NOTE ON GABRIEL'S THEOREM

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We work with a fixed algebraically closed field $\Bbbk$.

Motivated by the result of $\operatorname{rep} Q \cong \operatorname{mod} \Bbbk Q$, we present the well-known Gabriel's theorem, originally presented in [Gab72] which sparked the interest and development of quiver theory. In 1973 Herstein, Gel'fand, and Ponomarev [BGP73] presented a cleaner proof of the statement, which also utilized the Coxeter functor. Our approach of the proof is taken from [Sch14], expository notes which follows the Herstein technique can be found in [Cum11] and [Hal21].

The statement itself is incredibly powerful and quite surprising, it connects finite algebras to the study of quivers, complementing Morita equivalence nicely. The theorem references *Dynkin diagrams* which are shown in Figure 1.

Theorem 1 (Gabriel 1972)**.** *Let Q be a connected quiver.*

(1) *Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.*
(2) *If Q has underlying Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

Prior to presenting the proof we need to develop some mechanisms and tools, we begin with a few necessary algebraic varieties. Let Q be a connected quiver without orientated cycles, suppose $|Q_0| = n$, i.e., Q has n vertices. Fix some $\mathbf{d} = (d_i) \in \mathbb{Z}^n_{\geq 0}$ and define the space of all representations $M \in \operatorname{rep} Q$ with dimension vector $\mathbf{d}$ as $E_{\mathbf{d}} := \{M \in \operatorname{rep} Q \mid \underline{\dim} M = \mathbf{d}\}$. It follows that

$$E_{\mathbf{d}} = \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_{\Bbbk}\left(\Bbbk^{d_{s(\alpha)}}, \Bbbk^{d_{t(\alpha)}}\right).$$

Moreover, $E_{\mathbf{d}}$ is a $\Bbbk$-vector space with dimension $\sum_\alpha d_{s(\alpha)} d_{t(\alpha)}$.

We then define the group

$$G_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\Bbbk). \tag{1}$$

The group naturally acts on $E_{\mathbf{d}}$ via conjugation; if $g = (g_i) \in G_{\mathbf{d}}$, $M = (M_i, \varphi_\alpha) \in E_{\mathbf{d}}$, and $i \xrightarrow{\alpha} j \in Q_1$ then, $(g \cdot \varphi)_\alpha = g_j \varphi_\alpha g_i^{-1}$. We denote the orbit of $M \in E_{\mathbf{d}}$ under $G_{\mathbf{d}}$ by $\mathcal{O}_M := \{g \cdot M \mid g \in G_{\mathbf{d}}\}$. The following lemma shows that this orbit is in fact the isoclass of the representation M.

Lemma 2. *The orbit $\mathcal{O}_M$ is the isoclass of the representation M, that is,*

$$\mathcal{O}_M = \{M' \in \operatorname{rep} Q \mid M \cong M'\}.$$

$\mathbb{A}_n$ $\quad 1 - 2 - \cdots - n-1 - n$

$\mathbb{B}_n$ $\quad 1 = 2 - \cdots - n-1 - n$

$\mathbb{C}_n$ $\quad 1 - 2 - \cdots - n-1 = n$

$\mathbb{D}_n$
$$\begin{tikzcd}[arrows=dash] & & & & n-1 \\ 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-2 \arrow[ur] \arrow[dr] & \\ & & & & n \end{tikzcd}$$

$\mathbb{E}_6$
$$\begin{tikzcd}[arrows=dash] 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \\ & & 6 & & \end{tikzcd}$$

$\mathbb{E}_7$
$$\begin{tikzcd}[arrows=dash] 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[r] & 6 \\ & & 7 & & & \end{tikzcd}$$

$\mathbb{E}_8$
$$\begin{tikzcd}[arrows=dash] 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] \arrow[d] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \\ & & 8 & & & & \end{tikzcd}$$

$\mathbb{F}_4$ $\quad 1 - 2 = 3 - 4$

$\mathbb{G}_2$ $\quad 1 \equiv 2$

FIGURE 1. Dynkin diagrams.

