Project in Optimiziation Theory - TMA 4180

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Abstract

A truss is a mechanical structure constructed of straight elastic members connected at joints. In this project we have used optimization theory to maximize the stiffness of the structure.

Equations for question 1-3

$$\begin{aligned} \underset{(\boldsymbol{q},\boldsymbol{f}^{\text{supp}})}{\text{minimize}} \quad & \frac{1}{2}\sum_{j=1}^{m}\frac{\ell_j q_j^2}{E_j A_j} = g(\boldsymbol{q},\boldsymbol{f}^{\text{supp}}) \\ \text{subject to} \quad & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}} \end{aligned} \tag{1}$$

$$\begin{aligned} \boldsymbol{D}\boldsymbol{q} &= \boldsymbol{B}^T\boldsymbol{u} \\ \boldsymbol{B}\boldsymbol{q} &= \boldsymbol{I}_{\text{supp}}\boldsymbol{f}^{\text{supp}} + \boldsymbol{I}_{\text{ext}}\boldsymbol{f}^{\text{ext}} \\ \boldsymbol{I}_{\text{supp}}^T\boldsymbol{u} &= 0 \end{aligned} \tag{2}$$

Question 1

To prove that (2) constitutes the necessary and sufficient optimality conditions, we use Theorem 12.1 [1]. Since it is hard to know if LICQ holds in this case, we instead use another possible constraint qualification that is neither weaker nor stronger than LICQ. It says that the condition that all active constraints are linear, is enough.

Because the function $g(\boldsymbol{q},\boldsymbol{f}^{\text{supp}})$ and the constraints in (1) are continuously differentiable, the requirements for Thm 12.1 are met.

By using the first of the KKT-conditions,

$$\nabla_x\mathcal{L}(x^*,\lambda^*) = 0, \tag{3}$$

for both of the variables, namely $\boldsymbol{q}$ and $\boldsymbol{f}^{\text{supp}}$, we get two equations:

$$\begin{aligned} \nabla_{\boldsymbol{q}}\mathcal{L}(\boldsymbol{q}^*,\lambda^*) &= \nabla_{\boldsymbol{q}} g(\boldsymbol{q}^*) - \sum_{i\epsilon\mathcal{A}(\boldsymbol{q}^*)}\lambda_i^*\nabla_{\boldsymbol{q}} c_i(\boldsymbol{q}^*) \\ \nabla_{\boldsymbol{f}^{\text{supp}}}\mathcal{L}(\boldsymbol{f}^{\text{supp}^*},\lambda^*) &= \nabla_{\boldsymbol{q}} g(\boldsymbol{f}^{\text{supp}^*}) - \sum_{i\epsilon\mathcal{A}(\boldsymbol{f}^{\text{supp}^*})}\lambda_i^*\nabla_{\boldsymbol{f}^{\text{supp}}} c_i(\boldsymbol{f}^{\text{supp}^*}). \end{aligned} \tag{4}$$

This leads to the two equations:

$$
\begin{aligned}
\boldsymbol{D}\boldsymbol{q} &= \boldsymbol{B}^T\lambda \\
\boldsymbol{I}_{\text{supp}}^T\lambda &= 0
\end{aligned}
\tag{5}
$$

By choosing λ to be $\boldsymbol{u}$ we see that the KKT-conditions for (1) and the system (2) are equivalent, which is what we aimed to prove. Thm 12.1 is, however, only for proving necessary conditions, but since the KKT-conditions are sufficient under convexity, we can prove it is sufficient as well. For this to be true, $g(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}})$ must be convex and c_i affine. The first requirement holds since we have a quadratic function. Because the c_i's are linear functions, they are both convex and concave (affine). Hence we have proved that system (2) constitutes the necessary and sufficient optimality conditions for (1).

Question 2

To show that the problem (1) admits at least one solution, we use Theorem 2 in [2].

Since Ω is non-empty, we must also show that it is closed. From [4] we know that if the function is continuous, the hyperplane is closed. As we have already stated, our function $g(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}})$ is continuous. Hence Ω is both non-empty and closed.

Proving that $g(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}})$ is coercive, can be shown in this way,

$$
\lim_{\|\boldsymbol{q}\|\to\infty} \frac{1}{2}\sum_{j=1}^{m} \frac{\ell_j q_j^2}{E_j A_j} > \lim_{\|\boldsymbol{q}\|\to\infty} \max_{1\le \mathrm{j}\le \mathrm{m}} \frac{\ell_{\mathrm{j}} \|q_{\mathrm{j}}\|^2}{E_{\mathrm{j}} A_{\mathrm{j}}} \to \infty. \tag{6}
$$

The requirement that $g(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}})$ must be a lower semi-continuous function holds as all continuous functions are both lower- and upper semi-continuous. Hence from Theorem 2, we know that (1) admits at least one global minimum.

To conclude that (2) admits at least one solution under these assumptions we use the fact that (1) has a solution. Since we know that (2) is the optimal solution to (1), we then know that (2) has at least one solution.

Question 3

For proving that the problem (1) admits exactly one solution, we

References

[1] Jorge Nocedal, Stephen J. Wright, *Numerical Optimization*, 2nd Edition, 2006.

[2] Anton Evgrafov, *Optimization Theory*

[3] Anton Evgrafov, *Introduction to optimality conditions*

[4] California Institute of Technology, *Separating Hyperplane Theorems* http://www.hss.caltech.edu/~kcb/Notes/SeparatingHyperplane.pdf, 28. March 2014