Approach and Analysis of Modified Bessel Functions $I_m(x)$ and $K_m(x)$

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Our solution

- $\frac{\Phi_{\mathrm{dp}}(0,\frac{L}{2})}{\Phi_{\mathrm{disk}}} =$

$$2\sum_{n=1,odd}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{I_0\left(\frac{n\pi}{2}\right)}\left[\frac{1}{n\pi}I_0\left(\frac{n\pi}{2}\right)-\frac{1}{4}\left(I_0\left(\frac{n\pi}{2}\right)K_1\left(\frac{n\pi}{4}\right)+\ K_0\left(\frac{n\pi}{2}\right)I_1\left(\frac{n\pi}{4}\right)\right)\right]$$

Numerical Considerations

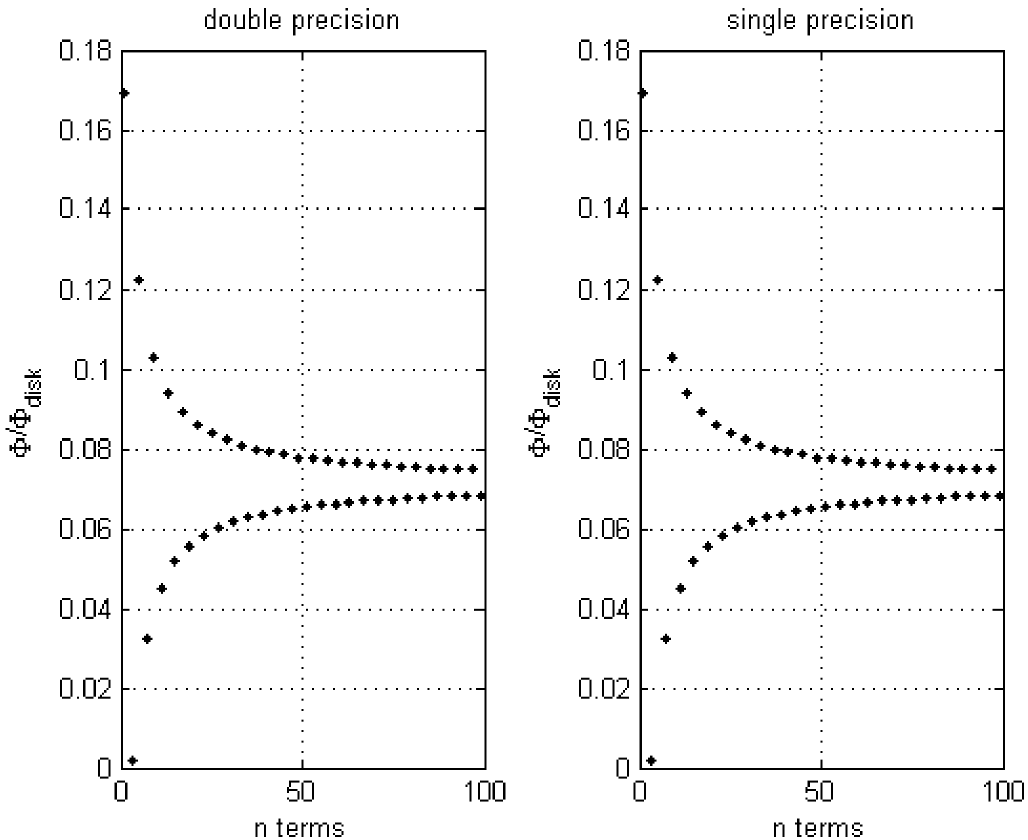
- Double vs. Single Precision
- MATLAB's functions
- Own calculation of I_m and K_m

What is double and single precision?

- Single floating point number has 8-bit exponent component.
- It can represent values as small as $2^{\wedge}(-2^7)$ and as large as $2^{\wedge}(2^7)$
- Double has 11-bit exponent, meaning it can go from $2^{\wedge}(-2^{10})$ to $2^{\wedge}(2^{10})$.

64 vs. 32 bit

- The single precision can get up to 23 significant digits in binary-> 7 significant digits
- Double can get up to 52 sig. digits in binary -> 15 significant digits