Proof. Suppose that $M = (M_i, \varphi_\alpha)$ and $M' = (M'_i, \varphi'_\alpha)$ are in the same orbit, then there exists some $g \in G_{\mathbf{d}}$ such that $g \cdot M = M'$. That is, for each arrow $i \xrightarrow{\alpha} j$ in Q the following diagram commutes:

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "g_i"'] & M_j \arrow[d, "g_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"'] & M'_j \end{tikzcd}$$

Therefore, g is a morphism of representations, moreover since each g_i is an element of $\mathrm{GL}_{d_i}(k)$ we have that it is invertible and thus an isomorphism. That is, $M \cong M'$.

It follows immediately that if $M \cong M'$ then there is a $g \in G_{\mathbf{d}}$ such that $g \cdot M = g(M) = M'$. $\square$

The *stabilizer* $\operatorname{Stab} M = \{g \in G_{\mathbf{d}} \mid g \cdot M = M\}$ corresponds with the automorphism group $\operatorname{Aut} M$ of the representation M. For any $\mathbf{d} \in \mathbb{Z}^n$ such that a representation M has dimension vector $\mathbf{d}$ we have the following equality of variety dimensions;

$$\dim \mathcal{O}_M = \dim G_{\mathbf{d}} - \dim \operatorname{Aut} M. \tag{2}$$

This follows from the natural bijection $(G_{\mathbf{d}}/\operatorname{Stab} M) \to \mathcal{O}_M$ defined by $\bar{g} \mapsto g \cdot M$, thus $\dim \mathcal{O}_M = \dim(G_{\mathbf{d}}/\operatorname{Stab} M)$ and of course $\dim(G_{\mathbf{d}}/\operatorname{Stab} M) = \dim G_{\mathbf{d}} - \dim \operatorname{Stab} M = \dim G_{\mathbf{d}} - \dim \operatorname{Aut} M$.

Another fact we can observe is that there is at most one orbit of O of codimension zero in $E_{\mathbf{d}}$, this follows since the algebraic variety $E_{\mathbf{d}}$ is irreducible and an orbit with codimension zero in $E_{\mathbf{d}}$ is open.

1. Tits Quadratic Form of a Quiver

We define the associated quadratic form of a quiver (sometimes referred to in literature as the Tits form or Tits quadratic form), building up to a key step in the proof of Gabriel's theorem (Theorem 6).

Definition 1.1. The *quadratic form* $q : \mathbb{Z}^n \to \mathbb{Z}$ of Q is defined as

$$q(x_1, \dots, x_n) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{t(\alpha)}.$$

Note that q has no dependency on α-orientation, i.e., q associates to the underlying Dynkin diagram Δ_Q of Q.

Example 1.1. Let Q be the quiver $1 \longrightarrow 2 \longleftarrow 3$, it's corresponding quadratic form is then

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3.$$

It follows that the value of a quadratic form relies only on the dimension vector of a representation and not the actual representation itself. That is, q is constant in the space $E_{\mathbf{d}}$. The following proposition interprets q in the language of representation theory. Prior to proving the next result, we need the following result regarding projective representations.

Theorem 3. *Let $M = (M_i, \varphi_\alpha) \in \operatorname{rep} Q$, then for any vertex $i \in Q_0$ there is an isomorphism of vector spaces:*

$$\operatorname{Hom}(\mathcal{P}(i), M) \cong M_i.$$

Lemma 4. *For any representation $M \in \operatorname{rep} Q$ with $\underline{\dim} M = \mathbf{d}$, we have*

$$q(\mathbf{d}) = \dim \operatorname{Hom}(M, M) - \dim \operatorname{Ext}^1(M, M).$$

Proof. Consider the standard projective resolution,

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} d_{s(\alpha)} \mathcal{P}(t(\alpha)) \xrightarrow{f} \bigoplus_{i \in Q_0} d_i \mathcal{P}(i) \xrightarrow{g} M \longrightarrow 0 .$$

