1a) $\underline{X} \sim B(10, .35)$ Declaration Signature: Adam v Ami

$$P_{\underline{X}}(x \geq 4) = 1 - P_{\underline{X}}(x \leq 3) = 1 - [P_{\underline{X}}(x=0) + P_{\underline{Y}}(x=1) + \ldots P_{\underline{Y}}(x=3)]$$

$$= 1 - \left[\sum_{i=0}^{3} \binom{10}{i}(.35)^i(.65)^{10-i}\right] = \boxed{.486173}$$

1b) $\underline{X} \sim NB(4, 0.35)$

$$E[\underline{X}] = \frac{r}{p} = \frac{4}{.35} = 11.4286 \rightarrow \boxed{\text{She should expect to call 12 people}}$$

1c) $\underline{X} \sim H(29, 17, 10)$

$$P_{\underline{X}}(x=5) = \frac{\binom{17}{5}\binom{12}{5}}{\binom{29}{10}} = \frac{3 \cdot 4 \cdot 4 \cdot 17}{5 \cdot 29 \cdot 23} = \boxed{.24468}.$$

2 Method 1)

$$\underline{X} = \sum_i \underline{X}_i, \quad \underline{X}_i \sim P(\lambda_i)$$

$$P_{\underline{X}_i}(x_i) = \frac{\lambda_i^{x_i}}{x_i!} e^{-\lambda_i} \qquad P_{\underline{X}}(\underbrace{x_1, x_2, x_3, \ldots, x_n}_{\text{independent}})$$

let $n=2 \rightarrow \underline{X}_1 = \underline{X}, \ \underline{X}_2 = \underline{Y}, \ \underline{X}_1 + \underline{X}_2 = \underline{Z}_1$

$$P_{\underline{Z}_1}(z) = P_{\underline{Z}}(\underline{X}=k, \underline{Y}=z-k) = \sum_{k=0}^{z} P(X=k) \cdot P(Y=z-k) = \sum_{k=0}^{z}\left(\frac{\lambda_x^k}{k!}e^{-\lambda_x} \cdot \frac{\lambda_y^{z-k}}{(z-k)!}e^{-\lambda_y}\right)$$

$$= e^{-\lambda_x}e^{-\lambda_y}\left(\sum_{k=0}^{z} \frac{\binom{z}{k}\lambda_x^k \lambda_y^{z-k}}{z!}\right) = \frac{e^{-\lambda_x}e^{-\lambda_y}}{z!}[\lambda_x + \lambda_y] = \frac{\prod_{i=1}^{n} e^{-\lambda_i}}{z!}\left[\sum_{j=1}^{n} \lambda_j\right]^z$$

2 Method 2)

$$\bar{X} = \sum_{i=1}^{n} \bar{X}_i \quad , \quad M_{\bar{X}_i}(t) = e^{\lambda_i (e^t - 1)}$$

since all $\bar{X}_i$ are independent,

$$M_{\bar{X}}(t) = \prod_{i=1}^{n} M_{\bar{X}_i}(t) = e^{\lambda_1(e^t-1)} e^{\lambda_2(e^t-1)} e^{\lambda_3(e^t-1)} \dots e^{\lambda_n(e^t-1)}$$

$$= e^{(e^t-1)(\lambda_1+\lambda_2+\dots\lambda_n)}$$

Therefore, $\bar{X}$ is also a Poisson-distributed R.V. with $\lambda = \sum_{i=1}^{n} \lambda_i$.

Thus, $$P_{\bar{X}}(x) = \frac{e^{-\lambda}}{x!} \lambda^x = \frac{e^{-(\lambda_1+\lambda_2+\dots\lambda_n)}}{x!} (\lambda_1+\lambda_2+\dots\lambda_n)^x$$

$$= \frac{e^{-\lambda_1} e^{-\lambda_2} \dots e^{-\lambda_n}}{x!} \left(\sum_{i=1}^{n} \lambda_i\right)^x = \underline{\frac{\prod_{i=1}^{n} e^{-\lambda_i}}{x!} \left(\sum_{i=1}^{n} \lambda_i\right)^x}$$

3a) $P_{\bar{X},N}(x,n) > 0$ for all values of $n \geq x$.

To have x successes, there must be at least x chances to achieve success. Moreover, if there are x chances to achive success, the probability of achieving x successes (if $p \neq 0$) is greater than zero.

3b) $$P_{\bar{X},N}(x,n) = P(x|n)P(n) = P(x|n)\left(\frac{\lambda^n}{n!} e^{-\lambda}\right) = \binom{n}{x} \frac{p^x (1-p)^{n-x} \lambda^n e^{-\lambda}}{n!}$$

