ECN 710 : Advanced Macroeconomics

Chapter 2: Solow Growth Model

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January 07, 2025

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Plan

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▷ Canada at the beginning of the 20th century!

▷ Evolution of Countries Over Time: A Historical Perspective

The Solow Model

- The previous video raises two fundamental questions:

  ① How can a country initiate a growth process that will lead it to a higher level of GDP per capita?

  ② Why are some countries rich and others poor?

- This chapter presents a model, the **Solow Model** (1956, QJE), that provides some initial answers.

- We will show how the long-term evolution of income and consumption per worker in a country is affected by:

  ☞ the savings rate and investment,

  ☞ technological progress and the population growth rate.

Economic Environment

- The Solow model is a dynamic general equilibrium model.

- The economy is assumed to be closed and decentralized.

- The agents:

    ☞ Households,

    ☞ Firms.

- There is a final good, which is produced with two factors of production: capital and labor.

- Time is discrete, $t \in \{0, 1, 2, ...\}$.

- Representative household

  - Demands the final good.

  - Owns the factors of production (i.e. capital and labor).

  - Saves a constant fraction **s** of disposable income.

- All firms have access to the same production technology:

    $\implies$ the economy admits a representative firm.

- The aggregate production function for the unique final good is:

$$Y(t) = F(K(t), L(t), A(t)) \tag{1}$$

- $K(t)$ and $L(t)$ represent the demand for capital and labor at time t.

- $A(t)$ is the technology at time t.

- Technology is free, publicly available as a non-exclusive and non-rival good.

Assumptions: We assume that $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is

- **continuous**, twice differentiable in K and L, and satisfies:

$$
\begin{aligned}
F_K(K,L,A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K,L,A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0 \\
F_{KK}(K,L,A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0
\end{aligned}
\tag{2}
$$

- with **constant returns to scale** in K and L, i.e., homogeneous of degree 1.

- The marginal products of capital and labor increase with the other factor, i.e., capital and labor are complementary:

$$
F_{KL}(K,L,A) = F_{LK}(K,L,A) \equiv \frac{\partial^2 F}{\partial K \partial L}(K,L,A) > 0
$$

- with $F_{KL}(K,L,A) = F_{LK}(K,L,A)$ by Schwarz's theorem.

Definition: Homogeneity

Let m be an integer. The function $F : \mathbb{R}^{p+2} \to \mathbb{R}$ is homogeneous of degree m in $(K, L) \in \mathbb{R}^2$ if and only if

$$F(\lambda K, \lambda L, A) = \lambda^m F(K, L, A), \qquad \forall \lambda \in \mathbb{R}_+ \text{ and } A \in \mathbb{R}^p. \tag{3}$$

Theorem 1 (Euler's Theorem).

Let $F : \mathbb{R}^{p+2} \to \mathbb{R}$ be a differentiable function with partial derivatives F_K and F_L, and **homogeneous** of degree m in K and L. Then:

① $mF(K, L, A) = F_K(K, L, A) \times K + F_L(K, L, A) \times L \quad \forall K, L \in \mathbb{R}, A \in \mathbb{R}^p.$

② $F_K(K, L, A)$ and $F_L(K, L, A)$ are homogeneous of degree $m - 1$ in (K, L).

Proof of Euler's Theorem:

Since F is homogeneous of degree m in (K, L), for all $\lambda \in \mathbb{R}_+$, we have:

$$F(\lambda K, \lambda L, A) = \lambda^m F(K, L, A). \tag{4}$$

Since F is differentiable, differentiating (4) with respect to λ gives:

$$m\lambda^{m-1} F(K, L, A) = F_K(\lambda K, \lambda L, A)K + F_L(\lambda K, \lambda L, A)L$$

Substituting $\lambda = 1$, we obtain the first relation.

Differentiating (4) with respect to K and L and dividing both relations by $\lambda > 0$, we get:

$$F_K(\lambda K, \lambda L, A) = \lambda^{m-1} F_K(K, L, A)$$

$$F_L(\lambda K, \lambda L, A) = \lambda^{m-1} F_L(K, L, A)$$

The function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ must satisfy the Inada conditions:

$$\lim_{K \to 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K \to \infty} F_K(K, L, A) = 0, \quad \forall L > 0$$

$$\lim_{L \to 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L \to \infty} F_L(K, L, A) = 0, \quad \forall K > 0 \tag{5}$$

$\implies$ This is important for ensuring the existence of interior equilibria.

