The Kuramoto-Sivashinsky Equation

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Abstract

This is the abstract. Write smart things here.

Introduction

The Kuramoto-Sivashinsky equation,

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0 \tag{1}$$

is one of the simplest partial differential equations that exhibits complicated dynamics in both time and space, which is why the equation has been the attention for a lot of research. The equation was developed by two scientists at the same time in 1977 [1]. Gregory Sivashinsky determined an equation for a laminar flame front, while Yoshiki Kuramoto modeled a diffusion-induced chaos using the same equation. Because of this, the equation is named Kuramoto-Sivashinsky. The KS-equation also models the motion of a fluid going down a vertical wall, e.g. solitary pulses in a falling thin film. [2]

The reason for the complex behaviour comes from the second- and fourth-order derivatives in (1). While the second-order term acts as an energy source and has a destabilizing effect, the fourth-order term has a stabilizing effect. In addition to this, the nonlinear term transfers energy from low to high wave numbers. [3] The KS-equation is a stiff equation, i.e. an equation where numerical methods for solving it are numerically unstable, unless the step size is extremely small. u_{xxxx} is the main reason for this as it leads to rapid variation in the solution.

Numerical results

Initial conditions

In the solution of the KS-equation we had periodic boundary conditions, i.e. $u(0,t) = u(L,t)$. We also used L-periodic initial conditions. We experienced that a common initial condition used in several other reports was

$$u(x,0) = \cos(\frac{x}{16})(1 + \sin(\frac{x}{16}). \tag{2}$$

We also tried the initial condition

$$u(x,0) = \frac{1}{\sqrt{2}}\sin(x) - \frac{1}{8}\sin(2x), \tag{3}$$

which worked well. The L-periodic initial conditions is customarily taken [4] to satisfy

$$\int_0^L f(x)\,\mathrm{d}x = 0, \tag{4}$$

which both of our initial conditions satisfy. The same article also states that for L-periodic initial data, a unique solution for (1) exits, and is bounded as $t \to \infty$. The bound has been proven to be smaller than $O(L^{8/5})$. In our numerical tests, with $t = 5000$, the initial condition (2) did indeed not exceed the bound, nor did (3).

Stability

Von Neumann stability analysis is a method based on Fourier decomposition of numerical error, and is used to check the stability of linear pde-s. Since the KS-equation has a non-linear term, $\frac{1}{2}(u^2)_x$, this has to be linearized before the method is applied. The KS-equation is stiff, and the non-linear term stabilizes the equation. Replacing it by $\frac{1}{2}(\rho(x)u)_x$, where $\rho(x) \approx u(x)$ is constant in time, we obtain

$$u_t = -u_{xx} - u_{xxxx} - \frac{1}{2}\left(\rho(x)u\right)_x. \tag{5}$$

This equation is not stable, so the schemes will not be stable, as we will see in the analysis. Still, an interesting result is attained to the bound of k/h^4 in the explicit scheme.

Stability analysis of implicit scheme

Choosing $\rho(x) = f(x,0)$, where $f(x,0) = U^0$, we obtain the linearized scheme where R is the matrix representing $\rho(x)$:

$$\left(I + \frac{k}{h^2}A + \frac{k}{h^4}A^2\right)U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n,$$

which written out becomes

$$\begin{aligned} U_m^{n+1} + \frac{k}{h^2}(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) + \frac{k}{h^4}(U_{m+2}^{n+1} - 4U_{m+1}^{n+1} + 6U_m^{n+1} - 4U_{m-1}^{+1}n + U_{m-2}^{n+1}) \\ = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n) \\ - \frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n) \end{aligned}$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ such that

$$
\begin{aligned}
\xi(1 + \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) &+ \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\
= 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) &- \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\
&\qquad - \frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})
\end{aligned}
$$

$$
\begin{aligned}
\xi &= \frac{1 - \frac{k}{h^2}(\cos(\beta h) - 1) - \frac{k}{2h^4}(6 - 8\cos(\beta h) + \cos(2\beta h)) - \frac{k}{2h} i \sin(2\beta h)}{1 + \frac{k}{h^2}(\cos(\beta h) - 1) + \frac{k}{2h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h))} \\
&= \frac{1 + \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) - \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right) - \frac{k}{2h}\sin^2(2\beta h)}{1 - \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) + \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right)}
\end{aligned}
$$

Let $q = \sin^2(\frac{\beta h}{2})$ and $r = \frac{k}{h^4}$. The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$.

$$
|\xi|^2 = \left(\frac{1 + 2rq(h^2 - 4q)}{1 - 2rq(h^2 - 4q)}\right)^2 + \frac{1}{4}\frac{krh^2\sin^2(2\beta h)}{(1 - 2rq(h^2 - 4q))^2}
$$

Maximizing $-2rq(4q - h^2)$ wrt q gives $q = h^2/8$, which replaced in the equation gives

$$
\begin{aligned}
|\xi|^2 &\leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + \frac{1}{4}kh^2 r\frac{\sin(2\beta h)}{(1 - rh^4/8)^2} \\
&\leq \left(\frac{1 + k/8}{1 - k/8}\right)^2 + k\frac{rh^2}{4(1 - k/8)^2}
\end{aligned}
$$

This expression is dependent on both k and h, so the method is not stable. This is expected, since the linearized KS-equation is unstable in numerical experiments.

Stability analysis of explicit scheme

The explicit scheme is stable when $k < r \cdot h^4$. Let us now use Von Neumann stability analysis on the explicit scheme with the same $\rho(x)$:

$$
U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n
$$

$$
\begin{aligned}
\xi = 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) &- \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\
&\qquad - \frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})
\end{aligned}
$$

Let $r = k/h^4$ as before.

$$|\xi|^2 = (1 - 4rh^2(\cos(\beta h) - 1) - 2r(3 - 4\cos(\beta h) + \cos(2\beta h)))^2 + k\frac{rh^2}{4}\sin^2(2\beta h)$$
$$= \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

We want $|\xi| \leq 1 + \mu k$, so we need

$$\psi = |1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)| \leq 1 + \tilde{\mu}k.$$

Let $q = \sin^2\left(\frac{\beta h}{2}\right)$ and assume that $(1 \leq 16rq^2 \leq 2)$ and $(0 \leq 4rxh^2 \leq 1)$

$$\psi = \left|1 + 4rh^2q - 16rq^2\right| \leq \left|1 - 16rq^2\right| \leq 1 \underset{0\leq x\leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

$$|\xi|^2 \leq |1 - 16rq^2|^2 + k\frac{rh^2}{4} = |16rq^2 - 1|^2 + k\frac{rh^2}{4}$$

insering $r = 1/8$

$$\leq |16\frac{1}{8}q^2 - 1|^2 + k\frac{h^2}{8\cdot 4} = |2q^2 - 1| + k\frac{h^2}{32}$$
$$\leq 1 + k\frac{h^2}{32}$$

Under the assumptions above, we have not shown stability, but have achieved an upper bound for r. The linearized scheme is, by numerical results, not stable, but numerical results also show that the non-linear explicit scheme provide approximately an upper bound of $r = k/h^4 = 1/8$.

References

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