Optimal Referral Auction

Presentation 2

Current strategies for Auctions

1. Classical Auction - Winner pays his bid, but information is passed to only a few of the possible bidders who are in contact with the seller i.e it is not incentive compatible. (Local Optimisation)

2. Network Auction(VCG mechanism) - Aim to make auctions more widespread and increase information flow among bidders so that maximum bid can be raised.

Terms Related to Network Auctions

1. Truthfulness - Reporting a bid equal to what the bidder actually thinks the product is worth (Not more nor less).
2. Feasible mechanism - A bidder cannot perform an action until he has received information about the auction. (Action set will be a null set.)
3. Efficient Allocation - Maximum value of sum of bids for available products is the bid(s) that win(s).

Definition 3. An allocation π is *efficient* if for all $\mathbf{a}' \in A$,

$$\pi \in \arg\max_{\pi' \in \Pi} \sum_{i \in N_{-s}, a'_i \neq null} \pi'_i(\mathbf{a}')v'_i$$

4. Utility in a network auction is defined as $u_i(a_i, \mathbf{a}', (\pi, p)) = \pi_i(\mathbf{a}')v_i - p_i(\mathbf{a}')$.

5. Individual Rationality - Utility is non negative if she truthfully reports her bid irrespective of who she tells or doesn't tell.

Definition 4. A mechanism (π, p) is *individually rational* (IR) if $u_i(a_i, ((v_i, r_i'), \mathbf{a}'_{-i}), (\pi, p)) \geq 0$ for all $i \in N_{-s}$, all $r_i' \in \mathcal{P}(r_i)$, and all $\mathbf{a}'_{-i} \in A_{-i}^{(v_i, r_i')}$.

6. Incentive Compatibility - Reporting her value truthfully and pssing on the information to all he neighbours is a dominant strategy.

7. Weakly budget balanced - Seller will never be in loss.

Definition 6. A mechanism $\mathcal{M} = (\pi, p)$ is *weakly budget balanced* if for all $\mathbf{a}' \in A$, $Rev^{\mathcal{M}}(\mathbf{a}') \geq 0$.

Diffusion Dominant Strategy Incentive Compatibility

A DA (g,p) on a graph G is DDSIC if

1. *every agent's utility is maximized by reporting her true valuation irrespective of the diffusing status of herself and the other agents, i.e., for every* $i \in N, \forall r_i, \hat{\theta}_{-i}$*, the following holds*

$$v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$$
$$\geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})), \forall v_i, v_i', \hat{\theta}_{-i}, r_i' \subseteq r_i, \text{ and,}$$

2. *for every true valuation, every agent's utility is maximized by diffusing to all its neighbors irrespective of the diffusion status of the other agents, i.e., for every* $i \in N, \forall r_i, \hat{\theta}_{-i}$*, the following holds*

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$$
$$\geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})), \forall v_i, \hat{\theta}_{-i}, r_i' \subseteq r_i.$$

Monotone and Forwarding-Friendliness

(a) the functions $g_i(f^G((v_i, r_i), \hat{\theta}_{-i}))$ are monotone non-decreasing in v_i, for all $r_i, \hat{\theta}_{-i}$, and $i \in N$, and for the given allocation function g, the payment p_i for each player $i \in N$ is such that, for every v_i, r_i, and $\hat{\theta}_{-i}$, the following two conditions hold.