$\implies$ All factors of production are necessary, i.e.

$$F(0, L, A) = F(K, 0, A) = 0. \tag{6}$$

$\implies$ Output is unbounded above, i.e.

$$\lim_{K \to \infty} F(K, L, A) = \infty \quad \text{and} \quad \lim_{L \to \infty} F(K, L, A) = \infty. \tag{7}$$

Let $y \equiv Y/L$ and $k \equiv K/L$ denote output and capital per worker, respectively.

- The production function F is homogeneous of degree 1, thus

$$y = F(K, L, A)/L = F(k, 1, A) \equiv f(k). \tag{8}$$

- By the definition of f and properties of F, we have the following:

  ☞ $f(0) = 0$ and f is twice differentiable,

  ☞ $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$.

  ☞ $F_K(K, L, A) = f'(k)$ $F_L(K, L, A) = f(k) - kf'(k)$.

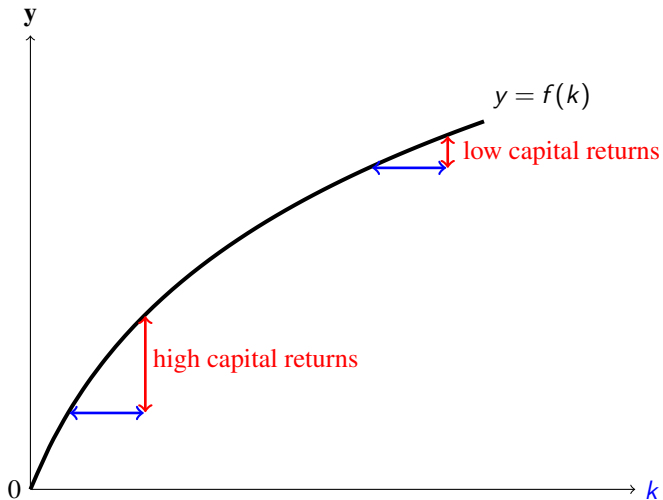 ☞ f is increasing, $f'(k) > 0$ and concave, $f''(k) < 0$.

Figure: Production function: diminishing marginal returns

- Competitive Markets

  ☞ Households and firms are price takers.

  ☞ Prices equilibrate the markets.

- At each date t, households have $\overline{L}(t)$ units of labor that they supply inelastically.

- The labor market equilibrium condition at time t can be expressed as follows:

$$L(t) = \overline{L}(t)$$

  where $L(t)$ is the labor demand by firms.

- Given the wage $w(t)$, this equilibrium condition can also be written as

$$L(t) \leq \overline{L}(t), \ w(t) > 0 \quad \text{and} \quad (L(t) - \overline{L}(t))w(t) = 0.$$

- Households also own capital and supply the quantity $\overline{K}(t) > 0$, with $K(0) > 0$ being the initial capital stock.

- The equilibrium condition for the capital market is:

$$K(t) = \overline{K}(t).$$

- Capital depreciates at rate δ, i.e., of 1 unit of capital used at time t, only $1 - \delta$ units remain for the next period.

- Let $R(t)$ be the real rent on capital at time t, and the interest rate received by households will be

$$r(t) = R(t) - \delta.$$

- The representative firm maximizes its profit by choosing the quantity of production factors

$$\max_{L(t)\geq 0, K(t)\geq 0} F(K(t), L(t), A(t)) - w(t)L(t) - R(t)K(t). \qquad (9)$$

- The first-order conditions imply

$$R(t) = F_K(K(t), L(t), A(t)) = f'(k(t)) \qquad (10)$$

$$w(t) = F_L(K(t), L(t), A(t)) = f(k(t)) - k(t)f'(k(t)) \qquad (11)$$

- At the equilibrium of the Solow model, firms make zero profit, and in particular,

$$Y(t) = w(t)L(t) + R(t)K(t)$$

This immediately follows from Euler's theorem for $m = 1$.

