Recall the following definitions.

Definition. A *category* $\mathcal{C}$ consists of; $\mathrm{Ob}(\mathcal{C})$ a class of objects, $\mathrm{Hom}_{\mathcal{C}}$ the class of morphisms between objects. Which has an operation *composition* satisfying $f : X \to Y$, $g : Y \to Z$ has composition of $g \circ f : X \to Z$. Furthermore we have the added properties;

(1) If $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ then $h \circ (g \circ f) = (h \circ g) \circ f$.
(2) If $X \in \mathrm{Ob}(\mathcal{C})$ then there exists $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that for all morphisms $f : X \to -$ and $g : - \to X$ we have $f \circ 1_X = f$ and $1_X \circ g = g$.

Definition. Let $\mathcal{C}, \mathcal{C}'$ be two $\Bbbk$-categories (morphisms act $\Bbbk$-linearly), a *covariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is a mapping s.t.

$$F : (X \in \mathcal{C}) \mapsto (F(X) \in \mathcal{C}')$$
$$F : (f : X \to Y) \mapsto (F(f) : F(X) \to F(Y))$$

and $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. A *contravariant functor* $F : \mathcal{C} \to \mathcal{C}'$ is a mapping s.t.

$$F : (x \in \mathcal{C}) \mapsto (F(X) \in \mathcal{C}')$$
$$F : (f : X \to Y) \mapsto (F(f) : F(Y) \to F(X))$$

and $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$.

Our main result which we want to prove today is.

Theorem. *For a fixed quiver Q, the category of finite-dimensional representations* $\operatorname{rep} Q$ *is equivalent to the category of finite-dimensional* $\Bbbk Q$*-modules* $\operatorname{mod} \Bbbk Q$*:*

$$\operatorname{rep} Q \cong \operatorname{mod} \Bbbk Q.$$

Proof. We prove the equivalence by constructing two functors F and G and show that they are inverses. Suppose Q is a finite quiver with n vertices, i.e. $n = |Q_0|$.

Let $F : \operatorname{rep} Q \to \operatorname{mod} \Bbbk Q$ such that for some representation $M \in \operatorname{rep} Q$ we have $F : M \mapsto V$ where $V := \oplus_{i \in Q_0} M_i$. It is easy to see that V is a graded $\Bbbk$-vector space. We first consider how F interacts with the vector spaces of $\operatorname{rep} Q$. Let

$$\iota_i : M_i \to V \qquad \pi_i : V \to M_i$$

be the canonical inclusion and projections, we define the module action for $v \in V$, $e_i, c = (i \mid \alpha_1, \dots, \alpha_t \mid j) \in \Bbbk Q$ of $\Bbbk Q$ on V as follows;

(i) $v \cdot e_i = \iota_i \circ \pi_i(v)$,
(ii) $v \cdot c = v \cdot (i \mid \alpha_1, \dots, \alpha_t \mid j) = \iota_j \circ (v_{\alpha_t} \cdots v_{\alpha_1}) \circ \pi_i(v)$.

Let $f : M \to N$ be a morphism of quiver representations M and N. From above, let $F(M) = V$ and $F(N) = W$, now consider the mapping

$$\psi : V \to W, \qquad \psi : (v_1, \dots, v_n) \mapsto (f_1(v_1), \dots, f_n(v_n)).$$

We now show that ψ is a $\Bbbk Q$ module homomorphism. Consider $v = (v_1, \dots, v_n), v' = (v'_1, \dots, v'_n) \in V$ then

$$\begin{aligned} \psi(v + v') = \psi(v_1 + v'_1, \dots, v_n + v'_n) &= (f_1(v_1 + v'_1), \dots, f_n(v_n + v'_n)) \\ &= (f_1(v_1), \dots, f_n(v_n)) + (f_1(v'_1), \dots, f_n(v'_n)) \\ &= \psi(v) + \psi(v') \end{aligned}$$

For $v = (v_1, \dots, v_n) \in V$ and $c = \alpha_1 \cdots \alpha_t \in \Bbbk Q$ we see that

$$\begin{aligned}
\psi(vc) &= \psi\left((v_1, \dots, v_n)(\alpha_1 \cdots \alpha_t)\right) \\
&= \psi\Big((0, \dots, 0, \underbrace{v_{\alpha_t} \cdots v_{\alpha_1}(v_{s(\alpha_1)})}_{t(\alpha_t)\text{th index}}, 0, \dots, 0)\Big) \\
&= \big(0, \dots, 0, f_{t(\alpha_t)}(v)(v_{\alpha_n} \cdots v_{\alpha_1}(v_{s(\alpha_1)})), 0, \dots, 0\big) \quad (1) \\
&= \big(0, \dots, 0, w_{\alpha_t} \cdots w_{\alpha_1}(f_{s(\alpha_1)}(v_{s(\alpha_1)})), 0, \dots, 0\big) \quad (2) \\
&= (f_1(v_1), \dots, f_n(v_n)) \cdot (\alpha_1 \cdots \alpha_t) \\
&= \psi\,(v_1, \dots, v_n) \cdot (\alpha_1 \cdots \alpha_t) = \psi(v) \cdot c
\end{aligned}$$

Where (1) to (2) occurs since the following diagram is commutative:

$$\begin{tikzcd}
M_{s(\alpha_1)} \arrow[r, "m_{\alpha_1}"] \arrow[d, "f_{s(\alpha_1)}"'] & M_{t(\alpha_1)} \arrow[r, "m_{\alpha_2}"] \arrow[d, "f_{t(\alpha_1)}"] & M_{t(\alpha_2)} \arrow[r] \arrow[d, "f_{t(\alpha_2)}"] & \cdots \arrow[r] & M_{t(\alpha_t)} \arrow[d, "f_{t(\alpha_t)}"] \\
N_{s(\alpha_1)} \arrow[r, "n_{\alpha_1}"] & N_{t(\alpha_1)} \arrow[r, "n_{\alpha_2}"] & N_{t(\alpha_2)} \arrow[r] & \cdots \arrow[r] & N_{t(\alpha_t)}
\end{tikzcd}$$