(b) For every $r'_i \subseteq r_i$, the following payment formula is satisfied.

$$p_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) = p_i(f^G((0, r'_i), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r'_i), \hat{\theta}_{-i})) - \int_0^{v_i} g_i(f^G((y, r'_i), \hat{\theta}_{-i}))\,\mathrm{d}y \tag{1}$$

(c) For every $r'_i \subseteq r_i$, the values of $p_i(f^G((0, r'_i), \hat{\theta}_{-i}))$ and $p_i(f^G((0, r_i), \hat{\theta}_{-i}))$ satisfies the following inequality.

$$p_i(f^G((0, r'_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) \geqslant \int_0^{v_i} \Big(g_i(f^G((y, r'_i), \hat{\theta}_{-i})) - g_i(f^G((y, r_i), \hat{\theta}_{-i})) \Big)\,\mathrm{d}y. \tag{2}$$

IC implies MFF - Proof

From IC condition, $v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i}) \geq v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i}) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}) \quad (3)$

Adding and subtracting $v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})$ on the RHS of Eqn. (3), we get:

$$
\begin{aligned}
& v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \\
& \geq v_i' g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \\
\implies & u_i((v_i, r_i), \hat{\theta}_{-i}) \geq u_i((v_i', r_i), \hat{\theta}_{-i}) + (v_i - v_i') g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))
\end{aligned}
$$

From the following equation we can say g_i will be a sub-gradient of the utility function if we can prove the utility function is convex. (Can prove graphicall)

Proving the utility function is convex

For brevity, we use the shorthand $h(v_i) := u_i((v_i, r_i), \hat{\theta}_{-i})$ and $\phi(v_i') := g_i(f^G((v_i', r_i), \hat{\theta}_{-i}))$. Because v_i and v_i' were arbitrary in the above inequality, we can choose arbitrary $x_i, z_i \in \mathcal{V}_i$ and define $y_i = \lambda x_i + (1-\lambda) z_i$ where $\lambda \in [0, 1]$. From the above inequality, we get

$$h(x_i) \geqslant h(y_i) + \phi(y_i)(x_i - y_i) \tag{4}$$

$$h(z_i) \geqslant h(y_i) + \phi(y_i)(z_i - y_i). \tag{5}$$

Multiplying Eqn. (4) by λ and Eqn. (5) by $(1-\lambda)$ and adding, we get $\lambda h(x_i) + (1-\lambda) h(z_i) \geqslant h(y_i)$, which proves that h or the utility u_i is convex, and ϕ or the allocation g_i is its sub-gradient. Since sub-gradient of a convex function is non-decreasing, we get the claimed implication.

IC implies the payment function

For the utility function(h) and its sub-gradient(∅) we have $h(y) = h(z) + \int_z^y \phi(t)dt$

From this we have,

$$u_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = u_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt$$

$$\Rightarrow v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = -p_i(f^G((0, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt$$

$$\Rightarrow p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) = p_i(f^G((0, r_i), \hat{\theta}_{-i})) + v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt$$

Which is condition b) of MFF.

Condition 3 of MFF from IC

Using the fact that IC implies, sharing information with all neighbours is a weakly dominant strategy and then substituting utility function and the then further substituting the payment functions in this equation we get,

$$\int_0^{v_i} g_i(f^G((y,r_i),\hat{\theta}_{-i}))dy - p_i(f((0,r_i),\hat{\theta}_{-i})) \geqslant \int_0^{v_i} g_i(f^G((y,r_i'),\hat{\theta}_{-i}))dy - p_i(f((0,r_i'),\hat{\theta}_{-i}))$$

$$p_i(f((0,r_i'),\hat{\theta}_{-i})) - p_i(f((0,r_i),\hat{\theta}_{-i})) \geqslant \int_0^{v_i} \left(g_i(f^G((y,r_i'),\hat{\theta}_{-i})) - g_i(f^G((y,r_i),\hat{\theta}_{-i}))\right) dy.$$

Which by rearranging the terms is condition 3 of MFF.

MFF implies DDSIC

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \tag{6}$$

$$= v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt,$$

and the utility when she misreports to v_i' is given by

$$v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i})) \tag{7}$$

$$= v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((0, r_i), \hat{\theta}_{-i})) - v_i' g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) + \int_0^{v_i'} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt.$$

Subtracting Eqn. (7) from Eqn. (6), we get

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - [v_i g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i), \hat{\theta}_{-i}))]$$

$$= (v_i' - v_i) g_i(f^G((v_i', r_i), \hat{\theta}_{-i})) + \int_{v_i'}^{v_i} g_i(f^G((t, r_i), \hat{\theta}_{-i}))dt. \tag{8}$$

Since u_i, is convex, we get that the RHS will always be non-negative.

