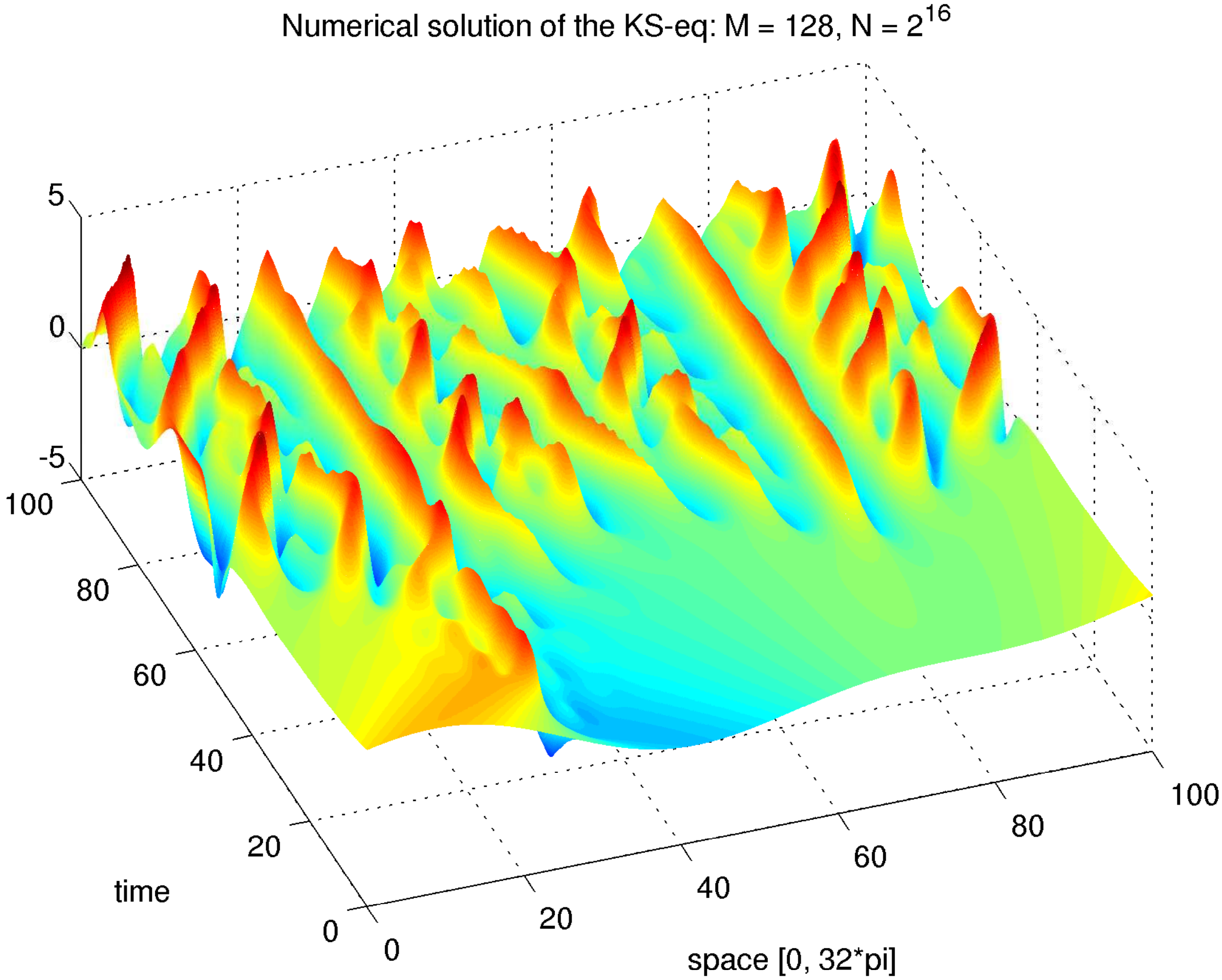
