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MODALITIES AND QUANTIFICATION

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1. The problems of modal logic. The purpose of this article is to give a survey of some results I have found in investigations concerning logical modalities. The results refer: (1) to semantical systems, i.e., symbolic language systems for which semantical rules of interpretation are laid down; (2) to corresponding calculi, i.e., syntactical systems with primitive sentences and a rule of inference; (3) to relations between a semantical system and the corresponding calculus.

The semantical systems to be dealt with are the following: propositional logic (PL), functional logic (FL), and the corresponding modal systems, viz. modal propositional logic (MPL) and modal functional logic (MFL). MPL is built out of PL by the addition of the symbol 'N' for logical necessity; likewise MFL out of FL. In terms of Lewis's symbol '$\Diamond$' for logical possibility, 'Np' means the same as '$\sim\Diamond\sim p$'. All other logical modalities can, of course, be defined on the basis of 'N'; e.g., impossibility by 'N$\sim p$', possibility by '$\sim$N$\sim p$', contingency by '$\sim$Np.$\sim$N$\sim p$', etc.

The calculi corresponding to these semantical systems are the following: the propositional calculus (PC), the functional calculus (FC), and the modal calculi (MPC and MFC) again constructed by the addition of 'N'.

Lewis's systems of strict implication[1] are forms of MPC. So far, no forms of MFC have been constructed, and the construction of such a system is our chief aim. The corresponding semantical systems MPL and MFL are constructed chiefly for the purpose of enabling us to show that the modal calculi MPC and MFC are adequate, i.e., that every sentence provable in them is L-true (analytic). With the help of a normal form, it can further be shown that for MPC the inverse holds also; MPC is complete in the sense that every sentence which is L-true in MPL is provable in MPC. The reduction to the normal form constitutes a decision method for MPC and MPL. For MFC likewise a method of reduction to a normal form will be given. This reduction removes all occurrences of 'N' of higher order, i.e., such that the scope of one 'N' contains another 'N'. A decision method for MFC is of course not possible; however, the reduction makes it possible to apply to MFC the known decision methods for special cases in FC.

The semantical systems FL and MFL contain an infinite number of individual constants. Therefore the representation of these systems requires a very strong metalanguage, dealing with classes of classes of sentences. Consequently, the semantical concepts defined, e.g., L-truth, are indefinite (non-effective) to a high degree. The chief reasons for constructing corresponding calculi are here, as usually in the case of logical calculi, the following two: (1) avoidance of any reference to the meanings of the signs and sentences, (2) use of basic concepts which are effective. The second purpose is here, as generally in the case of calculi without transfinite rules, fulfilled in the following sense. Al-

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[1] C. I. Lewis and C. H. Langford, ***Symbolic logic***, 1932; the systems are developed from those in Lewis's earlier book (1918).

though C-truth (provability) is not itself an effective concept, it is defined on the basis of two effective concepts, viz. the concept of primitive sentence, given by a finite list of primitive sentence schemata, and the concept of direct derivability, defined by the rule of inference.

For lack of space, this article will state only a few of the relevant theorems, most of them without proofs. For the same reason, this article will be restricted to the technical aspects of the systems dealt with and will not contain any discussion of the more general problems connected with logical modalities.

The guiding idea in our constructions of systems of modal logic is this: a proposition p is logically necessary if and only if a sentence expressing p is logically true. That is to say, the modal concept of the logical necessity of a proposition and the semantical concept of the logical truth or analyticity of a sentence correspond to each other. Both concepts have been used in logic and philosophy, mostly, however, without exact rules. If we succeed in explicating one of these two concepts, that is, in finding an exact concept, which we call the explicatum, to take the place of the given inexact concept, the explicandum, then this leads, on the basis of the parallelism stated between the two concepts, to an explication for the other concept. Now it is easy to give, with the help of the semantical concepts of state-description and range, an exact definition for 'L-true' as an explicatum for logical truth with respect to the systems PL and FL, as we shall see. Therefore it seems natural to interpret 'N' in such a way that the following convention is always fulfilled:

C1-1. If '$\cdots$' is any sentence in a system S containing 'N', then the corresponding sentence 'N($\cdots$)' is to be taken as true if and only if '$\cdots$' is L-true in S.

