The Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0$$
$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}(u^2)_x = 0$$

Difference scheme:

$$u_t \approx \frac{\Delta u}{k} = \frac{u^{n+1} - u^n}{k}$$
$$u_{xx} \approx \frac{\delta^2 u}{h^2} = \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} = \frac{1}{h^2}Au$$
$$u_{xxxx} \approx \frac{\delta^4 u}{h^4} = \frac{u_{m+2} - 4u_{m+1} + 6u_m - 4u_{m-1} + u_{m-2}}{h^4} = \frac{1}{h^4}AAu$$
$$(u^2)_x \approx \frac{\mu\delta u^2}{h} = \frac{(u_{m+1})^2 - (u_{m-1})^2}{2h} = \frac{1}{2h}D$$

$$U^{n+1} = U^n - \frac{k}{h^2}AU^n - \frac{k}{h^4}AAU^n - \frac{k}{4h}D(U^n \odot U^n)$$

$\odot$ = Element-wise multiplication

Consistency

$$
\begin{aligned}
u_t &= \frac{\Delta u}{k} + O(k) \\
u_{xx} &= \frac{\delta^2 u}{h^2} + O(h^2) \\
u_{xxxx} &= \frac{\delta^4 u}{h^4} + O(h^2) \\
\mu\delta[u(x)^2] &= \mu[u(x + \frac{h}{2}^2) - u(x - \frac{h}{2})^2] \\
&= 2huu_x + O(h^3) \\
uu_x &= \frac{\mu\delta u^2}{4h} + O(h^2)
\end{aligned}
$$

Local Truncation Error

$$
\tau^n = O(h^2) + O(k) \xrightarrow{k,h\to 0} 0 \tag{1}
$$

$\Rightarrow$ Consistent