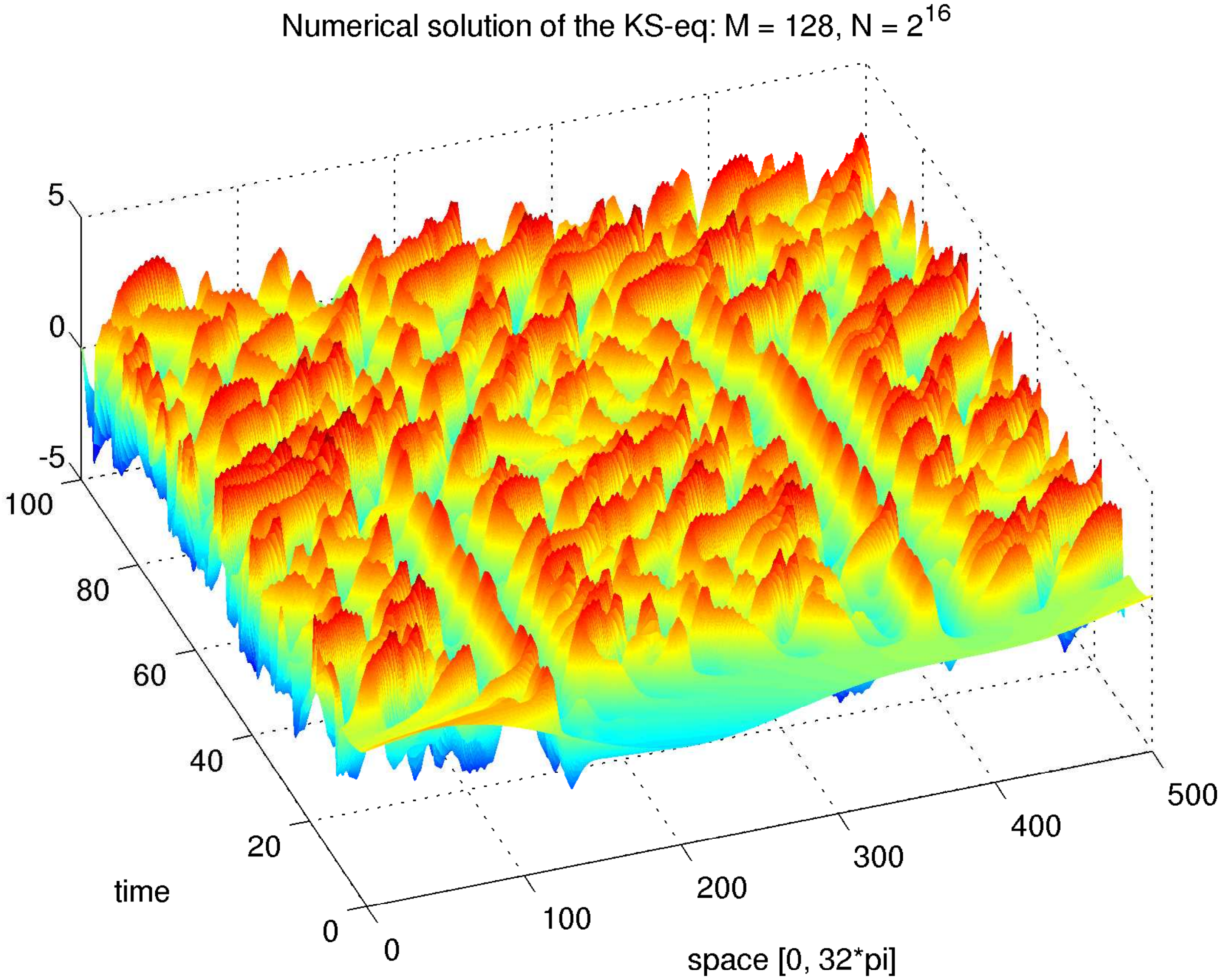
