Continuous Random Variables

Def $f : \mathbb{R} \to \mathbb{R}$ $\left((-\infty, \infty) \longrightarrow (-\infty, \infty) \right)$

is said to be a probability density function (pdf) if

(1) $f(x) \geq 0 \quad x \in \mathbb{R}$

(2) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\left(\text{Recall} \quad \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \lim_{L \to \infty} \int_{-L}^{L} f(x)\,dx \right)$

Def $\{\Omega, \Sigma, P\}$ a probability space,

$X : \Omega \to \mathbb{R}$ is called a continuous random variable if there exists f (or f_X) a pdf such that $P(X \in A) = \displaystyle\int_A f(x)\,dx$

$$\Rightarrow \quad P(a < \underline{X} \le b) = \int_a^b f_{\underline{X}}(x)\,dx$$

<u>Note</u>

$$P(\underline{X} = a) = \lim_{\varepsilon \to 0} P(a-\varepsilon < \underline{X} \le a+\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} f_{\underline{X}}(x)\,dx$$

$$= \int_a^a f_{\underline{X}}(x)\,dx = 0.$$

$\Rightarrow$ For a continuous r.v. - $P(\underline{X} = a) = 0$

$$\Rightarrow \quad P(\underline{X} < a) = P(\underline{X} \le a), \text{ etc.}$$

$$P(\underline{X} \le a) = P(\{\underline{X} < a\} \cup \{\underline{X} = a\})$$

$$= P(\underline{X} < a) + P(\underline{X} = a)$$

$$= P(\underline{X} < a).$$

So for a continuous r.v.

$$P(a < X \le b) = P(a \le X \le b)$$

$$= P(a \le X < b) = P(a < X < b)$$

etc.

What happens at a sigle pT
does not matter.

For a continuous r.v. we never ask

$P(X = a)$ — always zero.

Def Cumulative Distribution Function.

$$F_X(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f_X(t)\,dt.$$

Note

$$P(\bar{X} \leq b) = P(\{\bar{X} \leq a\} \cup \{a < \bar{X} \leq b\})$$

$$= P(\bar{X} \leq a) + P(a < \bar{X} \leq b)$$

$$\Rightarrow P(a < \bar{X} \leq b) = P(\bar{X} \leq b) - P(\bar{X} \leq a)$$

$$= F(b) - F(a)$$

$$\Rightarrow P(a \leq \bar{X} \leq b) = P(a < \bar{X} \leq b)$$

$$= P(a < \bar{X} \leq b) = P(a < \bar{X} < b)$$

$$= P(a \leq \bar{X} < b) = \underline{F(b) - F(a)}$$

$$\Rightarrow P(\bar{X} \in [a, b]) = P(\bar{X} \in (a, b])$$

$$= P(\bar{X} \in [a, b)] = P(\bar{X} \in (a, b])$$

$$= F(b) - F(a)$$

Expectation

Def For $X \sim f_X$ and $g: \mathbb{R} \to \mathbb{R}$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$\Longrightarrow \quad \mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\sigma_X^2 = E[(X-\mu_X)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx = \mathrm{Var}(X).$$

Note:

$$\mathrm{Var}[X] = \sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2 - 2x\mu_X + \mu_X^2) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - 2\mu_X \underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx}_{\mu_X} + \mu_X^2 \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

$$= E[X^2] - 2\mu_X^2 + \mu_X^2$$

$$= E[X^2] - \mu_X^2.$$

$$\Rightarrow Var[X] = \sigma_X^2 = E[X^2] - \mu_X^2$$

Moment Generating Function

$$M_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

as in discrete case

$$E[X^k] = M_X^{(k)}(0)$$

So $\mu_X = E(X) = M_X'(0)$

$$\sigma_X^2 = E[X^2] - \mu_X^2$$

$$= M_X''(0) - M_X'(0)^2.$$

Not As in Discrete Case -

☆ $E[aX+b] = a\,E(X) + b$

$$\Rightarrow \mu_{aX+b} = a\mu_X + b$$

☆ $Var[aX+b] = a^2 Var(X)$

☆ $\sigma_{aX+b} = \sqrt{Var[aX+b]} = \sqrt{a^2 Var(X)}$

$$= |a|\,\sigma_X.$$

Why does the MGF work –

Note

$$M_X'(0) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx \Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} e^{tx} f_X(x)\,dx \Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} x e^{tx} f_X(x)\,dx \Big|_{t=0}$$

$$= \int_{-\infty}^{\infty} x e^{0 \cdot x} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx = E[X].$$

