Kuramoto-Sivashinsky equation

2. April 2014

Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

About Kuramato-Sivashinsky equation

- Stiff
- Firefronts

Forward and central difference

$$
\begin{aligned}
u_t &\approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k} \\
u_{xx} &\approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} && = \frac{1}{h^2} A u \\
u_{xxxx} &\approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} && = \frac{1}{h^4} A^2 u \\
(u^2)_x &\approx \frac{\mu \delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} && = \frac{1}{2h} D
\end{aligned}
$$

Explicit scheme

$$U^{n+1} = U^n - \frac{k}{h^2} A U^n - \frac{k}{h^4} A^2 U^n - \frac{k}{4h} D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

- Unstable for $k > r \cdot h^4$

Crank-Nicolson

$$\left[\frac{U^{n+1} - U^n}{k} = -\frac{1}{2h^2}A(U^{n+1} + U^n) - \frac{1}{2h^4}A^2(U^{n+1} + U^n)\right] - \frac{1}{4h}D(U^n \odot U^n)$$

- Not applied to non-linear term

Implicit scheme

$$(I + \frac{k}{2h^2}A + \frac{k}{2h^4}A^2)U^{n+1} = (I - \frac{k}{2h^2}A - \frac{k}{2h^4}A^2)U^n - \frac{k}{4h}D(U^n \odot U^n)$$

- Crank-Nicholson
- Explicit non-linear term
- Stable

Consistency

$$
\begin{aligned}
u_t &= \frac{\Delta u}{k} + O(k) \\
u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\
u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\
\mu\delta\left[u(x)^2\right] &= \mu\left[u(x+\frac{h}{2}^2) - u(x-\frac{h}{2})^2\right] + O(h^3) \\
uu_x &= \frac{\mu\delta u^2}{4h} + O(h^2)
\end{aligned}
$$

Local Truncation Error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k,h\to 0} 0$$

$$\Rightarrow \text{Consistent}$$

Stability analysis - implicit linearized scheme

- Von Neumann on linearized equation
- $\frac{1}{2}(u^2)_x \approx \frac{1}{2}(\rho(x)u)_x$
- $\rho(x) = U_m^0 = \xi^0 e^{i\beta x_m}$

$$|\xi|^2 \leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + k\frac{rh^2}{4}$$

Not stable unless $r = \frac{k}{h^4} = 0$

Stability analysis - explicit linearized scheme

$$|\xi|^2 = \left(1 + 4rh^2\alpha - 16r\alpha^2\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

- $\alpha = \sin^2(\frac{\beta h}{2})$
- Assume: $(1 \leq 16r\alpha \leq 2)$ and $(4rh^2\alpha <= 1)$

$$\left|1 + 4rh^2\alpha - 16r\alpha^2\right| \leq 1 \underset{0\leq\alpha\leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

Stability analysis - explicit linearized scheme

Using $r = 1/8$

$$|\xi|^2 \leq \left(1 + \frac{h^4}{32}\right)^2 + k\frac{h^2}{32}$$

- Not stable
- $r < 1/8$ yields stability for non-linearized explicit scheme

Convergence of explicit scheme

- $\tau^n = O(h^2) + O(k)$
- $|\xi|^2 \leq \left(\frac{1+rh^4/8}{1-rh^4/8}\right)^2 + k\frac{rh^2}{4}$
- Experimentally the scheme is stable