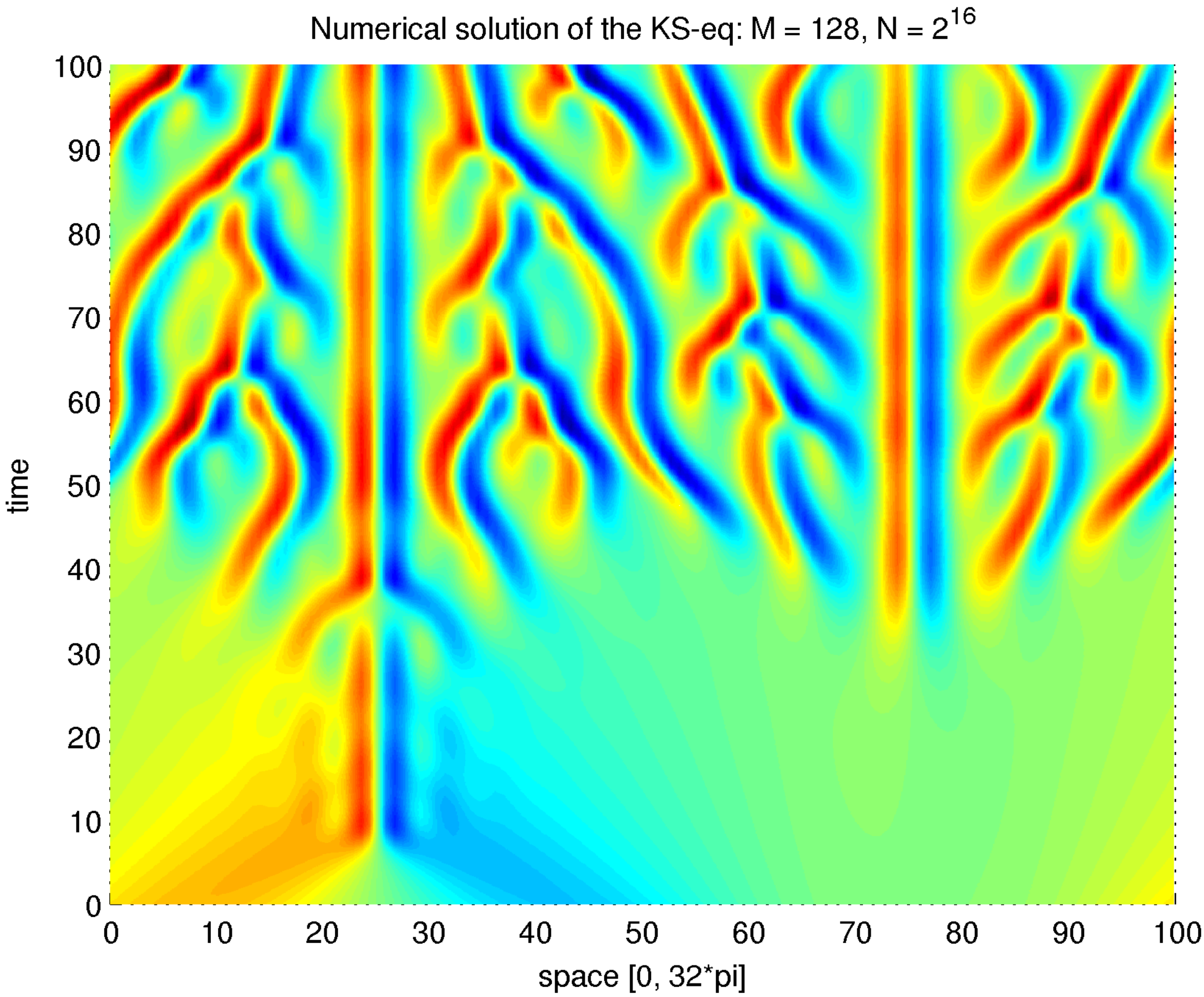