- The law of motion for the capital stock is

$$K(t+1) = (1-\delta)K(t) + I(t). \tag{12}$$

where $I(t)$ is the investment at time t.

- Since the economy is closed, the sum of global consumption and investment cannot exceed total production.

$$Y(t) = C(t) + I(t). \tag{13}$$

- Since the economy is closed (and there is no government spending),

$$S(t) := Y(t) - C(t) = I(t). \tag{14}$$

- Households save a constant fraction *s* of their income,

$$S(t) = sY(t). \tag{15}$$

$$C(t) = Y(t) - I(t) = (1-s)Y(t). \tag{16}$$

- Thus, the **fundamental equation of the Solow model** is

$$
\begin{aligned}
K(t+1) &= sY(t) + (1-\delta)K(t) \\
&= sF(K(t), L(t), A(t)) + (1-\delta)K(t). \tag{17}
\end{aligned}
$$

- This is a nonlinear difference equation.

- The equilibrium of the Solow model is described by this equation, along with the laws of motion for $L(t)$ and $A(t)$.

- The demographic growth rate, denoted n, is exogenous and constant.

- Thus, at each date t, the number of households is given by

$$\overline{L}(t) = (1+n)\overline{L}(t-1) \quad \Longleftrightarrow \quad L(t) = (1+n)^t\overline{L}(0)$$

with $\overline{L}(0)$ being the initial population level.

- We normalize $\overline{L}(0) = 1$.

Definition of Equilibrium

Given a sequence of $\{L(t), A(t)\}_t^\infty$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, production levels, consumption levels, wages, and rental rates of capital $\{K(t), Y(t), C(t), w(t), R(t)\}_t^\infty$ such that $K(t)$ satisfies the fundamental equation of the Solow model (17), $Y(t)$ is given by (1), $C(t)$ is given by (16), and $w(t)$ and $R(t)$ are given by (10) and (11), respectively.

- Equilibrium is defined as a complete trajectory of allocations and prices.

- Equilibrium does not refer to a static object, but rather specifies the trajectory of economic variables.

Steady State in the Solow Model

- Recall the fundamental dynamic equation of Solow

$$K(t+1) \;=\; sF(K(t), L(t), A(t)) + (1-\delta)K(t). \tag{18}$$

- Dividing this equation by $L(t)$, we obtain

$$(n+1)k(t+1) \;=\; sf(k(t)) + (1-\delta)k(t). \tag{19}$$

$\Rightarrow$ Saving serves to replace depreciated capital, endow new workers with capital, and potentially increase the capital per worker.

- Equation (19) is also called the difference equation of the Solow model equilibrium.

- The other equilibrium quantities can be derived from $k(t)$.

- Initially assume that $A(t) = A$ is constant.

Definition

A steady-state equilibrium (without technological progress), also called a regular state, is an equilibrium path in which

$$k(t) = k^*, \text{for all } t. \tag{20}$$

- The economy will tend toward this steady-state equilibrium over time (but it will never reach it in finite time).

- At the steady state, all aggregate variables ($L(t)$, $K(t)$, $Y(t)$, $C(t)$) and prices ($w(t)$, $R(t)$) grow at a constant rate, possibly zero.

The expressions for output and consumption per worker at the steady state are:

$$y^* = f(k^*) \quad \textbf{and} \quad c^* = (1-s)f(k^*)$$

Proposition

☞ k^* exists and is unique, satisfying

$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s} \tag{21}$$

☞ k^* and y^* increase with s and decrease with δ and n.

☞ c^* is non-monotonic with respect to s and decreases with δ and n.