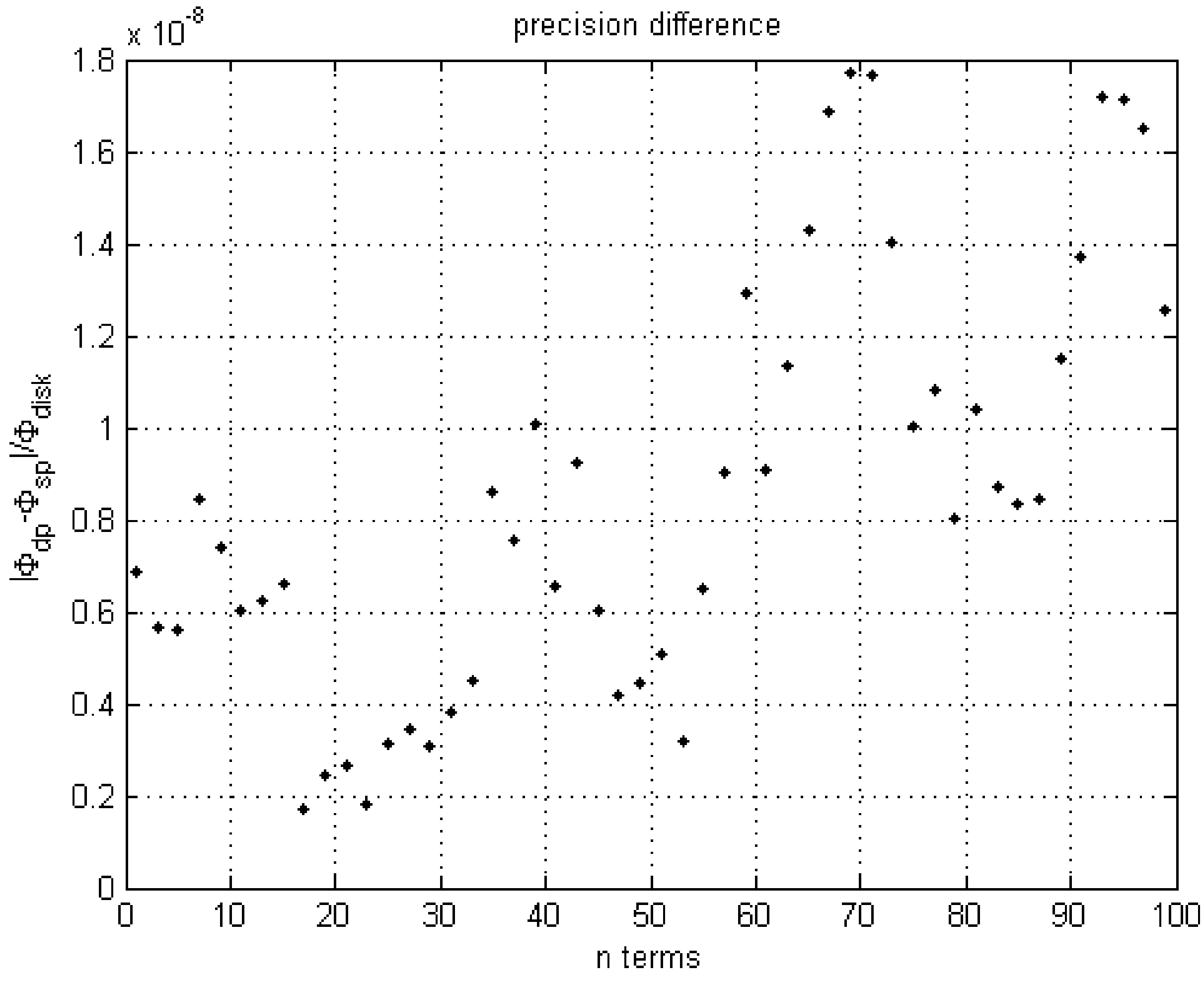


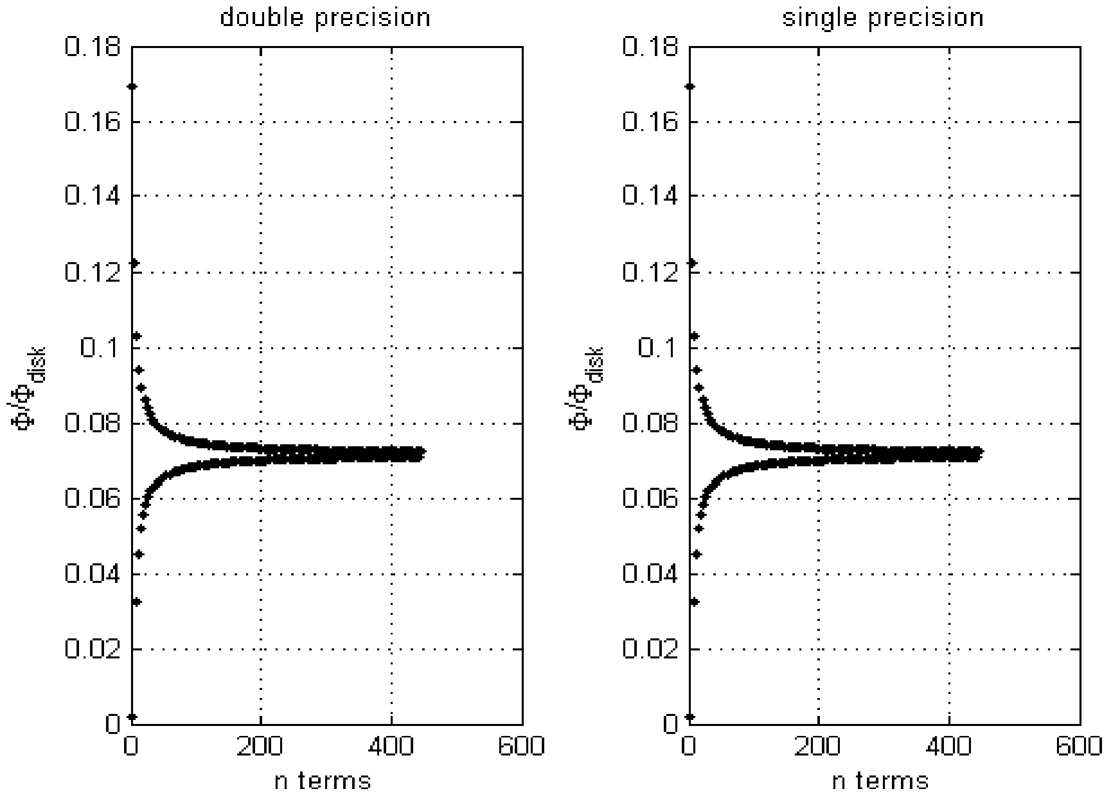
Up to n = 100

$\frac{\Phi_{dp}(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow$ 0.068346592176658

$\frac{\Phi_{sp}(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow$ 0.0683466