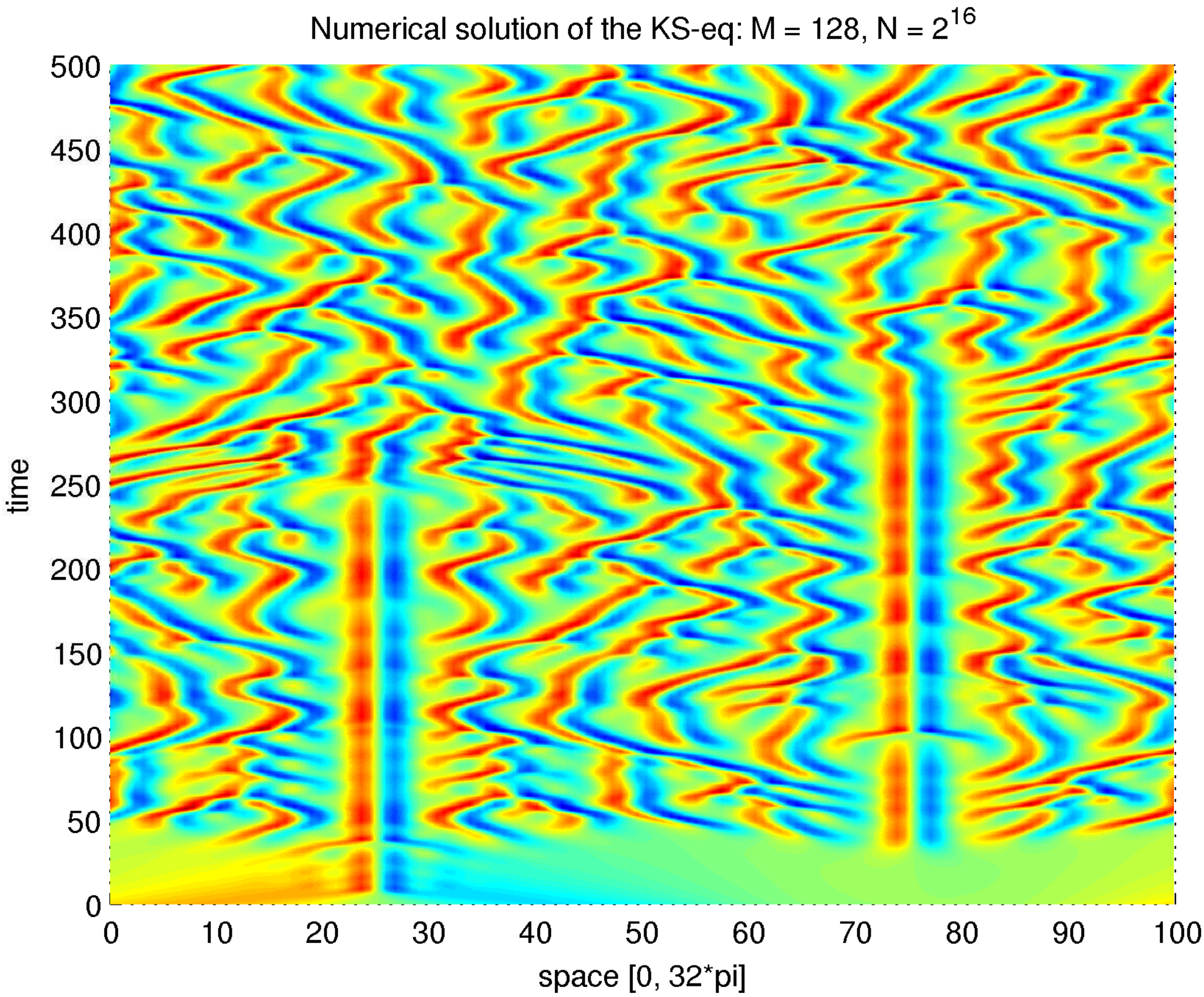
Numerical solution of the KS-eq: M = 128, N = 2^16
5
0
-5
100
80
60
40
20
0
time
0
20
40
60
80
100
space [0, 32*pi]



