Домашнее задание

1. $\int 2\sin^2\frac{x}{2}\,dx = 2\int \sin^2\frac{x}{2}\,dx = \left| \begin{array}{l} u = \frac{x}{2} \\ du = \frac{1}{2}dx \\ dx = 2du \end{array} \right. =$

$2\int \sin^2 u \, 2du = 4\int \sin^2 u\, du =$

$4\int \frac{1-\cos(2u)}{2}\,du = 4\int \frac{1}{2}du - \int \frac{\cos 2u}{2}\,du =$

$4\left(\frac{1}{2}u - \frac{1}{2}\int \cos\sigma \cdot \frac{d\sigma}{2}\right) = 4\left(\frac{1}{2}u - \frac{1}{4}\int \cos\sigma\, d\sigma\right) =$

$\left| \begin{array}{l} \sigma = 2u \\ d\sigma = 2du \\ du = \frac{d\sigma}{2} \end{array} \right.$

$2u - \sin\sigma + C = 2u - \sin 2u + C = x - \sin x + C$

2. $\int \frac{(1+x)^2}{x(1+x^2)}\,dx$

$\int \frac{1+2x+x^2}{x(1+x^2)}\,dx = \int\left(\frac{1}{x} + \frac{2}{(1+x^2)}\right)dx =$

$\int \frac{1}{x}\,dx + \int \frac{2\cdot 1}{(1+x^2)}\,dx =$

$\ln|x| + 2\,\mathrm{arctg}\,x + C$

3. $\int \frac{d(1+x^2)}{\sqrt{1+x^2}} = \ln\left|x + \sqrt{x^2+1}\right| + C$

4. $\int \frac{x^2\,dx}{(8x^3+27)^{2/3}} = \left| \begin{array}{l} u = 8x^3+27 \\ du = 24x^2\,dx \\ x^2dx = \frac{du}{24} \end{array} \right. = \int \frac{du}{24\, u^{2/3}} =$

$\frac{1}{24}\int \frac{1}{u^{2/3}}\,du = \frac{1}{\cancel{24}_{8}} \cdot \frac{\cancel{3}}{1} u^{\frac{1}{3}} + C = \frac{1}{8}(8x^3+27)^{\frac{1}{3}} + C$

5 $\int \frac{dx}{x\ln x\, \ln(\ln x)} = \left| \begin{array}{l} u = \ln(\ln(x)) \\ du = \frac{1}{x\cdot \ln(x)}\,dx \end{array} \right.$

$\int du \cdot \frac{1}{u} = \ln|u| + C = \ln|\ln(\ln(x))| + C$

6 $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx =$ $\left| \begin{array}{l} u = a^2 \sin^2 x + b^2 \cos^2 x \\ du = (a^2 \cdot 2 \cdot \sin x \cdot \cos x - 2b^2 \cos(x) \sin(x)) dx \end{array} \right.$

$dx = \frac{du}{2a^2 \sin x \cos x - 2b^2 \cos x \sin x}$

$\int - \frac{1}{2(b^2 - a^2)\sqrt{u}} du = -\frac{1}{+2b^2 - 2a^2} \int \frac{1}{\sqrt{u}} du =$

$\frac{1}{-(2b^2 - 2a^2)} \cdot \frac{2}{1} \cdot \sqrt{u} + C = -\frac{2\sqrt{u}}{2(b^2 - a^2)} + C = \frac{-\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}}{b^2 - a^2} + C$

$\int u \, dv = uv - \int v \, du$

7 $\int x \cdot 3^x dx$ $\left| \begin{array}{l} u = x \\ dv = 3^x dx \\ du = dx \\ v = \int 3^x dx = \frac{3^x}{\ln 3} \end{array} \right.$

$\frac{x \cdot 3^x}{\ln 3} - \int \frac{3^x}{\ln 3} dx = \frac{x \cdot 3^x}{\ln 3} - \frac{1}{\ln 3} \int 3^x dx = \frac{x \cdot 3^x}{\ln 3} - \frac{1}{\ln 3} \cdot \frac{3^x}{\ln 3} + C =$

$\frac{x \cdot 3^x}{\ln 3} - \frac{3^x}{\ln^2 3} + C = \frac{3^x (x \cdot \ln 3 - 1)}{\ln^2 3} + C$