Precision Difference

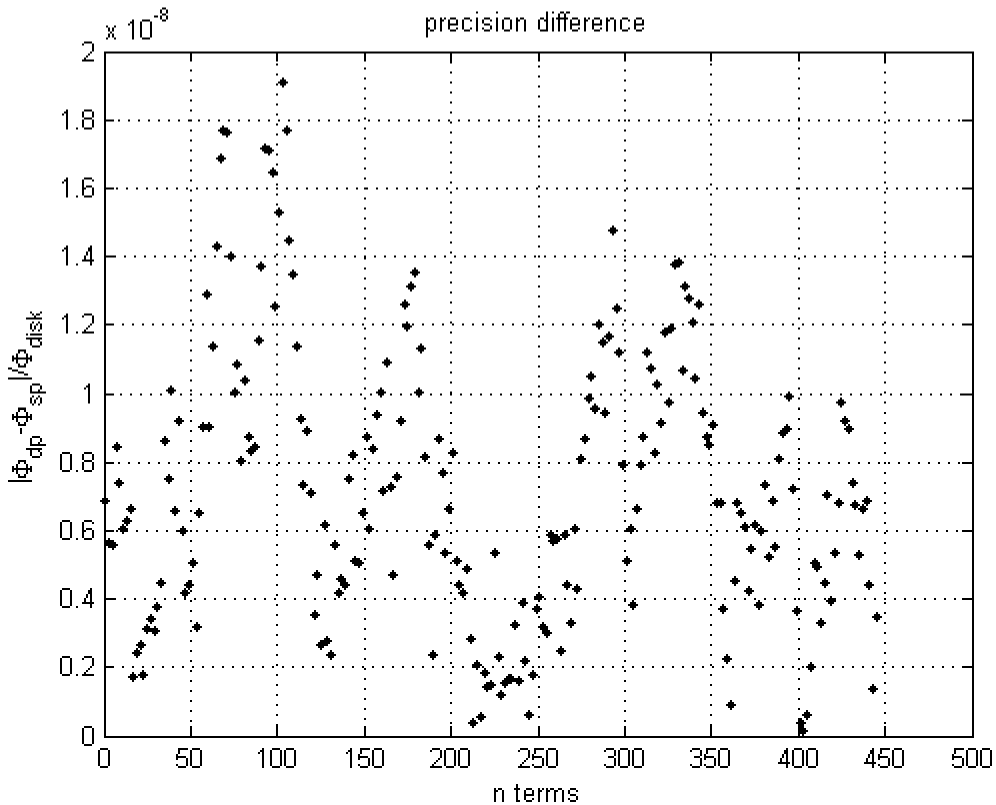




Up to n = 446

$\frac{\Phi_{dp}(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow$ 0.072243068596094

$\frac{\Phi_{sp}(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow$ 0.0722431