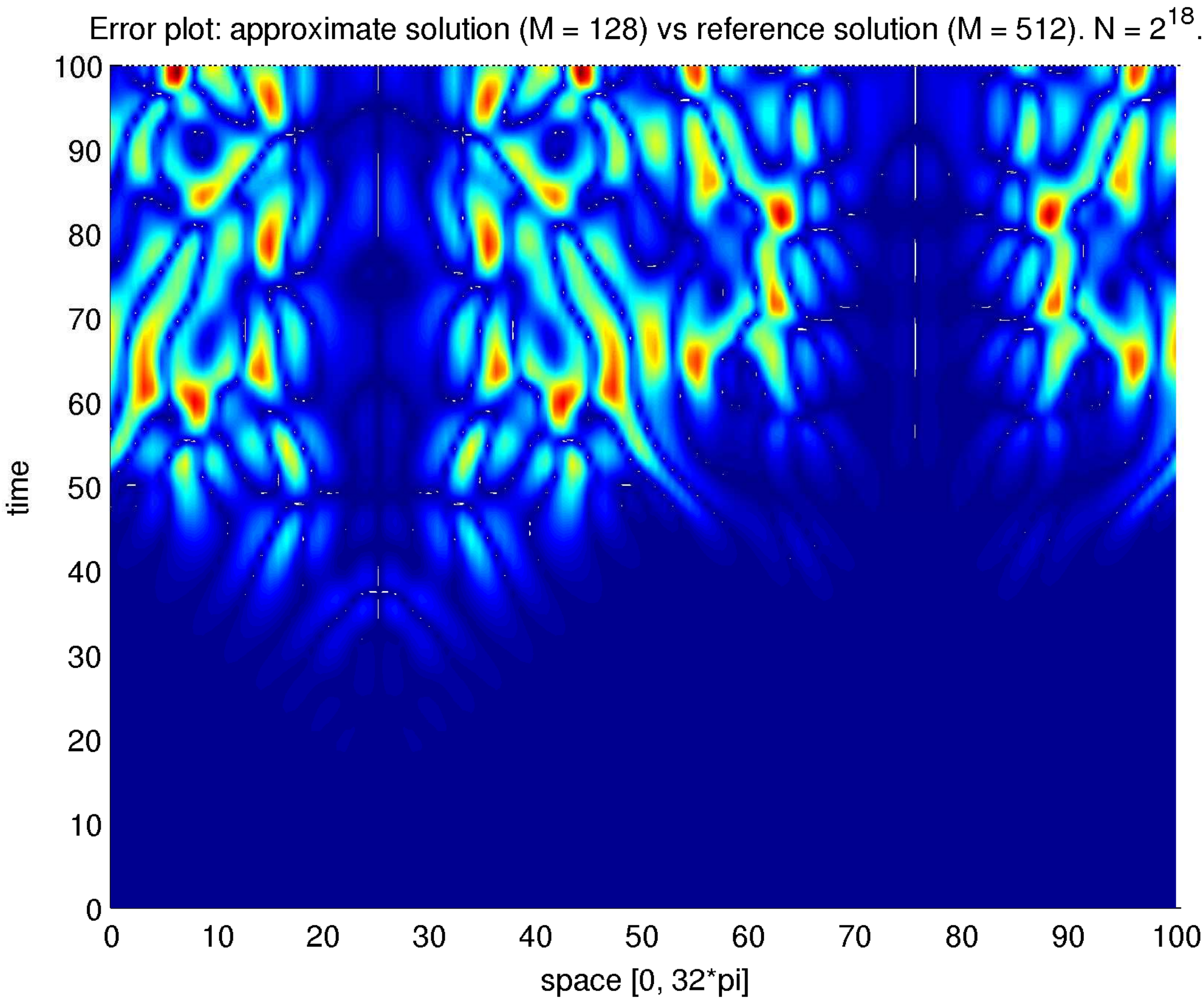
Numerical solution of the KS-eq: M = 128, N = 2^16
5
0
-5
100
80
60
40
20
0
time
0
100
200
300
400
500
space [0, 32*pi]