Applying the $\mathrm{Hom}(-, M)$ functor then yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}(M, M) \longrightarrow \bigoplus_{i\in Q_0} d_i \mathrm{Hom}(\mathcal{P}(i), M) \longrightarrow \bigoplus_{\alpha\in Q_0} d_{s(\alpha)} \mathrm{Hom}(\mathcal{P}(t(\alpha)), M) \longrightarrow \mathrm{Ext}^1(M, M) \longrightarrow 0$$

since each $\mathcal{P}(i)$ is projective the last term of the sequence is zero. Then we conclude that

$$\sum_{i\in Q_0} d_i \dim \mathrm{Hom}(\mathcal{P}(i), M) - \sum_{\alpha\in Q_1} d_{s(\alpha)} \dim \mathrm{Hom}(\mathcal{P}(t(\alpha)), M) = \sum_{i\in Q_0} d_i^2 - \sum_{\alpha\in Q_1} d_{s(\alpha)} d_{t(\alpha)} = q(\mathbf{d})$$

since we have the isomorphism $\mathrm{Hom}(\mathcal{P}(i), M) \cong M_i$ from Theorem 3. Hence,

$$\dim \mathrm{Hom}(M, M) - \dim \mathrm{Ext}^1(M, M) = \sum_{i\in Q_0} d_i^2 - \sum_{\alpha\in Q_1} d_{s(\alpha)} d_{t(\alpha)} = q(\mathbf{d})$$

as desired. □

Recall the simply laced Dynkin diagrams in Figure 1, in Figure 2 we present the Affine Dynkin diagrams sometimes referred to as Euclidean diagrams or extended Dynkin diagrams. These are constructed by adding one vertex to their corresponding Dynkin diagram and some amount of edges such that the diagram itself is not Dynkin, but by removing one vertex the resulting diagram is the union of Dynkin diagrams.

For a given quadratic form q, we say q is *positive definite* (resp. *positive semi-definite*) if $q(\mathbf{x}) > 0$ (resp. $q(\mathbf{x}) \geq 0$) for all non-zero $\mathbf{x} \in \mathbb{Z}^n$.

Lemma 5. *Let Q be a connected quiver with quadratic form q and $\mathbf{d} \in \mathbb{Z}^n \setminus \{0\}$ such that $(\mathbf{d}, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$. Then,*

1. *q is positive semi-definite.*
2. *For all i, $d_i \neq 0$.*
3. *$q(\mathbf{x}) = 0$ if and only if $\mathbf{x} = c\mathbf{d}$ where $c = a/b \in \mathbb{Q}$.*

Proof. Let n_{ij} denote the number of *edges* between i and j in Q (note that this is the number of arrows $i \to j \in Q$ and $j \to i \in Q$). We also assume a standard enumeration of vertices, i.e., $1, \dots, n$. Then we have

$$(1)\quad q(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j<i} n_{ij} x_i x_j \qquad \text{and} \qquad (\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} 2x_i y_i - \sum_{i=1}^{n} \sum_{j\neq i} n_{ij} x_i y_j.$$

Suppose $\mathbf{d} \in \mathbb{Z}^n \setminus \{0\}$ and $(\mathbf{d}, \mathbf{x}) = 0$ for all non-zero $\mathbf{x} \in \mathbb{Z}^n$, let $\mathbf{e}_i$ be the ith standard basis vector of $\mathbb{Z}^n$. Then, $0 = (\mathbf{d}, \mathbf{e}_i) = 2d_i - \sum_{j\neq i} n_{ij} d_j$ hence;

$$(2)\qquad d_i = \sum_{j<i} n_{ij} d_j.$$

Since $n_{ij} \geq 0$ we have that there is some vertex i such that $d_i = 0$, and for all its neighbors j we have $d_j = 0$. As Q is connected we have that this implies that $d_j = 0$ for all $j \in Q_0$, which contradicts our assumptions. This proves statement 2.

