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MODALITIES AND QUANTIFICATION

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1. The problems of modal logic. The purpose of this article is to give a survey of some results I have found in investigations concerning logical modalities. The results refer: (1) to semantical systems, i.e., symbolic language systems for which semantical rules of interpretation are laid down; (2) to corresponding calculi, i.e., syntactical systems with primitive sentences and a rule of inference; (3) to relations between a semantical system and the corresponding calculus.

The semantical systems to be dealt with are the following: propositional logic (PL), functional logic (FL), and the corresponding modal systems, viz. modal propositional logic (MPL) and modal functional logic (MFL). MPL is built out of PL by the addition of the symbol 'N' for logical necessity; likewise MFL out of FL. In terms of Lewis's symbol '$\Diamond$' for logical possibility, 'Np' means the same as '$\sim\Diamond\sim p$'. All other logical modalities can, of course, be defined on the basis of 'N'; e.g., impossibility by 'N$\sim p$', possibility by '$\sim$N$\sim p$', contingency by '$\sim$Np.$\sim$N$\sim p$', etc.

The calculi corresponding to these semantical systems are the following: the propositional calculus (PC), the functional calculus (FC), and the modal calculi (MPC and MFC) again constructed by the addition of 'N'.

Lewis's systems of strict implication[1] are forms of MPC. So far, no forms of MFC have been constructed, and the construction of such a system is our chief aim. The corresponding semantical systems MPL and MFL are constructed chiefly for the purpose of enabling us to show that the modal calculi MPC and MFC are adequate, i.e., that every sentence provable in them is L-true (analytic). With the help of a normal form, it can further be shown that for MPC the inverse holds also; MPC is complete in the sense that every sentence which is L-true in MPL is provable in MPC. The reduction to the normal form constitutes a decision method for MPC and MPL. For MFC likewise a method of reduction to a normal form will be given. This reduction removes all occurrences of 'N' of higher order, i.e., such that the scope of one 'N' contains another 'N'. A decision method for MFC is of course not possible; however, the reduction makes it possible to apply to MFC the known decision methods for special cases in FC.

The semantical systems FL and MFL contain an infinite number of individual constants. Therefore the representation of these systems requires a very strong metalanguage, dealing with classes of classes of sentences. Consequently, the semantical concepts defined, e.g., L-truth, are indefinite (non-effective) to a high degree. The chief reasons for constructing corresponding calculi are here, as usually in the case of logical calculi, the following two: (1) avoidance of any reference to the meanings of the signs and sentences, (2) use of basic concepts which are effective. The second purpose is here, as generally in the case of calculi without transfinite rules, fulfilled in the following sense. Al-

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[1] C. I. Lewis and C. H. Langford, ***Symbolic logic,*** 1932; the systems are developed from those in Lewis's earlier book (1918).

though C-truth (provability) is not itself an effective concept, it is defined on the basis of two effective concepts, viz. the concept of primitive sentence, given by a finite list of primitive sentence schemata, and the concept of direct derivability, defined by the rule of inference.

For lack of space, this article will state only a few of the relevant theorems, most of them without proofs. For the same reason, this article will be restricted to the technical aspects of the systems dealt with and will not contain any discussion of the more general problems connected with logical modalities.

The guiding idea in our constructions of systems of modal logic is this: a proposition p is logically necessary if and only if a sentence expressing p is logically true. That is to say, the modal concept of the logical necessity of a proposition and the semantical concept of the logical truth or analyticity of a sentence correspond to each other. Both concepts have been used in logic and philosophy, mostly, however, without exact rules. If we succeed in explicating one of these two concepts, that is, in finding an exact concept, which we call the explicatum, to take the place of the given inexact concept, the explicandum, then this leads, on the basis of the parallelism stated between the two concepts, to an explication for the other concept. Now it is easy to give, with the help of the semantical concepts of state-description and range, an exact definition for 'L-true' as an explicatum for logical truth with respect to the systems PL and FL, as we shall see. Therefore it seems natural to interpret 'N' in such a way that the following convention is always fulfilled:

C1-1. If '$\cdots$' is any sentence in a system S containing 'N', then the corresponding sentence 'N($\cdots$)' is to be taken as true if and only if '$\cdots$' is L-true in S.

This convention determines our interpretation of 'N', but it is not a definition for 'N'. The sentence 'N($\cdots$)' cannot be transformed by definition into the sentence "'$\cdots$' is L-true in S," because the first sentence belongs to the object-language S, the second to the metalanguage M; but the first sentence holds, according to the convention, if and only if the second holds. We shall not define 'N' (it cannot be defined on the basis of the ordinary truth-functional connectives and quantifiers for individuals) but shall take it as a primitive sign in MPL and MFL. However, we shall frame the semantical rules of these systems in such a manner that the convention is fulfilled.[2, 3]

C1-1 gives a sufficient and necessary condition for the truth of 'N($\cdots$)'. Now the following two questions remain: (1) if 'N($\cdots$)' is true, is it L-true? If so, 'NN($\cdots$)' is likewise true; in other words, '$\mathrm{N}p \supset \mathrm{NN}p$' is always true.

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[2] I shall hereafter refer to the following publications of mine by the signs in square brackets:

[Syntax] *The logical syntax of language,* (1934) 1937.

[I] *Introduction to semantics,* 1942.

[II] *Formalization of logic,* 1943.

[3] I have indicated the parallelism between the modal concept of the necessity of a proposition and the meta-concept of the analyticity of a sentence first in [Syntax] §69 (where, however, 'analytic' was still regarded as a syntactical term), and, more clearly, in [I] pp. 91 ff.

(2) If 'N(· · ·)' is false, is it L-false? ('L-false' is taken as the explicatum for 'logically false,' 'self-contradictory.') If so, '∼N(. . .)' is L-true and hence 'N∼N(· · ·)' is true; in other words, '∼Np ⊃ N∼Np' is always true.

At the present moment, these questions are not meant with respect to any given system, but as pre-systematic questions, concerning the inexact, pre-systematic explicandum rather than the exact explicatum. The purpose of the following considerations is merely to make the vague meaning of logical necessity or logical truth clearer to ourselves, so as to lead to a convention more specific than C1-1 concerning the use of 'N'. This convention will then later guide us in constructing our systems. Once the systems are constructed, the two questions can be answered in an exact way. At the present stage, however, our considerations, as always in tasks of self-clarification, are necessarily inexact and, in a certain sense, even circular.

In order to make clearer what is meant by the explicandum of logical, necessary truth, we will distinguish two kinds of data concerning any sentence 'C' as follows:

I. The meaning of 'C' is given, that is to say, the interpretation assigned to 'C' by the semantical rules. (In technical terms, the rules may either be formulated so as to determine the proposition expressed by 'C' or so as to determine the range of 'C'; the rules of our system will have the latter form.)

II. Information concerning the facts relevant for 'C' is given, that is to say, concerning the properties and relations of the individuals involved.

If the answer to a given question is merely dependent upon data of the kind I but independent of those of kind II, we call it a logical question; if in addition, data of the kind II are required, we call it a factual question. In particular, if a sentence 'C' is true in such a way that its truth is based on I alone, we regard it as logically true; if its truth is dependent upon II also, we regard it as factually, contingently true. This conception of the distinction between logical and factual truth as explicandum will guide our choice of the definition of 'L-true' as explicatum. It seems to me that this conception is in agreement with customary conceptions.

Let us take as an example the sentence 'Pa.∼Qb', which we abbreviate by 'A'. We learn from the semantical rules what individuals are named by 'a' and 'b' and what properties are designated by 'P' and 'Q'; we learn further that 'A' says that a is P and b is not Q. This is all we can obtain from data of the kind I. In order to establish the truth-value of 'A' we need data of the kind II, viz., information whether or not a is P and whether or not b is Q. Thus, 'A' is neither L-true nor L-false; we say that it is L-indeterminate or factual.

(i) Now consider the sentence 'A ∨ ∼A'. We can find that it is true by using merely the semantical rules for '∨' and '∼' (in our system, the rules of ranges D7-5c and b, which correspond to the customary truth-tables for the two connectives); we need no factual information concerning the individuals a and b occurring in the sentence. Therefore, 'A ∨ ∼A' is L-true. Hence, according to our convention C1-1, 'N(A ∨ ∼A)' is true. The question is whether it is L-true. Now we can easily see that it must be, because the truth of this N-sentence follows from those semantical rules by which we established the truth and hence the L-truth of 'A v ∼A' together with the semantical rule for 'N' which is to be laid down in accordance with C1-1. Thus no factual knowledge is required for establishing that 'N(A v ∼A)' is true; hence it is L-true.

(ii) Similarly, the falsity of 'A.∼A' can be established by the semantical rules alone. Therefore, this sentence is L-false and not L-true. Hence, according to C1-1, 'N(A.∼A)' is false and '∼N(A.∼A)' is true.

(iii) Finally, let us go back to the sentence 'A' itself, i.e., 'Pa.∼Qb'. We found that 'A' is neither L-true nor L-false by merely using semantical rules, not using any factual knowledge concerning the individuals occurring in 'A'. Therefore we see that, according to C1-1, 'N(A)' is false and '∼N(A)' is true. These results are based merely on the semantical rules for the signs occurring in 'A' and for 'N'. Therefore, 'N(A)' is L-false and '∼N(A)' is L-true.

The results found for these simple examples can be generalized. Let 'C' be an abbreviation for a given sentence of any form with or without 'N'.

(i) Suppose that 'N(C)' is true. Then, according to C1-1, 'C' must be L-true. Hence the truth of 'C' is determined by certain semantical rules. Then these same rules together with the rule for 'N' determine the truth of 'N(C)'. Therefore, 'N(C)' is L-true, and hence 'NN(C)' is true. Thus our earlier question (1) is answered in the affirmative.

(ii) Let us now suppose that 'C' is L-false and hence 'N(C)' is false. Then those semantical rules which determine the falsity of 'C' together with the rule for 'N' determine the falsity of 'N(C)'. Therefore, 'N(C)' is L-false, and '∼N(C)' is L-true.

(iii) Finally, let us suppose that 'N(C)' is false but 'C' is not L-false. Then 'C' is neither L-true nor L-false. The decisive question here is this: is the result that 'C' is not L-true determined by data I alone or are data II, i.e., factual knowledge, required? Data II are certainly relevant for the truth-value of 'C', but they cannot be relevant for the character of 'C' being L-indeterminate, factual, contingent. It would be absurd to assume such a relevance, to say, for example: " 'C' is contingent because the individual c happens to have the property Q; if this were not so then 'C' would not be contingent but L-true." Thus contingent facts, by being relevant for the contingency of 'C' would also be relevant for L-truth or L-falsity, in contradiction to our explanation of these concepts. Since now data I alone determine that 'C' is not L-true, they determine that 'N(C)' is false and '∼N(C)' is true. Therefore, 'N(C)' is L-false, and '∼N(C)' is L-true.

From (ii) and (iii) together we see that, if 'N(C)' is false, it is L-false. Thus our earlier question (2) is answered in the affirmative. Together with the result under (i), this leads to the following convention, which is more specific than C1-1.

C1-2. If '···' is L-true, 'N(···)' is L-true; otherwise 'N(···)' is L-false.

We shall later construct the rule for 'N' (D9-5i) in such a manner that this convention is fulfilled (T9-1).

In the preceding analysis, I have repeatedly referred to a certain result as "following from" or "determined by" certain data. This is not meant in the sense that the result can be derived from the data with the help of deductive means which are systematized in a given metalanguage; still less is it implied that there is an effective method for this derivation. What I mean is rather that, if the data hold, the result cannot possibly fail to hold. This is the wide, non-systematized concept of logical implication which logicians have in mind before they construct their systems and of which only a part can be grasped in any one fixed system. This concept is necessarily inexact; but it is clear enough for practical purposes of pre-systematic discussions. Logicians refer to it as their explicandum before they offer their explicatum in the form of a system with exact rules; so does, for example, Lewis when he explains that his explicatum 'strict implication' is intended to systematize the common concept of logic, implication, deducibility, entailment.

The opinions of logicians on the two questions (1) and (2) mentioned earlier seem to differ, with no clear arguments on either side. Our affirmative answers to these questions do not mean that a negative answer to either question is wrong but only that it must be based on an interpretation of 'N' different from ours. It seems to me that the usual discussions of the validity of various systems of modal logic are inconclusive because no clear interpretation of the modal signs are offered. The interpretation here suggested, based on the parallelism between the necessity of propositions and the L-truth of sentences and on the distinction between data of kinds I and II, leads to clear solutions of the controversial questions. This interpretation seems to be in agreement with customary conceptions; it is, indeed, nothing else than a clarification of these conceptions. I am not aware of any clear and simple interpretation which leads to one of the alternative systems, e.g., to a negative answer to either of the questions (1) or (2).

Since we intend to combine modalities with quantification, we have also to decide how to interpret a sentence of the form '$(x)[\mathrm{N}(\cdot\cdot x\cdot\cdot)]$'. This is done by the following convention.

C1-3. 'N' is to be interpreted in such a way that any sentence of the form '$(x)[\mathrm{N}(\cdot\cdot x\cdot\cdot)]$' is regarded as L-equivalent to (i.e., meaning the same as) the corresponding sentence '$\mathrm{N}[(x)(\cdot\cdot x\cdot\cdot)]$'.

The reasons for this convention (that is, "the motives for its choice," not "the proof of its validity") are as follows. We adopt the principle, which seems generally accepted, that any sentence with a universal quantifier, no matter whether it contains modal signs or not, is to be interpreted as a joint assertion for all values of the variable. Thus, if 'x' has only three values, say a, b, c, then '$(x)[\mathrm{N}(\cdot\cdot x\cdot\cdot)]$' means the same as '$\mathrm{N}(\cdot\cdot\mathrm{a}\cdot\cdot).\mathrm{N}(\cdot\cdot\mathrm{b}\cdot\cdot).\mathrm{N}(\cdot\cdot\mathrm{c}\cdot\cdot)$'. The latter sentence is L-equivalent to '$\mathrm{N}[(\cdot\cdot\mathrm{a}\cdot\cdot).(\cdot\cdot\mathrm{b}\cdot\cdot).(\cdot\cdot\mathrm{c}\cdot\cdot)]$' (see below, T4-1e), because a conjunction is L-true if and only if all of its components are L-true. Finally, the sentence last mentioned is L-equivalent to '$\mathrm{N}[(x)(\cdot\cdot x\cdot\cdot)]$', because, in virtue of the above principle, '$(x)(\cdot\cdot x\cdot\cdot)$' means the same as '$(\cdot\cdot\mathrm{a}\cdot\cdot).(\cdot\cdot\mathrm{b}\cdot\cdot).(\cdot\cdot\mathrm{c}\cdot\cdot)$'. It seems natural to transfer this result also to variables with a denumerably infinite range of values, as in MFL. I think that the same convention could be applied even to variables with a non-denumerable range, e.g., real number variables, although in this case not all values are expressible in the language; this problem, however, need not concern us in the present context. [It may be remarked that the foregoing discussion over-simplifies the situation. The actual situation is complicated by the fact that the values of individual variables in a modal system are not individuals but individual concepts; compare the remarks at the end of §12. However, for the present purposes we may leave aside this distinction.]

2. Propositional logic (PL) and propositional calculus (PC). We shall not construct PL as a separate system. Instead, we shall say of certain systems that they "contain PL" (roughly speaking, if they contain the ordinary connectives and semantical rules for them corresponding to the ordinary truth-tables) and of some of their sentences that they are "L-true by PL" (roughly speaking, if their truth can be shown by the truth tables alone).[4] For the sake of brevity, we omit here the exact definitions. The definition for "S contains PL" requires that S contain rules like D7-1a, b, c, D7-2c, d, e, D7-3, D7-5b, c, d, h. The definition for "L-true by PL" corresponds to D7-6a but is framed in such a way that it applies only if the universality of a range is based on the mentioned rules of ranges (D7-5b, c, d) which correspond to the ordinary truth-tables. Thus we shall see that FL, and likewise MFL, contain PL.