8. $\int x^n \ln x \, dx =$ $\left| \begin{array}{l} u = \ln x \\ dv = x^n dx \\ du = \frac{1}{x} \cdot dx \\ v = \int x^n dx = \frac{x^{n+1}}{n+1} \end{array} \right.$

$\frac{\ln x \cdot x^{n+1}}{n+1} - \int \frac{x^n}{n+1} dx = \frac{\ln x \cdot x^{n+1}}{n+1} - \frac{1}{n+1} \cdot \int x^n dx =$

$\frac{\ln x \cdot x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}(\ln x (n+1) - 1)}{(n+1)^2} + C$

9. $\int \sin \sqrt[3]{x} =$ $\left| \begin{array}{l} u = \sqrt[3]{x} \\ du = \frac{dx}{3 \cdot x^{\frac{2}{3}}} \\ dx = 3x^{\frac{2}{3}} du = 3u^2 du \end{array} \right.$

$$\int \sin u \cdot 3u^2\,du = 3\int u^2 \sin u\,du = \left| \begin{array}{l} t = u^2 \\ d\sigma = \sin u\,du \\ dt = 2u\,du \\ \sigma = \int \sin u\,du = -\cos u \end{array} \right.$$

$$-u^2\cos u + 2\int u\cos u\,du$$

$$\left| \begin{array}{l} a = u \\ db = \cos u\,du \\ da = du \\ b = \int \cos u\,du = \sin u \end{array} \right.$$

$$u\sin u - \int \sin u\,du = u\sin u + \cos u$$

$$3(-u^2\cos u + 2u\sin u + 2\cos u) = -3u^2\cos u + 6u\sin u + 6\cos u + C =$$

$$-3x^{\frac{2}{3}}\cos\sqrt[3]{x} + 6\sqrt[3]{x}\sin\sqrt[3]{x} + 6\cos\sqrt[3]{x} + C =$$

$$-3(\cos\sqrt[3]{x}\,(x^{\frac{2}{3}} - 2) - 2\sin\sqrt[3]{x}\cdot\sqrt[3]{x}) + C$$

10. $$\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}} = \int \frac{\sqrt{x+1} - \sqrt{x-1}}{2}\,dx =$$

$$\frac{1}{2}\int \sqrt{x+1}\,dx - \frac{1}{2}\int \sqrt{x-1}\,dx =$$

1) $$\int \sqrt{x+1}\,dx = \left| \begin{array}{l} u = x+1 \\ du = dx \end{array} \right. = \int \sqrt{u}\,du = \frac{2u^{\frac{3}{2}}}{3} = \frac{2}{3}\cdot(x+1)^{\frac{3}{2}}$$

2) $$\int \sqrt{x-1}\,dx = \left| \begin{array}{l} u = x-1 \\ du = dx \end{array} \right. = \int \sqrt{u}\,du = \frac{2}{3}(x-1)^{\frac{3}{2}}$$

$$\frac{1}{2}\cdot\frac{2}{3}\cdot(x+1)^{\frac{3}{2}} - \frac{1}{2}\cdot\frac{2}{3}\cdot(x-1)^{\frac{3}{2}} + C = \frac{(x+1)^{\frac{3}{2}}}{3} - \frac{(x-1)^{\frac{3}{2}}}{3} + C$$

11. $$\int \frac{dx}{4x^2+4x+5} = \int \frac{dx}{(2x+1)^2+4} = \left| \begin{array}{l} u = \frac{(2x+1)}{2} \\ du = \frac{1}{2}\cdot 2\,dx = dx \end{array} \right.$$

$$\int \frac{1}{4u^2+4}\,du = \int \frac{1}{4}\frac{1}{u^2+1}\,du = \frac{1}{4}\int \frac{1}{u^2+1}\,du =$$

$$\frac{\operatorname{arctg} u}{4} + C = \frac{\operatorname{arctg}\frac{2x+1}{2}}{4} + C$$

12. $\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C = \frac{3x\sin 3x + \cos 3x}{9} + C$

	D	I
+	x	$\cos 3x$
−	1	$\frac{1}{3}\sin 3x$
+	0	$-\frac{1}{9}\cos 3x$

$u = x$
$dv = \cos 3x\,dx$
$du = dx$
$v = \int \cos 3x\,dx = \frac{1}{3}\sin 3x$

$\frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C = \frac{3x\sin 3x + \cos 3x}{9} + C$

