HW #7

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4.46 A = passing, N = on-day

$P(A) = P(A|N)P(N) + P(A|N^c)P(N^c)$

$= P(A|N)(1/3) + P(A|N^c)(2/3)$

$P(A|N) = \sum_{i=\lceil n/2 \rceil}^{n} P(A=i|N) = \begin{cases} \sum_{i=3}^{5} \binom{n}{i} P_N^i (1-P_N)^{n-i}, & n=5 \\ \sum_{i=2}^{3} \binom{n}{i} P_N^i (1-P_N)^{n-i}, & n=3 \end{cases}$

$\sim B(n, P_N)$

$= \begin{cases} \binom{5}{3}(.8)^3(.2)^2 + \binom{5}{4}(.8)^4(.2)^1 + \binom{5}{5}(.8)^5 = .94208 \\ \binom{3}{2}(.8)^2(.2)^1 + \binom{3}{3}(.8)^3 = .896 \end{cases}$

$P(A|N^c) = \sum_{i=\lceil n/2 \rceil}^{n} P(A=i|N^c) = \begin{cases} \sum_{i=3}^{5} \binom{n}{i}(.4)^i(.6)^{n-i}, & n=5 \\ \sum_{i=2}^{3} \binom{n}{i}(.4)^i(.6)^{n-i}, & n=3 \end{cases}$

$= \begin{cases} \binom{5}{3}(.4)^3(.6)^2 + \binom{5}{4}(.4)^4(.6) + \binom{5}{5}(.4)^5 = .31744 \\ \binom{3}{2}(.4)^2(.6)^1 + \binom{3}{3}(.4)^3 = .352 \end{cases}$

$P(A_3) = 1/3(.94208) + 2/3(.31744) = .52565$

$P(A_5) = 1/3(.896) + 2/3(.352) = .416$

$P(A_3) > P(A_5)$

Thus, he should request an exam w/ 3 examiners.

4.4 (a) $\sum_{n=1}^{\infty} P\{X=n\} = \sum_{n=1}^{\infty} \frac{4}{n(n+1)(n+2)} = \frac{4}{1(2)(3)} + \frac{4}{2(3)(4)} + \frac{4}{(3)(4)(5)} + \cdots$

$= \frac{4}{6} + \frac{4}{24} + \frac{4}{60} + \frac{4}{120} + \frac{4}{210} + \cdots = \frac{2}{3} + \frac{1}{6} + \frac{1}{15} + \frac{1}{30} + \frac{2}{105} + \cdots$

$P(X=n) = \frac{4}{n(n+1)(n+2)} = 4\left[\frac{1}{n(n+1)} - \frac{1}{n(n+2)}\right] = 4\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n} - \frac{1}{n+2}\right)\frac{1}{2}\right]$

$\rightarrow \sum_{n=1}^{\infty} P(X=n) = \sum_{n=1}^{\infty} 4\left(\frac{1}{n} - \frac{1}{n+1}\right) - 2\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$= 4\sum_{n=1}^{\infty} \frac{1}{n} - 4\sum_{n=1}^{\infty} \frac{1}{n+1} - 2\sum_{n=1}^{\infty} \frac{1}{n} + 2\sum_{n=1}^{\infty} \frac{1}{n+2}$

$= 4(1) + 4\sum_{n=2}^{\infty} \frac{1}{n} - 4\sum_{n=1}^{\infty} \frac{1}{n+1} - 2\left(1 + \frac{1}{2}\right) - 2\sum_{n=3}^{\infty} \frac{1}{n} + 2\sum_{n=1}^{\infty} \frac{1}{n+2}$

$= 4 + (0) - 2(1.5) + (0) = 4 - 3$

$= \underline{1}$ $\blacksquare$

(b) $E[X] = \sum_{n=1}^{\infty} n \frac{4}{(n+1)(n+2)n} = \sum_{n=1}^{\infty} \frac{4}{(n+2)(n+1)} = \sum_{n=1}^{\infty} 4\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

$= 4\sum_{n=1}^{\infty} \frac{1}{n+1} - 4\sum_{n=1}^{\infty} \frac{1}{n+2} = 4\left(\frac{1}{2}\right) + (0) = \underline{2}$ $\blacksquare$

(c) $E[X^2] = \sum_a^b n^2 \frac{4}{n(n+1)(n+2)} = \sum_a^b \frac{4n}{\underbrace{(n+1)}_{\le 2n}\underbrace{(n+2)}_{\le 3n}} \rightarrow \sum_a^b \frac{4n}{(n+1)(n+2)} > \sum_a^b \frac{4n}{6n^2}$

$> \sum_a^b \frac{2}{3n}$

$\sum_a^b \frac{2}{3n} = \frac{2}{3} \sum_a^b \frac{1}{n} = \infty, \quad a: i=1, \ b = \infty$

$$\rightarrow \sum_{n=1}^{\infty} \frac{4n}{(n+1)(n+2)} = \infty \quad \checkmark \quad \blacksquare$$

4.20 $E[X^n] = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} j^n = \lambda \sum_{j=0}^{\infty} \frac{j^{n-1} j \lambda^{j-1} e^{-\lambda}}{j \cdot (j-1)!} = \lambda \sum_{j=0}^{\infty} \frac{j^{n-1} \lambda^{j-1} e^{-\lambda}}{(j-1)!}$

$= \lambda \sum_{j=1}^{\infty} \frac{j^{n-1} \lambda^j e^{-\lambda}}{(j-1)!} = \lambda \sum_{k=0}^{\infty} \frac{(k+1)^{n-1} \lambda^k e^{-\lambda}}{k!} = \lambda E[(X+1)^{n-1}] \quad \blacksquare$

$E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2 + 2X + 1] = \lambda(E[X^2] + E[2X] + E[1])$

$= \lambda([\lambda E[X+1]] + 2\lambda + 1)$

$= \lambda([\lambda[\lambda+1]] + 2\lambda + 1) = \lambda(\lambda^2 + \lambda + 2\lambda + 1) = \underline{\lambda^3 + 3\lambda^2 + \lambda}$

4.20 $E[1/X] = \sum_{k=1}^{\infty} \frac{p}{k}(1-p)^{k-1} = \frac{p}{1-p} \sum_{k=0}^{\infty} \frac{1}{k+1}(1-p)^{k+1}$

by hint, $\frac{p}{1-p} \sum_{k=0}^{\infty} \int_0^{1-p} x^k \, dx = \frac{p}{1-p} \int_0^{1-p} \sum_{k=0}^{\infty} x^k \, dx$

$\uparrow$ geom. series

$\Rightarrow \frac{p}{1-p} \int_0^{1-p} \frac{1}{1-x} \, dx = \frac{p}{1-p} \left(-\ln(1-x)\right)\Big|_0^{1-p}$

$= \frac{p}{p-1}[\ln p - \ln 1] = \frac{-p \log p}{1-p} \quad \blacksquare$