MATH299Q: Midterm Solutions
Quiver Representations

March 10, 2023

Let k be an algebraically closed field.

1. (20 points) M is not indecomposable. For example, $M \cong \mathcal{S}_1 \oplus \mathcal{P}_1 \oplus \mathcal{S}_4 \oplus \mathcal{S}_5$. Where the $\mathcal{S}(i)$, $\mathcal{P}(i)$, $\mathcal{I}(i)$ are the (indecomposable) simple, projective, and injective representations.

2. (20 points) We have that f is a section if there exists a morphism $h : M \to L$ such that $h \circ f = 1_L$. Likewise, g is a retraction if there exists a morphism $h : N \to M$ such that $g \circ h = 1_N$.

3. (a) (5 points) Classify all representations of a given quiver and all morphisms between them up to isomorphism.

   (b) (10 points) Let Q be a quiver and let $M \in \operatorname{rep} Q$. Then

   $$M \cong M_1 \oplus M_2 \oplus \cdots \oplus M_t$$

   where the $M_i \in \operatorname{rep} Q$ are indecomposable and unique up to order.

4. (a) (10 points) $0 \longrightarrow \mathcal{S}_2 \xrightarrow{f} M \xrightarrow{g} \mathcal{S}_1 \longrightarrow 0$.

   (b) (5 points) $0 \longrightarrow \mathcal{S}_2 \xrightarrow{f'} \mathcal{S}_1 \oplus \mathcal{S}_2 \xrightarrow{g'} \mathcal{S}_1 \longrightarrow 0$.

   (c) (10 points) Since $M \cong \mathcal{S}_1 \oplus \mathcal{S}_2$ we have that $f \cong f'$ and $g \cong g'$. Specifically, $f = (f_1, f_2) = (0, 1)$ and $g = (g_1, g_2) = (1, 0)$.

5. (a) (10 points) A category $\mathcal{C}$ is a collection of objects, $\operatorname{ob}(\mathcal{C})$, and morphisms, $\operatorname{Hom}_{\mathcal{C}}$. Such that for each $f \in \operatorname{Hom}_{\mathcal{C}}$, $f : X \to Y$ where $X, Y \in \operatorname{ob}(\mathcal{C})$, the set of all morphisms from X to Y is $\operatorname{Hom}_{\mathcal{C}}(X, Y)$. Moreover we have the operation

   $$\operatorname{Hom}_{\mathcal{C}}(X, Y) \times \operatorname{Hom}_{\mathcal{C}}(Y, Z) \to \operatorname{Hom}_{\mathcal{C}}(X, Z) \qquad (f, g) \mapsto g \circ f),$$

   satisfying:

   i. (associativity) $h \circ (g \circ f) = (h \circ g) \circ f$.

   ii. (identity) for every $X \in \operatorname{ob}(\mathcal{C})$ there is a morphism $1_X \in \operatorname{Hom}_{\mathcal{C}}(X, X)$ such that for $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(Z, X)$ we have

   $$f \circ 1_X = f \qquad 1_X \circ g = g.$$

   (b) (5 points) The category $\mathcal{C}$ has direct sums, and there exists a zero object $0 \in \operatorname{ob}(\mathcal{C})$ such that the identity morphism $1_0 \in \operatorname{Hom}_{\mathcal{C}}(0, 0)$ is the zero of the vector space $\operatorname{Hom}(0, 0)$.

   (c) (5 points) The most relevant example would be $\operatorname{rep} Q$.