As primitive signs in all our systems we shall use the connectives '$\sim$', '$\vee$', and '$\cdot$'; and also the tautologous sentence 't'. ('$\cdot$' could of course be defined; likewise 't', e.g., by 'Pa $\vee$ $\sim$Pa'. We take them, nevertheless, as primitive in order to be able to write the normal forms in primitive notation; see D3-1.) In the following discussions, we make use also of '$\supset$' and '$\equiv$'; they do not belong to the systems themselves but serve as shorthand in the customary manner. We adopt the customary conventions for the omission of parentheses; in particular, we write 'Np' instead of 'N(p)' (but not if a compound sentence takes the place of 'p').

We use *German letters* as signs of the metalanguage: '$\mathfrak{S}$' for sentences (all sentences in our system are closed, i.e., without free variables); '$\mathfrak{M}$' for matrices (by which we mean here always sentential matrices (Quine), sometimes called "sentential functions"; they include the sentences); '$\mathfrak{in}$' for individual constants; '$\mathfrak{i}$' for individual variables; and occasionally others. Any expression containing a German letter belongs to the metalanguage and denotes the corresponding expression of the object-language in the customary way; e.g., '$\mathfrak{S}_i \vee \mathfrak{S}_j$' denotes the disjunction with the components $\mathfrak{S}_i$ and $\mathfrak{S}_j$.

We say that $\mathfrak{S}_i$ is L-false by PL, that $\mathfrak{S}_i$ L-implies $\mathfrak{S}_j$ by PL, or that $\mathfrak{S}_i$ is L-equivalent to $\mathfrak{S}_j$ by PL if and only if $\sim\mathfrak{S}_i$, $\mathfrak{S}_i \supset \mathfrak{S}_j$, or $\mathfrak{S}_i \equiv \mathfrak{S}_j$, respectively, is L-true by PL. (These simple definitions can be used here because all sentences are closed.)

In analogy to PL, we do not take PC as a separate calculus, since our systems will not contain propositional variables. We define instead "the calculus K contains PC." The definition requires that K contain the signs, matrices and sentences required for PL; further the rule of implication (as D4-2e), and certain primitive sentences (e.g., 't' and all sentences formed by substitution from Bernays's[5] four axioms of the propositional calculus). We say that $\mathfrak{S}_i$ is C-true by PC in K if $\mathfrak{S}_i$ is C-true (provable) in a sub-calculus K' of K contain-

[4] For this concept, the term 'tautologous' is sometimes used; see W. V. Quine, ***Mathematical logic,*** 1940, p. 50, and [I], p. 240. In [II] D11-30, I have used the term 'L-true by NTT.'

[5] D. Hilbert and W. Ackermann, ***Grundzüge der theoretischen Logik,*** (1928) 1938. P. Bernays, ***Mathematische Zeitschrift,*** vol. 25 (1926).

ing all sentences of K but only the rule of implication and those primitive sentences required for PC.

Definitions for other C-concepts (which we shall not state here) are constructed in such a way that they lead to the following results: $\mathfrak{S}_i$ is C-false (refutable) by PC, $\mathfrak{S}_i$ C-implies $\mathfrak{S}_j$ ($\mathfrak{S}_j$ is derivable from $\mathfrak{S}_i$) by PC, or $\mathfrak{S}_i$ is C-equivalent to $\mathfrak{S}_j$ (mutually derivable) by PC if and only if $\sim\mathfrak{S}_i$, $\mathfrak{S}_i \supset \mathfrak{S}_j$, or $\mathfrak{S}_i \equiv \mathfrak{S}_j$, respectively, is C-true by PC.

The following theorem T2-1a is well known; it is easily proved by examining the primitive sentences and the rule of inference of PC.

T2-1. Let the semantical system S and the calculus K contain the same signs and the same sentences. Let S contain PL and K contain PC.

a. Any sentence which is C-true by PC is L-true by PL.

b. Whenever C-falsity by PC holds, then L-falsity by PL holds. Analogously with C-implication and L-implication, and with C-equivalence and L-equivalence. Hence, in a certain sense, PL is an L-true interpretation of PC ([I] D34-1, [II] §14). (We add "in a certain sense," because PL and PC have not been defined here as independent systems.)

3. P-reduction. We shall now lay down rules of reduction leading to a normal form (definition D3-1). This is, essentially, the customary transformation into a conjunctive normal form. The present procedure is, however, considerably simplified, because by the use of 't' we get rid of many parts of a sentence which otherwise make the transformation very cumbersome. This method of reduction is applicable both to PL and to PC; therefore we use the neutral term 'P-reduction.' The rules are formulated so as to apply not only to sentences but to all matrices; this will be needed later in FC and MFC (see D8-1a). We presuppose that the customary notation of multiple disjunctions and conjunctions is used; this simplifies the procedure. (Thus we presuppose, for example, that instead of '(A ∨ B) ∨ C', the simple form 'A ∨ B ∨ C' is written; and further, that 'A ∨ (B ∨ C)' is transformed, according to the associative law, into '(A ∨ B) ∨ C' and hence likewise simplified to 'A ∨ B ∨ C'.)

D3-1. The *P-reduction* of a matrix $\mathfrak{M}_i$ is its transformation according to the following rules. At any step in the transformation, the first of these rules that can be applied must be applied. The replacement applies to any part having the specified form, provided this part is either the whole matrix or one of those components out of which the whole is built up with connectives. The final result to which no rule is applicable any more is called the *P-reductum* of $\mathfrak{M}_i$.

a. Any disjunction containing among its components a matrix and its negation is replaced by 't'.

b. Any conjunction containing among its components a matrix and its negation is replaced by '$\sim$t'.

c. $\sim\sim\mathfrak{M}_k$ is replaced by $\mathfrak{M}_k$.

d. If a disjunction or a conjunction contains among its components several occurrences of the same matrix, then all occurrences except the first one are omitted. (This rule is inessential but simplifies the form.)

e. If 't' is a component of a disjunction, then all other components of this disjunction are omitted.

f. If '$\sim$t' is a component of a disjunction, it is omitted.

g. If 't' is a component of a conjunction, it is omitted.

h. If '$\sim$t' is a component of a conjunction, then all other components of this conjunction are omitted.

i. The negation of a disjunction is replaced by the conjunction of the negations of the components.

j. The negation of a conjunction is replaced by the disjunction of the negations of the components.

k. If a conjunction occurs as a component of a disjunction, it is distributed (e.g., $\mathfrak{M}_h \vee (\mathfrak{M}_{k1}.\mathfrak{M}_{k2}) \vee \mathfrak{M}_l$ is replaced by $(\mathfrak{M}_h \vee \mathfrak{M}_{k1} \vee \mathfrak{M}_l).(\mathfrak{M}_h \vee \mathfrak{M}_{k2} \vee \mathfrak{M}_l)$).

T3-1. Let S and K be as in T2-1. Let $\mathfrak{S}_i$ be any sentence in S and K, and let $\mathfrak{S}_j$ be its P-reductum; then the following holds.

a. $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are C-equivalent by PC. (This follows by induction from the fact that C-equivalence holds for each application of one of the rules of reduction in D3-1.)

b. $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are L-equivalent by PL. (From (a) and T2-1b.)

c. If $\mathfrak{S}_j$ is 't', $\mathfrak{S}_i$ is L-true by PL. (From (b).)

d. $\mathfrak{S}_i$ is L-true by PL if and only if $\mathfrak{S}_j$ is 't'.

Proof. Let $\mathfrak{S}_i$ be L-true by PL (tautologous). It is well known[6] that the application of the reduction rules D1c, i, j, k to $\mathfrak{S}_i$ leads to a conjunctive normal form, in which every disjunction contains a sentence and its negation as components. Therefore, the rules D1a and g lead to 't'.—The converse is (c).

e. If $\mathfrak{S}_j$ is 't', $\mathfrak{S}_i$ is C-true by PC. (From (a).)

f. *Completeness of PC.*[7] If $\mathfrak{S}_i$ is L-true by PL, it is C-true by PC. (From (d), (e).)

g. Whenever L-falsity by PL holds, then C-falsity by PC holds. Analogously with L-implication and C-implication, and with L-equivalence and C-equivalence. Hence, in a certain sense, PC is an L-exhaustive calculus for PL ([I] D36-3). (From (f).)

4. Modal propositional logic (MPL) and modal propositional calculus (MPC). MPL consists of PL with the addition of the modal symbol 'N' for logical necessity. Here again, we do not define MPL as an independent system but only: "The system S contains MPL." The definition (which will not be given here) requires the same rules as those for PL but with the following addition: the modal symbol 'N' is added (D9-1h, D9-2g); and the rule of ranges for it (as D9-5i) says that the range of N$\mathfrak{S}_i$ is the universal range if the range of $\mathfrak{S}_i$ is the universal range; otherwise it is the null range. This is in accord with our previous convention C1-2, since L-truth is defined by the universality of the range (D7-6a). Our system MFL (§9) will contain MPL.

In analogy to 'L-true by PL' we now define 'L-true by MPL':

D4-1. $\mathfrak{S}_i$ is *L-true by MPL* in $S =_{\mathrm{Df}} S$ contains MPL; $\mathfrak{S}_i$ belongs to S; every sentence formed out of $\mathfrak{S}_i$ in the following way is L-true in S: the ultimate MPL-components of $\mathfrak{S}_i$ (i.e., those sentences out of which $\mathfrak{S}_i$ is built up with

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[6] See, e.g., Hilbert and Ackermann, op.cit., Kap. I, §§3 and 4, or Hilbert and Bernays, *Grundlagen der Mathematik,* vol. I, 1934, pp. 53f.

[7] The completeness of PC was first proved by E. L. Post, *American journal of mathematics,* vol. 43 (1921). See W. V. Quine, this JOURNAL, vol. 3 (1938), pp. 37ff.

the help of connectives and 'N', which themselves, however, are not thus built up out of other sentences) are replaced by any sentences of S (occurrences of the same component to be replaced by occurrences of the same sentence).

L-falsity by MPL, L-implication by MPL, and L-equivalence by MPL are defined in analogy to the corresponding concepts for PL in §2.

Now we shall define: "The calculus K contains MPC." This will later be applied to MFC. We here make use of 'p', 'q', etc. as *auxiliary variables*; that is to say, these letters do not belong to the signs of our calculi but are merely used (following Quine) for the description of certain forms of sentences. We say, e.g., that a sentence of K has the form '$p \supset q$' if it is formed out of the auxiliary formula '$p \supset q$' (which does not belong to K) by substituting for 'p' and 'q' any sentences of K (which do not contain 'p', 'q', etc.); for instance, 'Pa $\supset$ N(Pb)'. We write '$p \;\underline{\supset}\; q$' and '$p \equiv q$' as abbreviations for 'N($p \supset q$)' and 'N($p \equiv q$)', respectively. Thus '$\underline{\supset}$' is a symbol of strict implication (corresponding to Lewis's '$-\!\!3$'); and '$\underline{\equiv}$' is a symbol of strict equivalence (or identity of propositions, corresponding to Lewis's '='). The essential features of the form of MPC here stated are due to M. Wajsberg.[8]

D4-2. K contains MPC $=_{\mathrm{Df}}$ K is a calculus fulfilling the following conditions.

a. The following *signs* are among the signs of K:

a1. Connectives: '$\sim$', '$\vee$', '.'.

a2. Parentheses: '(', ')'.

a3. 't'.

a4. 'N'.

b. If $\mathfrak{M}_i$ and $\mathfrak{M}_j$ are matrices in K, then all expressions of the following forms are *matrices* in K:

b1. 't'.

b2. $\sim(\mathfrak{M}_i)$.

b3. $(\mathfrak{M}_i) \vee (\mathfrak{M}_j)$.

b4. $(\mathfrak{M}_i).(\mathfrak{M}_j)$.

b5. $\mathrm{N}(\mathfrak{M}_i)$.

c. All the closed matrices in K, and only these, are *sentences* in K.

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[8] An earlier system MPC, which I constructed in 1940, was slightly different from the one here given; I constructed a proof for its completeness with the help of the reduction procedure explained in the next section. I found later that my system was equivalent to, but simpler than, Lewis's system S5. While writing this article, I found that M. Wajsberg had given a still simpler form (*Ein erweiterter Klassenkalkül*, ***Monatshefte für Mathematik und Physik***, vol. 40 (1933), pp. 113–126); therefore I now adopt (in D4-2d) his axioms, with the following two inessential changes. (1) I take (d1), where Wajsberg takes the four axioms of the propositional calculus of Hilbert and Ackermann with the symbol of necessity added to each. (2) In (d3) I have '$\supset$', while Wajsberg has a symbol corresponding to '$\underline{\supset}$'; this change is due to the fact that I do not use, like Wajsberg, a rule of strict implication but a rule of material implication in order to have MPC and MFC contain PC. Wajsberg's calculus is primarily intended as a class calculus with a symbol for class universality added to it, but he remarks that it can also be interpreted as an extended propositional calculus corresponding to Lewis's systems of strict implication. In this interpretation, Wajsberg's '$|X|$' corresponds to 'Np', and therefore his '$X < Y$' to my '$p \;\underline{\supset}\; q$' and to Lewis's '$p -\!\!3\; q$'. In the same paper, Wajsberg gave a proof for the completeness of his calculus (see below, T6-2f).

d. K contains among its primitive sentences all sentences of the following forms; we call them *primitive sentences of MPC* in K:

d1. N$\mathfrak{S}_i$, where $\mathfrak{S}_i$ is any sentence whose P-reductum is 't'.

d2. '$(p ⥽ q) ⥽ (\mathrm{N}p ⥽ \mathrm{N}q)$'.

d3. '$\mathrm{N}p \supset p$'.

d4. '$\sim\mathrm{N}p ⥽ \mathrm{N}\sim\mathrm{N}p$'.

e. K contains among its rules of inference the *rule of implication*: $\mathfrak{S}_j$ is a direct C-implicate of (directly derivable from) $\mathfrak{S}_i$ and $\mathfrak{S}_i \supset \mathfrak{S}_j$.

f. *Rule of refutation*: the class of all sentences is directly C-false in K.

(f) is the rule of refutation of the simplest form (see [II] §20). If we are willing to dispense with C-falsity, in order to have a form more similar to customary calculi, we may omit this rule.

C-truth (provability) by MPC is defined in the customary way with reference to the primitive sentences (d) and the rule of inference (e). C-falsity, C-implication, and C-equivalence are defined in such a way that the conditions mentioned in §2 are fulfilled.

We shall now state some theorems of MPC without proofs. We use again the auxiliary variables 'p' etc. for convenience and easier comparison with modal calculi of other authors. ('d1' etc. refer to D4-2d1 etc.)