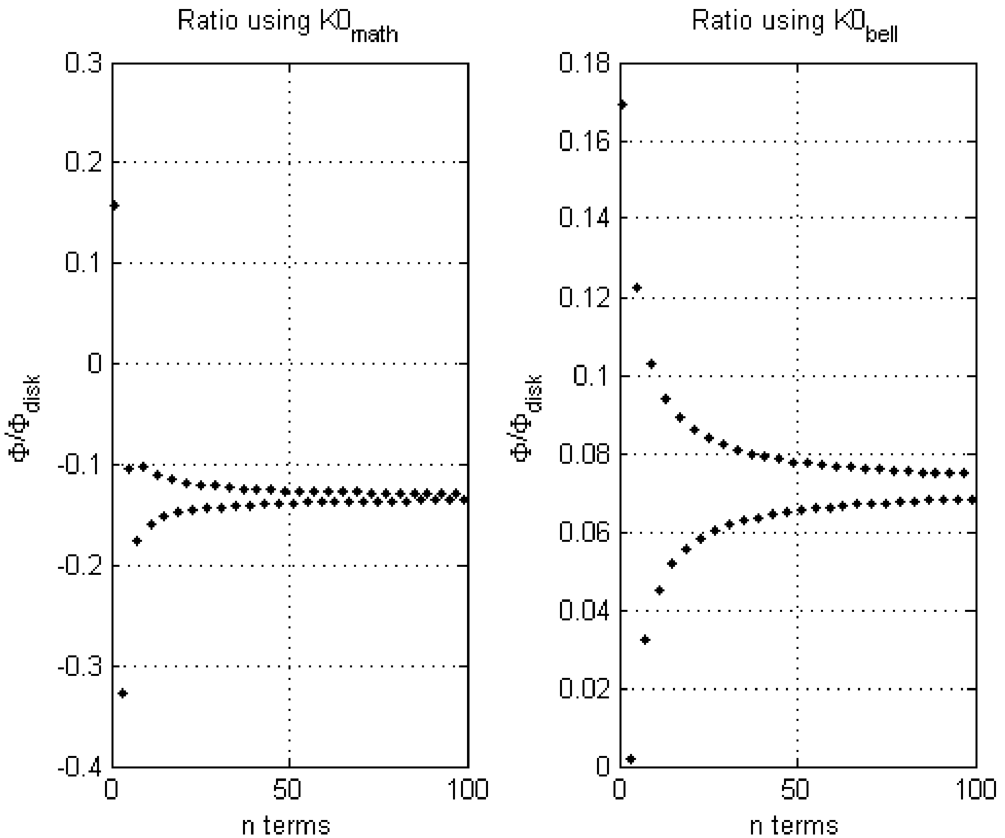
precision difference
x 10^-8
|Φdp-Φsp|/Φdisk
n terms
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
50
100
150
200
250
300
350
400
450
500

Manual

- $I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+m+1)} \left(\frac{x}{2}\right)^{m+2k}$
- $K_m(x) = \frac{\pi}{2} \frac{I_{-m} - I_m}{\sin(m\pi)}$ -> issue m = integer

Integral form

- From mathworld.wolfram.com
- For m >0, $K_m(x) = \frac{\Gamma(m+\frac{1}{2})(2x)^m}{\sqrt{\pi}} \int_0^\infty \frac{\cos(t)dt}{(t^2+x^2)^{m+1/2}}$
- $K_0(x) = \int_0^\infty \frac{\cos(xt)dt}{\sqrt{t^2+1}}$ call $K0_{math}$
- Gaato, M.A. and Seerv, J.B. “Numerical Evaluation of Modified Bessel Functions I and K.” Bell Laboratories (1980)
- $K_0(x) = \int_0^\infty e^{-x\cosh(t)}dt$ call $K0_{bell}$