13.
$\int (9x-1)\cos 2x\,dx = \frac{\sin 2x\,(9x-1)}{2} + \frac{9\cos 2x}{4} + C = \frac{(18x-2)\sin 2x + 9\cos 2x}{4} + C$

	D	I
+	$9x-1$	$\cos 2x$
−	9	$\frac{1}{2}\sin 2x$
+	0	$-\frac{1}{4}\cos 2x$

$u = 9x-1$
$dv = \cos 2x\,dx$
$du = 9\,dx$
$v = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$

$(9x-1)\cdot\frac{1}{2}\sin 2x - \frac{9}{2}\int \sin 2x\,dx = \frac{(9x-1)\sin 2x}{2} + \frac{9\cos 2x}{4} + C = \frac{(18x-2)\sin 2x + 9\cos 2x}{4} + C$

14. $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$

	D	I
+	x^2	$\cos x$
−	$2x$	$\sin x$
+	2	$-\cos x$
−	0	$-\sin x$

$u = x^2$

$dv = \cos x \, dx$

$du = 2x \, dx$

$v = \int \cos x \, dx = \sin x$

- $x^2 \sin x - 2\int x \sin x \, dx$

$u = x$

$dv = \sin x \, dx$

$du = dx$

$v = \int \sin x \, dx = -\cos x$

$-x\cos x + \int \cos x \, dx = -x\cos x + \sin x$

$= x^2 \sin x + 2x\cos x - 2\sin x + C$

15. $\int \frac{x-1}{4} \cos\frac{n\pi x}{2} \, dx = \frac{(x-1)\sin\left(\frac{n\pi x}{2}\right)}{2n\pi} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{n^2\pi^2} + C$

	D	I
+	$\frac{x-1}{4}$	$\cos\frac{n\pi x}{2}$
−	$\frac{1}{4}$	$\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$
+	0	$-\frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)$

$\int \cos\frac{n\pi x}{2} \, dx =$ $\left| \begin{array}{l} u = \frac{n\pi x}{2} \\ du = \frac{n\pi}{2} dx \\ dx = \frac{du \cdot 2}{n\pi} \end{array} \right.$ $= \int \cos u \, du \cdot \frac{2}{n\pi}$

$\frac{2}{n\pi} \sin\frac{n\pi x}{2}$

$u = \frac{x-1}{4}$

$dv = \cos\frac{n\pi x}{2}$

$du = \frac{1}{4} dx$

$v = \int \cos\frac{n\pi x}{2} = \frac{2}{n\pi} \sin\frac{n\pi x}{2}$

$\frac{(x-1)}{\cancel{4}_2} \cdot \frac{\cancel{2}\sin\left(\frac{n\pi x}{2}\right)}{n\pi} - \frac{2 \cdot 1}{n\pi \cdot 4} \int \sin\frac{n\pi x}{2} \, dx = \frac{(x-1)\sin\left(\frac{n\pi x}{2}\right)}{2n\pi} + \frac{\cos\left(\frac{n\pi x}{2}\right)}{n^2\pi^2} + C$

16. $\int (x-4) \operatorname{arcctg} 2x \, dx \ominus$

$u = \operatorname{arcctg} 2x$
$dv = (x-4)\,dx$
$du = -\frac{2}{4x^2+1}\,dx$
$v = \int (x-4)\,dx = \frac{x^2}{2} - 4x$

$\left(\frac{x^2}{2} - 4x\right)\operatorname{arcctg} 2x + \int \frac{\left(\frac{x^2}{2} - 4x\right)\cdot 2}{4x^2+1}\,dx =$

$\int \frac{x^2 - 8x}{4x^2+1}\,dx = \int \frac{x^2}{4x^2+1}\,dx - \int \frac{8x}{4x^2+1}\,dx$

1) $\int \frac{x^2}{4x^2+1}\,dx = \int \frac{\frac{1}{4}\cancel{(4x^2+1)}}{\cancel{(4x^2+1)}} - \frac{\frac{1}{4}}{4x^2+1}\,dx =$

$\int \frac{1}{4} - \frac{1}{4(4x^2+1)}\,dx = \frac{1}{4}x - \frac{1}{4}\int \frac{1}{4x^2+1}\,dx$

