Kuramoto-Sivashinsky equation

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Kuramoto-Sivashinsky equation

$$
\begin{aligned}
u_t + u_{xx} + u_{xxxx} + uu_x &= 0 \\
u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x &= 0 \\
u(x,0) &= f(x) \\
u(0,t) &= u(L,t)
\end{aligned}
$$

About Kuramato-Sivashinsky equation

- Kuramoto - Japan 1977
  - reaction-diffusion
- Sivashinsky - Israel 1977
  - flame fronts
- Chaos
- Stiff

Forward and central difference

$$
\begin{aligned}
u_t &\approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k} \\
u_{xx} &\approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} &&= \frac{1}{h^2} A u \\
u_{xxxx} &\approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} &&= \frac{1}{h^4} A^2 u \\
(u^2)_x &\approx \frac{\mu\delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} &&= \frac{1}{2h} D
\end{aligned}
$$

Explicit scheme

$$U^{n+1} = U^n - \frac{k}{h^2} A U^n - \frac{k}{h^4} A^2 U^n - \frac{k}{4h} D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

- Unstable for $k > r \cdot h^4$

Crank-Nicolson

- Trapezoidal rule
- Not applied to non-linear term

$$\left[\frac{U^{n+1} - U^n}{k} = -\frac{1}{2h^2}A(U^n + U^{n+1}) - \frac{1}{2h^4}A^2(U^n + U^{n+1})\right] - \frac{1}{4h}D(U^n \odot U^n)$$

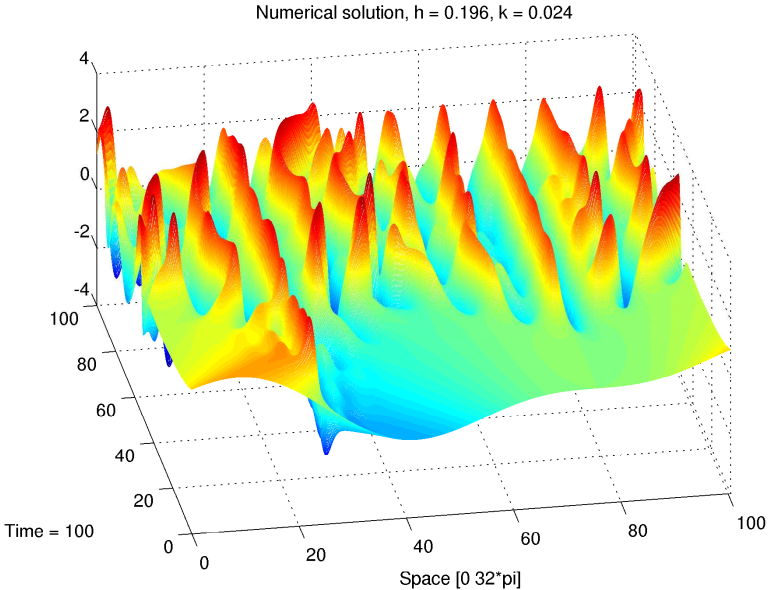
Implicit scheme

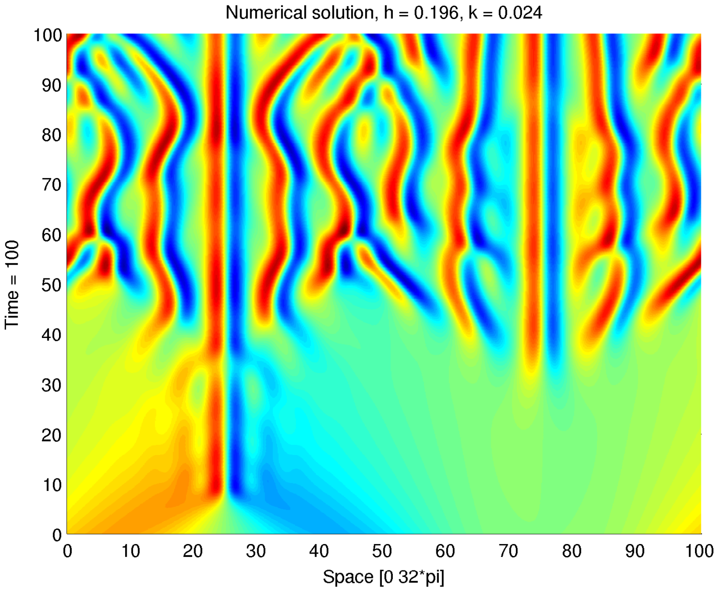
$$(I + \frac{k}{2h^2}A + \frac{k}{2h^4}A^2)U^{n+1} = (I - \frac{k}{2h^2}A - \frac{k}{2h^4}A^2)U^n - \frac{k}{4h}D(U^n \odot U^n)$$