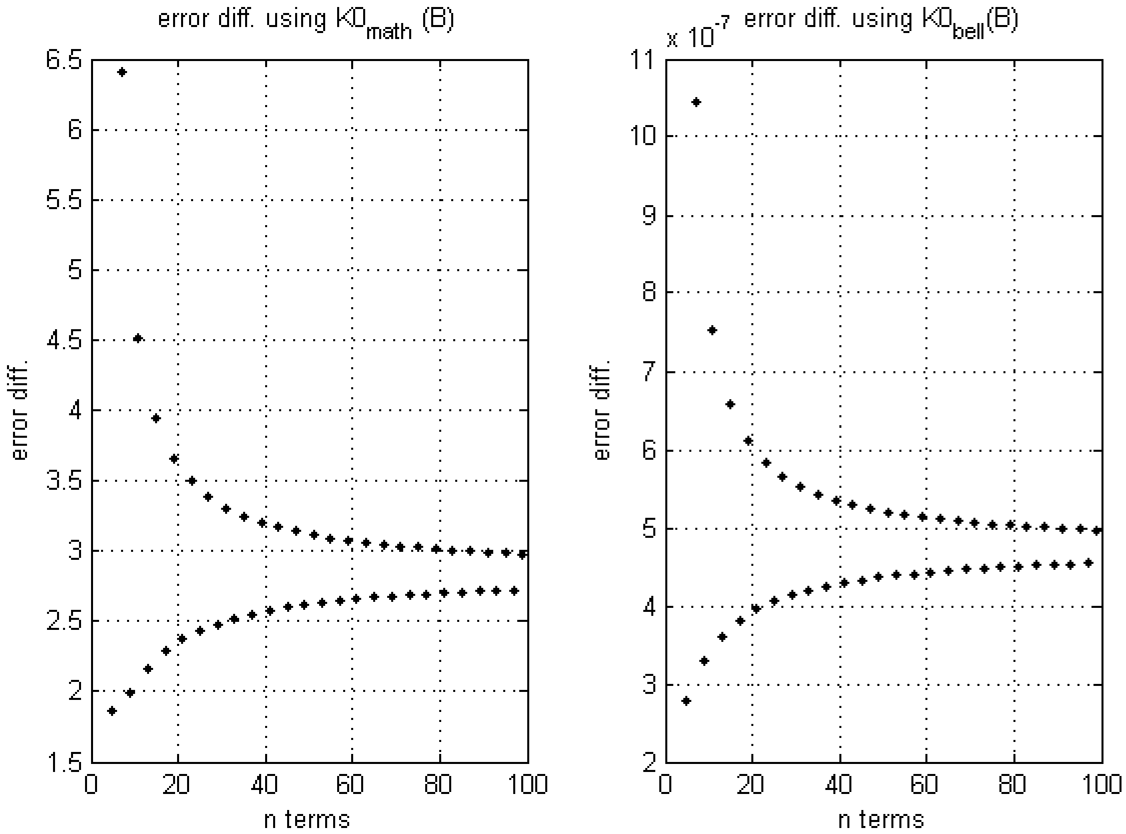


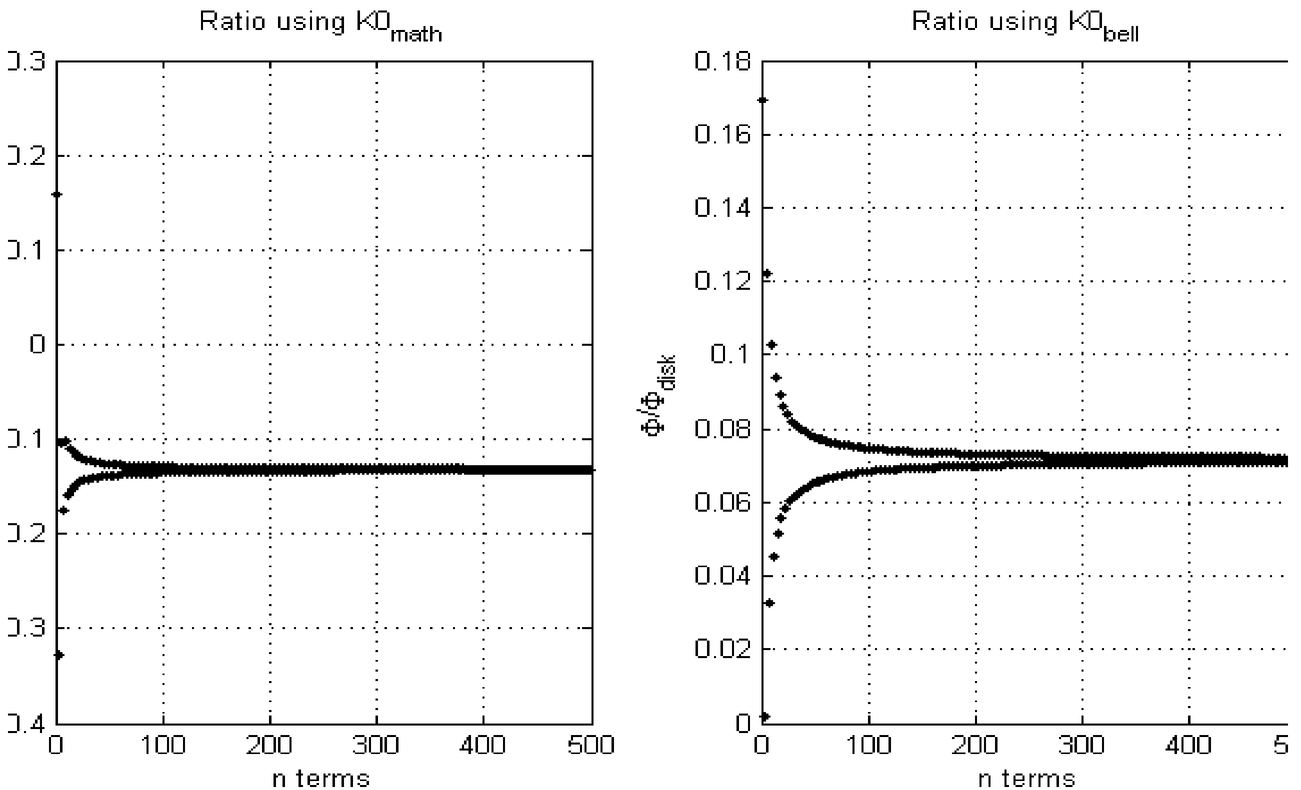
Up to n = 100

$$\frac{\Phi(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow -0.207367773951561$$

$$\frac{\Phi(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow 0.070893309053610$$