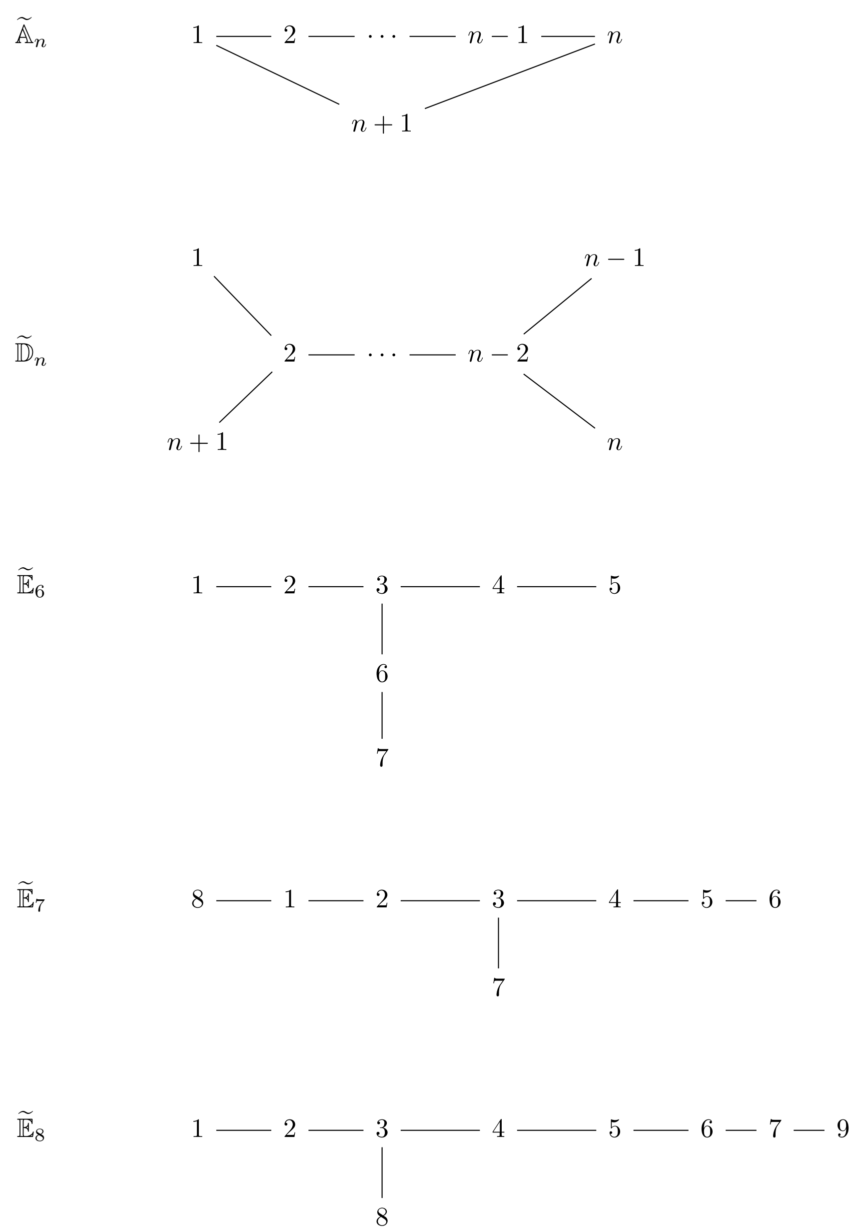

FIGURE 2. Affine Dynkin diagrams $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, and $\widetilde{\mathbb{E}}_{6,7,8}$.

Let $\mathbf{x} \in \mathbb{Z}^n$, by (2) we have

$$\sum_{i=1}^{n} x_i^2 = \sum_i \frac{x_i^2}{d_i} \sum_{j<i} n_{ij} d_j = \sum_i \sum_{j<i} n_{ij} d_j \frac{x_i^2}{d_i}$$

$$= \sum_i \sum_{j \neq i} \frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} \tag{3}$$

$$= \sum_i \sum_{j<i} \left(\frac{n_{ij} d_j}{2} \frac{x_i^2}{d_i} + \frac{n_{ij} d_i}{2} \frac{x_j^2}{d_j} \right) = \sum_i \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} \right) \tag{4}$$

Δ_Q	δ
$\widetilde{\mathbb{A}}_n$	$(1, 1, \ldots, 1, 1)$
$\widetilde{\mathbb{D}}_n$	$(1, 2, 2, \ldots, 2, 1, 1, 1)$
$\widetilde{\mathbb{E}}_6$	$(1, 2, 3, 2, 1, 2, 1)$
$\widetilde{\mathbb{E}}_7$	$(2, 3, 4, 3, 2, 1, 2, 1)$
$\widetilde{\mathbb{E}}_8$	$(2, 4, 6, 5, 4, 3, 2, 3, 1)$

FIGURE 3. Vectors δ such that $(\delta, \mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{Z}^n$, for Δ_Q Affine Dynkin.