- Crank-Nicholson
- Explicit non-linear term
- Stable

Numerical conditions

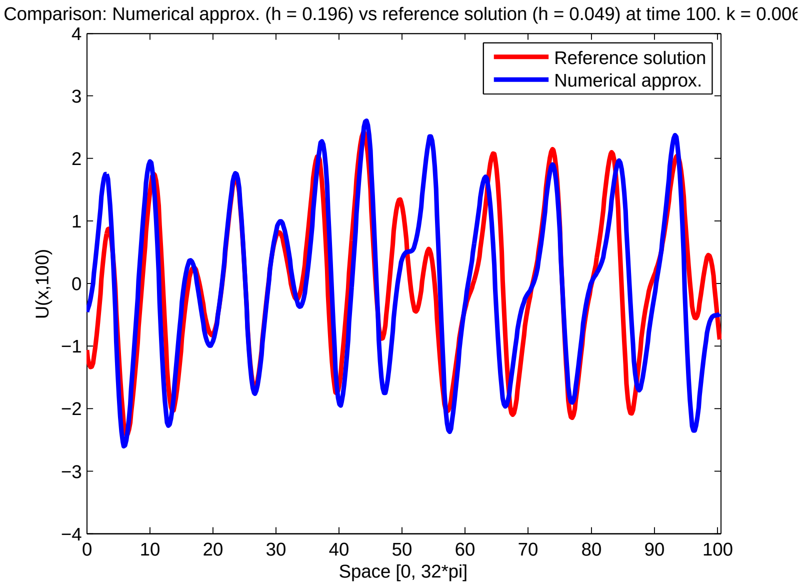
- Initial Condition: $\text{f}(\text{x}) = \cos(\frac{x}{16})(1 + \sin(\frac{x}{16}))$
- Interval of length: $L = 32\pi$

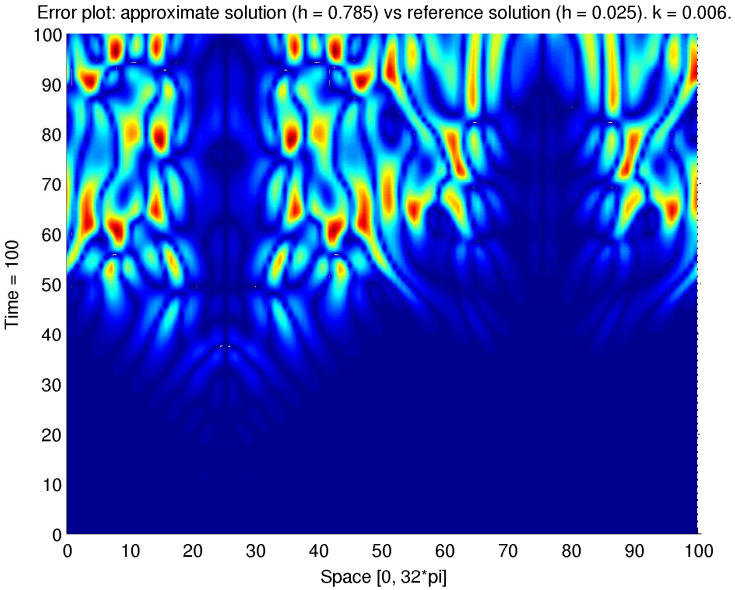

Numerical solution, h = 0.196, k = 0.024
4
2
0
-2
-4
100
80
60
40
20
0
Time = 100
0
20
40
60
80
100
Space [0 32*pi]

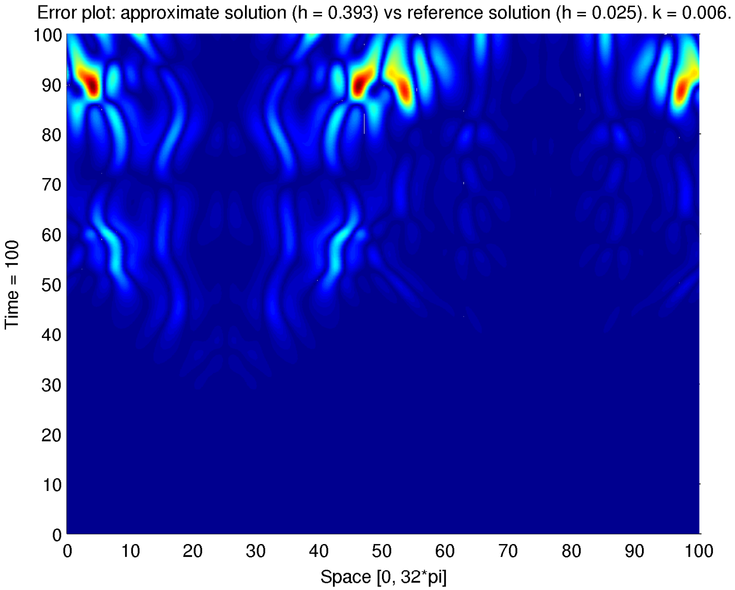

Numerical solution, h = 0.196, k = 0.024
Time = 100
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
Space [0 32*pi]

Reference solution

- No analytic solution
- Semi discretization and stiff system
- MATLAB ode15s


Comparison: Numerical approx. (h = 0.196) vs reference solution (h = 0.049) at time 100. k = 0.006
Reference solution
Numerical approx.
U(x,100)
Space [0, 32*pi]
4
3
2
1
0
−1
−2
−3
−4
0
10
20
30
40
50
60
70
80
90
100


