The Kuramoto-Sivashinsky Equation

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Abstract

This is the abstract. Write smart things here.

Introduction

The Kuramoto-Sivashinsky equation,

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0 \tag{1}$$

is one of the simplest partial differential equations that exhibits complicated dynamics in both time and space, which is why the equation has been the attention for a lot of research. The equation was developed by two scientists at the same time in 1977 [1]. Gregory Sivashinsky determined an equation for a laminar flame front, while Yoshiki Kuramoto modeled a diffusion-induced chaos using the same equation. Because of this, the equation is named Kuramoto-Sivashinsky. The KS-equation also models the motion of a fluid going down a vertical wall, e.g. solitary pulses in a falling thin film. [2]

The reason for the complex behaviour comes from the second- and fourth-order derivatives in (1). While the second-order term acts as an energy source and has a destabilizing effect, the fourth-order term has a stabilizing effect. In addition to this, the nonlinear term transfers energy from low to high wave numbers. [3] The KS-equation is a stiff equation, i.e. an equation where numerical methods for solving it are numerically unstable, unless the step size is extremely small. u_{xxxx} is the main reason for this as it leads to rapid variation in the solution.

Numerical results

Initial conditions

In the solution of the KS-equation we had periodic boundary conditions, i.e. $u(0,t) = u(L,t)$. We also used L-periodic initial conditions. We experienced that a common initial condition used in several other reports was

$$u(x,0) = \cos(\frac{x}{16})(1 + \sin(\frac{x}{16}). \tag{2}$$

We also tried the initial condition

$$u(x, 0) = \frac{1}{\sqrt{2}} \sin(x) - \frac{1}{8} \sin(2x), \tag{3}$$

which worked well. The L-periodic initial conditions is customarily taken [4] to satisfy

$$\int_0^L f(x)\,\mathrm{d}x = 0, \tag{4}$$

which both of our initial conditions satisfy. The same article also states that for L-periodic initial data, a unique solution for (1) exits, and is bounded as $t \to \infty$. The bound has been proven to be smaller than $O(L^{8/5})$. In our numerical tests, with $t = 5000$, the initial condition (2) did indeed not exceed the bound, nor did (3).

Plots of the function

The IMEX method produced figure 1.

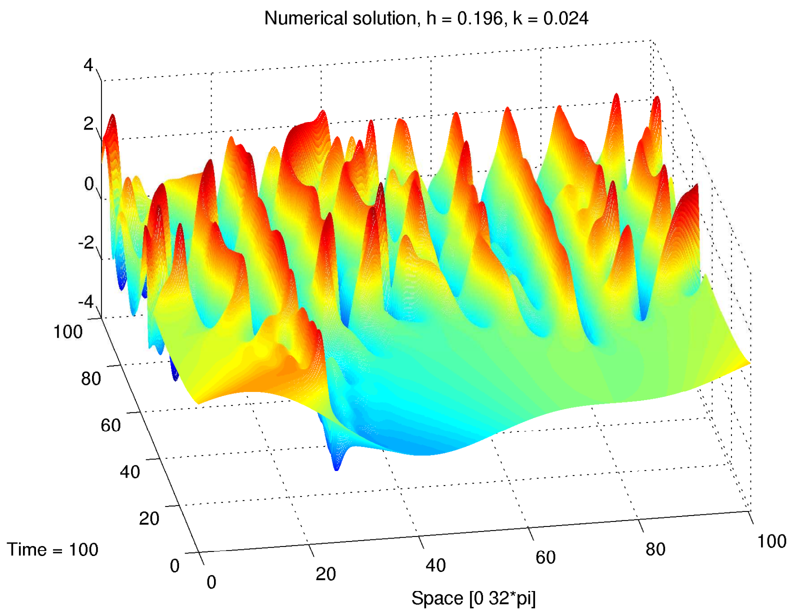


Figure 1: Surface plot of the solution u(x,t)

As we can see, there are two parallel lines where the solution is symmetric in an interval around them. This is easier seen from the contour plot, figure 2.