Proof: (For PhD Students): k^* is the steady-state capital if and only if it solves

$$(n+1)k^* = sf(k^*) + (1-\delta)k^* \tag{22}$$

This implies that

$$\phi(k^*) \equiv \frac{f(k^*)}{k^*} = \frac{\delta + n}{s}$$

The properties of f imply that ϕ is continuous and strictly decreasing, with

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$
$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

Thus, equation (22) has a unique solution:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right)$$

Recall that

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

By applying the formula for the derivative of the inverse function,

$$(\phi^{-1}(k))' = \frac{1}{\phi'[\phi^{-1}(k)]}$$

we find that k^* decreases with $(\delta + n)/s$ (since $\phi'(k) < 0$).
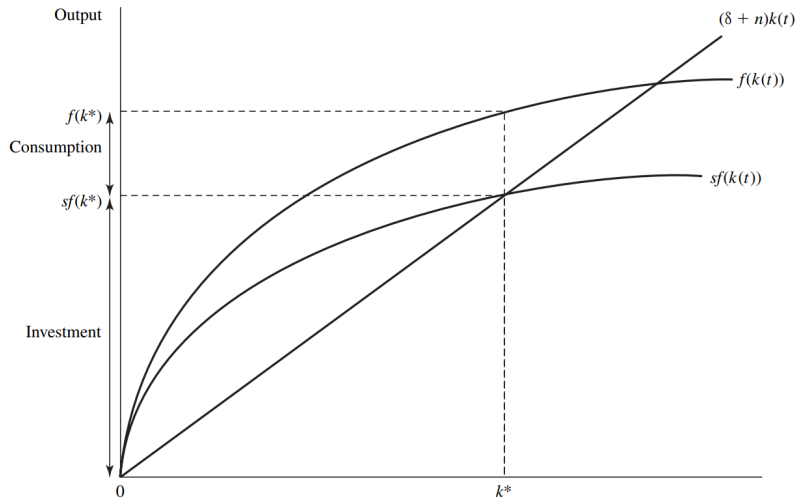
$y^* = f(k^*)$ and since f is increasing, y^* also decreases with $(\delta + n)/s$.

On the other hand, consumption is given by

$$c^* = (1-s)f(k^*)$$

Thus, c^* decreases with $\delta + n$, but the effect of s is ambiguous.

For each value of the savings rate s, there exists a unique value of k^*, and $\partial k^*(s)/\partial s > 0$. The steady-state consumption can be written as

$$c^*(s) = f(k^*(s)) - (n+\delta)k^*(s)$$

$c^*(s)$ is increasing for low values of s and decreasing for high values of s.

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - (n+\delta) \right] \frac{\partial k^*(s)}{\partial s}$$

Let s_{or} denote the savings rate that maximizes consumption, and k_{or} the corresponding capital level. Then

$$f'(k_{or}) = (n+\delta) \tag{23}$$

The equation (23) is called the golden rule of capital accumulation.

- In the Solow model, the savings rate is exogenous.

- An important question is what the appropriate savings rate should be.

- It is difficult to answer this until we have specified a detailed objective function (Lecture 3).

- One answer is to choose the savings rate that maximizes consumption, i.e., s_{or}.

- A savings rate $s \geq s_{or}$ is inefficient because higher consumption per capita could be achieved by reducing the savings rate.

- Suppose an economy with an saving rate $s_2 > s_{or}$, then $k_2^* > k_{or}$ and $c_2^* < c_{or}$.

- If we reduce s_2 to s_{or}, per capita consumption first increases and then decreases monotonically during the transition to its new steady-state value, c_{or}.

- Since $c_2^* < c_{or}$, consumption will be greater than c_2 at all moments during the transition and in the new steady-state.

- Therefore, when $s > s_{or}$, the economy is in a situation of over-saving in the sense that per capita consumption at all points in time could be increased by reducing the saving rate.

- An economy that over-saves is said to be dynamically inefficient because its per capita consumption trajectory is lower than potential alternative trajectories at any point in time.

- If the economy has a saving rate $s_1 < s_{or}$, then higher savings will increase per capita consumption in the steady state.

- However, this increase in the saving rate will reduce consumption today and for some time during the transition period.

- The result will therefore be seen as good or bad depending on how households weigh today's consumption against future consumption trajectories.