We now show that F is a functor, preserving identity and composition. Identity follows trivially since if f_i is the identity map for all vertices $i \in Q_0$ then $\psi(v_1, \dots, v_n) = (v_1, \dots, v_n)$. For composition let $f : L \to M$ and $g : M \to N$ be morphisms in $\operatorname{rep} Q$, by definition $f \circ g$ is another morphism $f \circ g : L \to N$, thus we need to show $F(f \circ g) = F(f) \circ F(g)$. Observe

$$\begin{aligned}
F(f \circ g)(l_1, \dots, l_n) &= (f_1 \circ g_1(l_1), \dots, f_n \circ g_n(l_n)) \\
&= F(f)\,(g_1(l_1), \dots, g_n(l_n)) = F(f) \circ F(g)(l_1, \dots, l_n).
\end{aligned}$$

It follows then that $F : \operatorname{rep} Q \to \operatorname{mod} \Bbbk Q$ is a functor.

Let $G : \operatorname{mod} \Bbbk Q \to \operatorname{rep} Q$ such that for some module $V \in \operatorname{mod} \Bbbk Q$ we have $G : V \mapsto M$ where $M = (Me_i, \varphi_\alpha)$ is defined as the collection $\Bbbk$-vector spaces $Me_i := \{ve_x \mid v \in V\}$ and $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ is defined as

$$\varphi_\alpha(ve_{s(\alpha)}) = (ve_{s(\alpha)}) \cdot \alpha = (m \cdot \alpha) = (m \cdot \alpha) \cdot e_{t(\alpha)} \in M_{t(\alpha)} = Ve_{t(\alpha)}.$$

Let $\phi : V \to W$ be a $\Bbbk Q$-module homomorphism, $G(V) = M$, and $G(W) = N$ consider the mapping

$$f : M \to N \qquad f_i : Ve_i \to We_i \qquad f_i : ve_i \mapsto \phi(v)e_i.$$

For f a morphism of representations of a quiver we need the following diagram to commute for all $\alpha \in Q_1$.

$$\begin{tikzcd}
M_{s(\alpha)} \arrow[r, "m_\alpha"] \arrow[d, "f_{s(\alpha)}"'] & M_{t(\alpha)} \arrow[d, "f_{t(\alpha)}"] \\
N_{s(\alpha)} \arrow[r, "n_\alpha"] & N_{t(\alpha)}
\end{tikzcd}$$

Let $ve_{s(\alpha)} \in M_{s(\alpha)}$, then

$$f_{t(\alpha)}\left(m_\alpha(ve_{s(\alpha)})\right) = f_{t(\alpha)}(v \cdot \alpha) = f_{t(\alpha)}\left((v \cdot \alpha) \cdot e_{t(\alpha)}\right) = f(v \cdot \alpha) \cdot e_{t(\alpha)} = f(v) \cdot \alpha$$

and

$$n_\alpha\left(f_{s(\alpha)}(ve_{s(\alpha)})\right) = n_\alpha\left(f(v)\cdot e_{s(\alpha)}\right) = \left(f(v)\cdot e_{s(\alpha)}\right)\cdot\alpha = f(v)\cdot\alpha.$$

So the diagram commutes and thus $G : \phi \mapsto f$ maps $\Bbbk Q$-module homomorphisms to quiver representation morphisms.

We now show that G is a functor. Identity follows trivially, let $\phi : U \to V$ and $\theta : V \to W$ be $\Bbbk Q$-module homomorphisms. By definition we have that $G(\phi) = f : G(U) \to G(V)$ where $f = (f_i)$ is the collection of $f_i : Ue_i \to Ve_i$ and $G(\theta) = g : G(V) \to G(W)$ and g has collection $g_i : Ve_i \to We_i$. We show that $G(\theta\circ\phi) = G(\theta)\circ G(\phi)$. For $i \in Q_0$ and $u \in U$ we have

$$G(\theta_i\circ\phi_i)(ue_i) = \theta\circ\phi(u)e_i = G(\theta_i)(\phi(u)e_i) = G(\theta_i)\circ G(\phi_i)(ue_i).$$

Thus $G : \operatorname{mod}\Bbbk Q \to \operatorname{rep} Q$ is a functor.

By construction, we see that $F\circ G = \mathrm{id}_{\operatorname{mod}\Bbbk Q}$ and $G\circ F = \mathrm{id}_{\operatorname{rep} Q}$, so F and G are inverses to each other thus the two categories are equivalent. $\square$

Thus, from here on it suffices to identify representations of Q with right modules over $\Bbbk Q$, and the category $\operatorname{rep} Q$ with $\operatorname{mod}\Bbbk Q$. We also have the natural extension of this theorem.

Theorem. *For a fixed and connected quiver Q and I an admissible ideal in $\Bbbk Q$ the category of finite-dimensional representation* $\operatorname{rep}(Q, I)$ *is equivalent to the category of finite-dimensional $\Bbbk Q$-modules* $\operatorname{mod}(\Bbbk Q/I)$*:*

$$\operatorname{rep}(Q, I) \cong \operatorname{mod}(\Bbbk Q/I).$$

Definition. An algebra A is called *hereditary* if each submodule of a projective module is projective.

The it should follow that path algebras of quivers without orientated cycles are hereditary since in the case of the quiver we have seen that quivers themselves are hereditary, this then follows from the theorems above.