Kuramoto-Sivashinsky equation

2. April 2014

Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$
$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

About Kuramato-Sivashinsky equation

- Stiff
- Firefronts

Forward and central difference

$$
\begin{aligned}
u_t &\approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k} \\
u_{xx} &\approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} &&= \frac{1}{h^2} A u \\
u_{xxxx} &\approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} &&= \frac{1}{h^4} A^2 u \\
(u^2)_x &\approx \frac{\mu\delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} &&= \frac{1}{2h} D
\end{aligned}
$$

Explicit scheme

$$U^{n+1} = U^n - \frac{k}{h^2} A U^n - \frac{k}{h^4} A^2 U^n - \frac{k}{4h} D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

- Unstable for $k > r \cdot h^4$

Crank-Nicolson

$$\left[\frac{U^{n+1}-U^n}{k} = -\frac{1}{2h^2}A(U^{n+1}+U^n) - \frac{1}{2h^4}A^2(U^{n+1}+U^n)\right] - \frac{1}{4h}D(U^n \odot U^n)$$

- Not applied to non-linear term

Implicit scheme

$$(I + \frac{k}{2h^2}A + \frac{k}{2h^4}A^2)U^{n+1} = (I - \frac{k}{2h^2}A - \frac{k}{2h^4}A^2)U^n - \frac{k}{4h}D(U^n \odot U^n)$$

- Crank-Nicholson
- Explicit non-linear term
- Stable

Consistency

$$
\begin{aligned}
u_t &= \frac{\Delta u}{k} + O(k) \\
u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\
u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\
\mu\delta\left[u(x)^2\right] &= \mu\left[u(x + \frac{h}{2}^2) - u(x - \frac{h}{2})^2\right] + O(h^3) \\
uu_x &= \frac{\mu\delta u^2}{4h} + O(h^2)
\end{aligned}
$$