where (3) holds since the change from pairs (i, j) such that $j < i$ to pairs (i, j) such that $j \neq i$ is accounted for by dividing by 2. Then (4) holds since changing back from $j \neq i$ to $j < i$ we add the second summand. Combining (1) with (4) we see that

$$
\begin{aligned}
q(\mathbf{x}) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \sum_{j<i} n_{ij} x_i x_j \\
&= \sum_{i} \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i^2}{d_i^2} + \frac{x_j^2}{d_j^2} - 2 \frac{x_i x_j}{d_i d_j} \right) = \sum_{i} \sum_{j<i} \frac{n_{ij} d_i d_j}{2} \left(\frac{x_i}{d_i} - \frac{x_j}{d_j} \right)^2 \qquad (5)
\end{aligned}
$$

This immediately asserts statement 1. since $q(\mathbf{x}) \geq 0$ as $d_i, d_j > 0$ and $n_{ij} \geq 0$. Furthermore, the last equality in (5) shows that $q(\mathbf{x}) = 0$ if and only if $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ for all vertices $i, j \in Q_0$, which shows statement 3. □

The following is the main result of this section and connects Dynkin diagrams to the corresponding quadratic form.

Theorem 6. *Let Q be a connected quiver with quadratic form q and Δ_Q the underlying diagram of Q, then*

(1) *q is positive definite if and only if Δ_Q is of Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, or $\mathbb{E}_{6,7,8}$.*
(2) *q is positive semi-definite if and only if Δ_Q is of Affine Dynkin type or Dynkin type $\mathbb{A}_n$, $\mathbb{D}_n$, or $\mathbb{E}_{6,7,8}$.*

Proof. We begin by showing that q is positive semi-definite if Δ_Q is of Affine Dynkin type, it suffices to find a dimension vector δ such that $(\delta, \mathbf{x}) = 0$ by Lemma 5, which are listed in Figure 3. It is easy to verify the desired condition holds.

Suppose q is positive semi-definite and Δ_Q is not simply laced Dynkin or Affine Dynkin. Then Q contains a subquiver Q' with $\Delta_{Q'}$ of Affine Dynkin type and q' as its quadratic form. Let δ be as in Figure 3. If $Q_0 = Q'_0$ then Q contains more arrows than Q' and then $q'(\delta) = 0 > q(\delta)$ a contradiction. If Q contains more vertices than Q', fix a vertex i_0 in Q which is connected by an arrow to a vertex j_0 in Q'. Define $\mathbf{x} = 2\delta$, then $x_{i_0} = 1$ and $x_j = 0$ for all other vertices $j \in Q_0$. So $q(\mathbf{x}) \geq q'(2\delta) + 1 - 2\delta_{j_0} = 1 - 2\delta_{j_0} < 0$, a contradiction. It then follows that if q is positive definite then Δ_Q must be simply laced Dynkin since for each Affine Dynkin diagram we have δ such that $q(\delta) = 0$. Hence, 2. is proven. □

Along with quadratic forms, we have the notion of roots. Let $\mathbf{x} \in \mathbb{Z}^n \setminus \{0\}$, if $q(\mathbf{x}) = 1$ then $\mathbf{x}$ is a *real root*, if $q(\mathbf{x}) = 0$ then $\mathbf{x}$ is an *imaginary root*. We denote the set of all roots by Φ, and say $\mathbf{x}$ is *positive* (resp. *negative*) if $x_i \geq 0$ (resp.

$x_i \leq 0$) for all $x_i \in \mathbf{x}$, the set of positive real roots is Φ_+ and the set of negative real roots is Φ_-.

2. Proof of Gabriel's Theorem

The following proposition relates the dimension of an orbit to the quadratic form.

Proposition 7. *Let Q be a connected quiver and $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$, then*

$$\operatorname{codim} \mathcal{O}_M = \dim \operatorname{End} M - q(\mathbf{d}) = \dim \operatorname{Ext}^1(M, M).$$

Proof. Observe,

$$\begin{aligned} \operatorname{codim} \mathcal{O}_M &= \dim E_{\mathbf{d}} - \dim \mathcal{O}_M \\ &= \dim E_{\mathbf{d}} - (\dim G_{\mathbf{d}} - \dim \operatorname{Aut} M) \qquad (2) \\ &= \dim E_{\mathbf{d}} - \sum_{i \in Q_0} d_i^2 + \dim \operatorname{End} M \qquad (3) \\ &= \sum_{\alpha \in Q_0} d_{s(\alpha)} d_{t(\alpha} - \sum_{i \in Q_0} d_i^2 + \dim \operatorname{End} M \end{aligned}$$