3c) $$P_{\bar{X}}(x) = \sum_n \binom{n}{x} p^x (1-p)^{n-x} \lambda^n e^{-\lambda}/n! = \sum_n \frac{n!}{n!\,x!\,(n-x)!} p^{n+(x-n)} (1-p)^{n-x} \lambda^n e^{-\lambda}$$

$$= \sum_n \frac{n!}{n!\,x!} \frac{p^x (1-p)^{n-x}}{(n-x)!} \lambda^{n-x} \lambda^x e^{-\lambda} = \sum_n \frac{n!}{x!\,n!} p^x \frac{(\lambda(1-p))^{n-x}}{(n-x)!} \lambda^x e^{-\lambda} = e^{\lambda} e^{-\lambda p} \sum_n \frac{p^x \lambda^x e^{-\lambda}}{x!}$$

$$= \frac{e^{\lambda}}{e^{\lambda}} e^{-\lambda p} (p\lambda)^x \sum_n \frac{1}{x!} = \boxed{\frac{(\lambda p)^x}{x!} e^{-\lambda p}}$$

3c continued) since $\underline{Y} \sim P(\lambda p)$ has $P_{\underline{Y}}(y) = \frac{(\lambda p)^y}{y!} e^{-\lambda p}$

and the marginalization for $\underline{X}$ has distribution $\frac{(\lambda p)^x}{x!} e^{-\lambda p}$,

it is clear that $\underline{X} \sim P(\lambda p)$.

4a)

$\mathbb{I}_a : P(\{R=2 \text{ or } R=3\}) = \frac{2}{6} \frac{6}{6} = \boxed{1/3 = p}$

$\mathbb{I}_b : P(\{R+G=7\}) = \frac{6}{36} = \boxed{1/6 = q}$

$\mathbb{I}_c : P(\{R \le 3 \text{ and } G \le 4\}) = \frac{3}{6} \frac{4}{6} = \boxed{1/3 = r}$

4b) $E[\mathbb{I}_a] = 0(2/3) + 1(1/3) = \boxed{1/3}$ $\quad Var[\mathbb{I}_a] = 1^2(1/3) + 0^2(2/3) = \boxed{1/3}$

$E[\mathbb{I}_b] = 0(5/6) + 1(1/6) = \boxed{1/6}$ $\quad Var[\mathbb{I}_b] = 1^2(1/6) + 0^2(5/6) = \boxed{1/6}$

$E[\mathbb{I}_c] = 0(2/3) + 1(1/3) = \boxed{1/3}$ $\quad Var[\mathbb{I}_c] = 1^2(1/3) + 0^2(2/3) = \boxed{1/3}$

4c) $E[\mathbb{I}_a + \mathbb{I}_b] = ? = E[\mathbb{I}_A] + E[\mathbb{I}_B] - E(\mathbb{I}_A, \mathbb{I}_B)$

$= 1/3 + 1/6 - 2/36 = 4/9$

$$\left.\begin{array}{l} R=1, G=6 \\ R=2, G \in \{1,\dots,6\} \\ R=3, G \in \{1,\dots 6\} \\ R=4, G=3 \\ R=5, G=2 \\ R=6, G=1 \end{array}\right\} \quad \frac{4+6+6}{36} = \frac{16}{36} = \boxed{\frac{4}{9}}$$

$Var[\mathbb{I}_A + \mathbb{I}_B] =$

$Var(\mathbb{I}_A) + Var(\mathbb{I}_B) + 2(Var(\mathbb{I}_A \mathbb{I}_B) - \mu_{\mathbb{I}_A}\mu_{\mathbb{I}_B})$

$= \frac{1}{3} + \frac{1}{6} + 2(Var(\mathbb{I}_A \mathbb{I}_B) - \frac{1}{18})$

$= \frac{1}{2} - \frac{1}{9} + 2(0^2(34/36) + 1^2(P(2|3,7)))$

$= \frac{1}{2} - \frac{1}{9} + 2(2/36) = \frac{1}{2} - \frac{1}{9} + \frac{1}{9} = \boxed{\frac{1}{2}}$

4d)

G

	11	12	13	14	15	16
R	21	22	23	24	25	26
	31	32	33	34	35	36
	41	42	43	44	45	46
	...					...
	61	...				66

event C

looking for 9/36 of overall grid

3/12 of C

$P(\mathbb{I}_D = 1) = \frac{1}{4}$ $\quad P(\mathbb{I}_D \mathbb{I}_C = 1) = P(\mathbb{I}_D = 1) P(\mathbb{I}_C = 1) = 1/12$

$P(\mathbb{I}_C = 1) = 1/3$

$\boxed{D = \{R \le 3, G \ge 4\}}$

4e) $\sigma_{x+y} = \sqrt{\sigma^2_{x+y}} = \sqrt{Var(X) + Var(Y) - 2Cov(X,Y)}$

$X = \mathbb{I}_C, Y = \mathbb{I}_D$

$\sigma_{x+y} = \sqrt{1/3 + 3/16 - 2(0)} = \sqrt{(16+9)/48} = \sqrt{25/48}$

$= 5/4\sqrt{3}$

$Var(Y) = E[Y^2] - E[Y]^2 = 1/4 - 1/16 = 3/16$

$E[Y] = 1(1/4) = 1/4$

$E[Y^2] = 1^2(1/4) = 1/4$

$Cov(X,Y) = E[XY] - E[X]E[Y] = 1(1/12) + 0(11/12) - (1/3)(1/4)$

$= 1/12 - 1/12 = 0$

$$\boxed{\sigma_{\mathbb{I}_C + \mathbb{I}_D} = \frac{5}{4\sqrt{3}}}$$