* Check higher derivatives and moments

Lemma If $\mathbb{X}$ a crv, $\mathbb{X} \geq 0$

Then

$$E[\mathbb{X}] = \int_0^\infty P(\mathbb{X} > x)\,dx$$

$$= \int_0^\infty 1 - F_{\mathbb{X}}(x)\,dx$$

Pf First note That

$$P(\mathbb{X} > x) = P(\{\mathbb{X} \leq x\}^c)$$

$$= 1 - P(\mathbb{X} \leq x)$$

$$= 1 - F_{\mathbb{X}}(x)$$

$$\Rightarrow \int_0^\infty P(\mathbb{X} > x)\,dx = \int_0^\infty 1 - F_{\mathbb{X}}(x)\,dx$$

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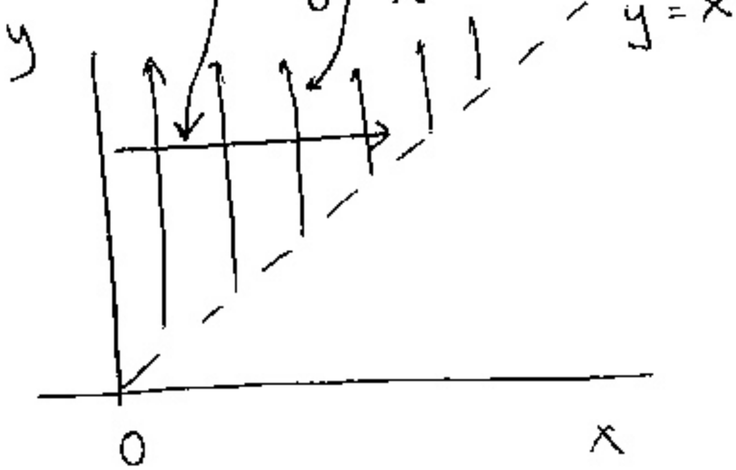
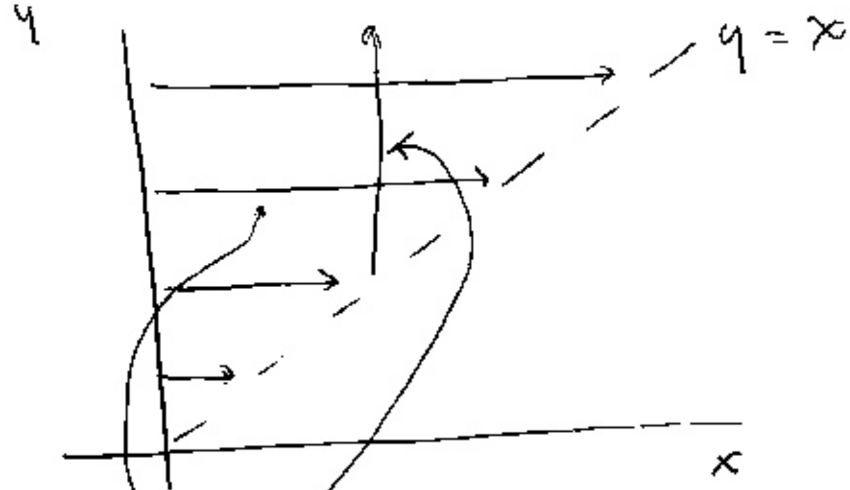
Now -

$$\int_0^\infty P(\bar{X} > x)\,dx = \int_0^\infty \int_x^\infty f_{\bar{X}}(y)\,dy\,dx$$



$$\Longleftrightarrow$$



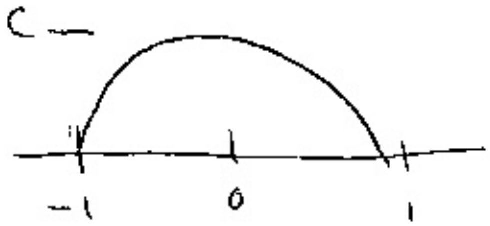
$$= \int_0^\infty \int_0^y f_{\bar{X}}(y)\,dx\,dy$$

$$= \int_0^\infty f_{\bar{X}}(y) \int_0^y dx\,dy = \int_0^\infty f_{\bar{X}}(y)\, x\Big|_0^y dy = \int_0^\infty y f_{\bar{X}}(y)\,dy$$

$$= E[\bar{X}].$$

Example

$$f_X(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$



1. What value of c will make f_X a pdf?

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c(1-x^2)\,dx + \int_{1}^{\infty} 0\,dx$$

$$= c\int_{-1}^{1}(1-x^2)\,dx = c\left\{x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= c\left\{1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right\}$$

$$= c\left(2 - \frac{1}{3} - \frac{1}{3}\right) = c\,\frac{4}{3}$$

$$\Rightarrow \quad c = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$\Rightarrow f_X(x) = \frac{3}{4}(1-x^2)\,\chi_{[-1,1]}(x) = \frac{3}{4}(1-x^2)\,\mathbb{I}_{[-1,1]}(x)$$