T4-1. Let K be a calculus containing MPC. Every sentence in K of any of the following forms is *C-true by MPC* in K.

a. '$(p ⥽ q) \supset (\mathrm{N}p ⥽ \mathrm{N}q)$. (From d2, d3.)

b. '$\sim\mathrm{N}p \equiv \mathrm{N}\sim\mathrm{N}p$'. (From d4, d3.)

c. '$(p ⥽ q) \supset (\mathrm{N}p \supset \mathrm{N}q)$'. (From (a), d3.)

d. '$(p ⥽ q) \supset [(r ⥽ p) \supset (r ⥽ q)]$'. (From d1, (c).)

e. '$\mathrm{N}p.\mathrm{N}q \equiv \mathrm{N}(p.q)$'. (From d1, (c).)

f. '$\mathrm{N}p \supset \mathrm{N}(q \supset p)$'. (From d1, (c).)

g. '$\mathrm{N}p \equiv \mathrm{NN}p$'. (From (a), (c), d4; d3.)

h. '$(\mathrm{N}p ⥽ q) \supset (\mathrm{N}p \supset \mathrm{N}q)$'. (From (c), (g).)

i. '$\sim p \supset \sim\mathrm{N}p$'. (From d3.)

j. '$q \supset \sim\mathrm{N}\sim q$'. (From (i).)

k. '$\mathrm{N}\sim p \supset \sim\mathrm{N}p$'. (From d3, (i).)

l. '$\mathrm{N}(\sim p_1 \vee \sim p_2 \vee \cdots \vee \sim p_n \vee q) \supset \sim\mathrm{N}p_1 \vee \sim\mathrm{N}p_2 \vee \cdots \vee \sim\mathrm{N}p_n \vee \mathrm{N}q$'. (From (c).)

m. '$\mathrm{N}(\sim p_1 \vee \cdots \vee \sim p_n) \supset \sim\mathrm{N}p_1 \vee \cdots \vee \sim\mathrm{N}p_n$'. (From (l), (k).)

n. '$\mathrm{N}(\mathrm{N}p \vee q) \supset \mathrm{N}p \vee \mathrm{N}q$'. (From d1, (c), (b).)

p. NN$\mathfrak{S}_i$ where $\mathfrak{S}_i$ is any sentence whose P-reductum is 't'. (From d1, (g).)

q. N$\mathfrak{S}_i \supset$ N$\mathfrak{S}_j$ where $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are such that the P-reductum of $\mathfrak{S}_i \supset \mathfrak{S}_j$ is 't'. (From d1, (c).)

r. '$\mathrm{N}p \supset \mathrm{N}(p \vee q)$'. (From (q).)

s. '$\mathrm{N}p \vee \mathrm{N}q \supset \mathrm{N}(p \vee q)$'. (From (r), (q).)

t. '$(\mathrm{N}p \supset \mathrm{N}q) \equiv (\mathrm{N}p ⥽ q)$'. (From (r), (b), (q); (h).)

u. '$\mathrm{N}p ⥽ p$'. (From (t).)

v. '$\mathrm{N}p \vee \mathrm{N}q \equiv \mathrm{N}(\mathrm{N}p \vee q)$'. (From (r), (b), (q); (n).)

x. '$\sim\mathrm{N}\sim\mathrm{N}p \supset \mathrm{N}p$'. (From (b).)

y. '$\mathrm{N}p \vee \mathrm{N}q \equiv \mathrm{N}(\mathrm{N}p \vee \mathrm{N}q)$'. (From (g), (v).)

For proofs of many of these theorems, see Wajsberg (op. cit.).

T4-2. Let K contain MPC. Let $\mathfrak{S}_i$ and $\sim\mathfrak{S}_j$ be C-true by MPC in K. Then the sentences of the following forms are likewise C-true by MPC in K.

a. $\mathrm{N}\mathfrak{S}_i$.

Proof. If we prefix 'N' to every sentence in the proof of $\mathfrak{S}_i$, then it can be shown, with the help of T4-1g, u, and a, that every sentence thus resulting is C-true by MPC.

b. $\mathfrak{S}_i \equiv \mathrm{N}\mathfrak{S}_i$. (From (a).)

c. $\mathrm{N}\sim\mathfrak{S}_j$. (From (a).)

d. $\sim\mathrm{N}\mathfrak{S}_j$. (From (c), T4-1i.)

e. $\sim\mathfrak{S}_j \equiv \mathrm{N}\sim\mathfrak{S}_j$. (From (c).)

f. $\sim\mathfrak{S}_j \equiv \sim\mathrm{N}\mathfrak{S}_j$. (From (d).)

g. $\mathfrak{S}_j \equiv \mathrm{N}\mathfrak{S}_j$. (From (f).)

h. $\mathfrak{S}_j \equiv \sim\mathrm{N}\sim\mathfrak{S}_j$. (From (e).)

5. MP-reduction. We shall now explain a reduction method which leads to a normal form for both MPL and MPC; therefore we call it MP-reduction. This method is an extension of P-reduction. It is similar to the method used by Wajsberg (op. cit.).

It is easy to construct a decision method for MPL if we require only that it be *theoretically* effective, i.e., that it lead to a decision in a finite number of steps, no matter how large. [If we have a modal sentence $\mathfrak{S}_i$ with n ultimate components, then they determine $p = 2^n$ smallest ranges. Thus there are $2^p - 1$ possibilities for none or some (but not all) of these ultimate ranges to be null (see the explanation of basic modal functions in §6). If we find, on the basis of the rules of ranges for MPL, that for each of these possibilities the range of $\mathfrak{S}_i$ is universal, then $\mathfrak{S}_i$ is L-true by MPL.] MP-reduction will yield a decision method which is, moreover, practicable, i.e., sufficiently short for modal sentences of ordinary length. This reduction method further leads to the result that MPC is complete in this sense: every sentence which is L-true by MPL is C-true by MPC.

D5-1. The *MP-reduction* of a matrix $\mathfrak{M}_i$ is its transformation according to the following rules. At any step in the transformation, the first of these rules that can be applied must be applied. The replacement applies to any part having the specified form, provided this part is either the whole matrix or one of those components out of which the whole is built up with connectives and 'N'. Here we have two kinds of rules: (a) to (n), and (o) to (p). If none of the rules (a) to (n) is any more applicable, the result is called the *first MP-reductum* of $\mathfrak{M}_i$. If none of all the rules (a) to (p) is any more applicable, the result is called the second MP-reductum or, briefly, the *MP-reductum* of $\mathfrak{M}_i$.

a to **k**, as in D3-1.

l. Omission of 'N'. $\mathrm{N}\mathfrak{M}_k$ is replaced by $\mathfrak{M}_k$ if $\mathfrak{M}_k$ has one of the following four forms: (1) $\mathrm{N}\mathfrak{M}_h$ (with any $\mathfrak{M}_h$), (2) $\sim\mathrm{N}\mathfrak{M}_h$, (3) 't', (4) '$\sim$t'.

m. $\mathrm{N}(\mathfrak{M}_{k1}.\cdots.\mathfrak{M}_{kn})$ is replaced by $\mathrm{N}(\mathfrak{M}_{k1}).\cdots.\mathrm{N}(\mathfrak{M}_{kn})$.

n. Suppose that $\mathrm{N}\mathfrak{M}_h$ occurs where $\mathfrak{M}_h$ is a disjunction with n components ($n \geqq 2$) such that there is at least one component of the form '$\mathrm{N}(\cdots)$' or '$\sim\mathrm{N}(\cdots)$'. Let $\mathfrak{M}_k$ be the first of the components having either of these forms. Let $\mathfrak{M}_l$ be either (if $n > 2$) the disjunction of the remaining components (in the order in which they occur in $\mathfrak{M}_h$) or (if $n = 2$) the one remaining component. Then $N\mathfrak{M}_h$ is replaced by $\mathfrak{M}_k \vee N\mathfrak{M}_l$.

These are the rules of the first kind; the following two are those of the second kind.

o. Suppose that $\mathfrak{M}_k$ is either the whole matrix or a conjunctive component of the whole, and has the form $\sim N\mathfrak{M}_{k1} \vee \cdots \vee \sim N\mathfrak{M}_{km}$ ($m \geqq 1$; for $m = 1$, $\mathfrak{M}_k$ is $\sim N\mathfrak{M}_{k1}$). Then $\mathfrak{M}_k$ is replaced by $N(\sim\mathfrak{M}_{k1} \vee \cdots \vee \sim\mathfrak{M}_{km})$.

p. Suppose that $\mathfrak{M}_h$ is either the whole matrix or a conjunctive component of the whole and is a disjunction (with $m + n + p$ components) which has the following form or can be brought into this form by merely changing the order of the components: $\sim N\mathfrak{M}_{j1} \vee \cdots \vee \sim N\mathfrak{M}_{jm} \vee N\mathfrak{M}_{k1} \vee \cdots \vee N\mathfrak{M}_{kn} \vee \mathfrak{M}_{l1} \vee \cdots \vee \mathfrak{M}_{lp}$, where the p $\mathfrak{M}_l$-components have neither the form '$\sim N(\cdots)$' nor '$N(\cdots)$', and where $m \geqq 1$, $n \geqq 0$, $p \geqq 0$, but $n + p \geqq 1$, hence $m + n + p \geqq 2$. Let $\mathfrak{M}_j$ be $\sim\mathfrak{M}_{j1} \vee \cdots \vee \sim\mathfrak{M}_{jm}$, and $\mathfrak{M}_l$ be $\mathfrak{M}_{l1} \vee \cdots \vee \mathfrak{M}_{lp}$. Then $\mathfrak{M}_h$ he replaced by $N(\mathfrak{M}_j \vee \mathfrak{M}_{k1}) \vee N(\mathfrak{M}_j \vee \mathfrak{M}_{k2}) \vee \cdots \vee N(\mathfrak{M}_j \vee \mathfrak{M}_{kn}) \vee N(\mathfrak{M}_j \vee \mathfrak{M}_l)$. (If $p = 0$, the last of these $n + 1$ components disappears; if $n = 0$, all components except the last disappear.)

(Concerning the difference between the rules of the first and the second kind, see T6-1e, f, g.)

With respect to this reduction method, the following theorems can be proved. The proofs cannot be given here.

T5-1. Let $\mathfrak{M}_k$ be a matrix in a semantical system containing MPL or in a calculus containing MPC, such that none of the rules of MP-reduction (D1a to o) can be applied to $\mathfrak{M}_k$. Then $\mathfrak{M}_k$ has exactly one of the following forms (a) to (h).

a. $\mathfrak{M}_k$ is 't'.

b. $\mathfrak{M}_k$ is '$\sim$t'.

c. $\mathfrak{M}_k$ has a form different from the five forms listed in D4-2b.

d. $\mathfrak{M}_k$ is $\sim\mathfrak{M}_h$, where $\mathfrak{M}_h$ has form (c).

e. $\mathfrak{M}_k$ is a disjunction of two or more components of the forms (c), (d); no component is the negation of another component; no two components are alike.

f. $\mathfrak{M}_k$ is $N(\mathfrak{M}_h)$, where $\mathfrak{M}_h$ has one of the forms (c), (d), (e).

g. $\mathfrak{M}_k$ is a disjunction of two or more components of the forms (c), (d), (f), at least one being of the form (f); no component is the negation of another component; no two components are alike.

h. $\mathfrak{M}_k$ is a conjunction of two or more components of the forms (c), (d), (e), (f), (g); no component is the negation of another component; no two components are alike.

6. Relations between MPC and MPL. If a calculus is constructed for the purpose of a formalization of a given logical or empirical theory, then two questions may be raised. (1) Is the calculus in accord with the given theory? In technical terms, is the theory a true (or L-true) interpretation for the calculus? (2) Is the calculus strong enough to yield all the statements of the theory? In technical terms, is the calculus an exhaustive (or L-exhaustive) calculus for the theory? An affirmative answer to the first question is certainly required, because otherwise the calculus would not fulfill its purpose. An affirmative answer to the second question is, although desirable, in general not required. If we use only the customary means (excluding so-called transfinite rules), then for certain logical systems no L-exhaustive calculus can be constructed. (We leave aside the question of the requirement of full formalization in a sense still stronger than that of L-exhaustiveness; this question has been discussed for PC and FC in [II].)

For PC and PL, it is well known that both questions can be answered affirmatively (see above, T2-1 and T3-1). The same result will now be stated for MPC and MPL; it was first found by Wajsberg. The affirmative answer to the first question is easily found (T6-1c, d); it is based on a simple examination of the primitive sentences of MPC (T6-1a) and its rule of inference (T6-1b). The proof of the completeness of MPC, however, is more complicated. It makes use of the method of MP-reduction (D5-1). The result is reached in two steps: (i) every L-true sentence is reducible to 't' (T6-2d); (ii) if a sentence is reducible to 't,' it is C-true (T6-2e); hence, if a sentence is L-true, it is C-true (T6-2f).

T6-1. Let the semantical system S and the calculus K contain the same signs and the same sentences. Let S contain MPL, and K contain MPC.

a. Every primitive sentence of MPC (D4-2d) is L-true by MPL.

b. In every instance of direct C-implication by MPC (that is, of an application of the rule of implication D4-2e), L-implication by MPL holds.

c. Any sentence which is C-true by MPC is L-true by MPL. (From (a), (b).)

d. Whenever C-falsity by MPC holds, then L-falsity by MPL holds; analogously with C-implication and L-implication, and with C-equivalence and L-equivalence. (From (c).) Thus, in a certain sense, MPL is an L-true interpretation of MPC.

e. If one application of any of the rules of MP-reduction of the first kind (D5-1a to n) to a sentence $\mathfrak{S}_i$ leads to $\mathfrak{S}_j$, then $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are C-equivalent by MPC and hence L-equivalent by MPL.

f. Let $\mathfrak{S}_j$ be formed from $\mathfrak{S}_i$ by an application of one of the rules of MP-reduction of the second kind (D5-1o or p). Then, if $\mathfrak{S}_i$ is L-true by MPL, $\mathfrak{S}_j$ is likewise. (The converse holds too but will not be needed. Note that, in contradistinction to (e), L-equivalence is here not asserted; it does not hold in general.)

g. Let $\mathfrak{S}_j$ and $\mathfrak{S}_i$ be as in (f). Then $\mathfrak{S}_j$ C-implies $\mathfrak{S}_i$ by MPC. (From T4-1m, D4-2d3.) [Between $\mathfrak{S}_i$ and $\mathfrak{S}_j$ C-implication by MPC does not hold generally, but only the weaker relation that, if $\mathfrak{S}_i$ is C-true by MPC, then so is $\mathfrak{S}_j$.]

T6-2. Let S and K be as in T6-1. Let $\mathfrak{S}_j$ be the (second) MP-reductum of $\mathfrak{S}_i$. Then the following holds.

a. If $\mathfrak{S}_i$ is L-true by MPL, then so is $\mathfrak{S}_j$. (From T6-1e, f.)

b. $\mathfrak{S}_j$ C-implies $\mathfrak{S}_i$ by MPC. (From T6-1e, g.)

c. $\mathfrak{S}_j$ L-implies $\mathfrak{S}_i$ by MPL. (From (b), T6-1d.)

d. If $\mathfrak{S}_i$ is L-true by MPL, then $\mathfrak{S}_j$ is 't'.

Proof. $\mathfrak{S}_j$ is L-true by MPL (a), and none of the rules of MP-reduction is applicable to $\mathfrak{S}_j$. An examination of the forms listed in T5-1 shows that all of them except the first, i.e., 't', are impossible in this case.

e. If $\mathfrak{S}_i$ is 't', then $\mathfrak{S}_i$ is C-true by MPC. (From (b), D4-2d1.)

f. *Completeness of MPC.* If $\mathfrak{S}_i$ is L-true by MPL, then it is C-true by MPC. (From (d), (e).) In this case, the rules of MP-reduction provide an effective method for the construction of a proof of $\mathfrak{S}_i$ by MPC (according to T6-1e, g).

g. L-truth by MPL holds if and only if C-truth by MPC holds. (From (f), T6-1c.) Analogously with L-falsity and C-falsity, with L-implication and C-implication, with L-equivalence and C-equivalence. Thus, in a certain sense, MPC is an L-exhaustive calculus for MPL.

h. A sufficient and necessary condition for $\mathfrak{S}_i$ to be L-true by MPL is that $\mathfrak{S}_i$ is 't', and likewise for $\mathfrak{S}_i$ to be C-true by MPC. (From (d), (e), (f), T6-1c.)

T6-2h shows that MP-reduction yields a *decision method* for both MPL and MPC. The rules of MP-reduction can be applied not only to a given sentence in S and K but also to any formula constructed out of auxiliary variables 'p', 'q', etc. with the help of connectives and 'N'. (In this case, the form T5-1c is a single variable.) The reduction of such a formula leads to 't' if and only if every sentence obtainable from the formula by substitutions is L-true by MPL and C-true by MPC. In the MP-reductum of any such formula, no 'N' occurs in the scope of another 'N'.

In the case of PL and PC, there are two well-known decision methods, viz. P-reduction (the conjunctive normal form) and the truth-table method. Analogously, in the case of MPL and MPC, there are, in addition to MP-reduction, decision methods using tables. These methods are likewise applicable both to sentences and to auxiliary formulas. If a sentence with n different ultimate MP-components (i.e., form T5-1c) or a formula with n different auxiliary variables is given, then first a truth-table of the customary kind with respect to these n components (hence containing 2^n lines) is constructed. Then a table of a new kind is to be formed which supplies the decision. It is easy to find a table method in which the second table consists of 2^{2^n} lines (thus, for $n = 3$, 256 lines); but this method is of course impracticable. I have found a method in which the second table has, in ordinary cases, only a few lines (for instance, for some theorems and postulates of Lewis's systems with $n = 3$ only three or four lines are needed). This method cannot be explained here.

