The Kuramoto-Sivashinsky Equation

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Abstract

This is the abstract. Write smart things here.

Introduction

The Kuramoto-Sivashinsky equation,

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0 \tag{1}$$

is one of the simplest partial differential equations that exhibits complicated dynamics in both time and space, which is why the equation has been the attention for a lot of research. The equation was developed by two scientists at the same time in 1977 [1]. Gregory Sivashinsky determined an equation for a laminar flame front, while Yoshiki Kuramoto modeled a diffusion-induced chaos using the same equation. Because of this, the equation is named Kuramoto-Sivashinsky. The KS-equation also models the motion of a fluid going down a vertical wall, e.g. solitary pulses in a falling thin film. [2]

The reason for the complex behaviour comes from the second- and fourth-order derivatives in (1). While the second-order term acts as an energy source and has a destabilizing effect, the fourth-order term has a stabilizing effect. In addition to this, the nonlinear term transfers energy from low to high wave numbers. [3] The KS-equation is a stiff equation, i.e. an equation where numerical methods for solving it are numerically unstable, unless the step size is extremely small. u_{xxxx} is the main reason for this as it leads to rapid variation in the solution.

Numerical results

Initial conditions

In the solution of the KS-equation we had periodic boundary conditions, i.e. $u(0,t) = u(L,t)$. We also used L-periodic initial conditions. We experienced that a common initial condition used in several other reports was

$$u(x,0) = \cos(\frac{x}{16})(1 + \sin(\frac{x}{16}). \tag{2}$$

We also tried the initial condition

$$u(x,0) = \frac{1}{\sqrt{2}}\sin(x) - \frac{1}{8}\sin(2x), \tag{3}$$

which worked well. The L-periodic initial conditions is customarily taken [4] to satisfy

$$\int_0^L f(x)\,\mathrm{d}x = 0, \tag{4}$$

which both of our initial conditions satisfy. The same article also states that for L-periodic initial data, a unique solution for (1) exits, and is bounded as $t \to \infty$. The bound has been proven to be smaller than $O(L^{8/5})$. In our numerical tests, with $t = 5000$, the initial condition (2) did indeed not exceed the bound, nor did (3).

Stability

Von Neumann stability analysis is a method based on Fourier decomposition of numerical error, and is used to check the stability of linear pde-s. Since the KS-equation has a non-linear term, $\frac{1}{2}(u^2)_x$, this has to be linearized before the method is applied. The KS-equation is stiff, and the non-linear term stabilizes the equation. Replacing it by $\frac{1}{2}(\rho(x)u)_x$, where $\rho(x) \approx u(x)$ is constant in time, we obtain

$$u_t = -u_{xx} - u_{xxxx} - \frac{1}{2}\left(\rho(x)u\right)_x. \tag{5}$$

Choosing $\rho(x) = f(x,0)$, where $f(x,0) = U^0$, we obtain the linearized scheme

$$\left(I + \frac{k}{h^2}A + \frac{k}{h^4}A^2\right)U^{n+1} = \left(I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}DR\right)U^n,$$

which written out becomes

$$\begin{aligned}
U_m^{n+1} + \frac{k}{h^2}(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) + \frac{k}{h^4}(U_{m+2}^{n+1} - 4U_{m+1}^{n+1} + 6U_m^{n+1} - 4U_{m-1}^{+1}n + U_{m-2}^{n+1})\\
= U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n)\\
-\frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n)
\end{aligned}$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ such that

$$\begin{aligned}
&\xi(1 + \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) + \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h})\\
&= 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h})\\
&\qquad -\frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})
\end{aligned}$$

$$\xi = \frac{1 - \frac{k}{h^2}(\cos(\beta h) - 1) - \frac{k}{2h^4}(6 - 8\cos(\beta h) + \cos(2\beta h)) - \frac{k}{2h} i \sin(2\beta h)}{1 + \frac{k}{h^2}(\cos(\beta h) - 1) + \frac{k}{2h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h))}$$
$$= \frac{1 + \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) - \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right) - \frac{k}{2h}\sin^2(2\beta h)}{1 - \frac{2k}{h^2}\sin^2\left(\frac{\beta h}{2}\right) + \frac{8k}{h^4}\sin^4\left(\frac{\beta h}{2}\right)}$$