From this payment function and eq.2 of MFF, by manipulation we get condition 3 of DDSIC as well.

DDSIC implies IC, This gives MFF,DDSIC,IC are equivalent

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i}))$$

$$\forall v_i, \hat{\theta}_{-i}, \forall i \in N,$$

and from condition 1 of Def. 2, we get

$$v_i g_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i'), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})),$$
$$\forall v_i, v_i', \hat{\theta}_{-i}, \forall i \in N.$$

Combining these two, we have

$$v_i g_i(f^G((v_i, r_i), \hat{\theta}_{-i})) - p_i(f^G((v_i, r_i), \hat{\theta}_{-i})) \geqslant v_i g_i(f^G((v_i', r_i'), \hat{\theta}_{-i})) - p_i(f^G((v_i', r_i'), \hat{\theta}_{-i})),$$
$$\forall v_i, v_i', \hat{\theta}_{-i}, \forall i \in N,$$

Algorithm 1: `LbLEV`

Input: reported types $\hat{\theta}_i = (\hat{v}_i, \hat{r}_i), \hat{r}_i \subseteq r_i, \ \forall i \in N$
Output: winner of the auction (which can be $\emptyset$), payments of each agent

1 **Preprocessing:** Since the underlying graph is a tree, let $\hat{T}$ be the sub-tree rooted at s induced from $\hat{r}_i, i \in N$. Pick $t \in \mathbb{R}^n_{>0}$ independent of the input
2 **if** $\hat{v}_i = 0, \forall\, i \in N$ **then**
3   Item is not sold and payment is set to zero for all agents, STOP
4 *Initialization:* all agents are `non-winners` and their `actual payments` are zeros, set $\texttt{offset} = 0$, $\texttt{level} = 1$, $\texttt{parent} = s$, $v_{\texttt{parent}} = 0$
5 In this level of $\hat{T}$:
6   **for** *each node* $i \in \texttt{children}(\texttt{parent})$ **do**
7   Set `effective valuation` $\rho_i := \max\{\hat{v}_j : j \in \hat{T}_i\} - \texttt{offset}$
8   Remove the nodes that have $\rho_i < 0$, denote the rest of the agents with $N_{\texttt{remain}}$
9   **if** $|N_{\texttt{remain}}| \geqslant 2$ **then**
10   Sort the nodes in decreasing order of $\rho_i^{t_i}$
11   Compute $z := \rho_\ell^{t_\ell / t_{i^*}}$, where i^* is the highest in this order and ℓ is the second highest node in the decreasing $\rho_i^{t_i}$ order
12   **else**
13   Set $z = 0$
14   **if** $v_{\texttt{parent}} \geqslant \texttt{offset} + z$ **then**
15   STOP and go to Step 23
16   Set the highest node i^* in this order as the `tentative winner` and its `effective payment` to be z
17   All nodes and their subtrees except i^* are declared `non-winners`
18   The `actual payment` of i^* to `parent` = `effective payment` + `offset`
19   `parent` = i^*, `offset` = `actual payment` of i^*
20   `level` = `level` + 1
21 Repeat Steps 5 to 20 with the updated `parent` and `offset` for the new level
22 STOP when no agent i has $\rho_i \geqslant 0$ OR the leaf nodes are reached
23 Set `tentative winner` as `final winner`; final payments are the `actual payments` that are paid to the respective parents of $\hat{T}$

LbLEV

Three types of buyer (like IDM as IDM is a special case of LbLEV)

1. Winner
2. On path winner
3. Not on path non-winner

We prove IC of LbLEV by satisfying conditions of MFF.