- We cannot judge whether an increase in the saving rate is appropriate in this situation until we have made specific assumptions about how agents discount the future.
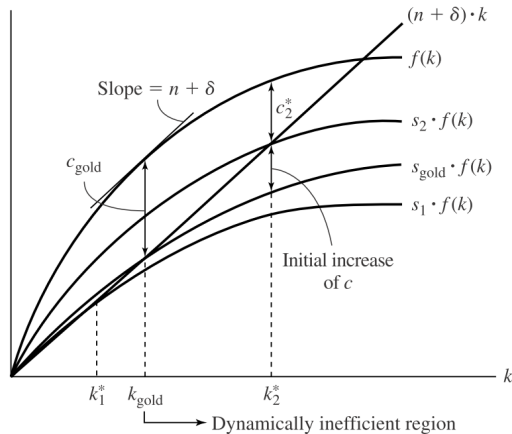
Figure: Golden Rule and Dynamic Inefficiency (Barro & Sala-i-Martin, p. 36)

The study of the transition dynamics answers the question:

Does an economy that is not at its steady state converge towards it?

In other words: **Will the economy eventually return to its steady state if an exogenous shock drives it away?**

Proposition

For any strictly positive initial level of capital per worker $k(0)$, the economy converges towards its steady-state level.

- The growth rate is positive and decreases to zero if $k(0) < k^*$;

- The growth rate is negative and increases to zero if $k(0) > k^*$.

Proof. Let h be the function

$$h(k) = \frac{sf(k) + (1-\delta)k}{n+1}$$

h is strictly increasing and concave (and continuous and differentiable):

$$h'(k) = \frac{sf'(k) + (1-\delta)}{n+1} > 0 \quad \text{and} \quad h''(k) = \frac{sf''(k)}{n+1} < 0$$

We show that if $k(0) < k^*$ then $k(t)$ increases and stays to the left of k^*, i.e.

$$k(t) < k^* \text{ and } \gamma(t) \equiv \frac{\Delta k(t)}{k(t)} > 0, \ \forall t.$$

Since h is increasing, it is enough to apply h t times to the inequality $k(0) < k^*$ and obtain

$$k(t) < k^*.$$

Moreover,

$$\gamma(t) = \frac{k(t+1) - k(t)}{k(t)} = \frac{s}{n+1} \left[\frac{f(k(t))}{k(t)} - \frac{n+\delta}{s} \right]$$

Recall that ϕ is decreasing.
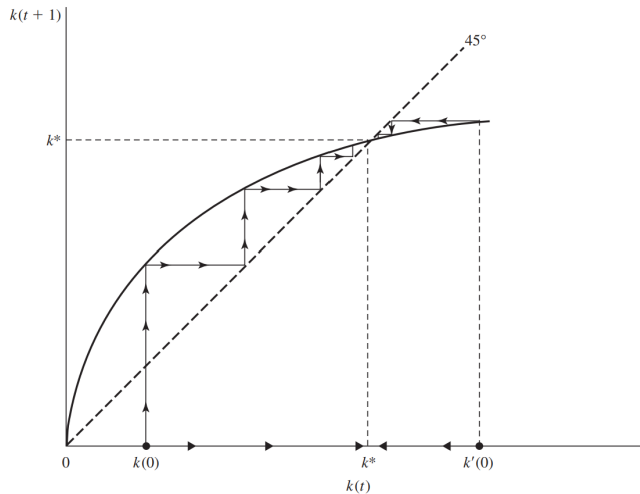
$$k(t) < k^* \quad \implies \quad \phi(k(t)) - \frac{n+\delta}{s} > \phi(k^*) - \frac{n+\delta}{s} = 0$$
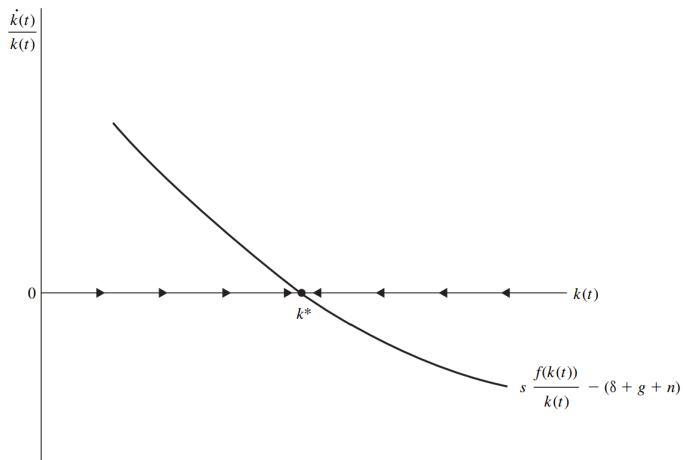
This shows that when $k(0) < k^*$, $k(t)$ is increasing.