Error difference

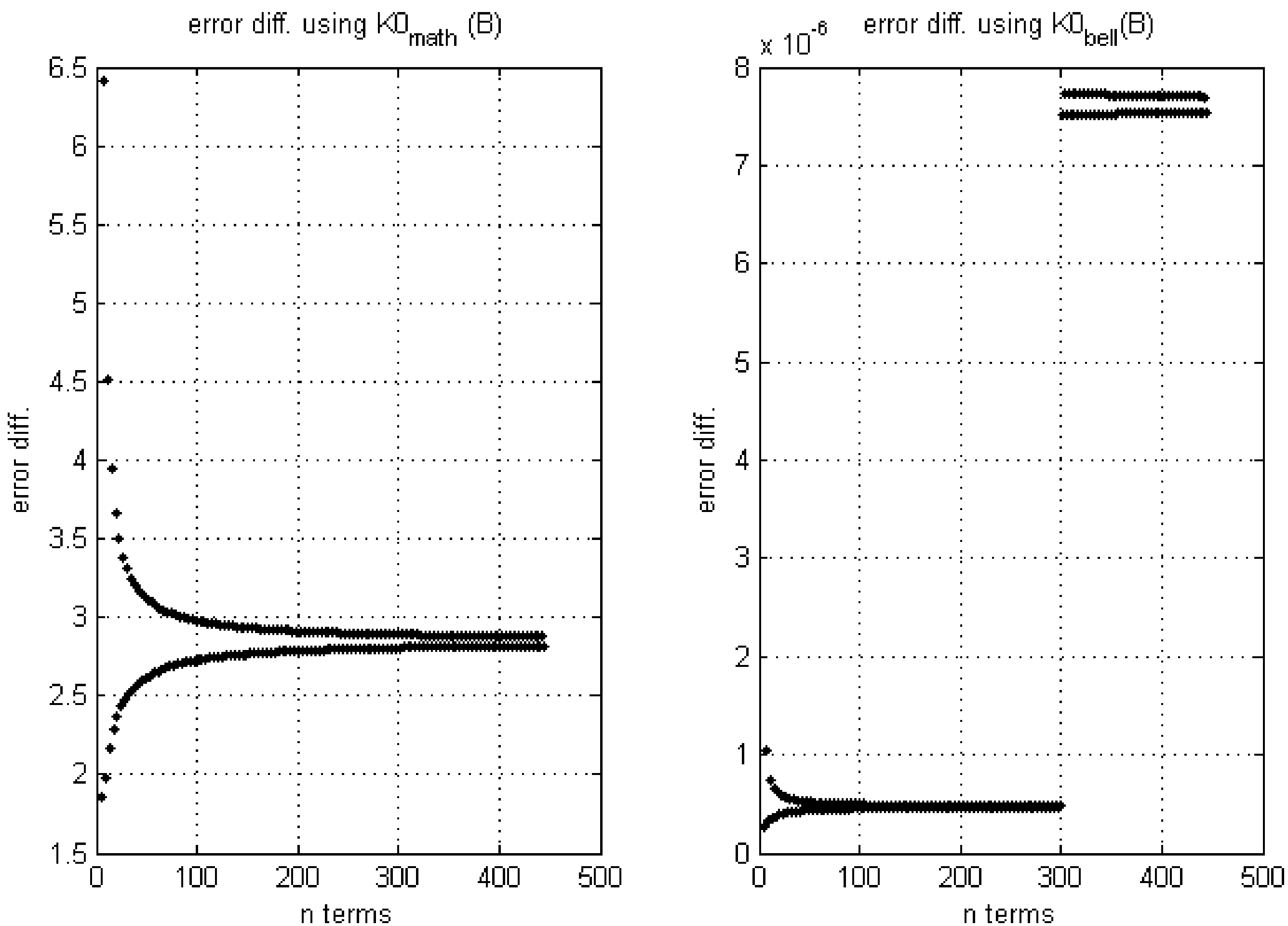




Up to n = 500

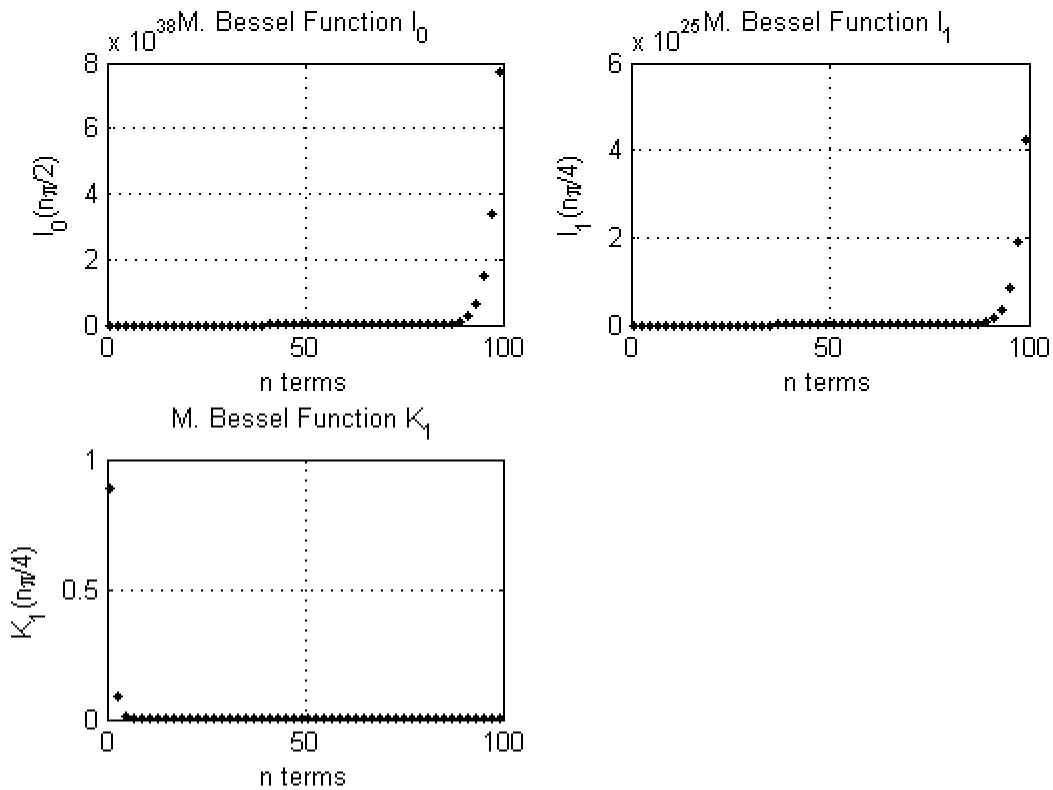
$$\frac{\Phi(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow -0.132856695881407$$