$u = 2x$
$du = 2dx$
$dx = \frac{1}{2}du$

$= \frac{1}{4}x - \frac{1}{8}\int \frac{1}{u^2+1}\,du = \frac{1}{4}x - \frac{1}{8}\cdot \operatorname{arctg} 2x$

2) $8\int \frac{x}{4x^2+1}\,dx = \left| \begin{array}{l} u = 4x^2+1 \\ du = 8x\,dx \\ dx = \frac{1}{8x}du \end{array} \right. = \cancel{8}\cdot\frac{1}{\cancel{8}}\int \frac{1}{u}\,du = \ln|u| =$

$\ln|4x^2+1|$

$\ominus \left(\frac{x^2}{2} - 4x\right)\operatorname{arcctg} 2x + \frac{1}{4}x - \frac{\operatorname{arctg} 2x}{8} - \ln|4x^2+1| + C$

17. $\int x \operatorname{arcctg}\sqrt{x^2-1}\,dx = \frac{x^2}{2}\cdot\operatorname{arcctg}\sqrt{x^2-1} + \sqrt{\frac{x^2-1}{2}} + C$

$u = \operatorname{arcctg}\sqrt{x^2-1}$
$dv = x\,dx$
$du = -\frac{1}{2x^2}\frac{2x}{\sqrt{x^2-1}}\cdot dx = -\frac{1}{x\sqrt{x^2-1}}\,dx$
$v = \int x\,dx = \frac{x^2}{2}$

$\frac{x^2}{2}\cdot\operatorname{arcctg}\sqrt{x^2-1} + \int \frac{x^2}{2}\cdot\frac{1}{x\sqrt{x^2-1}}\,dx =$

$$\int \frac{x}{2\sqrt{x^2-1}}\,dx = \left| \begin{array}{l} u = x^2-1 \\ du = 2x\,dx \\ dx = \frac{du}{2x} \end{array} \right. =$$

$$\int \frac{1\cdot \cancel{x}}{2\sqrt{u}\cdot 2\cancel{x}}\,du = \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{\cancel{4}_2}\cdot \cancel{2}\sqrt{u} = \frac{\sqrt{u}}{2} = \frac{\sqrt{x^2-1}}{2}$$

18. $$\int e^{ax}\sin(bx)\,dx$$

$$\left| \begin{array}{l} u = \sin(bx) \\ dv = e^{ax}\,dx \\ du = b\cos(bx)\,dx \\ v = \int e^{ax}\,dx = \frac{e^{ax}}{a} \end{array} \right.$$

$$\frac{e^{ax}\cdot\sin(bx)}{a} - \int \frac{be^{ax}\cos bx}{a}\,dx =$$

$$\frac{b}{a}\int e^{ax}\cos(bx)\,dx = \left| \begin{array}{l} u = \cos(bx) \\ dv = e^{ax}\,dx \\ du = -b\sin(bx) \\ v = \frac{e^{ax}}{a} \end{array} \right. = \frac{e^{ax}\cdot\cos(bx)}{a} + \frac{b}{a}\int e^{ax}\sin(bx)\,dx$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\cdot\sin bx}{a} - \frac{be^{ax}\cos(bx)}{a^2} - \frac{b^2}{a^2}\int e^{ax}\sin(bx)\,dx + C$$

$$\left(1+\frac{b^2}{a^2}\right)\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\sin(bx)}{a} - \frac{be^{ax}\cos(bx)}{a^2} + C \;\Big|:$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\sin(bx)\,a^{\cancel{2}}}{\cancel{a}\,(a^2+b^2)} - \frac{be^{ax}\cos(bx)\,\cancel{a^2}}{\cancel{a^2}\,(a^2+b^2)} + C =$$

19. (между 11 и 12) $$\int \frac{dx}{\sqrt{8+6x-9x^2}} = \int \frac{dx}{\sqrt{-(-8-6x+9x^2)}}$$

$$(3x-1)^2 = 9x^2-6x-1$$

$$\int \frac{dx}{\sqrt{-(3x-1)^2+9}} = \left| \begin{array}{l} u = \frac{3x-1}{3} \\ du = \frac{1}{3}\cdot 3\,dx \\ dx = du \end{array} \right. = \int \frac{du}{\sqrt{9-9u^2}} = \frac{1}{3}\int \frac{1\cdot du}{\sqrt{1-u^2}}$$

$$\frac{1}{3}\cdot\arcsin\frac{3x-1}{3} + C$$