In order to compare our system with those of other authors, let us consider a modification $\mathrm{MPC_v}$ of MPC: $\mathrm{MPC_v}$ is itself a calculus; it contains the variables 'p', 'q', etc.; these variables and 't' are the only ultimate components; it possesses the same rules as MPC, except that the rule of refutation is omitted and a rule of substitution for the variables is added. The method of MP-reduction is likewise applicable to $\mathrm{MPC_v}$. As mentioned earlier (footnote 8), this system $\mathrm{MPC_v}$ (except for containing 't', which corresponds to '$p \vee \sim p$') is equivalent to Lewis's system S5, which is the strongest of a series of systems investigated by Lewis.[9] Some of the reasons for which this system seems to me preferable to Lewis's other systems were indicated in §1. The completeness of this system is a further advantage. On the basis of the interpretation given by MPL, all sentences C-true (provable) in MPC and $\mathrm{MPC_v}$ are L-true, while those princi-

[9] See Lewis and Langford (op. cit.), especially Appendix II; for S5, see p. 501. Compare also W. T. Parry, *Modalities in the Survey system of strict implication*, this JOURNAL, vol. 4 (1939), pp. 137–154; concerning S5, see pp. 151 ff.

ples of other systems which go beyond Lewis's S5[10] are L-false. My chief reason for preferring the interpretation given by MPL is the simple parallelism between the modalities in this system and the semantical L-concepts, in particular between necessity and L-truth. It is true that these semantical concepts could be defined in a different way so as to correspond to a different conception of the modalities. However, the definition of L-truth here chosen, which is based on Wittgenstein's conception of the nature of logical truth, has the advantage of great simplicity, taking as criterion the universality of the range (D7-6a).

Let us consider all possible functions of n propositions $p_1, p_2, \cdots, p_n$. Any formula composed of n auxiliary variables with connectives and 'N' represents such a function. We will say that two such formulas $\mathfrak{M}_i$ and $\mathfrak{M}_j$ represent the *same* function if and only if they are L-equivalent, that is to say, if the P-reductum of $\mathfrak{M}_i \equiv \mathfrak{M}_j$ is 't' and therefore all sentences of this form are L-true by MPL and hence C-true by MPC. If a function can be represented by a formula without 'N', it is called a *truth-function* (or extensional function); otherwise we call it a *modal function*[11] (or intensional function). [N($p \vee \sim p$) represents a truth-function because it is L-equivalent to '$p \vee \sim p$' (T4-2b).] If a modal function can be represented by a formula in which all variables are under 'N' (i.e., within the scope of an 'N'), we call it a *purely modal function* (e.g., 'N$p \vee$ N$\sim p$'), otherwise a *mixed modal function* (e.g., '$p \vee$ N$\sim p$').

We shall now determine the number of functions of these kinds (without giving here exact proofs). The considerations are formulated for MPL (in terms of ranges); the results hold likewise for MPC. For n sentences 'A_1', $\cdots$, 'A_n', there are $p = 2^n$ conjunctions containing for every sentence in the given order either the sentence itself or its negation but not both; they correspond to the 2^n lines of the customary truth-table. Each of these conjunctions may be regarded as representing a basic truth-function of the n sentences. The truth-functions in general are represented by the disjunctions corresponding to the possible selections of some (or none or all) of these p conjunctions. (Here, by the disjunction of one sentence, the sentence itself is meant; by the null selection, any L-false sentence, e.g., '$A_1.\sim A_1.A_2.\cdots.A_n$'). Thus there are $q = 2^p = 2^{2^n}$ truth-functions, as is well known.

Let the p conjunctions mentioned be abbreviated by 'C_1', $\cdots$, 'C_p'. They have the smallest ranges expressible by the n sentences. If some (or none, but not all) of these ranges are null, but the others are not, a specific logical relation holds among the sentences; in other words, a basic modal function holds among the propositions. Each of these functions can be represented by a conjunction of p components, namely either 'N$\sim C_i$' (if the range of 'C_i' is null), or '$\sim$N$\sim C_i$' (if the range is not null) for $i = 1, 2, \cdots, p$; but the conjunction with all components of the form 'N$\sim C_i$' is excluded. Thus there are $2^p - 1$ or $q - 1$ basic modal functions. (The conjunction containing all components of the form 'N$\sim C_i$' says that all basic conjunctions are L-false, which is impossible; therefore this conjunction is L-equivalent to 'A.$\sim$A' and hence is not a modal

[10] See Parry, op. cit., pp. 152 ff.

[11] This use of the term 'modal function' is narrower than that of Parry, op. cit., p. 144, who takes it to include the truth-functions.

function.) The purely modal functions are the disjunctions of some (but not all) of these $q - 1$ basic modal functions. The disjunction of all of them must be excluded because it is L-true and hence not a modal function (it is L-equivalent to 'A v ∼A'); likewise the null disjunction (so to speak), which is L-false. Therefore the number of all purely modal functions is $2^{q-1} - 2$.

Now we shall determine the number of all functions of n propositions. We construct first the basic possibilities. If one of the p conjunctions is stated, say 'C_i', then this one cannot simultaneously be stated to be impossible (its range cannot be null); but any of the other $p - 1$ conjunctions may have a null range, possibly all of them or none of them. Thus there are in this case 2^{p-1} possibilities; each is represented by a conjunction containing 'C_i' and in addition, for every other conjunction 'C_j' either '$N{\sim}C_j$' or '${\sim}N{\sim}C_j$'. In the same way we may start with any other of the p conjunctions instead of 'C_i'. Let the conjunctions which we obtain in this way (e.g., '$C_1.N{\sim}C_2.\cdots$') be 'M_1',$\cdots$,'M_r'. Their number r is $p\cdot 2^{p-1} = 2^n\cdot 2^{p-1} = 2^{p+n-1}$. They represent the basic functions of n propositions. Every function of n propositions is represented by a disjunction of some (possibly all or none) of these r conjunctions 'M_i'. (Here again, by the null disjunction an L-false sentence is meant.) Thus the number of all functions of n propositions is 2^r. By subtraction we find the number of modal functions to be $2^r - q$, and the number of mixed modal functions to be $2^r - q - 2^{q-1} + 2$.

The Number of Functions of n Propositions
$(p = 2^n; q = 2^p; r = 2^{p+n-1}.)$

		$n=1$ ($p=2$; $q=4$; $r=4$)	$n=2$ ($p=4$; $q=16$; $r=32$)	
Purely modal functions:	$2^{q-1} - 2$	6	$2^{15} - 2 =$	32,766
Mixed modal functions:	$2^r - q - 2^{q-1} + 2$	6		4,294,934,514
Modal functions:	$2^r - q$	12		4,294,967,280
Truth-functions:	$q = 2^p$	4	$2^4 =$	16
All functions:	2^r	16	$2^{32} =$	4,294,967,296

Examples for $n = 1$. The four truth-functions are represented by '$p \vee {\sim}p$', 'p', '${\sim}p$', '$p.{\sim}p$'. The six purely modal functions are: 'Np' (p is necessary), '$N{\sim}p$' (impossible), '${\sim}Np.{\sim}N{\sim}p$' (contingent), '${\sim}Np$' (non-necessary), '${\sim}N{\sim}p$' (possible), '$Np \vee N{\sim}p$' (non-contingent). The six mixed modal functions are: '$p.{\sim}Np$', '${\sim}p.{\sim}N{\sim}p$', '$p \vee N{\sim}p$', '${\sim}p \vee Np$', '$Np \vee ({\sim}p.{\sim}N{\sim}p)$', '$N{\sim}p \vee (p.{\sim}Np)$'.

In spite of the finite number of functions in MPL, there is no finite characteristic value-table ("matrix") for MPL or MPC, i.e., a value-table with a finite number of values (so-called truth-values) which attributes to a sentence (or formula with auxiliary variables) one of a set of specified "designated" values if and only if that sentence is L-true by MPL and hence C-true by MPC.[12]

[12] See J. Dugundji, this JOURNAL, vol. 5 (1940), pp. 150 f., with references to Gödel and McKinsey.

7. Functional logic (FL). In preparation for the construction of the modal systems MFL and MFC, which will be the chief aim of this paper, we shall now briefly outline the ordinary, non-modal systems FL and FC. The term 'functional calculus' ('FC') is here used for a certain form of the lower functional calculus; in Church's terminology, it is a simple applied functional calculus of first order with identity. The individual variables are the only variables. The term 'functional logic' ('FL') is here used for a corresponding semantical system. The system FL applies to a universe of discourse containing a denumerable number of individuals. Every individual is denoted by an individual constant, and different individual constants denote different individuals. Therefore the semantical rules (D7-5f) are framed in such a way that '$a_1 = a_2$', for example, is L-false. [I have also studied other forms of functional logic and corresponding calculi. In one alternative FL′, sentences like '$a_1 = a_2$' are interpreted, not as L-false, but as factual, either true or false. In this system FL′ the state-descriptions must contain =-sentences and are more complicated than in FL (D7-4). In FL′, sentences which state that the cardinal number of the universe of discourse is n, or at least n, or at most n, or the like, are factual, although they are purely general (i.e., without non-logical constants). If we want a system in which individual descriptions ("the one individual which is ...") occur and are allowed to be substituted for individual variables, then a form like FL′ might be more suitable than the simpler form FL.—Other system forms contain functional variables.]

In the primitive notation, we use only universal quantifiers. The existential quantifier will occasionally be used as an abbreviation in the customary way; likewise '$x \neq y$' as short for '$\sim(x = y)$'.

The following definitions state the features of the semantical system FL.

D7-1. The *signs* in FL are the following:

a. Connectives: '$\sim$', '$\vee$', '$.$'.

b. Parentheses: '(', ')'.

c. 't'.

d. An infinite number of individual constants ($\mathfrak{in}$): 'a_1', 'a_2', etc. (or 'a', 'b', etc.).

e. An infinite number of individual variables ($\mathfrak{i}$): 'x_1', 'x_2', etc. (or 'x', 'y', etc.).

f. Any finite or infinite number of predicates (functional constants) of any degree.

g. The sign of identity: '='.

D7-2. *Matrices* ($\mathfrak{M}$) in FL are the expressions of the following forms:

a. Atomic matrix—a predicate of degree n, followed by n individual signs (constants or variables) (e.g., 'Rx_2b_4').

b. '=' preceded and followed by an individual sign.

c. $\sim(\mathfrak{M}_i)$.

d. $(\mathfrak{M}_i) \vee (\mathfrak{M}_j)$.

e. $(\mathfrak{M}_i) . (\mathfrak{M}_j)$.

f. Universal matrix— $(\mathfrak{i}_k)(\mathfrak{M}_i)$.

D7-3. *Sentences* ($\mathfrak{S}$) in FL are the closed matrices. (Atomic sentences are those of form D7-2a; universal sentences, D7-2f; general sentences, those containing variables.)

A state-description is a class of sentences which represents a possible specific state of affairs by giving a complete description of the universe of individuals with respect to all properties and relations designated by predicates in the system.

D7-4. A class of sentences $\mathfrak{K}_i$ is a *state-description* in FL $=_{\text{Df}}$ $\mathfrak{K}_i$ contains for every atomic sentence $\mathfrak{S}_i$ either $\mathfrak{S}_i$ itself or $\sim\mathfrak{S}_i$, but not both, and no other sentences.

The class of all state-descriptions is called the *universal range* and denoted by 'V_s'; the null class (of the same type) is called the *null range* and denoted by 'Λ_s'.

We shall now lay down rules which determine for every sentence $\mathfrak{S}_i$ of FL, in which state-descriptions $\mathfrak{S}_i$ holds; in other words, what is the *range* of $\mathfrak{S}_i$, i.e., the class of those state-descriptions in which $\mathfrak{S}_i$ holds. We shall write briefly '$\mathfrak{R}(\mathfrak{S}_i)$' for 'the range of $\mathfrak{S}_i$'. That $\mathfrak{S}_i$ holds in a given state-description means, in non-technical terms, that this state-description entails $\mathfrak{S}_i$; in other words, that $\mathfrak{S}_i$ would be true if this state-description were the description of the actual state of the universe. [I use the term 'true' here only occasionally in informal explanations. I shall not lay down a semantical definition of 'true in FL.' For the present purposes it is sufficient to have the concept of range which is recursively defined by D7-5. With its help the L-concepts can be defined (D7-6).]

D7-5. *Rules of ranges* ($\mathfrak{R}$) for FL.

a. If $\mathfrak{S}_i$ is atomic, $\mathfrak{R}(\mathfrak{S}_i)$ is the class of those state-descriptions to which $\mathfrak{S}_i$ belongs.

b. The range of $\sim\mathfrak{S}_i$ is $\mathrm{V}_s - \mathfrak{R}(\mathfrak{S}_i)$.

c. The range of $\mathfrak{S}_i \vee \mathfrak{S}_j$ is the class-sum of $\mathfrak{R}(\mathfrak{S}_i)$ and $\mathfrak{R}(\mathfrak{S}_j)$.

d. The range of $\mathfrak{S}_i . \mathfrak{S}_j$ is the class-product of $\mathfrak{R}(\mathfrak{S}_i)$ and $\mathfrak{R}(\mathfrak{S}_j)$.

e. The range of a =-sentence with two occurrences of the same individual constant (e.g., '$a_3 = a_3$') is V_s.

f. The range of a =-sentence with two different individual constants (e.g., '$a_3 = a_5$') is Λ_s.

g. The range of $(\mathrm{i}_k)(\mathfrak{M}_i)$ is the class-product of the ranges of the instances of $\mathfrak{M}_i$ (i.e., the sentences formed by substituting individual constants for the free occurrences of i_k in $\mathfrak{M}_i$; if $\mathfrak{M}_i$ is closed, $\mathfrak{M}_i$ itself is its only instance).

h. The range of a class $\mathfrak{K}_i$ of sentences is the class-product of the ranges of the sentences of $\mathfrak{K}_i$.

A sentence $\mathfrak{S}_i$ is usually regarded as logically true or logically necessary if it is true in every possible case. Therefore we call $\mathfrak{S}_i$ L-true if it holds in every state-description, in other words, if its range is V_s. Analogously, we use 'L-false' as explicatum for logical falsity or impossibility and define it by the null range. $\mathfrak{S}_j$ follows logically from $\mathfrak{S}_i$ if in every possible case in which $\mathfrak{S}_i$ holds, $\mathfrak{S}_j$ also holds. Therefore we define the explicatum 'L-implication' by the inclusion of the ranges. L-equivalence is meant as mutual L-implication; therefore it is defined by the identity of ranges.[13]

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[13] More detailed explanations and discussions of the L-concepts are given in [I] §§14 ff. For the definitions of these concepts with the help of 'range,' based on conceptions of

D7-6.
a. $\mathfrak{S}_i$ is L-true (in FL) $=_{Df}$ $\mathfrak{R}(\mathfrak{S}_i)$ is V_s.
b. $\mathfrak{S}_i$ is L-false $=_{Df}$ $\mathfrak{R}(\mathfrak{S}_i)$ is Λ_s.
c. $\mathfrak{S}_i$ L-implies $\mathfrak{S}_j$ $=_{Df}$ $\mathfrak{R}(\mathfrak{S}_i)$ is included in $\mathfrak{R}(\mathfrak{S}_j)$.
d. $\mathfrak{S}_i$ is L-equivalent to $\mathfrak{S}_j$ $=_{Df}$ $\mathfrak{S}_i$ and $\mathfrak{S}_j$ have the same range.

Since the range of a class of sentences has been defined (D7-5h), these L-concepts can likewise be applied to classes of sentences.

The many well-known theorems concerning FL need not be mentioned here. We shall state, without proofs, only a few theorems which seem less known but are important both here and later in MFL.

The following theorem allows the *variation of an individual constant* under certain conditions. Note that here, and analogously in T7-2, L-implication is not asserted but only the conditional relation with respect to L-truth.