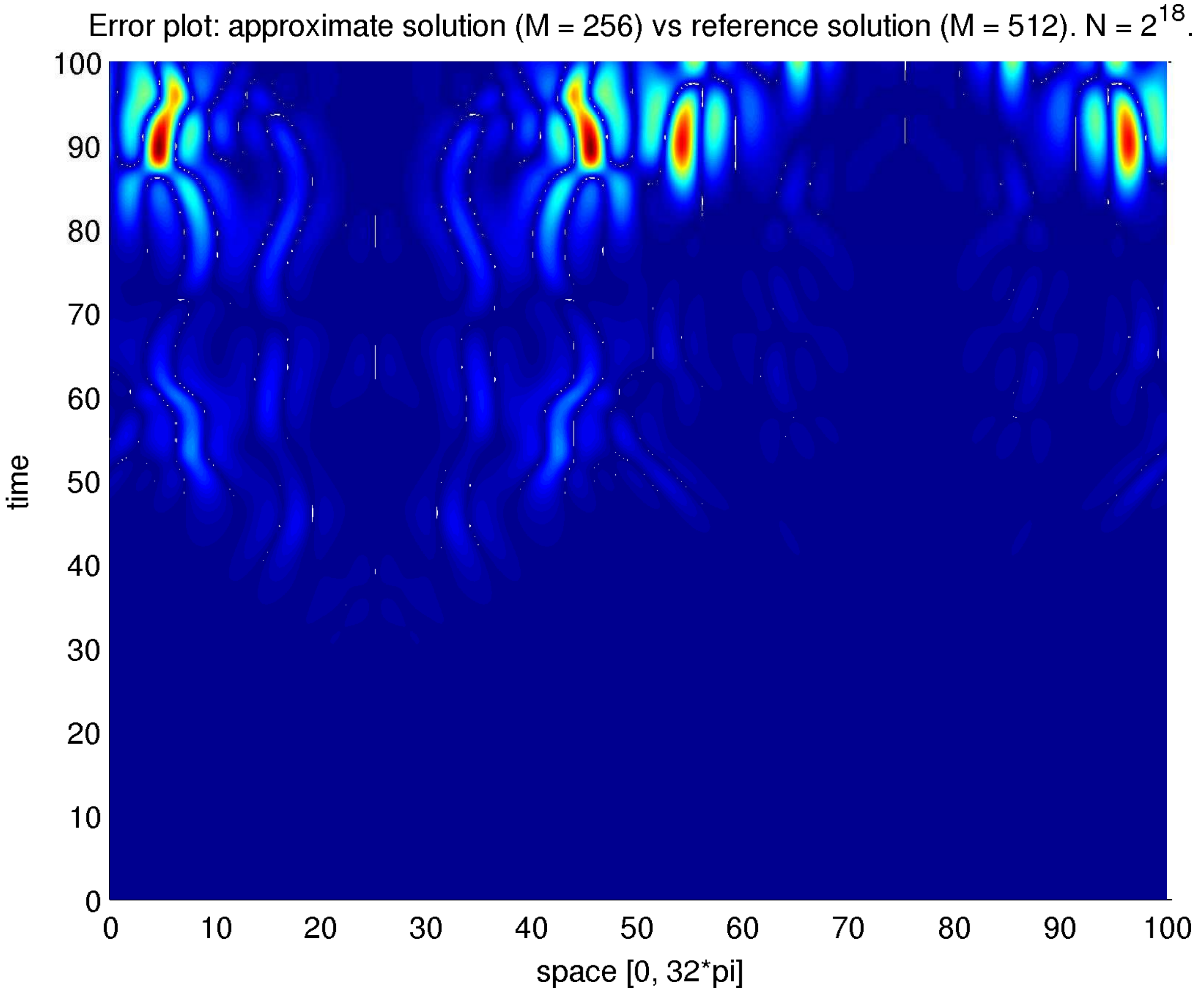


Numerical solution of the KS-eq: M = 128, N = 2^{16}
time
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
space [0, 32*pi]