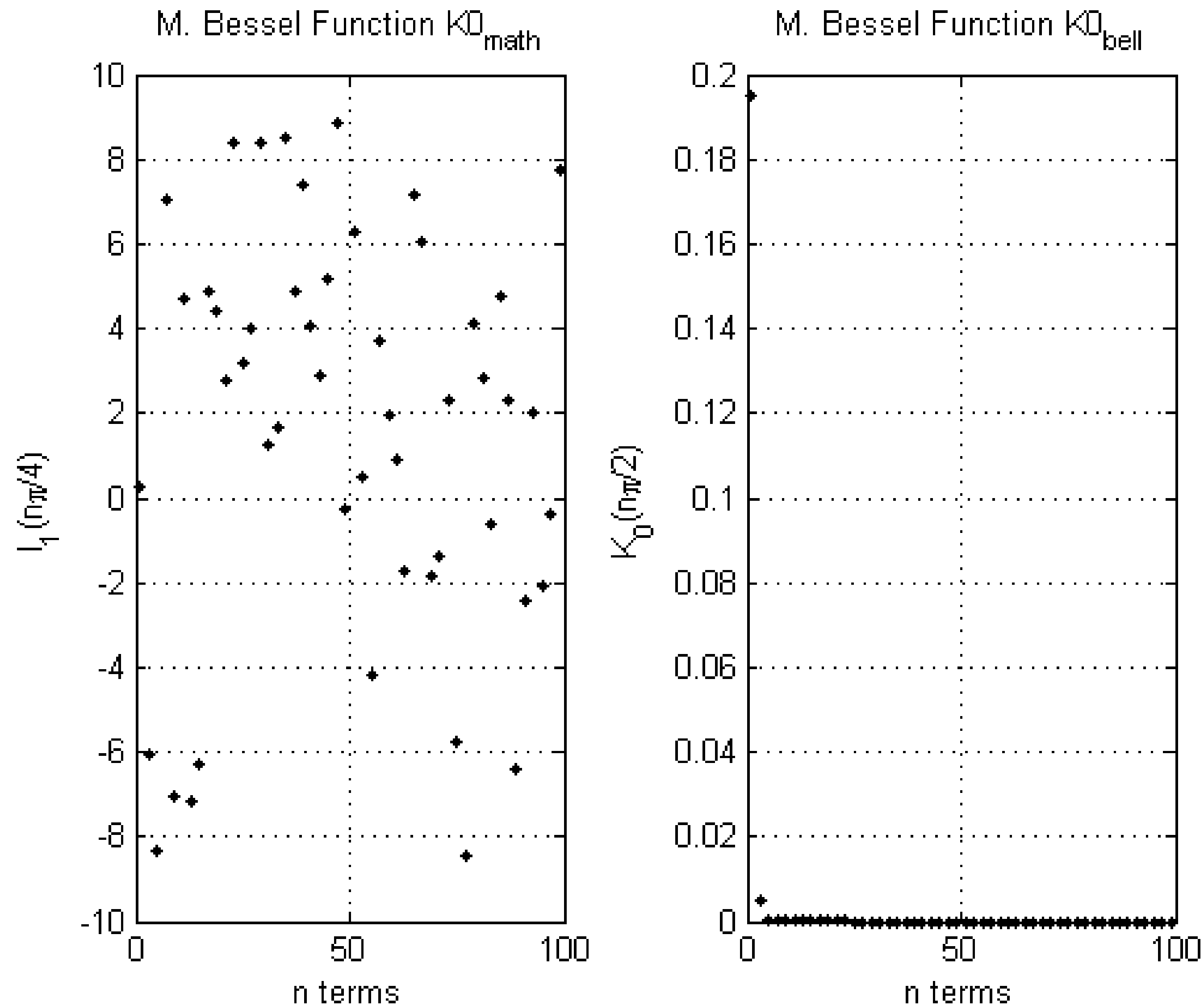
$$\frac{\Phi(0,\frac{L}{2})}{\Phi_{disk}} \rightarrow 0.070893309053610$$