Proof - Condition a)

Case 1: agent i is a *not-on-path non-winner*: in this case, $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0$. From the description of LbLEV, it is clear that for $v_i' > v_i$, either agent i can remain a *not-on-path non-winner*, or it can become a *winner*. It cannot become an *on-path non-winner* because it would imply that there was another agent in i's subtree that had a maximum valuation in this network at agent i's original valuation v_i and then agent i could not be a *not-on-path non-winner*. In both the cases where agent i is *not-on-path non-winner* or *winner*, $g_i(f^G(v_i', r_i), \hat{\theta}_{-i}) \geqslant 0$, hence it is monotone non-decreasing.

Case 2: agent i is an *on-path non-winner*: the allocation for agent i is $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 0$ in this case as well. This agent is *on-path non-winner* with bid v_i implies that there is an agent in $\hat{T}_i$ that has reported the winning bid. Now, if agent i bids v_i' which is higher than v_i, it can either continue to be an *on-path non-winner* or may become the new *winner* at a sufficiently high bid. In both these cases, the allocation probability is monotone non-decreasing.

Case 3: agent i is the *winner*: here $g_i(f^G(v_i, r_i), \hat{\theta}_{-i}) = 1$. We need only to show that for all $v_i' > v_i$, agent i continues to be the winner. This is fairly easy to see from Alg. 1. An agent can be the winner either when it is the parent node in line 14 or line 22.

Proof - Condition 2 of MFF

Not-on- non-winner : $VIPC_i$ turns out to be 0 by substituting the other 3 terms in the payment equation, which in turn satisfies condition c) trivially.

On-path non-winner : We find the $VIPC_i$ in this case to be $-\rho_\ell^{t_\ell/t_k}$. , which can be shown to be independent of v_i , by splitting p_l into v_l - offset(i) and showing both of these terms are independent of v_i.

Winner : Divide into 2 cases, step 14 and step 22. Simplify the last 2 terms into k_i, where k_i is the critical point at which i becomes the winner and satisfy the payment conditions according to the above cases to obtain $VIPC_i$ (Not sure about leaf node condition)

Condition 3 of MFF

For not-on-path non-winners and winners condition c) is trivially satisfied as both the RHS and LHS become 0 as g_i and $VIPC_i$ in these cases do not change with r_i.

- When i becomes a *not-on-path non-winner* by diffusing to $r_i' \subset r_i$, its utility becomes 0 (see *Case 1* of Part 2 above). However, an *on-path non-winner* draws utility $R_i - \pi(\hat{T}_i)$ where $R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k} = \pi(\hat{T}_i) + \rho_\ell^{t_\ell/t_k}$ (see *Case 2* of Part 2 above). As $\rho_\ell \geqslant 0$, hence utility being an *on-path non-winner* is non-negative and makes diffusion to r_i a weakly better option for i.
- In the other case, when i becomes a *winner* by strategic forwarding, its utility becomes $v_i - \pi(\hat{T}_i)$. Note that i is an *on-path non-winner* because it failed the `if` condition in line 14, hence,

$$v_i < \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i). \tag{12}$$

Now, as an *on-path non-winner*, i's utility is $R_i - \pi(\hat{T}_i)$ (since the allocation probability of i is zero by Eqn. (10)). However, the actual payment by $\texttt{children}(i)$ to i is

$$R_i = \texttt{offset}(i) + \rho_\ell^{t_\ell/t_k}, \text{ where } k = \texttt{winner}(i), \ell = \texttt{runnerup}(i). \tag{13}$$

Therefore, from Eqns. (12) and (13) we get, $v_i < R_i$. Agent i pays $\pi(\hat{T}_i)$ to its parent regardless of whether it forwards to r_i or not. Therefore, if agent i forwards to r_i, it gets an utility of $R_i - \pi(\hat{T}_i) = \rho_\ell^{t_\ell/t_k} \geqslant 0$ which is larger than the utility $v_i - \pi(\hat{T}_i)$ of i when it diffuses to some $r_i' \subset r_i$ and becomes the *winner*. Therefore, in this case as well, forwarding to r_i is better than any partial forwarding to $r_i' \subset r_i$ for agent i.