T7-1. Let $\mathfrak{S}_i$ be a sentence in FL, and $\mathfrak{S}_j$ be formed out of $\mathfrak{S}_i$ by replacing all occurrences of $\mathfrak{in}_i$ by $\mathfrak{in}_j$.

a. Let $\mathfrak{S}_i$ not contain $\mathfrak{in}_j$ (but, in distinction to (b), '=' may occur). If $\mathfrak{S}_i$ is L-true, then $\mathfrak{S}_j$ is L-true, and vice versa.

b. Let $\mathfrak{S}_i$ not contain '=' (but, in distinction to (a), $\mathfrak{in}_j$ may occur). If $\mathfrak{S}_i$ is L-true then $\mathfrak{S}_j$ is L-true. (The converse does not generally hold.)

That the restrictions in T7-1a and b are necessary is seen from the following counter-example: '$\sim$(a = b)' is L-true, but '$\sim$(a = a)' is not. As a counter-example for the converse of (b), let $\mathfrak{S}_i$ be 'Pa $\supset$ Pb', and $\mathfrak{S}_j$ 'Pa $\supset$ Pa'; then $\mathfrak{S}_j$ is L-true but $\mathfrak{S}_i$ is not.

A theorem analogous to T7-1 holds for the variation of a predicate.

The following theorem allows the *generalization of an individual constant* under certain conditions.

T7-2. Let $\mathfrak{M}_i$ be a matrix in FL with $\mathfrak{i}_k$ as the only free variable, and $\mathfrak{S}_j$ be formed out of $\mathfrak{M}_i$ by the substitution of $\mathfrak{in}_j$ for $\mathfrak{i}_k$ (i.e., for all free occurrences of $\mathfrak{i}_k$).

a. Let $\mathfrak{in}_j$ and '=' not occur in $\mathfrak{M}_i$. If $\mathfrak{S}_j$ is L-true, then the universal sentence $(\mathfrak{i}_k)(\mathfrak{M}_i)$ is L-true. (The converse is obvious.) (From T7-1b.)

b. Let $\mathfrak{M}_i$ not contain any individual constant (but '=' may occur). If $\mathfrak{S}_j$ is L-true, then $(\mathfrak{i}_k)(\mathfrak{M}_i)$ is L-true. (From T7-1a.)

'$(\ )(\mathfrak{M}_k)$' denotes the *closure* of $\mathfrak{M}_k$, i.e., the sentence formed out of $\mathfrak{M}_k$ by prefixing universal quantifiers for all variables occurring freely in $\mathfrak{M}_k$, in their inverse alphabetical order. (If $\mathfrak{M}_k$ is closed, '$(\ )(\mathfrak{M}_k)$' denotes $\mathfrak{M}_k$ itself.)

T7-3. Let $\mathfrak{M}_i$ be a matrix in FL with n free variables ($n \geqq 2$) (i.e., any number of occurrences of n different variable-designs). Let $\mathfrak{S}_j$ be formed out of $\mathfrak{M}_i$ by the substitution of n different individual constants for the variables. Let $\mathfrak{M}_h$ be a disjunction of =-matrices, one for any two different ones of the n variables. If $\mathfrak{S}_j$ is L-true, then $(\ )(\mathfrak{M}_h \vee \mathfrak{M}_i)$ is L-true.

T7-2 and 3 say in effect the following. If either no '=' occurs or only one individual constant occurs, simple generalization is allowed (T7-2). If two or

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Wittgenstein, see [I] §§18 and 19; the method used in the present paper is similar to procedure E in §19, but it can take a simpler form here because FL contains atomic sentences for all atomic propositions.

more individual constants occur, generalization with added =-matrices is allowed. The necessity of the restrictions is seen from the following example. 'a $\neq$ b' is L-true; however, '$(y)(x)[x \neq y]$' is not L-true, but only $(y)(x)[x = y \vee x \neq y]$'.

8. Functional calculus (FC). FC is a formalization for FL. The signs, matrices, and sentences in FC are the same as in FL (D7-1, 2, 3). The only rule of inference is the *rule of implication* (as D4-2e). Further a rule of refutation is laid down (as D4-2f). This rule may be omitted if one wishes the form of the calculus to be more similar to the customary form. We add the rule in order to be able to compare C-falsity in FC with L-falsity in FL.

D8-1. The *primitive sentences* of FC are the sentences of the following forms. [$(\ )(\mathfrak{M}_k)$ is the closure of $\mathfrak{M}_k$; see explanation to T7-3.]

a. $(\ )(\mathfrak{M}_i)$, where $\mathfrak{M}_i$ is any matrix whose P-reductum (D3-1) is 't'. (This means in effect that $\mathfrak{M}_i$ has a tautologous form; we could state here instead a sufficient number of particular forms, e.g., the four forms stated by Hilbert and Ackermann as axioms of their propositional calculus.[5])

b. $(\ )[(\mathfrak{i}_i)(\mathfrak{M}_j \supset \mathfrak{M}_k) \supset ((\mathfrak{i}_i)(\mathfrak{M}_j) \supset (\mathfrak{i}_i)(\mathfrak{M}_k))]$.

c. $(\ )[\mathfrak{M}_j \supset (\mathfrak{i}_k)(\mathfrak{M}_j)]$, where $\mathfrak{i}_k$ does not occur freely in $\mathfrak{M}_j$.

d. $(\ )[(\mathfrak{i}_k)(\mathfrak{M}_k) \supset \mathfrak{M}_j]$, where $\mathfrak{M}_j$ is like $\mathfrak{M}_k$ except for containing free occurrences of $\mathfrak{i}_j$ wherever $\mathfrak{M}_k$ contains free occurrences of $\mathfrak{i}_k$.

e. As (d), but here $\mathfrak{M}_j$ is like $\mathfrak{M}_k$ except for containing occurrences of the individual constant $\mathfrak{in}_j$, wherever $\mathfrak{M}_k$ contains free occurrences of $\mathfrak{i}_k$.

f. $(\mathfrak{i}_k)(\mathfrak{i}_k = \mathfrak{i}_k)$.

g. $(\ )[(\mathfrak{i}_k = \mathfrak{i}_j) \supset (\mathfrak{M}_k \supset \mathfrak{M}_j)]$, where $\mathfrak{M}_j$ is like $\mathfrak{M}_k$ except for containing free occurrences of $\mathfrak{i}_j$ wherever $\mathfrak{M}_k$ contains free occurrences of $\mathfrak{i}_k$.

h. $\sim(\mathfrak{in}_i = \mathfrak{in}_j)$, where $\mathfrak{in}_i$ and $\mathfrak{in}_j$ are different individual constants.

D8-1a to d are four of Quine's axiom schemata of quantification.[14] To these we add the schema (e) for the substitution of an individual constant and the schemata (f), (g), (h) for identity.

Thus in our calculus FC, Quine's theorems of quantification[15] hold; that is to say, the sentences of the forms specified by Quine are C-true in FC. Further, all theorems on identity stated by Hilbert and Bernays[16] hold in FC, because they have been proved by these authors on the basis of two axioms corresponding to the schemata D8-1f and g here.

Schema D8-1h serves as a kind of axiom of infinity. With its help, the negations of those sentences can be proved which say that the number of all individuals is at most n, for any finite n (e.g., for $n = 2$, '$\sim(x)(y)(z)[x = y \vee x = z \vee y = z]$' is C-true).

The following theorem says in effect that FC is in accord with FL, as it was intended to be.

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[14] Quine, op. cit., p. 88, *100, 102, 103, 104. G. Berry has shown (this JOURNAL, vol. 6 (1941), pp. 23–27) that Quine's schema *101 may be omitted if the closure of a matrix is defined by referring to the inverse alphabetical order of the quantifiers. This simplification has here been adopted for FC (and MFC).

[15] Quine, op. cit., §§17–21.

[16] Hilbert and Bernays, op. cit., pp. 179 ff.

T8-1.

a. Every primitive sentence of FC is L-true in FL.

b. Every C-true sentence in FC is L-true in FL. (This follows from (a) together with the fact that for every instance of application of the rule of implication, L-implication holds.)

c. FL is an L-true interpretation of FC. (This follows from (a) and (b) and the fact that the rule of refutation represents an instance of L-falsity.)

It is not clear whether FC is complete in the sense that every sentence which is L-true in FL is C-true in FC. Gödel's theorem of the completeness of the ordinary functional calculus of first order cannot be directly applied to FC because of the following difference. In ordinary functional logic, a sentence is regarded as L-true (or a matrix as universal ("allgemeingültig")) if it is L-true not only in the infinite universe of individuals but, in addition, in every finite (non-null) universe. In FL, on the other hand, more sentences are regarded as L-true, viz. all those which are L-true in the infinite universe. [For example, '$(\exists x)(\exists y)(x \neq y)$' says that there are at least two individuals. Thus this sentence holds in the infinite universe but not in the universe with only one individual. Therefore, it is L-true in FL (D7-5f) (and, moreover, C-true in FC, in virtue of D8-1h), but not L-true in the ordinary functional logic (and not provable in the ordinary functional calculus).] However, the following restricted theorem holds; it would be of interest to investigate the question whether the restriction can be weakened or even eliminated.

T8-2. If $\mathfrak{S}_i$ is an L-true sentence in FL without '=', then $\mathfrak{S}_i$ is C-true in FC.

Proof. Let the conditions be fulfilled. Let $\mathfrak{M}_i$ be formed from $\mathfrak{S}_i$ by replacing all individual constants with individual variables not occurring in $\mathfrak{S}_i$ and all predicates by predicate variables (like constants to be replaced by like variables, different by different ones). Then $(\ )[\mathfrak{M}_i]$ (where the closure applies only to the individual variables) is L-universal (i.e., L-true for all values of the predicate variables) in the infinite universe (compare T7-2a) and also in every non-null finite universe, because '=' does not occur. Therefore, according to Gödel's completeness theorem, it is provable in the ordinary functional calculus. Therefore $\mathfrak{S}_i$ is C-true in FC.

9. Modal functional logic (MFL). Now we shall come to our aim, the construction of the semantical and syntactical modal systems with quantification, viz., modal functional logic (MFL) and modal functional calculus (MFC). MFL is built from FL by the addition of the modal sign 'N'; likewise MFC from FC.

D9-1. The *signs* in MFL are the following:

a to **g**, as D7-1a to g.

h. 'N'.

D9-2. The *matrices* in MFL are the expressions of the following forms:

a to **f**, as D7-2a to f.

g. $N(\mathfrak{M}_i)$. (N-matrix).

D9-3. *Sentences* in MFL are the closed matrices. (Modal sentences are those containing 'N'.)

D9-4. The *state-descriptions* in MFL are the same as in FL (D7-4).

D9-5. *Rules of ranges* for MFL:

a to **h**, as D7-5a to h.

i. The range of $N(\mathfrak{S}_i)$ is V_s if the range of $\mathfrak{S}_i$ is V_s, and otherwise Λ_s.

L-truth, L-falsity, L-implication and L-equivalence in MFL are defined as in D7-6.

We shall now state some theorems concerning MFL. The proofs, in most cases, cannot be given here, because they presuppose a number of theorems concerning state-descriptions and ranges which have not been stated here.

T9-1.

a. $N\mathfrak{S}_i$ is L-true if and only if $\mathfrak{S}_i$ is L-true.

b. $N\mathfrak{S}_i$ is L-false if and only if $\mathfrak{S}_i$ is not L-true.

c. $N\mathfrak{S}_i$ is L-true or L-false. (From D9-5i.)

Exchange of 'N' and universal quantifier:

T9-2. Every sentence of the following form is L-true in MFL:

$N(\ )[(\mathfrak{i}_k)N(\mathfrak{M}_i) \equiv N(\mathfrak{i}_k)(\mathfrak{M}_i)]$.

T9-1 and 2 show that the rules of ranges give to 'N' the interpretation intended (see §1, especially conventions C1-2 and C1-3).

Variation of individual constants under 'N' (analogous to T7-1):

T9-3. Let $\mathfrak{S}_j$ be formed from $\mathfrak{S}_i$ by replacing all occurrences of $\mathfrak{in}_i$ by $\mathfrak{in}_j$.

a. Let $\mathfrak{S}_i$ not contain $\mathfrak{in}_j$ (but it may contain '=' and 'N'). Then $N\mathfrak{S}_i$ is L-equivalent to $N\mathfrak{S}_j$.

b. Let $\mathfrak{S}_i$ contain neither '=' nor 'N' (but it may contain $\mathfrak{in}_j$). Then $N\mathfrak{S}_i$ L-implies $N\mathfrak{S}_j$. (The converse does not generally hold.)

10. Modal functional calculus (MFC). MFC is constructed as a formalization of MFL. It is built from FC by the addition of 'N'.

The signs, matrices and sentences in MFC are the same as in MFL (D9-1, 2, 3). We shall occasionally use as abbreviations, not belonging to the systems themselves, existential quantifiers and the modal sign of possibility '$\Diamond$'; '$\Diamond p$' is short for '$\sim N \sim p$'.

D10-1. The *primitive sentences* of MFC are those sentences in MFC which have one of the following forms:

a. $N(\ )(\mathfrak{M}_i)$, where $\mathfrak{M}_i$ is any matrix whose P-reductum (D3-1) is 't'. (See remark on D8-1a.)

b. $N(\ )[(\mathfrak{M}_i \supset \mathfrak{M}_j) \supset (N\mathfrak{M}_i \supset N\mathfrak{M}_j)]$.

c. $(\ )[N\mathfrak{M}_i \supset \mathfrak{M}_i]$.

d. $N(\ )[N\mathfrak{M}_i \vee N{\sim}N\mathfrak{M}_i]$.

e to **k.** $N\mathfrak{S}_i$, where $\mathfrak{S}_i$ has one of the forms D8-1b to h, respectively (primitive sentences of FC).

l. $N(\ )[(\mathfrak{i}_k)N(\mathfrak{M}_i) \supset N(\mathfrak{i}_k)(\mathfrak{M}_i)]$.

m. $N(\ )[N(\mathfrak{i}_k)(\mathfrak{M}_i) \supset (\mathfrak{i}_k)N(\mathfrak{M}_i)]$.

n. (Assimilation.) $N(\ )[\mathfrak{i}_i = \mathfrak{i}_{k1} \vee \mathfrak{i}_i = \mathfrak{i}_{k2} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{i}_{kn} \vee {\sim}N\mathfrak{M}_i \vee N\mathfrak{M}_j$; here $\mathfrak{M}_i$ does not contain '=', 'N', any quantifier with $\mathfrak{i}_j$, or any individual constant; it contains free occurrences of $\mathfrak{i}_i$, $\mathfrak{i}_j$, $\mathfrak{i}_{k1}$, $\cdots$, $\mathfrak{i}_{kn}$ but of no other variables; $\mathfrak{M}_j$ is like $\mathfrak{M}_i$ except for containing free occurrences of $\mathfrak{i}_j$ wherever $\mathfrak{M}_i$ contains free occurrences of $\mathfrak{i}_i$.

o. (Variation and Generalization.) $N(\;)[\mathfrak{i}_i = \mathfrak{i}_{k1} \vee \mathfrak{i}_i = \mathfrak{i}_{k2} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{i}_{kn} \vee \sim N\mathfrak{M}_i \vee \mathfrak{i}_j = \mathfrak{i}_{k1} \vee \cdots \vee \mathfrak{i}_j = \mathfrak{i}_{kn} \vee N\mathfrak{M}_j]$; here $\mathfrak{M}_i$ contains no individual constant and contains $\mathfrak{i}_i, \mathfrak{i}_{k1}, \mathfrak{i}_{k2}, \cdots, \mathfrak{i}_{kn}$ as the only free variables, and $\mathfrak{M}_j$ is like $\mathfrak{M}_i$ except for containing free occurrences of $\mathfrak{i}_j$ wherever $\mathfrak{M}_i$ contains free occurrences of $\mathfrak{i}_i$.

p. (Substitution for predicate.) $N(\;)[N\mathfrak{M}_i \supset N\mathfrak{M}_j]$. Here $\mathfrak{M}_i$ contains a predicate $\mathfrak{pr}_k$ of degree m but no 'N'; $\mathfrak{M}_k$ is any matrix containing the m alphabetically first variables ($\mathfrak{i}_1, \cdots, \mathfrak{i}_m$) as the only free variables and not containing a quantifier with any variable occurring in $\mathfrak{M}_i$ ($\mathfrak{M}_k$ may contain 'N' and $\mathfrak{pr}_k$); $\mathfrak{M}_j$ is formed from $\mathfrak{M}_i$ by replacing every atomic matrix containing $\mathfrak{pr}_k$ by the corresponding substitution form of $\mathfrak{M}_k$ (i.e., $\mathfrak{pr}_k\mathfrak{a}_{k1} \cdots \mathfrak{a}_{km}$ is replaced by the matrix resulting from $\mathfrak{M}_k$ by substituting $\mathfrak{a}_{k1}$ for $\mathfrak{i}_1, \cdots, \mathfrak{a}_{km}$ for $\mathfrak{i}_m$).