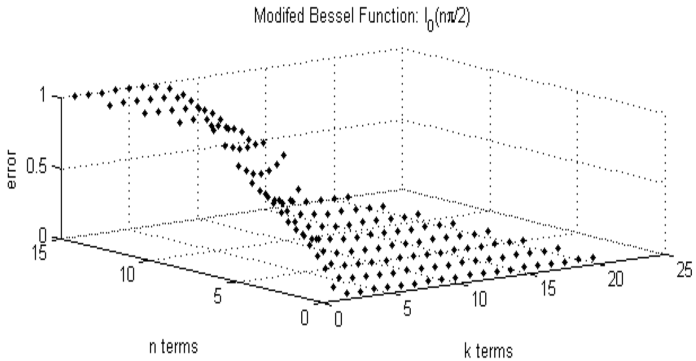


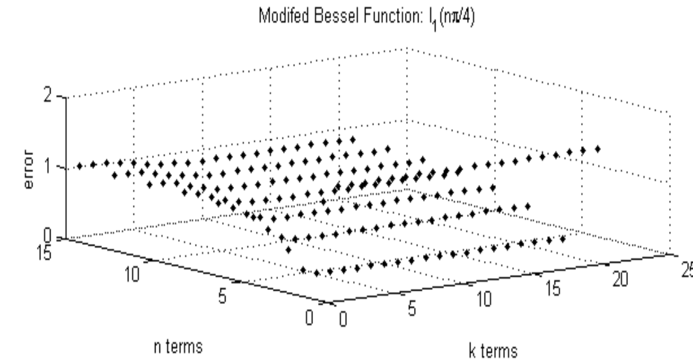
error diff. using $K0_{math}$ (B)
error diff.
n terms
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
0
100
200
300
400
500
error diff. using $K0_{bell}$(B)
x 10^{-6}
error diff.
n terms
8
7
6
5
4
3
2
1
0
0
100
200
300
400
500

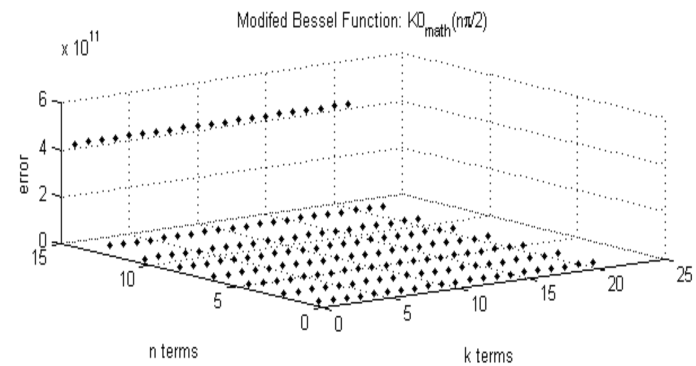
Why the large error?

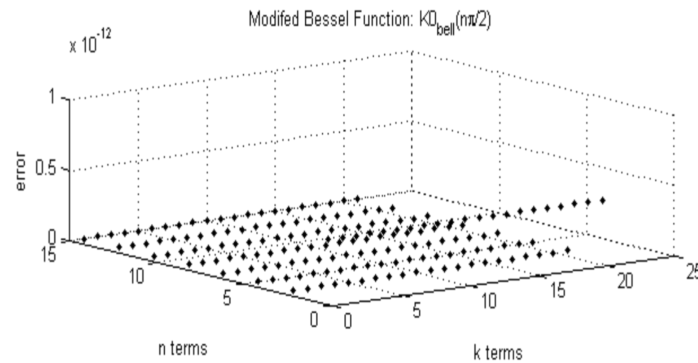


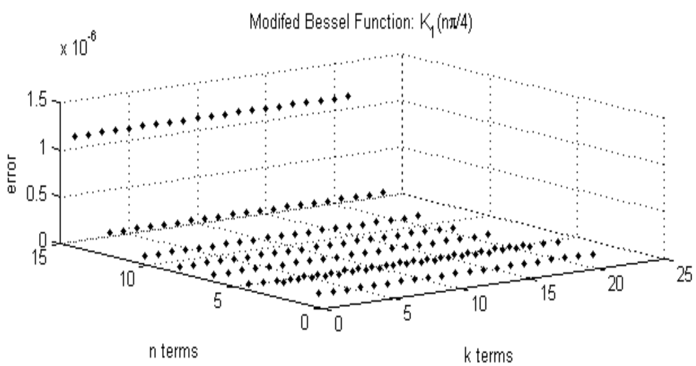

M. Bessel Function $K0_{math}$
$I_1(n\pi/4)$
n terms
M. Bessel Function $K0_{bell}$
$K_0(n\pi/2)$
n terms


Modifed Bessel Function: $I_0(n\pi/2)$
error
1
0.5
0
15
10
5
0
n terms
0
5
10
15
20
25
k terms


Modifed Bessel Function: $I_1(n\pi/4)$
error
2
1
0
15
10
5
0
n terms
0
5
10
15
20
25
k terms


Modifed Bessel Function: $K0_{math}(n\pi/2)$
x 10^{11}
error
6
4
2
0
15
10
5
0
n terms
0
5
10
15
20
25
k terms


Modifed Bessel Function: $K0_{bell}(n\pi/2)$
x 10^{-12}
error
1
0.5
0
15
10
5
0
n terms
0
5
10
15
20
25
k terms


Modifed Bessel Function: $K_1(n\pi/4)$
x 10^{-6}
error
1.5
1
0.5
0
15
10
5
0
n terms
0
5
10
15
20
25
k terms