Where (2) follows from (2) and (3) follows from $\operatorname{Aut} M$ being an open subgroup of $\operatorname{End} M$. The second equality then follows from Lemma 4. □

Suppose $\mathbf{d} \in \mathbb{Z}^n$ is such that $q(\mathbf{d}) \leq 0$ and $M \in \operatorname{rep} Q$ where $\underline{\dim} M = \mathbf{d}$. Then we see that $\operatorname{codim} \mathcal{O}_M \geq \dim \operatorname{End} M \geq 1$ therefore the dimension of $\operatorname{End} M$ is strictly greater than the dimension of any orbit $\mathcal{O}_M$. Hence the number of orbits is infinite, it follows then that there are infinitely many isoclasses of representations with dimension vector $\mathbf{d}$.

We may now prove Gabriel's theorem. The theorem is stated in two parts, we first present a proof of part two and then the proof of part one follows nicely.

Theorem 8. *Let Q be a connected quiver.*

2. *If Q has underlying Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ then the dimension vector induces a bijection from isoclasses of indecomposable representations of Q to the set of positive roots:*

$$\Psi : \operatorname{ind} Q \to \Phi_+ \qquad \Psi : M \mapsto \underline{\dim} M.$$

Proof. To prove our statement we show that Ψ is well-defined, then that Ψ is injective, and lastly that Ψ is surjective.

Let M be an indecomposable representation of Q, we need to show that $q(\underline{\dim} M) = 1$, of which it suffices to show $\operatorname{End} M \cong \Bbbk$ and $\dim \operatorname{Ext}^1(M, M) = 0$. We first show that $\operatorname{End} M \cong \Bbbk$, we proceed by induction on the dimension of M. If M is a simple representation, then it follows immediately. Suppose M has dimension strictly greater than 1, since M is indecomposable this implies that for all $f \in \operatorname{End} M$, $f = \lambda 1_M + g$ where $\lambda \in k$ and $g \in \operatorname{End} M$ is a nilpotent endomorphism. Since g is nilpotent, without loss of generality, we assume that $g^2 = 0$, moreover we choose g such that $\dim(\operatorname{im} g)$ is minimal. Then, $\operatorname{im} g \subset \ker g$ therefore there exists some indecomposable subrepresentation L such that $\operatorname{im} g \cap L$ is non-zero.

Let $\pi : \ker g \to L$ be the canonical projection and i the non-zero morphism given by the incl : $\operatorname{im} g \to \ker g$ and π. That is,

$$\begin{tikzcd} \operatorname{im} g \arrow[r, "\text{incl}"] \arrow[rr, bend right, "i"'] & \ker g \arrow[r, "\pi"] & L \end{tikzcd}$$

This implies the composition $M \xrightarrow{g} \operatorname{im} g \xrightarrow{i} L \xrightarrow{\text{incl}} M$ is a non-zero endomorphism whose square is zero. Then, the image is $i(\operatorname{im} g)$ and since g is taken to be minimal we have that $\dim i(img) \geq \dim(\operatorname{im} g)$ and thus i is injective. So the short exact sequence

$$0 \longrightarrow \operatorname{im} g \xrightarrow{\quad i \quad} L \longrightarrow \operatorname{coker} i \longrightarrow 0$$

can have the $\operatorname{Hom}(-, L)$ functor applied to it, resulting in the following surjective morphism

$$\operatorname{Ext}^1(L, L) \longrightarrow \operatorname{Ext}^1(\operatorname{im} g, L) \longrightarrow 0\ .$$

Then, by induction, $\dim \operatorname{Hom}(L, L) = 1$, and q is positive definite thus $\dim \operatorname{Ext}^1(L, L) = 0$ so the above equation shows that $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$.