The only rule of inference for MFC is the *rule of implication* (as D4-2e). We add further a rule of refutation, as D4-2f (see remark on this rule for FC, §8).

Examples for D10-1. The following examples show either primitive sentences in MFC or sentences easily provable with their help, or cases of simple derivations made possible by the primitive sentences. ('$\cdot\cdot x\cdot\cdot$' and '$--x--$' are meant to indicate matrices containing 'x' as the only free variable.)

(a) to (d) correspond to D4-2d1 to 4, respectively; hence any primitive sentence of MPC in MFC is a primitive sentence of MFC. However, (a) to (d) admit also sentences like the following examples, which are not primitive sentences of MPC.

a. '$N(x)(Px \vee \sim Px)$'.

b. '$N(x)[(\cdot\cdot x\cdot\cdot \;⥽\; --x--) \supset (N(\cdot\cdot x\cdot\cdot) \;⥽\; N(--x--))]$'.

c. '$(x)[N(\cdot\cdot x\cdot\cdot) \supset (\cdot\cdot x\cdot\cdot)]$'. $N\mathfrak{S}_i$ C-implies $\mathfrak{S}_i$.

d. '$(x)[\sim N(\cdot\cdot x\cdot\cdot) \supset N\sim N(\cdot\cdot x\cdot\cdot)]$'. $\sim N\mathfrak{S}_i$ C-implies $N\sim N\mathfrak{S}_i$.

(a), (e) to (k) yield, with the help of (c), all the primitive sentences of FC.

(l) and (m), which could of course be combined into a sentence with '$\equiv$', allow the exchange of a universal quantifier and 'N', e.g., the transformation of '$(x)N(\cdot\cdot x\cdot\cdot)$' into '$N(x)(\cdot\cdot x\cdot\cdot)$' and vice versa, and hence also any change in the order of a sequence consisting of universal quantifiers and occurrences of 'N'.

n. '$N(z)(y)(x)[x = z \vee \sim N(\cdot\cdot x\cdot\cdot y\cdot\cdot z\cdot\cdot) \vee N(\cdot\cdot y\cdot\cdot y\cdot\cdot z\cdot\cdot)]$'; hence '$(z)(y)(x)[x \neq z.N(\cdot\cdot x\cdot\cdot y\cdot\cdot z\cdot\cdot) \supset N(\cdot\cdot y\cdot\cdot y\cdot\cdot z\cdot\cdot)]$'. Thus, '$N(\cdot\cdot a\cdot\cdot b\cdot\cdot c\cdot\cdot)$' C-implies '$N(\cdot\cdot b\cdot\cdot b\cdot\cdot c\cdot\cdot)$'. In this way, an *assimilation* under 'N' is made possible, i.e., the change of an individual constant ('a') to another individual constant ('b') already occurring.

o. '$N(z)(y)(x)[x = z \vee \sim N(\cdot\cdot x\cdot\cdot z\cdot\cdot) \vee y = z \vee N(\cdot\cdot y\cdot\cdot z\cdot\cdot)]$'; hence '$(z)(y)(x)[x \neq z.N(\cdot\cdot x\cdot\cdot z\cdot\cdot) \supset (y \neq z \supset N(\cdot\cdot y\cdot\cdot z\cdot\cdot))]$'; hence (1) '$N(\cdot\cdot a\cdot\cdot c\cdot\cdot) \supset (y)[y \neq c \supset N(\cdot\cdot y\cdot\cdot c\cdot\cdot)]$', and (2) '$N(\cdot\cdot a\cdot\cdot c\cdot\cdot) \supset N(\cdot\cdot b\cdot\cdot c\cdot\cdot)$'. (1) shows the possibility of a *generalization* under 'N', (2) that of a *variation* under 'N'. (o) permits only the variation into an individual constant otherwise not occurring. The variation into an individual constant already occurring, which we call assimilation, is possible only under the more restricting conditions of (n).

p. '$N(u)(v)[\text{—}Rva\text{—}Rbu\text{—}]$' C-implies '$N(u)(v)[\text{—}(\cdot\cdot v\cdot\cdot a\cdot\cdot)\text{—}(\cdot\cdot b\cdot\cdot u\cdot\cdot)\text{—}]$', e.g., '$N(u)(v)[\text{—}(Pv.(\exists w)(Saw))\text{—}(Pb.(\exists w)(Suw))\text{—}]$'; $\mathfrak{M}_k$ is here '$Px.(\exists w)(Syw)$'.

Some theorems on MFC follow; the proofs are omitted or only briefly indicated.

T10-1.

a. If $\mathfrak{S}_i$ is C-true by PC or by MPC or in FC, both $\mathfrak{S}_i$ and $N\mathfrak{S}_i$ are C-true in MFC.

b. $N\mathfrak{S}_i$ is C-true in MFC if and only if $\mathfrak{S}_i$ is C-true in MFC.

c. *Deduction theorem.* $\mathfrak{S}_i$ C-implies $\mathfrak{S}_j$ (in MFC) if and only if $\mathfrak{S}_i \supset \mathfrak{S}_j$ is C-true.

Proof. Since the rule of implication is the only rule of inference and all sentences are closed, this follows from [II] T6-14b.

d. $\mathfrak{S}_i$ is C-equivalent to $\mathfrak{S}_j$ (in MFC) if and only if $\mathfrak{S}_i \equiv \mathfrak{S}_j$ is C-true. (From (c).)

In what follows, 'ψ' is used to denote any context in which a given expression may occur; thus '$\psi(\mathfrak{M}_i)$' denotes any matrix containing $\mathfrak{M}_i$, and then '$\psi(\mathfrak{M}_j)$' denotes any of those matrices formed from $\psi(\mathfrak{M}_i)$ by replacing one or several (not necessarily all) occurrences of $\mathfrak{M}_i$ by $\mathfrak{M}_j$. $\mathfrak{M}_i$ and $\mathfrak{M}_j$ are called *C-interchangeable* if for any context ψ, $(\ )[\psi(\mathfrak{M}_i) \equiv \psi(\mathfrak{M}_j)]$ is C-true.

T10-2. *Theorems of replacement* in MFC.

a. If $\psi(\mathfrak{M}_i)$ and $\mathfrak{M}_j$ contain no 'N', $(\ )[(\mathfrak{M}_i \equiv \mathfrak{M}_j) \supset (\psi(\mathfrak{M}_i) \equiv \psi(\mathfrak{M}_j))]$ is C-true. (From FC, MPC, T10-1a.)

b. $(\ )[(\mathfrak{M}_i \equiv \mathfrak{M}_j) \supset (\psi(\mathfrak{M}_i) \equiv \psi(\mathfrak{M}_j))]$ is C-true (here 'N' may occur). (From FC, MPC, T10-1a.)

c. If $(\ )[\mathfrak{M}_i \equiv \mathfrak{M}_j]$ is C-true, then $\mathfrak{M}_i$ and $\mathfrak{M}_j$ are C-interchangeable. (From (b).)

d. If $\mathfrak{S}_i \equiv \mathfrak{S}_j$ is C-true, $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are C-interchangeable. (From T10-1b, (c).)

e. If $(\ )(\mathfrak{M}_i)$ is C-true, $\mathfrak{M}_i$ is C-interchangeable with 't'. (From (c).)

f. If $(\ )(\sim\mathfrak{M}_i)$ is C-true, $\mathfrak{M}_i$ is C-interchangeable with '$\sim$t'. (From (e).)

T10-3. Sentences of the following forms are C-true in MFC.

a. $(\ )[(\exists \mathfrak{i}_k)N(\mathfrak{M}_k) \supset N(\exists \mathfrak{i}_k)(\mathfrak{M}_k)]$. (The converse implication does not generally hold. 'N' is here analogous to a universal quantifier.) (From FC, T10-1a, D10-1m, D10-1b.)

b. $(\ )[\Diamond(\mathfrak{i}_k)(\mathfrak{M}_k) \supset (\mathfrak{i}_k)\Diamond(\mathfrak{M}_k)]$. (The converse implication does not generally hold. '$\Diamond$' is here analogous to an existential quantifier.) (From (a).)

c. $(\ )[(\exists \mathfrak{i}_i)(\mathfrak{i}_i \neq \mathfrak{i}_{k1} . \cdots . \mathfrak{i}_i \neq \mathfrak{i}_{kn} . N\mathfrak{M}_i) \supset (\mathfrak{i}_i)(\mathfrak{i}_i \neq \mathfrak{i}_{k1} . \cdots . \mathfrak{i}_i \neq \mathfrak{i}_{kn} \supset N\mathfrak{M}_i)]$, where $\mathfrak{M}_i$ contains $\mathfrak{i}_i$, $\mathfrak{i}_{k1}$, $\mathfrak{i}_{k2}$, $\cdots$, $\mathfrak{i}_{kn}$ as the only free variables and no individual constants. (From D10-1o.)

d. $(\ )[(\mathfrak{in}_l \neq \mathfrak{i}_{k1} . \cdots . \mathfrak{in}_l \neq \mathfrak{i}_{kn} . N\mathfrak{M}_l) \supset (\mathfrak{i}_i)(\mathfrak{i}_i \neq \mathfrak{i}_{k1} . \cdots . \mathfrak{i}_i \neq \mathfrak{i}_{kn} \supset N\mathfrak{M}_i)]$, where $\mathfrak{M}_i$ is as in (c), and $\mathfrak{M}_l$ is formed from $\mathfrak{M}_i$ by substituting $\mathfrak{in}_l$ for $\mathfrak{i}_i$. (From (c).)

e. Generalization and specification under 'N'. $(\ )[\mathfrak{in}_l = \mathfrak{i}_{km} \vee \psi_m(\mathfrak{in}_l = \mathfrak{i}_{k,m-1} \vee \psi_{m-1}(\cdots \vee \psi_2(\mathfrak{in}_l = \mathfrak{i}_{k1} \vee \psi_1[(\mathfrak{i}_i)(\mathfrak{i}_i = \mathfrak{i}_{k1} \vee \mathfrak{i}_i = \mathfrak{i}_{k2} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{i}_{km} \vee \mathfrak{i}_i = \mathfrak{in}_{j1} \vee \mathfrak{i}_i = \mathfrak{in}_{j2} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{in}_{jn} \vee \mathfrak{M}_h \vee N\mathfrak{M}_i)]))\cdots) \equiv (\mathfrak{in}_l = \mathfrak{i}_{km} \vee \psi_m(\cdots \vee \psi_1[(\mathfrak{i}_i)(\mathfrak{i}_i = \mathfrak{i}_{k1} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{i}_{km} \vee \mathfrak{i}_i = \mathfrak{in}_{j1} \vee \cdots \vee \mathfrak{i}_i = \mathfrak{in}_{jn} \vee \mathfrak{M}_h \vee N\mathfrak{M}_l)]))\cdots)]$, (the right side is like the left side except for $\mathfrak{M}_l$ instead of $\mathfrak{M}_i$) where $\mathfrak{M}_i$ contains $\mathfrak{i}_i$, $\mathfrak{i}_{k1}$, $\mathfrak{i}_{k2}$, $\cdots$, $\mathfrak{i}_{km}$ ($m \geqq 0$) as the only free variables and $\mathfrak{in}_{j1}$, $\mathfrak{in}_{j2}$, $\cdots$, $\mathfrak{in}_{jn}$ ($n \geqq 0$) as the only individual constants, and $\mathfrak{M}_l$ is formed from $\mathfrak{M}_i$ by substituting for $\mathfrak{i}_i$ $\mathfrak{in}_l$, which does not belong to the constants mentioned, and $\mathfrak{M}_h$ contains $\mathfrak{i}_i$ as free variable; $\psi_1, \cdots, \psi_m$ are arbitrary contexts. (From D10-1o, T10-2b.) (For examples of specification under 'N', see the later examples of applications of D11-1y (1).)

f. $(\ )[\mathfrak{M}_i \equiv \mathrm{N}\mathfrak{M}_i]$, where $\mathfrak{M}_i$ is built out of N-matrices and =-matrices with the help of connectives and quantifiers. (From D8-1f, g, MPC, T10-1a, D10-1m, (a).)

g. $(\ )[\mathrm{N}(\mathfrak{M}_i \vee \mathfrak{M}_j) \equiv \mathfrak{M}_i \vee \mathrm{N}\mathfrak{M}_j]$, where $\mathfrak{M}_i$ is as in (f). (From (f).)

h. $\sim\mathrm{N}\mathfrak{S}_i$, where $\mathfrak{S}_i$ is a disjunction of n components ($n \geqq 1$), each being an atomic sentence or a negation of an atomic sentence, but no atomic sentence occurring together with its negation.

Proof. We form $\mathfrak{S}_j$ from $\mathfrak{S}_i$ by substituting certain matrices for all atomic matrices occurring in the manner explained in D10-1p. The matrices to be substituted are built out of =-matrices with connectives; they can be chosen in such a way that for every disjunctive component $\mathfrak{S}_k$ of $\mathfrak{S}_j$, $\sim\mathfrak{S}_k$ is C-true; hence $\sim\mathfrak{S}_j$ is C-true, hence likewise $\mathrm{N}\sim\mathfrak{S}_j$ (T10-2b), and $\sim\mathrm{N}\mathfrak{S}_j$ (T4-1k). $\mathrm{N}\mathfrak{S}_i \supset \mathrm{N}\mathfrak{S}_j$ is C-true (D10-1p), hence also $\sim\mathrm{N}\mathfrak{S}_i$.

i. $\sim\mathrm{N}\mathfrak{S}_j$, where $\mathfrak{S}_j$ is a conjunction of m components ($m \geqq 1$), each having the form described for $\mathfrak{S}_i$ in (h). (From (h), T4-1e.)

11. MF-reduction. We shall describe a method for transforming sentences of MFL or MFC into a normal form which is both L-equivalent in MFL and C-equivalent in MFC to the given sentence. We call this method MF-reduction. It is an extension of MP-reduction. Here likewise, as in the earlier case, the fact that the reduction of a sentence $\mathfrak{S}_i$ leads to 't' is a sufficient condition for $\mathfrak{S}_i$ being both L-true (in MFL) and C-true (in MFC). Here, however, in distinction to the previous method, this result is not a necessary condition. And moreover, it is not possible to construct a method of reduction of such a kind that it leads to 't' if and only if the given sentence is C-true (or L-true); such a method would be a decision method for MFC (and MFL); this, however, is impossible because there is no decision method for FC (or FL), as Church has shown.

Thus MF-reduction has the more modest aim of leading to 't' in many cases of L-true and C-true sentences, and of leading in general to a sentence which shows the logical nature of the given sentence by simplifying its structure. Obviously, this can be done in a more or less thoroughgoing way; to any set of reduction rules of this kind it is always possible to add further rules which bring about a simplification, and in particular also a reduction to 't', in some additional cases. Now our aim will be to effect such a simplification of the structure of the sentence in two important respects: if an expression $\mathrm{N}\mathfrak{M}_i$ occurs in the reductum then $\mathfrak{M}_i$ shall always fulfill the following two conditions: (i) $\mathfrak{M}_i$ contains no 'N', (ii) $\mathfrak{M}_i$ is closed, i.e., $\mathfrak{M}_i$ contains no variable which is free in $\mathfrak{M}_i$; in other words, $\mathfrak{M}_i$ is a sentence. (ii) means that an expression of the form '$(x)[\cdots\mathrm{N}(\cdot\cdot x\cdot\cdot)\cdots]$' does not occur; however, quantifiers and hence bound variables may occur in $\mathfrak{M}_i$, e.g., '$\mathrm{N}[\cdots(x)(\cdot\cdot x\cdot\cdot)\cdots]$'. (i) and (ii) together say that every scope of 'N' is a non-modal sentence, and hence a sentence in FL and FC.