Numerical solution of the KS-eq: M = 128, N = 2^16
time
space [0, 32*pi]
0
50
100
150
200
250
300
350
400
450
500
0
10
20
30
40
50
60
70
80
90
100


Error plot: approximate solution (M = 128) vs reference solution (M = 512). N = 2^18.
time
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
space [0, 32*pi]



Error plot: approximate solution (M = 256) vs reference solution (M = 512). N = 2^{18}.
time
space [0, 32*pi]
0
10
20
30
40
50
60
70
80
90
100

Difference Scheme

The Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$
$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

Difference scheme:

$$\begin{aligned}
u_t &\approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k} \\
u_{xx} &\approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} = \frac{1}{h^2}Au \\
u_{xxxx} &\approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} = \frac{1}{h^4}AAu \\
(u^2)_x &\approx \frac{\mu\delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} = \frac{1}{2h}D
\end{aligned}$$

$$U^{n+1} = U^n - \frac{k}{h^2}AU^n - \frac{k}{h^4}AAU^n - \frac{k}{4h}D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

Consistency

Consistency

$$
\begin{aligned}
u_t &= \frac{\Delta u}{k} + O(k) \\
u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\
u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\
\mu\delta[u(x)^2] &= \mu[u(x + \frac{h}{2}^2) - u(x - \frac{h}{2})^2] \\
&= 2huu_x + O(h^3) \\
uu_x &= \frac{\mu\delta u^2}{4h} + O(h^2)
\end{aligned}
$$

Local Truncation Error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k,h\to 0} 0 \tag{1}$$

$\Rightarrow$ Consistent

Stability

Stability analysis of the KS-equation

Using Von Neumann stability analysis on a linearized version of the equation where the $\frac{1}{2}(u^2)_x$ term is replaced by $\frac{1}{2}(\rho(x)u)_x$. $\rho(x) = U^0$ is the initial condition $f(x,0)$. We will look at the scheme

$$U^{n+1} = (I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}D(RU^n))$$

$$U_m^{n+1} = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n)$$
$$-\frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n)$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ s.t.

$$\xi = 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h})$$
$$-\frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})$$
$$= 1 - \frac{2k}{h^2}(\cos(\beta h) - 1) - \frac{k}{h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h)) - \frac{k}{4h}2i\sin(2\beta h)$$

The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$. Let $r = r/k^4$.

$$|\xi|^2 = (1 - 4rh^2(\cos(\beta h) - 1) - 2r(3 - 4\cos(\beta h) + \cos(2\beta h)))^2 + k\frac{rh^2}{4}\sin^2(2\beta h)$$
$$= \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

We want $|\xi| \leq 1 + \mu k$, so we need $\psi = |1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)| \leq 1 + \tilde{\mu}k$. Let $q = \sin^2\left(\frac{\beta h}{2}\right)$. For the case $(1 \leq 16rq^2 \leq 2)$ we achieve

$$\psi = \left|1 + 4rh^2 q - 16rq^2\right| \leq 1 \underset{0\leq x\leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

For the case $(0 \leq 16rq^2 \leq 1)$ we get no furher result. Let us now assume that $(1 \leq 16rq^2 \leq 2)$ and $r = 1/8$.

$$|\xi|^2 = \left(1 + \frac{4}{8}qh^2 - \frac{16}{8}q^2\right)^2 + k\frac{h^2}{4 \cdot 8}\sin(2\beta h)$$

$$\max\left(1 + \frac{xh^2}{2} - 2x^2\right)|(0 \leq x \leq 1)\text{gives } x = \frac{h^2}{8}$$

$$=$$