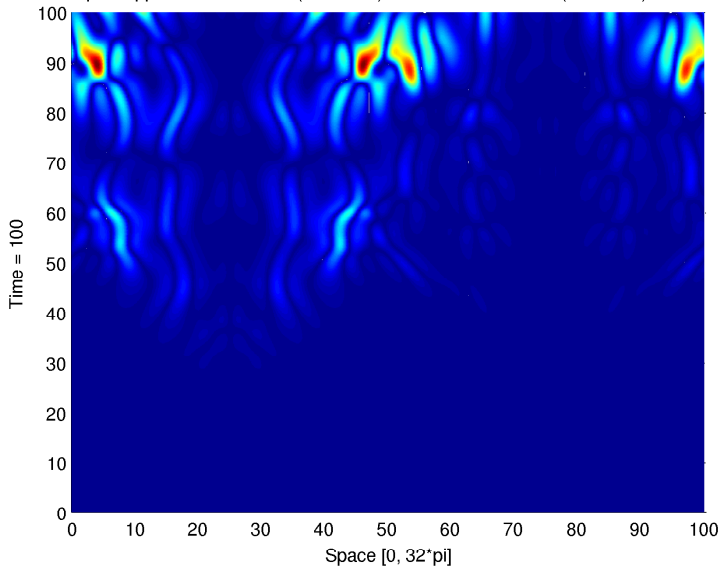
Comparison: Numerical approx. ($h = 0.196$) vs reference solution ($h = 0.049$) at time 100. $k = 0.006$



Error plot: approximate solution ($h = 0.785$) vs reference solution ($h = 0.025$). $k = 0.006$.



Error plot: approximate solution ($h = 0.393$) vs reference solution ($h = 0.025$). $k = 0.006$.



Consistency

$$u_t = \frac{\Delta u}{k} + O(k)$$

$$u_{xx} = \frac{\delta^2 u}{h^2} + O(h^2)$$

$$u_{xxxx} = \frac{\delta^4 u}{h^4} + O(h^2)$$

$$uu_x = \frac{\mu \delta u^2}{4h} + O(h^2)$$

Local Truncation Error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k, h \rightarrow 0} 0$$

\Rightarrow Consistent

Stability analysis - implicit linearized scheme

- ▶ Von Neumann on linearized equation
- ▶ $\frac{1}{2}(u^2)_x \approx \frac{1}{2}(\rho(x)u)_x$
- ▶ $\rho(x) = U_m^0 = \xi^0 e^{i\beta x_m}$

$$|\xi|^2 \leq \left(\frac{1 + k/8}{1 - k/8} \right)^2 + \frac{k}{4h^2}k$$

Not stable

Stability analysis - explicit linearized scheme

- ▶ $\alpha = \sin^2(\frac{\beta h}{2})$
- ▶ Assume: $(1 \leq 16r\alpha \leq 2)$ and $(4rh^2\alpha \leq 1)$

$$|1 - 16r\alpha^2| \leq 1 \xRightarrow{0 \leq \alpha \leq 1} r \leq 1/8$$

$$|\xi|^2 \leq \left(1 + \frac{h^4}{32}\right)^2 + k \frac{h^2}{32}$$

- ▶ Not stable
- ▶ experimentally $r < 1/8$ stable for non-linear scheme

Convergence of implicit scheme

- ▶ $\tau^n = O(h^2) + O(k)$
- ▶ $|\xi|^2 \leq \left(\frac{1+rh^4/8}{1-rh^4/8} \right)^2 + k \frac{rh^2}{4}$
- ▶ Experimentally the scheme is stable