The requirements (i) and (ii) just mentioned are closely connected. It can easily be shown that, if no variables occur, the requirement (i) can be fulfilled very simply. Suppose that $N[\cdots N\mathfrak{S}_i\cdots]$ occurs where $\mathfrak{S}_i$ contains no 'N' and no variable. Then there is a simple effective method for determining whether $\mathfrak{S}_i$ is L-true or not with the help of P-reduction (T3-1d); we may then replace $N\mathfrak{S}_i$ in the first case by 't', in the second by '$\sim$t'.

If variables occur, the situation is more complicated. Suppose that all rules of MP-reduction (D5-1) which are applicable have been applied. Consider the possible forms resulting, as listed in T5-1; form (c) is either an atomic matrix or a universal matrix. We see that one 'N' can occur under another 'N' (i.e., within its scope) only if the one 'N' stands under a quantifier which in turn stands under the other 'N', e.g., $N[\cdots(x)(\cdots N\mathfrak{M}_i\cdots)\cdots]$. Here we have to distinguish two possibilities. 1. Suppose that $\mathfrak{M}_i$ is closed. In this case we can place $N\mathfrak{M}_i$ outside the operand of the quantifier '(x)' and likewise outside the operand of any other quantifier standing between the two 'N' (this is done by rules like D11-1o, p, q below). Then, by applying MP-reduction again, the one 'N' which stands under the other 'N' will disappear. 2. Suppose that 'x' occurs free in $\mathfrak{M}_i$: '$N[\cdots(x)(\cdots N(\cdot\cdot x\cdot\cdot)\cdots)\cdots]$'. This situation represents the most serious difficulty we have to overcome to reach our aim, the fulfillment of requirement (i). We must find a way of eliminating free variables under 'N'. Consider the two sentences '$(y)[y \neq \mathrm{c} \supset N(\cdot\cdot y\cdot\cdot \mathrm{c}\cdot\cdot)]$' and '$N(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{c}\cdot\cdot)$', where '$\cdot\cdot y\cdot\cdot \mathrm{c}\cdot\cdot$' contains '$y$' as the only free variable and 'c' as the only individual constant and '$\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{c}\cdot\cdot$' is the result of substituting 'a' for 'y' in '$\cdot\cdot y\cdot\cdot \mathrm{c}\cdot\cdot$'. It is clear that the first sentence C-implies the second; the second can be derived from the first by specification (D10-1g corresponding to D8-1d) and the use of '$\mathrm{a} \neq \mathrm{c}$' (D10-1k corresponding to D8-1h). Now the important fact is that, moreover, the second sentence C-implies the first by what we have called generalization under 'N'; see the example for D10-1o. Thus the two sentences are C-equivalent. Therefore we may lay down a rule of reduction which permits the replacement of the first by the second and thereby the elimination of the free variable 'y' under 'N'. This will be done in a more general way by the rule D11-1y(1). Other reduction rules are necessary for bringing the sentence into a form where this rule can be applied. In this way both aims will be reached, first (ii) and then (i), as we shall see (T11-1h).

D11-1. The *MF-reduction* of a matrix $\mathfrak{M}_i$ is its transformation according to the following rules. At any step in the transformation, the first of these rules that can be applied must be applied. (This is especially important in the case of an application of rules (r), (s), (v), and (y).) The replacement applies to any part having the specified form (without any restriction like those in D3-1 and D5-1). The final result to which none of the rules is applicable any more is called the *MF-reductum* of $\mathfrak{M}_i$.

a to **k**, as D3-1a to k, respectively.

l. Omission of 'N'. $N\mathfrak{M}_k$ is replaced by $\mathfrak{M}_k$ if $\mathfrak{M}_k$ has one of the following forms: (1) 't'; (2) '$\sim$t'; (3) $N\mathfrak{M}_h$ (with any $\mathfrak{M}_h$); (4) any $=$-matrix; (5) $\mathfrak{M}_k$ is built out of one or several matrices of the forms (3) and (4) with the help of connectives and quantifiers.

m, as D5-1m.

n. (1). A =-matrix with two occurrences of the same individual constant or variable is replaced by 't'.

(2). A =-matrix with two different individual constants is replaced by '$\sim$t'.

o. Omission of quantifier. $(\mathfrak{i}_k)(\mathfrak{M}_k)$, where $\mathfrak{i}_k$ does not occur free in $\mathfrak{M}_k$, is replaced by $\mathfrak{M}_k$.

p. Distribution of quantifier in conjunction. $(\mathfrak{i}_k)(\mathfrak{M}_{k1}.\mathfrak{M}_{k2}.\cdots.\mathfrak{M}_{kn})$ $(n \geqq 2)$ is replaced by $(\mathfrak{i}_k)(\mathfrak{M}_{k1}).(\mathfrak{i}_k)(\mathfrak{M}_{k2}).\cdots.(\mathfrak{i}_k)(\mathfrak{M}_{kn})$.

q. Quantifier with disjunction. Suppose that $(\mathfrak{i}_k)(\mathfrak{M}_k)$ occurs where $\mathfrak{M}_k$ is a disjunction among whose n components $(n \geqq 2)$ there is at least one in which $\mathfrak{i}_k$ occurs freely and at least one in which it does not. Let $\mathfrak{M}_h$ be the disjunction of the components of the first kind (in their original order) and $\mathfrak{M}_j$ the disjunction of the components of the second kind (in their original order). $(\mathfrak{i}_k)(\mathfrak{M}_k)$ is replaced by $(\mathfrak{i}_k)(\mathfrak{M}_h) \vee \mathfrak{M}_j$.

r. Infinity. $(\mathfrak{i}_k)(\mathfrak{M}_k)$, where $\mathfrak{M}_k$ is either a =-matrix or a disjunction of =-matrices, is replaced by '$\sim$t'. [Note that after application of (q) and (m), every component of $\mathfrak{M}_k$ contains $\mathfrak{i}_k$ and, in addition, either a variable different from $\mathfrak{i}_k$ or an individual constant.]

s. Elimination of $\neq$-matrices.

(1). $(\mathfrak{i}_k)(\sim\mathfrak{M}_k)$, where $\mathfrak{M}_k$ is a =-matrix (which contains $\mathfrak{i}_k$, according to (o)), is replaced by '$\sim$t'.

(2). Suppose that $(\mathfrak{i}_k)(\mathfrak{M}_k)$ occurs where $\mathfrak{M}_k$ is a disjunction among whose n components $(n \geqq 2)$ there is at least one of the form $\sim\mathfrak{M}_j$, where $\mathfrak{M}_j$ is a =-matrix containing $\mathfrak{i}_k$ and another individual sign. $(\mathfrak{i}_k)(\mathfrak{M}_k)$ is replaced by the matrix formed from $\mathfrak{M}_k$ by first omitting the first component of the kind described and then substituting for $\mathfrak{i}_k$ the other individual sign mentioned.

t. Quantifier under 'N'. Suppose that N$\mathfrak{S}_i$ occurs ($\mathfrak{S}_i$ is a sentence and hence closed!), where either $\mathfrak{S}_i$ itself or the first disjunctive component in $\mathfrak{S}_i$ of universal form is $(\mathfrak{i}_k)(\mathfrak{M}_k)$. Then this quantifier $(\mathfrak{i}_k)$ is shifted before the 'N' (that is to say, N$\mathfrak{S}_i$ is replaced by $(\mathfrak{i}_k)[\mathrm{N}\mathfrak{M}_j]$, where $\mathfrak{M}_j$ results from $\mathfrak{S}_i$ by omitting the occurrence in question of the quantifier $(\mathfrak{i}_k)$). [After this, rule (w) becomes applicable.]

u. Suppose that a sentence $\mathfrak{S}_i$ (closed!) of the form $(\mathfrak{i}_{k1})(\mathfrak{i}_{k2})\cdots(\mathfrak{i}_{kn})[\mathrm{N}\mathfrak{M}_k]$ occurs $(n \geqq 2)$. $\mathfrak{S}_i$ is replaced by $(\mathfrak{i}_{k1})[\mathrm{N}(\mathfrak{i}_{k2})\cdots(\mathfrak{i}_{kn})(\mathfrak{M}_k)]$. [After this, rule (w) becomes applicable, and then (t).]

v. Factual sentence under 'N'. N$(\mathfrak{S}_i)$ is replaced by '$\sim$t' if $\mathfrak{S}_i$ has one of the following forms: (1) an atomic sentence, (2) the negation of an atomic sentence, (3) a disjunction of sentences of forms (1) and (2). [Note that, because of (a), a disjunction of the form (3) does not contain any atomic sentence together with its negation.]

w. (Some simple cases of elimination of a free variable under 'N'.) Suppose that $\mathfrak{S}_i$ of the form $(\mathfrak{i}_k)[\mathrm{N}\mathfrak{M}_k]$ or $\mathfrak{S}_j$ of the form $(\mathfrak{i}_k)[\sim\mathrm{N}\mathfrak{M}_k]$ occurs. Let the individual constants occurring in $\mathfrak{M}_k$ be (in alphabetical order) $\mathfrak{in}_{k1}$, $\mathfrak{in}_{k2}$, $\cdots$, $\mathfrak{in}_{kn}$ $(n \geqq 0)$; let $\mathfrak{in}_l$ be an individual constant (the first in alphabetical order) not occurring in $\mathfrak{M}_k$. Let $\mathfrak{S}_{k1}$, $\mathfrak{S}_{k2}$, $\cdots$, $\mathfrak{S}_{kn}$, and $\mathfrak{S}_l$ be formed from $\mathfrak{M}_k$ by substituting for $\mathfrak{i}_k$ $\mathfrak{in}_{k1}$, $\mathfrak{in}_{k2}$, $\cdots$, $\mathfrak{in}_{kn}$, and $\mathfrak{in}_l$, respectively.

(**1**). Let $\mathfrak{M}_k$ contain no individual constant. Then $\mathfrak{S}_i$ is replaced by $N\mathfrak{S}_l$, and $\mathfrak{S}_j$ by $\sim N\mathfrak{S}_l$.

(**2**). Let $\mathfrak{M}_k$ contain neither '=' nor 'N'. Then $\mathfrak{S}_i$ is replaced by $N\mathfrak{S}_l$, and $\mathfrak{S}_j$ by $\sim N\mathfrak{S}_{k1}.\sim N\mathfrak{S}_{k2}.\cdots.\sim N\mathfrak{S}_{kn}$.

(**3**). (To be applied only if (1) and (2) are not applicable.) $\mathfrak{S}_i$ is replaced by $N\mathfrak{S}_{k1}.N\mathfrak{S}_{k2}.\cdots.N\mathfrak{S}_{kn}.N\mathfrak{S}_l$, and $\mathfrak{S}_j$ by $\sim N\mathfrak{S}_{k1}.\sim N\mathfrak{S}_{k2}.\cdots.\sim N\mathfrak{S}_{kn}.\sim N\mathfrak{S}_l$.

x. Disjunction under 'N'. Suppose that $N\mathfrak{M}_h$ occurs where $\mathfrak{M}_h$ is a disjunction of n components ($n \geqq 2$) of which at least one has one of the forms (3), (4), (5) described in (1). Let $\mathfrak{M}_j$ be the disjunction of all components of this kind (in their original order) and $\mathfrak{M}_l$ be the disjunction of the remaining components (in their original order). [There is at least one remaining component because of (1).] Then $N\mathfrak{M}_h$ is replaced by $\mathfrak{M}_j \vee N\mathfrak{M}_l$.

y. Elimination of a free variable under 'N'. Let $N\mathfrak{M}_k$ be the first occurrence of an N-matrix such that $\mathfrak{M}_k$ contains a free variable but no 'N'. Consider the universal quantifiers binding the free variables in $\mathfrak{M}_k$; let the last one of these quantifiers contain $\mathfrak{i}_j$, and let $\mathfrak{K}_k$ be the class of the remaining free variables in $\mathfrak{M}_k$. Let $\mathfrak{M}_j$ be the operand of the quantifier with $\mathfrak{i}_j$ just mentioned (hence the occurrence of $N\mathfrak{M}_k$ in question is a part of $\mathfrak{M}_j$). Let $\mathfrak{K}_l$ be the class of the individual constants occurring in $\mathfrak{M}_k$, and let $\mathfrak{in}_l$ be an individual constant (the alphabetically first one) not occurring in $\mathfrak{M}_k$. ($\mathfrak{K}_k$ and $\mathfrak{K}_l$ may be empty.) We examine whether or not the following conditions (α) and (β) are fulfilled. Then one and only one of the subsequent rules (1), (2), (3) is applicable.

(α). $\mathfrak{M}_j$ is a disjunction with two or more components such that for every sign $\mathfrak{a}_i$ (variable or individual constant) in $\mathfrak{K}_k$ and in $\mathfrak{K}_l$ there is a component $\mathfrak{i}_j = \mathfrak{a}_i$ or $\mathfrak{a}_i = \mathfrak{i}_j$.

(β). For every variable $\mathfrak{i}_k$ in $\mathfrak{K}_k$, the operand (which contains the occurrence of $N\mathfrak{M}_k$ in question) of the quantifier with $\mathfrak{i}_k$ contains as (proper or improper) part a disjunction of which one component is $\mathfrak{i}_k = \mathfrak{in}_l$ or $\mathfrak{in}_l = \mathfrak{i}_k$ and another component contains the occurrence of $(\mathfrak{i}_j)(\mathfrak{M}_j)$ in question.

(**1**). *Specification* under 'N'. Suppose that the conditions (α) and (β) are fulfilled. Then the occurrence of $N\mathfrak{M}_k$ in question is replaced by $N\mathfrak{M}_l$, where $\mathfrak{M}_l$ is formed from $\mathfrak{M}_k$ by substituting $\mathfrak{in}_l$ for $\mathfrak{i}_j$.

(**2**). Suppose that the condition (α) is not fulfilled for all signs in $\mathfrak{K}_k$ and $\mathfrak{K}_l$. Let the signs for which it is not fulfilled be (in alphabetical order) $\mathfrak{a}_{h1}, \mathfrak{a}_{h2}, \cdots, \mathfrak{a}_{hn}$ ($n \geqq 1$). Let $\mathfrak{M}_{h1}, \mathfrak{M}_{h2}, \cdots, \mathfrak{M}_{hn}$ be formed from $\mathfrak{M}_j$ by substituting for $\mathfrak{i}_j$ $\mathfrak{a}_{h1}, \cdots, \mathfrak{a}_{hn}$, respectively. Then the occurrence of $(\mathfrak{i}_j)(\mathfrak{M}_j)$ in question is replaced by $(\mathfrak{i}_j)[\mathfrak{i}_j = \mathfrak{a}_{h1} \vee \mathfrak{i}_j = \mathfrak{a}_{h2} \vee \cdots \vee \mathfrak{i}_j = \mathfrak{a}_{hn} \vee \mathfrak{M}_j].[\mathfrak{M}_{h1}.\mathfrak{M}_{h2}.\cdots.\mathfrak{M}_{hn}]$.

(**3**). Suppose that condition (α) is fulfilled but not (β). Let the last one among those quantifiers which do not fulfill (β) contain the variable $\mathfrak{i}_h$ and have the operand $\mathfrak{M}_h$. Then this occurrence of $(\mathfrak{i}_h)(\mathfrak{M}_h)$ is replaced by $(\mathfrak{i}_h)[\mathfrak{i}_h = \mathfrak{in}_l \vee \mathfrak{M}_h].\mathfrak{M}_l$, where $\mathfrak{M}_l$ is formed from $\mathfrak{M}_h$ by substituting $\mathfrak{in}_l$ for $\mathfrak{i}_h$.