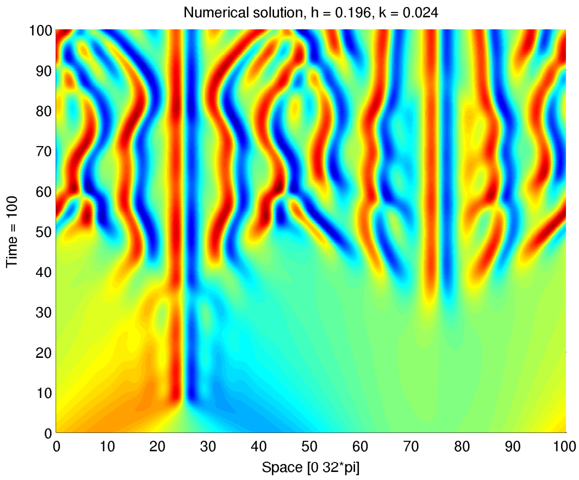
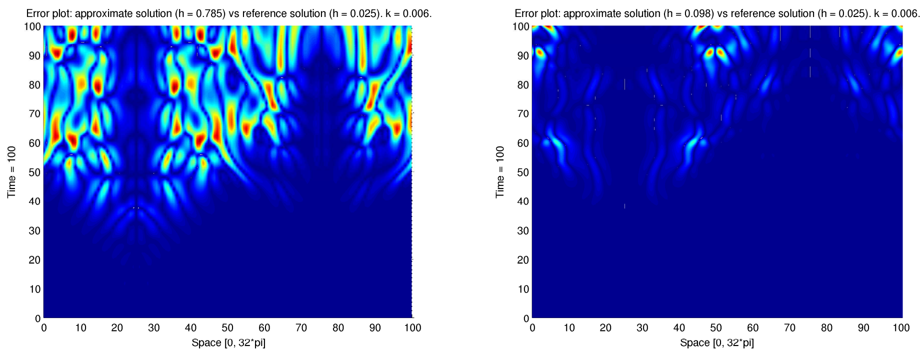


Figure 2: Contour plot of the solution u(x,t)

Although the lines are parallel from time $t = [0, 100]$, this ends after a time $t \approx 250$, and it becomes even more chaotic.

Because (1) has no analytical solution, we constructed a reference solution. Since our equation is stiff, we used the ODE15s solver to compute the solution, as this is particularly good for stiff systems. A semi-discretization, i.e. only discretization in space, was used in the solver. By using low values for k and h, typically $k = 0.006$ and $h = 0.025$, we are confident that the solver produces a good approximation of the result. To see exactly how well our numerical solution is compared to the reference solution, we plotted the error between them. This produced figure 3.



(a) Numerical solution, $h = 0.785$

(b) Numerical solution, $h = 0.098$

Figure 3: Comparison of the error between the reference solution and the numerical approximation for different h-values. Reference solution: $h = 0.025$, $k = 0.006$. Blue color shows low error, while red shows high error.

As we can see, the error decreases when the h-value is decreased. A plot of the reference solution vs. our numerical solution, figure 4, explains in a good way why some points have higher error than others. At the points where the reference solution and the numerical solution are out of phase, the error will naturally be large. Worth noting is the low error at the two parallel lines, $x \approx 25$ and $x \approx 75$, which can also be seen in figure 3. This means a worst case error will be the sum of the amplitudes of the solutions.

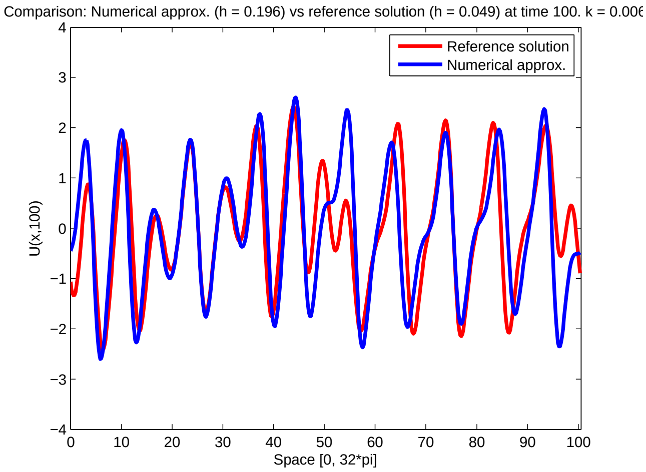


Figure 4: Plot of $u(x, 0)$ for the reference solution and the numerical solution