Thus, when $k(0) < k^*$, $k(t)$ grows asymptotically towards k^*.

Since $k(t+1) = h(k(t))$, and $k(t)$ converges, its limit is the unique fixed point of h, which is nothing but k^* by definition.

Symmetrically, it can be shown that if $k(0) > k^*$, then $k(t)$ decreases asymptotically towards k^*.

- Consider the difference equation

$$x(t+1) - x(t) = g(x(t)). \tag{24}$$

where for each unit of time $t = 0, 1, 2, \ldots$, $g(x(t))$ describes the absolute growth of x between t and $t+1$.

- Equation (24) describes the variation of the variable x between two discrete points in time, t and $t+1$.

- Now consider the following approximation for $\Delta t \in [0, 1]$

$$x(t+\Delta t) - x(t) \simeq \Delta t g(x(t)). \tag{25}$$

- If $\Delta t = 0$, equation (25) is just an identity, and if $\Delta t = 1$, we obtain equation (24).

- This relation is a linear approximation between $\Delta t = 0$ and $\Delta t = 1$.

- This approximation will be relatively accurate if the distance between t and $t+1$ is not very large, so that $g(x) = g(x(t))$ for all $x \in [x(t), x(t+1)]$.

- By dividing both sides of (25) by Δt and taking the limit, we get:

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \dot{x}(t) \approx g(x(t)) \qquad (26)$$

where $\dot{x}(t) \equiv \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$.

- Equation (26) is a differential equation representing the same dynamics as the difference equation (24).

- Nothing has changed on the production side,

  ☞ equations (10) and (11) still give the factor prices,

  ☞ now $w(t)$ and $R(t)$ are interpreted as the instantaneous wage rate and the rental rate of capital.

- Savings is again given by

$$S(t) = s\,Y(t).$$

- Consumption is given by

$$C(t) = (1 - s)\,Y(t). \tag{27}$$

- The labor factor (population) in continuous time,

$$L(t) = L(0)\exp(nt). \tag{28}$$

- Recall that the capital stock per worker is

$$k(t) \equiv \frac{K(t)}{L(t)}$$

- Then

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n. \tag{29}$$

- The fundamental equation of the Solow model, cf. Equation (18), in continuous time becomes:

$$\dot{K}(t) = sF(K(t), L(t), A) - \delta K(t) \tag{30}$$

- Using the properties of F, we get

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + n). \tag{31}$$

Definition

Given the initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, production levels, consumption levels, wages, and capital rental rates $\{K(t), L(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $L(t)$ satisfies (28), $k(t) \equiv K(t)/L(t)$ satisfies (31), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (27), and $w(t)$ and $R(t)$ are given by (10) and (11).

- As before, the steady-state equilibrium implies that $k(t)$ remains constant at a certain level k^*.

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right) \quad \text{with} \quad \phi(k) = \frac{f(k)}{k}. \tag{32}$$

Property

For an initial capital level $k(0) > 0$, the economy described by the Solow model in continuous time without technological change converges asymptotically to its steady state.

- $\dot{k}(t)/k(t) > 0$ and decreases toward zero if $k(0) < k^*$;

- $\dot{k}(t)/k(t) < 0$ and increases toward zero if $k(0) > k^*$.

The proof is identical to that provided for the discrete time model.

The growth rate of output is

$$\frac{\dot{y}}{y} = \frac{f'(k)}{f(k)} k = sf'(k) - (\delta + n)\alpha_k \tag{33}$$

where $\alpha_k(t) \equiv f'(k(t))k(t)/f(k(t))$ is the share of capital income.

By differentiating equation (33) with respect to k, we find

$$\partial(\dot{y}/y)/\partial k = \left[\frac{f''(k) \cdot k}{f(k)}\right](\dot{k}/k) - \frac{(n+\delta)f'(k)}{f(k)}(1-\alpha_k) \tag{34}$$

- If $k(0) < k^*$, then $\dot{k}/k > 0$ and thus $\partial(\dot{y}/y)/\partial k \geq 0$.