T11-1. Let $\mathfrak{S}_j$ be a sentence in MFL and MFC to which none of the rules of MF-reduction (D11-1) is any more applicable. Then every matrix occurring in $\mathfrak{S}_j$, including $\mathfrak{S}_j$ itself, has one of the following forms.

a. Atomic matrix.

b. Negation of (a).

c. =-matrix. This matrix contains either a variable and a constant or two different variables. It is a disjunctive component of an operand whose quantifier contains a variable occurring in this matrix.

d. Disjunction with two or more components. Every component has one of the forms (a), (b), (c), (f), (g), (h). No component is the negation of another component. No two components are alike.

e. Conjunction with two or more components. Every component has one of the forms (a), (b), (d), (f), (g), (h). No component is the negation of another component. No two components are alike. This form occurs only as the whole sentence $\mathfrak{S}_i$ (or as a partial conjunction of the whole conjunction).

f. $(\mathfrak{i}_k)(\mathfrak{M}_k)$. $\mathfrak{M}_k$ has one of the forms (a), (b), (d), (f), (g).

g. $\sim(\mathfrak{i}_k)(\mathfrak{M}_k)$. $\mathfrak{M}_k$ is as in (f).

h. $\mathrm{N}\mathfrak{S}_k$ or $\sim\mathrm{N}\mathfrak{S}_k$. $\mathfrak{S}_k$ is closed, contains a predicate and a quantifier, but no 'N'. $\mathfrak{S}_k$ has either form (g) or (d); if a disjunction (d), then every component has one of the forms (a), (b), (g), and at least one has (g). The forms (h) do not occur in an operand of a quantifier, but only either (1) as $\mathfrak{S}_i$ itself or (2) as a conjunctive component of $\mathfrak{S}_i$ or (3) as a disjunctive component of $\mathfrak{S}_i$ or of a conjunctive component of $\mathfrak{S}_i$.

i. 't' or '$\sim$t'. This occurs only if the whole sentence $\mathfrak{S}_i$ is 't' or '$\sim$t'.

Examples for rule D11-1y, *elimination of free variables under 'N'*. 1. Let the sentence '$(y)(z)[z = y \vee \mathrm{N}(\cdot\cdot y\cdot\cdot z\cdot\cdot)]$' be given, where '$\cdot\cdot y\cdot\cdot z\cdot\cdot$' indicates a matrix containing 'y' and 'z' as the only free variables but containing no individual constants and no 'N', and having such a form that no rule preceding D11-1y is applicable. $\mathfrak{M}_k$ is here '$\cdot\cdot y\cdot\cdot z\cdot\cdot$'; $\mathfrak{i}_j$ is 'z'; $\mathfrak{K}_k$ contains only 'y'; $\mathfrak{K}_l$ is empty; $\mathfrak{in}_l$ is 'a'. (α) is fulfilled because '$z = y$' occurs. (β) is not fulfilled because '$y =$ a' does not occur. Therefore rule (y)(3) is applied, with 'y' as $\mathfrak{i}_h$. The result is: '$(y)[y = \mathrm{a} \vee (z)[z = y \vee \mathrm{N}(\cdot\cdot y\cdot\cdot z\cdot\cdot)]].(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. Now we apply rule (y) to the first N-matrix. This time (y)(1) is applicable; the result is: '$(y)[y = \mathrm{a} \vee (z)[z = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]].(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. We apply rule (y) again to the first N-matrix. (α) is not fulfilled because '$z =$ a' does not occur. Therefore rule (y)(2) is applied, with 'a' as $\mathfrak{a}_{h1}$: '$(y)[y = \mathrm{a} \vee [(z)(z = \mathrm{a} \vee z = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)).(\mathrm{a} = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{a}\cdot\cdot))]].(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. Now rule (j) is applied, distribution with respect to the first sign of conjunction; then the quantifier with 'y' is distributed (rule (p)): '$(y)[y = \mathrm{a} \vee (z)(z = \mathrm{a} \vee z = \mathrm{y} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot))].(y)[y = \mathrm{a} \vee \mathrm{a} = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{a}\cdot\cdot)].(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. Now the second N-matrix is moved out of the operand of the quantifier (rule (q)); thereby a disjunctive component of the form '$(y)[y = \mathrm{a} \vee \mathrm{a} = y]$' is produced, which is then transformed into '$\sim$t' (rule (r)) and then omitted (rule (e)). The result is: '$(y)[y = \mathrm{a} \vee (z)(z = \mathrm{a} \vee z = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot))].\mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{a}\cdot\cdot).(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. Now (y)(1) can be applied to the first N-matrix, with 'b' as $\mathfrak{in}_l$: '$(y)[y = \mathrm{a} \vee (z)(z = \mathrm{a} \vee z = y \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{b}\cdot\cdot))].\mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{a}\cdot\cdot).(z)[z = \mathrm{a} \vee \mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)]$'. As previously, rules (q), (r), and (e) are applied, first to the first quantifier with 'z' and then to that with 'y', so that both quantifiers disappear; thus the first conjunctive component becomes '$\mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{b}\cdot\cdot)$'. Finally we transform the last conjunctive component. According to (y)(1), the matrix '$\mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot z\cdot\cdot)$' is replaced by '$\mathrm{N}(\cdot\cdot \mathrm{a}\cdot\cdot \mathrm{b}\cdot\cdot)$'. Now the rules (q), (r), and (e) are applied again thereby the quantifier with 'z' disappears. Thus the component becomes 'N(··a··b··)' and disappears (rule (d)). Hence the result of the whole reduction is 'N(··a··b··).N(··a··a··)'.

2. Let '– – –' indicate the original sentence of the first example. Consider 'N(u)N[(x)(Rxu) ∨ – – –]'. Here, the first 'N' is of third order. The expression preceding the square bracket becomes '(u)NN' (rule (t)) and then '(u)N' (rule (l)(3)). Then rule (x) yields '(u)[N(x)(Rxu) ∨ – – –]' because '– – –' has form (l)(5). Rule (q) yields '(u)[N(x)(Rxu)] ∨ – – –'; (w)(1), 'N(x)(Rxa) ∨ – – –'; (t), '(x)N(Rxa) ∨ – – –'; (w)(2), 'N(Rba) ∨ – – –'. The first disjunctive component becomes '∼t' (rule (v)(1)) and disappears (rule (e)). Thus the reductum of the whole is the same as in the first example.

12. Relations between MFC and MFL. The first question concerning the calculus MFC is whether it is not too strong, that is, whether it is indeed a formalization of MFL and hence justified by MFL as an L-true interpretation of it. The affirmative answer to this question is given by T12-1d. Then the second question, whether MFC is strong enough for certain purposes, will be examined.

T12-1.

a. Every primitive sentence of MFC is L-true in MFL. (The proof is based on complicated theorems concerning state-descriptions, which have not been stated here.)

b. Every C-true sentence in MFC is L-true in MFL. (From (a); see T8-1b.)

c. Whenever C-falsity in MFC holds, then L-falsity in MFL holds. Analogously with C-implication and L-implication, and with C-equivalence and L-equivalence. (From (b).)

d. MFL is an *L-true interpretation* of MFC. (From (a) and (b); see T8-1c.)

We shall now show that MFC is strong enough to cover the transformation by MF-reduction (T12-2b). However, the further question whether it is also strong enough to yield all L-true sentences remains open, as we shall see.

T12-2. Let $\mathfrak{S}_i$ be any sentence in MFC and MFL, and $\mathfrak{S}_j$ its *MF-reductum*. Then the following holds:

a. If a sentence is transformed into another sentence by an application of the first rule of MF-reduction which is applicable, then the two sentences are C-equivalent in MFC.

Proof. For the rules (a) to (k) in D11-1 this follows from T6-1e, T10-1a. For rules (l)(1), (2), and (3), from D5-1l(3), (4), (1), T6-1e, T10-1a. For (l)(4), from D10-1i, j (corresponding to D8-1f, g), and m. For (l)(5), from T10-3f. For (m) (= D5-1m), from T4-1e, T10-1a. For (n)(1), from D8-1f, T10-1a, T10-2e. For (n)(2), from D8-1h. For (o), from D8-1c, d. For (p) and (q), from FC, T10-1a. For (r), from D8-1h. For (s)(1), from D8-1f. For (s)(2), from D8-1g. For (t) and (u), from D10-1l, m, and FC. For (v), from T10-3h. For (w), from D10-1n and o. For (x), from T10-3g. For (y), from T10-3e.

b. $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are C-equivalent in MFC. (From (a).)

c. $\mathfrak{S}_i$ and $\mathfrak{S}_j$ are L-equivalent in MFL. (From (b), T12-1c.)

d. If $\mathfrak{S}_j$ is 't', $\mathfrak{S}_i$ is C-true in MFC and L-true in MFL. (From (b), (c).)

e. If $\mathfrak{S}_j$ is '∼t', $\mathfrak{S}_i$ is C-false in MFC and L-false in MFL. (From (b), (c).) —The converses of (d) and (e) do not hold generally. For instance, 'N[∼(x)(Px) ∨ Pa]' is L-true and C-true, but is an MF-reductum.

f. If $\mathfrak{S}_i$ contains no predicate, $\mathfrak{S}_i$ is either 't' or '$\sim$t'. (From T11-1.)

g. If $\mathfrak{S}_i$ contains no quantifier under an 'N', then $\mathfrak{S}_i$ contains no 'N'.

Proof. In the reduction procedure, no quantifier comes under 'N' except by rule D11-1u; but these quantifiers are then removed by rules (w) and (t). Therefore, $\mathfrak{S}_i$ contains no quantifier under 'N', and hence no 'N' (T11-1h).

Since there is no general decision method for FL (i.e., no effective method which decides for any given sentence in a finite number of steps whether or not it is L-true in FL), there is, of course, none for MFL. However, the method of MF-reduction makes it possible to apply partial solutions for FL, that is to say, decision methods for restricted classes of sentences in FL, to MFL in the following manner. A sentence is L-true in MFL if and only if its MF-reductum is L-true (T12-2c). The latter has in general the form of a conjunction of disjunctions (T11-1). A conjunction is L-true if and only if each of its components is L-true. Each component is in general a disjunction of the form $\sim N\mathfrak{S}_{i1} \vee \cdots \vee \sim N\mathfrak{S}_{im} \vee N\mathfrak{S}_{j1} \vee \cdots \vee N\mathfrak{S}_{jn} \vee \mathfrak{S}_k$, where $\mathfrak{S}_k$ is a disjunction of p N-free sentences ($m \geqq 0, n \geqq 0, p \geqq 0; m + n + p \geqq 1$). Since the components with '$\sim$N' and 'N' are either L-true or L-false, the whole disjunction is L-true if and only if either one of these components or $\mathfrak{S}_k$ is L-true, in other words, if and only if either one of the sentences $\mathfrak{S}_{i1}, \cdots, \mathfrak{S}_{im}$ is non-L-true or one of the sentences $\mathfrak{S}_{j1}, \cdots, \mathfrak{S}_{jn}, \mathfrak{S}_k$ is L-true. All these sentences are N-free and hence belong to FL also. Therefore, if all of them belong to classes for which decision methods for FL are available, then their application leads to a decision for the whole disjunction in MFL.

As has been remarked earlier (see the discussion preceding T8-2), it is not known whether FC is complete. Therefore, it is likewise an open question whether or not MFC is complete in the sense that every sentence which is L-true in MFL is C-true in MFC. The following theorem gives a partial answer to this question.

T12-3. Let $\mathfrak{S}_i$ be an MF-reductum in which no '=' and no sentence of the form '$\sim$N($\cdots$)' occurs. If $\mathfrak{S}_i$ is L-true in MFL, it is C-true in MFC.

Proof. Let the conditions be fulfilled. Then, disregarding simpler cases, in which the theorem is obvious, $\mathfrak{S}_i$ is a conjunction of which all components are L-true. Let $\mathfrak{S}_h$ be such a component. $\mathfrak{S}_h$ has the form $N\mathfrak{S}_{j1} \vee N\mathfrak{S}_{j2} \vee \cdots \vee N\mathfrak{S}_{jn} \vee \mathfrak{S}_k$, where $\mathfrak{S}_k$ is a disjunction of N-free sentences. At least one of the sentences $\mathfrak{S}_{j1}, \cdots, \mathfrak{S}_{jn}, \mathfrak{S}_k$ must be L-true in MFL and also in FL, since all these sentences are N-free. If $\mathfrak{S}_k$ is L-true in FL, it is C-true in FC (T8-2) and hence in MFC (T10-1a), and hence likewise the whole disjunction. If $\mathfrak{S}_{jr}$ ($r = 1, \cdots, n$) is L-true in FL, it is C-true in FC and in MFC, and hence likewise $N\mathfrak{S}_{jr}$ (T10-1a) and again the whole disjunction. Thus every component of the conjunction $\mathfrak{S}_i$ is C-true in MFC, and hence $\mathfrak{S}_i$ itself.

Whether the two restricting conditions in this theorem can be eliminated remains to be seen. If they can, MFC is complete. The restriction with respect to '=' is a problem concerning FC, as earlier discussed. With respect to MFC, the problem remains whether an MF-reductum of the form $\sim N\mathfrak{S}_i$ is C-true if it is L-true in MFL, in other words, if $\mathfrak{S}_i$ is not L-true in MFL. If $\mathfrak{S}_i$ is L-false, the proof of $\sim N\mathfrak{S}_i$ is simple (with T4-1k). If $\mathfrak{S}_i$ is factual, $\sim N\mathfrak{S}_i$ can often be proved in the following way. We construct an L-false sentence $\mathfrak{S}_j$ from $\mathfrak{S}_i$ by suitable substitutions for primitive predicates. Then D10-1p yields $N\mathfrak{S}_i \supset N\mathfrak{S}_j$ and hence $\sim N\mathfrak{S}_j \supset \sim N\mathfrak{S}_i$. Since $\sim N\mathfrak{S}_j$ is C-true, so is $\sim N\mathfrak{S}_i$. As examples, see T10-3h and i, which lead to the following theorem (in which $\mathfrak{S}_k$ is not required to be a reductum).

T12-4. If $\mathfrak{S}_k$ is a factual sentence in MFL without variables, then $\sim N\mathfrak{S}_k$ and $\sim N\sim\mathfrak{S}_k$ (in other symbols, $\Diamond\sim\mathfrak{S}_k$ and $\Diamond\mathfrak{S}_k$) are C-true in MFC. (From T10-3h and i.)

Even if variables occur, the procedure of substitution for predicates yields often a proof for $\sim N\mathfrak{S}_i$ with factual $\mathfrak{S}_i$. The decisive question remains whether this or another procedure will assure C-truth in *all* such cases.

This article has been restricted to an exhibition of some modal systems and to an explanation of some of their technical features. At another place[17] I shall discuss certain fundamental problems connected with modalities which are, so to speak, of a pre-technical nature. These discussions will not presuppose the systems but, on the contrary, try to prepare the ground for the construction of modal systems by clarifying some basic logical and semantical concepts. The problems concern, in particular, the relation of denotation and the nature of denoted entities; further, the concepts of the extension and the intension of linguistic expressions. It is customary to distinguish, for example, between the extension of a predicate (of degree one), which is a class, and its intension, which is a property. Further, we may distinguish between the extension of a sentence, which is a truth-value, and its intension, which is a proposition. I shall try to show that, in an analogous way, we have to distinguish between the extension of an individual expression (e.g., an individual description or a constant abbreviating it), which is an individual (e.g., a physical thing or a space-time point), and its intension, which is a concept of a special kind which we may call an individual concept. The distinction in this case as in the two other cases, becomes essential when the expressions occur in non-extensional contexts, e.g., in modal sentences. It will be shown that this distinction also helps to clarify and to overcome the difficulties which Quine[18] has pointed out for sentences combining modalities and quantification. Without an elimination of these difficulties, no modal functional logic could be constructed. Further, some related problems which have been raised by Church will be discussed.

The approach in the present article, which leaves aside all these fundamental problems, may appear to be uncritical and dogmatic. This appearance, however, is not due to an actual neglect of these problems but merely to the fact that for the sake of brevity this article had to be restricted to the technical aspects of modal systems.

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[17] *Meaning and necessity: A study in semantics and modal logic.* (To appear soon.)

[18] W. V. Quine, *Notes on existence and necessity.* ***The journal of philosophy,*** vol. 40 (1943), pp. 113–127.