# MATH 299Q: Homework 1 Solutions

## Quiver Representations

Let $k$ denote an algebraically-closed field.

1. (a) We have $\operatorname{im} T := \{w \in W \mid w = T(v), v \in V\}$ and $\ker T := \{v \in V \mid T(v) = 0\}$.

   (b) There are various ways of describing these. For instance, $T$ is injective if and only if $\ker T = \{0\}$, $T$ is surjective if $\operatorname{im} T = W$. If $T$ is an isomorphism then $\dim V = n = m = \dim W$.

   (c) *Proof.* Let $f, g \in \operatorname{Hom}(V, W)$. Define $k$-vector space structure of $\operatorname{Hom}(V, W)$ to be

   $$(f+g)(v) = f(v) + g(v) \qquad (\lambda f)(v) = \lambda(f(v))$$

   where $v \in V$ and $\lambda \in k$. It follows that since $f$ and $g$ are linear operators, the two operations preserve linearity and thus $\operatorname{Hom}(V, W)$ is a $k$-vector space. $\square$

2. For example,

   (a)

$$\begin{tikzcd}
 & & 3 \arrow[dr, "\delta"] & \\
1 \arrow[r, "\alpha"] & 2 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & 5 \arrow[ll, "\zeta"'] \\
 & & 4 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

   (b)

$$\begin{tikzcd}
 & & k^2 \arrow[dr, "\delta"] & \\
k^2 \arrow[r, "\alpha"] & k^3 \arrow[ur, "\beta"] \arrow[dr, "\gamma"'] & & k \arrow[ll, "\zeta"'] \\
 & & k \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

   With appropriate linear maps $\alpha, \beta, \gamma, \delta, \epsilon$.

3. *Proof.* It follows that a morphism $f : M \to N$ is a choice of scalars $a, b, c, d, e \in k$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & k \arrow[dr, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[ddd, bend right, "\begin{bmatrix}a\end{bmatrix}"'] & & \\
M \arrow[ddd] & & k^2 \arrow[ddd, "\begin{bmatrix}c & d\end{bmatrix}"] & k \arrow[l, "\begin{bmatrix}1\\1\end{bmatrix}"'] \arrow[ddd, "\begin{bmatrix}e\end{bmatrix}"] \\
 & k \arrow[ur, "\begin{bmatrix}0\\1\end{bmatrix}"'] \arrow[ddd, bend right, "\begin{bmatrix}b\end{bmatrix}"'] & & \\
 & k \arrow[dr, "1"] & & \\
N & & k & k \arrow[l, "1"'] \\
 & k \arrow[ur, "1"'] & &
\end{tikzcd}
$$

We then get the following relations

$$a = c, \quad b = d, \quad c + d = e.$$

Therefore, $a, b \in k$ completely determines the morphism. Since every choice of $a$ and $b$ yields a unique morphism we have that $\mathrm{Hom}(M, N) \cong k^2$. $\square$