- If $k(0) > k^*$, the sign of $\partial(\dot{y}/y)/\partial k$ is ambiguous, but it will be negative as we approach the steady state.

- Production function (Uzawa's Theorems):

$$Y(t) = F(K(t), A(t)L(t)) \tag{35}$$

where technological progress makes labor more productive.

- Technological progress evolves at the rate $g > 0$, such that:

$$g = \frac{\dot{A}(t)}{A(t)}. \tag{36}$$

- The population continues to grow at the rate $n > 0$, with $n = \dot{L}(t)/L(t)$.
- Therefore, the fundamental equation of the Solow model becomes:

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t) \tag{37}$$

- Let us now define $k(t)$ as capital per effective unit of labor, that is,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{38}$$

- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{39}$$

- The output per effective unit of labor can be written as follows:

$$\begin{aligned}
\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\
&\equiv f(k(t)).
\end{aligned}$$

- The income per capita is then:

$$y(t) \equiv Y(t)/L(t) = A(t)\hat{y}(t) = A(t)f(k(t)) \tag{40}$$

- Thus, if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will increase over time because $A(t)$ is increasing.

- We can no longer talk about a steady state where income per capita is constant.

- We now seek a balanced growth path, where income per capita increases at a constant rate.

- Some transformed variables, such as $\hat{y}(t)$ or $k(t)$ in (38), remain constant along the balanced growth path.

- The terms "steady state", "regular state", and "balanced growth path" are used interchangeably.

- The equations (37) and (39) combined imply:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n). \tag{41}$$

- Which can also be written as:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{42}$$

- The only difference is the presence of g. Thus, k is no longer capital per labor, but capital per effective labor.

Proposition

A steady-state equilibrium (with technological progress and population growth) is a balanced path where

$$k(t) = k^\star \tag{43}$$

where $k^\star$ satisfies the following relation for all t:

$$\frac{f(k^\star)}{k^\star} = \frac{\delta + n + g}{s}. \tag{44}$$

Production and consumption per worker grow at the rate g.

Proposition

For an initial level of capital per effective unit of labor $k(0) > 0$, the economy described by the Solow model with technological progress and population growth will asymptotically converge to its steady state:

$$k(t) \rightarrow k^{\star}.$$

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- This model provides a simple and workable framework to discuss capital accumulation and the implications of technological progress.

- It shows that without technological progress, there will be no sustained growth.

- It can generate growth in output per capita through technological progress, but only in an exogenous way.

- Technological progress is a "black box."

- Capital accumulation: determined by the savings rate, depreciation rate, and population growth rate. All these factors are exogenous.

- More work is needed to understand what lies within these "black boxes."

Although the basic Solow model is old, it continues to be used in research.

✍ Brock, W. A., & Taylor, M. S. (2010). **The green Solow model**. *Journal of Economic Growth*, 15, 127-153.,

Incorporates technological progress in emission reduction into the Solow model to estimate the environmental Kuznets curve.

✍ Durlauf, S. N., Kourtellos, A., & Minkin, A. (2001). **The local Solow growth model.** *European Economic Review*, 45(4-6), 928-940.

Generalizes the empirical analysis of the Solow model by relaxing the assumption that all countries have the same production function.

✍ Ding, S., & Knight, J. (2009). **Can the augmented Solow model explain China's remarkable economic growth? A cross-country panel data analysis.** *Journal of Comparative Economics*, 37(3), 432-452.

Uses panel data from 146 countries during 1980-2004 to examine how well the augmented Solow model explains China's rapid growth and the significant growth gap between China and other countries.

- McDonald, S., & Roberts, J. (2002). **Growth and multiple forms of human capital in an augmented Solow model: A panel data investigation.** *Economics Letters*, 74(2), 271-276.

  They show that omitting health capital from augmented Solow growth models creates specification biases and that health capital significantly impacts economic growth rates.

- Bräuninger, M., & Pannenberg, M. (2002). **Unemployment and productivity growth: An empirical analysis within an augmented Solow model**. *Economic Modelling*, 19(1), 105-120.

  Integrates unemployment into the Solow model. Using panel data from 13 OECD countries between 1960 and 1990, they find that an increase in unemployment reduces long-term productivity levels.