LbLEV is IR

Not on-path non-winner : Has an allocation probability of 0. Hence its payment is 0.

On path non-winner : Always has a utility of $-\rho_\ell^{t_\ell/t_k}$. which is always greater than equal to 0 as p_l is always greater than equal to 0.

Winner : We prove p_i^t_i >= p_j^t_j and then prove that v_i > p(i) as v_i for a winner is offset(parent(i)) + p_i

LbLEV vs IDM

As LbLEV mechanism uses expected values to determine the value of t_i , hence if the true winner and runner-up aren't from the same sub-trees with respect to the expected value choosing λ to be 1 would be suboptimal compared to IDM. By conducting experiments we get that λ is a function of n and σ. When both n and σ are very large LbLEV decreases to IDM.

Bayesian setup and Optimal Auction

Optimal RA for i.i.d valuations

Transformed Auction :

Definition 9 (Transformed Auction) *A* transformed auction *(*$\mathtt{TA}$*) of an* $\mathtt{RA}$ *is the auction where each subtree* $\hat{T}_i, i \in \mathtt{children}(s)$ *is replaced with a node with a valuation of* $\max_{j \in \hat{T}_i} v_j$*, and the allocation and payments are given by* $(\mathcal{g}_i, \mathcal{p}_i), i \in \mathtt{children}(s)$.

Revenue earned by RA is identical to its TA.

Proof: Note that in an $\mathtt{RA}$, the net payment received by the seller s comes directly from the nodes in the first level of the tree. The `offset` is zero, and the payment is calculated based on the maximum valuation in the subtree of the agents in the first level. The rest of the payments in the tree are internally adjusted within the nodes and does not reach the seller. Therefore, the total revenue earned by an $\mathtt{RA}$ can be simulated by transforming every first-level nodes with their valuations replaced with the maximum valuation of their subtree and applying $(\mathcal{g}, \mathcal{p})$ on those nodes. Hence, we have the lemma. ■

The maximum of a finite number of i.i.d variables, each of which satisfy the MHR condition, also satisfies the MHR condition.

Proof: Suppose, there are n i.i.d. random variables given by $X_1, X_2, \ldots, X_n$, and their distribution is given by F. Let $Y := \max\{X_1, X_2, \ldots, X_n\}$. It is given that F satisfies MHR condition, i.e., $f(x)/(1-F(x))$ is monotone non-decreasing. Denote the distribution and density of Y by $\tilde{F}$ and $\tilde{f}$ respectively. Now,

$$\begin{aligned}\tilde{F}(x) &= P(Y \leqslant x) = P(\max\{X_1, X_2, \ldots, X_n\} \leqslant x) \\ &= P(\cap_{i=1}^{n}\{X_i \leqslant x\}) = \prod_{i=1}^{n} P(X_i \leqslant x) = F^n(x).\end{aligned}$$

$$\text{Hence, } \tilde{f}(x) = nF^{n-1}(x)f(x).$$

$$\begin{aligned}\text{Therefore, } \frac{\tilde{f}(x)}{1-\tilde{F}(x)} &= \frac{nF^{n-1}(x)f(x)}{1-F^n(x)} \\ &= \frac{f(x)}{1-F(x)} \cdot \left(\frac{n}{1+\frac{1}{F(x)}+\frac{1}{F^2(x)}+\ldots+\frac{1}{F^{n-1}(x)}}\right).\end{aligned}$$

Since $F(x)$ is non-decreasing, $1/F(x)$ is non-increasing. Hence, the denominator of the last term in the last expression is also non-increasing, leading the expression to be non-decreasing. Since $\frac{f(x)}{1-F(x)}$ is non-decreasing as well, we conclude that $\frac{\tilde{f}(x)}{1-\tilde{F}(x)}$ is also non-decreasing and hence Y satisfies