This convention determines our interpretation of 'N', but it is not a definition for 'N'. The sentence 'N($\cdots$)' cannot be transformed by definition into the sentence "'$\cdots$' is L-true in S," because the first sentence belongs to the object-language S, the second to the metalanguage M; but the first sentence holds, according to the convention, if and only if the second holds. We shall not define 'N' (it cannot be defined on the basis of the ordinary truth-functional connectives and quantifiers for individuals) but shall take it as a primitive sign in MPL and MFL. However, we shall frame the semantical rules of these systems in such a manner that the convention is fulfilled.[2, 3]

C1-1 gives a sufficient and necessary condition for the truth of 'N($\cdots$)'. Now the following two questions remain: (1) if 'N($\cdots$)' is true, is it L-true? If so, 'NN($\cdots$)' is likewise true; in other words, '$\mathrm{N}p \supset \mathrm{NN}p$' is always true.

[2] I shall hereafter refer to the following publications of mine by the signs in square brackets:

[Syntax] *The logical syntax of language,* (1934) 1937.

[I] *Introduction to semantics,* 1942.

[II] *Formalization of logic,* 1943.

[3] I have indicated the parallelism between the modal concept of the necessity of a proposition and the meta-concept of the analyticity of a sentence first in [Syntax] §69 (where, however, 'analytic' was still regarded as a syntactical term), and, more clearly, in [I] pp. 91 ff.

(2) If 'N(· · ·)' is false, is it L-false? ('L-false' is taken as the explicatum for 'logically false,' 'self-contradictory.') If so, '∼N(. . .)' is L-true and hence 'N∼N(· · ·)' is true; in other words, '∼Np ⊃ N∼Np' is always true.

At the present moment, these questions are not meant with respect to any given system, but as pre-systematic questions, concerning the inexact, pre-systematic explicandum rather than the exact explicatum. The purpose of the following considerations is merely to make the vague meaning of logical necessity or logical truth clearer to ourselves, so as to lead to a convention more specific than C1-1 concerning the use of 'N'. This convention will then later guide us in constructing our systems. Once the systems are constructed, the two questions can be answered in an exact way. At the present stage, however, our considerations, as always in tasks of self-clarification, are necessarily inexact and, in a certain sense, even circular.

In order to make clearer what is meant by the explicandum of logical, necessary truth, we will distinguish two kinds of data concerning any sentence 'C' as follows:

I. The meaning of 'C' is given, that is to say, the interpretation assigned to 'C' by the semantical rules. (In technical terms, the rules may either be formulated so as to determine the proposition expressed by 'C' or so as to determine the range of 'C'; the rules of our system will have the latter form.)

II. Information concerning the facts relevant for 'C' is given, that is to say, concerning the properties and relations of the individuals involved.

If the answer to a given question is merely dependent upon data of the kind I but independent of those of kind II, we call it a logical question; if in addition, data of the kind II are required, we call it a factual question. In particular, if a sentence 'C' is true in such a way that its truth is based on I alone, we regard it as logically true; if its truth is dependent upon II also, we regard it as factually, contingently true. This conception of the distinction between logical and factual truth as explicandum will guide our choice of the definition of 'L-true' as explicatum. It seems to me that this conception is in agreement with customary conceptions.

Let us take as an example the sentence 'Pa.∼Qb', which we abbreviate by 'A'. We learn from the semantical rules what individuals are named by 'a' and 'b' and what properties are designated by 'P' and 'Q'; we learn further that 'A' says that a is P and b is not Q. This is all we can obtain from data of the kind I. In order to establish the truth-value of 'A' we need data of the kind II, viz., information whether or not a is P and whether or not b is Q. Thus, 'A' is neither L-true nor L-false; we say that it is L-indeterminate or factual.