Consider the commutative diagram, whose rows are exact, and the bottom row is a push out of the top row along the morphism π.

$$\begin{tikzcd} 0 \arrow[r] & \ker g \arrow[r, "u"] \arrow[d, "\pi"'] & M \arrow[r, "g"] \arrow[d, "j_2"'] & \operatorname{im} g \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "j_1"'] & X \arrow[r] & \operatorname{im} g \arrow[r] & 0 \end{tikzcd}$$

Since $\operatorname{Ext}^1(\operatorname{im} g, L) = 0$ this implies that the bottom row splits so there exists some morphism $h : X \to L$ such that $hj_1 = 1_L$. Let $\nu : L \to \ker g$ be the inclusion of the direct summand, so $\pi\nu = 1_L$. We then construct $hj_2 : M \to L$ and $u\nu : L \to M$ such that $hj_2u\nu = hj_1\pi\nu = 1_L1_L = 1_L$ and thus L is a direct summand of M. Thus, M is indecomposable so L must be either 0 or M. However, $L \neq 0$ since $\operatorname{im} g \cap L$ is non-zero and $L \neq M$ since $L \subset \ker g$ and $g \neq 0$. Therefore we arrive at a contradiction and $\dim \operatorname{End}(M) = 1$, q is positive definite, $\dim \operatorname{Ext}^1(M, M) = 0$, and $q(\underline{\dim} M) = 1$. Hence $\underline{\dim} M$ is a positive root and Ψ is well-defined.

We now show that Ψ is injective. Let $M, M' \in \operatorname{rep} Q$ such that they are both indecomposable and $\underline{\dim} M = \underline{\dim} M'$. We know that for Dynkin types $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}$ the indecomposable representations have no self-extensions. Therefore, the orbits $\mathcal{O}_M$ and $\mathcal{O}_{M'}$ both have codimension zero, which occurs when $M \cong M'$. This shows that Ψ is injective.

Now we show that Ψ is surjective. Let Q be of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, $\mathbf{d}$ a positive root, $M \in \operatorname{rep} Q$ such that $\underline{\dim} M = \mathbf{d}$ and $\mathcal{O}_M$ of maximum dimension in $E_{\mathbf{d}}$. We need to show that M is indecomposable. Let $M = M_1 \oplus M_2$, we will first show that $\operatorname{Ext}^1(M_1, M_2) = \operatorname{Ext}^1(M_2, M_1) = 0$. Suppose that $\operatorname{Ext}^1(M_1, M_2) \neq 0$, then this implies that there exists a non-split short exact sequence of the form

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

here $\underline{\dim}E = \underline{\dim}M$. Then a previous result implies that $\dim \mathcal{O}_M < \dim \mathcal{O}_E$, a contradiction of the maximality of $\mathcal{O}_M$. Thus, $\mathrm{Ext}^1(M_1, M_2) = 0$ and by symmetry we see that $\mathrm{Ext}^1(M_2, M_1) = 0$. Since $q(\mathbf{d}) = \dim \mathrm{Hom}(M, M) - \dim \mathrm{Ext}^1(M, M)$, we see that

$$1 = q(\mathbf{d}) = \dim \mathrm{Hom}(M_1 \oplus M_2, M_1 \oplus M_2) \geq 2$$

and arrive at a contradiction. Thus M is indecomposable, $\Psi(M) = \mathbf{d}$, and Ψ is surjective. $\square$

We can now prove Theorem 1 part 1.

Theorem 9. *Let Q be a connected quiver.*

(1) *Then Q is of finite representation type if and only if the underlying Dynkin diagram, Δ_Q, of Q is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$.*

Proof. Suppose Q is not of Dynkin type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$, then there exists some $\mathbf{d} \neq 0$ such that $q(\mathbf{d}) \leq 0$, so by the Corollary there are infinitely many isoclasses of representations with dimension vector $\mathbf{d}$. Each representation is a finite direct sum of indecomposable representations, therefore the number of isoclasses of indecomposable representations is infinite, thus the proof is complete. $\square$

We end this chapter by mentioning another well-established result in early quiver theory. In 1982 Kac extended the correspondence of indecomposable representations of a quiver to Kac-Moody Lie algebras [Kac82]. We present his theorem here, but omit a proof. An expository note by Lennen [Len19] details a proof of a weaker statement.

Theorem 10 (Kac 1982)**.** *Let Q be a quiver.*

(1) *There exists an indecomposable representation of dimension $\mathbf{d}$ if and only if $\mathbf{d} \in \Phi_+$.*
(2) *If $\mathbf{d} \in \Phi_+$ is real, then there exists a unique indecomposable representation of dimension $\mathbf{d}$.*
(3) *If $\mathbf{d} \in \Phi_+$ is imaginary, then there are infinitely many indecomposable representations of dimension $\mathbf{d}$.*

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