Error plot: approximate solution (h = 0.785) vs reference solution (h = 0.025). k = 0.006.
Time = 100
100
90
80
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
80
90
100
Space [0, 32*pi]


Error plot: approximate solution (h = 0.393) vs reference solution (h = 0.025). k = 0.006.
Time = 100
Space [0, 32*pi]
0
10
20
30
40
50
60
70
80
90
100

Consistency

- Space discretizations

$$u_{xx} = \frac{\delta^2 u}{h^2} + O(h^2)$$
$$u_{xxxx} = \frac{\delta^4 u}{h^4} + O(h^2)$$
$$uu_x = \frac{\mu\delta u^2}{4h} + O(h^2)$$

Consistency

- Trapezoidal rule in time with explicit nonlinear term: O(k)
- Local truncation error

$$\tau^n = O(h^2) + O(k) \xrightarrow{k,h\to 0} 0$$

$$\Rightarrow \text{Consistent}$$

Stability analysis - linearized implicit scheme

- Von Neumann on linearized equation
- $\frac{1}{2}(u^2)_x \approx \frac{1}{2}(\rho(x)u)_x$
- $\rho(x) = U_m^0 = \xi^0 e^{i\beta x_m}$

$$|\xi|^2 \leq \left(\frac{1 + k/8}{1 - k/8}\right)^2 + \frac{k}{4h^2}k$$

Linearized scheme not stable

Stability analysis - linearized explicit scheme

- $\alpha = \sin^2(\frac{\beta h}{2})$
- Assume: $(1 \leq 16r\alpha \leq 2)$ and $(4rh^2\alpha \leq 1)$

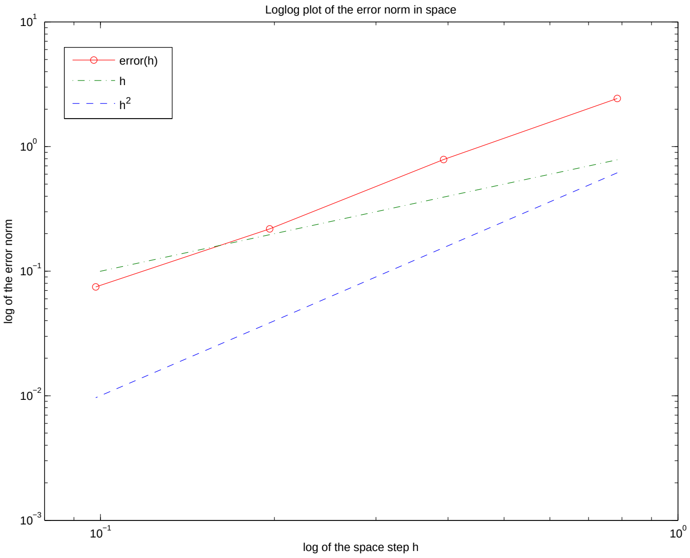
$$\left|1 - 16r\alpha^2\right| \leq 1 \underset{0\leq\alpha\leq 1}{\Longrightarrow} r \leq 1/8$$

$$|\xi|^2 \leq \left(1 + \frac{h^4}{32}\right)^2 + k\frac{h^2}{32}$$

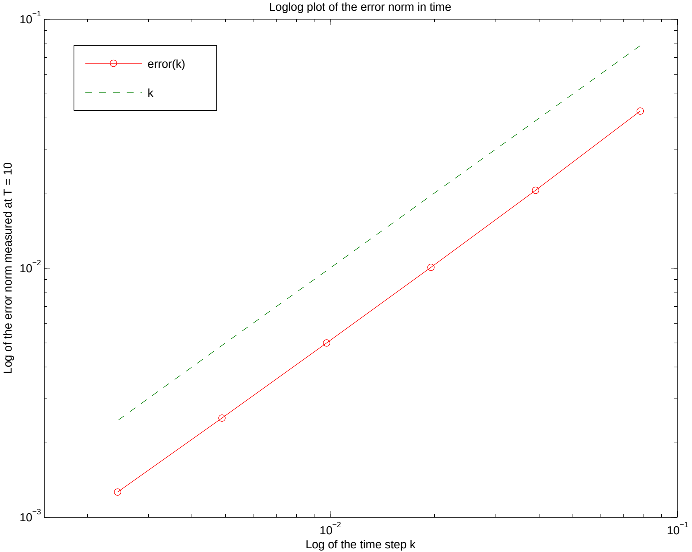
- Linearized scheme not stable
- experimentally $r < 1/8$ stable for non-linear scheme

Convergence of implicit scheme

- Lax' equivalence theorem
- $\tau^n = O(h^2) + O(k)$
- $|\xi|^2 \leq \left(\frac{1+rh^4/8}{1-rh^4/8}\right)^2 + k\frac{rh^2}{4}$
- Experimentally the scheme is stable



Loglog plot of the error norm in space
error(h)
h
h2
10^1
10^0
10^-1
10^-2
10^-3
10^-1
10^0
log of the error norm
log of the space step h



Loglog plot of the error norm in time
error(k)
k
Log of the error norm measured at T = 10
Log of the time step k
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}