Stability

Von Neumann stability analysis is a method based on Fourier decomposition of numerical error, and is used to check the stability of linear pde-s. Since the KS-equation has a non-linear term, $\frac{1}{2}(u^2)_x$, this has to be linearized before the method is applied. The KS-equation is stiff, and the non-linear term stabilizes the equation. Replacing it by $\frac{1}{2}(\rho(x)u)_x$, where $\rho(x) \approx u(x)$ is constant in time, we obtain

$$u_t = -u_{xx} - u_{xxxx} - \frac{1}{2}\left(\rho(x)u\right)_x . \tag{5}$$

Choosing $\rho(x) = f(x, 0)$, where $f(x, 0) = U^0$, we obtain the linearized scheme

$$\left(I + \frac{k}{h^2}A + \frac{k}{h^4}A^2\right)U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n,$$

which written out becomes

$$\begin{aligned} U_m^{n+1} + \frac{k}{h^2}(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) + \frac{k}{h^4}(U_{m+2}^{n+1} - 4U_{m+1}^{n+1} + 6U_m^{n+1} - 4U_{m-1}^{+1}n + U_{m-2}^{n+1}) \\ = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n) \\ -\frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n) \end{aligned}$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ such that

$$\begin{aligned}
&\xi(1 + \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) + \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\
&= 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h}) \\
&\qquad - \frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})
\end{aligned}$$

$$\begin{aligned}
\xi &= \frac{1 - \frac{k}{h^2}(\cos(\beta h) - 1) - \frac{k}{2h^4}(6 - 8\cos(\beta h) + \cos(2\beta h)) - \frac{k}{2h} i \sin(2\beta h)}{1 + \frac{k}{h^2}(\cos(\beta h) - 1) + \frac{k}{2h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h))} \\
&= \frac{1 + \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) - \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right) - \frac{k}{2h}\sin^2(2\beta h)}{1 - \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) + \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right)}
\end{aligned}$$

Let $q = \sin^2(\frac{\beta h}{2})$ and $r = \frac{k}{h^4}$. The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$.

$$|\xi|^2 = \left(\frac{1 + 2rq(h^2 - 4q)}{1 - 2rq(h^2 - 4q)}\right)^2 + \frac{1}{4}\frac{krh^2\sin^2(2\beta h)}{(1 - 2rq(h^2 - 4q))^2}$$

Maximizing $-2rq(4q - h^2)$ wrt q gives $q = h^2/8$, which replaced in the equation gives

$$\begin{aligned}
|\xi|^2 &\leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + \frac{1}{4}kh^2 r\frac{\sin(2\beta h)}{(1 - rh^4/8)^2} \\
&\leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + k\frac{rh^2}{4}
\end{aligned}$$

This expression is dependent on both k and h, so the method is not stable. This is expected, since the linearized KS-equation is unstable in numerical experiments.

$$U^{n+1} = (I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}D(RU^n))$$

$$\begin{aligned}
U_m^{n+1} = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n) \\
- \frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n)
\end{aligned}$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ s.t.

$$\xi = 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h})$$
$$-\frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})$$
$$= 1 - \frac{2k}{h^2}(\cos(\beta h) - 1) - \frac{k}{h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h)) - \frac{k}{4h}2i\sin(2\beta h)$$

The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$. Let $r = r/k^4$.

$$|\xi|^2 = (1 - 4rh^2(\cos(\beta h) - 1) - 2r(3 - 4\cos(\beta h) + \cos(2\beta h)))^2 + k\frac{rh^2}{4}\sin^2(2\beta h)$$
$$= \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

We want $|\xi| \leq 1 + \mu k$, so we need $\psi = |1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)| \leq 1 + \tilde{\mu}k$. Let $q = \sin^2\left(\frac{\beta h}{2}\right)$.
For the case $(1 \leq 16rq^2 \leq 2)$ we achieve

$$\psi = \left|1 + 4rh^2 q - 16rq^2\right| \leq 1 \underset{0 \leq x \leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

For the case $(0 \leq 16rq^2 \leq 1)$ we get no furher result. Let us now assume that $(1 \leq 16rq^2 \leq 2)$ and $r = 1/8$.

$$|\xi|^2 = \left(1 + \frac{4}{8}qh^2 - \frac{16}{8}q^2\right)^2 + k\frac{h^2}{4 \cdot 8}\sin(2\beta h)$$

$$\max\left(1 + \frac{xh^2}{2} - 2x^2\right)|(0 \leq x \leq 1)\text{gives } x = \frac{h^2}{8}$$
$$=$$

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