Let $q = \sin^2(\frac{\beta h}{2})$ and $r = \frac{k}{h^4}$. The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$.

$$|\xi|^2 = \left(\frac{1 + 2rq(h^2 - 4q)}{1 - 2rq(h^2 - 4q)}\right)^2 + \frac{1}{4}\frac{krh^2\sin^2(2\beta h)}{(1 - 2rq(h^2 - 4q))^2}$$

Maximizing $-2rq(4q - h^2)$ wrt q gives $q = h^2/8$, which replaced in the equation gives

$$|\xi|^2 \leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + \frac{1}{4}kh^2 r\frac{\sin(2\beta h)}{(1 - rh^4/8)^2}$$
$$\leq \left(\frac{1 + rh^4/8}{1 - rh^4/8}\right)^2 + k\frac{rh^2}{4}$$

This expression is dependent on both k and h, so the method is not stable. This is expected, since the linearized KS-equation is unstable in numerical experiments.

$$U^{n+1} = (I - \frac{k}{h^2}A - \frac{k}{h^4}A^2 - \frac{k}{4h}D(RU^n))$$

$$U_m^{n+1} = U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) - \frac{k}{h^4}(U_{m+2}^n - 4U_{m+1}^n + 6U_m^n - 4U_{m-1}^n + U_{m-2}^n)$$
$$-\frac{k}{4h}(U_{m+1}^0 U_{m+1}^n - U_{m-1}^0 U_{m-1}^n)$$

Let $U_m^n = \xi^n e^{i\beta x_m}$ and $U_m^0 = \xi^0 e^{i\beta x_m} = e^{i\beta x_m}$ s.t.

$$\xi = 1 - \frac{k}{h^2}(e^{i\beta h} - 2 + e^{-i\beta h}) - \frac{k}{h^4}(e^{2i\beta h} - 4e^{i\beta h} + 6 - 4e^{-i\beta h} + e^{-2i\beta h})$$
$$-\frac{k}{4h}(e^{i\beta h}e^{i\beta h} - e^{-i\beta h}e^{-i\beta h})$$
$$= 1 - \frac{2k}{h^2}(\cos(\beta h) - 1) - \frac{k}{h^4}(6 - 8\cos(\beta h) + 2\cos(2\beta h)) - \frac{k}{4h}2i\sin(2\beta h)$$

The Von Neumann's stability criterion claims there is a constant $\mu \geq 0$ such that $|\xi| \leq 1 + \mu k$. Let $r = r/k^4$.

$$|\xi|^2 = (1 - 4rh^2(\cos(\beta h) - 1) - 2r(3 - 4\cos(\beta h) + \cos(2\beta h)))^2 + k\frac{rh^2}{4}\sin^2(2\beta h)$$
$$= \left(1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)\right)^2 + k\left(\frac{rh^2}{4}\sin^2(2\beta h)\right)$$

We want $|\xi| \leq 1 + \mu k$, so we need $\psi = |1 + 4rh^2\sin^2\left(\frac{\beta h}{2}\right) - 16r\sin^4\left(\frac{\beta h}{2}\right)| \leq 1 + \tilde{\mu}k$. Let $q = \sin^2\left(\frac{\beta h}{2}\right)$.
For the case $(1 \leq 16rq^2 \leq 2)$ we achieve

$$\psi = \left|1 + 4rh^2q - 16rq^2\right| \leq 1 \underset{0\leq x\leq 1}{\Longrightarrow} (1/16 \leq r \leq 1/8)$$

For the case $(0 \leq 16rq^2 \leq 1)$ we get no furher result. Let us now assume that $(1 \leq 16rq^2 \leq 2)$ and $r = 1/8$.

$$|\xi|^2 = \left(1 + \frac{4}{8}qh^2 - \frac{16}{8}q^2\right)^2 + k\frac{h^2}{4\cdot 8}\sin(2\beta h)$$

$$\max\left(1 + \frac{xh^2}{2} - 2x^2\right)|(0 \leq x \leq 1)\text{gives } x = \frac{h^2}{8}$$
$$=$$

References

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