2. $P\left(-\frac{3}{2} < X < \frac{1}{2}\right)$

$$= \int_{-\frac{3}{2}}^{\frac{1}{2}} f_X(x)\,dx = \int_{-\frac{3}{2}}^{-1} 0\,dx + \int_{-1}^{\frac{1}{2}} \frac{3}{4}(1-x^2)\,dx$$

$$= \frac{3}{4}\int_{-1}^{\frac{1}{2}} 1 - x^2\,dx$$

$$= \frac{3}{4}\left(x - \frac{x^3}{3}\right)_{-1}^{\frac{1}{2}}$$

$$= \frac{3}{4}\left\{\left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3}\right) - \left(-1 - \frac{-1}{3}\right)\right\}$$

$$= \frac{3}{4}\left\{\left(\frac{1}{2} - \frac{1}{24}\right) - \left(-\frac{2}{3}\right)\right\}$$

$$= \frac{3}{4}\left\{\frac{11}{24} + \frac{16}{24}\right\}$$

$$= \frac{3}{4}\left\{\frac{27}{24}\right\} = \frac{27}{32}.$$

3. $$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy$$

$$= \begin{cases} \int_{-\infty}^{x} 0\,dy = 0 & -\infty < x \le -1 \\ \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{x} \frac{3}{4}(1-y^2)\,dy & -1 < x \le 1 \\ 1 & 1 < x < \infty \end{cases}$$

$$= \int_{-1}^{x} \frac{3}{4}(1-y^2)\,dy$$

$$= \frac{3}{4}\left(y - \frac{y^3}{3}\right)\Big|_{-1}^{x}$$

$$= \frac{3}{4}\left(x - \frac{x^3}{3}\right) - \frac{3}{4}\left(-1 - \frac{(-1)^3}{3}\right)$$

$$= \frac{3}{4}(x+1) - \frac{3}{4}\left(\frac{x^3}{3} + \frac{1}{3}\right)$$

$$F_X(x) = \begin{cases} 0 & x \le -1 \\ \frac{3}{4}(x+1) - \frac{3}{4}\left(\frac{x^3}{3} + \frac{1}{3}\right) & -1 < x < 1 \\ 1 & x > 1 \end{cases}$$

4. $$E[X] = \int_{-1}^{1} x \tfrac{3}{4}(1-x^2)\,dx$$

$$= \int_{-1}^{1} \tfrac{3}{4}x - \tfrac{3}{4}x^3\,dx$$

$$= \tfrac{3}{4}\tfrac{x^2}{2}\Big|_{-1}^{1} - \tfrac{3}{4}\tfrac{x^4}{4}\Big|_{-1}^{1}$$

$$= \tfrac{3}{8} - \tfrac{3}{8} - \tfrac{3}{4}\left(\tfrac{1}{4} - \tfrac{1}{4}\right) = 0 = \mu_X.$$

$$E[X^2] = \int_{-1}^{1} x^2 \tfrac{3}{4}(1-x^2)\,dx$$

$$= \tfrac{3}{4}\int_{-1}^{1} x^2\,dx - \tfrac{3}{4}\int_{-1}^{1} x^4\,dx$$

$$= \tfrac{3}{4}\tfrac{x^3}{3}\Big|_{-1}^{1} - \tfrac{3}{4}\tfrac{x^5}{5}\Big|_{-1}^{1}$$

$$= \tfrac{3}{4}\left(\tfrac{2}{3}\right) - \tfrac{3}{4}\left(\tfrac{2}{5}\right) = \tfrac{6}{12} - \tfrac{6}{20} = \tfrac{60-36}{120}$$

$$= \tfrac{24}{120} = \tfrac{1}{5}.$$

$$\Rightarrow \operatorname{Var}(X) = E[X^2] - 0^2$$

$$= \frac{1}{5} - 0 = \frac{1}{5}$$

$$\sigma_X = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$

5. $$M_X(t) = E[e^{tX}]$$

$$= \frac{3}{4}\int_{-1}^{1} e^{tx}(1-x^2)\,dx$$

$$= \begin{cases} \frac{3}{4}\left(\frac{e^t - e^{-t}}{t}\right) - \frac{3}{4}\left(\frac{e^t - e^t}{t} - \frac{2}{t^2}(e^t + e^{-t}) + 2\left(\frac{e^t - e^t}{t^3}\right)\right) & t \neq 0 \\ 1 & t = 0 \end{cases}$$

A Dictionary of Common crv's

1. Uniform
2. Exponential
3. Normal
4. Gamma
5. Beta
6. χ^2 (Chi squared)
7. Cauchy
8. Pareta.