MATH 299Q: Homework 1

Quiver Representations

Let k denote an algebraically-closed field.

1. Suppose V and W are two finite-dimensional k-vector spaces, with $\dim V = n$ and $\dim W = m$. Let $T \in \operatorname{Hom}(V, W)$, that is $T : V \to W$.

   (a) Define $\operatorname{im} T$ and $\ker T$.

   (b) What does it mean for T to be *injective*? What about *surjective*? What conditions on n and m must we have for T to be an *isomorphism*?

   (c) Show that $\operatorname{Hom}(V, W)$ has a natural k-vector space structure.
   [*Hint: For* $f, g \in \operatorname{Hom}(V, W)$ *show that* $f + \lambda g \in \operatorname{Hom}(V, W)$ *for* $\lambda \in k$.]

2. (a) Draw your favorite (finite) quiver!

   (b) Give a non-trivial representation of the quiver you drew in (a).

3. Let Q be the quiver

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"] & & \\
& 3 & 4 \arrow[l, "\gamma"'] \\
2 \arrow[ur, "\beta"'] & &
\end{tikzcd}$$

and consider the representations M and N:

$$M: \begin{tikzcd}
k \arrow[dr, "\begin{bmatrix}1\\0\end{bmatrix}"] & & \\
& k^2 & k \arrow[l, "\begin{bmatrix}1\\1\end{bmatrix}"'] \\
k \arrow[ur, "\begin{bmatrix}0\\1\end{bmatrix}"'] & &
\end{tikzcd}
\qquad
N: \begin{tikzcd}
k \arrow[dr, "1"] & & \\
& k & k \arrow[l, "1"'] \\
k \arrow[ur, "1"'] & &
\end{tikzcd}$$

Prove that $\operatorname{Hom}(M, N) \cong k^2$. [*Hint: Draw a morphism between* M *and* N.]