(i) Now consider the sentence 'A ∨ ∼A'. We can find that it is true by using merely the semantical rules for '∨' and '∼' (in our system, the rules of ranges D7-5c and b, which correspond to the customary truth-tables for the two connectives); we need no factual information concerning the individuals a and b occurring in the sentence. Therefore, 'A ∨ ∼A' is L-true. Hence, according to our convention C1-1, 'N(A ∨ ∼A)' is true. The question is whether it is L-true. Now we can easily see that it must be, because the truth of this N-sentence follows from those semantical rules by which we established the truth and hence the L-truth of 'A ∨ ∼A' together with the semantical rule for 'N' which is to be laid down in accordance with C1-1. Thus no factual knowledge is required for establishing that 'N(A ∨ ∼A)' is true; hence it is L-true.

(ii) Similarly, the falsity of 'A.∼A' can be established by the semantical rules alone. Therefore, this sentence is L-false and not L-true. Hence, according to C1-1, 'N(A.∼A)' is false and '∼N(A.∼A)' is true.

(iii) Finally, let us go back to the sentence 'A' itself, i.e., 'Pa.∼Qb'. We found that 'A' is neither L-true nor L-false by merely using semantical rules, not using any factual knowledge concerning the individuals occurring in 'A'. Therefore we see that, according to C1-1, 'N(A)' is false and '∼N(A)' is true. These results are based merely on the semantical rules for the signs occurring in 'A' and for 'N'. Therefore, 'N(A)' is L-false and '∼N(A)' is L-true.

The results found for these simple examples can be generalized. Let 'C' be an abbreviation for a given sentence of any form with or without 'N'.

(i) Suppose that 'N(C)' is true. Then, according to C1-1, 'C' must be L-true. Hence the truth of 'C' is determined by certain semantical rules. Then these same rules together with the rule for 'N' determine the truth of 'N(C)'. Therefore, 'N(C)' is L-true, and hence 'NN(C)' is true. Thus our earlier question (1) is answered in the affirmative.

(ii) Let us now suppose that 'C' is L-false and hence 'N(C)' is false. Then those semantical rules which determine the falsity of 'C' together with the rule for 'N' determine the falsity of 'N(C)'. Therefore, 'N(C)' is L-false, and '∼N(C)' is L-true.

(iii) Finally, let us suppose that 'N(C)' is false but 'C' is not L-false. Then 'C' is neither L-true nor L-false. The decisive question here is this: is the result that 'C' is not L-true determined by data I alone or are data II, i.e., factual knowledge, required? Data II are certainly relevant for the truth-value of 'C', but they cannot be relevant for the character of 'C' being L-indeterminate, factual, contingent. It would be absurd to assume such a relevance, to say, for example: " 'C' is contingent because the individual c happens to have the property Q; if this were not so then 'C' would not be contingent but L-true." Thus contingent facts, by being relevant for the contingency of 'C' would also be relevant for L-truth or L-falsity, in contradiction to our explanation of these concepts. Since now data I alone determine that 'C' is not L-true, they determine that 'N(C)' is false and '∼N(C)' is true. Therefore, 'N(C)' is L-false, and '∼N(C)' is L-true.

From (ii) and (iii) together we see that, if 'N(C)' is false, it is L-false. Thus our earlier question (2) is answered in the affirmative. Together with the result under (i), this leads to the following convention, which is more specific than C1-1.

C1-2. If '···' is L-true, 'N(···)' is L-true; otherwise 'N(···)' is L-false.

We shall later construct the rule for 'N' (D9-5i) in such a manner that this convention is fulfilled (T9-1).

In the preceding analysis, I have repeatedly referred to a certain result as "following from" or "determined by" certain data. This is not meant in the sense that the result can be derived from the data with the help of deductive means which are systematized in a given metalanguage; still less is it implied that there is an effective method for this derivation. What I mean is rather