Maximising Revenue

To maximise revenue, VIPC is set to 0 and as it is a transformed auction $p_l = v_l$. Hence we have to maximise the following function :

$$\begin{aligned} \max \quad & \sum_{\ell \in \tilde{N}} \int_{v_\ell = 0}^{b_\ell} \mathtt{pay}_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ \text{s.t.} \quad & \mathcal{g} \text{ is monotone non-decreasing and deterministic} \end{aligned}$$

Which by change of limits, reduces to

$$\begin{aligned} & \int_{v_\ell = 0}^{b_\ell} \mathtt{pay}_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ &= \int_0^{b_\ell} w_\ell(v_\ell) \alpha_\ell(v_\ell) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ &= \int_0^{b_\ell} w_\ell(v_\ell) \left(\int_{v_{-\ell}} \mathcal{g}_\ell(v_\ell, v_{-\ell}) f_{-\ell}(v_{-\ell})\, \mathrm{d}v_{-\ell} \right) f_\ell(v_\ell)\, \mathrm{d}v_\ell \\ &= \int_v w_\ell(v_\ell) \mathcal{g}_\ell(v_\ell, v_{-\ell}) f(v)\, \mathrm{d}v. \end{aligned}$$

The objective function of Eqn. (19) can therefore be written as

$$\int_v \left(\sum_{\ell \in \tilde{N}} w_\ell(v_\ell) g_\ell(v_\ell, v_{-\ell}) \right) f(v)\, \mathrm{d}v.$$

The solution to the unconstrained version of the optimization problem given by Eqn. (19) is rather simple.

$$\begin{aligned} &\text{if } w_\ell(v_\ell) < 0, \forall \ell \in \tilde{N}, \text{ then } g_\ell(v_\ell, v_{-\ell}) = 0, \forall \ell \in \tilde{N} \\ &\text{else } g_\ell(v_\ell, v_{-\ell}) = \begin{cases} 1 & \text{if } w_\ell(v_\ell) \geqslant w_k(v_k), \forall k \in \tilde{N} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{20}$$

The ties in $w_\ell(v_\ell)$ are broken arbitrarily. Since the distributions of $v_\ell, \ell \in \tilde{N}$ satisfy MHR, the virtual valuations, w_ℓ, are monotone non-decreasing. Also, since this mechanism breaks the tie arbitrarily in favor of an agent, the allocation is also deterministic. Therefore, the optimal solution of the unconstrained problem of Eqn. (19) also happens to be the optimal solution of the constrained problem. We find the payments of the winner from Eqn. (18) as follows.

$$\begin{aligned} \text{define } \kappa_\ell^*(v_{-\ell}) &= \inf\{y : g_\ell(y, v_{-\ell}) = 1\}, \\ p_\ell(v_\ell, v_{-\ell}) &= \kappa_\ell^*(v_{-\ell}) \cdot g_\ell(v_\ell, v_{-\ell}), \end{aligned} \tag{21}$$

where $\kappa_\ell^*(v_{-\ell})$ is the minimum valuation of agent ℓ to become the winner. Formally, we define the auction as follows.

Maximum Virtual Valuation Auction(maxViVa)

Definition 10 (Maximum Virtual Valuation Auction (`maxViVa`)) *The* maximum virtual valuation *auction is a subclass of* `RA`*, where the* `TA`*s of that subclass follow the allocation and payments given by Eqns.* (20) *and* (21) *respectively.*

We consolidate the arguments above in the form of the following theorem.

Theorem 9 *For agents having i.i.d. MHR valuations, the revenue-optimal* `RA` *is* `maxViVa`.

Since multiple `RA`s can reduce to the same `TA`, the revenue optimal `RA` is a class of auctions, all belonging to `RA`, that has the same `TA` given by Def. 10. Note that neither IDM nor `LbLEV` is `maxViVa` because they do not use any priors. Therefore, the revenue-maximizing auctions in this setting are a new class of mechanisms that have not been explored in the literature.